Central Extensions of Topological Groups and the Cohomology of Classifying Spaces

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Certificate

This is to certify that this dissertation entitled 'Central Extensions of Topological Groups and the Cohomology of Classifying Spaces' towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Pranjal Jain at the Indian Institute of Science Education and Research, Pune under the supervision of Steven Spallone, Professor, Department of Mathematics, during the academic year 2024-2025.

Steven Spallone

Steven Spallone

Committee:
Steven Spallone
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This one is for my parents,
whose love and support let my passion shine.

Declaration

I hereby declare that the matter embodied in the report entitled ‘Central Extensions of Topological Groups and the Cohomology of Classifying Spaces’ is the result of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune under the supervision of Steven Spallone, and the same has not been submitted elsewhere for any other degree.

Pranjal Jain

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Abstract

For a group G and an abelian group A, the theory of group cohomology gives an isomorphism $\mathbb{E}(G, A) \to H^2(BG, A)$, where $\mathbb{E}(G, A)$ is the group of central extensions of G by A. We generalize this construction to the case where G and A are (sufficiently nice) topological groups by producing a map $\alpha : \mathbb{E}(G, A) \to H^2(BG, A)$. Here, $\mathbb{E}(G, A)$ consists of central extensions which are also principal A-bundles, and $H^2(BG, A)$ is defined using the Ω-spectrum $A, BA, B^2A, \cdots$.

The study of $\ker \alpha$ naturally leads us to define certain maps $\alpha^n : H^n_{\mathrm{c}}(G, A) \to H^n(BG, A)$, where $H^*_{\mathrm{c}}(G, A)$ is the homology of the chain complex of continuous inhomogeneous cochains. When G and A are discrete, α^n agrees with the classical isomorphism between group cohomology $H^n_{\mathrm{gp}}(G, A)$ and $H^n(BG, A)$. Contingent on a conjecture regarding the cohomology of the Milgram–Steenrod filtration (equivalently, Milnor's filtration) of BG, we obtain the following satisfactory characterization of $\ker \alpha^n$: a cohomology class lies in $\ker \alpha^n$ if and only if the algebraic information it contains can be killed by homotopy, loosely speaking. The special case $n = 2$ gives a similar characterization of the extensions contained in $\ker \alpha$. We demonstrate several examples where $\ker \alpha^n$ and $\ker \alpha$ can be characterized independent of the conjecture.

The study of α^n is of independent interest, since it generalizes the homotopy-theoretic approach to classical group cohomology. Furthermore, it complements the analytic and categorical lenses employed in existing literature on continuous group cohomology.

Contents

Chapter 1

Introduction

Let G and A be groups with A abelian. Classically, there are several ways to interpret the group cohomology $H^*_{\text{gp}}(G, A)$ (with G acting trivially on A). Algebraically, the low-dimensional cohomology groups $H^1_{\text{gp}}(G, A)$ and $H^2_{\text{gp}}(G, A)$ are understood through their isomorphisms with group homomorphisms $\text{Hom}(G, A)$ and central extensions $\mathbb{E}(G, A)$ respectively. Topologically, $H^*_{\text{gp}}(G, A)$ can be studied using its isomorphism with the singular cohomology $H^*(BG, A)$ of the classifying space BG. Combining these perspectives yields bijections

$$\text{Hom}(G, A) \xrightarrow{\sim} H^1(BG, A) \text{ and} \tag{1.1}$$

$$\mathbb{E}(G, A) \xrightarrow{\sim} H^2(BG, A). \tag{1.2}$$

The domains and codomains of these two bijections make sense even when G is a (sufficiently nice) topological group and A is a discrete abelian group — by $\text{Hom}(G, A)$ we mean the group of continuous homomorphisms $G \to A$, by $\mathbb{E}(G, A)$ we mean the group (under Baer sums) of central extensions of G by A which are A-sheeted covers of G, and classifying spaces exist for sufficiently nice topological groups. Hence, it is natural to ask whether the bijections themselves also generalize to this context. This is immediate for (1.1) — $\text{Hom}(G, A) \approx \text{Hom}(\pi_0(G), A)$ since A is discrete and $H^1(BG, A) \approx \text{Hom}(\pi_1(BG), A) \approx \text{Hom}(\pi_0(G), A)$ by Hurewicz's Theorem, the Universal Coefficients Theorem, and the long exact sequence of homotopy groups for the universal G-bundle $EG \to BG$. Generalizing (1.2) is not as straightforward, and is considered by Joshi–Spallone in [9]. They produce a natural injection $\alpha : \mathbb{E}(G, A) \to H^2(BG, A)$ and show that it is an isomorphism in several cases (most importantly, when G is connected).

Further generalization is possible. If both G and A are sufficiently nice[1] topological groups, then $\mathrm{Hom}(G, A)$ can still be interpreted as the group of continuous homomorphisms $G \to A$ and $\mathbb{E}(G, A)$ as the group of central extensions of G by A which are principal A-bundles over G.[2] Making sense of $H^*(BG, A)$ is less straightforward. Let B denote the Milgram–Steenrod construction of the classifying space functor, described in [16]. This construction has the property that for an (sufficiently nice) abelian topological group A', the classifying space BA' itself is an abelian topological group.[3] Hence, we obtain a sequence

$$A, BA, B^2A, \cdots \tag{1.3}$$

of abelian topological groups. There is a weak homotopy equivalence $A' \to \Omega BA'$ (this holds even if A' is not abelian; see **Lemma 2.8.1**), so (1.3) is an Ω-spectrum. Furthermore, B^nA is a $K(A, n)$-space when A is discrete. This yields a reduced cohomology theory $X \mapsto H^n(X, A) := [X, B^nA]_*$ which agrees with reduced singular cohomology when A is discrete (see [8, Theorem 4.57]).[4] In particular, $H^1(BG, A) = [BG, BA]_*$ and $H^2(BG, A) = [BG, B^2A]_*$. With these definitions, we produce maps

$$B : \mathrm{Hom}(G, A) \to H^1(BG, A); f \mapsto Bf \text{ and}$$
$$\alpha : \mathbb{E}(G, A) \to H^2(BG, A)$$

which reduce to those previously discussed when A is discrete. In this generality, one cannot expect these maps to be isomorphisms, or even injections. For instance, if $G = A = \mathbb{R}$, then clearly $\mathrm{Hom}(\mathbb{R}, \mathbb{R})$ is non-trivial but $H^1(B\mathbb{R}, \mathbb{R})$ is trivial since $\mathbb{R}$ is contractible. Similarly, $\mathbb{E}(\mathbb{R}^2, \mathbb{R})$ is not trivial[5] but $H^2(B(\mathbb{R}^2), \mathbb{R})$ is. In light of these examples, one might hope that although B and α are not injective, perhaps they do still capture information which cannot be killed by homotopy. One of the main goals of this thesis is to study the kernels of these maps and show that this hope does materialize in a certain precise sense. We do so in

[1]For the purposes of the introduction, 'sufficiently nice' can be understood as G being a CW complex and A being well-pointed, i.e., $1_A \hookrightarrow A$ is a cofibration.

[2]$\mathbb{E}(G, A)$ is isomorphic to the second "locally continuous" cohomology $H^2_{\mathrm{lc}}(G, A)$ (see [18, Remark 1.3], for instance). This cocycle-centric approach is more suitable when working with sheaves.

[3]Actually, BA' may not be a topological group — the multiplication $BA' \times BA' \to BA'$ will only be continuous when the domain is given the compactly generated topology. We ignore this technicality for now.

[4]**Caution.** Generally, an expression like $H^*(X, \mathbb{R})$ refers to the cohomology of X with coefficients given by $\mathbb{R}$ as a *discrete* group. However, in our notation, the topology of $\mathbb{R}$ comes into play and $H^*(X, \mathbb{R}) = 0$ since $\mathbb{R}$ is contractible (with the Euclidean topology).

[5]The Heisenberg group, which consists of 3×3 real matrices with all diagonal entries 1, is a non-trivial extension of $\mathbb{R}^2$ by $\mathbb{R}$.

several increments. During this journey, we also encounter a generalization of the classical isomorphism $H^*_{\mathrm{gp}}(G, A) \xrightarrow{\sim} H^n(BG, A)$ from the case of G and A discrete. Henceforth, BG will always denote the Milgram–Steenrod construction of the classifying space for G.

Remark 1.0.1. This thesis will not study the images of α and B, except in the case of A discrete (see Chapter 8). Note that B and α are trivial when G is discrete and $A = BA'$ for some (sufficiently nice) abelian topological group A' (see **Example 3.1.3**), so B and α are not surjective in general. We discuss an important aspect of the failure of surjectivity in Section 10.2. □

1.1 Understanding $\ker \alpha$

The most obvious question to ask is whether α detects the bundle structure of extensions. We answer this in the affirmative in Chapter 3:

Theorem 1.1.1. *Every extension in* $\ker \alpha$ *is trivial as an* A*-bundle over* G*.*

This is **Corollary 3.1.2** in the main text, which follows from the generalization **Theorem 3.1.1** of [9, Proposition 7.1].

In light of the above, the natural next step for understanding $\ker \alpha$ is to demand a systematic way of analyzing extensions which are trivial as bundles. Analogous to the classical isomorphism $H^2_{\mathrm{gp}}(G, A) \xrightarrow{\sim} \mathbb{E}(G, A)$ from the case of G and A discrete, one can use a (inhomogeneous) continuous 2-cocycle $f : G \times G \to A$ to produce a continuous multiplication on $G \times A$ given by

$$(g, a) \cdot (g', a') := (gg', aa' f(g, g')) \,,$$

yielding a central extension of G by A. Two 2-cocycles f and f' yield isomorphic extensions if and only if they differ by the coboundary of a continuous 1-cochain $G \to A$. Furthermore, every central extension of G by A which is trivial as an A-bundle comes from a 2-cocycle in this way. Writing $H^*_{\mathrm{c}}(G, A)$ for *continuous group cohomology*, given by continuous cocycles modulo coboundaries of continuous cochains, we obtain an injection $H^2_{\mathrm{c}}(G, A) \hookrightarrow \mathbb{E}(G, A)$ whose image contains precisely those extensions which are trivial as A-bundles. This can be stated succinctly as a short exact sequence (see (2.17)). Henceforth, we will identify $H^2_{\mathrm{c}}(G, A)$ as a subgroup of $\mathbb{E}(G, A)$ in this way.

By **Theorem 1.1.1**, the study of $\ker \alpha$ reduces to the study of the kernel of the restriction $\alpha : H^2_{\mathrm{c}}(G, A) \to H^2(BG, A)$. For this, we use a natural filtration $B_1 G \subset B_2 G \subset \ldots \subset BG$

(with BG the direct limit) of the Milgram–Steenrod construction. This filtration has the property that the successive quotients $B_nG/B_{n-1}G$ are homeomorphic to $\Sigma^n G^{\wedge n}$, the n-fold reduced suspension of the n-fold smash product of G with itself. This allows us to obtain an explicit description of $\alpha[f]$ in terms of a given normalized[6] 2-cocycle $f : G^{\wedge 2} \to A$ — the restriction of $\alpha[f]$ to B_2G is given by the image of the homotopy class of f under the composition

$$H^0(G^{\wedge 2}, A) \xrightarrow{\approx} H^2\left(\Sigma^2 G^{\wedge 2}, A\right) \longrightarrow H^2(B_2G, A); \tag{1.4}$$

this is **Theorem 5.2.1**. The isomorphism comes from the fact that $H^*(-, A)$ is a reduced cohomology theory and the second map is induced by the quotient map $B_2G \to \Sigma^2 G^{\wedge 2}$. This has the following implication for $\ker \alpha$, where ι_n is the inclusion $B_nG \hookrightarrow BG$.

Theorem 1.1.2. *Every cohomology class in $\ker \alpha \subset H^2_{\mathrm{c}}(G, A)$ has a null-homotopic representative. In fact, a cohomology class lies in $\ker(\iota_2^* \circ \alpha)$ if and only if it has a null-homotopic representative.*

Remark 1.1.3. When A is discrete, we have $H^2_{\mathrm{c}}(G, A) \approx H^2_{\mathrm{gp}}(\pi_0(G), A)$. There is only one null-homotopic map $\pi_0(G) \times \pi_0(G) \to A$, namely the constant map at the identity of A. Hence, the injectivity result of [9] follows from the above theorem. □

This theorem provides a better upper bound for $\ker \alpha$ than **Theorem 1.1.1**, combining the algebraic and topological aspects of the set-up in a way which fits the hope we set out with. However, it has an obvious limitation — it only uses the information captured by $\iota_2^* \circ \alpha$, the restriction of α to B_2G. The natural next step would be to start with a null-homotopic 2-cocycle $f : G^{\wedge 2} \to A$ and try to obtain a necessary condition for $\iota_3^* \circ \alpha[f]$ to be trivial. From **Theorem 1.1.2** and the long exact sequence of cohomology for the pair (B_3G, B_2G), we know that $\iota_3^* \circ \alpha[f]$ lies in the image of the map

$$H^2\left(\Sigma^3 G^{\wedge 3}, A\right) \to H^2(B_3G, A)$$

induced by the quotient map $B_3G \to B_3G/B_2G \cong \Sigma^3 G^{\wedge 3}$. Note that $\Omega^3 B^2 A$ is weakly homotopy equivalent to ΩA (since (1.3) is an Ω-spectrum), so

$$H^2\left(\Sigma^3 G^{\wedge 3}, A\right) = \left[\Sigma^3 G^{\wedge 3}, B^2 A\right]_* \approx \left[G^{\wedge 3}, \Omega A\right]_* = H^0\left(G^{\wedge 3}, \Omega A\right).$$

[6] An inhomogeneous n-cocycle $G^n \to A$ is said to be normalized if it factors through $G^{\wedge n}$. Throughout the introduction, we will assume cocycles are normalized whenever convenient. This is justified by **Proposition 2.4.2**.

Hence, $\iota_3^* \circ \alpha[f]$ lies in the image of the map

$$H^0\left(G^{\wedge 3}, \Omega A\right) \to H^2(B_3 G, A). \tag{1.5}$$

Comparing with (1.4) suggests that there might exist a normalized 3-cocycle $f' : G^{\wedge 3} \to \Omega A$ whose homotopy class maps to $\iota_3^* \circ \alpha[f]$ under (1.5). A natural guess for such a 3-cocycle is as follows. Since f is null-homotopic, it has a lift $\tilde{f} : G^{\wedge 2} \to PA$ to PA, the path space of A. Although $\tilde{f}$ may not be a 2-cocycle, its composition with the evaluation map $e_1 : PA \to A; \gamma \mapsto \gamma(1)$ is a 2-cocycle (indeed, $e_1 \circ \tilde{f} = f$). Hence, the image of $\delta\tilde{f}$, the coboundary of $\tilde{f}$, lies in $\Omega A \subset PA$. Now $\delta^2 \tilde{f} = 0$, so $\delta\tilde{f} : G^{\wedge 3} \to \Omega A$ is a 3-cocycle.[7] Our guess for f' is then $\delta\tilde{f}$.

We will return to the topic of the correctness of this guess later; for now, suppose it is indeed correct. With some work, perhaps one could then show that $\iota_3^* \circ \alpha[f] = 0$ if and only if the cohomology class $[f'] \in H^3_{\mathrm{c}}(G, \Omega A)$ has a null-homotopic representative, which is an improvement on **Theorem 1.1.2**. Furthermore, the techniques used so far suggest that if f' is null-homotopic, one could choose a null-homotopy $\tilde{f}'$ and obtain a normalized 4-cocycle $f'' : G^{\wedge 4} \to \Omega^2 A$ such that $\iota_4^* \circ \alpha[f]$ is the image of the homotopy class of f'' under the analogue

$$H^0\left(G^{\wedge 4}, \Omega^2 A\right) \to H^2(B_4 G, A)$$

of (1.5). It is clear how this algorithmic procedure can be repeated *ad infinitum*, hopefully giving a complete description of $\ker \alpha$. Of course, this is based on a lot of guesses, particularly that our construction of f', f'', etc. has the desired properties. The need for systematizing this procedure and proving the requisite intermediate results brings us to the next part of this thesis, which is to generalize the classical isomorphisms $H^*_{\mathrm{gp}}(G, A) \xrightarrow{\sim} H^*(BG, A)$ from the discrete case.

1.2 Analogues for α in higher degrees

In Chapter 7, we construct and study maps

$$\alpha^n : H^n_{\mathrm{c}}(BG, A) \to H^n(BG, A)$$

[7] Although $\delta\tilde{f}$ is a coboundary in PA, it may not be a coboundary in ΩA.

which generalize the isomorphisms $H^n_{\mathrm{gp}}(G, A) \xrightarrow{\sim} H^n(BG, A)$ from the setting of G and A discrete. In order to convey the full significance of this construction, we first recall the construction of the classical isomorphism $H^n_{\mathrm{gp}}(G, A) \xrightarrow{\sim} H^n(\bar{B}G, A)$ for G and A discrete. Here, $\bar{B}G$ is Milnor's construction of the classifying space of G. The isomorphism is based on the observation that the inhomogeneous chain complex for group cohomology is isomorphic to the simplicial chain complex of $\bar{B}G$ (viewed as a Δ-complex in the sense of Hatcher [8]). Similarly, one observes that the normalized inhomogeneous chain complex for group cohomology is isomorphic to the cellular chain complex of BG (which is a CW complex with d-skeleton B_dG when G is discrete). This yields an isomorphism $H^n_{\mathrm{gp}}(G, A) \xrightarrow{\sim} H^n_{\mathrm{CW}}(BG, A)$ (the notation highlights that the cohomology of spaces being used here is the cellular kind).

In order to identify $H^n_{\mathrm{CW}}(BG, A)$ with $H^n(BG, A)$ as defined using an Ω-spectrum, we must start with a cellular n-cocycle f_{CW} for BG with coefficients in A and construct a based map $\phi : BG \to B^nA$. An outline of the standard approach for this is given in this MathOverflow post, with the upshot that ϕ is easy to define on B_nG whereas the process of extending its definition to $B_{n+1}G, B_{n+2}G, \cdots$ is cellular and non-constructive. Hence, this approach is not feasible when G and A may not be discrete.

In Chapter 7, we show that this extension process can instead be done constructively and purely algebraically using combinatorial techniques. This insight allows us to construct the maps $\alpha^n : H^n_{\mathrm{c}}(BG, A) \to H^n(BG, A)$, although some might consider the insight more important than the maps themselves. As one might expect, α^2 is the restriction of α to $H^2_{\mathrm{c}}(G, A)$ and $\alpha^1 = B : \mathrm{Hom}(G, A) = H^1_{\mathrm{c}}(G, A) \to H^1(BG, A)$ (see **Proposition 7.2.2** and **Proposition 7.2.3**).

The restriction $\iota_n^* \circ \alpha^n$ of α^n to B_nG has a description in terms of cocycles analogous to that of $\iota_2^* \circ \alpha$. For $f : G^{\wedge n} \to A$ a normalized n-cocycle, $\iota_n^* \alpha^n[f]$ is the image of the homotopy class of f under the analogue

$$H^0(G^{\wedge n}, A) \xrightarrow{\approx} H^n\left(\Sigma^n G^{\wedge n}, A\right) \longrightarrow H^n(B_nG, A)$$

of (1.4). The corresponding analogue of **Theorem 1.1.2** is **Theorem 1.2.3** below, which requires the following conjecture:

Conjecture 1.2.1. *For A' a discrete abelian group, the restriction maps $H^d(B_{n-1}G, A') \to H^d(B_{n-2}G, A')$ and $H^d(BG, A') \to H^d(B_{n-2}G, A')$ have the same image.*

Remark 1.2.2. We show in Section 10.1.1 that the above conjecture is equivalent to the corresponding statement for Milnor's filtration of the classifying space obtained using joins (see **Conjecture 10.1.2**).

Theorem 1.2.3. *For given G and $n \geq 1$, suppose **Conjecture 1.2.1** holds for all A' and $d \geq 0$. Every cohomology class in $\ker \alpha^n \subset H^n_c(G, A)$ has a null-homotopic representative. In fact, a cohomology class lies in $\ker(\iota_n^* \circ \alpha^n)$ if and only if it has a null-homotopic representative.*

Just as **Theorem 1.1.2** only uses the information contained in the restriction of α to B_2G, **Theorem 1.2.3** only uses the information contained in the restriction of α^n to B_nG. The procedure for linking $\ker \alpha$ to null-homotopic cocycles in degrees higher than 2 (discussed in Section 1.1) is systematized and generalized by the following theorem, whose notation we explain below.

Theorem 1.2.4. *The following commutes up to a sign of $(-1)^n$.*

$$\begin{tikzcd}
H^n_c(G, PA, \Omega A) \arrow[r, "\delta^n"] \arrow[dd, "J_*"] & H^{n+1}_c(G, \Omega A) \arrow[d, "\alpha^{n+1}"] \\
 & H^{n+1}(BG, \Omega A) \arrow[d, hook, "(\theta_A)_*"] \\
H^n_c(G, A) \arrow[r, "\alpha^n"] & H^n(BG, A)
\end{tikzcd}$$

Here,

- $H^*_c(G, PA, \Omega A)$ is the homology of the chain complex of continuous null-homotopic cochains,
- J_* is the map induced by the inclusion of the above-mentioned chain complex in the chain complex of continuous cochains,
- δ^n is defined using the natural generalization of the lifting procedure described in Section 1.1,[8] and
- the map $(\theta_A)_*$ is induced by a homotopy equivalence $\theta_A : B\Omega A \to A^\circ$.

In particular, setting $n = 2$ and $n = 3$ respectively in this theorem shows that our guess for f' in Section 1.1 is correct, whereas our guess for f'' is off by a sign. This issue of signs

[8]The notation δ^n is justified by the fact that this map is the connecting morphism from a certain long exact sequence of homology induced by a short exact sequence of chain complexes (see Section 2.4.1).

does not affect the algorithmic procedure for determining $\ker\alpha$ that was discussed, although we now see that **Conjecture 1.2.1** must be assumed for the procedure to work. More generally, **Theorem 1.2.4** implies that an analogue of this algorithmic procedure works for determining $\ker\alpha^n$ for all $n \geq 1$ (once again, contingent on **Conjecture 1.2.1**).

At the ends of Chapters 6 and 7, we provide several examples where $\ker\alpha^n$ can be characterized without assuming **Conjecture 1.2.1**.

Remark 1.2.5. Although the continuous cohomology groups $H^n_c(G, A)$ have been studied extensively, our homotopy-theoretic approach using α^n is novel. Existing literature has generally focused on techniques that 'see' the entire group $H^n_c(G, A)$, whereas our approach ignores those classes in $H^n_c(G, A)$ which can be 'killed by homotopy'.

Stasheff [15] provides a thorough exposition of the work done on continuous cohomology until the 1970's by relating it to various other cohomology theories (including some homotopy-theoretic ones). More recent work includes [18], which relates continuous cohomology and its "locally continuous" counterpart to various other cohomology theories, [7], which shows that the continuous and locally continuous cohomologies are isomorphic when the coefficient group is contractible through group homomorphisms, and [2], which uses techniques from Lie theory (such as connections with the cohomology of Lie algebras) when the coefficient group is a vector space. □

1.3 Layout of the thesis

- We begin by covering various preliminaries in Chapter 2, of which Section 2.2 (details of the Milgram–Steenrod construction), Section 2.4 (an introduction to continuous cohomology), Section 2.5 (an introduction to central extensions of topological groups), and Section 2.8 (defining the cohomology theory $H^*(-, A)$) are the most important. The reader who does not wish to wade into the weeds of topological technicalities may read only these sections of Chapter 2 and still follow most of the thesis.

- In Chapter 3, we define the map $\alpha : \mathbb{E}(G, A) \to H^2(BG, A)$ and show that it detects the underlying bundle structure of extensions. The main tool for this is a certain model of the classifying space for A, written $X_{\mathcal{E}}$, whose structure encodes the information contained in a given extension $\mathcal{E} \in \mathbb{E}(G, A)$. Hence, we conclude that $\ker\alpha \subset H^2_c(G, A)$.

- Chapter 4 uses the Dold–Thom Theorem to relate the cohomology theory coming from the Ω-spectrum $A, BA, B^2A, \cdots$ to singular cohomology. Section 4.4 explores the relation between the homotopy types of A and $B\Omega A$.

- In order to further understand $\ker \alpha$, Chapter 5 provides an explicit formula for $\iota_2^* \circ \alpha[f]$, where f is a cocycle in $Z^2_{\mathrm{c}}(G, A)$. Section 5.3 uses this formula to show that α agrees with the classical isomorphism $H^2_{\mathrm{gp}}(G, A) \xrightarrow{\sim} H^2(BG, A)$ when G and A are discrete.

- The above-mentioned formula easily generalizes to give a map $\alpha_n : C^n_{\mathrm{c}}(G, A) \to H^n(B_n G, A)$, which we explore in Chapter 6. **Theorem 1.2.3** follows from **Theorem 6.3.2**. In particular, this gives **Theorem 1.1.2** as **Corollary 6.3.5**.

- In Chapter 7, everything comes together to yield our main results. We define the maps $\alpha^n : H^n_{\mathrm{c}}(G, A) \to H^n(BG, A)$, with $\iota_n^* \circ \alpha^n$ equal to the restriction of α_n to cocycles. Furthermore, $\alpha^1 = B$ (**Proposition 7.2.2**) and $\alpha^2 = \alpha$ (**Proposition 7.2.3**). **Theorem 1.2.4** is proved as **Corollary 7.3.3**, which links $\ker \alpha^n_{G,A}$ to $\ker \alpha^{n+1}_{G,\Omega A}$. The algorithmic description of $\ker \alpha^n$ then follows easily.

- Chapter 8 is largely independent of the rest of the thesis, and looks at some partial results for the surjectivity of α when A is discrete. The main result **Theorem 8.0.6** says that α is an isomorphism when A is discrete and $H^3_{\mathrm{gp}}(\pi_0(G), A)$ is trivial.

1.4 Original contributions

- Our definition of and results regarding pCW complexes (Sections 2.6 and 9.1) are novel, although most of the proofs are natural generalizations of well-known techniques used for CW complexes. The most important consequence of this work is **Corollary 2.6.9**, which asserts that BG has the homotopy type of a CW complex when G is a CW complex.

- The contents of Chapter 3 are original, although several of our ideas are inspired by [9]. A more precise description of how we drew inspiration from [9] can be inferred from the remarks which relate our results to theirs.

- The main definitions and theorems in Chapters 5 to 7 are novel. These include the definitions of α_n and α^n, Theorems 5.2.1, 5.3.1 and 6.3.2, **Corollary 7.3.3**, and the algorithmic description of $\ker \alpha^n$ in Section 7.4.

- Chapter 8 gives a breakdown of the original contributions it contains.

- The author is unaware whether **Proposition 9.2.1** is novel, but suspects that it might follow from known necessary conditions for a given space to have the homotopy type of a loop space.

- **Proposition 9.3.3** and the examples of k-rings in Section 9.3 are original.

Chapter 2

Preliminaries

For all unexplained notation, we refer to [8]. By 'space', we will always mean topological space. A map between topological spaces is understood to be a continuous function. If the continuity condition is to be relaxed, we will make this explicit by saying 'set-map'. $I := [0,1]$ is the compact unit interval. $\mathbb{N}$ and $\mathbb{N}_0$ are the sets of positive and non-negative integers respectively. For $n \in \mathbb{N}_0$, let $[n] = \{1, \cdots, n\}$ and $[n]_0 = [n] \cup \{0\}$. In particular, $[0] = \emptyset$ and $[0]_0 = \{0\}$. The n-skeleton of a CW complex X is written as $X^{(n)}$.

For spaces X and Y, the set of homotopy classes of maps from X to Y is $[X, Y]$. If base points are chosen, then $[X, Y]_*$ is the set of based homotopy classes. Write $X \cong Y$ if X and Y are homeomorphic, and $X \approx Y$ if X and Y are homotopy equivalent (likewise for pairs of spaces). For maps $f, f' : X \to Y$, write $f \approx f'$ if f and f' are homotopic. If f is constant and $f(x) = y$ for all $x \in X$, then write $f \equiv y$.

