A survey of IBN property of Leavitt Path Algebras

A thesis submitted to
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by
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Certificate

This is to certify that this thesis entitled "*A survey of IBN property of Leavitt Path Algebras*" submitted towards the partial fulfilment of the Mathematics M.Sc Degree Program at the Indian Institute of Science Education and Research Pune represents work carried out by *Drishti Sunder Phukon* under the supervision of *Dr. Rabeya Basu.*

Dr. Rabeya Basu
Master's Thesis Supervisor

Declaration

I hereby declare that the matter embodied in the report entitled A survey of IBN property of Leavitt Path Algebras are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research (IISER) Pune, under the supervision of Dr. Rabeya Basu and the same has not been submitted elsewhere for any other degree.

Drishti Sunder Phukon

Drishti Sunder Phukon

This thesis is dedicated to my parents

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Abstract

This thesis presents a survey of the Invariant Basis Number (IBN) property within the context of Leavitt Path Algebras (LPAs), which lie at the intersection of ring theory, graph theory, and noncommutative algebra. Originating from W.G. Leavitt's work on rings lacking the IBN property, LPAs are constructed from directed graphs and exhibit rich structural properties linked to their underlying graphs. The primary focus is investigating the conditions under which an LPA possesses the IBN property. We explore the fundamental structure, definitions, and key examples of LPAs, contrasting them with related concepts like Cohn path algebras. Utilizing monoid-theoretic techniques (specifically the graph monoid M_E and its group completion) and matrix-based formulations derived from the graph's incidence matrix (following the work of T.G. Nam and N.T. Phuc), we establish explicit criteria for determining if $L_k(E)$ satisfies IBN property. These criteria are then applied to analyze the IBN property for LPAs associated with specific graph constructions arising from finite groups. We examine Cayley graphs, demonstrating that their LPAs have IBN if and only if the generating set for the group contains a single element. Furthermore, we investigate the power graphs of cyclic groups of prime power order ($\mathbb{Z}_{p^m}$). This work aims to provide a self-contained exploration of the IBN property for Leavitt path algebras, combining theoretical developments with concrete examples and graphical constructions, primarily based on the findings presented in Nam and Phuc (2019) [9].

Contents

1 Introduction

The study of Leavitt path algebras lies at a fascinating intersection of ring theory, graph theory, and noncommutative algebra. Originating from the groundbreaking work of W.G. Leavitt in the 1960s, these algebras provide concrete examples of rings without the Invariant Basis Number (IBN) property—an essential concept in module theory. Constructed from directed graphs, Leavitt path algebras exhibit a rich blend of algebraic and combinatorial structure, offering insights into both abstract algebra and the underlying graph-theoretic data.

Leavitt path algebras have attracted increasing interest in recent years due to their connections with symbolic dynamics, C*-algebras, and K-theory, and because they often serve as test cases or counterexamples in ring-theoretic contexts. Their construction involves real and ghost edges, and is governed by Cuntz–Krieger relations, which make these algebras particularly amenable to analysis through graphical techniques.

This thesis investigates the fundamental structure of Leavitt path algebras, including their universal properties, examples, and algebraic behavior. A significant focus is placed on the Invariant Basis Number property, with an emphasis on how certain graph-theoretic configurations lead to the failure or preservation of this property. In particular, we study how algebraic constructions such as Cohn path algebras and graphs arising from finite groups influence IBN behavior. Additionally, the thesis explores how monoid-theoretic techniques and matrix-based formulations can be used to analyze and classify these algebras.

By combining concrete examples, theoretical developments, and graphical constructions, this work aims to provide a self-contained exploration of Leavitt path algebras and their role in understanding deeper questions in noncommutative algebra. This thesis is primarily based on a paper by TG Nam [9].

2 Motivations and definitions

This chapter contains the basic definitions required for Leavitt path algebras. Also it includes the motivation behind the construction of such algebras.

2.1 The Leavitt algebras

Students are typically introduced to rings through fundamental examples such as fields, polynomial ring over a field and the set of integers, matrix rings over fields. A shared feature among these rings is the Invariant Basis Number (IBN) property.

Definition 2.1.1. IBN: A ring R has the Invariant Basis Number property if, for any positive integers m and n, the isomorphism of free left R-modules $R^m \cong R^n$ implies that $m = n$.

A ring has the IBN property (or is IBN) if every pair of bases of a finitely generated free left R-module contain the same number of elements. Many familiar types of rings, such as noetherian and commutative rings, satisfy this property. This includes the standard examples typically introduced to students, like the field of real numbers, which has IBN property as it is a field.

Lemma 2.1.2. *A ring A is IBN if*

1. *A is commutative.*
2. *A is local.*
3. *$A \neq 0$ and is noetherian.*

Proof.
1. Suppose $A^m \cong A^n$ for some positive integers m and n. Let $\mathfrak{p}$ be a maximal ideal of A, then the quotient $A/\mathfrak{p} = k$ is a field. Since tensor product distributes over direct sums, we have:

$$k^m \cong (A/\mathfrak{p})^m \cong (A/\mathfrak{p})^n \cong k^n$$

   Therefore, as isomorphic vector spaces over a field must have equal dimensions, we conclude $m = n$.

2. Let $A^m \cong A^n$. If A is a local ring with maximal ideal $\mathfrak{p}$, then $A/\mathfrak{p}$ is a division ring D. Since division rings have the IBN, applying the same reasoning as 1 gives $D^m \cong D^n$, and thus $m = n$.

3. Assume without loss of generality that $m \geq n$ and that there is an isomorphism $f : A^n \to A^m$. Consider the natural projection map $\pi : A^m \to A^n$. Then $f \circ \pi : A^m \to A^m$ is a surjective A-module endomorphism. As A is noetherian, this implies $f \circ \pi$ is also injective, and hence an isomorphism. Therefore, $\ker(f \circ \pi) = \{0\}$, which gives $\ker(\pi) = \{0\}$, so $m \leq n$, which implies $m = n$.

□

Unfortunately since during the early development of this subject, such rings are introduced, it leaves a wrong impression that all the rings have IBN property. An example of such a non-IBN ring.

Let K be a field, and let $\mathbb{K}^{(\mathbb{N})}$ represent the infinite direct sum

$$\mathbb{K}^{(\mathbb{N})} = \mathbb{K} \oplus \mathbb{K} \oplus \cdots \oplus \mathbb{K} \oplus \cdots$$

Then taking $R = \mathrm{End}_{\mathbb{K}}(\mathbb{K}^{(\mathbb{N})})$ we find

$$R^2 \cong \mathrm{End}_{\mathbb{K}}(\mathbb{K}^{(\mathbb{N})}) \oplus \mathrm{End}_{\mathbb{K}}(\mathbb{K}^{(\mathbb{N})}) \cong \mathrm{Hom}(\mathbb{K}^{(\mathbb{N})}, \mathbb{K}^{(\mathbb{N})}) \oplus \mathrm{Hom}(\mathbb{K}^{(\mathbb{N})}, \mathbb{K}^{(\mathbb{N})})$$

$$\cong \mathrm{Hom}(\mathbb{K}^{\mathbb{N}}, \mathbb{K}^{\mathbb{N}} \oplus \mathbb{K}^{\mathbb{N}}) \cong \mathrm{Hom}(\mathbb{K}^{\mathbb{N}}, \mathbb{K} \oplus \mathbb{K} \oplus \cdots \oplus \mathbb{K} \oplus \cdots \oplus \mathbb{K} \oplus \mathbb{K} \oplus \cdots \oplus \mathbb{K} \oplus \cdots)$$

$$\cong \mathrm{Hom}(\mathbb{K}^{\mathbb{N}}, \mathbb{K}^{\mathbb{N}}) \cong R$$

So, $R^2 \cong R$. And in fact using the same technique we can see that $R^m \cong R^n$ **for all** $n, m \in \mathbb{Z}^+$.

Notation 2.1.3. For a ring R without IBN, let m be the smallest natural number such that $R^m \cong R^l$ for some $l > m$. Then, for this m, let $n > m$ be the smallest natural number so that $R^m \cong R^n$. Then, R is said to have a module type (m, n).

Example 2.1.4. *$End_R(\mathbb{F}^{(\mathbb{N})})$ has module type* $(1, 2)$.

2.2 Rings of type $(\mathbf{1}, \mathbf{n})$ without Invariant Basis Number

The structure of rings without IBN can be quite complex, but for rings of type $(1, n)$, where $n > 1$, the analysis of their structure is relatively straightforward.

Let R be a ring without the IBN property of type $(1, n)$, with $n > 1$. Then $R \cong R^n$. This implies the existence of isomorphisms of free modules

$$\psi \in \mathrm{Hom}(R^n, R) \text{ and } \phi \in \mathrm{Hom}(R, R^n)$$

such that

$$\psi \circ \phi = \mathrm{Id}_{R^n} \text{ and } \phi \circ \psi = \mathrm{Id}_R$$

Then by using the matrix representation of a homomorphism for a unital ring, we get $n \times 1$ and $1 \times n$ vectors over R

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \text{ and } \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

such that

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} . \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} = \begin{pmatrix} 1_R & 0 & \cdots & 0 \\ 0 & 1_R & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1_R \end{pmatrix} \text{ and } \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} . \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = (1_R)$$

i.e. $R \cong R^n$ if and only if there are $2n$ elements of R such that

$$y_i x_i = \delta_{ij} 1_R \text{ (for all } 1 \leq i, j \leq n) \text{ and } \sum_{i=1}^{n} x_i y_i = 1_R$$

This actually characterises all the algebras of type $(1, n)$. Let k be a field, and let

$$S = k\langle X_1, \ldots, X_n, Y_1, \ldots, Y_n \rangle$$

Let T be the free associative k-algebra in $2n$ variables that do not commute. Let J denote the ideal of T generated by the relations.

$$J = \langle \sum_{i=1}^{n} X_i Y_i - 1,\ Y_i X_j - \delta_{ij} 1 \ : \ 1 \leq i, j \leq n \rangle$$

and let

$$A = T/J$$

Thus, $\{x_i = \overline{X_i},\ y_j = \overline{Y_j} \mid i, j = 1, 2, \ldots, n\}$ functions as intended in the construction, ensuring that $A \cong A^n$ as left A-modules.

For the ring $R = \mathrm{End}_k(\mathbb{F}^{(\mathbb{N})})$ of type $(1, 2)$, we can easily to identify a set of 4 elements, namely $\{x_1, x_2, y_1, y_2\}$, in R that exhibit this behavior.

Leavitt proved the following groundbreaking and fundamental result.

Theorem 2.2.1. *For any positive integers $n > m$ and a field k, there is a unique unital k-algebra $L_k(m, n)$, up to k-algebra isomorphism, such that:*

1. *$L_k(m, n)$ has module type (m, n), and*

2. *for every k-algebra A with identity with module type (m, n), a unit-preserving k-algebra homomorphism $\phi : L_k(m, n) \to A$ exists, which satisfies certain compatibility requirements.*

Definition 2.2.2. Let k be a field and $n > 1$ be an integer. The *Leavitt k-algebra of type* $(1, n)$, denoted by $L_k(1, n)$, is the quotient of the k-algebra

$$k\langle X_1, \ldots, X_n, Y_1, \ldots, Y_n\rangle / \langle \sum_{i=1}^{n} X_i Y_i - 1,\ Y_i X_j - \delta_{ij} 1 \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n\rangle$$

.

2.3 Directed graph

Definition 2.3.1. A *directed graph* $E = (E_0, E_1, r, s)$ is defined by two sets, E^0 and E^1, along with two functions $r, s : E^1 \to E^0$. The elements of E^0 are referred to as vertices, while the elements of E^1 are called edges. The function r maps each edge to its target vertex, and s maps it to its source vertex. Throughout this thesis, the term 'a graph' will refer to a directed graph unless otherwise specified.

The next diagram is an example of a directed graph. The direction of the arrow signifies the direction of the edge.

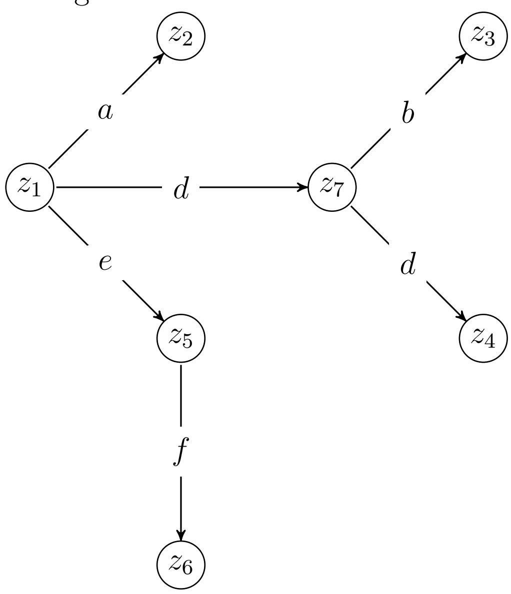


Figure 1: a directed graph

In this graph

$$E^0 = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7\}$$
$$E^1 = \{a, b, c, d, e, f\}$$

And some relations like

$$r(a) = x_2,\ s(c) = x_7,\ s(a) = x_1,\ r(d) = x_7$$

Definition 2.3.2. 1. **Sink**: A vertex v is termed a sink if no edges originate from it, meaning

$$s^{-1}(v) = \emptyset$$

The collection of all sink vertices is represented as $\text{Sink}(E)$.

2. **Source**: A vertex v is called a source if no edges terminate at it, i.e.

$$r^{-1}(v) = \emptyset$$

The collection of source vertices is denoted by $\text{Source}(E)$.

3. **Isolated vertex**: A vertex that is both a sink and a source is called isolated.

4. **Infinite emitter**: A vertex v is an infinite emitter if an infinite number of edges emanate from it, i.e.

$$|s^{-1}(v)| = \infty$$

The collection of vertices that are infinite emitters is denoted by $\text{Inf}(E)$.

5. **Regular vertex**: A vertex v is considered regular if it is neither a sink nor an infinite emitter, i.e.

$$0 < |s^{-1}(v)| < \infty$$

$\text{Reg}(E)$ is the set of regular vertices.

6. **Singular vertex**: A vertex that is not regular is referred to as a singular vertex.

The following notation means that n edges going from the vertex x to the vertex y.

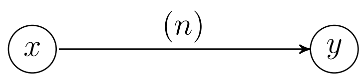


Figure 2: n edges between vertices

For the following graph, we can see the different terms we just defined.

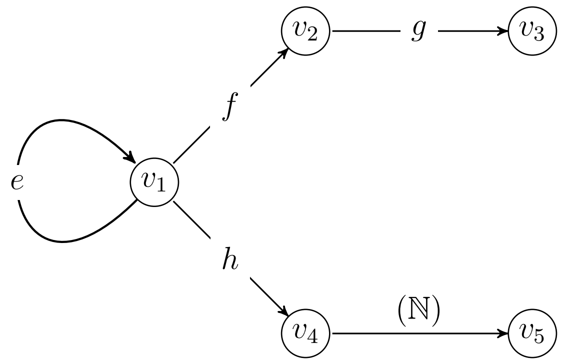


Figure 3: a directed graph

In this graph

$$E^0 = \{v_1, v_2, v_3, v_4, v_5\}$$

$$E^1 = \{e, f, g, h, \text{(all the infinite edges from } v_4 \text{ to } v_5)\}$$

$$\text{Sink(E)} = \{v_3, v_4\}$$

$$\text{Source(E)} = \phi$$

$$\text{Inf(E)} = \{v_4\}$$

$$\text{Reg(E)} = \{v_1, v_2\}$$

$$\text{Singular vertex} = \{v_3, v_4, v_5\}$$

Definition 2.3.3. A *path* in a graph E is a sequence of edges, denoted $p = e_1e_2 \ldots e_n$, where each edge e_i is connected to the next through the relationship $r(e_i) = s(e_{i+1})$ for all $i = 1, 2, \ldots, n-1$.

1. The path p originates at the vertex $s(p) = s(e_1)$ and terminates at $r(p) := r(e_n)$.

2. The length of the path p is denoted by $|p| := n$, the number of edges in the path.

3. $p^0 = \{s(e_i), r(e_i) \mid i = 1, 2, \ldots, n\}$ is the set of vertices associated with the path p.

4. A path p is considered *closed* if it originates and ends at the same vertex, i.e., $s(p) = r(p)$. If this closed path does not revisit any vertex except for the starting point, it is called a *cycle*.

A cycle f is classified as a *source cycle* if for every vertex $v \in f^0$, the condition $|r^{-1}(v)| = 1$ holds true.

A graph without any cycles is called *acyclic*.