For (X, x_0) a based space, CX and ΣX denote its reduced cone and reduced suspension respectively. Explicitly, CX is the quotient of $X \times I$ by the relation $(x, 0) \sim (x_0, t)$ for all $(x, t) \in X \times I$, and ΣX is the quotient of CX by the relation $(x, 1) \sim (x_0, 1)$ for all $x \in X$. The unreduced cone $\tilde{C}X$ and unreduced suspension $\tilde{\Sigma}X$ are defined analogously. For both cones and both suspensions, we take the base point to be $(x_0, 1)$. There is an action $((x, s), t) \mapsto (x, st)$ of I (as a monoid under multiplication) on CX and $\tilde{C}X$. X is identified as a subspace of CX and $\tilde{C}X$ as $x \mapsto (x, 1)$. For a pair of spaces (X, X'), we often write $X \cup CX'$ for the pushout of $X' \hookrightarrow X$ and $X' \hookrightarrow CX'$ (likewise $X \cup \tilde{C}X'$).

For A a discrete abelian group, write $H^*_{\text{sing}}(-, A)$ for singular cohomology with coefficient group A. $H^*_{\text{CW}}(-, A)$ and $H^*_{\Delta}(-, A)$ will denote the cellular and simplicial cohomologies for CW complexes and Δ-complexes respectively. For discrete abelian groups A and A', write $A \approx A'$ if they are isomorphic.

2.1 Conventions for compactly generated topologies

The category of compactly generated spaces is often convenient for doing algebraic topology in, and we will use it extensively in this thesis. This section provides a brief exposition of our conventions and notations regarding it, and a detailed exposition can be found in [17]. A space X is said to be *compactly generated* (CG for short) if $U \subset X$ is open if and only if $U \cap K$ is open in K for every compact subspace $K \subset X$. In other words, the topology on X is the finest topology which makes the inclusion $K \hookrightarrow X$ continuous for all compact subspaces $K \subset X$.

From a space X we obtain a CG space kX, the *k-ification* of X, by taking an obvious refinement of the topology on X. The set-theoretic identity map $kX \to X$ is continuous, and $kX = X$ if and only if X is CG. Furthermore, k-ification is functorial — corresponding to a map $f : X \to Y$ we obtain a k-ified map $kf : kX \to kY$, with $kf = f$ as set-maps. In other words, k is a functor from the category **Top** of topological spaces to the category k**Top** of CG spaces (morphisms being continuous functions).

For spaces X and Y, write $X \times_\tau Y$ for their product space (with the product topology) and write $X \times Y$ for their *k-product* given by $k(X \times_\tau Y)$. Write X^n for the n-fold k-product of X with itself and $X^{\times_\tau n}$ for the n-fold τ-product of X with itself. Note that if X and Y are CG, then $X \times Y$ is the product of X and Y in k**Top**. Furthermore, if X and Y are CG and $\overset{X}{\sim}$ and $\overset{Y}{\sim}$ are equivalence relations on X and Y respectively, then natural the set-map

$$\frac{X \times Y}{\overset{X}{\sim} \times \overset{Y}{\sim}} \to \frac{X}{\overset{X}{\sim}} \times \frac{Y}{\overset{Y}{\sim}}$$

is a homeomorphism (apply [17, Proposition 2.17] twice). This property, which would not hold if the k-product were replaced by the τ-product, is the main reason for working in k**Top**.

If base points $x_0 \in X$ and $y_0 \in Y$ are chosen, then $X \wedge Y$ denotes the *smash product* of X and Y, defined as

$$\frac{X \times Y}{(\{x_0\} \times Y) \cup (X \times \{y_0\})}.$$

The smash product is associative for CG spaces — if X, Y, Z are based CG spaces, then the natural set-map $X \wedge (Y \wedge Z) \to (X \wedge Y) \wedge Z$ is a homeomorphism. This allows us to define $X^{\wedge n} := X \wedge X \wedge \ldots \wedge X$ (n times) unambiguously.

All CW complexes are CG. For CW complexes X and Y, a natural product cell structure can be given to $X \times Y$. If both X and Y have countably many cells, then $X \times Y = X \times_\tau Y$

(see [8, Theorem A.6]).

2.1.1 Spaces of maps

For spaces X and Y, $\mathrm{Map}(X, Y)$ is the space of all maps from X to Y, topologized using the k-ification of the compact-open topology. If X and Y are based spaces, then $\mathrm{Map}_*(X, Y)$ denotes the analogous space of based maps. In particular, if X is a based space, then $\Omega X := \mathrm{Map}_*(S^1, X)$ is its *loop space* and $PX := \mathrm{Map}_*((I, 0), X)$ is its *path space*. The base point for $\mathrm{Map}(X, Y)$ and $\mathrm{Map}_*(X, Y)$ is the constant map at the base point of Y.

For pairs of spaces (X, X') and (Y, Y'), write $\mathrm{Map}((X, X'), (Y, Y'))$ for the space of maps of pairs topologized as above. If X has base point x_0, then we define $\mathrm{Map}(X, (Y, Y')) := \mathrm{Map}((X, x_0), (Y, Y'))$. Likewise for $\mathrm{Map}((X, X'), Y)$ if Y is a based space.

For $x \in X$, the *evaluation map* $e_x : \mathrm{Map}(X, Y) \to Y; f \mapsto f(x)$ is continuous (likewise for spaces of based maps and maps of pairs). In particular, if X is a based space then

$$\Omega X \hookrightarrow PX \xrightarrow{e_1} X$$

is the *path space fibration*.

2.1.2 Currying

Let X, Y, Z be CG spaces with Y locally compact and Hausdorff. The *currying* of a map $f : X \times Y \to Z$ is the map $\hat{f} : X \to \mathrm{Map}(Y, Z); x \mapsto f(x, -)$, and the *uncurrying* of a map $\hat{f}' : X \to \mathrm{Map}(Y, Z)$ is $f' : X \times Y \to Z; (x, y) \mapsto F(x)(y)$. It is a standard result that currying and uncurrying preserve continuity. In fact, [17, Proposition 2.12] states that

$$\mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z)); f \mapsto (x \mapsto f(x, -)) \tag{2.1}$$

is a homeomorphism (we will not need this). Currying and uncurrying preserve homotopy classes, so we obtain a bijection $[X \times Y, Z] \to [X, \mathrm{Map}(Y, Z)]$. Certain factoring and base point restrictions on a map $X \times Y \to Z$ easily translate to restrictions on its currying:

Proposition 2.1.1. *Let X, Y, Z be based CG spaces with Y locally compact and Hausdorff.*

Let $Y' \subset Y$ be a subspace containing the base point. Currying yields natural set-bijections

$$\mathrm{Map}_*(X \wedge Y, Z) \to \mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z)) \text{ and } [X \wedge Y, Z]_* \to [X, \mathrm{Map}_*(Y, Z)]_*,$$

$$\mathrm{Map}_*\left(\frac{X \times Y}{X}, Z\right) \to \mathrm{Map}(X, \mathrm{Map}_*(Y, Z)) \text{ and } \left[\frac{X \times Y}{X}, Z\right]_* \to [X, \mathrm{Map}_*(Y, Z)], \text{ and}$$

$$\mathrm{Map}_*\left(\frac{X \times Y}{X \times Y'}, Z\right) \to \mathrm{Map}\left(X, \mathrm{Map}_*\left(\frac{Y}{Y'}, Z\right)\right) \text{ and } \left[\frac{X \times Y}{X \times Y'}, Z\right]_* \to \left[X, \mathrm{Map}_*\left(\frac{Y}{Y'}, Z\right)\right]$$

with inverses given by uncurrying.

2.1.3 Groups, monoids, and H-spaces

A Hausdorff CG space G is said to be a *k-group* if

- it is a group,
- the multiplication $G \times G \to G$ and inverse $G \to G$ are continuous, and
- it is well-pointed, i.e., $(G, 1_G)$ is a cofibration.

Remark 2.1.2. Since G is well-pointed, 1_G has a neighborhood U such that the inclusion $U \hookrightarrow G$ is null-homotopic. In particular, U lies in the identity component G° of G. Taking shifts, we see that every point in G has a neighborhood in its path component. Hence, all path components of G are open. In particular, G is path-connected if and only if it is connected. □

A *τ-group* is defined analogously, except G need not be CG and the multiplication map must be continuous with domain $G \times_\tau G$. Similarly, we define *τ-monoids* and *k-monoids*. When such objects are viewed as based spaces, the base point is always assumed to be the identity. Every τ-group (monoid) that is CG is also a k-group (monoid), although the converse does not hold.

We will make extensive use of the following technical facts.

Lemma 2.1.3. *For G a k-group and X a compact Hausdorff based space, $\mathrm{Map}_*(X, G)$ is a k-group under pointwise multiplication of maps.*

Corollary 2.1.4. *For G a k-group, PG and ΩG are k-groups.*

*Proof of **Lemma 2.1.3**.* We only need to check that $\tilde{G} := \mathrm{Map}_*(X, G)$ is well-pointed. First, recall that G being well-pointed is equivalent to the existence of maps $u : G \to I$ and $h : G \times I \to G$ such that

- $u^{-1}(0) = \{1_G\}$,
- $h(g, 0) = g$ for $g \in G$, and
- $h(g, t) = 1_G$ whenever $u(g) < t$.

Now, define

$$\tilde{u} : \tilde{G} \to I; f \mapsto \sup_{x \in X} uf(x) \text{ and}$$
$$\tilde{h} : \tilde{G} \times I \to \tilde{G}; (f, t) \mapsto (x \mapsto h(f(x), t)).$$

$\tilde{u}$ is continuous since X is compact, and continuity of $\tilde{h}$ follows from the continuity of its uncurrying (here we used that (2.1) is a bijection). We also have

- $\tilde{u}^{-1}(0) = \{1_{\tilde{G}}\}$,
- $\tilde{h}(f, 0) = f$ for $f \in \tilde{G}$, and
- $\tilde{h}(f, t) = 1_{\tilde{G}}$ whenever $\tilde{u}(f) < t$,

so the lemma follows. □

A k-group (monoid) is said to be a *CW group* (monoid) if the underlying space is a CW complex. Henceforth, G will always denote a CW group and A will always denote an abelian k-group (unless explicitly mentioned otherwise). The identity of a CW group is always assumed to be a 0-cell.

Remark 2.1.5. A CW group (monoid) with countably many cells is a τ-group (see [8, Theorem A.6]). □

A *H-space* is a well-pointed CG space $(L, 1_L)$ with a multiplication map $L \times L \to L$ such that $1_L \cdot 1_L = 1_L$ and the composition $L \hookrightarrow L \times L \to L$ is homotopic to id_L rel 1_L for both axial inclusions $L \hookrightarrow L \times L$.

2.1.4 Bundles

Let G be any k-group. Our definition of (principal) G-bundles is the standard one, except that local trivializations and the continuity of the G-action on the total space are interpreted in the CG sense. To be precise, a G-bundle is a map $p : E \to B$ such that

- there is a free and continuous right G-action $E \times G \to E$, and
- for each $b \in B$ there exists a neighborhood $U \subset B$ of b and a G-equivariant homeomorphism $p^{-1}(U) \to U \times G$ which makes

$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d] & U \times G \arrow[dl] \\ U & \end{tikzcd}$$

commute, where the diagonal map is the first projection.

Write $\mathbf{Bun}_B(G)$ for the collection of G-bundles over B and $\mathrm{Bun}_B(G)$ for the set of isomorphism classes of these bundles. $\mathrm{Bun}_B(G)$ forms an abelian group under *Baer sums*.

2.2 The Milgram–Steenrod construction of the classifying space

We give a brief summary of the important aspects of the Milgram–Steenrod classifying space construction. For G any k-group, [16] constructs a contractible k-group EG with G a closed subgroup. The coset space is $BG := EG/G$. Additionally, the projection $p_G : EG \to BG$ is a numerable G-bundle, so that BG is the classifying space for G. This construction is functorial, i.e.,

- E is a functor from the category $k\mathbf{Grp}$ of k-groups and continuous homomorphisms to itself,
- the inclusion $G \hookrightarrow EG$ is a natural transformation from the identity functor on $k\mathbf{Grp}$ to E,
- B is a functor from $k\mathbf{Grp}$ to $k\mathbf{Top}_*$, the category of based CG spaces, and
- $G \mapsto p_G$ is a natural transformation from E to B, where the codomain of E is viewed as $k\mathbf{Top}_*$ instead of $k\mathbf{Grp}$.

Remark 2.2.1. Here, we are using the fact that a pair (X, A) is an NDR in the sense of [16] if and only if it is a closed cofibration. This can be seen as follows. Puppe [12, Satz 1] shows that (X, A) is a closed cofibration if and only if there exist maps $v, w : X \to I$ and $\psi : X \times I \to X$ satisfying

- $w^{-1}(0) = A$,
- $v(A) = \{0\}$,
- $\psi(x, 1) \in A$ for $x \in X$ with $v(x) < 1$, and
- $\psi(a, t) = a$ for $(a, t) \in A \times I$.

The latter of these is clearly weaker than [16]'s NDR condition (take $v = w = u$ and $\psi = k$), and it is also stronger since setting $u = \max(v, w)$ and $k = \psi$ in [16]'s definition works. $\square$

As a group, EG is generated by the set $G \times I$ subject to the following relations for $g, g' \in G$ and $0 \leq t' \leq t \leq 1$.

$$\begin{aligned}(g, 0) = (1_G, t) &= 1_{EG} \\ (g, t)(g', t) &= (gg', t) \\ (g, t)(g', t') &= (gg'g^{-1}, t')(g, t)\end{aligned} \tag{2.2}$$

Consequently, each non-trivial element of EG is represented by a unique word of the form $(g_1, t_1) \cdots (g_j, t_j)$ with $j \geq 1$, $0 < t_1 < \ldots < t_j \leq 1$, and $g_i \neq 1_G$ for all i (see [16, §7]). Such words, together with the empty word, are said to be in normal form. Non-empty words in normal form are non-trivial elements of EG. G is embedded in EG as $g \mapsto (g, 1)$.

We will refer to the spaces labeled D_n in [16] as D_nG so that the dependence on G is explicit. Recall that we have natural inclusions $D_0G \hookrightarrow D_1G \hookrightarrow \ldots$, with EG the colimit. This filtration can be understood in terms of the above group structure — D_nG consists of all elements which have a normal form representation with at most n words (see [16, Theorem 7.6]). In other words, D_nG contains all those elements of EG which are represented by length n words (not necessarily in normal form).

$$D_nG = \{(g_1, t_1) \cdots (g_n, t_n) \mid g_i \in G, t_i \in I\} \subset EG \tag{2.3}$$

A word is said to be in semi-normal form if it is empty or has the form $(g_1, t_1) \cdots (g_j, t_j)$ with $0 \leq t_1 \leq \ldots \leq t_j \leq 1$ (the g_i's are allowed to be trivial). We can reduce the redundancy in (2.3) by restricting to words in semi-normal form:

$$D_nG = \{(g_1, t_1) \cdots (g_n, t_n) \mid g_i \in G, 0 \leq t_1 \leq \ldots t_n \leq 1\}. \tag{2.4}$$

The space

$$\Delta_n := \{(t_1, \cdots, t_n) \mid 0 \leq t_1 \leq \ldots \leq t_n \leq 1\} \subset I^n$$

is an n-simplex, and (2.4) determines a surjection $k_n : G^n \times \Delta_n \to D_n G$. This is a quotient map by [16, Theorem 5.1]. G^n is embedded in G^{n+1} by fixing the last coordinate to 1_G and Δ_n is embedded in Δ_{n+1} by fixing the last coordinate to 1, so k_{n+1} restricts to k_n on $G^n \times \Delta_n$.

The space $p_G^{-1}(p_G(D_n G))$, the union of all left G-cosets which intersect $D_n G$, is given by

$$E_n G = \{(g_1, t_1) \cdots (g_n, t_n)(g_{n+1}, 1) \mid g_i \in G, 0 \leq t_1 \leq \ldots \leq t_n \leq 1\} \subset D_{n+1} G.$$

The restriction of k_{n+1} to $G^{n+1} \times \Delta_n$ is a G-equivariant surjection onto $E_n G$, where G acts on $G^{n+1} \times \Delta_n$ by multiplication with the rightmost coordinate. Note that when G is discrete, $E_n G$ is the n-skeleton of EG (as a CW complex). This follows from the fact that $k_{n+1}^{-1}(E_n G)$ is the n-skeleton of $G^{n+1} \times \Delta_{n+1}$. Consequently, the n-skeleton of BG is given by

$$B_n G := p_G(E_n G).$$

Write ι_n for the inclusion $B_n G \hookrightarrow BG$. If G is a CW group with cellular multiplication, then $D_n G, E_n G$ and $B_n G$ are CW complexes (with the obvious subcomplex relations) and the projection $E_n G \to B_n G$ is cellular. In particular, EG and BG are CW complexes and p_G is cellular. The multiplication on EG is also cellular. See [16, §9].

General points of BG will always be written as

$$(g_1, t_1) \cdots (g_n, t_n),$$

with the understanding that this expression is in semi-normal form, i.e., $g_i \in G$ and $0 \leq t_1 \leq \ldots \leq t_n \leq 1$.

2.2.1 Iterating B

Since A is abelian, so is EA. Hence, the coset space $BA = EA/A$ also becomes a k-group. BA is generated by $A \times I$, subject to the relations (2.2) and $(a, 1) = 1_{BA}$ for $a \in A$. Applying the functor B again, we see that BBA is generated by elements of the form $((a, t), t')$ for $a \in A$ and $t, t' \in I$. We drop the inner brackets for simplicity, so BBA is the abelian group generated by $A \times I^2$ with the following relations for $a, a_i \in A$ and $t, t' \in I$.

$$\begin{aligned} (a_1, t, t')(a_2, t, t') &= (a_1 a_2, t, t') \\ (a, t, t') &= 1_{BBA} \text{ if } \{t, t'\} \cap \{0, 1\} \neq \emptyset \end{aligned}$$

These relations have obvious analogues for $B^n A$.

Now, suppose A were discrete. Then $B^n A$ is a $K(A,n)$-space, so it is desirable to fix an identification of $\pi_n(B^n A)$ with A. A simple guess is that $a \in A$ can be identified with the class of the map

$$(I^n, \partial I^n) \to (B^n A, 1_{B^n A}); (t_1, \cdots, t_n) \mapsto (a, t_1, \cdots, t_n). \tag{2.5}$$

Indeed this works, and it matches with the isomorphism $\pi_n(B^n A) \to A$ that the below sequence yields (this follows from a routine check). The maps ∂ are the appropriate connecting morphisms from long exact sequences of bundles, and they are all isomorphisms.

$$\ldots \longrightarrow \pi_n(B^n A) \xrightarrow{\partial} \pi_{n-1}(B^{n-1}A) \longrightarrow \ldots \longrightarrow \pi_1(BA) \xrightarrow{\partial} \pi_0(A) = A$$

Additionally, since discrete groups are CW groups with cellular multiplication, BA is a CW group. The multiplication on BA is cellular by [16, Theorem 9.6]. Indeed, by induction on n, the same theorem yields that $B^n A$ is a CW group with cellular multiplication for all $n \geq 0$.

2.3 Milnor versus Milgram–Steenrod

Throughout this section, G is discrete. We recall Milnor's construction of the classifying space of G, denoted by $\bar{B}G$. The universal cover $\bar{E}G$, a G-space, is a Δ-complex with 0-cells the elements of G and j-cells given by $(j+1)$-tuples in G^{j+1}. The cell corresponding to $(g_0, \cdots, g_j)$ is glued to the cell corresponding to $(g_0, \cdots, \hat{g}_i, \cdots, g_j)$ in the obvious way, and the right-action of G is the diagonal action.

$$(g_0, \cdots, g_j) \cdot g = (g_0 g, \cdots, g_j g)$$

Since the G-action is simplicial, $\bar{B}G := \bar{E}G/G$ becomes a Δ-complex such that the map $\bar{p}_G : \bar{E}G \to \bar{B}G$ is simplicial and a covering.

A general point in $\bar{E}G$, lying in the cell corresponding to $(g_0, \cdots, g_j)$, has barycentric coordinates $(s_0, \cdots, s_j) \in I^{j+1}$ with $\sum\limits_i s_i = 1$. We will represent this point with the notation

$$[g_0, s_0, \cdots, g_j, s_j]. \tag{2.6}$$

The 'i-th coordinate' of such a representation refers to the tuple (g_i, s_i). The gluing of the various cells is captured by the heuristic that if $s_i = 0$ for some i, then the i-th coordinate can be ignored. We now use this to make the simplicial structure more explicit.

For $n \geq 0$, let

$$\Gamma^n = \bigcup_{j=0}^{n} G^{j+1} \times \Delta^j,$$

where $\Delta^j \subset I^{j+1}$ is the standard j-simplex given by

$$\Delta^j = \left\{ (s_0, \cdots, s_j) \in I^{j+1} \,\middle|\, \sum_i s_i = 1 \right\}.$$

Following are homeomorphisms between Δ_j and Δ^j (they are inverses of each other).

$$\Delta_j \to \Delta^j; (t_1, \cdots, t_j) \mapsto (t_1, t_2 - t_1, \cdots, t_j - t_{j-1}, 1 - t_j) \tag{2.7}$$

$$\Delta^j \to \Delta_j; (s_0, \cdots, s_j) \mapsto (s_0, s_0 + s_1, \cdots, s_0 + s_1 + \ldots + s_{j-1}) \tag{2.8}$$

When writing out the coordinates of a point in $G^{j+1} \times \Delta^j$, we will interleave the g's and s's as in (2.6). Consider the equivalence relation on Γ_n generated by

$$(g_0, s_0, \cdots, g_i, 0, \cdots, g_j, s_j) \sim (g_0, s_0, \cdots, \hat{g}_i, \hat{0}, \cdots, g_j, s_j).$$

The n-skeleton of $\bar{E}G$ is precisely $\Gamma^n / \sim$, so taking a colimit yields $\bar{E}G$. The equivalence class of $(g_0, s_0, \cdots, g_j, s_j)$ in this colimit is precisely $[g_0, s_0, \cdots, g_j, s_j]$.

The simplex which is the image of $(g_0, \cdots, g_n) \times \Delta^n$ in $\bar{B}G$ will be said to be the n-simplex with vertices $g_0, \cdots, g_n$. The characteristic map of this simplex is taken to be

$$\Delta_n \to \bar{B}G; (t_1, \cdots, t_n) \mapsto [g_0, t_0, g_1, t_1 - t_0, \cdots, g_{n-1}, t_n - t_{n-1}, g_n, 1 - t_n]. \tag{2.9}$$

This simplex is the same as that with vertices $g_0 g_n^{-1}, \cdots, g_{n-1} g_n^{-1}, 1_G$. Note that $\bar{B}G$ has only one 0-cell, so, strictly speaking, it does not make sense to specify simplices in $\bar{B}G$ by referring to their vertices. However, the proposed convention alludes to the fact that the simplex with vertices $g_0, \cdots, g_n$ lifts to the simplex in $\bar{E}G$ with vertices $[g_0, 1], \cdots, [g_n, 1]$, and n-simplices can be uniquely specified using their vertices in $\bar{E}G$. Of course, this lift is not unique; this can be remedied by demanding that the last vertex of the lift be $[1_G, 1]$. We will not do this, however.

2.3.1 A homotopy equivalence

We will now produce a G-equivariant surjection $\Psi : \bar{E}G \to EG$ which descends to a homotopy equivalence $\bar{\Psi} : \bar{B}G \to BG$.[1] The map is analogous to the isomorphism between the homogeneous and inhomogeneous cochain complexes used to calculate group cohomology — EG is analogous to the inhomogeneous complex and $\bar{E}G$ to the homogeneous complex.

First we define a map $\tilde{\Psi}_n : G^{n+1} \times \Delta^n \to G^{n+1} \times \Delta_n$ using (2.8) as follows, where all coordinates are interleaved as in (2.6).

$$(g_0, s_0, \cdots, g_n, s_n) \mapsto (g_0 g_1^{-1}, s_0, g_1 g_2^{-1}, s_0 + s_1, \cdots, g_{n-1} g_n^{-1}, s_0 + \ldots + s_{n-1}, g_n, 1)$$

Here, Δ_n has been realized as its embedding in Δ_{n+1}. The image of $k_{n+1} \circ \tilde{\Psi}_n$ is $E_n G$, the $(n+1)$-fold join of G. Furthermore, $k_{n+1} \circ \tilde{\Psi}_n$ factors through the n-skeleton $\bar{E}_n G = \Gamma^n / \sim$. This yields a surjection $\Psi_n : \bar{E}_n G \to E_n G$. We define Ψ to be the colimit of Ψ_*. It is clear that $\tilde{\Psi}_n$ is G-equivariant, so Ψ is too.

To see that $\bar{\Psi}$ is a homotopy equivalence, it suffices to show that $\pi_1(\bar{\Psi})$ is an isomorphism (since BG and $\bar{B}G$ are both CW models of $K(G,1)$). For this, observe that $\bar{\Psi}$ sends the loop $\bar{\gamma}_g : t \mapsto [g, t, 1_G, 1-t]$ in $\bar{B}G$ to the loop $\gamma_g : t \mapsto (g, t)$ in BG. The unique lift $\tilde{\bar{\gamma}}_g$ of $\bar{\gamma}_g$ to $\bar{E}G$, starting at $[1_G, 1]$, has endpoint $[g, 1]$. Likewise, the unique lift $\tilde{\gamma}_g$ of γ_g to EG, starting at 1_{EG}, has endpoint $(g, 1)$. Hence, the following diagram commutes, where the maps ∂ are the appropriate maps from the long exact sequence of homotopy groups for $\bar{E}G \to \bar{B}G$ and $EG \to BG$.

$$\begin{tikzcd}
\pi_1(\bar{B}G) \arrow[r, "\partial", "\approx"'] \arrow[d, "\bar{\Psi}"'] & \pi_0(G) = G \arrow[d, equal] \\
\pi_1(BG) \arrow[r, "\partial", "\approx"'] & \pi_0(G) = G
\end{tikzcd}$$

This completes the proof of $\bar{\Psi}$ being a homotopy equivalence. In Section 10.1.1, we briefly sketch why $\bar{\Psi}$ is a homotopy equivalence even when G is a CW group.

2.4 Continuous group cohomology

First, we recall the classical construction of the cohomology groups of a discrete group G with discrete coefficient group A, acted on trivially by G. Let $\hat{C}^n_{\mathrm{gp}}(G, A)$ $(n \geq 0)$ be the

[1] In Section 10.1.1, we will show that this map is also a homotopy equivalence when G is a CW group.

group of set-maps from G^n to A. The coboundary maps $\delta^n : \hat{C}^n_{\mathrm{gp}}(G, A) \to \hat{C}^{n+1}_{\mathrm{gp}}(G, A)$ are defined as follows for $n \geq 1$.

$$\delta^n f(g_1, \cdots, g_{n+1}) = f(g_2, \cdots, g_{n+1}) + \sum_{i=1}^{n} f(g_1, \cdots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \cdots, g_{n+1}) \\ + (-1)^{n+1} f(g_1, \cdots, g_n)$$

For $n = 0$, we set $\delta^0 = 0$. This yields the *inhomogeneous cochain complex*

$$0 \longrightarrow \hat{C}^0_{\mathrm{gp}}(G, A) \longrightarrow \dots \longrightarrow \hat{C}^n_{\mathrm{gp}}(G, A) \xrightarrow{\delta^n} \hat{C}^{n+1}_{\mathrm{gp}}(G, A) \longrightarrow \dots \tag{2.10}$$

whose homology groups are defined to be the cohomology groups $H^*_{\mathrm{gp}}(G, A)$.