For $m \geq 2$, let E^m be the collection of all paths of the graph of length m. Let Path(E) denote all the paths in the graph. Then

$$\text{Path}(E) = \bigcup_{n \geq 0} E^n$$

For a path $e_1 e_2 \dots e_n$,

$$r(e_1 e_2 \dots e_n) = e_n \text{ and } s(e_1 e_2 \dots e_n) = e_1$$

3 Leavitt Path Algebras

3.1 Formation of Leavitt Path Algebra

Now that we have established all the necessary definitions and notations, we have the necessary definitions to define Leavitt Path Algebras.

Definition 3.1.1. (Leavitt Path Algebras) Let E be an arbitrary directed graph and k any field. We define a set

$$(E^1)^* = \{e^* \mid e \in E^1\}$$

The *Leavitt path algebra* of E with coefficients in k, denoted by $L_k(E)$ is the free associative k-algebra generated by the set $E^0 \cup E^1 \cup (E^1)^*$ constrained by the relations:

$$\begin{aligned}
(V) \quad & vv' = \delta_{vv'}v \text{ for all } v \in E^0 \\
(E1) \quad & s(e)e = e = er(e) \text{ for all } e \in E^1 \\
(E2) \quad & r(e)e^* = e^* = e^*s(e) \text{ for all } e \in E^1 \\
(CK1) \quad & e^*e' = \delta_{ee'}r(e) \text{ for all } e, e' \in E^1 \\
(CK2) \quad & v = \sum_{\{e \in E^1 \mid s(e) = v\}} ee^* \text{ for all } v \in \mathrm{Reg}(E)
\end{aligned}$$

In essence, $L_k(E)$ can be described as the free associative k-algebra built from the set of generators $E^0 \cup E^1 \cup (E^1)^*$, where the construction is constrained by the five defining relations previously established, which encode the combinatorial structure of the graph E into the algebra.

For the remainder of the thesis, k will denote a field and E a directed graph, unless otherwise stated.

Leavitt path algebra defined in this way may seem complex at first. So here are some ground rules that can make this construction easier to understand.

1. By *ghost edges*, we mean the set $(E^1)^*$, while real edges mean the set of edges in E^1. Essentially the edges in $(E_1)^*$ are the same edges in E^1 but their direction is reversed.

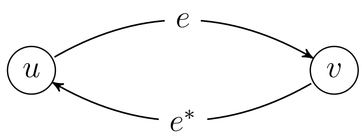


Figure 4: real edge and ghost edge

2. The vertices are orthogonally idempotent, i.e., if u and v are distinct vertices, then $uv = 0$. And for any vertex v in E^0, $v^2 = v$.

3. The multiplication of a vertex and an edge is nonzero and results in an edge only if the vertex is either the source or the target of the edge. The main For example, the previous Fig 4,

$$ue = e = ev$$

This same rule applies for a ghost edge as well.

$$ve^* = e^* = e^*u$$

And the multiplication of an edge with a vertex that is not its starting or ending vertex give zero since if e is an edge and v is a vertex such that is neither its starting nor ending point, i.e. $r(e) \neq v$ and $s(e) \neq v$, then

$$ev = (er(e))v = e(r(e)v) = e0 = 0 \text{ and } ve = v(s(e)e) = (vs(e))e = 0e = 0$$

4. The multiplication of a ghost edge with its real edge with having the ghost edge on the left yields the range of e, i.e. in Fig 4

$$e^*e = v$$

However the multiplication of them with the real edge on the left does not simplify, it remains as it is. And the multiplication of a ghost edge e^* (of a real edge e) with another real edge f yields 0, that is,

$$e^*f = 0$$

For example in

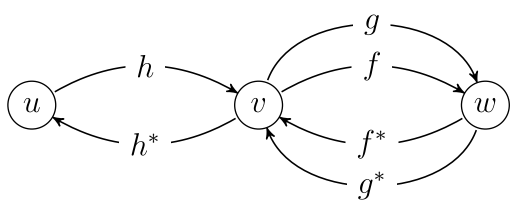


Figure 5: a directed graph

we have

$$f^*f = w \text{ and } f^*g = 0$$

5. The final rule is relatively straightforward. For example in Fig 5, we have for the regular vertex v,

$$v = f^*f + g^*g$$

The following is an example of the properties. Consider the graph E

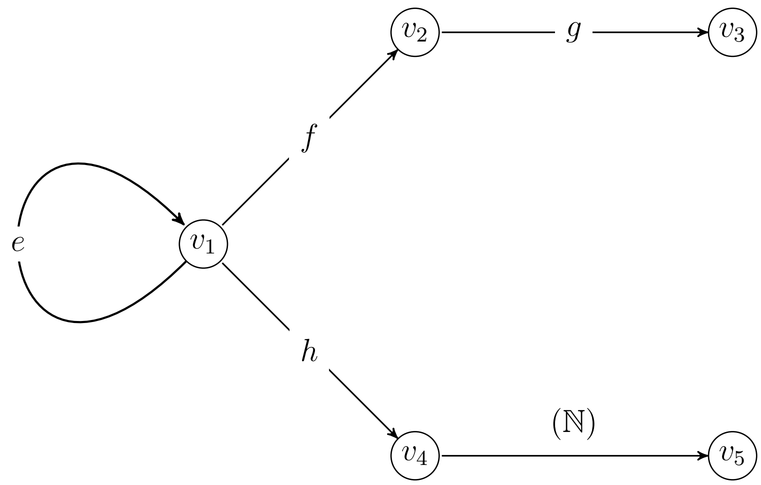


Figure 6: a directed graph

Then the graph with the ghost edges is

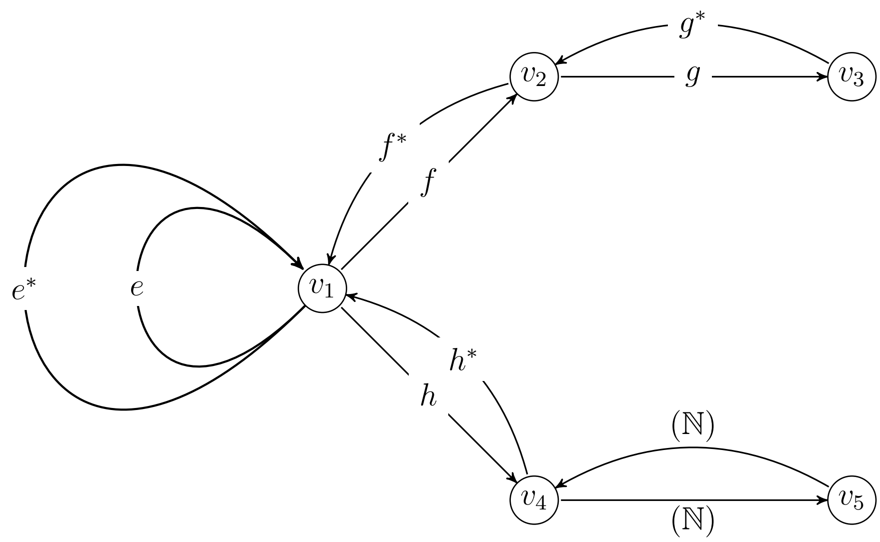


Figure 7: graph with the ghost edges drawn

Here we see some relations as

$$g^*g = v_3,\ g^*f = 0,\ f^*f = v_2$$

$$v_1 = ee^* + hh^* + ff^* \text{ and } v_2 = gg^*$$

$$v_1^2 = v,\ v_2v_3 = 0$$

$$gv_3 = g,\ v_1f = f,\ v_1h = h,\ f^*v_1 = f^*$$

But here we see that ee^* remains the same, we cannot simplify it any further.

Theorem 3.1.2. ***(Universal Property of*** $\mathbf{L_k(E)}$***)*** *Let A be a k-algebra that includes a set of pairwise idempotents $\{x_v \mid v \in E^0\}$, along with two families of elements $\{x_e \mid e \in E^1\}$ and $\{b_e \mid e \in E^1\}$ that satisfy the following conditions:*

1. $x_{s(e)}x_e = x_ex_{r(e)} = x_e$ *and* $x_{r(e)}y_e = y_ex_{s(e)} = y_e$ *for all* $e \in E^1$.
2. $y_fx_e = \delta_{ef}x_{r(e)}$ *for all* $e, f \in E^1$.
3. *For every regular vertex* v, $x_v = \sum_{\{} e \in E^1 \mid s(e) = v\}x_ey_e$.

Then, there exists a unique k-algebra isomorphism $\psi : L_k(E) \to A$ such that $\psi(v) = x_v$, $\psi(e) = x_e$, and $\psi(e^*) = y_e$ for all $e \in E^1$ and $v \in E^0$.

3.2 Properties of Leavitt Path Algebra

Definition 3.2.1. An associative ring R is said to have a collection of *local units* F if F consists of idempotent elements in R with the property that, for any finite set of elements $r_1, \dots, r_n \in R$, there is an element $f \in F$ such that $fr_if = r_i$ for all $1 \le i \le n$. In other words, a set of idempotents $F \subseteq R$ serves as a set of local units for R if every finite subset of R can be embedded in a (unital) subring of the form fRf for some $f \in F$.

An associative ring R is said to possess enough idempotents if there is a collection of nonzero orthogonal idempotents E in R, such that the set F of finite sums of distinct elements from E forms a set of local units for R.

Lemma 3.2.2 ([2], Lemma 1.2.12)**.** *Let x, y, z, w be elements of Path(E). Then for $L_k(E)$, the following hold:*

1. *The product of monomials is computed as:*

$$(xy^*)(zw^*) = \begin{cases} xpw^*, & \text{if } z = yp \text{ for some } p \in Path(E), \\ xq^*w^*, & \text{if } y = zq \text{ for some } q \in Path(E), \\ 0, & \text{otherwise.} \end{cases}$$

2. *The algebra* $L_k(E)$ *is spanned as a* k*-vector space by the set of monomials of the form:*
$$\{xy^* \mid x, y \in Path(E) \text{ such that } r(x) = r(y)\}.$$
That is, every element $a \in L_k(E)$ *can be written as:*
$$a = \sum_{i=1}^{n} k_i x_i y_i^*.$$
However, this set does not form a basis, as the representation of elements is not unique, except in simpler cases.

3. $L_k(E)$ *has a unit element if and only if* E^0 *is finite. Then the unit element is:*
$$1_{L_k(E)} = \sum_{v \in E^0} v.$$
This is evident because multiplying any vertex v *by* $\sum_{v\in E^0} v$ *results in only* $v^2 = v$*, with all other vertex products cancelling out. Similarly, for any edge* e*, the identity holds since* $e = er(e)$*.*

4. $L_k(E)$ *is a ring with enough idempotents, and hence it is a ring having local units. Therefore, a graph with infinitely many vertices will have local units.*

3.3 Some examples of Leavitt Path Algebras

Here we will see that Leavitt path algebras are not some foreign objects. They can give rise to very exotic algebras and at the same time, can give rise to well known rings like matrix rings and $k[x, x^{-1}]$.

Definition 3.3.1. The n-rose petal, denoted by R_n, is a graph consisting of one vertex and n loops attached to that vertex.

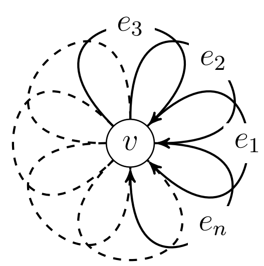


Figure 8: R_n

We will be particularly working here with the 1-petal rose R_1.

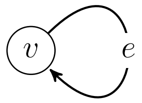


Figure 9: R_1

The graph A_n consists of n vertices with $n-1$ edges joining every vertex to its subsequent vertex.

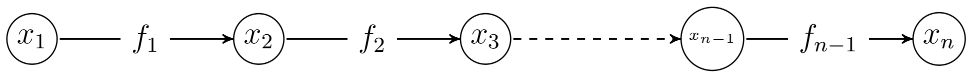


Figure 10: A_n

Theorem 3.3.2.

$$L_k(R_n) \cong L_k(1, n)$$

Proof. From the characterisation of Leavitt algebras having type $(1, n)$, we know that there exist elements from the algebra, $x_1, \ldots, x_n$ and $y_1 \ldots, y_n$ such that

$$\sum_{i=1}^{n} x_i y_i = 1_R \text{ and } y_i x_i = \delta_{ij} 1_R \text{ (for all } 1 \leq i, j \leq n)$$

Then map the elements from $L_k(1, n)$ to elements of $L_k(R_n)$ as

$$x_i \to e_i$$

$$y_i \to e_i^*$$

Then these elements behave similarly in their respective sets since v is a regular vertex, and also v acts as the identity element of $L_k(R_n)$ we have

$$v = e_1 e_1^* + e_2 e_2^* + \cdots + e_n e_n^*$$

and

$$e_i^* e_j = \delta_{ij} v$$

just like x_i and y_i. □

Definition 3.3.3. The algebra of Laurent polynomials over k, generated by x and y with the commutation relation $xy = yx = 1$, is denoted by $k[x, y]$ or $k[x, x^{-1}]$.

Theorem 3.3.4.
$$L_k(R_1) \cong k[x, x^{-1}]$$

Proof. Just like in the previous theorem, we map the elements as

$$x \to e$$

$$x^{-1} \to e^*$$

Then we have the relations $ee^* = v = 1_{L_k(R_1)}$ by (CK2) and $e^*e = v = 1_{L_k(R_1)}$ by (CK1) □

Now the next theorem shows that not all Leavitt path algebras are exotic, some of them are well known and well known matrix rings $M_n(k)$.

Theorem 3.3.5.
$$L_k(A_n) \cong M_n(k)$$

Proof. Consider the set $\{f_{i,j} \mid i, j = 1, 2, \ldots, n\}$, where each $f_{i,j}$ represents an $n \times n$ matrix that has 1 in the (i, j)th position and 0 else. These matrix units obey the following relation:
$$f_{i,j} f_{k,l} = \delta_{j,k} f_{i,l}$$
Then map the elements from $L_k(A_n)$ to $M_n(k)$ as

$$v_i \mapsto f_{i,i}$$

$$e_i \mapsto f_{i,i+1}$$
$$e_i^* \mapsto f_{i+1,i}$$

i.e. we are mapping the vertices to the diagonal matrix unit elements, the real edges to the upper diagonal and the ghost edges to the lower diagonal. Then we see that the behaviour of these elements is alike.

1. (V) $v_i v_j = \delta_{i,j} v_i \equiv f_{i,i} f_{j,j} = \delta i, j f_{i,j} = \delta_{i,j} f_{i,i}$

2. (E1) $s(e_i)e_i = e_i = e_i r(e_i) \equiv v_i e_i = e_i = e_i v_{i+1} \equiv f_{i,i} f_{i,i+1} = f_{i,i+1} = f_{i,i+1} f_{i+1,i}$

3. (E2) $r(e_i)e_i^* = e_i^* = e_i^* s(e_i) \equiv v_{i+1} e_i^* = e_i^* = e_i^* v_i \equiv f_{i+1,i+1} f_{i+1,i} = f_{i+1,i} = f_{i+1,i} f_{i,i}$

4. (CK1) $e_i^* e_j = \delta_{i,j} r(e_i) \equiv e_i^* e_j = \delta v_{i+1} \equiv f_{i+1,i} f j, j+1 = \delta_{i,j} f_{i+1,j+1} = \delta_{i,j} f_{i+1,i+1}$

5. (CK2) $e_i e_i^* = v_i \equiv f_{i,i+1} f_{i+1,i} = f_{i,i}$

□

This theorem also shows that not all Leavitt path algebras are of non-IBN type, some of them give rise to IBN rings too.

4 Making new graphs from existing graphs

In this chapter we try to construct new graphs from existing graphs and study the behavior of the new graph in relation to the original one.

4.1 $E(v_0, n)$ and $E(e_0, n)$

Definition 4.1.1. Let n be a natural number, and let $v_0 \in E^0$. Define the graph $E(v_0, n)$ as:

$$E(v_0, n)^0 = E^0 \cup \{v_1, v_2, \ldots, v_n\}$$
$$E(v_0, n)^1 = E^1 \cup \{e_1, e_2, \ldots, e_n\}$$

The functions $r_{E(v_0,n)}$ and $s_{E(v_0,n)}$ are defined to be the same as r_E and s_E, respectively. Furthermore, for $i = 1, \ldots, n$, we define:

$$r_{E(v_0,n)}(e_i) = v_{i-1} \quad \text{and} \quad s_{E(v_0,n)}(e_i) = v_i$$

Definition 4.1.2. Let $e_0 \in E^1$ be an edge, and let $n \in \mathbb{N}$. Define the graph $E(e_0, n)$ as:

$$E(e_0, n)^0 = E^0 \cup \{v_1, v_2, \ldots, v_n\}$$
$$E(e_0, n)^1 = (E^1 \setminus \{e_0\}) \cup \{e_1, e_2, \ldots, e_{n+1}\}$$

The functions $r_{E(e_0,n)}$ and $s_{E(e_0,n)}$ are the same as r_E and s_E, respectively. Additionally, we define:

$$r_{E(e_0,n)}(e_1) = r_E(e_0) \quad \text{and} \quad s_{E(e_0,n)}(e_{n+1}) = s_E(e_0)$$
$$s_{E(e_0,n)}(e_i) = v_i \quad \text{for} \quad i = 1, \ldots, n$$
$$r_{E(e_0,n)}(e_i) = v_{i-1} \quad \text{for} \quad i = 2, \ldots, n+1$$

Example 4.1.3. *Consider*

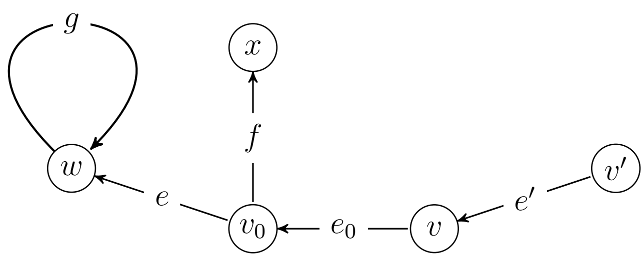


Figure 11: graph E

Then $E(v_0, 2)$ would be the graph

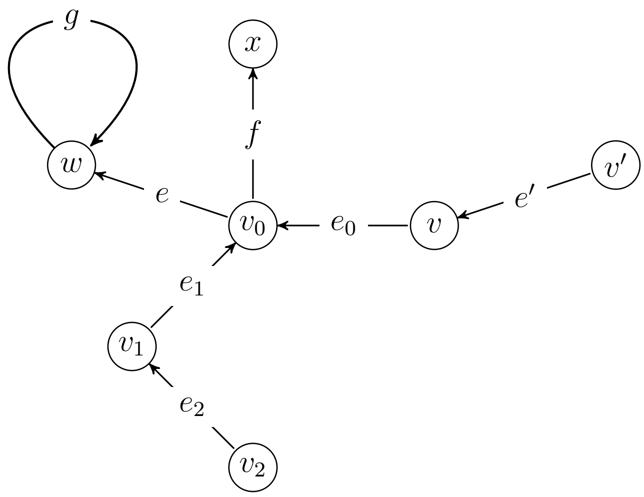


Figure 12: graph $E(v_0, 2)$

And $E(e_0, 3)$ would be the graph

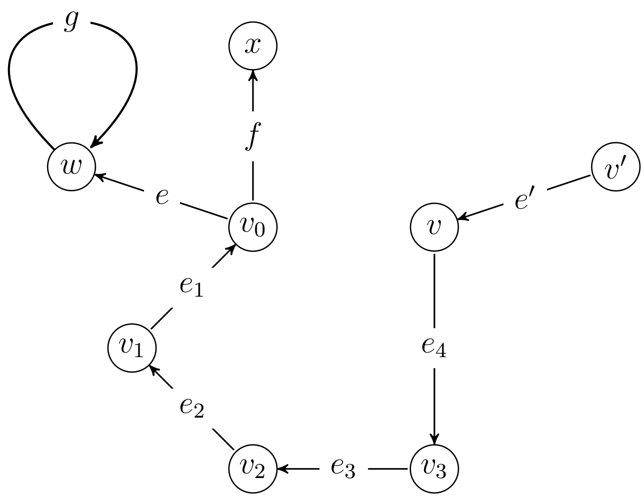


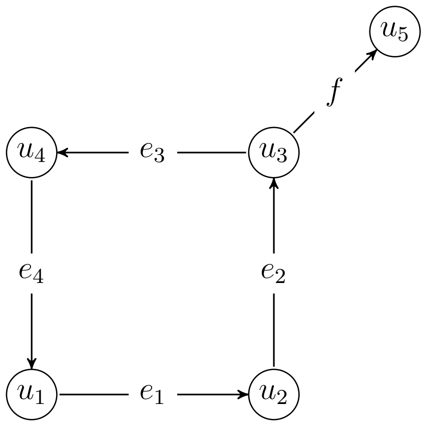
Figure 13: graph $E(e_0, 3)$

Note: In layman's terms, to construct the graph $E(v_0, n)$ from E, we adjoin a path of length n to the vertex v_0 with the edge connecting this path to v_0 being named e_1 which continues down the path. And to construct the graph $E(e_n, n)$ from E, we again adjoin a path having length n, similarly as $E(v_0, n)$, to the range of e_0, with the exception that we also join the end vertex of this path to the source of e_0, and then we delete the edge e_0. So in essence, we are replacing the edge e_0 by a path of length n.

Definition 4.1.4. An *exit* for a path $p = e_1e_2\dots e_n$ is an edge f if there exists some e_i within the path such that $s(f) = s(e_i)$, but $f \neq e_i$.

A graph E is called a *no exit* graph if there are no exits present in any cycle of E.

An example of an exit f in a cycle $c = e_1e_2e_3e_4e_5$ would be

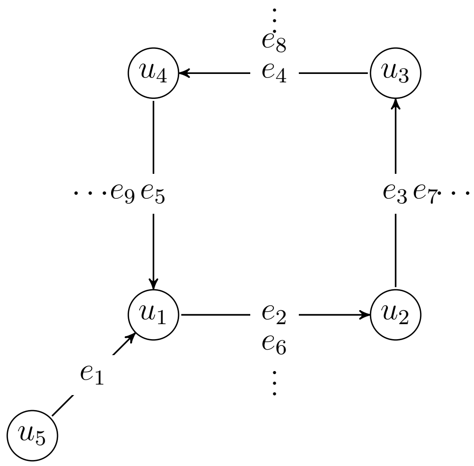


Definition 4.1.5. A *trail* in E is defined as:

1. A finite sequence of edges $\tau = e_1e_2\dots e_n$ (where n may be zero) such that the range of the last edge, $r(t) = r(e_n)$, is a sink. In other words, there are no edges leaving $r(e_n)$, i.e., $s^{-1}(r(e_n)) = \emptyset$.

2. An infinite sequence of edges $\tau = e_1e_2\dots e_n\dots$ in which the range of one edge is the same as the source of the next, i.e., $r(e_n) = s(e_{n+1})$ for all $n \in \mathbb{N}$.

Definition 4.1.6. An infinite trail $\tau = e_1e_2\dots$ in E is called periodic, if there are natural numbers p, q so that $e_{n+q} = e_n$ for all $n \geq p$. For example



In this scenario, we observe that $q = 4$ and $p = 2$. It's evident that the path $\rho = e_j \dots e_{j+k-1}$ forms a closed cycle.

By selecting p, q so that $p + q$ is minimized, and defining the paths

$$\alpha = e_1 e_2 \dots e_{j-1}$$

$$\lambda = e_j \dots e_{j+k-1}$$

we call the pair (α, λ) the *seed* of τ. Then λ represents a closed path and is referred to as the *period* of τ.

A trail τ is considered periodic if its period is a closed path without exits, meaning λ is a cycle that does not contain any exits. This type of trail is known as an *infinite discrete essentially aperiodic trail.*

Such a trail can be described by the seed (α, λ_α), indicating that the cycle λ_α begins and terminates at the vertex $r(\alpha)$. The path α is then referred to as a *distinguished path.*

Definition 4.1.7. A *Cuntz-Krieger E-family* in F is a collection of elements of a k-algebra in the graph F that follow the five properties that define a Leavitt Path Algebra.

Let A be a k-algebra with a Cuntz-Krieger E-family. By the Universal Homomorphism Property of $L_k(E)$, there exists a unique k-algebra homomorphism from $L_k(E)$ to A that sends the generators of $L_k(E)$ to their corresponding elements in A. Such a family is also called CK E-family in F.

Definition 4.1.8. For paths γ and δ in Path (E), we say that $\gamma \leq \delta$ if $\delta = \gamma\gamma'$ for some path γ'.

Definition 4.1.9. $y \in L_k(E)$ is called *normal* if $yy^* = y^*y$.

Lemma 4.1.10 ([8], Lemma 4.3)**.** *A cycle γ has no exits if and only if $\gamma\gamma^* = s(\gamma)$.*

Proposition 4.1.11 ([8], Proposition 4.5)**.** Let $\gamma, \delta \in \text{Path}(E)$ with $r(\gamma) = r(\delta)$. The generator $\gamma\delta^*$ is normal in $L_k(E)$ if and only if one of the following conditions holds:

1. $\gamma = \delta$
2. $\delta \leq \gamma$ and δ is a distinguished path, i.e., $\gamma = \delta\lambda_\delta$ where λ_δ is a cycle without exits.
3. $\gamma \leq \delta$ and γ is a distinguished path, i.e., $\delta = \gamma\lambda_\gamma$ where λ_γ is a cycle without exits.

Proof. Let $x = \gamma\delta^*$ be an element such that it satisfies property

(1) If $\gamma = \delta$ then clearly x is normal since

$$xx^* = \gamma\delta^*(\gamma\delta^*)^* = \gamma\gamma^*(\gamma\gamma^*)^* = \gamma\gamma^*\gamma\gamma^* = (\gamma\gamma^*)^*\gamma\gamma^* = x^*x$$

(2) If $\delta \leq \gamma$, it follows that $\gamma = \delta\lambda$, where λ represents a cycle without exits. According to Lemma 4.1.10, we have the relation $\lambda\lambda^* = s(\lambda) = r(\delta)$. So

$$xx^* = \gamma\delta^*(\gamma\delta^*)^* = \gamma\delta^*\delta\gamma^* = \delta\lambda\delta^*\delta\lambda^*\delta^* = \delta\lambda r(\delta)\lambda^*\delta^*$$

$$= \delta\lambda s(\lambda)\lambda^*\delta^* = \delta\lambda\lambda^*\delta^* = \delta s(\lambda)\delta^* = \delta\delta^*$$

and

$$x^*x = (\gamma\delta^*)^*\gamma\delta^* = \delta\gamma^*\gamma\delta^* = \delta r(\gamma)\delta^* = \delta\delta^*$$

Hence $xx* = x^*x$.

(3) Can be done in similar way to (2).

Conversely, assume that $0 \neq x = \gamma\delta^*$ is normal element of $L_k(E)$. Since $x \neq 0$ we have that $r(\gamma) = r(\delta)$ and also we have $xx^* = x^*x$, we get that

$$\begin{aligned}
&xx^* = x^*x \\
\Rightarrow&\gamma\delta^*(\gamma\delta^*)^* = (\gamma\delta^*)^*\gamma\delta^* \\
\Rightarrow&\gamma\delta^*\delta\gamma^* = \delta\gamma^*\gamma\delta^* \\
\Rightarrow&\gamma r(\delta)\gamma^* = \delta r(\gamma)\delta^* \\
\Rightarrow&\gamma\gamma^* = \delta\delta^* && \text{(since } r(\gamma) = r(\delta)) \\
\Rightarrow&s(\gamma) = s(\delta)
\end{aligned}$$

Now we see that

$$
\begin{aligned}
(xx^*)^2 &= xx^*xx^* \\
&= \gamma\delta^*(\gamma\delta^*)^*\gamma\delta^*(\gamma\delta^*)^* \\
&= \gamma\delta^*\delta\gamma^*\gamma\delta^*\delta\gamma^* \\
&= \gamma(\delta^*\delta)(\gamma^*\gamma)(\delta^*\delta)\gamma^* \\
&= \gamma r(\delta)r(\gamma)r(\delta)\gamma^* \\
&= \gamma\gamma^* && \text{(since } r(\gamma) = r(\delta)\text{)} \\
&\neq 0
\end{aligned}
$$

Thus

$$
\begin{aligned}
0 \neq (xx^*)^2 = xx^*xx^* = x^*x^2x^* \\
\Rightarrow x^2 \neq 0 \\
\Rightarrow (\gamma\delta^*)(\gamma\delta^*) \neq 0
\end{aligned}
$$

Then by the way that the elements of the algebra look, we must have

$$\delta \leq \gamma \text{ or } \gamma \leq \delta$$

Assuming $\delta \leq \gamma$ and using the facts discussed, we have $\gamma = \delta\lambda$ where λ is a closed path. If λ has an exit, then by Lemma 4.1.10 we get

$$\lambda\lambda^* \neq s(\lambda) = r(\delta)$$

and so

$$xx^* = \gamma\delta^*\delta\gamma^* = \gamma\gamma^* = \delta\lambda\lambda^*\delta^* \neq \delta\delta^* = x^*x,$$

which leads to a contradiction. Therefore, λ must be a cycle without exits. Similarly, if we assume $\gamma \leq \delta$, we obtain condition (3).

□

Lemma 4.1.12 ([10], proposition 3.4)**.** *$L_k(E)$ is such that the elements of the set $\{v, e, e^* \mid v \in E^0, e \in E^0\}$ are all non zero. Moreover*

$$L_k(E) = span_k\{(\gamma\delta^*) : \gamma, \delta \in \mathit{Path}(E) \text{ and } r(\gamma) = r(\delta)\}$$

and $rv \neq 0$ for all $v \in E^0$ and for all $r \in k \setminus \{0\}$.

Definition 4.1.13. Let G_M^E be the set of all normal generators of the form $\alpha\beta^*$ for some paths $\alpha, \beta \in L_k(E)$.

The subalgebra $M_k(E) = \langle G_M^E \rangle$ generated by all the elements of G_M^E is called the *commutative core* of $L_k(E)$.

Definition 4.1.14. Define a set

$$G_E^\Delta = \{\mu\mu^* \mid \mu \in \text{Path}(E)\}$$

This set G_E^Δ is termed the *standard diagonal generator set* and the k-algebra generated by G_E^Δ, i.e., $\langle G_E^\Delta \rangle$ is termed the *diagonal algebra associated with E* and is denoted by $\Delta(E)$.

Note: By proposition 4.1.11 (1), every element of the form $\mu\mu^*$ for some path μ is normal, hence $\mu\mu^* \in M_k(E)$. Thus, $\Delta(E) \subseteq M_k(E)$.

Proposition 4.1.15 ([8], Proposition 4.5)**.** The commutative core of $L_k(E)$ is commutative.

Proposition 4.1.16 ([8], Remark 4.7)**.** Consider a distinguished path α. Let λ_α be the cycle without exits starting and terminating at $r(\alpha)$. Define the element $\omega_\alpha := \alpha\lambda_\alpha\alpha^*$. Then the k-algebra generated by ω_α, denoted $\langle \omega_\alpha \rangle$, satisfies $\langle \omega_\alpha \rangle \cong k[x, x^{-1}]$.

Proof. By proposition 4.1.11 we see that $\omega_\alpha = \alpha\lambda_\alpha\alpha^*$ is of the form $\alpha(\lambda_\alpha\alpha^*)$, hence $\omega_\alpha \in G_M^E$, and thus, the k-algebra generated by ω_α, $\langle \omega_\alpha \rangle \subseteq M_k(E)$. And this algebra is unital with unit $\alpha\alpha^*$. And also ω_α is invertible since

$$\omega_\alpha^*\omega_\alpha = (\alpha\lambda_\alpha\alpha^*)^*\alpha\lambda_\alpha\alpha^* = \alpha\lambda_\alpha^*\alpha^*\alpha\lambda_\alpha\alpha^* = \alpha\lambda_\alpha^*\lambda_\alpha\alpha^* = \alpha\alpha^*$$

and

$$\omega_\alpha\omega_\alpha^* = \alpha\lambda_\alpha\alpha^*(\alpha\lambda_\alpha\alpha^*)^* = \alpha\lambda_\alpha\alpha^*\alpha\lambda_\alpha^*\alpha^* = \alpha\lambda_\alpha\lambda_\alpha^*\alpha^* = \alpha\alpha^*$$

Then the powers of ω_α can be defined, i.e., $(\omega_\alpha)^n$ for $n \in \mathbb{Z}$ as

$$\text{if } n < 0 \text{ then } \omega_\alpha^n = (\omega_\alpha^*)^{-n}$$

and

$$\omega_\alpha^0 = \alpha\alpha^*.$$

After this we define the homomorphism $\phi : \langle \omega_\alpha \rangle \to k[x, x^{-1}]$ as

$$\phi(\omega_\alpha) = x$$

$$\phi(\omega_\alpha^*) = x^{-1}$$

$$\phi(\alpha\alpha^*) = 1$$

Then this homomorphism becomes an isomorphism. ☐

Following is a theorem from *Leavitt Path Algebra Book* [2], with restriction of the arbitrary ring R to a field k.