In our context, it will be more convenient to work with a normalized version of the above chain complex. Let $C^n_{\mathrm{gp}}(G, A)$ be the subgroup of $\hat{C}^n_{\mathrm{gp}}(G, A)$ consisting of set-maps $G^{\wedge n} \to A$, which are set-maps $G^n \to A$ which vanish on tuples $(g_1, \cdots, g_n)$ with $g_i = 1_G$ for some i. We take $C^0_{\mathrm{gp}}(G, A) = \hat{C}^0_{\mathrm{gp}}(G, A)$. This yields a subcomplex

$$0 \longrightarrow C^0_{\mathrm{gp}}(G, A) \longrightarrow \dots \longrightarrow C^n_{\mathrm{gp}}(G, A) \xrightarrow{\delta^n} C^{n+1}_{\mathrm{gp}}(G, A) \longrightarrow \dots \tag{2.11}$$

of the inhomogeneous cochain complex (2.10), called the *normalized inhomogeneous cochain complex.*

Proposition 2.4.1. *The homology groups of* (2.11) *are isomorphic to $H^*_{\mathrm{gp}}(G, A)$, with isomorphism induced by the inclusion of* (2.11) *in* (2.10)*.*

Sketch of proof. It is well-known that the simplicial cochain complex of the Δ-complex $\bar{B}G$ is naturally isomorphic homogeneous cochain complex of G with coefficients A. The latter is naturally isomorphic to the inhomogeneous cochain complex $\hat{C}^*_{\mathrm{gp}}(G, A)$ (for instance, see [5, §17.2, Exercise 2]).[2] Similarly, $C^*_{\mathrm{gp}}(G, A)$ is naturally isomorphic to the cellular cochain complex of the CW complex BG.

With the above isomorphisms treated as identifications, the map from the cellular cochain complex of BG to the simplicial cochain complex of $\bar{B}G$ induced by the cellular map $\bar{\Psi} : \bar{B}G \to BG$ (see Section 2.3.1) is the inclusion $C^*_{\mathrm{gp}}(G, A) \hookrightarrow \hat{C}^*_{\mathrm{gp}}(G, A)$. This must induce an isomorphism on homology since Ψ is a homotopy equivalence. $\square$

Remark. **Proposition 2.4.1** is a special case of **Proposition 2.4.2** below, and our proof of

[2] The composition of these isomorphisms is described in some detail in Section 5.3.

the prior does not work for the latter. Nonetheless, the above proof is a nice demonstration of the technique of doing algebra using topology. □

For G a CW group and A an abelian k-group, we define the *continuous group cohomology* $H^*_{\mathrm{c}}(G, A)$ completely analogously. Let $\hat{C}^n_{\mathrm{c}}(G, A) \subset \hat{C}^n_{\mathrm{gp}}(G, A)$ be the group of continuous maps $G^n \to A$. The continuous cohomology groups $H^*_{\mathrm{c}}(G, A)$ are defined to be the homology groups of the following subcomplex of (2.10).

$$0 \longrightarrow \hat{C}^0_{\mathrm{c}}(G, A) \longrightarrow \dots \longrightarrow \hat{C}^n_{\mathrm{c}}(G, A) \xrightarrow{\delta^n} \hat{C}^{n+1}_{\mathrm{c}}(G, A) \longrightarrow \dots \tag{2.12}$$

We also have a normalized version of this, with $C^n_{\mathrm{c}}(G, A) \subset C^n_{\mathrm{gp}}(G, A)$ the group of continuous maps $G^{\wedge n} \to A$.

$$0 \longrightarrow C^0_{\mathrm{c}}(G, A) \longrightarrow \dots \longrightarrow C^n_{\mathrm{c}}(G, A) \xrightarrow{\delta^n} C^{n+1}_{\mathrm{c}}(G, A) \longrightarrow \dots \tag{2.13}$$

Proposition 2.4.2. *The homology groups of* (2.13) *are isomorphic to $H^*_{\mathrm{c}}(G, A)$, with isomorphism induced by the inclusion of* (2.13) *in* (2.12)*.*

Proof. See Eilenberg–MacLane's [6, Lemmas 6.1 & 6.2]. Note that they work in a purely algebraic set-up (i.e., G and A discrete), but their proof works in the continuous set-up too. Indeed, in their Equation 6.3, continuity of f implies continuity of g_i for all i, which in turn implies continuity of f_n. □

In light of the above, we will view the continuous cohomology groups as the homology groups of (2.13) throughout this thesis. Elements of $C^n_{\mathrm{c}}(G, A)$ will be called *continuous n-cochains* (n-cochains if continuity is clear from context), and elements of $\hat{C}^n_{\mathrm{c}}(G, A)$ will be called *non-normalized continuous n-cochains*.

The group of *continuous cocycles* is denoted by $Z^n_{\mathrm{c}}(G, A) := \ker \delta^n \subset C^n_{\mathrm{c}}(G, A)$ and the group of *continuous coboundaries* by $B^n_{\mathrm{c}}(G, A) := \operatorname{im} \delta^{n-1} \subset C^n_{\mathrm{c}}(G, A)$. Consequently, $H^*_{\mathrm{c}}(G, A) = Z^*_{\mathrm{c}}(G, A)/B^*_{\mathrm{c}}(G, A)$.

Of course, $C^*_{\mathrm{gp}}(G, A) = C^*_{\mathrm{c}}(G, A)$ when G and A are discrete, so $H^*_{\mathrm{gp}}(G, A) = H^*_{\mathrm{c}}(G, A)$ in this case.

Proposition 2.4.3. *When A is discrete, the quotient map $G \to \pi_0(G)$ induces an isomorphism*

$$H^n_{\mathrm{gp}}(\pi_0(G), A) = H^n_{\mathrm{c}}(\pi_0(G), A) \xrightarrow{\sim} H^n_{\mathrm{c}}(G, A).$$

Proof. The map $C^n_{\mathrm{gp}}(\pi_0(G), A) = C^n_{\mathrm{c}}(\pi_0(G), A) \to C^n_{\mathrm{c}}(G, A)$ induced by the quotient $G \to \pi_0(G)$ is an isomorphism since every continuous map $G^{\wedge n} \to A$ factors through $\pi_0(G^{\wedge n}) = \pi_0(G)^{\wedge n}$. $\square$

2.4.1 A long exact sequence

When G is discrete, a short exact sequence

$$1 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 1$$

of discrete abelian groups induces a short exact sequence

$$1 \longrightarrow C^*_{\mathrm{gp}}(G, A_1) \longrightarrow C^*_{\mathrm{gp}}(G, A_2) \longrightarrow C^*_{\mathrm{gp}}(G, A_3) \longrightarrow 1 \tag{2.14}$$

of chain complexes. This produces a long exact sequence

$$\dots \longrightarrow H^n_{\mathrm{gp}}(G, A_1) \longrightarrow H^n_{\mathrm{gp}}(G, A_2) \longrightarrow H^n_{\mathrm{gp}}(G, A_3) \longrightarrow H^{n+1}_{\mathrm{gp}}(G, A_1) \longrightarrow \dots \tag{2.15}$$

of cohomology groups. However, for continuous group cohomology with G a CW group and A_1, A_2, A_3 all k-groups, right-exactness of (2.14) fails. Consequently, there is no immediate analogue of (2.15). To remedy this, we define a 'relative' version of continuous group cohomology which fits into a long exact sequence analogous to that for relative singular homology.

Let G now be a CW group and $A' \subset A$ be abelian k-groups with A/A' also a k-group. A continuous relative n-cochain is an n-cochain $G^{\wedge n} \to A/A'$ which lifts to A, and $C^n_{\mathrm{c}}(G, A, A')$ is the group of all continuous relative n-cochains. In other words, $C^n_{\mathrm{c}}(G, A, A')$ is the image of the map $C^n_{\mathrm{c}}(G, A) \to C^n_{\mathrm{c}}(G, A/A')$. Clearly, $C^*_{\mathrm{c}}(G, A, A')$ is a subcomplex of $C^*_{\mathrm{c}}(G, A/A')$. Its homology groups are $H^*_{\mathrm{c}}(G, A, A')$. Furthermore, there is a short exact sequence

$$1 \longrightarrow C^*_{\mathrm{c}}(G, A') \longrightarrow C^*_{\mathrm{c}}(G, A) \longrightarrow C^*_{\mathrm{c}}(G, A, A') \longrightarrow 1$$

of chain complexes, yielding a long exact sequence

$$\dots \longrightarrow H^n_{\mathrm{c}}(G, A') \longrightarrow H^n_{\mathrm{c}}(G, A) \longrightarrow H^n_{\mathrm{c}}(G, A, A') \xrightarrow{\delta^n} H^{n+1}_{\mathrm{c}}(G, A') \longrightarrow \dots$$

of cohomology groups. The inclusion $J : C^*_{\mathrm{c}}(G, A, A') \hookrightarrow C^*_{\mathrm{c}}(G, A/A')$ induces a map

$$J_* : H^*_{\mathrm{c}}(G, A, A') \to H^*_{\mathrm{c}}(G, A/A')$$

of cohomology groups.

Of particular interest to us will be the case of the short exact sequence

$$1 \longrightarrow \Omega A \longrightarrow PA \xrightarrow{e_1} A^\circ \longrightarrow 1 \ ,$$

in which case $C^n_{\mathrm{c}}(G, PA, \Omega A)$ is the group of null-homotopic n-cochains $G^{\wedge n} \to A$.

2.5 Central extensions

A *central extension* of G by A is a tuple (E, μ, p) such that

- $p : E \to G$ is an A-bundle,
- $\mu : E \times E \to E$ is a multiplication map which makes E a k-group,
- p is a group homomorphism, and
- The A-action on E is compatible with μ, i.e.,

$$\mu(e \cdot a, e' \cdot a') = \mu(e, e') \cdot aa' \ \forall \, e, e' \in E, a, a' \in A.$$

Remark. The compatibility condition implies that the fiber inclusion $A \hookrightarrow E; a \mapsto 1_E \cdot a$ is a group homomorphism with image contained in the center of E. □

Write $\mathbf{E}(G, A)$ for the collection of all central extensions of G by A. Two such extensions $(E_i, \mu_i, p_i) \in \mathbf{E}(G, A)$ $(i = 1, 2)$ are said to be equivalent if there is an isomorphism $E_1 \to E_2$ of k-groups which is also an A-bundle isomorphism. Denote the collection of isomorphism classes by $\mathbb{E}(G, A)$. The *Baer sum* makes $\mathbb{E}(G, A)$ an abelian group with identity the trivial extension $G \times A$. There is also a 'forgetful' map

$$F_{G,A} : \mathbb{E}(G, A) \to \mathrm{Bun}_G(A),$$

which simply forgets the group structure. Clearly, this is a group homomorphism under the

Baer sum. We also have a map

$$T_{G,A} : H^2_{\mathrm{c}}(G, A) \to \mathbb{E}(G, A), \tag{2.16}$$

analogous to the standard isomorphism $H^2_{\mathrm{gp}}(G, A) \xrightarrow{\sim} \mathbb{E}(G, A)$ from the case when G and A are discrete, which 'twists' the component-wise multiplication on $G \times A$ (see [5, §17.4]). For a continuous cocycle $f : G^{\wedge 2} \to A$, we define $T_{G,A}([f])$ to be the class of $(G \times A, \mu_f, p) \in \mathbf{E}(G, A)$, where p is the first projection and μ_f is defined as

$$\mu_f((g, a), (h, b)) = (gh, abf(g, h)).$$

The identity element of this extension is $(1_G, 1_A)$. Clearly, $F \circ T = 0$. The standard proof of the fact that T is an isomorphism when G and A are discrete generalizes immediately to show that T is a group homomorphism and the sequence

$$0 \longrightarrow H^2_{\mathrm{c}}(G, A) \xrightarrow{T} \mathbb{E}(G, A) \xrightarrow{F} \mathrm{Bun}_G(A) \tag{2.17}$$

is exact.

Next, observe that we have a double-sided action of G on $\mathrm{Bun}_G(A)$ given as follows. For $\mathcal{X} = (X, p) \in \mathbf{Bun}_G(A)$, we define

$$g \cdot \mathcal{X} \cdot h = (X, g \cdot p \cdot h) \;\forall\, g, h \in G.$$

The image of F is fixed under both these actions by virtue of the group structure of central extensions. Furthermore, $g \cdot \mathcal{X} \cdot h$ is the pullback of $\mathcal{X}$ under the map $G \to G; x \mapsto g^{-1}xh^{-1}$. The homotopy class of $x \mapsto g^{-1}xh^{-1}$ depends only on the connected components in which g and h lie, so the actions of G on $\mathrm{Bun}_G(A)$ factor through $\pi_0(G)$ (here, we used that G is paracompact). Hence (2.17) can be refined to say that the following sequence is exact, where $\mathrm{Bun}_G(A)^{\pi_0(G)}$ is the collection of fixed points of the double-sided action of $\pi_0(G)$.

$$0 \longrightarrow H^2_{\mathrm{c}}(G, A) \xrightarrow{T} \mathbb{E}(G, A) \xrightarrow{F} \mathrm{Bun}_G(A)^{\pi_0(G)} \tag{2.18}$$

G also acts on $\mathrm{Bun}_{G^0}(A)^{\pi_0(G)}$ as

$$g \cdot \mathcal{X} = (X, g \cdot p \cdot g^{-1})$$

for $g \in G$ and $\mathcal{X} = (X, p) \in \mathbf{Bun}_{G^0}(A)$, and as before one checks that this action factors through $\pi_0(G)$. Restriction gives a map

$$\mathrm{Res}_{G,A} : \mathrm{Bun}_G(A)^{\pi_0(G)} \to \mathrm{Bun}_{G^0}(A)^{\pi_0(G)},$$

which we claim is an isomorphism. To construct its inverse consider a bundle $\mathcal{X} = (X, p) \in \mathbf{Bun}_{G^0}(A)^{\pi_0(G)}$ and fix a choice of coset representatives $s : \pi_0(G) \to G$. Define $\mathcal{X}' = (\pi_0(G) \times X, p')$ with $p' : \pi_0(G) \times X \to G; ([g], x) \mapsto s([g])p(x)$. Since the isomorphism class of $\mathcal{X}$ is fixed under the action of $\pi_0(G)$, it follows that the isomorphism class of $\mathcal{X}'$ is fixed under the double-sided action of $\pi_0(G)$ and does not depend on the choice of s. The inverse of Res is then defined to take the class of $\mathcal{X}$ to that of $\mathcal{X}'$. Let $F' = \mathrm{Res} \circ F$, so that the discussion so far can be summarized as follows.

Theorem 2.5.1. *The following is exact.*

$$0 \longrightarrow H^2_c(G, A) \xrightarrow{T} \mathbb{E}(G, A) \xrightarrow{F'} \mathrm{Bun}_{G^0}(A)^{\pi_0(G)}.$$

Remark 2.5.2. When A is discrete, $H^2_c(G, A) = H^2_{\mathrm{gp}}(\pi_0(G), A) \approx \mathbb{E}(\pi_0(G), A)$ and $\mathrm{Bun}_{G^0}(A)^{\pi_0(G)} \approx \mathbb{E}(G^0, A)^{\pi_0(G)}$ (see [9, Theorem 4.9]). Hence, **Theorem 2.5.1** reduces to [9, (5)] in this case. In particular, [9, Example 3.7] shows that (2.18) need not be right-exact. $\square$

2.6 pCW complexes

In general, EG and BG need not be CW complexes when G is. In this section, we will define a class of spaces called pCW complexes (short for pseudo-CW complexes) such that EG and BG are pCW complexes (for G a CW group). Additionally, we will show that all pCW complexes are homotopy equivalent to CW complexes. Section 9.1 provides pCW analogues of the computation of homotopy and (co)homology groups of CW complexes using skeleta. A space X is said to be a *pCW complex* if there exist

- subspaces $X_0 \subset X_1 \subset \ldots \subset X$, with X_0 a CW complex and $X = \varinjlim_m X_m$,
- CW complexes $Y_1, Y_2, \cdots$ and respective subcomplexes $Z_m \subset Y_m$, and

- maps $\ell_m : Z_m \to X_{m-1}$ such that

$$\begin{tikzcd} Z_m \arrow[r, hook] \arrow[d, "\ell_m"'] & Y_m \arrow[d] \\ X_{m-1} \arrow[r, hook] & X_m \end{tikzcd}$$

  is a pushout square.

Remark 2.6.1. $X_{m-1} \hookrightarrow X_m$ is a closed cofibration because $Z_m \hookrightarrow Y_m$ is. Consequently, $X_m \hookrightarrow X$ is a closed cofibration. □

These spaces and maps give a *pCW structure* on X. If there also exists $M : \mathbb{N} \to \mathbb{N}$ such that the n-skeleton of Y_m/Z_m is a point for all $m > M(n)$, then we say that X is a *good* pCW complex. It is not hard to see that $CX, \Sigma X, \tilde{C}X$, and $\tilde{\Sigma}X$ have natural pCW structures coming from that of X, and they are good if X is.

Every CW complex B is a pCW complex in a trivial way, by letting $B = X_0 = X_1 = \cdots$. However, it will be more convenient to give the pCW structure in which X_n is the n-skeleton and Y_n is a disjoint union of n-cells. Hence, CW complexes are good pCW complexes with $M(n) = n$.

Remark 2.6.2. Given a CW complex B with a subcomplex B', one can also give a good pCW structure to B by taking X_m to be the union of B' and the m-skeleton of B (as a CW complex). We will not use this in the present discussion. □

Henceforth in this section, X denotes a pCW complex with the above pCW complex structure. If X is good, then the n-skeleton of X is defined to be $X_{M(n)}$. This agrees with the standard notion of 'n-skeleton' for CW complexes. Note that $X/X_{M(n)}$ is a good pCW complex with n-skeleton a point.

Remark 2.6.3. A single pCW structure can admit several good pCW structures, and it could happen that the n-skeleton of a good pCW complex X contains the $(n+1)$-skeleton as a proper subset (since M need not be strictly increasing). □

Proposition 2.6.4. *BG is a good pCW complex with n-skeleton B_nG.*

Proof. Set $X_n = B_nG$ and $Y_n = G^n \times \Delta_n$. Let Z_n be the subspace of Y_n consisting of points $(g_1, \cdots, g_n, t_1, \cdots, t_n)$ with

- $g_i = 1_G$ for at least one i, or

- $(t_1, \cdots, t_n) \in \partial\Delta_n$.

Clearly, Z_n is a subcomplex of Y_n, where Y_n is given the standard product cell structure (recall that 1_G is a 0-cell of G). Now, recall the quotient map $k_n : G^n \times \Delta_n \to D_nG$ studied in Section 2.2. Since $p_G \circ k_n : Y_n \to X_n$ is surjective and its restriction to $Y_n - Z_n$ is injective, gluing Y_n to X_{n-1} along $\ell_n := p_G \circ k_n|_{Z_n}$ gives X_n.

Any cell of Y_n which is not in Z_n must contain the n-cell of Δ_n as a factor, so the $(n-1)$-skeleton of Y_n/Z_n is a point. Hence, we may take $M(n) = n$ so that the n-skeleton is $X_n = B_nG$. □

Proposition 2.6.5. *pCW complexes are paracompact. In particular, BG is paracompact.*

Proof. Use Theorem 4.1 and Proposition 4.2 here. □

To show that X has the homotopy type of a CW complex, we will mimic the proof of Whitehead's Theorem [8, Theorem 4.5] on a CW approximation for X to show that this CW approximation is actually a homotopy equivalence. For this, we first rephrase the Compression Lemma [8, Lemma 4.6] accordingly.

Lemma 2.6.6 (Compression Lemma)**.** *Let V be a space, (Y, Z) a CW pair, $\theta : Z \to V$ a map, and (Q, R) a pair of spaces. Let $U = V \sqcup_\theta Y$ be the pushout and $\xi : (U, V) \to (Q, R)$ a map. If $\pi_i(Q, R, r)$ is trivial for all $i \geq 0, r \in R$, then ξ is homotopic* rel V *to a map $U \to R$.*

Remark. The condition that $\pi_0(Q, R, r)$ is trivial for all $r \in R$ is understood as saying that R meets every path component of Q. □

Proof. Completely analogous to the proof of [8, Lemma 4.6]. □

Corollary 2.6.7. *Let (Q, R) be a pair of spaces with $\pi_i(Q, R, r)$ trivial for all $i \geq 0, r \in R$. Any map $\xi : (X, X_0) \to (Q, R)$ is homotopic* rel X_0 *to a map $X \to R$.*

Proof. Let $\xi_m : (X_m, X_0) \to (Q, R)$ be the restriction of ξ to X_m, and set $\xi_0' = \xi_0$, $\xi_0'' = \xi$. Suppose, inductively, that $\xi_{m-1}' : X_{m-1} \to R$ extends to $\xi_{m-1}'' : (X, X_{m-1}) \to (Q, R)$. By **Lemma 2.6.6**, $\xi_{m-1}''|_{X_m}$ is homotopic rel X_{m-1} to a map $\xi_m' : X_m \to R$. Since $X_m \hookrightarrow X$ is a cofibration, this extends to a homotopy from ξ_{m-1}'' to a map $\xi_m'' : (X, X_m) \to (Q, R)$. This completes the inductive construction of ξ_m' and ξ_m'' for all $m \geq 0$.

Let $\xi' : X \to R$ be the direct limit of the maps ξ_m', which makes sense since $\xi_m'|_{X_p} = \xi_p'$ for $p < m$. A homotopy from ξ to ξ' is obtained by playing out the homotopy from $\xi = \xi_0''$ to ξ_1'' in $[0, 1/2]$, then that from ξ_1'' to ξ_2'' in $[1/2, 3/4]$, and so on. At time 1 we define the homotopy to be ξ'. Continuity at 1 follows from the fact that the restriction of this homotopy to X_m is constant in $\left[\frac{2^m-1}{2^m}, 1\right]$. □

Theorem 2.6.8. *If $f : B \to X$ is a CW approximation for X, then it is a homotopy equivalence.*

Proof. As in the proof of Whitehead's Theorem in [8], the high-level strategy is to show that the mapping cylinder M_f deforms onto B using **Corollary 2.6.7** with $(Q, R) = (M_f, B)$. There are two ways to go about this, each with a counterpart in [8]. The first is to argue that f is homotopic to a cellular map, i.e., one which takes the n-skeleton of B to the n-skeleton of X. Hence, we may assume that f is cellular without loss of generality. This allows us to give a pCW structure on the mapping cylinder M_f, and then the theorem follows immediately from **Corollary 2.6.7**. This is left as an exercise for the inquisitive reader.

The second way is more direct. The inclusion $(B \sqcup X, B) \hookrightarrow (M_f, B)$ is homotopic rel B to a map $B \sqcup X \to B$ (by **Corollary 2.6.7**). Since $B \sqcup X \hookrightarrow M_f$ is a cofibration, this extends to a homotopy from the identity $M_f \to M_f$ to a map $g : (M_f, B \cup X) \to (M_f, B)$. Then we apply **Lemma 2.6.6** to the composition

$$(B \times I \sqcup X, B \times \partial I \sqcup X) \longrightarrow (M_f, B \sqcup X) \xrightarrow{g} (M_f, B) \ ,$$

and the resulting homotopy factors through M_f to yield a deformation of M_f onto X. □

Corollary 2.6.9. *BG has the homotopy type of a CW complex.*

2.6.1 Subcomplexes

A subcomplex of a pCW complex is defined analogously to that of a CW complex. A subspace $X' \subset X$ is is a subcomplex if there exist

- a subcomplex $X'_0 \subset X_0$,
- subspaces $X'_m \subset X_m$ with $X'_0 \subset X'_1 \subset \ldots$, and
- subcomplexes $Y'_m \subset Y_m$,

such that

- the restriction ℓ'_m of ℓ_m to $Z'_m = Z_m \cap Y'_m$ has image in X'_{m-1},
- X'_m is the pushout $X_{m-1} \sqcup_{\ell'_m} Y'_m$, and
- $X' = \bigcup_{m \geq 1} X'_m$.

The subspace topology on X' coming from X is the same as the pCW complex topology. $X'_m \hookrightarrow X_m$ can be seen to be a closed cofibration by induction on m. Consequently, $X'_m \hookrightarrow X$ and $X' \hookrightarrow X$ are also closed cofibrations. Collapsing X' in X, the space $X'' := X/X'$ has a natural pCW structure. Note that X' and X'' are good if X is.

Remark 2.6.10. With the above definitions, the excision axiom for generalized cohomology theories can be stated for *pCW pairs*. This allows us to talk about cohomology theories on pCW complexes, which we discuss in the next section. □

Proposition 2.6.11. *A pCW pair (X, X') is a good pair in the sense of [8], i.e., X' has a neighborhood in X which deforms onto X'.*

Proof. The proof of this statement for CW pairs (e.g., [8, Proposition A.5]) generalizes to the pCW case without much difficulty. □

Theorem 2.6.12 (Blakers–Massey Excision Theorem)**.** *If a based pCW pair (X, X') is r-connected and X' is s-connected ($r, s \geq 0$), then the map*

$$\pi_i(X, X') \to \pi_i(X/X')$$

is an isomorphism for $1 \leq i \leq r+s$ and a surjection for $i = r+s+1$.

Proof. Let $U \subset X$ be a neighborhood of X' which deforms onto X' (such U exists by **Proposition 2.6.11**). Set

$$\begin{aligned} Y &= X \cup \tilde{C}X', \\ Y_1 &= \tilde{C}X' \cup U \\ Y_2 &= X \cup (1/2, 1] \times X', \text{ and} \\ Y_0 &= Y_1 \cap Y_2 = U \cup (1/2, 1] \times X'. \end{aligned}$$

Hence, $\{Y_1, Y_2\}$ is an open cover of Y. Observe that Y_1 is contractible, Y_2 deforms onto X, and Y_0 deforms onto X'. From the long exact sequence of relative homotopy, we see that the pair (Y_1, Y_0) is $(s+1)$-connected. Also, $(Y_2, Y_0) \approx (X, X')$ is r-connected. Hence, [4, Theorem 6.4.1] yields that the map

$$\pi_i(Y_2, Y_0) \to \pi_i(Y, Y_1) \tag{2.19}$$

is an isomorphism for $1 \le i \le r+s$ and a surjection for $i = r+s+1$. We also have the commutative diagram

$$\begin{tikzcd}
\pi_i(Y_2, Y_0) \arrow[r] \arrow[d] & \pi_i(Y, Y_1) \arrow[d, "\approx"] \\
\pi_i(Y_2/Y_0) \arrow[r, "\approx"] & \pi_i(Y/Y_1)
\end{tikzcd},$$

with the isomorphisms coming from the facts that Y_1 is contractible and $Y_2/Y_0 \cong Y/Y_1$. Hence, the preceding observations about (2.19) show that

$$\pi_i(Y_2, Y_0) \to \pi_i(Y_2/Y_0)$$

is an isomorphism for $1 \le i \le r+s$ and a surjection for $i = r+s+1$. Using [8, Proposition 0.17] together with the fact that U deforms onto X' (equivalently, U/X' deforms onto the point X'/X') now proves the theorem. $\square$

2.7 Extraordinary cohomology theories

This section provides a brief account of the extraordinary cohomology theory coming from an Ω-spectrum. We adapt the treatment given in [8, §4.3] to our context of pCW complexes (in light of **Theorem 2.6.8**, there is no essential difference between the theories for CW complexes and pCW complexes).

An *Ω-spectrum* $\mathbb{K}$ is a sequence of based spaces $(K_n)_n$ (where n generally runs over either the integers or the non-negative integers) together with weak homotopy equivalences $K_n \to \Omega K_{n+1}$. Hence, for any based space X with the homotopy type of a CW complex, $[X, K_n]_* \approx [X, \Omega^2 K_{n+2}]_*$ is an abelian group (in particular, we may take X to be a pCW complex). The cohomology theory coming from $\mathbb{K}$ is the sequence of functors $H^n(-, \mathbb{K}) := [-, K_n]_*$ from the category of based pCW complexes and based maps to the category of abelian groups. If A is a discrete abelian group and $\mathbb{K}$ is the Eilenberg-MacLane spectrum $K_n = K(A, n)$, then $H^*(-, \mathbb{K})$ is the reduced singular cohomology theory with coefficient group A.