Theorem 4.1.17 ([2], Theorem 2.2.11)**.** *For any non-zero element $a \in L_k(E)$, there exist paths $\nu, \mu \in Path(E)$ such that one of the following holds:*

1. *There exists a non-zero scalar $r \in k \setminus \{0\}$ and a vertex $v \in E^0$ such that $0 \neq \mu^* a v = rv$.*

2. *There exists a cycle λ without exits such that $0 \neq \mu^* a v = p(\lambda)$, where $p(x)$ is a non-zero polynomial in $k[x, x^{-1}]$.*

Theorem 4.1.18 ([8], Theorem 5.2)**.** *(Generalised Uniqueness Theorem for Leavitt Path Algebras) Suppose there is a ring homomorphism $\phi : L_k(E) \to \mathcal{A}$, where $L_k(E)$ is the Leavitt path algebra associated with E, and $\mathcal{A}$ is some k-algebra. Then, TFAE:*

(i) ϕ is injective.

(ii) The restriction of ϕ to $M_k(E)$, the subalgebra of $L_k(E)$ generated by the set of matrices corresponding to paths in E, is also injective.

(iii) Both of the following conditions are satisfied:

(a) For each vertex $v \in E^0$ and each non-zero scalar $r \in k$, we have $\phi(rv) \neq 0$, which ensures that ϕ maps non-zero scalar multiples of vertices in E to non-zero elements of $\mathcal{A}$.

(b) For each distinguished path $\alpha \in Path(E)$, the k-algebra generated by $\phi(\omega_\alpha)$, where $\omega_\alpha = \alpha\lambda_\alpha\alpha^*$ and λ_α is a cycle without exits, is isomorphic to the Laurent polynomial algebra $k[x, x^{-1}]$.

Proof. (i)⇒(ii) is obvious.

(ii)⇒(iii) We will prove (a) first. Let $r \in k \setminus \{0\}$ and $v \in E^0$ be arbitrary. Then $rv = rvv^* \in \Delta(E) \subseteq M_k(E)$. By lemma 4.1.12 $rv \neq 0$ and since ϕ is injective when restricted to $M_k(E)$, we get $\phi(rvv^*) \neq 0 \Rightarrow \phi(rv) \neq 0$.

Now we will prove (b). We know that for all distinguished paths α, $\omega_\alpha = \alpha\lambda_\alpha\alpha^*$ and for all $j \in \mathbb{Z}$ we have $\omega_\alpha^i \in M_k(E)$ by proposition 4.1.11. And since restriction of ϕ to $M_k(E)$ is injective, $\phi(\omega_\alpha^j) \neq 0$. Then we have a natural k-homomorphism $\langle\omega_\alpha\rangle \cong \langle\phi(\omega_\alpha)\rangle$. And from proposition 4.1.16 we have $\langle\omega_\alpha\rangle \cong k[x, x^{-1}]$. Thus

$$\langle\phi(\omega_\alpha)\rangle \cong k[x, x^{-1}].$$

(iii)⇒(i) Suppose, if possible, ϕ be not injective. Then there exits $0 \neq a \in L_k(E)$ such that $a \in \ker\phi$. Then by theorem 4.1.17, there exist path μ, ν in Path(E) such that there are two possibilities.

If $0 \neq \mu^* a \nu = rv$ for some $r \in k \setminus \{0\}$ and $v \in E^0$, we would have that $\phi(rv) \neq 0$ by condition (a). However, since $a \in \ker\phi$, we also have $\phi(a) = 0$. This leads to the contradiction that $\phi(rv) = \phi(\mu^* a\nu) = 0$, implying $rv \in \ker\phi$, which violates condition (a).

Consequently, the only scenario left to analyze is when $0 \neq \mu^* a\nu = p(\lambda)$, for some cycle λ without exits and a non-zero polynomial $p(x) \in k[x, x^{-1}]$. Given that $a \in \ker\phi$, we must have $p(\lambda) \in \ker\phi$, which leads to $\phi(p(\lambda)) = 0$.

Choose a vertex $v \in \lambda^0$ belonging to the cycle λ. Since λ possesses no exits, v qualifies as a distinguished vertex, and λ coincides with λ_v. Hypothesis (b) then guarantees the existence of a k-algebra homomorphism

$$g : \langle\phi(\lambda)\rangle \to k[x, x^{-1}].$$

From $\phi(p(\lambda)) = 0$, it follows that $g(\phi(p(\lambda))) = 0$. Let us express the polynomial as $p(x) = r_{-m}x^{-m} + \cdots + r_n x^n$, where $m, n \in \mathbb{N}$. We can then compute:

$$\begin{aligned} 0 = g(\phi(p(\lambda))) = g\left(\phi(r_{-m}\lambda^{-m} + \cdots + r_n\lambda^n)\right) &= g(\phi(r_{-m}v)\phi(\lambda^*)^m + \cdots \\ &\quad +\phi(r_n v)\phi(\lambda)^n) \\ &= g(\phi(r_{-m}v))x^{-m} + \cdots + g(\phi(r_n v))x^n. \end{aligned}$$

Because the powers of x are linearly independent in $k[x, x^{-1}]$, this result implies $g(\phi(r_i v)) = 0$ for every index i in the range $\{-m, \cdots, n\}$. This, in turn, necessitates $\phi(r_i v) = 0$ for all such i, thereby contradicting assumption (a).

□

4.2 LPA of $E(v_0, n)$ and $E(e_0, n)$

The following theorem links the newly constructed graphs $E(v_0, n)$ and $E(e_0, n)$ and their LPAs.

Theorem 4.2.1. *Let e_0 be an edge, and let $n \in \mathbb{N}$. Define $v_0 = r_E(e_0)$. Then*

$$L_k(E(e_0, n)) \cong L_k(E(v_0, n))$$

Proof. We will construct a homomorphism from $L_k(E(v_0, n))$ to $L_k(E(e_0, n))$ using the universal property of Leavitt Path Algebra.

For $v_0 \in E(v_0, n)^0$ and for $e \in E(v_0, n)^1$ define respectively

$$Q_v = v$$

$$T_e = \begin{cases} e, & \text{if } e \neq e_0 \\ e_{n+1}e_n \ldots e_1, & \text{if } e = e_0. \end{cases}$$

$$T_e^* = T_{e^*}$$

Note here that Q_v and T_e are elements in $E(e_0, n)$.

Our aim is to show that $\{Q_v, T_e, T_e^*\}$ is a CK $E(v_0, n)$-family in $E(e_0, n)$. We show the properties step by step.

1. (V) Clearly for any $v, v' \in E(v_0, n)^0$, we have

$$Q_v Q'_v = vv' = \delta_{vv'} v = \delta_{Q_v Q'_v} Q_v$$

2. (E1) Let $e \in E(v_0, n)^1$. If $e \neq e_0$ then we see that $s_{E(v_0,n)}(e) = s_E(e) = s_{E(e_0,n)}(e)$ and $r_{E(v_0,n)}(e) = r_E(e) = r_{E(e_0,n)}(e)$. So we have

$$Q_{s_{E(v_0,n)}(e)} T_e = s_{E(v_0,n)}(e) e = e r_{E(v_0,n)}(e) = T_e Q_{s_{E(v_0,n)}(e)}$$

and if $e = e_0$ then $T_e = T_{e_0} = e_{n+1} e_n \ldots e_1$. And so $s_{E(v_0,n)}(e_0) = s_E(e_0) = s_{E(e_0,n)}(e_{n+1})$. Thus

$$\begin{aligned}
Q_{s_{E(v_0,n)}(e_0)} T_{e_0} &= s_{E(v_0,n)}(e_0) e_{n+1} e_n \ldots e_1 \\
&= s_{E(e_0,n)}(e_{n+1}) e_{n+1} e_n \ldots e_1 \\
&= (s_{E(e_0,n)}(e_{n+1}) e_{n+1}) e_n \ldots e_1 \\
&= (e_{n+1} r_{E(e_0,n)}(e_{n+1})) e_n \ldots e_1 \\
&= e_{n+1} v_n e_n \ldots e_1 \\
&= e_{n+1} (v_n e_n) e_{n-1} \ldots e_1 \\
&= e_{n+1} (e_n v_{n-1}) e_{n-1} \ldots e_1 \\
&\quad \vdots \\
&= e_{n+1} e_n \ldots v_1 e_1 (= e_{n+1} e_n \ldots e_1 = T_{e_0}) \\
&= e_{n+1} e_n \ldots e_1 r_{E(e_0,n)}(e_1) \\
&= e_{n+1} e_n \ldots e_1 r_E(e_0) \\
&= e_{n+1} e_n \ldots e_1 r_{E(v_0,n)}(e_0) \\
&= T_{e_0} Q_{r_{E(v_0,n)}(e_0)}
\end{aligned}$$

3. (E2) Let $e \in E(v_0, n)^1$. If $e \neq e_0$ then we see that $s_{E(v_0,n)}(e^*) = s_E(e^*) = s_{E(e_0,n)}(e^*)$ and $r_{E(v_0,n)}(e^*) = r_E(e^*) = r_{E(e_0,n)}(e^*)$. So we have

$$Q_{r_{E(v_0,n)}(e^*)} T_e^* = r_{E(v_0,n)}(e^*) e^* = e^* s_{E(v_0,n)}(e^*) = T_e^* Q_{s_{E(v_0,n)}(e^*)}$$

and if $e = e_0$ then $T_e^* = T_{e_0}^* = (e_{n+1} e_n \ldots e_1)^* = e_1^* e_2^* \ldots e_{n+1}^*$. And so

$r_{E(v_0,n)}(e_0^*) = r_E(e_0^*) = r_{E(e_0,n)}(e_1^*)$. Thus

$$\begin{aligned}
Q_{r_{E(v_0,n)}(e_0^*)}T_{e_0}^* &= r_{E(v_0,n)}(e_0^*)e_1^*e_2^*\dots e_{n+1}^* \\
&= r_{E(e_0,n)}(e_1^*)e_1^*e_2^*\dots e_{n+1}^* \\
&= (r_{E(e_0,n)}(e_1^*)e_1^*)e_2^*\dots e_{n+1}^* \\
&= (e_1^* s_{E(e_0,n)}(e_1^*))e_2^*\dots e_{n+1}^* \\
&= e_1^* v_2 e_2^* \dots e_{n+1}^* \\
&= e_1^*(v_2 e_2^*)e_3^* \dots e_{n+1}^* \\
&= e_1^*(e_2^* v_{n-1})e_2^* \dots e_{n+1}^* \\
&\vdots \\
&= e_1^*e_2^*\dots v_{n+1}e_{n+1}^* (= e_1^*e_2^*\dots e_{n+1}^* = T_{e_0}^*) \\
&= e_1^*e_2^*\dots e_{n+1}^* s_{E(e_0,n)}(e_{n+1}^*) \\
&= e_1^*e_2^*\dots e_{n+1}^* s_E(e_0^*) \\
&= e_1^*e_2^*\dots e_{n+1}^* s_{E(v_0,n)}(e_0^*) \\
&= T_{e_0}^* Q_{s_{E(v_0,n)}(e_0^*)}
\end{aligned}$$

4. (CK1) Let $f, e \in E(v_0, n)^1$. Suppose $\neq e$. Then

$$\begin{aligned}
T_e^*T_f &= \begin{cases} e^*f, & \text{if } e \neq e_0 \text{ and } f \neq e_0 \\ e_1^*e_2^*\dots e_{n+1}^*f, & \text{if } e = e_0 \\ e^*e_{n+1}e_n\dots e_1, & \text{if } f = e_0. \end{cases} \\
&= \begin{cases} 0, & \text{if } e \neq e_0 \text{ and } f \neq e_0 \\ e_1^*e_2^*\dots e_{n+1}^*f, & \text{if } e = e_0 \\ e^*e_{n+1}e_n\dots e_1, & \text{if } f = e_0. \end{cases}
\end{aligned}$$

Now for any edge $g \in E(v_0, n)^1$, $g \neq e_{n+1}$, hence $e_{n+1}^*f = 0$ and $e^*e_{n+1} = 0$. So we get

$$T_e^*T_f^* = 0 \text{ when } e \neq f$$

Now let e be an edge in E, then

$$\begin{aligned}
T_e^*T_e &= \begin{cases} e^*e, & \text{if } e \neq e_0 \\ e_1^*e_2^*\dots e_{n+1}^*e_{n+1}e_n\dots e_1, & \text{if } e = e_0 \end{cases} \\
&= \begin{cases} r_{E(e_0,n)}(e), & \text{if } e \neq e_0 \\ e_1^*e_2^*\dots e_n^* r_{E(e_0,n)}(e_{n+1})e_n\dots e_1, & \text{if } e = e_0 \end{cases} \\
&= \begin{cases} r_{E(e_0,n)}(e), & \text{if } e \neq e_0 \\ e_1^*e_2^*\dots e_n^* v_n e_n\dots e_1, & \text{if } e = e_0 \end{cases} \\
&= \begin{cases} r_{E(e_0,n)}(e), & \text{if } e \neq e_0 \\ e_1^*e_2^*\dots e_n^* e_n\dots e_1, & \text{if } e = e_0 \end{cases} \\
&\vdots \\
&= \begin{cases} r_{E(e_0,n)}(e), & \text{if } e \neq e_0 \\ e_1^*e_1, & \text{if } e = e_0 \end{cases} \\
&= \begin{cases} r_{E(e_0,n)}(e), & \text{if } e \neq e_0 \\ r_{E(e_0,n)}(e_1), & \text{if } e = e_0 \end{cases} \\
&= r_{E(v_0,n)}(e) \\
&= Q_{r_{E(v_0,n)}(e)}
\end{aligned}$$

5. (CK2) Let $v \in E(v_0,n)^0$ be a regular vertex. Then we observe that in $E(e_0,n)$, v is also a regular vertex. We break it into cases according to whether $v = v_i$ for $i = 1, 2, \dots n$ or not.

   Case 1: $v = v_i$ for some $i = 1, 2, \dots, n$.

   Then $s^{-1}_{E(e_0,n)}(v_i) = \{e_i\} = s^{-1}_{E(v_0,n)}$. So from CK2 of $L_k(E)$ we have

$$Q_v = Q_{v_i} = v_i = e_ie_i^* = T_{e_i}T_{e_i}^*$$

   Case 2: $v \neq v_i$ for any $1 \leq i \leq n$.

   This can further be broken down into two subcases:

   Subcase 1: $e_{n+1} \notin s^{-1}_{E(e_0,n)}(v)$

   Since $v \neq v_i$ for $i = 1, 2, \dots, n$ and $v \neq s_E(e_0)$, we have $s^{-1}_{E(v_0,n)}(v) \cap \{e_0, e_1 \dots, e_n\} = \emptyset$ and $s^{-1}_{E(e_0,n)}(v) \cap \{e_1, e_2 \dots, e_{n+1}\} = \emptyset$, i.e. none of the edges $e_0, e_1, \dots, e_n, e_{n+1}$ have no interaction with the vertex v in their appropriate graphs. Thus,

$$s^{-1}_{E(v_0,n)}(v) = s^{-1}_E(v) = s^{-1}_{E(e_0,n)}(v)$$

Hence

$$Q_v = v = \sum_{e \in s_E^{-1}(v)} ee^* = \sum_{e \in s_{E(v_0,n)}^{-1}(v)} ee^* = \sum_{e \in s_{E(v_0,n)}^{-1}(v)} T_e T_e^*$$

Subcase 2: $e_{n+1} \in s_{E(e_0,n)}^{-1}(v)$

Then we see that $s_{E(e_0,n)}(e_{n+1}) = s_{E(v_0,n)}(e_0) = s_E(e_0)$, so $e_0 \in s_{E(v_0,n)}^{-1}$. We also see that $e_i e_i^* = v_i$ for all $i = 1, 2, \ldots, n$.