For our purposes, the most important property of such cohomology theories is the excision property — for a based pCW pair (X, X') with inclusion $\iota : X' \hookrightarrow X$ and quotient map $q : X \to X/X'$, there is a natural long exact sequence of cohomology groups

$$\dots \longrightarrow H^n(X/X', \mathbb{K}) \xrightarrow{q^*} H^n(X, \mathbb{K}) \xrightarrow{\iota^*} H^n(X', \mathbb{K}) \xrightarrow{\delta^n} H^{n+1}(X/X', \mathbb{K}) \longrightarrow \dots .$$

The *connecting morphism* δ^n is defined using a Puppe sequence, and is best understood by first considering the case of the pair (CY, Y). In this case, the connecting morphism $\delta^n : H^n(Y, \mathbb{K}) \to H^{n+1}(\Sigma Y, \mathbb{K})$ is defined to be the composition

$$[Y, K_n]_* \xrightarrow{\approx} [Y, \Omega K_{n+1}]_* \xrightarrow{\approx} [\Sigma Y, K_{n+1}]_*.$$

Here, the first isomorphism comes from the weak homotopy equivalence $K_n \to \Omega K_{n+1}$ and the second comes from the adjoint relation between Σ and Ω. To define δ^n for an arbitrary pCW pair (X, X'), we start by fixing a homotopy inverse $h : X/X' \to X \cup CX'$ for the map $X \cup CX' \to X/X'$ which collapses CX' (this is a homotopy equivalence by [8, Proposition 0.17]). We define δ^n to be the composition

$$[X', K_n]_* \xrightarrow[\approx]{\delta^n} [\Sigma X', K_{n+1}]_* \longrightarrow [X \cup CX', K_{n+1}]_* \xrightarrow[\approx]{h^*} [X/X', K_{n+1}]_*, \tag{2.20}$$

where the first arrow is the connecting morphism for the pair (CX', X') and the second arrow is induced by the map $X \cup CX' \to \Sigma X'$ which collapses X.

Remark 2.7.1. Of course, the above discussion works just as well with the unreduced cone and unreduced suspension replacing their reduced counterparts. □

2.7.1 Morphisms of spectra and cohomology theories

A morphism $\mathbb{K} \to \mathbb{L}$ of Ω-spectra is a sequence of maps $K_n \to L_n$ such that the resulting square

$$\begin{array}{ccc} K_n & \longrightarrow & \Omega K_{n+1} \\ \downarrow & & \downarrow \\ L_n & \longrightarrow & \Omega L_{n+1} \end{array}$$

commutes. This induces a natural transformation $H^*(-, \mathbb{K}) \to H^*(-, \mathbb{L})$ of cohomology theories, and the commutativity of the above square ensures that

$$\begin{array}{ccc} H^n(X', \mathbb{K}) & \xrightarrow{\delta^n} & H^{n+1}(X/X', \mathbb{K}) \\ \downarrow & & \downarrow \\ H^n(X', \mathbb{L}) & \xrightarrow{\delta^n} & H^{n+1}(X/X', \mathbb{L}) \end{array} \tag{2.21}$$

commutes for all pCW pairs (X, X').

More generally, a natural transformation between two cohomology theories (as functors from the category of based pCW complexes to the category of graded abelian groups) is said to be a *morphism of cohomology theories* if it respects the connecting morphism (as in (2.21)). Injectivity and surjectivity of such morphisms is defined in the obvious way. A notion of exact sequences of cohomology theories is now immediate. A *split* short exact sequence of cohomology theories is a short exact sequence of cohomology theories which splits via a morphism of cohomology theories. A *weakly split* short exact sequence of cohomology theories is a short exact sequence of cohomology theories which splits via a functor which may not necessarily be a morphism of cohomology theories (i.e., this functor may not respect the connecting morphism).

In categorical language, most of the above discussion can be summarized as follows. There is a category of Ω-spectra and their morphisms, an abelian category of cohomology theories and their morphisms, and a functor $\mathbb{K} \mapsto H^*(-, \mathbb{K})$ from the prior to the latter.

2.8 ΩB

The classifying space functor B is a right-inverse for the loop space functor Ω in the following sense:

Lemma 2.8.1. *Let G be a k-group. There is a weak homotopy equivalence $\phi_G : G \to \Omega BG$ such that the following triangle commutes.*

$$\begin{tikzcd} \pi_n(G) \arrow[rr, "\pi_n(\phi_G)"] & & \pi_n(\Omega BG) \\ & \pi_{n+1}(BG) \arrow[ul, "\partial", "\approx"'] \arrow[ur, "\approx"'] & \end{tikzcd}$$

If G is abelian, then ϕ_G can be chosen to be a group homomorphism (with ΩBG a group under pointwise multiplication of loops).

Remark 2.8.2. Let $p : (X, x_0) \to (B, b_0)$ be a G-bundle. In our setup, it will be convenient to define the connecting morphism $\partial : \pi_{n+1}(B, b_0) \to \pi_n(G, 1_G)$ by lifting maps $(I^{n+1}, \partial I^{n+1}) \to (B, b_0)$ to $(I^{n+1}, I^n \times \{1\}, \partial I^{n+1} - I^n \times \{1\}) \to (X, x_0 \cdot G, e_0)$. This is slightly different from the convention described in [8, P. 344]. □

Proof. The inclusion $G \hookrightarrow EG$ is null-homotopic (since EG is contractible), so there is a homotopy $H : G \times I \to EG$ with $H(\cdot, 0) \equiv 1_{EG}$ and $H(\cdot, 1)$ the inclusion of G in EG. Composing with $p_G : EG \to BG$, we see that $p_G \circ H(\cdot, 0) = p_G \circ H(\cdot, 1) \equiv p_G(1_{EG})$. Hence

$p_G \circ H$ induces a map $\phi_G : G \to \Omega BG$. Commutativity of the triangle follows from a routine check using the definition of ∂. In particular, this yields that $\pi_n(\phi_G)$ is an isomorphism and hence ϕ_G is a weak homotopy equivalence.

Recalling the construction of EG, an explicit choice for H is $(g, t) \mapsto (g, t)$. The induced map $G \to PG$ is a group homomorphism, since $(g_1, t)(g_2, t) = (g_1 g_2, t)$. When G is abelian, this choice of H ensures that ϕ_G is also group homomorphism. $\square$

Remark 2.8.3. The homotopy class of ϕ_G is independent of the choice of H. One way to see this is using the fact that concatenating one such null-homotopy with the reverse of another yields a map $G \to \Omega EG$, which is null-homotopic since ΩEG is contractible. $\square$

Corollary 2.8.4. *The sequence* $\mathbb{A} := (A, BA, B^2A, \cdots)$ *forms an* Ω*-spectrum.*

The map obtained by slightly modifying the definition of ϕ_G as

$$\phi'_G : G \to \Omega BG; g \mapsto (t \mapsto (g, 1 - t))$$

is also important. ϕ'_G is a homomorphism of H-spaces, i.e., the two maps $G \times G \to \Omega BG$ given by

$$\begin{aligned} (g_1, g_2) &\mapsto \phi'_G(g_1 g_2) \text{ and} \\ (g_1, g_2) &\mapsto \phi'_G(g_1) * \phi'_G(g_2) \end{aligned}$$

(where '$*$' denotes concatenation of loops) are homotopic. This can be seen from the fact that the map

$$G^2 \times \partial\Delta_2 \to BG; (g_1, g_2, t_1, t_2) \mapsto (g_1, t_1)(g_2, t_2)$$

extends to $G^2 \times \Delta_2$ (an extension is given by $p_G \circ k_2$). Since $g \mapsto \phi_G(g) * \phi'_G(g)$ is null-homotopic, ϕ'_G is a weak homotopy equivalence. Furthermore, the map $g \mapsto \phi_G(g^{-1})$ is homotopic to ϕ'_G.

Remark 2.8.5. $\pi_0(\phi'_G)$ is a group homomorphism only when $\pi_0(G)$ is abelian. Indeed, $\pi_0(\phi'_G)$ is an *antihomomorphism* since $\pi_0(\phi_G)$ is an isomorphism and $\pi_0(\phi'_G)(x) = \pi_0(\phi_G)(x)^{-1}$. $\square$

2.8.1 The cohomology theory $H^*(-, A)$

We define $H^*(-, A)$ to be the cohomology theory $H^*(-, \mathbb{A})$ coming from $\mathbb{A}$ (in the sense of Section 2.7). When A is discrete, $\mathbb{A}$ is the Eilenberg-MacLane spectrum. Consequently,

$H^*(-,A)$ is the reduced singular cohomology theory with coefficient group A (up to choice of isomorphism — see [8, Theorem 4.57]).[3] This construction is is also natural in A — a continuous homomorphism $A \to A'$ of abelian k-groups induces a morphism $B^*A \to B^*A'$ of Ω-spectra, which in turn induces a morphism $H^*(-,A) \to H^*(-,A')$ of cohomology theories.

Remark 2.8.6. There are two natural group operations on these cohomology groups. First is the usual group operation defined using the fact that $\mathbb{A}$ is an Ω-spectrum, and second is the pointwise-addition of maps using the group operation on B^nA. It is a standard exercise to check that these operations coincide. Due to its simplicity, we will treat the latter operation as the 'standard' choice. $\square$

Remark 2.8.7. For X a well-pointed space and G any path-connected k-group, the base-point-forgetting map $[X,G]_* \to [X,G]$ is an isomorphism of groups (group operations being pointwise multiplication of maps, cf. [8, §4.A, Exercise 1]). In particular, this allows us to make the identification $H^n(-,A) = [-,B^nA]$ for $n \geq 1$. $\square$

[3]**Caution:** For A discrete, $H^*(-,A)$ is the singular cohomology theory in the notation of [8], whereas it is the reduced singular cohomology theory in our notation.

Chapter 3

A first encounter with α

In this chapter, we will construct the map $\alpha_{G,A} : \mathbb{E}(G,A) \to H^2(BG, \mathbb{A})$, whose study motivates much of this thesis. For any space B and abelian k-group A', we have an isomorphism

$$\eta_{B,A'} : \mathrm{Bun}_B(A') \xrightarrow{\sim} [B, BA'] = H^1(B, A')$$

with inverse given by the pullback construction $f \mapsto f^*EA'$. For $B = BG$ and $A' = BA$, this yields

$$\eta_{BG,BA} : \mathrm{Bun}_{BG}(BA) \xrightarrow{\sim} [BG, BBA] = H^2(BG, BA).$$

Next, an extension $\mathcal{E} = (E, \mu, p) \in \mathbf{E}(G,A)$ gives a BA-bundle $Bp : BE \to BG$ (by [11, Lemma 7.4, Theorem 7.7]). Equivalent extensions in $\mathbf{E}(G,A)$ yield equivalent bundles in $\mathrm{Bun}_{BG}(BA)$, so we have a map $B : \mathbb{E}(G,A) \to \mathrm{Bun}_{BG}(BA)$. We define

$$\alpha_{G,A} := \eta_{BG,BA} \circ B.$$

This definition coincides with that given in [9] when A is discrete, up to choice of isomorphism between $H^2(BG, A)$ and $H^2_{\mathrm{sing}}(BG, A)$.

Having generalized the definition of α from [9], we now generalize one of their key results which many of their proofs rely on, namely Proposition 7.1. The obvious challenge in doing so is that there is no obvious analogue for Hurewicz's Theorem and Hurewicz's map when working with extraordinary cohomology theories. To address this, we interpret the proposition in terms of bundles and their classifying maps. However, we still state the result in terms of an anticommuting square.

3.1 Constructing the square

Let $f : BG \to BBA$ be any map. By **Lemma 2.8.1** and [8, Proposition 4.22], there exists a map $\tilde{f} : G \to BA$ (unique up to homotopy) such that the following diagram commutes.[1]

$$\begin{tikzcd} \Omega BG \arrow[r, "\Omega f"] & \Omega BBA \\ G \arrow[u, "\phi_G", "\sim"'] \arrow[r, dashed, "\tilde{f}"] & BA \arrow[u, "\sim", "\phi_{BA}"'] \end{tikzcd}$$

$f \mapsto \tilde{f}$ yields a map

$$\omega : [BG, BBA] \to [G, BA].$$

One checks that this is a homomorphism of groups using the last statement of **Lemma 2.8.1**.

Hence, given an extension $\mathcal{E} \in \mathbf{E}(G, A)$, we can use ω to construct an A bundle over G by pulling back $EA \to BA$ along $\omega \circ \alpha(\mathcal{E})$. This yields the following square.

$$\begin{tikzcd} \mathbb{E}(G, A) \arrow[r, "F"] \arrow[d, "\alpha"'] & \mathrm{Bun}_G(A) \arrow[d, "\eta"] \\ H^2(BG, A) \arrow[r, "\omega"] & H^1(G, A) \end{tikzcd}$$

Generalizing [9, Proposition 7.1], we claim that this square anticommutes.

Theorem 3.1.1. *$\omega \circ \alpha + \eta \circ F : \mathbb{E}(G, A) \to H^1(G, A)$ is the trivial map.*

The proof of the above spans the next two sections. Combining this theorem with the exactness of (2.17) yields the important corollary that extensions in $\ker \alpha$ come from continuous cocycles.

Corollary 3.1.2. *$\ker \alpha \subset \operatorname{im} T$, i.e., $\ker \alpha$ can be identified with a subgroup of $H^2_c(G, A)$.*

Example 3.1.3. If $f : G \wedge G \to A$ is a 2-cocycle which is null-homotopic through cocycles, then f has a lift $\tilde{f} : G \wedge G \to PA$ which is also a cocycle. By naturality of $\alpha \circ T$, we have $(e_1)_* \circ \alpha_{G,PA} \circ T_{G,PA}[\tilde{f}] = \alpha_{G,A} \circ T_{G,A} \circ (e_1)_*[\tilde{f}]$. The prior is 0 (since PA is contractible) and the latter is $\alpha_{G,A} \circ T_{G,A}[f]$, so $[f] \in \ker \alpha_{G,A}$ under the identification mentioned in **Corollary 3.1.2**.

[1]Since f is not assumed to be based, a priori Ωf need not be defined. However, by Remark 2.8.7, this is not an issue since BBA is connected.

The hypothesis on f is satisfied when G is discrete and $A = BA'$ for some k-group A'. This is because f must lift to a cocycle $G \wedge G \to EA'$ and EA' is contractible through group homomorphisms. $\square$

3.2 A heuristic involving path space bundles

Let (B, b_0) be a based CW complex and $p : (X, x_0) \to (B, b_0)$ a based G-bundle. Pick a representative $f_X \in \eta_{B,A}(X)$, so $X \cong f^*EG$ as G-bundles. Let $\tilde{f}_X : X \to EG$ be the corresponding overmap.

Lemma 3.2.1. *The projection $Pp : PX \to PB$ of path spaces is a PG-bundle, where PG is viewed as a group under pointwise multiplication of paths.*

Proof. It is clear that PG acts freely on PX with orbits given by fibers of Pp. It remains to produce local sections of Pp. Fix a path $\gamma \in PB$, i.e., $\gamma : I \to B$ with $\gamma(0) = b_0$. The image of γ is compact, so there exists a partition

$$0 = a_0 < a_1 < \ldots < a_k = 1$$

of I and open sets $U_1, \cdots, U_k \subset B$ such that $\gamma([a_{i-1}, a_i]) \subset U_i$ and there is a section $s_i : U_i \to p^{-1}(U_i)$ of p. Assume without loss of generality that $s_1(b_0) = x_0$. Let $\mathcal{U} \subset PB$ be the open neighborhood of γ consisting of all paths $\gamma' \in PB$ s.t.

$$\gamma'([a_{i-1}, a_i]) \subset U_i \ \forall\, i.$$

Let $\tau_i : U_i \cap U_{i+1} \to G$ be the transition functions of these sections, satisfying $s_i = s_{i+1} \cdot \tau_i$. We will now construct a section $s : \mathcal{U} \to Pp^{-1}(\mathcal{U})$. For $\gamma' \in \mathcal{U}$ define

$$s(\gamma')(t) = \begin{cases} s_1 \circ \gamma'(t) & t \in [0, a_1] \\ s_2 \circ \gamma'(t) \cdot \tau_1 \circ \gamma'(t) & t \in [a_1, a_2] \\ s_3 \circ \gamma'(t) \cdot \tau_2 \circ \gamma'(t) \cdot \tau_1 \circ \gamma'(t) & t \in [a_2, a_3] \\ \vdots & \end{cases}$$

Clearly, $s(\gamma')$ is a well-defined and continuous path. To show that s is continuous, we note that the uncurried map

$$\hat{s} : \mathcal{U} \times I \to X; (\gamma', t) \mapsto s(\gamma')(t)$$

is continuous (by the Pasting Lemma). $\square$

Since PB is contractible, it is tempting to conclude that every principal bundle over PB (in particular, $Pp : PX \to PB$) is trivial. However, we recall that drawing this conclusion would require the bundle Pp to be numerable. The author is unaware of any reasonable conditions on p which guarantee that Pp is numerable. Notwithstanding this technical issue, assuming the following conjecture is a useful heuristic.

Conjecture 3.2.2. *The bundle $Pp : PX \to PB$ admits a section $s_X : PB \to PX$.*

Lemma 3.2.3. *Assuming **Conjecture 3.2.2**, there exists a map $\eth_X : \Omega B \to G$ such that the following diagrams commute in the group and homotopy categories respectively, where ∂ is the connecting morphism from the long exact sequence of homotopy groups associated to the bundle $X \to B$ (see Remark 2.8.2).*

$$\begin{tikzcd} \pi_n(\Omega B) \arrow[rr, "\pi_n(\eth_X)"] \arrow[dr, "\approx"'] & & \pi_n(G) \\ & \pi_{n+1}(B) \arrow[ur, "\partial_{n+1}"'] & \end{tikzcd} \qquad \begin{tikzcd} \Omega B \arrow[rr, "\Omega f_X"] \arrow[dr, "\eth_X"'] & & \Omega BG \arrow[dl, "\eth_{EG}"] \\ & G & \end{tikzcd}$$

Furthermore, the composition $\Omega B \xrightarrow{\eth_X} G \hookrightarrow X$ is null-homotopic.

Remark. $\eth$ is an Old English letter, pronounced 'eth'.

Proof. Identifying $p^{-1}(b_0)$ with G as $x_0 \cdot g \sim g$, the section s_X allows us to define

$$\eth_X(\gamma) := s_X(\gamma)(1) \in G.$$

Continuity of $\eth_X$ is clear. Commutativity of the first triangle follows from a routine check using the definition of ∂. Commutativity of the second triangle follows from the fact that given a loop $\gamma \in \Omega B$, a lift of $f_X \circ \gamma \in \Omega BG$ to $P(EG)$ can be obtained by first lifting γ to PX and then composing with overmap $\tilde{f}_X$.

With $\iota_X : G \to X; g \mapsto x_0 \cdot g$ a fiber inclusion, a null-homotopy of $\iota_X \circ \eth_X$ is given by $(t, \gamma) \mapsto s_X(\gamma)(t)$. $\square$

Corollary 3.2.4. *Assuming **Conjecture 3.2.2**, $\eth_{EG}$ is a left homotopy inverse for ϕ_G.*

Proof. The composition $\eth_{EG} \circ \phi_G : G \to G$ induces identity on the homotopy groups of G by **Lemma 2.8.1** and **Lemma 3.2.3**, so the claim follows by Whitehead's Theorem. $\square$

An analogue of **Lemma 3.2.3** which does not rely on **Conjecture 3.2.2** can be obtained using CW approximation. Let ωB be a CW approximation for ΩB, with $\psi_B : \omega B \to \Omega B$ a weak homotopy equivalence. For B' another based CW complex and a map $h : B \to B'$, there exists a map $\omega h : \omega B \to \omega B'$ (unique up to homotopy) such that the following commutes in the homotopy category (by [8, Proposition 4.22]).

$$\begin{tikzcd}
\Omega B' \arrow[r, "\Omega h"] & \Omega B \\
\omega B' \arrow[u, "\psi_{B'}", "\sim"'] \arrow[r, dashed, "\omega h", "\exists!"'] & \omega B \arrow[u, "\sim", "\psi_B"']
\end{tikzcd}$$

This gives a map $\omega : [B, B'] \to [\omega B, \omega B']$. We adopt the convention $\omega BG = G$ and $\psi_{BG} = \phi_G$, so that this definition does not clash with the map

$$\omega : [BG, BBA] \to [G, BA]$$

defined previously. Now we define $\eth'_X := \omega f_X : \omega B \to \omega BG = G$, i.e.,

$$[\eth'_X] = \omega \circ \eta_{B,G}(X). \tag{3.1}$$

Lemma 3.2.5. *The following diagram commutes, where ∂ is the connecting morphism from the long exact sequence of homotopy groups associated to the bundle $X \to B$.*

$$\begin{tikzcd}
\pi_n(\omega B) \arrow[r, "\pi_n(\eth'_X)"] \arrow[d, "\pi_n(\psi_B)"', "\approx"] & \pi_n(G) \\
\pi_n(\Omega B) \arrow[r, "\approx"'] & \pi_{n+1}(B) \arrow[u, "\partial"]
\end{tikzcd}$$

Furthermore, the composition $\omega B \overset{\eth'_X}{\to} G \hookrightarrow X$ is null-homotopic.

Proof. The lemma essentially follows using an alternate definition of $\eth'_X$ which is closer in spirit to the definition of $\eth_X$. The composition

$$\omega B \xrightarrow{\psi_X} \Omega B \hookrightarrow PB$$

is null-homotopic (since PB is contractible), so the PG-bundle over ωB obtained by pulling back $PX \to PB$ is trivial (since ωB is paracompact and Hausdorff). The existence of a section for this bundle translates to a lift $\tilde{\psi}_B : \omega B \to PX$ of the above composition.

$$\begin{tikzcd} & & PX \arrow[d, "Pp"] \\ \omega B \arrow[r, "\psi_B"'] \arrow[urr, dashed, "\tilde{\psi}_B"] & \Omega B \arrow[r, hook] & PB \end{tikzcd}$$

The image of this lift is contained in $Pp^{-1}(\Omega B)$, so

$$\tilde{\psi}_B(y)(1) \in p^{-1}(b_0) \ \forall\, y \in \omega B.$$

Identifying $p^{-1}(b_0)$ with G as $x_0 \cdot g \sim g$, this yields a map

$$\eth''_X : \omega B \to G; y \mapsto \tilde{\psi}_B(y)(1).$$

To see that $\eth''_X \approx \eth'_X$, it suffices to show that the following commutes in the homotopy category.

$$\begin{tikzcd} \Omega B \arrow[r, "\Omega f_X"] & \Omega BG \\ \omega B \arrow[u, "\psi_B", "\sim"'] \arrow[r, "\eth''_X"] & \omega BG = G \arrow[u, "\sim", "\psi_{BG} = \phi_G"'] \end{tikzcd} \tag{3.2}$$

For this, let $H : G \times I \to EG$ be a null-homotopy of $G \hookrightarrow EG$ as in the proof of **Lemma 2.8.1** and

$$\hat{H} : G \to Pp_G^{-1}(\Omega BG); g \mapsto (t \mapsto H(g,t))$$

be the currying of H. We have the following diagram.

$$\begin{tikzcd} & G \arrow[dr, dashed, bend left, "\hat{H}"] & \\ & Pp^{-1}(\Omega B) \arrow[u, "q"] \arrow[r, "Pf_X"] \arrow[d, "Pp"] & Pp_G^{-1}(\Omega BG) \arrow[ul, "r"] \arrow[d, "Pp_G"] \\ \omega B \arrow[ur, "\tilde{\psi}_B"] \arrow[r, "\psi_B"] & \Omega B \arrow[r, "\Omega f_X"] & \Omega BG \end{tikzcd} \tag{3.3}$$

The maps temporarily labeled q and r are defined as $\gamma \mapsto \gamma(1)$, where G is identified with the fibers of the base points of B and BG as usual. One checks that the square and two triangles formed by the solid arrows commute. By definition of $\eth''_X$ and ϕ_G we have

$$\eth''_X = q \circ \tilde{\psi}_B,$$
$$\phi_G = Pp_G \circ \hat{H}.$$

Hence, the commutativity of (3.2) reduces to checking that the two maps $\omega B \to \Omega BG$ obtained by following the outermost paths in (3.3) are homotopic. By commutativity of the square and triangles in (3.3), this reduces to showing that $\hat{H} \circ r$ is homotopic to the identity on $Pp_G^{-1}(\Omega BG)$. For this, consider the map

$$Pp_G^{-1}(\Omega BG) \to \Omega EG; \gamma \mapsto \left(t \mapsto \gamma(t) \cdot \hat{H}(r(\gamma))(t)^{-1}\right),$$

where '$\cdot$' and '${}^{-1}$' are interpreted in the group EG. Since EG is contractible, this map is null-homotopic. This shows that $\hat{H} \circ r$ is homotopic to the identity on $Pp_G^{-1}(\Omega BG)$,[2] completing the proof of the fact that $\eth''_X \approx \eth'_X$. The lemma now follows using arguments analogous to **Lemma 3.2.3**. □

3.3 Combining several bundles into one

We now work towards a proof of **Theorem 3.1.1**. Fix an extension $\mathcal{E} = (E, \mu, p) \in \mathbb{E}(G, A)$. Our main tool will be the object $X_{\mathcal{E}} := EG \times_{BG} BE$ which fits into the pullback square

$$\begin{tikzcd} X_{\mathcal{E}} \arrow[r] \arrow[d] & BE \arrow[d, "Bp"] \\ EG \arrow[r, "p_G"] & BG \end{tikzcd}.$$

The first projection $X_{\mathcal{E}} \to EG$ is a BA-bundle and the second projection $X_{\mathcal{E}} \to BE$ is a G-bundle, so the diagonal composition $X_{\mathcal{E}} \to BG$ is a $(G \times BA)$-bundle. Furthermore, the map $Ep \times p_E : EE \to X_{\mathcal{E}}$ is an A-bundle.[3] This allows us to view $X_{\mathcal{E}}$ as the quotient group EE/A, since $A \subset EA$ is contained in the center of EE. The fiber inclusion $\iota_\times : G \times BA \hookrightarrow X_{\mathcal{E}}$ then becomes a group homomorphism. Let $\iota_G : G \hookrightarrow X_{\mathcal{E}}$ and $\iota_{BA} : BA \hookrightarrow X_{\mathcal{E}}$ be the components of $\iota_\times$.

[2] Here we are using the fact that $\Omega EG \leq Pp_G^{-1}(\Omega BG)$ as groups under pointwise multiplication of paths.
[3] This follows from the fact that $p_E : EE \to BE$ is an E-bundle and $p : E \to G$ is an A-bundle.