Thus

$$\begin{aligned}
Q_v = v &= \sum_{e \in s_{E(e_0,n)}^{-1}(v)} ee^* \\
&= \left(\sum_{e \in s_{E(e_0,n)}^{-1}(v) \setminus \{e_{n+1}\}} ee^* \right) + e_{n+1} e_{n+1}^* \\
&= \left(\sum_{e \in s_E^{-1}(v) \setminus \{e_0\}} ee^* \right) + e_{n+1} v_n e_{n+1}^* \\
&= \left(\sum_{e \in s_E^{-1}(v) \setminus \{e_0\}} ee^* \right) + e_{n+1} e_n e_n^* e_{n+1}^* \\
&\vdots \\
&= \left(\sum_{e \in s_E^{-1}(v) \setminus \{e_0\}} ee^* \right) + e_{n+1} e_n \ldots e_1 e_1 * \ldots e_n^* e_{n+1}^* \\
&= \left(\sum_{e \in s_E^{-1}(v) \setminus \{e_0\}} T_e T_e^* \right) + T_{e_0} T_{e_0}^* \\
&= \sum_{e \in s_{E(v_0,n)}^{-1}(v)} T_e T_e^*
\end{aligned}$$

This shows that $\{Q_v, T_e, T_e^* | v \in E(v_0,n)^0, e \in E(v_0,n)^1\}$ is a Cuntz-Kreiger $E(v_0,n)$ family in $E(e_0,n)$ since Q_v, T_e and T_e^* are elements in $L_k(E(e_0,n))$. Then by the universal property of Leavitt Path Algebra, there exists a k-homomorphism $\psi : L_k(E(v_0,n)) \to L_k(E(e_0,n))$

$$\psi(v) = Q_v$$

$$\psi(e) = T_e$$
$$\psi(e^*) = T_e^* \text{ ,for all } v \in E(v_0, n)^0 \text{ and } e \in E(v_0, n)^1$$

To show that this map is indeed a homomorphism, it suffices to show that the sole generator of $L_k(E(e_0, n))$ not present in

$$\{Q_v, T_e, T_e^* \mid v \in E(v_0, n)^0, e \in E(v_0, n)^1\}$$

i.e. e_{n+1}, can be constructed using these elements. We see that

$$\begin{aligned}
T_{e_0}T_{e_1}^*T_{e_2}^* \dots T_{e_n}^* &= e_{n+1}e_n \dots e_1 e_1^* e_2^* \dots e_n^* \\
&= e_{n+1}e_n \dots e_2 v_1 e_2^* e_2^* \dots e_n^* \\
&= e_{n+1}e_n \dots e_2 r_{E(e_0,n)(e_2)} e_2^* e_2^* \dots e_n^* \\
&= e_{n+1}e_n \dots e_2 e_2^* \dots e_n^* \\
&\vdots \\
&= e_{n+1}e_1 e_1^* \\
&= e_{n+1}v_n \\
&= e_{n+1} r_{E(e_0,n)(e_{n+1})} \\
&= e_{n+1}
\end{aligned}$$

Hence, $e_{n+1} \in \psi(L_k(E(v_0, n)))$, and this proves that ψ is indeed surjective.

Now we will prove the injectivity of this map.

Consider α, a distinguished path within $E(v_0, n)$, and define the element $\omega_\alpha = \alpha\lambda_\alpha\alpha^*$. Here, λ_α represents a cycle without exits based at the vertex $r_{E(v_0,n)}(\alpha)$. Let the decomposition of this cycle be $\lambda_\alpha = f_1 f_2 \dots f_m$. Assume that for every edge f_i $(i = 1, 2, \dots, m)$, the condition $s_{E(v_0,n)}(f_i) \neq s_{E(v_0,n)}(e_0)$ holds. This ensures that each f_i remains unchanged in the graph $E(e_0, n)$ and is distinct from e_0. Consequently, $\psi(\lambda_\alpha) = \lambda_\alpha$ constitutes a cycle without exits within $E(e_0, n)$. Applying Proposition 4.1.16, we deduce that the $*$-k-algebra $\langle\psi(\omega_\alpha)\rangle$, which is generated by $\psi(\omega_\alpha)$, is $*$-isomorphic to $k[x, x^{-1}]$.

Alternatively, suppose there exists some index $i \in \{1, 2, \dots, m\}$ such that $s_{E(v_0,n)}(f_i) = s_{E(v_0,n)}(e_0)$. Because $\lambda_\alpha = f_1 f_2 \dots f_m$ is specifically a cycle without exits, this condition necessitates that $f_i = e_0$. The cycle thus takes the form:

$$\lambda_\alpha = f_1 f_2 \dots f_{i-1} e_0 f_{i+1} \dots f_m$$

Applying the map ψ yields:

$$\psi(\lambda_\alpha) = \psi(f_1 f_2 \dots f_{i-1} e_0 f_{i+1} \dots f_m) = f_1 f_2 \dots f_{i-1} e_{n+1} e_n \dots e_1 f_{i+1} \dots f_m$$

This resulting path is also a cycle without exits in $E(e_0, n)$, owing to the fact that the newly introduced edges $e_1, e_2, \ldots, e_{n+1}$ lack any exits. Therefore, invoking Proposition 4.1.16 once more, we find that $\langle \psi(\omega_\alpha) \rangle$ is isomorphic to $k[x, x^{-1}]$.

Furthermore, let us consider an arbitrary vertex $v \in E(v_0, n)^0$ and any non-zero scalar $r \in k \setminus \{0\}$. We observe that

$$\psi(rv) = rQ_v = rv \neq 0 \text{ (from lemma 4.1.12)}$$

Invoking the Generalised Uniqueness Theorem 4.1.18, this observation allows us to conclude that the map ψ must be injective. Consequently, ψ is established as a bijection, leading to the isomorphism:

$$L_k(E(e_0, n)) \cong L_k(E(v_0, n)).$$

□

5 IBN property for Leavitt Path Algebras through matrices

5.1 Group completion of a monoid

Definition 5.1.1. An *abelian monoid* is a set M with a binary operation '$+$' that is associative, has an identity element 0 and is abelian wrt this operation.

A monoid map from a monoid M to another monoid N is a function $f : M \to N$ such that

$$f(0) = 0 \text{ and } f(m + m') = f(m) + f(m')$$

Example 5.1.2. $\mathbb{N} = \{0, 1, 2, \ldots\}$ *is a classical example of an abelian monoid.*

The *group completion* of an abelian monoid M is an abelian group $M^{-1}M$ together with a monoid map

$$[\,] : M \to M^{-1}M$$

such that if there is a monoid map $\alpha : M \to A$ for some abelian group A then there is an unique group homomorphism $\overline{\alpha} : M \to A$ such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & A \\ {\scriptstyle []}\downarrow & \nearrow_{\overline{\alpha}} & \\ M^{-1}M & & \end{array}$$

i.e.

$$\alpha(m) = \overline{\alpha}([m])$$

Example 5.1.3. *The group completion of* $\mathbb{N}$ *is* $\mathbb{Z}$*.*

It is a known property that any abelian monoid M possesses a group completion. An alternative viewpoint for constructing this completion involves initially forming the free abelian group $F(M)$, which utilizes a basis comprising elements $[m]$ for every $m \in M$. Subsequently, attention is turned to the subgroup $R(M)$, defined as the subgroup generated by all relations taking the form $[m+n]-[m]-[n]$. In terms of this construction, the group completion of the monoid M is then identified as

$$\frac{F(M)}{R(M)}$$

Proposition 5.1.4 ([12], Proposition 1.1)**.** Let M be an abelian monoid, then

1. Every element of $M^{-1}M$ is of the form $[m] - [n]$ for some $m, n \in M$.

2. If $m, n \in M$ then $[m] = [n]$ if and only if $m + p = n + p$ for some $p \in M$.

Definition 5.1.5. For a ring A, define $\mathcal{V}(A)$ to be the set of isomorphism classes $[P]$ of finitely generated projective right A-modules. This set becomes an abelian monoid under the operation:

$$[P] + [Q] = [P \oplus Q]$$

for all $[P], [Q] \in \mathcal{V}(A)$.

Lemma 5.1.6. *The following are equivalent for any ring A:*

1. *A has IBN property;*

2. *For all $m, n \in \mathbb{N}$, the identity $m[A] = n[A]$ in $\mathcal{V}(A)$ implies $m = n$;*

3. *If $X \in M_{m \times n}(A)$ and $Y \in M_{n \times m}(A)$ satisfy $XY = I_m$ and $YX = I_n$, then $m = n$.*

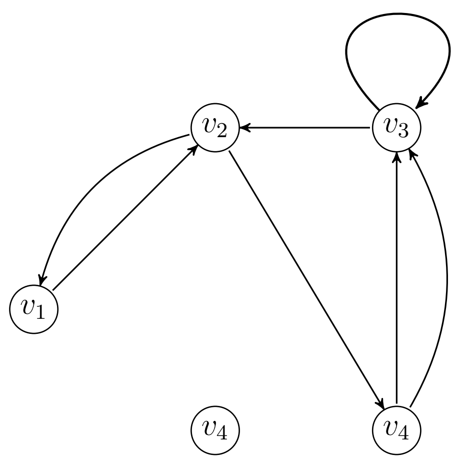
5.2 IBN property and Leavitt Path Algebras

Definition 5.2.1 (Incidence Matrix)**.** The $n \times n$ matrix, $A_E = (a_{ij})$ in $M_n(\mathbb{Z})$ with

$$a_{ij} = \text{number of edges from vertex } v_i \text{ to vertex } v_j$$

is called the Incidence Matrix of the graph E.

Example 5.2.2. *Consider the graph*



Then the matrix of incidence for this graph would be

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We construct a monoid of isomorphism classes of finitely generated projective modules of Leavitt Path Algebras as given by Ara, Moreno and Pardo in [7]. For a directed graph E let T be the free abelian monoid generated by generators $\{v \mid v \in E^0\}$.

We begin by defining a key relation, commonly referred to as relation $(*)$, which plays a fundamental role in the construction of the monoid associated with a graph E. For every regular vertex $v \in E^0$ we impose the following relation:

$$v = \sum_{e \in s^{-1}(v)} r(e) \qquad (*)$$

This equation intuitively expresses the idea that the vertex v can be "decomposed" into the collection of range vertices of all edges that emerge from it.

Let T denote the free abelian monoid generated by the vertices in E^0, and let $\sim_E$ denote the congruence relation on T that is generated by all such (M)-type relations corresponding to regular vertices in the graph.

Using this congruence relation, we define the **graph monoid** M_E as the quotient:

$$M_E = T / \sim_E$$

That is, M_E is obtained by identifying elements of the free monoid T according to the equivalence classes induced by the (M) relations.

An element in M_E is denoted by $[x]$, where $x \in T$ represents a formal sum of vertices in E^0. This notation reflects the class of x modulo the relation $\sim_E$, capturing the essential behavior of how vertices relate through their outgoing edges in the structure of the graph.

Theorem 5.2.3 ([7], Theorem 3.5)**.** *Consider the mapping defined by the rule*

$$[v] \mapsto [vL_k(E)].$$

This correspondence gives rise to an isomorphism of abelian monoids, establishing that $M_E \cong \mathcal{V}(L_k(E))$.

A particular consequence of this isomorphism is the identification

$$\left[\sum_{v \in E^0} v\right] \mapsto [L_k(E)].$$

Thus from Theorem 5.2.3 and Lemma 5.1.6 we get the corollary.

Corollary 5.2.4. *The following conditions are equivalent:*

1. *$L_k(E)$ has IBN property.*
2. *For $m, n \in \mathbb{Z}^+$,*

$$\text{if } m\left[\sum_{v\in E^0} v\right] = n\left[\sum_{v\in E^0} v\right] \text{ in } M_E, \text{ then } m = n.$$

Now we fix some notations. Throughout the remainder of this chapter we assume that E is a always a row finite graph and the set of vertices of E is

$$\{v_i \,|\, 1 \leq i \leq h\}$$

and the set of regular vertices is

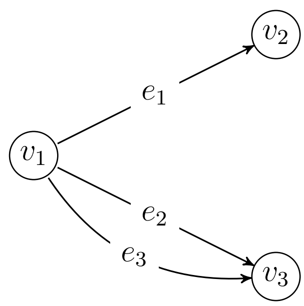
$$\{v_i \,|\, i = 1, 2, \ldots, z\} \text{ with } z \leq h.$$

For $i = 1, 2, \ldots, z$ we define a function

$$M_i : T \to T$$

It acts on an element $x = n_1v_1 + n_2v_2 + \ldots n_hv_h$ of T by applying the relation M on it. An example with make it easier to see.

Example 5.2.5. *Consider the graph*



Then relation $(*)$ is

$$v_1 = v_2 + 2v_3$$

Suppose

$$x = 2v_1 + 3v_2 + 4v_3 \in T$$

Then

$$\begin{aligned} M_1(x) &= M_1(2v_1 + 3v_2 + 4v_3) \\ &= M_1(v_1 + v_1 + 3v_2 + 4v_3) \\ &= v_1 + (v_2 + 2v_3) + 3v_2 + 4v_3 \\ &= v_1 + 4v_2 + 6v_3 \end{aligned}$$

Thus it is clear from the example that the effect of M_i on v_j for $x = n_1v_1 + n_2v_2 + \ldots n_hv_h$ is as follows

1. If v_j is a regular vertex, then

   (a) subtract 1 from coefficient of v_j, i.e. replace n_j by $n_j - 1$ if $n_j > 0$, otherwise do nothing

   (b) add a_{ij} to the coefficient of v_j for each $1 \leq i \leq h$

   The final term v_j will be

$$(n_j - 1 + \sum_{i=1}^{h} a_{ij})v_j$$

2. If v_j is not a regular vertex, then

   (a) add a_{ij} to the coefficient of v_j for each $1 \leq i \leq h$

   The final term v_j will be

$$(n_j + \sum_{i=1}^{h} a_{ij})v_j$$

Let σ be a finite sequence from $\{1, 2, \ldots, z\}$. Then for any $y \in T$, let $\Gamma_\sigma(y) \in T$ be the element that we get when we apply M_i's to y in the order specified by σ.

Lemma 5.2.6 (Confluence Lemma, [7], Lemma 4.3)**.** *Given any two elements $x, y \in T$, the equality $[x] = [y]$ holds within the monoid M_E precisely when sequences σ and σ', composed of elements from $\{1, 2, \ldots, z\}$, can be found such that the condition $\Gamma_\sigma(x) = \Gamma'_\sigma(y)$ is satisfied in T.*

We are now equipped to state the principal result of this chapter — a complete characterization of when the Leavitt Path Algebra of a graph satisfies the IBN property, expressed explicitly through matrix-theoretic conditions. This theorem bridges the structural aspects of the graph with the algebraic behavior of its associated LPA, offering a precise and computable criterion for the IBN property to hold.

Theorem 5.2.7. *Let E be a finite graph with vertices*

$$\{v_i \mid 1 \leq i \leq h\}$$

where the regular vertices are $v_1, \dots, v_z$. Define

$$J_E = \begin{pmatrix} I_z & 0 \\ 0 & 0 \end{pmatrix} \in M_h(\mathbb{N}) \quad \text{and} \quad b = [1 \ \dots \ 1]^t \in M_{h\times 1}(\mathbb{N}),$$

and let $[A_E^t - J_E \quad b]$ denote the matrix obtained by adding the column b to $A_E^t - J_E$. For any field K, the Leavitt path algebra $L_K(E)$ has the Invariant Basis Number property if and only if

$$rk(A_E^t - J_E) < rk([A_E^t - J_E \quad b]).$$

Proof. ($\Leftarrow$) Let $\text{rk}(A_E^t - J_E) < \text{rk}([A_E^t - J_E \quad b])$. We want to show that $L_k(E)$ has IBN property.

To show this, assume that

$$L_k(E)^m \cong L_k(E)^n$$

for some integers m, n, then we need to show that $m = n$.

Let $X := \sum_{i=1}^h v_i$. Then by Corollary 5.2.4, it is clear that it is sufficient to prove that

$$m[X] = n[X] \Rightarrow m = n$$

Since $m[X] = n[X]$ in M_E, by Confluence Lemma, there are two sequences $\sigma = (j_1, \dots, j_p)$ and $\sigma' = (j'_1, \dots, j'_{p'})$ of indices of regular vertices ($1 \leq j_r, j'_s \leq z$) such that applying the corresponding substitution rules M_j leads to the same element in the *free* monoid T:

$$\Lambda_\sigma(mX) = \Lambda_{\sigma'}(nX) = \gamma \in T$$

Let k_j be the number of times M_j appears in σ and k'_j be the number of times that M_j appears in σ'. Then we calculate the term $\Lambda_\sigma(mX)$. As we have seen already the effect of applying *one* M_j on a term v_i, doing it k many times will replicate that effect k times. We find that

$$\text{coeff}(v_i, \Lambda_\sigma(mX)) = \begin{cases} m - k_i + \sum_{j=1}^z a_{ji}k_j & \text{if } i = 1, 2, \dots, z \\ m + \sum_{j=1}^z a_{ji}k_j & \text{if } i = z+1, \dots, h \end{cases}$$

and similarly

$$\text{coeff}(v_i, \Lambda_{\sigma'}(nX)) = \begin{cases} n + \sum_{j=1}^{z} a_{ji}k'_j - k'_i & \text{if } i = 1, 2, \dots, z \\ n + \sum_{j=1}^{z} a_{ji}k'_j & \text{if } i = z+1, \dots, h \end{cases}$$

Thus we get the final calculation as

$$\begin{aligned}
\gamma &= \Lambda_{\sigma'}\left(m \sum_{i=1}^{h} v_i\right) \\
&= \left((m - k_1) + a_{11}k_1 + a_{21}k_2 + \cdots + a_{z1}k_z\right) v_1 \\
&\quad + \left((m - k_2) + a_{12}k_1 + a_{22}k_2 + \cdots + a_{z2}k_z\right) v_2 \\
&\qquad \vdots \\
&\quad + \left((m - k_z) + a_{1z}k_1 + a_{2z}k_2 + \cdots + a_{zz}k_z\right) v_z \\
&\quad + \left(m + a_{1(z+1)}k_1 + a_{2(z+1)}k_2 + \cdots + a_{z(z+1)}k_z\right) v_{z+1} \\
&\qquad \vdots \\
&\quad + \left(m + a_{1h}k_1 + a_{2h}k_2 + \cdots + a_{zh}k_z\right) v_h.
\end{aligned}$$

and

$$\begin{aligned}
\gamma &= \Lambda_{\sigma'}\left(n \sum_{i=1}^{h} v_i\right) \\
&= \left((n - k'_1) + a_{11}k'_1 + a_{21}k'_2 + \cdots + a_{z1}k'_z\right) v_1 \\
&\quad + \left((n - k'_2) + a_{12}k'_1 + a_{22}k'_2 + \cdots + a_{z2}k'_z\right) v_2 \\
&\qquad \vdots \\
&\quad + \left((n - k'_z) + a_{1z}k'_1 + a_{2z}k'_2 + \cdots + a_{zz}k'_z\right) v_z \\
&\quad + \left(n + a_{1(z+1)}k'_1 + a_{2(z+1)}k'_2 + \cdots + a_{z(z+1)}k'_z\right) v_{z+1} \\
&\qquad \vdots \\
&\quad + \left(n + a_{1h}k'_1 + a_{2h}k'_2 + \cdots + a_{zh}k'_z\right) v_h.
\end{aligned}$$

We now equate the coefficients of the terms v_j to get

- **Case 1:** $1 \le i \le z$ **(Regular vertex** v_i**)**

$$m - k_i + \sum_{j=1}^{z} a_{ji}k_j = n - k'_i + \sum_{j=1}^{z} a_{ji}k'_j$$

$$\Rightarrow \sum_{j=1}^{z} a_{ji}(k'_j - k_j) - (k'_i - k_i) = m - n$$

Let $m_j = k'_j - k_j$. Then

$$\sum_{j=1}^{z} a_{ji} m_j - m_i = m - n$$

$$\Rightarrow \sum_{j=1}^{z} a_{ji}(m_j - \delta_{ij}) = m - n$$

- **Case 2:** $z < i \leq h$ **(Sink** v_i**)**

$$m + \sum_{j=1}^{z} a_{ji} k_j = n + \sum_{j=1}^{z} a_{ji} k'_j$$

$$\Rightarrow \sum_{j=1}^{z} a_{ji}(k'_j - k_j) = m - n$$

Let $m_j = k'_j - k_j$. Then

$$\sum_{j=1}^{z} a_{ji} m_j = m - n$$

$$\Rightarrow \sum_{j=1}^{z} a_{ji} m_j = m - n$$

Combining these two we get

$$\begin{cases} m - n = (a_{11} - 1)m_1 + a_{21}m_2 + \cdots + a_{z1}m_z, \\ m - n = a_{12}m_1 + (a_{22} - 1)m_2 + \cdots + a_{z2}m_z, \\ \quad \vdots \\ m - n = a_{1z}m_1 + a_{2z}m_2 + \cdots + (a_{zz} - 1)m_z, \\ m - n = a_{1(z+1)}m_1 + a_{2(z+1)}m_2 + \cdots + a_{z(z+1)}m_z, \\ \quad \vdots \\ m - n = a_{1h}m_1 + a_{2h}m_2 + \cdots + a_{zh}m_z. \end{cases}$$

which is equivalent to the system

$$(A_E^t - J_E)x = (m - n)b$$

with a solution

$$x = \begin{pmatrix} m_1 \\ \vdots \\ m_z \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{Z}^h$$

This gives us

$$\text{rk}(A_E^t - J_E) = \text{rk}([A_E^t - J_E \quad (m-n)b])$$

by Rouché-Capelli theorem.

If $m \neq n$ then $m - n \neq 0$, thus

$$\text{rk}([A_E^t - J_E \quad (m-n)b]) = \text{rk}([A_E^t - J_E \quad b])$$

and hence

$$\text{rk}(A_E^t - J_E) = \text{rk}([A_E^t - J_E \quad b])$$

which contradicts our assumption. Thus we must have $m = n$.

($\Rightarrow$) Let $\text{rk}(A_E^t - J_E) = \text{rk}([A_E^t - J_E \quad b])$. We will show that $L_k(E)$ does not have the IBN property. To do so we need to find distinct integers m and n such that

$$m[\sum_{i=1}^{h} v_i] = n[\sum_{i=1}^{h} v_i]$$

Now, just as the last half of the proof, we can proceed and get distinct integers n and m such that

$$\begin{cases} m - n = (a_{11} - 1)m_1 + a_{21}m_2 + \cdots + a_{z1}m_z, \\ m - n = a_{12}m_1 + (a_{22} - 1)m_2 + \cdots + a_{z2}m_z, \\ \quad \vdots \\ m - n = a_{1z}m_1 + a_{2z}m_2 + \cdots + (a_{zz} - 1)m_z, \\ m - n = a_{1(z+1)}m_1 + a_{2(z+1)}m_2 + \cdots + a_{z(z+1)}m_z, \\ \quad \vdots \\ m - n = a_{1h}m_1 + a_{2h}m_2 + \cdots + a_{zh}m_z. \end{cases}$$

where $m_j = k'_j - k_j$ for all $1 \leq j \leq z$. Then we can write them as

$$(A_E^t - J_E)x = (m-n)b$$

where

$$x = \begin{pmatrix} m_1 \\ \vdots \\ m_z \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{Z}^h$$

We construct the integers m and n as follows:

1. By using $\mathrm{rk}(A_E^t - J_E) = \mathrm{rk}([A_E^t - J_E \quad b]) =: r \leq z$, we can reduce the matrix $[A_E^t - J_E \quad b]$ to its row echelon form and it will be of the form

$$\begin{pmatrix} 0 & \cdots & 0 & a_{1j_1} & \cdots & a_{1(j_2-1)} & 0 & a_{1(j_2+1)} & \cdots & a_{1(j_r-1} & 0 & \cdots & b_1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & a_{2j_2} & a_{2(j_2+1)} & \cdots & a_{2j_r-1} & 0 & \cdots & b_2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{rj_r} & \cdots & b_r \\ 0 & \cdots & 0 & 0 & 0\cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0\cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

   and here $j_1 < j_2 < \cdots < j_r$, $a_{1j_1}, a_{2j_2}, \cdots, a_{rj_r} \neq 0$ and $b_i's$ are not all zero.

2. We choose m_j, n and m with the rule:

$$m_j = \begin{cases} \frac{b_j |a_{1j_1} a_{2j_2} \cdots a_{rj_r}|}{a_{ij_i}}, & \text{if } j = j_i (1 \leq i \leq r) \\ 0, & \text{otherwise.} \end{cases}$$

$$n = \max\{|m_j|| j = 1, \ldots, h\} \text{ and } m = |a_{1j_1} a_{2j_2} \cdots a_{rj_r}| + n$$

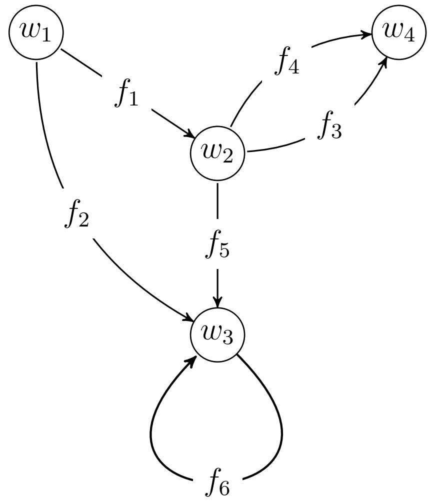
3. And k_j and k'_j $(1 \leq j \leq t)$ are given by:

$$(k'_j, k_j) = \begin{cases} (0,0), & \text{if } m_j = 0 \\ (m_j, 0), & \text{if } m_j > 0 \\ (0, -m_j), & \text{if } m_j < 0. \end{cases}$$

After this, calculations show that these are the m and n we need and these integers indeed solve the system of equations. □

Now let us see some examples where we use this theorem.

Example 5.2.8. *Consider E to be the graph*



Then

$$A_E = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are 3 regular vertices $\{v_1, v_2, v_3\}$, hence

$$J_E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and thus

$$A_E^t - J_E = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \text{ which has echelon form } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Also

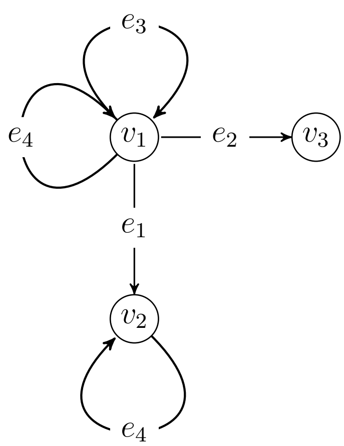
$$[A_E^t - J_E \quad b] = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 \end{pmatrix} \text{ which has echelon form } \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So

$$rk(A_E^t - J_E) = 2 < 3 = rk([A_E^t - J_E \quad b])$$

Therefore $L_k(E)$ has IBN property.

Example 5.2.9. *Consider the graph E as follows*



Then

$$A_E = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

There are 2 regular vertices $\{v_1, v_2\}$, hence

$$J_E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and thus

$$A_E^t - J_E = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ which has echelon form } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Also

$$[A_E^t - J_E \;\; b] = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \text{ which has echelon form } \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So

$$rk(A_E^t - J_E) = 1 = rk([A_E^t - J_E \quad b])$$

Therefore $L_k(E)$ is non-IBN.

To calculate m and n we follow the construction in the theorem.

We see that $j_1 = 1, a_{11} = 1, b_1 = 1$, *so*

$$m_1 = \frac{b_1|a_{11}|}{a_{11}} = 1$$

$$n = \max\{m_1\} = 1 \textit{ and } m = |a_{11}| + n = 2$$

Thus we get $L_k(E) \cong L_k(E)^2$.

5.3 Cohn Path Algebra

Definition 5.3.1. We define the set $(E^1)^* = \{e^* \mid e \in E^1\}$ to consist of formal symbols corresponding to the edges of E. The *Cohn path algebra* of E over the field k, denoted by $C_k(E)$, is the free associative k-algebra generated by the disjoint union of the sets E^0, E^1, and $(E^1)^*$, subject to the following relations:

- (V) For all $v, w \in E^0$, $vw = \delta_{v,w}v$.
- (E1) For all $e \in E^1$, $s(e)e = e = er(e)$.
- (E2) For all $e \in E^1$, $r(e)e^* = e^* = e^*s(e)$.
- (CK1) For all $e, f \in E^1$, $e^*f = \delta_{e,f}r(e)$.

These relations mirror those used during the construction of Leavitt path algebras, except that the Cohn path algebra does not impose the relation (CK2), which is essential for defining $L_k(E)$. Thus, $C_k(E)$ can be realised as a natural intermediate construction capturing partial structure of $L_k(E)$ without enforcing full Cuntz-Krieger conditions.

Proposition 5.3.2. Let $C_k(E)$ be the Cohn path algebra of the graph E and let

$$I = \langle \{v - \sum_{s(e)=v} ee^* | v \in \text{Reg(E)}\}\rangle$$

then

$$C_k(E)/I \cong L_k(E)$$

Unlike Leavitt path algebras, the set $\{xy^*|y, y \in \text{Path}(E) \text{ for which } r(\mu)\text{=}r(\lambda)\}$ actually forms a basis of the Cohn path algebras.

Proposition 5.3.3. As a k-vector space

$$\mathcal{B} = \{\lambda\mu^*|\lambda\mu \in \text{Path}(E),\, r(\lambda) = r(\mu)\}$$

is a basis of $C_k(E)$.

Proof. Consider A to be the K-vector space with basis $\mathcal{B}$. We define a bilinear product on A as follows:

$$(\lambda_1 v_1^*)(\lambda_2 v_2^*) = \begin{cases} \lambda_1 \lambda_2' v_2^* & \text{if } \lambda_2 = v_1 \lambda_2' \text{ for some } \lambda_2' \in \text{Path}(E), \\ \lambda_1 (v_1')^* v_2^* & \text{if } v_1 = \lambda_2 v_1' \text{ for some } v_1' \in \text{Path}(E), \\ 0 & \text{otherwise.} \end{cases}$$

To show that this defines an associative K-algebra structure on A, it suffices to verify the associativity condition, that is, to show $x = y$, where

$$x = (\lambda_1 v_1^*)((\lambda_2 v_2^*)(\lambda_3 v_3^*)) \quad \text{and} \quad y = ((\lambda_1 v_1^*)(\lambda_2 v_2^*))(\lambda_3 v_3^*).$$

After carrying out the multiplications step by step using the defining relations of the algebra, we find that:

$$x = y = \begin{cases} \lambda_1 \lambda_2' \lambda_3' v_3^* & \text{if } \lambda_3 = v_2 \lambda_3' \text{ and } \lambda_2 = v_1 \lambda_2', \\ \lambda_1 \lambda_2 \lambda_3' v_3^* & \text{if } \lambda_3 = v_2 \lambda_3'' \lambda_3' \text{ and } v_1 = \lambda_2 \lambda_2', \\ \lambda_1 (v_1')^* \lambda_3' v_3^* & \text{if } \lambda_3 = v_2 \lambda_3' \text{ and } v_1 = \lambda_2 \lambda_2' v_1', \\ \lambda_1 \lambda_2' (v_2')^* v_3^* & \text{if } v_2 = \lambda_3 v_2' \text{ and } \lambda_2 = v_1 \lambda_2', \\ \lambda_1 (v_1')^* (v_2')^* v_3^* & \text{if } v_2 = \lambda_3 v_2' \text{ and } v_1 = \lambda_2 \lambda_2' v_1', \\ 0 & \text{otherwise.} \end{cases}$$

This proves what we wanted.

□

5.4 Relation of Cohn path algebra with Leavitt path algebra

Definition 5.4.1. Let $X \subseteq \text{Reg(E)}$. For $v \in \text{Reg(E)}$, define $q_v = v - \sum_{e \in s^{-1}(v)} ee^*$. Then take

$$I^* = \text{ideal of } C_k(E) \text{ generated by } \{q_v | v \in \text{Reg}(E)\}$$

Then the Cohn Path Algebra of E relative to X, denoted by $C_k^X(E)$ is

$$C_k(E)/I$$

Theorem 5.4.2. ***(Universal property of Cohn path algebra)*** *Suppose E is a graph, X is a subset of* Reg(E)*, and A is a k-algebra that contains a set of pairwise orthogonal idempotents $\{x_v \mid v \in E^0\}$ and two sets $\{x_e \mid e \in E^1\}$, $\{y_e \mid e \in E^1\}$ satisfying the following:*

(i) $x_{s(e)} x_e = x_e x_{r(e)} = x_e$ *and* $x_{r(e)} y_e = y_e x_{s(e)} = y_e$ *for all* $e \in E^1$*,*

(ii) $y_f x_e = \delta_{ef} x_{r(e)}$ *for all* $e, f \in E^1$*, and*

(iii) $x_v = \sum_{\{e \in E^1 | s(e)=v\}} x_e y_e$ *for every vertex* $v \in X$*.*

Using the relations that define the relative Cohn path algebra, there is a unique k*-algebra homomorphism* $\varphi\colon C_k^X(E) \to A$ *such that* $\varphi(v) = x_v$*,* $\varphi(e) = x_e$*, and* $\varphi(e^*) = y_e$ *for all* $v \in E^0$ *and* $e \in E^1$*.*

We also see that

$$C_k^{\mathrm{Reg}(E)}(E) \cong L_k(E) \text{ and } C_k(E) \cong C_k^{\phi}(E)$$

A significant advantage offered by Cohn path algebras is the ability to transform a given graph E into a modified graph whose associated Leavitt path algebra is isomorphic to that of the original. Leveraging this property, we will now present the methodology for constructing a new graph, designated $E(X)$, based on an initial graph E.

Definition 5.4.3. Let X be subset of $\mathrm{Reg}(E)$. Take $Y = \mathrm{Reg}(E) \setminus X$. Let

$$Y' = \{v' | v \in Y\}$$

For $v \in Y$ and $e \in r^{-1}(v)$ we take a symbol e'.

Then $E(X)$ is the graph with

$$E(X)^0 = E^0 \sqcup Y'$$

$$E(X)^1 = E^1 \sqcup \{e' | r(e) \in Y\}$$

For $e \in E^1$, the relations amongst the edges and the vertices do not change in $E(X)$. For $e' \in E(X)^1$, set $s_{E(X)}(e') = s_E(e)$ and $e_{E(X)}(e') = r_E(e)'$.