Lemma 3.3.1. *The restriction of $EE \to X_{\mathcal{E}}$ to $G, BA \subset X_{\mathcal{E}}$ yields the extensions (and hence A-bundles) $p : E \to G$ and $p_A : EA \to BA$ respectively.*

Proof. The claim follows by considering the preimages of G and BA under $Ep \times p_E : EE \to X_{\mathcal{E}}$. □

Corollary 3.3.2. *ι_{BA} is a homotopy equivalence.*

Proof. EE is contractible, so $Ep \times p_E : EE \to X_{\mathcal{E}}$ is a universal A-bundle. Since $\iota_{BA}^* EE \cong EA$ is also a universal A-bundle, the claim follows. □

Corollary 3.3.3. *$(\iota_{BA})_* : [B, BA] \to [B, X_{\mathcal{E}}]$ is a group isomorphism under pointwise multiplication of maps.*

Proof. $(\iota_{BA})_*$ is a homomorphism since ι_{BA} is a homomorphism, and it is a bijection since ι_{BA} is a homotopy equivalence. □

Corollary 3.3.4. *Under $(\iota_{BA})_* \circ \eta \circ F : \mathbb{E}(G, A) \to [G, X_{\mathcal{E}}]$, the class of $\mathcal{E}$ maps to $[\iota_G]$.*

Proof. The pullback of $EA \cong \iota_{BA}^* EE$ under $\eta \circ F(\mathcal{E})$ is the A-bundle $E \cong \iota_G^* EE$. Since there is a unique class in $[G, X_{\mathcal{E}}]$ which pulls EE back to E (by universality), we must have $[\iota_G] = (\iota_{BA})_* \circ \eta \circ F(\mathcal{E})$. □

*Proof of **Theorem 3.1.1**.* The bundle $X_{\mathcal{E}} \to BG$ yields a map

$$\eth'_{X_{\mathcal{E}}} : \omega BG = G \to G \times BA.$$

Using the fact that $B(G \times BA) \cong BG \times BBA$ (see [16, §6]), one checks that

$$\eth'_{X_{\mathcal{E}}} \approx \eth'_{EG} \times \eth'_{BE}.$$

Remark. The above can also be shown using the alternate construction of $\eth'$ described in the proof of **Lemma 3.2.5**. □

Now $\eth'_{EG} = \omega\, \mathrm{id}_{BG} = \mathrm{id}_G$, since $EG \cong \mathrm{id}_{BG}^* EG$ (this is analogous to **Corollary 3.2.4**). Hence we have

$$\begin{aligned} \eth'_{X_{\mathcal{E}}} &\approx \mathrm{id}_G \times \eth'_{BE} \\ &\in [\mathrm{id}_G] \times \omega \circ \eta_{BG,BA}(BE) \text{ (by (3.1))} \\ &= [\mathrm{id}_G] \times \omega \circ \eta_{BG,BA} \circ B(\mathcal{E}) \\ &= [\mathrm{id}_G] \times \omega \circ \alpha(\mathcal{E}). \end{aligned} \tag{3.4}$$

Using the group structure on $G \times BA$, the right side can be written as the product of two classes in $[G, G \times BA]$ as follows, where 1_{BA} and 1_G denote the respective constant maps.

$$[\mathrm{id}_G] \times \omega \circ \alpha(\mathcal{E}) = [\mathrm{id}_G \times 1_{BA}] \cdot [1_G] \times \omega \circ \alpha(\mathcal{E})$$

Composing (3.4) with $\iota_\times$ yields

$$\begin{aligned} \iota_\times \circ \eth'_{X_\mathcal{E}} &\in [\iota_G] \cdot (\iota_{BA})_* \circ \omega \circ \alpha(\mathcal{E}) \\ &= (\iota_{BA})_* \circ \eta \circ F(\mathcal{E}) \cdot (\iota_{BA})_* \circ \omega \circ \alpha(\mathcal{E}) \text{ (by } \textbf{Corollary 3.3.4}\text{).} \end{aligned}$$

The left side is null-homotopic since $\iota_\times \circ \eth'_{X_\mathcal{E}}$ is null-homotopic (by **Lemma 3.2.5**), so

$$(\eta \circ F + \omega \circ \alpha)(\mathcal{E})$$

is in the kernel of $(\iota_{BA})_*$. The result now follows by **Corollary 3.3.3**. $\square$

Chapter 4

The Dold–Thom Theorem and $\mathbb{A}$

This chapter is a brief excursion meant to understand the consequences of the Dold–Thom Theorem for the Ω-spectrum $\mathbb{A}$ and the cohomology theory it births.

4.1 The classical and the CG

In this section, we adapt the exposition of the Dold–Thom Theorem given in [8] (Theorem 4K.6) and its consequences for abelian topological groups (Corollary 4K.7) to the CG context. For a based CG space (X, x_0), let $SP_n^\tau(X)$ denote the n-fold symmetric product of X in the sense of [8], i.e., $SP_n^\tau(X)$ is the quotient of $X^{\times_\tau n}$ by the coordinate-shuffling action of the symmetric group. We have an inclusion $SP_n^\tau(X) \hookrightarrow SP_{n+1}^\tau(X)$ by setting the $(n+1)$-st coordinate to x_0, and the direct limit is defined to be $SP^\tau(X)$. This is the symmetric product denoted by $SP(X)$ in [8]. Let $SP_n^k(X)$ and $SP^k(X)$ be the analogues of the above constructions with the τ-product replaced by the k-product. These constructions are functorial and homotopy-preserving. If X is a CW complex, then so are $SP_n^k(X)$ and $SP^k(X)$, since the action of the symmetric group on X^n is cellular.

Lemma 4.1.1. *The quotient map $X^{\times_\tau n} \to SP_n^\tau(X)$ is proper.*

Proof. Let $K \subset SP_n^\tau(X)$ be compact and $\tilde{K}$ be its preimage in $X^{\times_\tau n}$. Hence, $\tilde{K} = C^{\times_\tau n}$ for some $C \subset X$. We will show that C is compact. Let $\mathcal{U} = \{U_i \mid i \in \mathcal{I}\}$ be an open cover of C. The set $U_i^{\times_\tau n}$ is a saturated open set in $X^{\times_\tau n}$, so its image $V_i \subset SP_n^\tau(X)$ is open. Now $\{V_i \mid i \in \mathcal{I}\}$ is an open cover of K, so it has a finite subcover. The corresponding subcover of $\mathcal{U}$ is a finite cover of C. $\square$

Lemma 4.1.2. *Let Y be a topological space equipped with an equivalence relation $\sim$. If the quotient map $Y \to {}^{Y}\!/\!\sim$ is proper, then the induced map $kY \to k\left({}^{Y}\!/\!\sim\right)$ is also a proper quotient map.*

Proof. $kY \to k\left({}^{Y}\!/\!\sim\right)$ factors through ${}^{kY}\!/\!\sim \to k\left({}^{Y}\!/\!\sim\right)$, which is a continuous bijection. The compact sets in both ${}^{kY}\!/\!\sim$ and $k\left({}^{Y}\!/\!\sim\right)$ are images of compact sets in Y, so ${}^{kY}\!/\!\sim \to k\left({}^{Y}\!/\!\sim\right)$ is a homeomorphism. $kY \to {}^{kY}\!/\!\sim$ is proper by a similar argument, so the lemma follows. □

From the above lemmas and the fact that every compact set in $SP^{\tau}(X)$ is contained in $SP^{\tau}_n(X)$ for some finite n, we obtain

Corollary 4.1.3. $SP^k_n(X) = k\,SP^{\tau}_n(X)$ *and* $SP^k(X) = k\,SP^{\tau}(X)$.

Corollary 4.1.4. *The maps $SP^{\tau}_n(X) \to SP^k_n(X)$ and $SP^{\tau}(X) \to SP^k(X)$ are weak homotopy equivalences.*

This last corollary allows us to state the following as a consequence of [8, Theorem 4K.6].

Theorem 4.1.5 (Dold–Thom Theorem)**.** *For X a CW complex, there are natural isomorphisms $H_i(X, \mathbb{Z}) \approx \pi_i(SP^k(X))$, $i \geq 1$.*

Using the following lemma, we can replace 'H-space' by 'k-H-space' in [8, Corollary 4K.7].

Lemma 4.1.6. *Let $(X_i, x_i), i \geq 1$ be a sequence of based CG spaces. The obvious set-map*

$$SP^k\left(\bigvee_{i\geq 1} X_i\right) \to \varinjlim_n \prod_{i=1}^{n} SP^k(X_i)$$

is a homeomorphism.

We relegate the proof to the end of this section.

Corollary 4.1.7. *Let A be a connected abelian k-monoid. There exist abelian CW monoids $A_1, A_2, \cdots$ such that*

- *A_n is a $K(\pi_n(A), n)$-space, and*
- *there exists a continuous homomorphism*

$$\varinjlim_n \prod_{i=1}^{n} A_i \to A$$

of monoids, which is also a weak homotopy equivalence.

Proof. Morally, the proof is identical to that of [8, Corollary 4K.7], with **Theorem 4.1.5** used in place of [8, Theorem 4K.6]. We take

$$A_n = SP^k(M(\pi_n(A), n)),$$

where $M(-,-)$ denotes the standard CW realization of Moore spaces. The technical details which differ from [8] are covered by **Lemma 4.1.6**. □

*Proof of **Lemma 4.1.6**.* We start with the following general fact about iterated direct limits of topological spaces. Let Y be a space with subspaces $Y_{m,n}$, indexed by $m, n \geq 1$. Suppose there are inclusions $Y_{m,n} \subset Y_{m+1,n}$ and $Y_{m,n} \subset Y_{m,n+1}$. If

$$Y \cong \varinjlim_m \varinjlim_n Y_{m,n},$$

then

$$Y \cong \varinjlim_n \varinjlim_m Y_{m,n} \cong \varinjlim_{(m,n)} Y_{m,n}.$$

This allows us to write

$$SP^k\left(\bigvee_{i\geq 1} X_i\right) \cong \varinjlim_n SP^k\left(\bigvee_{i=1}^n X_i\right).$$

Hence, the lemma reduces to showing that the obvious set-map

$$SP^k\left(\bigvee_{i=1}^n X_i\right) \to \prod_{i=1}^n SP^k(X_i)$$

is a homeomorphism. Induction on n reduces this to the $n = 2$ case, i.e.,

$$SP^k(X_1 \vee X_2) \to SP^k(X_1) \times SP^k(X_2) \tag{4.1}$$

is a homeomorphism. Continuity and bijectivity are easy to check, so it suffices to show that (4.1) is proper. Any compact subset $K \subset SP^k(X_1) \times SP^k(X_2)$ is contained in $SP^k_i(X_1) \times SP^k_i(X_2)$ for some $i \geq 1$. The quotient maps $X_1^i \to SP^k_i(X_1)$ and $X_2^i \to SP^k_i(X_2)$ are proper (by **Lemma 4.1.2** and **Corollary 4.1.3**), so the preimage $K' \subset X_1^i \times X_2^i$ of K under their product is compact. Hence, the image of K' under

$$X_1^i \times X_2^i \to (X_1 \vee X_2)^{2i} \to SP^k(X_1 \vee X_2)$$

is compact. This image is precisely the preimage of K under (4.1). $\square$

4.2 The connected case

A will be assumed to be a connected abelian k-group throughout this section.

Lemma 4.2.1. *Let $A_1, A_2, \cdots$ be abelian k-monoids. The obvious set-theoretic map*

$$\varinjlim_n \prod_{i=1}^n BA_i \to B \varinjlim_n \prod_{i=1}^n A_i \tag{4.2}$$

is an isomorphism of k-monoids.

Here, B still refers to the Milgram–Steenrod construction from [16]; the description of this construction for abelian k-monoids is much the same as that for abelian k-groups.

Proof. One checks directly that (4.2) is an isomorphism of abstract monoids. We now show that it is continuous and proper. From the discussion in [16, §6], it is clear that the map

$$\prod_{i=1}^n BA_i \to B \prod_{i=1}^n A_i \tag{4.3}$$

is a homeomorphism. In particular, (4.2) is continuous. Every compact subset of $B \varinjlim_n \prod_{i=1}^n A_i$ is contained in the image of (4.3) for some n, so (4.2) is also proper. $\square$

Lemma 4.2.2. *If A is a connected abelian k-monoid which is a $K(A', n)$-space, then BA is a $K(A', n+1)$-space.*

Proof. This is immediate from [16, Theorem 8.1]. $\square$

We now show that the functor $H^*(-, A)$ can be expressed in terms of shifts of singular cohomology with various coefficient groups.

Proposition 4.2.3. *Suppose A is connected. There is an isomorphism*

$$H^*(-, A) \approx \prod_{i \geq 1} H^{*+i}(-, \pi_i(A))$$

of cohomology theories (in the sense of Section 2.7.1).

Proof. Let $A_1, A_2, \cdots$ be as in **Corollary 4.1.7**, and set

$$A' = \lim_{\substack{\rightarrow \\ n}} \prod_{i=1}^{n} A_i \text{ and}$$
$$A''_n = \prod_{i \geq 1} B^n A_i.$$

There is a weak homotopy equivalence $A' \to A$ which is also a homomorphism of monoids. Hence, we have

$$H^n(X, A) \approx [X, B^n A']_*$$

for X a based pCW complex. Furthermore, using **Lemma 4.2.1**, it is immediate that the inclusion $B^n A' \hookrightarrow A''_n$ is continuous and a weak homotopy equivalence. Hence, we also have

$$\begin{aligned} [X, B^n A']_* &\approx [X, A''_n]_* \\ &\approx \prod_{i \geq 1} [X, B^n A_i]_* \\ &\approx \prod_{i \geq 1} [X, B^{n+i} \pi_i(A)]_* \\ &= \prod_{i \geq 1} H^{n+i}(X, \pi_i(A)) \end{aligned}$$

for every based pCW complex X, where the third isomorphism comes from **Lemma 4.2.2** and the definition of the A_i's. Combining the above isomorphisms proves the proposition. $\square$

For $d \geq 0$ and A not necessarily connected, say that A is of *type* d if $\pi_n(A)$ is trivial for $n > d$. Say that A is of *finite type* if A is of type d for some $d \geq 0$. When A is connected and of finite type, the direct product in **Proposition 4.2.3** becomes a direct sum.

4.3 The general case

We no longer assume that A is connected.

Lemma 4.3.1. *The short exact sequence*

$$1 \longrightarrow A^\circ \longrightarrow A \longrightarrow \pi_0(A) \longrightarrow 1$$

induces a weakly split short exact sequence

$$0 \longrightarrow H^*(-, A^\circ) \longrightarrow H^*(-, A) \longrightarrow H^*(-, \pi_0(A)) \longrightarrow 0$$

of cohomology theories.

Proof. Let $A' = \pi_0(A)$. For each $n \geq 0$, we have a B^nA°-bundle $B^nA \to B^nA'$. The lemma will follow if we show that this bundle is trivial, i.e., there is a section $B^nA' \to B^nA$ (not necessarily a group homomorphism). Since B^nA' is a CW complex, this can be done using a cell-by-cell lifting argument. In particular, it suffices to show that given any map $f : D^d \to B^nA'$ and a lift $\tilde{f}' : S^{d-1} \to B^nA$ of its restriction to the boundary, $\tilde{f}'$ extends to a lift $\tilde{f}$ of f. This follows from the facts that bundles are Serre fibrations and that the connecting morphism $\partial : \pi_d(B^nA') \to \pi_{d-1}(B^nA^\circ)$ from the long exact sequence of homotopy groups for the bundle $B^nA \to B^nA'$ is trivial. $\square$

Combining **Proposition 4.2.3** and **Lemma 4.3.1** yields

Theorem 4.3.2. *There is a weakly split short exact sequence*

$$0 \longrightarrow \prod_{i \geq 1} H^{*+i}(-, \pi_i(A)) \longrightarrow H^*(-, A) \longrightarrow H^*(-, \pi_0(A)) \longrightarrow 0$$

of cohomology theories.

4.4 $B\Omega$

Loosely speaking, **Lemma 2.8.1** says that the functor Ω is a left-inverse for B up to weak homotopy equivalence. We will now show that for connected abelian k-groups, it is in fact a two-sided inverse in this sense.

There is a natural homomorphism

$$\theta_A : B\Omega A \to A$$

generated by $(\gamma, t) \mapsto \gamma(t)$. It is easy to see that this is well-defined and continuous.[1] Through routine arguments, one checks the following.

[1]The fact that A is abelian is essential here.

Lemma 4.4.1. *For $n \geq 1$, the following triangle commutes.*

$$\begin{tikzcd}
\pi_n(B\Omega A) \arrow[rr, "\partial", "\approx"'] \arrow[dr, "\pi_n(\theta_A)"'] & & \pi_{n-1}(A) \\
& \pi_n(A) \arrow[ur, "\approx"'] &
\end{tikzcd}$$

In particular, $\theta_A : B\Omega A \to A^\circ$ is a weak homotopy equivalence.

Together with **Lemma 4.3.1**, the above yields

Corollary 4.4.2. *The following is a weakly split short exact sequence of cohomology theories.*

$$0 \longrightarrow H^*(-, B\Omega A) \xrightarrow{(\theta_A)_*} H^*(-, A) \longrightarrow H^*(-, \pi_0(A)) \longrightarrow 0$$

It will also be useful to know how ϕ_A interacts with θ_A:

Lemma 4.4.3. *The composition*

$$BA \xrightarrow{B\phi_A} B\Omega BA \xrightarrow{\theta_{BA}} BA$$

is the identity.

Proof. It suffices to check that the composition is the identity on generators $(a, t) \in BA$ for $a \in A, t \in I$. First, recall that $\phi_A(a)$ is the loop $\gamma_a : I \to BA; s \mapsto (a, s)$. Hence, we have

$$B\phi_A(a, t) = (\gamma_a, t).$$

Consequently, we have

$$\begin{aligned} \theta_{BA} \circ B\phi_A(a, t) &= \theta_{BA}(\gamma_a, t) \\ &= \gamma_a(t) \\ &= (a, t). \end{aligned}$$

□

Chapter 5

α in terms of cocycles

In light of **Corollary 3.1.2**, studying $\ker\alpha$ requires understanding how α acts on extensions coming from continuous 2-cocycles $G \wedge G \to A$. This chapter gives a partial explicit description of this action, essentially providing a formula for the restriction $\iota_2^* \circ \alpha[f]$ in terms of the cocycle f. Furthermore, the ideas of this chapter form the backbone of our subsequent construction of the maps $\alpha^n : H_c^n(G, A) \to H^n(BG, A)$.

5.1 D_1G, D_2G, and their images in BG

We begin with some simple technical lemmas.

Lemma 5.1.1. *Let $p : E \to B$ be a fiber bundle, $X \subset E$ and $Y = p(X)$. Then $p|_X : X \to Y$ is a quotient map.*

Proof. The claim is easy to prove when p is a trivial bundle. The general claim follows by locally reducing to the trivial case using local trivializations. □

Lemma 5.1.2. *Let X, Y and Z be spaces with a continuous map $f : X \to Y$ and a set-map $g : Y \to Z$. Furthermore, suppose that f and $g \circ f$ are quotient maps. Then g is also a quotient map.*

Proof. The lemma follows from routine arguments. □

Now we proceed with studying the objects mentioned in the title of this section. Since $k_1 : G \times I \to D_1G$ is a quotient map, D_1G is the reduced cone CG of G by (2.4). Consequently, B_1G is ΣG, the reduced suspension of G. Note that here we are using both of the above lemmas as follows. $p_G|_{D_1G} : D_1G \to B_1G$ is a quotient map by **Lemma 5.1.1**, and there is also a set-theoretic bijection between B_1G and ΣG. This yields the following commuting diagram, where the dotted arrow denotes a set-map which is not a priori continuous and solid arrows denote quotient maps.

$$\begin{tikzcd} G \times I \arrow[r] \arrow[dr] & D_1G \arrow[r] & B_1G \arrow[dl, dashed] \\ & \Sigma G & \end{tikzcd}$$

By **Lemma 5.1.2** (with $X = G \times I$, $Y = D_1G$ and $Z = \Sigma G$), we now see that the dotted map is a homeomorphism. Henceforth we will identify B_1G with ΣG, and hence we have a natural inclusion $\iota_1 : \Sigma G \hookrightarrow BG$. This identification also makes the isomorphism $[X, G]_* \approx [\Sigma X, BG]_*$ (for X a based pCW complex) easier to understand explicitly. Originally, this isomorphism comes from the weak homotopy equivalence $\phi_G : G \to \Omega BG$ (see **Lemma 2.8.1**), but we also have the following commutative triangle.

$$\begin{tikzcd} {[X, G]_*} \arrow[r, "\approx"] \arrow[d, "\Sigma"'] & {[\Sigma X, BG]_*} \\ {[\Sigma X, \Sigma G]_*} \arrow[ur, "(\iota_1)_*"'] & \end{tikzcd} \tag{5.1}$$

Remark 5.1.3. Commutativity of the triangle follows from the commutativity of

$$\begin{tikzcd} {[X, G]_*} \arrow[r, "\phi_G", "\approx"'] \arrow[d, "\Sigma"'] & {[X, \Omega BG]_*} \\ {[\Sigma X, \Sigma G]_*} \arrow[r, "(\iota_1)_*"] & {[\Sigma X, BG]_*} \arrow[u, "\approx"'] \end{tikzcd},$$

which is easier to see. $\square$

Now we will do a similar analysis for D_2G. First, we have a quotient map

$$q'' : G^2 \times \Delta_2 \to \Sigma^2(G \wedge G); (g_1, g_2, t_1, t_2) \mapsto \begin{cases} \left(g_1, g_2, \frac{t_1}{t_2}, t_2\right) & t_2 \neq 0 \\ (1_G, 1_G, 0, 0) & t_2 = 0 \end{cases}. \tag{5.2}$$

Here, $\Sigma^2(G \wedge G)$ is viewed as a quotient of $G^2 \times I^2$. Next, note that q'' factors through $p_G \circ k_2$ (which is a quotient map, by **Lemma 5.1.1**), yielding a map $q' : B_2G \to \Sigma^2(G \wedge G)$ with $q'' = q' \circ p_G \circ k_2$. By **Lemma 5.1.2**, we see that q' is a quotient map. In fact, q' is the map which collapses $\Sigma G = B_1G \subset B_2G$. Hence we have a homeomorphism $B_2G/\Sigma G \cong \Sigma^2(G \wedge G)$, and the corresponding quotient map

$$q : B_2G \to \Sigma^2(G \wedge G).$$

5.2 A partial explicit description of $\alpha \circ T$

Throughout the rest of this section, we will identify $H^2_c(G, A)$ as a subgroup of $\mathbb{E}(G, A)$ using T (see (2.16) and (2.17)). Hence, we may apply α directly on 2-cocycles by defining $\alpha f := \alpha[f]$, where $[f]$ denotes the cohomology class of f. Combining the various maps in this section yields a peculiar square.

$$\begin{tikzcd}
Z^2_c(G,A) \arrow[r, "\alpha"] \arrow[d, "F_c"'] & H^2(BG,A) \arrow[d, "\iota_2^*"] \\
H^0(G\wedge G,A) \arrow[r, dashed, "q^*"] \arrow[d, "\approx"'] & H^2(B_2G,A) \\
H^2(\Sigma^2(G\wedge G),A) \arrow[ur, "q^*"'] &
\end{tikzcd} \tag{5.3}$$

Here, F_c is the forgetful map obtained by looking at the homotopy class of a cocycle, the vertical isomorphism comes from excision for $H(-, A)$, and the dotted arrow is defined so that the triangle commutes. We claim that the square commutes.

Theorem 5.2.1. *The two maps $\iota_2^* \circ \alpha, q_* \circ F_c : Z^2_c(G, A) \to H^2(B_2G, A)$ are equal.*

Given a 2-cocycle f, this theorem essentially yields, explicitly in terms of f, the restriction to B_2G of a representative of αf. This can be seen by examining $q^* \circ F_c$ as follows. Applying (5.1) twice (first with A in place of G, then with BA in place of G), we see that the vertical isomorphism in (5.3) takes $f : G \wedge G \to A$ to the map

$$\Sigma^2 f : \Sigma^2(G \wedge G) \to B^2A; (g_1, g_2, t_1, t_2) \mapsto (f(g_1, g_2), t_1, t_2).$$

Composing with q, we obtain the following representative of $q^* \circ F_c(f)$.

$$B_2G \to B^2A; (g_1, t_1)(g_2, t_2) \mapsto \left(f(g_1, g_2), \tfrac{t_1}{t_2}, t_2\right) \tag{5.4}$$

Remark 5.2.2. The above expression is not problematic when $t_2 = 0$ because $(-, -, 0) = (1_A, 0, 0)$ regardless of what is substituted for '$-$'. This is made precise by the fact that

$(t_1, t_2) \mapsto \left(\frac{t_1}{t_2}, t_2\right)$ defines a homeomorphism $\Delta_2/\partial\Delta_2 \to I^2/\partial I^2$. $\square$

Using (5.4) together with **Theorem 5.2.1** is the most important step in the proof of **Theorem 5.3.1**, which makes precise the agreement of α with the classical isomorphism $H^2_{\mathrm{gp}}(G, A) \xrightarrow{\sim} H^2_{\mathrm{sing}}(BG, A)$ when G and A are discrete. Another application of **Theorem 5.2.1** is **Corollary 6.3.5**, which provides a complete description of $\ker(\iota_2^* \circ \alpha)$.

*Proof of **Theorem 5.2.1**.* Fix a 2-cocycle f and let $\mathcal{E} = (E, \mu, p)$ be the corresponding extension. Hence, $E = G \times A$ as a topological space, p is the first projection, and multiplication in E is given by

$$\mu((g, a), (g', a')) = (gg', aa' f(g, g')).$$

Recall that $1_E = (1_G, 1_A)$. Also recall the topological group $X_{\mathcal{E}} = EE/A$ from Section 3.3, and the fact that the inclusion $\iota_{BA} : BA \hookrightarrow X_{\mathcal{E}}$ is a homotopy equivalence (see **Corollary 3.3.2**). Hence, $B\iota_{BA} : B^2A \to BX_{\mathcal{E}}$ is a weak homotopy equivalence. Consequently, it suffices to show that

$$(B\iota_{BA})_* \circ \iota_2^* \circ \alpha(f) = (B\iota_{BA})_* \circ q^* \circ F_c(f).$$

$(B\iota_{BA})_*$ and ι_2^* commute (the prior acts by left-composition and the latter acts by restriction/right-composition), so we will instead prove that

$$\iota_2^* \circ (B\iota_{BA})_* \circ \alpha(f) = (B\iota_{BA})_* \circ q^* \circ F_c(f). \tag{5.5}$$

(5.4) yields the following representative R of the right side of (5.5). The conventions used to write elements of $BX_{\mathcal{E}}$ are analogous to those used for B^2A.

$$R : B_2G \to BX_{\mathcal{E}}; (g_1, t_1)(g_2, t_2) \mapsto \left((1_G, f(g_1, g_2)), \frac{t_1}{t_2}, t_2\right)$$

For the left side of (5.5), first recall that $\iota_G : G \hookrightarrow X_{\mathcal{E}}$ pulls back EE to E (as extensions by A). Hence, $B\iota_G : BG \hookrightarrow BX_{\mathcal{E}}$ pulls back BEE to BE (as BA-bundles). The homotopy class $(B\iota_{BA})_* \circ \alpha(f) \in [BG, BX_{\mathcal{E}}]$ also pulls back BEE to BE. The BA-bundle $BEE \to BX_{\mathcal{E}}$ is universal (since BEE is contractible), so we must have

$$[B\iota_G] = (B\iota_{BA})_* \circ \alpha(f). \tag{5.6}$$

ι_G has the straightforward description

$$\iota_G : G \to X_{\mathcal{E}}; g \mapsto ((g, 1_A), 1),$$

so $B\iota_G$ is given by

$$B\iota_G : BG \to BX_{\mathcal{E}}; (g_1, t_1)\cdots(g_n, t_n) \mapsto ((g_1, 1_A), 1, t_1)\cdots((g_1, 1_A), 1, t_n). \tag{5.7}$$

In toto, the left side of (5.5) is represented by

$$L : B_2G \to BX_{\mathcal{E}}; (g_1, t_1)(g_2, t_2) \mapsto ((g_1, 1_A), 1, t_1)((g_2, 1_A), 1, t_2).$$

It remains to show that L and R homotopic. For this, we first observe that while $BX_{\mathcal{E}}$ is not a group, it is nonetheless acted on by B^2A on the right (it is a B^2A-bundle over BEG, since $X_{\mathcal{E}}$ is an extension of EG by BA). In this light, we can multiply two maps $h_1, h_2 : X \to BX_{\mathcal{E}}$ pointwise to obtain $h_1 \cdot h_2$, where X is any space and the image of h_2 is contained in $B^2A \hookrightarrow BX_{\mathcal{E}}$. Likewise, h_2 can be inverted pointwise to obtain h_2^{-1}. Hence, to show that L and R are homotopic, it suffices to prove that $L \cdot R^{-1}$ is null-homotopic. Following is a null-homotopy, with $H_1 = L \cdot R^{-1}$ and H_0 constant.

$$H_s : (g_1, t_1)(g_2, t_2) \mapsto ((g_1, 1_A), s, t_1)((g_2, 1_A), s, t_2)\left(\left(1_G, f(g_1, g_2)^{-1}\right), s\tfrac{t_1}{t_2}, t_2\right)$$

In order to check that this is well-defined and continuous, it is left to the reader to verify the following for $g_i \in G, t \in I$, and $0 \le t_3 \le t_1 \le t_2 \le 1$.

- $H_s((g_1, t_1)(1_G, t_2)) = H_s((1_G, t_3)(g_1, t_1)) = H_s((g_1, t_1)(1_G, 1))$.
- $H_s((g_1, 0)(g_2, t_2)) = H_s((g_2, t_2)(g_3, 1)) = H_s((g_2, t_2)(1_G, 1))$.
- $H_s((g_1, t)(g_2, t)) = H_s((g_1g_2, t)(1_G, 1))$. □

5.3 α when G and A are discrete

In this section, we will use **Theorem 5.2.1** to show that when BG and $\bar{B}G$ are identified (up to homotopy) using $\bar{\Psi}$, the maps $\alpha : \mathbb{E}(G, A) \to H^2(BG, A)$ and the classical isomorphism $H^2_{\mathrm{gp}}(G, A) \xrightarrow{\sim} H^2_{\Delta}(\bar{B}G, A)$ are 'the same'. In this sense, α generalizes the classical isomorphism $\mathbb{E}(G, A) \xrightarrow{\sim} H^2(BG, A)$ to the case of G a CW group and A an abelian k-group.

First, we fix an isomorphism $H^n(X, A) \xrightarrow{\sim} H^n_{\mathrm{CW}}(X, A)$ for $n \geq 1$ and X a CW complex. The Hurewicz map $\pi_n(B^nA) \to H_n(B^nA)$ is an isomorphism, so the Universal Coefficient Theorem yields an isomorphism $H^n_{\mathrm{CW}}(B^nA, A) \xrightarrow{\sim} \mathrm{Hom}(\pi_n(A), A)$. Under this identification, let $\varepsilon_n \in H^n_{\mathrm{CW}}(B^nA, A)$ be the class corresponding to the isomorphism $\pi_n(B^nA) \to A$ given by (2.5). By [8, Theorem 4.57], the map $\varepsilon_n^* : [X, B^nA] \to H^n_{\mathrm{CW}}(X, A); \phi \mapsto \phi^*(\varepsilon_n)$ is an isomorphism.

The inverse of the above isomorphism can be constructed as follows. Given a cellular cocycle $f : H^n_{\mathrm{CW}}(X^n/X^{n-1}) \to A$, consider the map $\phi_n : X^n \to B^nA$ which sends X^{n-1} to 1_{B^nA} and sends a n-cell $e : I^n \to X$ of X to the representative

$$I^n \to B^nA; (t_1, \cdots, t_n) \mapsto (f(e), t_1, \cdots, t_n)$$

of the class in $\pi_n(B^nA)$ corresponding to $f(e) \in A$. The fact that f is a cocycle implies (in fact, is equivalent to) the existence of an extension $\phi_{n+1} : X^{n+1} \to B^nA$ of ϕ_n. Since $\pi_i(B^nA) \approx 0$ for $i > n$, ϕ_{n+1} can now be extended cell-by-cell (uniquely, up to homotopy) to a map $\phi : X \to B^nA$. One checks that $\phi^*(\varepsilon_n) = [f]$ in $H^n_{\mathrm{CW}}(X, A)$, so this construction indeed gives a representative for $(\varepsilon_n^*)^{-1}[f]$. An outline of a direct proof that this construction is an isomorphism (without alluding to [8, Theorem 4.57]) can be found in this MathOverflow post.

Theorem 5.3.1. *The following commutes for G and A discrete.*

$$\begin{tikzcd}
\mathbb{E}(G, A) \arrow[r, "\alpha"] \arrow[d, "\approx"'] & H^2(BG, A) \arrow[r, "\bar{\Psi}^*"] & H^2(\bar{B}G, A) \arrow[d, "\approx"', "\varepsilon_2^*"] \\
H^2_{\mathrm{gp}}(G, A) \arrow[r, "\approx"] & H^2_{\Delta}(\bar{B}G, A) \arrow[r, "\approx"] & H^2_{\mathrm{CW}}(\bar{B}G, A)
\end{tikzcd}$$

Corollary 5.3.2. *α is an isomorphism when G and A are discrete.*

In the theorem, the isomorphism $H^2_{\mathrm{gp}}(G, A) \to \mathbb{E}(G, A)$ is the standard one. The isomorphism between simplicial and cellular cohomology of a Δ-complex X is obtained by regarding each characteristic map $\Delta_n \to X$ given by the Δ-complex structure as a characteristic map for an n-cell. The isomorphism between group cohomology and simplicial cohomology of $\bar{B}G$ is as follows.

Let $F_n = \mathbb{Z}G^{n+1}$, with G acting on the rightmost component from the right, be the inhomogeneous free resolution of $\mathbb{Z}$ as a simple G-module. F_n has basis G^{n+1} as an abelian group, and this basis is in G-equivariant bijection with the n-simplices of $\bar{E}G$ — $(g_0, \cdots, g_{n+1})$ corresponds to the simplex in $\bar{E}G$ with vertices $[g_0 \dots g_{n+1}, 1], [g_1 \dots g_n, 1], \cdots, [g_{n+1}, 1]$. This yields a G-equivariant isomorphism between F_n and the group $C_n^{\Delta}(\bar{E}G)$ of simplicial n-chains (with $\mathbb{Z}$-coefficients) in $\bar{E}G$.

$\mathbb{Z}G$-module morphisms $F_n \to A$ are constant on all G-orbits (since G acts trivially on A), and the group of G-equivariant simplicial cocycles in $C^n_{\Delta}(\bar{E}G, A)$ is isomorphic to $C^n_{\Delta}(\bar{B}G, A)$. Hence, the isomorphism of the preceding paragraph yields

$$\mathrm{Hom}_G(F_n, A) \approx C^n_{\Delta}(\bar{B}G, A). \tag{5.8}$$

As depicted in the below diagram, we now have two complexes, with the upper producing $H^*_{\mathrm{gp}}(G, A)$ and the lower producing $H^*(\bar{B}G, A)$. This yields the isomorphism between $H^2_{\mathrm{gp}}(G, A)$ and $H^2(\bar{B}G, A)$ used in **Theorem 5.3.1**.

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{Hom}_G(F_n, A) \arrow[r] \arrow[d, "\approx"'] & \mathrm{Hom}_G(F_{n+1}, A) \arrow[r] \arrow[d, "\approx"'] & \dots \\
\dots \arrow[r] & C^n_{\Delta}(\bar{B}G, A) \arrow[r] & C^{n+1}_{\Delta}(\bar{B}G, A) \arrow[r] & \dots
\end{tikzcd}$$

A basis of F_n as a $\mathbb{Z}G$-module is $G^n \times \{1_G\} \subset G^{n+1}$, so $\mathrm{Hom}_G(F_n, A)$ is isomorphic to the abelian group of set-maps $G^n \to A$. We will use this identification throughout, so that the simplicial cochain in $C^n_{\Delta}(\bar{B}G, A)$ corresponding (under (5.8)) to $f : G^n \to A$ is that which takes the simplex with vertices $g_0, \cdots, g_n$ to $f(g_0 g_1^{-1}, \cdots, g_{n-1} g_n^{-1})$.

Proof of ***Theorem 5.3.1****.* Let $f : G \wedge G \to A$ be a 2-cocycle. Under the lower horizontal composition in the diagram, the image of the class of f is represented by the cellular cocycle f_{CW} which sends the simplex with vertices g_0, g_1, g_2 (viewed as a 2-cell) to $f(g_0 g_1^{-1}, g_1 g_2^{-1})$. Now we will construct a representative $\phi : \bar{B}G \to B^2 A$ of $(\varepsilon_2^*)^{-1}[f_{\mathrm{CW}}] \in [\bar{B}G, B^2 A]$. For this we first need to describe the cells of $\bar{B}G$ using characteristic maps $I^n \to \bar{B}G$ instead of the more familiar $\Delta_n \to \bar{B}G$ (see (2.9)). This can be done by composing with the map

$$r : I^n \to \Delta_n; (t_1, \cdots, t_n) \mapsto (t_1 \dots t_n, t_2 \dots t_n, \cdots, t_n),$$

so that the characteristic map of the n-cell with vertices $g_0, \cdots, g_n$ is given by composing (2.9) with r. The construction of $(\varepsilon_n^*)^{-1}$ described previously yields that the restriction $\phi_2 = \phi|_{\bar{B}_2 G}$ is

$$\phi_2 : \bar{B}_2 G \to B^2 A; [g_0, s_0, g_1, s_1, g_2, s_2] \mapsto \left(f(g_0 g_1^{-1}, g_1 g_2^{-1}), \tfrac{s_0}{s_0 + s_1}, s_0 + s_1 \right).$$

Now, suppose $\phi' : BG \to B^2A$ is a representative for $\alpha(f)$. The theorem will follow if we show that $\phi' \circ \bar{\Psi}|_{\bar{B}_2G} \approx \phi_2$. Since $\bar{\Psi}(\bar{B}_2G) = B_2G$, it suffices to prove that

$$\phi_2' \circ \bar{\Psi}_2 \approx \phi_2, \tag{5.9}$$

where $\bar{\Psi}_2 = \bar{\Psi}|_{\bar{B}_2G}$ and $\phi_2' = \phi'|_{B_2G}$. By **Theorem 5.2.1** and (5.4), we may choose ϕ' so that ϕ_2' is given by

$$\phi_2' : B_2G \to B^2A; (g_1, t_1)(g_2, t_2) \mapsto \left(f(g_1, g_2), \tfrac{t_1}{t_2}, t_2\right).$$

Composing with $\bar{\Psi}_2$ yields

$$\phi_2' \circ \bar{\Psi}_2 : \bar{B}_2G \to B^2A; [g_0, s_0, g_1, s_1, g_2, s_2] \mapsto \left(f(g_0g_1^{-1}, g_1g_2^{-1}), \tfrac{s_0}{s_0+s_1}, s_0 + s_1\right),$$

which in fact gives equality in (5.9). $\square$

Chapter 6

Analogues of $\iota_2^* \circ \alpha$ in higher degrees

6.1 The successive quotients $B_nG/B_{n-1}G$

In Section 5.1, we observed that there are homeomorphisms

$$B_1G \cong \Sigma G, \text{ and}$$
$$B_2G/B_1G \cong \Sigma^2(G \wedge G).$$

This generalizes — we have

$$B_nG/B_{n-1}G \cong \Sigma^n G^{\wedge n}, \tag{6.1}$$

where $G^{\wedge n}$ is the n-fold smash product of G with itself. This holds for $n = 1$ too, since B_0G is a point. A homeomorphism can be constructed as follows, directly generalizing the form that (5.2) takes. Consider the homeomorphism

$$\mu_n : \Delta_n/\partial\Delta_n \to I^n/\partial I^n; (t_1, \cdots, t_n) \mapsto \left(\frac{t_1}{t_2}, \cdots, \frac{t_{n-1}}{t_n}, t_n\right), \tag{6.2}$$

and define

$$q_n : B_nG \to \Sigma^n G^{\wedge n}; (g_1, t_1) \cdots (g_n, t_n) \mapsto \left(\vec{g}, \mu_n(\vec{t})\right).$$

Here, $\vec{g} = (g_1, \cdots, g_n) \in G^n$ and $\vec{t} = (t_1, \cdots, t_n) \in \Delta^n$. It is left to the reader to verify that this map is a quotient map, as was done for $q = q_2$ in Section 5.1. It is clear that q_n collapses $B_{n-1}G$, and hence factors through (6.1).

6.2 A topological counterpart to the coboundary operator

Given an n-cochain $f : G^{\wedge n} \to A$, write $\Sigma^n f$ for the map

$$\Sigma^n G^{\wedge n} \to B^n A; (\vec{g}, \vec{t}) \mapsto \big(f(\vec{g}), \mu_n(\vec{t})\big) .$$

The homotopy class of $\Sigma^n f$ is the image of f under the composition

$$C^n_{\mathrm{c}}(G, A) \longrightarrow [G^{\wedge n}, A]_* \xrightarrow{\approx} [\Sigma^n G^{\wedge n}, B^n A] \ ,$$

where the first map takes homotopy classes and the second uses the isomorphism $[-, A]_* \xrightarrow{\sim} [\Sigma -, BA]$ n times (see (5.1)). Composing with q_n then gives a map

$$\alpha_n : C^n_{\mathrm{c}}(G, A) \to H^n(B_n G, A).$$

With this notation, **Theorem 5.2.1** can be rephrased as saying that $\iota_2^* \circ \alpha$ is the restriction of α_2 to cocycles. For an n-cochain f, we also write $\alpha_n f$ for the map

$$B_n G \to B^n A; (g_1, t_1) \cdots (g_n, t_n) \mapsto \big(f(\vec{g}), \mu_n(\vec{t})\big) .$$

With $\delta^n : H^n(B_n G, A) \to H^{n+1}(\Sigma^{n+1} G^{\wedge(n+1)}, A)$ the connecting morphism from the long exact sequence of cohomology for the pair $(B_{n+1}G, B_n G)$, we obtain the following square.

$$\begin{tikzcd} C^n_{\mathrm{c}}(G, A) \arrow[r, "\delta^n"] \arrow[d, "\alpha_n"'] & B^{n+1}_{\mathrm{c}}(G, A) \arrow[d, "\Sigma^{n+1}"] \\ H^n(B_n G, A) \arrow[r, "\delta^n"] & H^{n+1}(\Sigma^{n+1} G^{\wedge(n+1)}, A) \end{tikzcd} \tag{6.3}$$

Theorem 6.2.1. *For $n \geq 1$, the square (6.3) commutes up to a sign $\epsilon_n \in \{-1, 1\}$ (independent of G and A), i.e., $\Sigma^{n+1} \circ \delta^n = \epsilon_n \delta^n \circ \alpha_n$.*

Remark 6.2.2. The proof of the theorem will show that, with sufficient labor, it is possible to determine ϵ_n (see Remark 6.2.12). We will not do this, however, since the precise value of ϵ_n is immaterial for our purposes. □

This theorem essentially gives an explicit formula for the connecting morphism δ^n : $H^n(B_n G, A) \to H^{n+1}(\Sigma^{n+1} G^{\wedge(n+1)}, A)$. Consequently, proving the theorem would require us to make explicit the data conveyed by the statement '$(B_{n+1}G, B_nG)$ is a cofibration'. This is extremely difficult to do directly, so we instead translate the problem to one of giving an explicit formula for the connecting morphism for a different cofibration which is easier to work with.

Recall the quotient map $k_{n+1} : G^{n+1} \times \Delta_{n+1} \to D_{n+1}G$ defined in [16], which satisfies $p_G \circ k_{n+1}(G^n \times \partial\Delta_{n+1}) = B_nG$. Consider the following diagram, whose upper square is (6.3).

$$\begin{tikzcd}
C^n_{\mathrm{c}}(G,A) \arrow[r, "\delta^n"] \arrow[d, "\alpha_n"'] & B^{n+1}_{\mathrm{c}}(G,A) \arrow[d, "\Sigma^{n+1}"] \\
H^n(B_nG, A) \arrow[r, "\delta^n"] \arrow[d, "(p_G\circ k_{n+1})^*"'] & H^{n+1}(\Sigma^{n+1}G^{\wedge(n+1)}, A) \arrow[d] \\
H^n\left(G^{n+1} \times \partial\Delta_{n+1}, A\right) \arrow[r, "\delta^n"] & H^{n+1}\left(\frac{G^{n+1}\times\Delta_{n+1}}{G^{n+1}\times\partial\Delta_{n+1}}, A\right)
\end{tikzcd}$$

Here, the lowermost arrow denoted by δ^n comes from the long exact sequence of cohomology for the pair

$$\left(G^{n+1} \times \Delta_{n+1}, G^{n+1} \times \partial\Delta_{n+1}\right).$$

The lower right vertical arrow is induced by the quotient map

$$G^{n+1} \times \Delta_{n+1} \to \Sigma^{n+1}G^{\wedge(n+1)}; (\vec{g}, \vec{t}) \mapsto \left(\vec{g}, \mu_{n+1}(\vec{t})\right).$$

The lower square commutes (by naturality of the connecting morphism). If the lower-right vertical arrow were injective, then showing that the upper square commutes up to a sign would reduce to showing that the outer square (shown below) commutes up to a sign.

$$\begin{tikzcd}
C^n_{\mathrm{c}}(G,A) \arrow[r, "\delta^n"] \arrow[d] & B^{n+1}_{\mathrm{c}}(G,A) \arrow[d] \\
H^n\left(G^{n+1} \times \partial\Delta_{n+1}, A\right) \arrow[r, "\delta^n"] & H^{n+1}\left(\frac{G^{n+1}\times\Delta_{n+1}}{G^{n+1}\times\partial\Delta_{n+1}}, A\right)
\end{tikzcd}$$

A moment's thought should make the commutativity of this square believable — the left vertical arrow 'acts like' the coboundary operator, and the lower horizontal arrow simply raises the dimension by converting n-spheres to $(n+1)$-spheres. We provide a rigorous proof at the end of this section. First, we produce a chain of lemmas to show that the requisite injectivity indeed holds. Throughout this section, for based CG spaces (X, x_0) and (Y, y_0), we identify X and Y as subspaces of $X \times Y$ as $x \sim (x, y_0)$ and $y \sim (x_0, y)$ respectively.

Definition 6.2.3 (Fat wedge)**.** For a based space X and integers $0 \leq m \leq n$, the n-fold m-fat wedge of X is the subspace $\mathrm{Fat}^n_m(X) \subset X^n$ consisting of points $(x_1, \cdots, x_n) \in X^n$ with $x_i = x_0$ for at least $n - m$ values of i. Note that $\mathrm{Fat}^n_m(X)$ is a CW complex if X is. $\square$

Remark 6.2.4. The quotient map $X^n \to X^{\wedge n}$ induces a natural homeomorphism $\frac{X^n}{\mathrm{Fat}^n_{n-1}(X)} \xrightarrow{\sim} X^{\wedge n}$. $\square$

Lemma 6.2.5. *Let X be a based space. Every based map $\mathrm{Fat}^n_m(X) \to A$ extends to X^n.*

Proof. Let $x_0 \in X$ be the base point. For each set $S \subset [n]$ of size m, let $p_S : X^n \to \mathrm{Fat}^n_m(X)$ be the map

$$(x_1, \cdots, x_n) \mapsto (y_1, \cdots, y_n)$$

with $y_i = x_i$ if $i \in S$ and $y_i = x_0$ otherwise. Clearly, p_S is continuous.

The proof of the lemma is by induction on $m \in [n]_0$. For $m = 0$ the claim is trivial. Now suppose the claim is true for some $m \geq 0$ and we will prove it for $m + 1$. Let $\phi : \mathrm{Fat}^n_{m+1}(X) \to A$ be based map and $\psi : \mathrm{Fat}^n_m \to A$ be its restriction. Set

$$\tilde{\phi} := \prod_{\substack{S \subset [n] \\ |S| = m+1}} \phi \circ p_S : X^n \to A, \tag{6.4}$$

and let $\tilde{\psi}$ be an extension of ψ to X^n.

Note that if $\psi \equiv 1_A$, then $\tilde{\phi}$ an extension of ϕ to X^n (this follows directly from (6.4)). We can use this to construct an extension of ϕ in general as follows. Let

$$\phi' = \phi \cdot \tilde{\psi}|^{-1}_{\mathrm{Fat}^n_{m+1}(X)}$$

(where '$\cdot$' and '${}^{-1}$' are interpreted as pointwise operations done in A), so that

$$\phi'|_{\mathrm{Fat}^n_m(X)} \equiv 1_A.$$

Hence, ϕ' has an extension $\tilde{\phi}' : X^n \to A$ as noted above. Now, $\tilde{\phi}' \cdot \tilde{\psi}$ is an extension of ϕ. $\square$

Corollary 6.2.6. *For X a based pCW complex, the restriction map $H^*(X^n, A) \to H^*(\mathrm{Fat}^n_m(X), A)$ is surjective in all degrees.*

Corollary 6.2.7. *For X a based pCW complex, the map $H^*(X^{\wedge n}, A) \to H^*(X^n, A)$ induced by the quotient map $X^n \to X^{\wedge n}$ is injective in all degrees.*

Proof. Consider the following snippet of the long exact sequence of cohomology for the pair $(X^n, \mathrm{Fat}^n_{n-1}(X))$, keeping in mind that $X^{\wedge n} := \frac{X^n}{\mathrm{Fat}^n_{n-1}(X)}$.

$$H^{*-1}(X^n, A) \longrightarrow H^{*-1}(\mathrm{Fat}^n_{n-1}(X), A) \longrightarrow H^*(X^{\wedge n}, A) \longrightarrow H^*(X^n, A)$$

The leftmost arrow is surjective by **Corollary 6.2.6**, so the rightmost arrow is injective. □

Lemma 6.2.8. *Let X and Y be based pCW complexes. The map $\frac{X \times Y}{Y} \cup CX \to \Sigma X$ which collapses $\frac{X\times Y}{Y}$ is null-homotopic.*

Proof. Let $p_X : \frac{X\times Y}{Y} \to X$ be the projection and consider the following maps.

$$H : \frac{X \times Y}{Y} \times I \to \Sigma X; (z,t) \mapsto (p_X(z), t)$$
$$H' : CX \times I \to \Sigma X; (z,t) \mapsto tz$$

The disjoint union $H \sqcup H' : \left(\frac{X\times Y}{Y} \sqcup CX\right) \times I \to \Sigma X$ factors through $\left(\frac{X\times Y}{Y} \cup CX\right) \times I$, and this factor map is the desired null-homotopy. □

Corollary 6.2.9. *In the notation of the preceding lemma, the connecting morphism δ : $H^*(X, A) \to H^*(X \wedge Y, A)$ from the long exact sequence of the pair $\left(\frac{X\times Y}{Y}, X\right)$ is trivial.*

Proof. Use the definition of the connecting morphism (see (2.20)) and the preceding lemma. □

Corollary 6.2.10. *Let X be a based pCW complex, $d, n \geq 1$ and $0 \leq m \leq n$. The map*

$$H^*\left(\frac{X^n \times S^d}{X^n}, A\right) \to H^*\left(\frac{\mathrm{Fat}^n_m(X) \times S^d}{\mathrm{Fat}^n_m(X)}, A\right)$$

induced by the natural inclusion

$$\frac{\mathrm{Fat}^n_m(X) \times S^d}{\mathrm{Fat}^n_m(X)} \hookrightarrow \frac{X^n \times S^d}{X^n}$$

is surjective in all degrees.

Proof. The rows of the following commutative diagram are snippets of the long exact sequences corresponding to the pairs $\left(\frac{X^n\times S^d}{X^n}, S^d\right)$ and $\left(\frac{\mathrm{Fat}^n_m(X)\times S^d}{\mathrm{Fat}^n_m(X)}, S^d\right)$, where we use the fact that $Y \wedge S^d \cong \Sigma^d Y$ for all based CG spaces Y. The vertical arrows are induced by the obvious inclusion of pairs.

$$\begin{tikzcd}
H^*(\Sigma^d X^n, A) \arrow[r] \arrow[d] & H^*\left(\frac{X^n \times S^d}{X^n}, A\right) \arrow[r] \arrow[d] & H^*(S^d, A) \arrow[r] \arrow[d, equal] & H^{*+1}(\Sigma^d X^n, A) \arrow[d] \\
H^*(\Sigma^d \operatorname{Fat}^n_m(X), A) \arrow[r] & H^*\left(\frac{\operatorname{Fat}^n_m(X) \times S^d}{\operatorname{Fat}^n_m(X)}, A\right) \arrow[r] & H^*(S^d, A) \arrow[r] & H^{*+1}(\Sigma^d \operatorname{Fat}^n_m(X), A)
\end{tikzcd}$$

The rightmost arrows of both rows are trivial by **Corollary 6.2.9**. The leftmost vertical arrow is a surjection by **Corollary 6.2.6**, so the claim follows by the Four Lemma. □

Corollary 6.2.11. *Let X be a based pCW complex and $n, d \geq 1$. The map*

$$H^*(\Sigma^d X^{\wedge n}, A) \to H^*\left(\frac{X^n \times \Delta_d}{X^n \times \partial\Delta_d}, A\right)$$

induced by the quotient map

$$\frac{X^n \times \Delta_d}{X^n \times \partial\Delta_d} \to \Sigma^d X^{\wedge n}; \left(\vec{x}, \vec{t}\right) \mapsto \left(\vec{x}, \mu_d(\vec{t})\right)$$

is injective in all degrees.

Proof. We begin by observing that a choice of homeomorphism $\frac{\Delta_d}{\partial\Delta_d} \cong S^d$ induces a natural homeomorphism

$$\frac{Y \times \Delta_d}{Y \times \partial\Delta_d} \cong \frac{Y \times S^d}{Y} \tag{6.5}$$

for all CG spaces Y. Furthermore, there is a natural homeomorphism

$$\frac{\left(\frac{X^n \times S^d}{X^n}\right)}{\left(\frac{\operatorname{Fat}^n_{n-1}(X) \times S^d}{\operatorname{Fat}^n_{n-1}(X)}\right)} \cong \Sigma^d X^{\wedge n},$$

where $\frac{\operatorname{Fat}^n_{n-1}(X) \times S^d}{\operatorname{Fat}^n_{n-1}(X)}$ is identified as a subspace of $\frac{X^n \times S^d}{X^n}$ as in **Corollary 6.2.10** (cf. Remark 6.2.4). Hence, there is an exact sequence

$$H^{*-1}\left(\frac{X^n \times S^d}{X^n}, A\right) \longrightarrow H^{*-1}\left(\frac{\operatorname{Fat}^n_{n-1}(X) \times S^d}{\operatorname{Fat}^n_{n-1}(X)}, A\right) \longrightarrow H^*(\Sigma^d X^n, A) \longrightarrow H^*\left(\frac{X^n \times S^d}{X^n}, A\right).$$

The leftmost arrow is surjective by **Corollary 6.2.10**, so the rightmost arrow is injective. □

*Proof of **Theorem 6.2.1**.* As discussed in the beginning of this section, it suffices to show that the following square commutes up to a sign (the requisite injectivity follows from **Corollary 6.2.11**). We replace n by $n-1$ for convenience, so $n \geq 2$.

$$\begin{tikzcd} C_{\mathrm{c}}^{n-1}(G,A) \arrow[r, "\delta^{n-1}"] \arrow[d] & B_{\mathrm{c}}^{n}(G,A) \arrow[d] \\ H^{n-1}\left(G^n \times \partial\Delta_n, A\right) \arrow[r, "\delta^{n-1}"] & H^n\left(\frac{G^n\times\Delta_n}{G^n\times\partial\Delta_n}, A\right) \end{tikzcd} \tag{6.6}$$

Fix a continuous $(n-1)$-cochain $f : G^{\wedge(n-1)} \to A$. We break the proof into several parts, analyzing the journey of f along the various arrows in the above diagram. $\vec{g} = (g_1, \cdots, g_n) \in G^n$ and $\vec{t} = (t_1, \cdots, t_n) \in \Delta_n$ denote general points. For $1 \leq i \leq n$, write $\vec{t_i}$ for $(t_1, \cdots, \hat{t_i}, \cdots, t_n)$. Similarly,

$$\vec{g}_i = \begin{cases} (g_2, \cdots, g_n) & i = 0 \\ (g_1, \cdots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \cdots, g_n) & 0 < i < n \\ (g_1, \cdots, g_{n-1}) & i = n \end{cases} .$$

(a) **Right-down:** The right-down composition in the square takes f to the homotopy class of the map $f_1 : \frac{G^n\times\Delta_n}{G^n\times\partial\Delta_n} \to B^nA$ given by

$$(\vec{g}, \vec{t}) \mapsto \left(\delta f(\vec{g}), \mu_n(\vec{t})\right).$$

(b) For $0 \leq i \leq n$, let $\sigma_i : \Delta_{n-1} \to \partial\Delta_n$ be the i-th face map, given by

$$(t_1 \cdots, t_{n-1}) \mapsto \begin{cases} (0, t_1, \cdots, t_{n-1}) & i = 0 \\ (t_1, \cdots, t_i, t_i, \cdots, t_{n-1}) & 0 < i < n \\ (t_1, \cdots, t_{n-1}, 1) & i = n \end{cases} .$$

We also use σ_i to denote the image of the map σ_i.