What this construction essentially does is the following: for a given graph E, consider the set of regular vertices that are not in X, denoted by $\mathrm{Reg}(E) \setminus X$. For each vertex v in this set, we introduce a new vertex v' into the graph. Then, for every edge e in E with range $r(e) = v$ and source $s(e) = u$, we add a new edge from u to v'. In other words, we duplicate the incoming edges of v, redirecting them to a new copy v', while leaving the original structure of the graph intact. This modification is best understood through an example, which we now present.

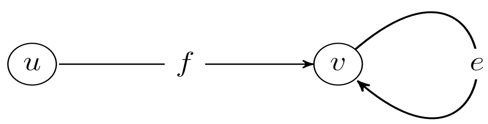


Figure 14: a graph

Then we see that $\text{Reg}(E) = \{u, v\}$.

First take $X = \phi$, then we have to add new vertices corresponding to $\text{Reg}(E) \setminus X = \{u, v\}$. Then there are two edges going into v, viz. e and f. Thus we make new edges e' and f' from v and u respectively. There is no edge going into u, hence we do not draw any edge to u'. Finally we see that the result is

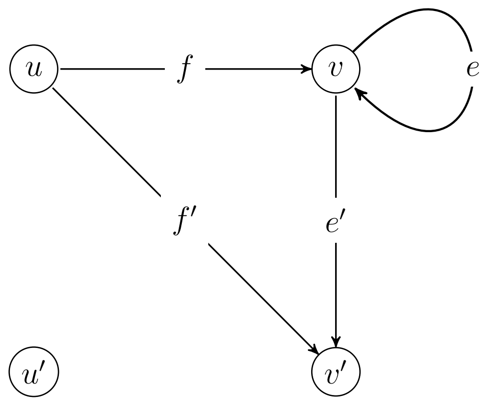


Figure 15: $E(X)$ when $X = \phi$

Next, take $X = \{u\}$, then we have to add new vertices corresponding to $\text{Reg}(E) \setminus X = \{v\}$. Then there are two edges going into v, viz. e and f. Thus we make new edges e' and f' from v and u respectively. Then we get

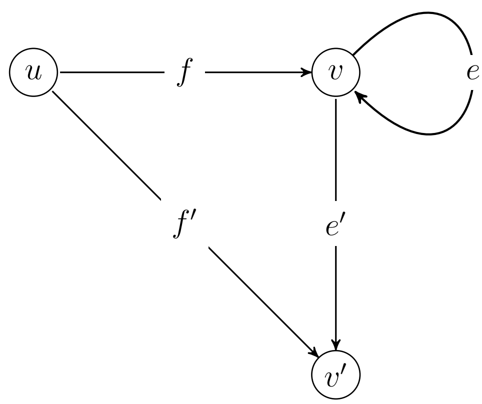


Figure 16: $E(X)$ when $X = \{u\}$

Finally, take $X = \{u, v\}$, then $\text{Reg}(E) \setminus X = \phi$. Hence we add no new vertices, and the graph remains the same.

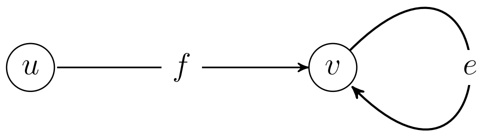


Figure 17: $E(X)$ when $X = \{u, v\}$

Building upon the prior observation that every Leavitt path algebra admits a realization as a relative Cohn path algebra, concretely via the isomorphism $L_k(E) \cong C_k^{\text{Reg}(E)}(E)$, the forthcoming theorem demonstrates the converse. Specifically, it shows that any given relative Cohn path algebra can indeed be expressed as the Leavitt path algebra corresponding to an appropriately constructed graph.

Theorem 5.4.4 ([2], Theorem 1.5.18)**.** *Let $X \subseteq$ Reg(E), and let $E(X)$ be the graph as defined above, then*

$$C_k^X(E) \cong L_k(E(X))$$

5.5 Cohn Path Algebra has IBN Property

Gene Abrams and Müge Kanuni prove in the paper [4] that Cohn path algebras have IBN property. Here is another proof of the same with a matrix theoretic approach.

Theorem 5.5.1. *For a graph E, the Cohn Path Algebra $C_k(E)$ has IBN property.*

Proof. From theorem 5.4.4, we see that when $X = \emptyset$ then $L_k(E(X)) \cong C_k(E)$. We make the following observations about the new graph $E(X)$:

1. There are $h + z$ vertices in $E(X)$. Label them as $\{v_1, v_2, \ldots, v_z, v_{z+1}, \ldots, v_h, v'_1, v'_2 \ldots, v'_z\}$, where $v_1, \ldots, v_z$ are the regular vertices in E (we maintain this ordering).

2. The newly added vertices v'_i have no vertices emerging from them, i.e. they are sinks.

3. The number of edges going into a new vertex v_i' is same as the number of edges that go into the vertex v_i.

We see that

$$A_E = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1z} & a_{1(z+1)} & \cdots & a_{1h} \\ a_{21} & a_{22} & \cdots & a_{2z} & a_{2(z+1)} & \cdots & a_{2h} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{z1} & a_{z2} & \cdots & a_{zz} & a_{z(z+1)} & \cdots & a_{zh} \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(h\times h)}$$

Then keeping the observations in mind, we see that

$$A_{E(X)} \begin{pmatrix} a_{11} & \cdots & a_{1z} & a_{1(z+1)} & \cdots & a_{1h} & a_{11} & \cdots & a_{1z} \\ a_{21} & \cdots & a_{2z} & a_{2(z+1)} & \cdots & a_{2h} & a_{21} & \cdots & a_{2z} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{z1} & \cdots & a_{zz} & a_{z(z+1)} & \cdots & a_{zh} & a_{z1} & \cdots & a_{zz} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}_{((h+z))\times(h+z))}$$

And now

$$[A_E^t - J_E \quad b] = \begin{pmatrix} a_{11}-1 & a_{21} & \cdots & a_{z1} & 0 & \cdots & 0 & 1 \\ a_{12} & a_{22}-1 & \cdots & a_{z2} & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{1z} & a_{2z} & \cdots & a_{zz}-1 & 0 & \cdots & 0 & 1 \\ a_{1(z+1)} & a_{2(z+1)} & \cdots & a_{z(z+1)} & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{1h} & a_{2h} & \cdots & a_{zh} & 0 & \cdots & 0 & 1 \\ a_{11} & a_{21} & \cdots & a_{z1} & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{1z} & a_{2z} & \cdots & a_{zz} & 0 & \cdots & 0 & 1 \end{pmatrix}$$

To simplify this, for each $i = 1, \ldots, z$, we subtract row i from row $h+i$ in this matrix which is equivalent to:

$$\begin{pmatrix} a_{11}-1 & a_{21} & \cdots & a_{z1} & 0 & \cdots & 0 & 1 \\ a_{12} & a_{22}-1 & \cdots & a_{z2} & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{1z} & a_{2z} & \cdots & a_{zz}-1 & 0 & \cdots & 0 & 1 \\ a_{1(z+1)} & a_{2(z+1)} & \cdots & a_{z(z+1)} & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{1h} & a_{2h} & \cdots & a_{zh} & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Next we use the identity matrix in the lower left portion of the matrix to cancel out other rows and get the equivalent matrix:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Thus we find that

$$\operatorname{rk}(A_E^t - J_E) = z < z+1 = \operatorname{rk}([A_E^t - J_E \quad b])$$

And by theorem 5.2.7 we conclude that $L_k(E(X))$ has invariant basis property and in turn $C_k(E)$ has invariant basis property. □

6 The IBN property for graphs constructed from finite groups

The present chapter is devoted to examining the structure of Leavitt Path Algebras corresponding to certain graphs that are derived from finite groups.

First we take a look at *Cayley graph* of finite groups.

6.1 Cayley graph

Definition 6.1.1 (Cayley graph)**.** Let G be a finite group and S be a generating set for G. The *Cayley graph* of G with respect to S, denoted by $E_{G,S}$ has the following properties:

1. $E^0_{G,S} = \{v_g \mid g \in G\}$
2. There is an edge from a vertex v_g to v_h if there is an element $s \in S$ such that $h = gs$.

Some examples will make the Cayley graph more clear.

Example 6.1.2. *Let $G = \mathbb{Z}_n$ where $n \geq 2$. We know that the singleton set $\{1\}$ can generate $\mathbb{Z}_n$. We take $S = \{1, j\}$ where j is a fixed integer $j = 0, 1, \ldots, n-1$. Then the Cayley graph we obtain from this group is denoted by C^j_n.*

It has n vertices, viz., $\{v_0, v_1, \ldots, v_{n-1}\}$ and $2n$ edges $\{e_0, e_1, \ldots, e_{n-1}, f_0, f_1, \ldots, f_{n-1}\}$ when $j \neq 1$, and n edges $\{e_0, e_1, \ldots, e_{n-1}\}$ when $j = 1$. The rules for the edges are

$$s(e_i) = v_i\ ,\ s(f_i) = v_i\ ,\ \ r(e_i) = v_{i+1},\ r(f_i) = v_{i+j}$$

Following are the graphs $C^0_4, C^1_4, C^2_4, C^3_4$.

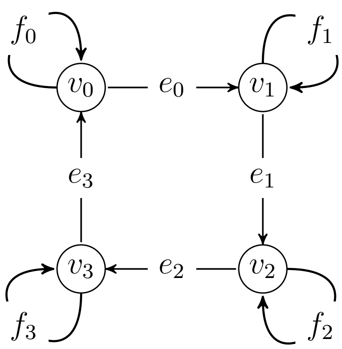


Figure 18: C^0_4

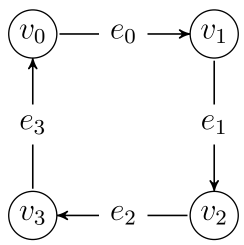


Figure 19: C^1_4

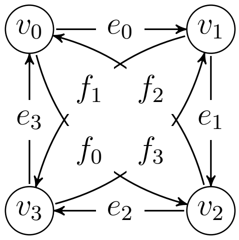


Figure 20: C_4^2

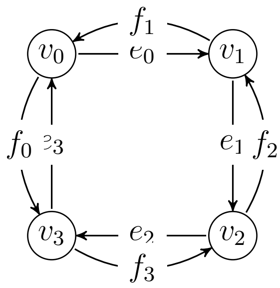


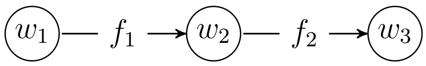
Figure 21: C_4^3

Definition 6.1.3. A subset H of a graph E called *hereditary* if for all $v \in H$ and $w \in E^0$, $v \geq w$ implies that $w \in H$.

In other words, a subset of E^0 is said to be hereditary if whenever a vertex v is in H then any vertex w reachable from v via a path must also be in H.

Example 6.1.4. *Some examples of hereditary subsets are:*

1. *the empty set* $\emptyset$ *is always hereditary.*
2. *the set of all vertices* E^0 *is always hereditary.*
3. *consider the graph*



Then the sets $\{v_3\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}$ *are hereditary but the set* $\{v_1\}$ *is not hereditary since* v_2 *can be reached by* v_1 *but* $v_2 \notin \{v_1\}$.

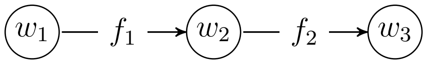
Definition 6.1.5. A subset H of E^0 is said to be *saturated* if whenever for any $v \in E^0$, $s^{-1}(v) \neq \emptyset$ and $\{r(e)|s(e) = v\} \subseteq H$, then $v \in H$.

In other words a subset H is said to be saturated if for every vertex v in the graph with the property that every edge coming out of v goes to a vertex within H, then v must be in H.

Example 6.1.6. *Some examples of saturated subsets are:*

1. *the empty set* $\emptyset$ *is always saturated.*
2. *the set of all vertices* E^0 *is always saturated.*

3. consider the graph



Then the subsets $\{v_1\}, \{v_1, v_2\}, \{v_1, v_2, v_3\}$ are saturated but $\{v_3\}$ is not saturated since all edges coming out of v_2 land into some vertex in $\{v_3\}$ but $v_2 \notin \{v_3\}$.

6.2 Properties of Cayley graphs

Definition 6.2.1. A unital k-algebra A is said to be *purely infinite simple* if

1. A is not a division ring
2. for every non zero element $x \in A$, there exist p and q in A such that $pxq = 1_A$

Definition 6.2.2. An algebra A is said to be *simple* if it does not have any non-trivial two sided ideal.

Example 6.2.3. *For any field $\mathbb{F}$ the algebra $M_n(\mathbb{F})$ is a simple algebra.*

The following is an interesting fact that was proved by Artin and Wedderburn that essentially characterizes all finite dimensional simple algebras over a field.

Theorem 6.2.4 ([11], Theorem 7.1.1)**.** *Let $\mathbb{F}$ be a field and B a finite-dimensional $\mathbb{F}$-algebra. Then B is simple if and only if $B \simeq M_n(D)$ where $n \geq 1$ and D is a finite-dimensional division $\mathbb{F}$-algebra.*

For more insight into simple algebra refer to chapter 7 of [11].

For the next theorem we will state two theorems from the cited sources.

Theorem 6.2.5 ([5], Theorem 3.11)**.** *Suppose E represents a row-finite graph. The associated Leavitt path algebra $L_k(E)$ qualifies as simple precisely when the graph E adheres to the following two stipulations:*

1. *Within the vertex set E^0, the only subsets that are simultaneously hereditary and saturated are the empty set $\emptyset$ and E^0 itself; and*
2. *An exit is present for every cycle contained within E.*

Theorem 6.2.6 ([6], Theorem 11)**.** *A graph E gives rise to a purely infinite simple Leavitt path algebra $L_k(E)$ iff the graph E adheres to the following set of characteristics:*

1. *The only subsets of E^0 that are both hereditary and saturated are the empty set $\emptyset$ and the entire set E^0.*

2. *Within E, every cycle must possess an exit.*

3. *Each vertex in E has a path connecting it to some cycle.*

We denote an arbitrary finite group by G and $S \subseteq G$ such that $\langle S \rangle$.

Theorem 6.2.7. *The following conditions are equivalent:*

1. *$L_k(E_{G,S})$ is purely infinitely simple*

2. *$L_k(E_{G,S})$ is simple*

3. *$|S| \geq 2$.*

Proof. (1) $\Rightarrow$ (2): Let $L_k(E_{G,S})$ be purely infinite simple. Let I be a non-trivial two sided ideal of $L_k(E_{G,S})$. Let $0 \neq x \in I$. Then by the definition of purely infinite simple, there are a and b in E such that

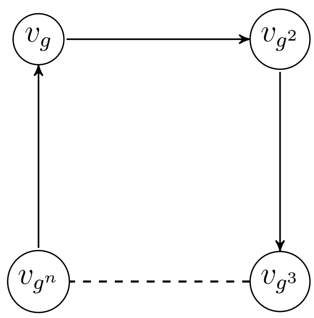
$$axb = 1$$

As I is a two sided ideal, $axb = 1 \in I \Rightarrow I = L_k(E_{G,S})$. And this is a contradiction since we assumed that I is non-trivial, hence $L_k(E_{G,S})$ is simple.

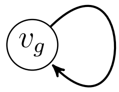
(2) $\Rightarrow$ (3): Let $L_k(E_{G,S})$ be simple. Let, if possible, $|S| = 1$, i.e., $S = \{g\}$. Then G is cyclic with generator g. Let $|G| = n$. Then

$$G = \{g^i \mid i = 1, 2, \ldots n\}$$

and hence $E^0_{G,S} = \{v_{g^i} \mid i = 1, 2, \ldots n\}$ and an edge is present from v_{g^i} to $v_{g^{i+1}}$ and when $n \geq 2$ the graph $E_{G,S}$ is of the form:



and when $n = 1$ then we have:



Thus we see that in either case none of the cycles have any exit, hence by theorem 6.2.5 $L_k(E_{G,S})$ is not simple, a contradiction.

(3) $\Rightarrow$ (1): Let $|S| \geq 2$, so assume that $S = \{s_1, s_2, \ldots, s_k\}$ with $k \geq 2$. Let H be a non-empty saturated and hereditary subset of $E^0_{G,S}$. We first prove that $H = E^0_{G,S}$.