(c) **Leftmost vertical:** The leftmost vertical arrow in the square takes f to the homotopy class of the map $f_2 : G^n \times \partial\Delta_n \to B^{n-1}A$ given by

$$(\vec{g}, \vec{t}) \mapsto \begin{cases} \big(f(\vec{g}_0), \mu_{n-1}(\vec{t}_1)\big) & t \in \sigma_0 \\ \big(f(\vec{g}_i), \mu_{n-1}(\vec{t}_i)\big) & t \in \sigma_i \text{ for } 0 < i \leq n \end{cases}.$$

(d) For X a based space and $\zeta : (\Delta_{n-1}, \partial\Delta_{n-1}) \to X$ any map, write $(\sigma_i)_*\zeta$ for the unique map $(\partial\Delta_n, \partial\Delta_n - \sigma_i) \to X$ satisfying $\zeta = ((\sigma_i)_*\zeta) \circ \sigma_i$. For $X_0 = \Delta_{n-1}/\partial\Delta_{n-1}$ and $\zeta_0 : (\Delta_{n-1}, \partial\Delta_{n-1}) \to X_0$ the quotient map, $(\sigma_i)_*\zeta_0$ can be viewed as a map from one $(n-1)$-sphere to another. An analysis of local degrees shows that the resulting elements $[(\sigma_i)_*\zeta_0] \in \pi_{n-1}(S^{n-1}) \approx H_{n-1}(S^{n-1})$ satisfy

$$[(\sigma_i)_*\zeta_0] = (-1)^{i-j}[(\sigma_j)_*\zeta_0]$$

(see [8, Proposition 2.30]).

(e) Let $(\partial\Delta_n)^{n-2}$ denote the $(n-2)$-skeleton of $\partial\Delta_n$. Let A' be an abelian k-group and $\xi : (\partial\Delta_n, (\partial\Delta_n)^{n-2}) \to A'$ a map. Set $\xi_i = \xi \circ \sigma_i$, so that

$$\xi = \prod_{i=0}^{n} (\sigma_i)_*\xi_i$$

with the product interpreted in A'. From part (d), it follows that ξ is homotopic to

$$(\sigma_0)_* \prod_{i=0}^{n} (\xi_i)^{(-1)^i}$$

with the exponents and product interpreted in A'. This construction is universal in the sense that the set-map

$$\mathrm{Map}\left((\partial\Delta_n, (\partial\Delta_n)^{(n-2)}), A'\right) \to \mathrm{Map}_*\left(\partial\Delta_n, A'\right); \xi \mapsto (\sigma_0)_* \prod_{i=0}^{n} (\xi_i)^{(-1)^i}$$

is continuous and homotopic to the inclusion $\mathrm{Map}\left((\partial\Delta_n, (\partial\Delta_n)^{(n-2)}), A'\right) \hookrightarrow \mathrm{Map}_*\left(\partial\Delta_n, A'\right)$.

(f) **Alternate description of leftmost vertical:** Part (e) yields that f_2 is homotopic to the map $f_3 : G^n \times \partial\Delta_n \to B^{n-1}A$ given by

$$(\vec{g}, \vec{t}) \mapsto \begin{cases} \left(\delta f(\vec{g}), \mu_{n-1}(\vec{t}_1)\right) & \vec{t} \in \sigma_0 \\ 1_{B^{n-1}A} & \vec{t} \notin \sigma_0 \end{cases}.$$

(g) We have homeomorphisms

$$\phi_1 : \tilde{C}\partial\Delta_n \to \Delta_n; (\vec{t}, t) \mapsto t \cdot \vec{t} + (1-t) \cdot \left(\frac{1}{n+1}, \frac{2}{n+1}, \cdots, \frac{n}{n+1}\right),$$

$$\phi_2 : \tilde{\Sigma}\partial\Delta_n \to \Delta_n \cup \tilde{C}\partial\Delta_n; (\vec{t}, t) \mapsto \begin{cases} \phi_1(\vec{t}, 2t) & t \leq \frac{1}{2} \\ (\vec{t}, 2-2t) & t \geq \frac{1}{2} \end{cases}, \text{ and}$$

$$\psi := \phi_2 \circ \phi_1^{-1} : \frac{\Delta_n}{\partial\Delta_n} \to \tilde{\Sigma}\partial\Delta_n \to \Delta_n \cup \tilde{C}\partial\Delta_n.$$

Remark. $\left(\frac{1}{n+1}, \frac{2}{n+1}, \cdots, \frac{n}{n+1}\right)$ is the image of the barycenter of Δ^n under (2.8).

Clearly, ψ is a homotopy inverse of the map $\Delta_n \cup \tilde{C}\partial\Delta_n \to \frac{\Delta_n}{\partial\Delta_n}$ which collapses $\tilde{C}\partial\Delta_n$. Let $R : \Delta_n \to I$ be 'radial' component of ϕ_1^{-1}, given by the composition

$$\Delta_n \xrightarrow{\phi_1^{-1}} \tilde{C}\partial\Delta_n \longrightarrow I.$$

Similarly, write $T(\vec{t}) \in \partial\Delta_n$ for the 'transverse' component of $\phi_1^{-1}(\vec{t})$ when $R(\vec{t}) > 0$. Hence,

$$\phi_1^{-1}(\vec{t}) = (T(\vec{t}), R(\vec{t}))$$

when $R(\vec{t}) > 0$.

(h) To make the connecting morphism δ^{n-1} in (6.6) explicit, we must produce an explicit homotopy inverse

$$h : \frac{G^n \times \Delta_n}{G^n \times \partial\Delta_n} \to (G^n \times \Delta_n) \cup \tilde{C}(G^n \times \partial\Delta_n)$$

of the map

$$(G^n \times \Delta_n) \cup \tilde{C}(G^n \times \partial\Delta_n) \to \frac{G^n \times \Delta_n}{G^n \times \partial\Delta_n} \tag{6.7}$$

which collapses $\tilde{C}(G^n \times \partial\Delta_n)$ (see (2.20)). Viewing $(G^n \times \Delta_n) \cup \tilde{C}(G^n \times \partial\Delta_n)$ as a quotient of $G^n \times (\Delta_n \cup \tilde{C}\partial\Delta_n)$, we define

$$h : (\vec{g}, \vec{t}) \mapsto (\vec{g}, \psi(\vec{t})).$$

It is not hard to see that h is a homotopy inverse of (6.7) and also a homeomorphism.

(i) **Down-right:** We wish to produce a representative for the class $\delta^{n-1}[f_3]$. This is the composition

$$f_4 : \frac{G^n \times \Delta_n}{G^n \times \partial\Delta_n} \xrightarrow{h} (G^n \times \Delta_n) \cup \tilde{C}(G^n \times \partial\Delta_n) \longrightarrow \tilde{\Sigma}(G^n \times \partial\Delta_n) \xrightarrow{\Sigma f_3} B^n A.$$

Explicitly, we have

$$\begin{aligned} f_4 : (\vec{g}, \vec{t}) \mapsto & \begin{cases} 1_{B^n A} & R(\vec{t}) \leq \frac{1}{2} \\ \left(f_3\left(\vec{g}, T(\vec{t})\right), 2 - 2R(\vec{t})\right) & R(\vec{t}) \geq \frac{1}{2} \end{cases} \\ = & \begin{cases} 1_{B^n A} & R(\vec{t}) \leq \frac{1}{2} \text{ or } T(\vec{t}) \notin \sigma_0 \\ \left(\delta f(\vec{g}), \mu_{n-1}(T(\vec{t})_1), 2 - 2R(\vec{t})\right) & R(\vec{t}) \geq \frac{1}{2} \text{ and } T(\vec{t}) \in \sigma_0 \end{cases}, \end{aligned}$$

where $T(\vec{t})_1$ is $T(\vec{t})$ with the first coordinate omitted.

(j) Consider the map

$$\rho : \Delta_n/\partial\Delta_n \to I^n/\partial I^n; \vec{t} \mapsto \begin{cases} \partial I^n/\partial I_n & R(\vec{t}) \leq \frac{1}{2} \text{ or } T(\vec{t}) \notin \sigma_0 \\ \left(\mu_{n-1}(T(\vec{t})_1), 2 - 2R(\vec{t})\right) & R(\vec{t}) \geq \frac{1}{2} \text{ and } T(\vec{t}) \in \sigma_0 \end{cases}.$$

ρ is injective on $\rho^{-1}(I^n - \partial I^n)$ and μ_n is injective on $\mu_n^{-1}(I^n - \partial I^n)$, so either $[\rho] = [\mu_n]$ or $[\rho] = -[\mu_n]$ as classes in $\pi_n(I^n/\partial I^n)$ (see [8, Proposition 2.30]). Let $\epsilon_{n-1} \in \{\pm 1\}$ so that $[\rho] = \epsilon_{n-1}[\mu_n]$.

Remark 6.2.12. To calculate ϵ_{n-1}, one must compare the signs of the determinants of the derivatives of ρ and μ_n (viewed as linear maps $\mathbb{R}^n \to \mathbb{R}^n$ in the obvious way) at a generic point $\vec{t} \in \Delta_n - \partial\Delta_n$ with $R(\vec{t}) \geq \frac{1}{2}$ and $T(\vec{t}) \in \sigma_0$. Here, a 'generic' point is one at which the derivatives of ρ and μ_n are non-singular (such a point exists by Sard's Theorem). $\square$

From part (j), it follows that $\epsilon_{n-1}[f_1] = [f_4]$ as classes in $H^n\left(\frac{G^n \times \Delta_n}{G^n \times \partial\Delta_n}, A\right)$. $\square$

Corollary 6.2.13. *α_n is trivial on n-cochains which are homotopic to a coboundary. In particular, restricting and factoring α_n yields a map $\alpha_n : H^n_{\mathrm{c}}(G,A) \to H^n(B_nG,A)$ whose kernel contains cohomology classes with null-homotopic representatives.*

Proof. The proof follows by chasing the following diagram, which commutes up to sign (by **Theorem 6.2.1**) and has an exact lower row.

$$\begin{tikzcd}
C^{n-1}_{\mathrm{c}}(G,A) \arrow[r, two heads, "\delta^{n-1}"] \arrow[d, "\alpha_{n-1}"'] & B^n_{\mathrm{c}}(G,A) \arrow[r, hook] \arrow[d, "\Sigma^n"'] & C^n_{\mathrm{c}}(G,A) \arrow[d, "\alpha_n"'] \arrow[dl, "\Sigma^n"'] \\
H^{n-1}(B_{n-1}G,A) \arrow[r, "\delta^{n-1}"] & H^n(\Sigma^n G^{\wedge n},A) \arrow[r, "q_n^*"] & H^n(B_nG,A)
\end{tikzcd}$$
□

6.3 Determining $\ker \alpha_n$

Corollary 6.2.13 says that

$$H^n_{\mathrm{c}}(G,PA,\Omega A) \longrightarrow H^n_{\mathrm{c}}(G,A) \xrightarrow{\alpha_n} H^n(BG,A) \tag{6.8}$$

is a chain complex. Phrased this way, it is natural to ask whether the above sequence is also exact — does every class in $\ker \alpha_n \subset H^n_{\mathrm{c}}(G,A)$ have a null-homotopic representative? We show that an affirmative answer is equivalent to the following conjecture with $A' = B^{n-1}A$.

Conjecture 6.3.1. *A map $B_{n-2}G \to A'$ extends to $B_{n-1}G$ if and only if it extends to B_nG.*

Theorem 6.3.2. *For fixed G, A, and $n \geq 2$, **Conjecture 6.3.1** holds with $A' = B^{n-1}A$ if and only if every n-cochain in $\ker \alpha_n$ is homotopic to a coboundary. In particular, **Conjecture 6.3.1** implies that (6.8) is exact — a cohomology class in $H^n_{\mathrm{c}}(G,A)$ lies in $\ker \alpha_n$ if and only if it has a null-homotopic representative.*

Remark 6.3.3. It is immediate that every 1-cochain in $\ker \alpha_1$ is homotopic to a coboundary since $q_1 : B_1G \to \Sigma G$ is a homeomorphism.

An alternative formulation of **Conjecture 6.3.1** is

Conjecture 6.3.4. *A map $B_{n-2}G \to A'$ extends to $B_{n-1}G$ if and only if it extends to BG.*

For fixed G and A', **Conjecture 6.3.1** holds for all $n \geq 2$ if and only if **Conjecture 6.3.4** holds for all $n \geq 2$. However, if n is also fixed, then **Conjecture 6.3.4** is stronger than **Conjecture 6.3.1**.

*Proof of **Theorem 6.3.2**.* The proof essentially follows from chasing the diagram

$$\begin{tikzcd}
 & C_{\mathrm{c}}^{n-1}(G,A) \arrow[dl, two heads, "\Sigma^{n-1}"'] \arrow[d, "\alpha_{n-1}"'] \arrow[rr, "\delta"] & & C_{\mathrm{c}}^{n}(G,A) \arrow[d, two heads, "\Sigma^{n}"'] \arrow[dr, "\alpha_n"] & \\
H^{n-1}(\Sigma^{n-1}G^{\wedge(n-1)}, A) \arrow[r, "q_{n-1}^*"] & H^{n-1}(B_{n-1}G, A) \arrow[rr, "\delta^{n-1}"] & & H^n(\Sigma^n G^{\wedge n}, A) \arrow[r, "q_n^*"] & H^n(B_nG, A)
\end{tikzcd},$$

which commutes up to sign by **Theorem 6.2.1**. Here, we write δ for the coboundary operator and δ^{n-1} for the connecting morphism to disambiguate notation. Since $\ker \Sigma^n$ consists of precisely the null-homotopic n-cochains (see (5.1)), we wish to show that

$$\ker \alpha_n = \ker \Sigma^n + \operatorname{im} \delta \tag{6.9}$$

if and only if **Conjecture 6.3.1** holds. By commutativity, (6.9) is equivalent to

$$\ker q_n^* = \operatorname{im} \delta \circ \Sigma^n. \tag{6.10}$$

The bottom row is exact at $H^n(\Sigma^n G^{\wedge n}, A)$, so (6.10) is equivalent to

$$\operatorname{im} \delta^{n-1} = \operatorname{im} \delta \circ \Sigma^n. \tag{6.11}$$

Once again alluding to commutativity, together with the surjectivity of Σ^{n-1}, (6.11) is equivalent to

$$\operatorname{im} \delta^{n-1} = \operatorname{im} \delta^{n-1} \circ q_{n-1}^*,$$

which is further equivalent to

$$\ker \delta^{n-1} + \operatorname{im} q_{n-1}^* = H^{n-1}(B_{n-1}G, A). \tag{6.12}$$

Now, let $\bar{\iota}_j : B_jG \hookrightarrow B_{j+1}G$ be the inclusion. Using the long exact sequences of cohomology for the pairs $(B_{n-2}G, B_{n-1}G)$ and $(B_{n-1}G, B_nG)$, (6.12) can be seen to be equivalent to

$$\operatorname{im} \bar{\iota}_{n-1}^* + \ker \bar{\iota}_{n-2}^* = H^{n-1}(B_{n-1}G, A).$$

Upon applying $\bar{\iota}_{n-2}^*$ to both sides, the theorem follows. ☐

Corollary 6.3.5. *A cohomology class lies in* $\ker(\iota_2^* \circ \alpha) \subset H_{\mathrm{c}}^2(G, A)$ *if and only if it has a null-homotopic representative.*

Proof. **Conjecture 6.3.1** is vacuously true for $n = 2$, so the corollary follows from **Theorem 6.3.2** and the fact that $\iota_2^* \circ \alpha = \alpha_2$ on 2-cocycles. $\square$

6.4 Some simple cases of the conjecture

By **Theorem 4.3.2**, proving **Theorem 4.3.2** for all abelian k-groups A' reduces to proving it in the discrete case. To be precise, for given G and $n \geq 2$, proving the following for all discrete abelian groups A and all $d \geq 0$ would prove **Conjecture 6.3.1**.

Conjecture 6.4.1. *For A a discrete abelian group, the restriction maps $H^d(B_{n-1}G, A) \to H^d(B_{n-2}G, A)$ and $H^d(B_nG, A) \to H^d(B_{n-2}G, A)$ have the same image.*

Here are some cases where the above is immediate.

Theorem 6.4.2. ***Conjecture 6.4.1*** *holds when*

1. *G is discrete,*
2. *$G = S^1$ or $G = S^3$, or*
3. *$d \leq n - 2$.*

In particular, ***Conjecture 6.3.1*** *and* ***Conjecture 6.3.4*** *hold when G is discrete, $G = S^1$, or $G = S^3$.*

Proof. The case of G discrete is immediate upon considering the cases $d \leq n-3$, $d = n-2$, and $d \geq n-1$ separately.

If $G = S^1$, then $B_nG \cong \mathbb{C}P^n$ has a cell in each even dimension $\leq 2n$. Likewise, if $G = S^3$, then $B_nG \cong \mathbb{H}P^n$ has a cell in each dimension $\leq 4n$ that is a multiple of 4. Hence, the restriction map $H^d(B_mG, A) \to H^d(B_{m-1}G, A)$ is either 0 or an isomorphism for each $m \geq 1$.

If $d \leq n-2$, then both the maps in **Conjecture 6.4.1** are isomorphisms by **Corollary 9.1.4**. $\square$

What **Conjecture 6.3.4** is to **Conjecture 6.3.1**, the following is to **Conjecture 6.4.1**.

Conjecture 6.4.3. *For A a discrete abelian group, the restriction maps $H^d(B_{n-1}G, A) \to H^d(B_{n-2}G, A)$ and $H^d(BG, A) \to H^d(B_{n-2}G, A)$ have the same image.*

To be precise, for fixed G, A, and $d \geq 0$, **Conjecture 6.4.1** holds for all $n \geq 2$ if and only if **Conjecture 6.4.3** holds for all $n \geq 2$. However, if n is also fixed, then **Conjecture 6.4.3** is stronger than **Conjecture 6.4.1**.

6.5 Some examples

Example 6.5.1. Suppose $G = S^1$ or $G = S^3$, i.e., the underlying space of G is a sphere. Hence, $G^{\wedge(n-1)}$ and $G^{\wedge n}$ are spheres of different dimensions. In particular, for any $d \geq 0$ and discrete abelian group A', at least one of $H^d(G^{\wedge(n-1)}, A')$ and $H^d(G^{\wedge n}, A')$ must be trivial. Consequently, **Theorem 4.3.2** yields that all n-coboundaries are null-homotopic. By **Theorem 6.3.2** and **Theorem 6.4.2**, $\ker \alpha_n \subset C^n_{\mathrm{c}}(G, A)$ consists of precisely those n-cochains which are null-homotopic. In other words,

$$\ker \alpha_n = C^n_{\mathrm{c}}(G, PA, \Omega A). \quad \square$$

Example 6.5.2. Let us examine the map induced by a non-normalized 2-cocycle $f : G\times G \to A$ on homotopy groups. Fix $n \geq 1$ and let $\phi_1, \phi_2 : \pi_n(G) \to \pi_n(A)$ be the maps induced by

$$G \hookrightarrow G \times G \xrightarrow{f} A \ ,$$

where $G \hookrightarrow G \times G$ varies over the two axial inclusions. The cocycle condition

$$f(x, y) + f(xy, z) = f(y, z) + f(x, yz)$$

yields the following for all $a, b, c \in \pi_n(G)$.

$$\phi_1(a) + \phi_2(b) + \phi_1(a + b) + \phi_2(c) = \phi_1(b) + \phi_2(c) + \phi_1(a) + \phi_2(b + c).$$

Setting $b = c = 0$ yields $\phi_1(a) = 0$ for all a, and setting $a = b = 0$ yields $\phi_2(c) = 0$ for all c. Hence, $\pi_n(f) = 0$ — f is trivial on all homotopy groups.

When A has a weakly contractible universal covering and G is connected, this implies that f must be null-homotopic. By **Corollary 6.2.7**, all normalized 2-cocycles must also be null-homotopic in this case. Hence, $\iota_2 \circ \alpha$ is trivial on continuous cohomology in this case (by **Corollary 6.3.5**). $\square$

Example 6.5.3. Suppose $G = BA_1$ and $A = B^2A_2$ for discrete abelian groups A_1 and A_2, with A_2 written additively. We will find the image of the map $B^2_{\mathrm{c}}(G, A) \to [G \wedge G, A]_*$. This will be used in the next example to produce an element of $H^2_{\mathrm{c}}(G, A)$ which is not in $\ker \alpha$ (for certain choices of A_1 and A_2).

First, we examine the homotopy classes of maps $G \to A$. We have

$$\begin{aligned}[G, A]_* &= H^2(BA_1, A_2) \\ &\approx H^2_{\mathrm{gp}}(A_1, A_2),\end{aligned}$$

with an isomorphism given by $\alpha_{A_1,A_2} : H^2_{\mathrm{gp}}(A_1, A_2) \to H^2(BA_1, A_2)$ (see **Corollary 5.3.2**). Hence, every 1-cochain $f : G \to A$ is homotopic to a map $f' : G \to A$ whose restriction to the 2-skeleton B_2A_1 of G is

$$B_2A_1 \to A; (a_1, t_1)(a_2, t_2) \mapsto (h(a_1, a_2), \mu_2(t_1, t_2)), \tag{6.13}$$

for some 2-cocycle $h \in Z^2_{\mathrm{gp}}(A_1, A_2)$. We wish to understand the homotopy class of $\delta f \approx \delta f'$, for which it suffices to examine $(\delta f')|_{B_1A_1 \wedge B_1A_1}$ (since $B_1A_1 \wedge B_1A_1$ is the 2-skeleton of $G \wedge G$).

$B_1A_1 = \Sigma A_1$ is a wedge of circles (one circle for each non-trivial element of A_1), so $B_1A_1 \wedge B_1A_1$ is a wedge of 2-spheres (one 2-sphere for each ordered pair of non-trivial elements of A_1). The characteristic map for the 2-cell corresponding to $(a_1, a_2) \in A_1 \times A_1$ is

$$e^2_{a,b} : I^2 \to BA_1 \wedge BA_1; (s_1, s_2) \mapsto ((a_1, s_1), (a_2, s_2)).$$

Hence, it suffices to examine the map

$$(\delta f') \circ e^2_{a,b} : I^2/\partial I^2 \to A$$

for non-trivial $a, b \in A_1$. By (6.13), we have

$$\begin{aligned}(\delta f') \circ e^2_{a,b}(s_1, s_2) &= \delta f'((a_1, s_1), (a_2, s_2)) \\ &= f'(a_1, s_1) + f'(a_2, s_2) - f'((a_1, s_1)(a_2, s_2)) \\ &= \begin{cases} (-h(a_1, a_2), \mu_2(s_1, s_2)) & s_1 \leq s_2 \\ (-h(a_2, a_1), \mu_2(s_2, s_1)) & s_1 \geq s_2 \end{cases}.\end{aligned} \tag{6.14}$$

It is not hard to see that the homotopy classes of the two maps $I^2/\partial I^2 \to I^2/\partial I^2$ given by

$$(s_1, s_2) \mapsto \begin{cases} \mu_2(s_1, s_2) & s_1 \leq s_2 \\ \partial I^2/\partial I^2 & s_1 \geq s_2 \end{cases} \text{ and} \tag{6.15}$$

$$(s_1, s_2) \mapsto \begin{cases} \partial I^2/\partial I^2 & s_1 \leq s_2 \\ \mu_2(s_2, s_1) & s_1 \geq s_2 \end{cases}$$

are negatives of each other (in $\pi_2(I^2/\partial I^2)$). Also, (6.15) is homotopic to $\mathrm{id}_{I^2/\partial I^2}$. Hence, (6.14) yields that $\delta f' \circ e^2_{a,b}$ is homotopic to the map

$$(s_1, s_2) \mapsto (h(a_2, a_1) - h(a_1, a_2), s_1, s_2).$$

Putting this together over all 2-cells of G shows that the homotopy class of $(\delta f')|_{B_1A_1 \wedge B_1A_1}$ is the image of the 2-cocycle[1]

$$A_1 \wedge A_1 \to A_2; (a_1, a_2) \mapsto h(a_2, a_1) - h(a_1, a_2) \tag{6.16}$$

under the map

$$W : Z^2_{\mathrm{gp}}(A_1, A_2) \to [B_1A_1 \wedge B_1A_1, A]_*; h' \mapsto [((a_1, s_1), (a_2, s_2)) \mapsto (h'(a_1, a_2), s_1, s_2)].$$

Here is a succinct reformulation of the above. Write ρ for the involution $Z^2_{\mathrm{gp}}(A_1, A_2) \to Z^2_{\mathrm{gp}}(A_1, A_2)$ induced by interchanging the two coordinates of $A_1 \wedge A_1$. The following commutes.

$$\begin{tikzcd} Z^2_{\mathrm{gp}}(A_1, A_2) \arrow[r, "\rho-\mathrm{id}"] \arrow[d, two heads] & Z^2_{\mathrm{gp}}(A_1, A_2) \arrow[r, hook, "W"] & {[B_1A_1 \wedge B_1A_1, A]_*} \\ H^2_{\mathrm{gp}}(A_1, A_2) \arrow[r, "\alpha_{A_1,A_2}", "\approx"'] & {[G, A]_*} \arrow[r, "\delta^2"] & {[G \wedge G, A]_*} \arrow[u, hook] \end{tikzcd} \tag{6.17}$$

Here, δ^2 denotes the coboundary operator acting on homotopy classes as $[f] \mapsto [\delta f]$, the vertical inclusion restricts to the 2-skeleton, and W is injective because $h' \mapsto \pi_2(W(h'))$ is. $\square$

Example 6.5.4. Let $A' = \mathbb{Z}/n\mathbb{Z}$ ($n \geq 2$) or $A' = \mathbb{Z}$. Set $G = BA'$ and $A = B^2A'$. We will produce a 2-cocycle $f : G \wedge G \to A$ which is not homotopic to a coboundary, so that $f \notin \ker \alpha$

[1] h being a cocycle implies that $(a_1, a_2) \mapsto h(a_2, a_1)$ is a cocycle since A_1 is abelian. Hence, (6.16) is also a cocycle.

by **Corollary 6.3.5**. The construction builds on the 2-cocycle $A' \wedge A' \to A'; (a, b) \mapsto ab$ (where ab is the product in A' as a ring).

We define f as follows, recalling that every element of BA' can be written as $\prod_{i=1}^{\ell}(a_i, s_i)$ for some $\ell \geq 0$ and $(a_i, s_i) \in A' \times I$ (we do not require that $s_1 \leq \ldots \leq s_\ell$ since BA' is abelian).

$$f : \left(\prod_{i=1}^{\ell}(a_i, s_i), \prod_{j=1}^{m}(b_j, t_j)\right) \mapsto \prod_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}} (a_i b_j, s_i, t_j).$$

To see that f is well-defined and continuous, observe that the following hold for all $a, b \in A'$ and $s, t \in I$.

$$\begin{aligned}
f(x(0, s), y) &= f(x, y), \\
f(x, y(0, s)) &= f(x, y), \\
f(x(a, 0), y) &= f(x(a, 1), y) = f(x, y), \\
f(x, y(a, 0)) &= f(x, y(a, 1)) = f(x, y), \\
f(x(a, s)(b, s), y) &= f(x(a + b, s), y), \\
f(x, y(a, s)(b, s)) &= f(x, y(a + b, s)), \text{ and} \\
f(x, 1_G) &= f(1_G, y) = 1_A.
\end{aligned}$$

A straightforward computation shows that f is also a cocycle. To show that f is not homotopic to a coboundary, first observe that all central extensions of A' by itself are abelian. Hence, the involution $\rho : Z^2_{\mathrm{gp}}(A', A') \to Z^2_{\mathrm{gp}}(A', A')$ from **Example 6.5.3** is the identity map and all coboundaries in $B^2_{\mathrm{c}}(G, A)$ are null-homotopic (by the commutativity of (6.17)). Hence, f is not homotopic to a coboundary if and only if it is not null-homotopic.

We will now show that $\pi_2(f)$ is non-trivial. Indeed, consider the following representative of a class in $\pi_2(G \wedge G)$.

$$\lambda : I^2 \to G \wedge G; (s, t) \mapsto ((1, s), (1, t)).$$

$f \circ \lambda$ represents the image of $1 \in A'$ under the isomorphism $A' \xrightarrow{\sim} \pi_2(B^2 A')$ (see (2.5)), so the claim follows. $\square$

Example 6.5.5. Let A' be a discrete abelian group (written additively) such that there exist $a_0, b_0 \in A'$ with $a_0 \otimes b_0 \neq -b_0 \otimes a_0$ in $A' \otimes A'$ (for instance, A' being finitely generated with order at least 3 works). Set $G = BA'$ and $A = B^2(A' \otimes A')$. We will produce a 2-cocycle $f : G \wedge G \to A$ which is not homotopic to a coboundary, so that $f \notin \ker \alpha$ by **Corollary 6.3.5**. The construction builds on the 2-cocycle $A' \wedge A' \to A' \otimes A'; (a, b) \mapsto a \otimes b$.