Let $v_h \in H$. Since G is a group and S is a generating set for G, we have

$$h^{-1} = s_{i_1} s_{i_2} \cdots s_{i_m}$$

where $1 \leq i_j \leq k$, s_{i_j} are not necessarily distinct and m is some positive integer. So

$$1_G = h s_{i_1} s_{i_2} \cdots s_{i_m}$$

We now show that $v_{1_G} \in H$. We show that there is a path from v_h to v_{1_G}. We verify the steps one by one using the definition of the graph $E_{G,S}$

$$\begin{gathered}\text{there is an edge } e_1 \text{ from } h \text{ to } hs_{i_1} \\ \text{there is an edge } e_2 \text{ from } hs_{i_1} \text{ to } hs_{i_1}s_{i_2} \\ \vdots \\ \text{there is an edge } e_{m-1} \text{ from } hs_{i_1}s_{i_2}\cdots s_{i_{m-1}} \text{ to } hs_{i_1}s_{i_2}\cdots s_{i_m}\end{gathered}$$

Thus there is a path $e_1e_2\cdots e_{m-1}$ from v_h to $v_{hs_{i_1}s_{i_2}\cdots s_{i_m}} = v_{1_G}$. And since $v_h \in H$ which is hereditary, thus $v_{1_G} \in H$.

Now let v_g be an arbitrary vertex of $E_{G,S}$. We show that $v_g \in H$. Since $1_G.g = g$ and $g = s_{i_1}s_{i_2}\cdots s_{i_m}$ for some positive integer m, by the same argument as above there is a path joining v_{1_G} to v_g. And since $v \in H$ and H is hereditary, $v_g \in H$. Thus $H = L_{G,S}$.

From these points we see that any vertex v_g we consider there is a path joining v_{1_G} to v_g and there is a path joining v_g to v_{1_G}, hence there is always a cycle based at v_g.

And for every vertex v_g, there is exactly one edge going from v_g to v_{gg_i} for all $1 \leq i \leq k$, and these are vertices v_{gg_i} are pairwise distinct. Thus we find that every cycle in $E_{G,S}$ has an exit. So we can invoke theorem 6.2.6 and we obtain that $L_k(E_{G,S})$ is purely infinite simple. □

Subsequently, we shall present the criteria that determine when the LPA associated with the Cayley graph of a finite group satisfies the IBN Property.

Theorem 6.2.8. *The following conditions are equivalent:*

1. *$L_k(E_{G,S})$ has IBN property*
2. *G is cyclic, i.e, S has only one element*
3. *$E_{G,S}$ consists of a single cycle with $|G|$ vertices.*

Proof. (1) ⇒ (2): Let $L_k(E_{G,S})$ have IBN property. Let, if possible, S have more than one element, i.e., $S = \{s_1, s_2, \dots, s_k\}$ where $k \geq 2$. Then for each $i = 1, \dots, k$, since G is a finite group, we notice that

$$Gg_i := \{gg_i | g \in G\} = G$$

By the construction of $E_{G,S}$ we see that there is exactly one edge from v_g to v_{gg_i} for each $1 \leq i \leq k$. Also a vertex v_g only emits edges to v_{gg_i} for $1 \leq i \leq k$ and no more. This means that

$$[v_g] = [v_{gg_1}] + [v_{gg_2}] + \cdots + [v_{gg_k}] \text{ in } M_{E_{G,S}}$$

and thus

$$[\sum_{g\in G} v_g] = [\sum_{g\in G} v_{gg_1}] + [\sum_{g\in G} v_{gg_2}] + \cdots + [\sum_{g\in G} v_{gg_k}] \text{ in } M_{E_{G,S}}.$$

Also, for each $1 \leq i \leq k$, using $Gg_i = G$ we also have that

$$[\sum_{g\in G} v_g] = [\sum_{g\in G} v_{gg_i}] \text{ in } M_{E_{G,S}}$$

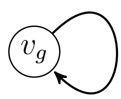
And from these observations we get that

$$[\sum_{g\in G} v_g] = \sum_{i=1}^{k} \sum_{g\in G} v_{gg_i} = k[\sum_{g\in G} v_g] \text{ in } M_{E_{G,S}}$$

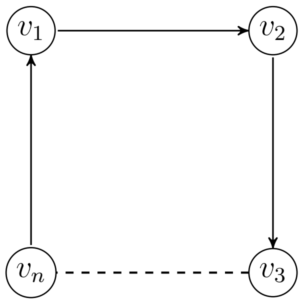
which gives us that $L_k(E_{G,S})$ does not have IBN property by corollary 5.2.4, which is a contradiction.

(2) ⇒ (3): Let $|S| = 1$ then by the construction given in theorem 6.2.7, we immediately see that $E_{G,S}$ is a graph with only one cycle with $|G|$ vertices.

(3) ⇒ (1): Let $E_{G,S}$ be a graph with only one cycle with $|G| = n$ vertices. We make cases, first we tackle the easy case when $n = 1$. Then $E_{G,S}$ is the graph:



Then $L_k(E_{G,S}) \cong k[x, x^{-1}]$ by theorem 3.3.4, and it has IBN property. Now let $n \geq 2$. Then the graph is of the form:



We notice that every vertex of this graph is a regular vertex. So

$$A_E = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

And thus

$$[A_E^t - J_E \;\; b] = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 1 & -1 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & -1 & \cdots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -1 & 1 \end{pmatrix}$$

Now to convert it to its Echelon form, we first add first row to the second row and get

$$\begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & -1 & 0 & \cdots & 0 & 1 & 2 \\ 0 & 1 & -1 & \cdots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -1 & 1 \end{pmatrix}$$

Then add second row to the third row and get

$$\begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & -1 & 0 & \cdots & 0 & 1 & 2 \\ 0 & 0 & -1 & \cdots & 0 & 1 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -1 & 1 \end{pmatrix}$$

And continuing like this we get the final form as

$$\begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & -1 & 0 & \cdots & 0 & 1 & 2 \\ 0 & 0 & -1 & \cdots & 0 & 1 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 & n-1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & n \end{pmatrix}$$

So $\text{rk}([A_E^t - J_E \ \ b]) = n$ and $\text{rk}(A_E^t - J_E) = n - 1$. So

$$\text{rk}(A_E^t - J_E) = n - 1 < n = \text{rk}([A_E^t - J_E \ \ b])$$

Thus, by theorem 5.2.7, $L_k(E_{G,S})$ has IBN property. □

Definition 6.2.9. A ring R possesses the *Unbounded Generating Number* (UGN) property when for any choice of $m, n \in \mathbb{N}$ and an arbitrary right R-module K, the isomorphism $R^n \cong R^m$ necessarily implies $n \geq m$.

$$R^n \cong R^m \Rightarrow n \geq m.$$

We designate the ring R as *stably finite* provided that for every $n \in \mathbb{N}$, the existence of an isomorphism $R^n \cong R^n \oplus K$ for some right R-module K forces K to be the zero module.

$$R^n \cong R^n \oplus K \Rightarrow K = 0.$$

The designation *Hermite ring* applies to R whenever for any $m, n \in \mathbb{N}$ and any right R-module K, the conditions $R^n \cong R^m \oplus K \cong R^{n-m}$ and K lead to the conclusion $n \geq m$.

$$R^n \cong R^m \oplus K \Rightarrow n \geq m \text{ and } R^{n-m} \cong K.$$

Finally, R exhibits *cancellation of projectives* should for any pair of finitely generated projective right R-modules Q and Q', an isomorphism $Q \oplus R \cong Q' \oplus R$ guarantee that $Q \cong Q'$.

$$Q \oplus R \cong Q' \oplus R \Rightarrow Q \cong Q'.$$

Lemma 6.2.10. *For a ring R with unity,*

$$\text{Cancellation of projectives} \Rightarrow \text{Hermite} \Rightarrow \text{Stably finite} \Rightarrow$$

$$\text{UGN} \Rightarrow \text{IBN}$$

Proof. Cancellation of projectives $\Rightarrow$ Hermite:
Let $R^n \cong R^m \oplus K$. Since K is direct summand of a free R module, K is projective. Let, if possible, $m > n$, then

$$R^n \cong R^n \oplus (R^{m-n} \oplus K)$$

And by cancellation of projectives, we have

$$0 \cong R^{m-n} \oplus K$$

which gives us $m = n$, a contradiction. Hence $n \geq m$.
Thus we get

$$R^{n-m} \oplus R^m \cong R^m \oplus K$$

And by cancellation of projectives, we get $K \cong R^{n-m}$.

Hermite $\Rightarrow$ Stably finite:
Let $R^n \cong R^n \oplus K$, then from the Hermite property we have $K \cong R^{n-n} \cong 0$.

Stably finite $\Rightarrow$ UGN:
Let $R^n \cong R^m \oplus K$. Let, if possible, $m > n$. Then we have

$$R^n \cong R^n \oplus (R^{m-n} \oplus K)$$

And by stably finite property we get $R^{m-n} \oplus K$ which gives us $m = n$, a contradiction. So $n \geq m$.

UGN $\Rightarrow$ IBN:
Let $R^n \cong R^m$. Then we have the equivalent forms as

$$R^n \cong R^m \oplus 0 \text{ and } R^m \cong R^n \oplus 0$$

So by UGN property $m \geq n$ and $n \geq m$, so that $m = n$. □

Corollary 6.2.11. *Regarding the LPA $L_k(E_{G,S})$, the characteristics listed subsequently are equivalent to one another:*

1. *$L_k(E_{G,S})$ has cancellation of projectives*
2. *$L_k(E_{G,S})$ is Hermite*

3. $L_k(E_{G,S})$ *is stably finite*

4. $L_k(E_{G,S})$ *has Unbounded Generating Number*

5. $L_k(E_{G,S})$ *has IBN*

6. $|S| = 1$

7. $L_k(E_{G,S})$ *is Noetherian*

Proof. The implications (1)⇒(2)⇒(3)⇒(4)⇒(5)⇒(6) follow directly from lemma 6.2.10 and theorem 6.2.8. And the rest follow from the fact that $L_k(E)$ has cancellation of projectives iff $L_k(E)$ is Noetherian. (using [1], theorem 3.10 and [3], theorem 4.2). □

6.3 IBN property for Power Graphs of semigroups

Here we will define a new kind of graph that arises from a weaker structure than groups, viz., semigroups.

Definition 6.3.1. For a semigroup S we construct the power graph of S, denoted by Pow(S) as follows:

1. Pow(S)=$\{v_s \ : \ g \in S\}$

2. There is an edge from v_g to $v_{g'}$ if $g \neq g'$ and $g' = g^n$ for some $n \in \mathbb{Z}^+$.

In other words for every element of S there is a vertex in Pow(S). And we take any vertex v_s and then we compute s^n for $n = 1, 2, \ldots$, and we draw an edge from s to s^n for all $n = 1, 2, \ldots$. Following are some examples of power graphs for some easy semigroups (groups).

Example 6.3.2. *Let* $\mathbb{Z}_n = \{1, 2, \ldots, n\}$ *and then* $Pow(\mathbb{Z}_n)^0 = \{v_i \ : \ i = 1, 2, \ldots, n\}$. *We see that for every generator* k *of* $\mathbb{Z}_n$ *there is an edge from* v_k *to every vertex of* $Pow(\mathbb{Z}_n)$. *And also the vertex* v_n *is a sink and there is an edge from every other vertex to* v_n. *Here are some power graphs*

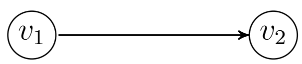


Figure 22: Pow($\mathbb{Z}_2$)

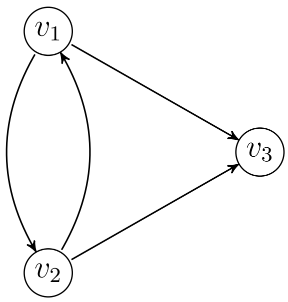


Figure 23: Pow($\mathbb{Z}_3$)

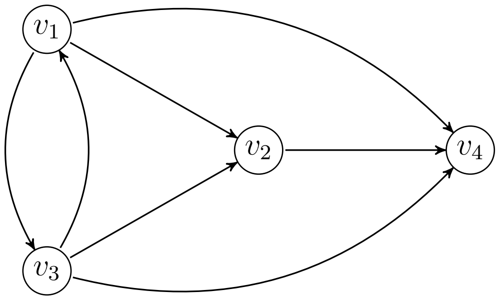


Figure 24: Pow($\mathbb{Z}_4$)

Lemma 6.3.3 ([9], Lemma 4.5)**.** *Consider the power graph Pow(Z_n) for $n \geq 2$. For any two vertices v_a and v_b such that $1 \leq a, b \leq n-1$ and $a \neq b$, the existence of a directed edge $a \to b$ is equivalent to the condition that* $\gcd(a, n)$ *divides* $\gcd(b, n)$*.*

Then using this lemma we can make the incidence matrix of power graphs.

Take n, i, k to be positive integers such that $1 \leq i, j \leq n$. Let

$$C(i) = \{v_k \in \text{Pow}(\mathbb{Z}_n) \ : \ \gcd(k, n) = i\}$$

Then using lemma 6.3.3 we find that for any distinct vertices v_p and v_q in $C(i)$, there is exactly one edge from v_p to v_q and exactly one edge from v_q to v_p. Also for $v_p \in C(i)$ and $v_q \in C(k)$ there is an edge from v_p to v_q iff i divides k.

Now let $\{d_1, d_2, \ldots, d_m\}$ be the set of divisors of n arranged in non-decreasing order. Then

$$\text{Pow}(\mathbb{Z}_n)^0 = C(d_1) \sqcup \cdots \sqcup C(d_m)$$

Then renumber the vertices as follows

$$\begin{aligned} C(d_1) &= \{v_1, \cdots, v_{k_1}\} \\ C(d_2) &= \{v_{k_1+1}, \cdots, v_{k_1+k_2}\} \\ &\vdots \\ C(d_{m-1}) &= \{v_{\sum_{i=1}^{m-2} k_i+1}, \cdots, v_{\sum_{i=1}^{m-1} k_i}\} \\ C(d_m) &= \{v_n\} \end{aligned}$$

where $k_i := |C(d_i)|$. Then

$$X_{\text{Pow}(Z_n)} = \begin{pmatrix} Z_{11} & Y_{12} & Y_{13} & \cdots & Y_{1(m-1)} & r_1^t \\ 0 & Z_{22} & Y_{23} & \cdots & Y_{2(m-1)} & r_2^t \\ 0 & 0 & Z_{33} & \cdots & Y_{3(m-1)} & r_3^t \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Z_{(m-1)(m-1)} & r_{(m-1)}^t \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where $r_i^t = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{k_i \times 1}$ $(i = 1, 2, \ldots, m-1)$, and the blocks Z_{ii} and Y_{ij} (for $i < j$) are defined as:

$$Z_{ii} := \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}_{k_i \times k_i} \quad \text{if } k_i > 1, \quad Z_{ii} := (0)_{1\times 1} \quad \text{if } k_i = 1,$$

and

$$Y_{ij} := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{k_i \times k_j} \quad \text{if } d_i \text{ divides } d_j, \quad Y_{ij} := (0)_{k_i \times k_j} \quad \text{otherwise.}$$

Now we see an example of this

Example 6.3.4. *We construct the power graph of* $\mathbb{Z}_6$*. We see that* $d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 6$*. Then*

$$\begin{aligned} C(d_1) &= \{v_1, v_2\} \\ C(d_2) &= \{v_3, v_4\} \\ C(d_3) &= \{v_5\} \\ C(d_4) &= \{v_6\} \end{aligned}$$

and the graph is

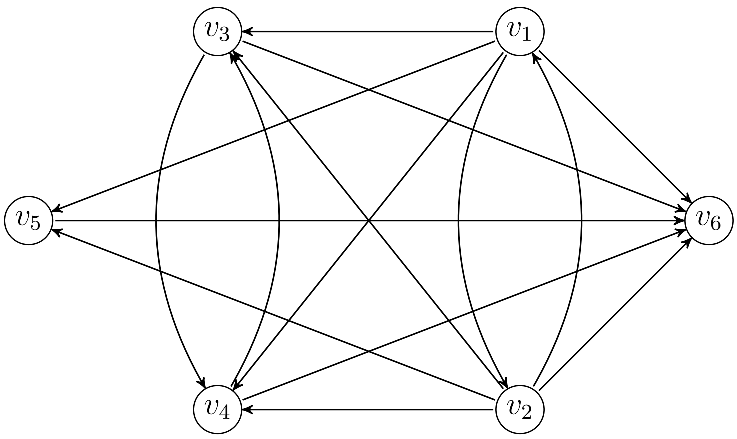


Figure 25: Pow($\mathbb{Z}_6$)

and

$$A_{\text{Pow}(\mathbb{Z}_6)}= \left(\begin{array}{cc|cc|c|c} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The next theorem by Nam proves that LPA of power graph of $\mathbb{Z}_{p^n}$ for some prime p has IBN property. The proof of which is a nice application of the matrix theoretic version of condition of IBN property, which can be seen in [9].

Theorem 6.3.5 ([9], Theorem 4.7)**.** *Given any positive integer m, an arbitrary prime number p, and any field k, the Leavitt path algebra $L_k(Pow(Z_{p^m}))$ possesses the IBN property.*

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