Analogous to the preceding example, we define

$$f : \left(\prod_{i=1}^{\ell} (a_i, s_i), \prod_{j=1}^{m} (b_j, t_j) \right) \mapsto \prod_{\substack{1 \le i \le \ell \\ 1 \le j \le m}} (a_i \otimes b_j, s_i, t_j).$$

To see that f is well-defined and continuous, observe that the following hold for all $a, b \in A'$ and $s, t \in I$.

$$\begin{aligned} f(x(0_{A'}, s), y) &= f(x, y), \\ f(x, y(0_{A'}, s)) &= f(x, y), \\ f(x(a, 0), y) &= f(x(a, 1), y) = f(x, y), \\ f(x, y(a, 0)) &= f(x, y(a, 1)) = f(x, y), \\ f(x(a, s)(b, s), y) &= f(x(ab, s), y), \\ f(x, y(a, s)(b, s)) &= f(x, y(ab, s)), \text{ and} \\ f(x, 1_G) &= f(1_G, y) = 1_A. \end{aligned}$$

A straightforward computation shows that f is also a cocycle.[2] To show that f is not homotopic to a coboundary, first observe that 2-coboundaries $G \wedge G \to A$ are fixed under composition with the involution $r : G \wedge G \to G \wedge G; (g_1, g_2) \mapsto (g_2, g_1)$ (since G is abelian). Hence, f is not homotopic to a 2-coboundary if $f \not\approx f \circ r$.

We will now show that $\pi_2(f) \neq \pi_2(f \circ r)$. Indeed, consider the following representative of a class in $\pi_2(G \wedge G)$.

$$\lambda : I^2 \to G \wedge G; (s, t) \mapsto ((a_0, s), (b_0, t)).$$

$f \circ \lambda$ represents the image of $a_0 \otimes b_0$ under the isomorphism $A' \otimes A' \xrightarrow{\sim} \pi_2 \left(B^2(A' \otimes A') \right)$ (see (2.5)). Likewise, $f \circ r \circ \lambda$ represents the image of $-b_0 \otimes a_0$ under said isomorphism. Hence, the claim follows from the fact that $a_0 \otimes b_0 \neq -b_0 \otimes a_0$. $\square$

Example 6.5.6. The general form taken by G and A in the preceding example, namely

[2]Actually, $BA' \otimes BA'$ and $B^2(A' \otimes A')$ are isomorphic as abstract groups — $x \otimes y \mapsto f(x, y)$ is an isomorphism. Hence, f is is also of the form $(a, b) \mapsto a \otimes b$.

$G = BA'$ and $A = B^2(A' \otimes A')$, can be generalized. The techniques used in that example also work for $G = B^d A'$ and $A = B^{2d}(A' \otimes A')$ with d odd. □

Example 6.5.7. Let $G = \mathbb{Z}/m\mathbb{Z}$ ($m \geq 2$) and $A = S^1$. Let $f : G \to A; x \mapsto e^{\frac{2\pi i x}{m}}$ be the canonical inclusion. Clearly, f is a null-homotopic group homomorphism and hence lies in the kernel of $\alpha_1 = \iota_1^* \circ B : H^1_{\mathrm{c}}(G, A) \to H^1(B_1 G, A)$. However, $[Bf] \in H^1(BG, A)$ is not trivial — see **Example 7.5.2**. □

Chapter 7

Analogues of α in higher degrees

When G and A are discrete, it is not hard to see, using the techniques of Section 5.3, that the composition

$$H^n_{\mathrm{gp}}(G, A) \xrightarrow{\approx} H^n(\bar{B}G, A) \xrightarrow{(\bar{\Psi}^*)^{-1}} H^n(BG, A) \hookrightarrow H^n(B_nG, A)$$

is the same as α_n. In particular, for $f \in Z^n_{\mathrm{c}}(G, A)$ a cocycle, the cohomology class $\alpha_n f \in H^n(B_nG, A)$ extends to BG. In this chapter, we will show that this holds in general, without the assumption that G and A are discrete.

When G and A are discrete, the restriction map $H^n(BG, A) \to H^n(B_{n+1}G, A)$ is an isomorphism. Hence, in this case, $\alpha_n f$ (as above) extends to BG if and only if it extends to $B_{n+1}G$. This line of reasoning fails for general G and A, but nonetheless it is instructive to first try extending $\alpha_n f$ to $B_{n+1}G$ in the general set up. We begin with the *ansatz* that the desired extension of $\alpha_n f$ to $B_{n+1}G$ takes the form

$$F : (g_1, t_1) \cdots (g_{n+1}, t_{n+1}) \mapsto \prod_{i=1}^{n+1} \big(x_i(\vec{g}), \mu_n(\vec{t_i})\big)$$

for some continuous maps $x_i : G^{\wedge(n+1)} \to A$. Here, we have borrowed notation from Chapter 6 — $\vec{g} = (g_1, \cdots, g_{n+1}) \in G^{n+1}$ and $\vec{t} = (t_1, \cdots, t_{n+1}) \in \Delta_{n+1}$ are general points. Since F

extends $\alpha_n f$, we must have

$$F(\vec{g},\vec{t}) = \begin{cases} \alpha_n f\left(\vec{g_0},\vec{t_1}\right) & 0 = t_1 \\ \alpha_n f\left(\vec{g_i},\vec{t_i}\right) & t_i = t_{i+1} \text{ for } 0 < i < n+1 \,, \\ \alpha_n f\left(\vec{g}_{n+1},\vec{t}_{n+1}\right) & t_{n+1} = 1 \end{cases} \tag{7.1}$$

where points in B_nG and $B_{n+1}G$ are viewed as points in $G^n \times \Delta_n$ and $G^{n+1} \times \Delta_{n+1}$ respectively for notational simplicity. This imposes the following equations on the x_i's.

$$\begin{aligned} x_1\left(\vec{g}\right) &= f\left(\vec{g_0}\right), \\ x_i\left(\vec{g}\right) x_{i+1}\left(\vec{g}\right) &= f\left(\vec{g_i}\right) \text{ for } 0 < i < n+1, \text{ and} \\ x_{n+1}\left(\vec{g}\right) &= f\left(\vec{g}_{n+1}\right). \end{aligned} \tag{7.2}$$

This is essentially a system of $n+1$ linear equations in n unknowns, so it is over-determined. However, it has a unique solution given by

$$x_i\left(\vec{g}\right) := \prod_{j=0}^{i-1} f\left(\vec{g_j}\right)^{(-1)^{i+j+1}} \text{ for } 0 \leq i \leq n+1.$$

The fact that this solution works is equivalent to the cocycle condition on f:

$$x_{n+1}\left(\vec{g}\right) = f\left(\vec{g}_{n+1}\right) = \prod_{j=0}^{n} f\left(\vec{g_j}\right)^{(-1)^{n+j}}.$$

One checks that the resulting map $F : B_{n+1}G \to B^nA$ is well-defined. In particular, if $g_i = 1_G$ for some $i \in [n+1]$, then

$$F\left(\vec{g},\vec{t}\right) = \alpha_n f\left(\vec{g_i},\vec{t_i}\right).$$

One could similarly use the ansatz

$$(g_1,t_1)\cdots(g_{n+1},t_{n+2}) \mapsto \prod_{1\leq i<j\leq n+2} \left(x_{i,j}(\vec{g}), \mu_n(\vec{t}_{i,j})\right) \tag{7.3}$$

for an extension of F to $B_{n+2}G$, where $\vec{t}_{i,j} := (t_1, \cdots, \hat{t_i}, \cdots, \hat{t_j}, \cdots, t_{n+2})$. The analogue of (7.1) then yields a system of linear equations for the $x_{i,j}$'s in terms of the x_i's (analogous to (7.2)), although it is once again over-determined. Hence, a non-trivial check needs to be done to ensure that a solution exists and that the resulting map (7.3) is well-defined. This process gets harder when one tries to extend to $B_{n+3}G, B_{n+4}G, \cdots$. This calls for a systematization of the above ansatz, which we now do.

7.1 Formalizing the ansatz

Throughout this section, A is written additively. Let S denote an arbitrary finite (possibly empty) subset of $\mathbb{N}_0$, the non-negative integers. If $i \in S$, write $\mathrm{dsc}_i S$ for the *i-th descent* of S, defined as

$$\mathrm{dsc}_i S := \{j \in S \mid j < i\} \cup \{j - 1 \mid j \in S, j > i\}.$$

If $i \in S$ and $i - 1 \notin S$, write $\mathrm{rep}_i S$ for the *i-th replacement* of S, defined as

$$\mathrm{rep}_i S := S - \{i\} \cup \{i - 1\}.$$

If $|S| \geq 2$, write $D(S)$ for the difference between the largest and second-largest elements of S. If S is singleton, then $D(S)$ will denote the sole element of S. For S non-empty, let $M(S)$ and $m(S)$ be the maximum and minimum elements of S respectively. Let $S' = S - \{M(S)\} = \mathrm{dsc}_{M(S)} S$.

Let $\mathcal{G}$ be the subspace of $G^{\mathbb{N}_0}$ consisting of tuples with all but finitely many coordinates trivial. For $i \geq 0$, define

$$d_i : \mathcal{G} \to \mathcal{G}; (g_0, g_1, \cdots) \mapsto (g_1, \cdots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \cdots).$$

Note that

$$d_i \circ d_j = d_{j-1} \circ d_i \text{ for } 0 \leq i < j. \tag{7.4}$$

To each finite $S \subset \mathbb{N}_0$, we associate an 'unknown' continuous map $x_S : \mathcal{G} \to A$ on which the following equations are imposed for $i \geq 1$.

$$x_S = x_{\mathrm{dsc}_i S} \circ d_{i-1} \text{ if } i, i - 1 \in S, \text{ and} \tag{7.5}$$

$$x_S + x_{\mathrm{rep}_i S} = x_{\mathrm{dsc}_i S} \circ d_{i-1} \text{ if } i \in S \text{ and } i - 1 \notin S. \tag{7.6}$$

Remark 7.1.1. For $m \geq 1$, these equations come from the ansatz

$$(g_1, t_1) \cdots (g_{n+m}, t_{n+m}) \mapsto \prod_{\substack{S \subset [n+m] \\ |S|=m}} \big(x_S(\vec{g}), \mu_n(\vec{t}_S)\big)$$

for an extension of $\alpha_n f$ to $B_{n+m}G$, and are analogous to Equation (7.2). Here, $\vec{t}_S$ denotes $\vec{t}$ with coordinates indexed by elements of S dropped. In analogy with (7.1), (7.5) and (7.6) come from the case of $t_{i-1} = t_i$ (which is interpreted as $0 = t_1$ for $i = 1$ and $t_{n+m} = 1$ for $i = n + m + 1$). This remark will be formalized in Section 7.2, but for now the reader shall regard it only as motivation for the algebra that follows. □

Proposition 7.1.2. *If $x_\emptyset$ is fixed and $x_{\{0\}} \equiv 0$, then for each finite $S \subset \mathbb{N}_0$ there exists unique x_S so that (7.5) and (7.6) are satisfied.*

Proof. We will first define x_S by inducting on $|S|$, and then show that (7.5) and (7.6) are satisfied. Uniqueness will be clear from the fact that our definition of the x_S is forced on us by special cases of (7.5) and (7.6) (see **Case 1** below). For S non-empty, we define x_S inductively as

$$x_S := \sum_{j=1}^{D(S)} (-1)^{j-1} x_{S'} \circ d_{M(S)-j}. \tag{7.7}$$

We now show that this definition satisfies (7.5) and (7.6) by inducting on $|S|$, with the base case of $|S| = 1$ being easy to verify (here one uses that $x_{\{0\}} = 0$). Hence, suppose $|S| > 1$ and $i \in S$ is positive. We will show that x_S satisfies (7.5) if $i - 1 \in S$ and (7.6) otherwise.

- **Case 1:** $i = M(S)$.
  In this case, the claim is immediate from the definition of x_S.

- **Case 2:** $i < M(S)$ and $i - 1 \in S$.
  First, observe that

$$\begin{aligned} (\mathrm{dsc}_i S)' &= \mathrm{dsc}_i S', \\ D(\mathrm{dsc}_i S) &= D(S), \text{ and} \\ M(\mathrm{dsc}_i S) &= M(S) - 1. \end{aligned} \tag{7.8}$$

Next, (7.7) yields

$$\begin{aligned}
x_S &= \sum_{j=1}^{D(S)} (-1)^{j-1} x_{S'} \circ d_{M(S)-j} \\
&= \sum_{j=1}^{D(S)} (-1)^{j-1} x_{\mathrm{dsc}_i S'} \circ d_{i-1} \circ d_{M(S)-j} \text{ (using (7.5) for } S' \text{ and } i) \\
&= \sum_{j=1}^{D(S)} (-1)^{j-1} x_{\mathrm{dsc}_i S'} \circ d_{M(S)-j-1} \circ d_{i-1} \text{ (by (7.4))} \\
&= \sum_{j=1}^{D(\mathrm{dsc}_i S)} (-1)^{j-1} x_{(\mathrm{dsc}_i S)'} \circ d_{M(\mathrm{dsc}_i S)-j} \circ d_{i-1} \text{ (by (7.8))} \\
&= x_{\mathrm{dsc}_i S} \circ d_{i-1} \text{ (by (7.7) for } \mathrm{dsc}_i S \text{, which is non-empty)},
\end{aligned}$$

as desired.

- **Case 3:** $i < M(S) - D(S)$ (i.e., i is neither the largest nor the second-largest element of S) and $i - 1 \notin S$.
  First, observe that

$$\begin{aligned}
(\mathrm{rep}_i S)' &= \mathrm{rep}_i S', \\
D(\mathrm{rep}_i S) &= D(S), \\
(\mathrm{dsc}_i S)' &= \mathrm{dsc}_i S', \\
M(\mathrm{dsc}_i S) &= M(S) - 1, \text{ and} \\
D(\mathrm{dsc}_i S) &= D(S).
\end{aligned} \tag{7.9}$$

Hence, (7.7) yields

$$\begin{aligned}
x_S + x_{\mathrm{rep}_i S} &= \sum_{j=1}^{D(S)} (-1)^{j-1} \left(x_{S'} + x_{\mathrm{rep}_i S'} \right) \circ d_{M(S)-j} \\
&= \sum_{j=1}^{D(S)} (-1)^{j-1} x_{\mathrm{dsc}_i S'} \circ d_{i-1} \circ d_{M(S)-j} \text{ (using (7.6) for } S' \text{ and } i) \\
&= \sum_{j=1}^{D(S)} (-1)^{j-1} x_{\mathrm{dsc}_i S'} \circ d_{M(S)-j-1} \circ d_{i-1} \text{ (using (7.4))} \\
&= \sum_{j=1}^{D(\mathrm{dsc}_i S)} (-1)^{j-1} x_{(\mathrm{dsc}_i S)'} \circ d_{M(\mathrm{dsc}_i S)-j} \circ d_{i-1} \text{ (using (7.9))} \\
&= x_{\mathrm{dsc}_i S} \circ d_{i-1} \text{ (by (7.7) for } \mathrm{dsc}_i S \text{, which is non-empty),}
\end{aligned}$$

as desired.

- **Case 4:** $i = M(S) - D(S)$ (i.e., i is the second-largest element of S) and $i - 1 \notin S$. First, observe that

$$\begin{aligned}
(\mathrm{rep}_i S)' &= \mathrm{rep}_i S', \\
D(\mathrm{rep}_i S) &= D(S) + 1, \\
(\mathrm{dsc}_i S)' &= \mathrm{dsc}_i S', \\
M(\mathrm{dsc}_i S) &= M(S) - 1, \\
D(\mathrm{dsc}_i S) &= D(S) + D(S') - 1 \geq D(S), \text{ and} \\
M(\mathrm{rep}_i S) - D(\mathrm{rep}_i S) &= i - 1.
\end{aligned} \tag{7.10}$$

Hence, (7.7) yields

$$\begin{aligned}
x_S + x_{\mathrm{rep}_i S} &= \sum_{j=1}^{D(S)} (-1)^{j-1} x_{S'} \circ d_{M(S)-j} + \sum_{j=1}^{D(\mathrm{rep}_i S)} (-1)^{j-1} x_{\mathrm{rep}_i S'} \circ d_{M(\mathrm{rep}_i S)-j} \\
&= \sum_{j=1}^{D(S)} (-1)^{j-1} \left(x_{S'} + x_{\mathrm{rep}_i S'} \right) \circ d_{M(S)-j} + (-1)^{D(S)} x_{\mathrm{rep}_i S'} \circ d_{i-1} \text{ (by (7.10)).}
\end{aligned} \tag{7.11}$$

We temporarily denote the summation in the above expression by T. We have

$$\begin{aligned}
T &= \sum_{j=1}^{D(S)} (-1)^{j-1} x_{\mathrm{dsc}_i S'} \circ d_{i-1} \circ d_{M(S)-j} \text{ (using (7.6) for } S' \text{ and } i) \\
&= \sum_{j=1}^{D(S)} (-1)^{j-1} x_{\mathrm{dsc}_i S'} \circ d_{M(S)-j-1} \circ d_{i-1} \text{ (using (7.4))} \\
&= \sum_{j=1}^{D(\mathrm{dsc}_i S)} (-1)^{j-1} x_{(\mathrm{dsc}_i S)'} \circ d_{M(\mathrm{dsc}_i S)-j} \circ d_{i-1} \\
&\quad - \sum_{j=D(S)+1}^{D(\mathrm{dsc}_i S)} (-1)^{j-1} x_{\mathrm{dsc}_i S'} \circ d_{M(\mathrm{dsc}_i S)-j} \circ d_{i-1} \text{ (by (7.10))}.
\end{aligned}$$

Using (7.7) for $\mathrm{dsc}_i S$ (which is non-empty) on the first summation and making a change of variable in the second summation, we obtain

$$\begin{aligned}
T &= x_{\mathrm{dsc}_i S} \circ d_{i-1} - (-1)^{D(S)} \sum_{j=1}^{D(\mathrm{dsc}_i S)-D(S)} (-1)^{j-1} x_{\mathrm{dsc}_i S'} \circ d_{M(\mathrm{dsc}_i S)-D(S)-j} \circ d_{i-1} \\
&= x_{\mathrm{dsc}_i S} \circ d_{i-1} - (-1)^{D(S)} \sum_{j=1}^{D(S')-1} (-1)^{j-1} x_{\mathrm{dsc}_i S'} \circ d_{M(\mathrm{dsc}_i S)-D(S)-j} \circ d_{i-1} \text{ (by (7.10))}.
\end{aligned} \tag{7.12}$$

We have

$$\begin{aligned}
M(\mathrm{dsc}_i S) - D(S) &= M(S) - 1 - D(S) \text{ (by (7.10))} \\
&= i - 1 \text{ (by definition of } D(S) \text{ and choice of } i) \\
&= M(\mathrm{rep}_i S'), \\
\mathrm{dsc}_i S' &= (\mathrm{rep}_i S')', \text{ and} \\
D(S') - 1 &= D(\mathrm{rep}_i S'),
\end{aligned}$$

so (7.12) yields

$$\begin{aligned}
T &= x_{\mathrm{dsc}_i S} \circ d_{i-1} - (-1)^{D(S)} \sum_{j=1}^{D(\mathrm{rep}_i S')} (-1)^{j-1} x_{(\mathrm{rep}_i S')'} \circ d_{M(\mathrm{rep}_i S')-j} \circ d_{i-1} \\
&= x_{\mathrm{dsc}_i S} \circ d_{i-1} - (-1)^{D(S)} x_{\mathrm{rep}_i S'} \circ d_{i-1} \text{ (using (7.7) on } \mathrm{rep}_i S' \text{, which is non-empty)}.
\end{aligned}$$

Plugging the above into (7.11) yields the desired result. $\square$

Corollary 7.1.3. *If $x_\emptyset$ is fixed and $x_{\{0\}} \equiv 0$, then $x_S \equiv 0$ whenever $0 \in S$.*

Proof. The claim is vacuous for $|S| = 1$. Hence, the corollary follows from (7.7) using induction. $\square$

Now fix $n \geq 1$ and a cochain $f \in C^n_{\mathrm{c}}(G, A)$. Write $x_S f$ for the map x_S given by (7.7) when $x_{\{0\}} \equiv 0$ and $x_\emptyset$ is chosen as

$$x_\emptyset : \mathcal{G} \to A; (g_0, g_1, \cdots) \mapsto f(g_1, \cdots, g_n).$$

By induction on $|S|$ (with base case $|S| = 0$), it is easy to see that $x_S f$ is only a function of the coordinates indexed by $[n + |S|]$, i.e.,

$$x_S f(g_0, g_1, \cdots) = x_S f(1_G, g_1, g_2, \cdots, g_{n+|S|}, 1_G, 1_G, \cdots).$$

Hence, we will view $x_S f$ as a map $G^{n+|S|} \to A$ by embedding $G^{n+|S|}$ in $\mathcal{G}$ as

$$(g_1, \cdots, g_{n+|S|}) \mapsto (1_G, g_1, \cdots, g_{n+|S|}, 1_G, 1_G, \cdots).$$

Note that the set-map $C^n_{\mathrm{c}}(G, A) \to \mathrm{Map}(G^{n+|S|}, A); f \mapsto x_S f$ is a group homomorphism (the topology of $\mathrm{Map}(G^{n+|S|}, A)$ is irrelevant here).

Lemma 7.1.4. *If $M(S) = n + |S| + 1 \in S$, then*

$$x_S f = (-1)^{n+1} x_{S'}(\delta f).$$

Proof. The proof proceeds by induction on $|S|$.

- **Base case:** $|S| = 1$.
  We have $S = \{n + 2\}$, so (7.7) yields

$$\begin{aligned} x_S f &= \sum_{j=1}^{n+2} (-1)^{j-1} x_\emptyset f \circ d_{n+2-j} \\ &= (-1)^{n+1} \delta f \\ &= (-1)^{n+1} x_\emptyset(\delta f). \end{aligned}$$

- **Induction step:** $|S| > 1$.

  We use induction on $m(S)$, with the base case of $m(S) = 0$ following from **Corollary 7.1.3**. For $m(S) > 1$, we have

$$\begin{aligned} x_S f &= x_{\mathrm{dsc}_{m(S)} S} f \circ d_{m(S)-1} - x_{\mathrm{rep}_{m(S)} S} f \text{ (by (7.6))} \\ &= (-1)^{n+1} \left(x_{\mathrm{dsc}_{m(S)} S'}(\delta f) \circ d_{m(S)-1} - x_{\mathrm{rep}_{m(S)} S'}(\delta f) \right) \text{ (by the induction hypotheses)} \\ &= (-1)^{n+1} x_{S'}(\delta f) \text{ (by (7.6))}. \end{aligned}$$

  Here we have used that $(\mathrm{dsc}_{m(S)} S)' = \mathrm{dsc}_{m(S)} S'$ and $(\mathrm{rep}_{m(S)} S)' = \mathrm{rep}_{m(S)} S'$, which follow from the fact that $|S| > 1$. □

Corollary 7.1.5. *If f is a cocycle and $M(S) = n + |S| + 1$, then $x_S \equiv 0$.*

Proof. Use **Lemma 7.1.4** and the linearity of $f \mapsto x_S f$. □

Lemma 7.1.6. *If $g_i = 1_G$ for some $i \in [n + |S|]$, then*

$$x_S f(g_1, \cdots, g_{n+|S|}) = \begin{cases} x_{\mathrm{dsc}_i S} f \circ d_{i-1}(g_1, \cdots, g_{n+|S|}) & i \in S \\ 0 & i \notin S \end{cases}.$$

Proof. The proof proceeds by induction on $|S|$, with the base case of $|S| = 0$ immediate from fact that f has domain $G^{\wedge n}$. Now suppose $|S| \geq 1$.

- **Case 1:** $i = M(S)$.

  We have

$$x_{S'} f \circ d_{M(S)-j}(g_1, \cdots, g_{n+|S|}) = 0 \text{ for } j > 1$$

  by the induction hypothesis, so the claim follows by (7.7).

- **Case 2:** $i < M(S) - D(S)$ and $i \notin S$.

  We have

$$x_{S'} f \circ d_{M(S)-j}(g_1, \cdots, g_{n+|S|}) = 0 \text{ for } 1 \leq j \leq D(S)$$

  by the induction hypothesis, so the claim follows by (7.7).

- **Case 3:** $i < M(S) - D(S)$ and $i \in S$.

  We have

$$x_{S'} f \circ d_{M(S)-j}(g_1, \cdots, g_{n+|S|}) = x_{\mathrm{dsc}_i S'} \circ d_{i-1} \circ d_{M(S)-j}(g_1, \cdots, g_{n+|S|}) \text{ for } 1 \leq j \leq D(S)$$

by the induction hypothesis, so the claim follows by (7.4) and (7.7) (with S replaced by $\mathrm{dsc}_i S$).

- **Case 4:** $M(S) - D(S) < i < M(S)$.
(7.7) yields

$$x_S(g_1, \cdots, g_{n+|S|}) = \sum_{j=1}^{D(S)} (-1)^{j-1} x_{S'} \circ d_{M(S)-j}(g_1, \cdots, g_{n+|S|}).$$

The $j = M(S) - i$ and $j = M(S) - i + 1$ terms cancel, so the claim follows from the induction hypothesis.

- **Case 5:** $i = M(S) - D(S)$. We have $d_i(g_1, \cdots, g_{n+|S|}) = d_{i-1}(g_1, \cdots, g_{n+|S|})$, so the claim follows from (7.7) and the induction hypothesis. □

7.2 Extending α_n to BG

We are now ready to produce an extension of $\alpha_n f$ to BG for $f \in Z^n_c(G, A)$ a cocycle. First, for $m \geq n$ we define

$$\alpha^n_m f : B_m G \to B^n A; (g_1, t_1) \cdots (g_m, t_m) \mapsto \prod_{\substack{S \subset [m] \\ |S| = m-n}} \left(x_S f(\vec{g}), \mu_n(\vec{t}_S) \right),$$

where $\vec{g} = (g_1, \cdots, g_m)$ and $\vec{t}_S$ is the tuple obtained by omitting the coordinates in $(t_1, \cdots, t_m)$ which are indexed by S. To see that this is well-defined and $\alpha^n_m f|_{B_{m-1}G} = \alpha^n_{m-1} f$ for $m > n$, we make the following checks.

- If $t_1 = 0$, then

$$\begin{aligned}
\alpha_m^n f\left((g_1,t_1)\cdots(g_m,t_m)\right) &= \prod_{\substack{S\subset[m]\\|S|=m-n\\1\in S}} \left(x_S f(\vec{g}), \mu_n(\vec{t}_S)\right) \\
&= \prod_{\substack{S\subset[m]\\|S|=m-n\\1\in S}} \left(x_S f(\vec{g}) + x_{\mathrm{rep}_1 S} f(\vec{g}), \mu_n(\vec{t}_S)\right) \text{ (by \textbf{Corollary 7.1.3})} \\
&= \prod_{\substack{S\subset[m]\\|S|=m-n\\1\in S}} \left(x_{\mathrm{dsc}_1 S} f\circ d_0(\vec{g}), \mu_n(\vec{t}_S)\right) \text{ (by (7.6))} \\
&= \prod_{\substack{S\subset[m-1]\\|S|=m-n-1}} \left(x_S f\circ d_0(\vec{g}), \mu_n(\vec{t}_{\mathrm{dsc}_1^{-1} S})\right) \\
&= \alpha_{m-1}^n f\left((g_2,t_2)\cdots(g_m,t_m)\right). \qquad (7.13)
\end{aligned}$$

- If $t_{i-1} = t_i$ for some i with $1 < i \leq m$, then

$$\begin{aligned}
\alpha_m^n f\left((g_1,t_1)\cdots(g_m,t_m)\right) &= \prod_{\substack{S\subset[m]\\|S|=m-n\\\{i-1,i\}\cap S\neq\emptyset}} \left(x_S f(\vec{g}), \mu_n(\vec{t}_S)\right) \\
&= \prod_{\substack{S\subset[m]\\|S|=m-n\\i-1,i\in S}} \left(x_S f(\vec{g}), \mu_n(\vec{t}_S)\right) \prod_{\substack{S\subset[m]\\|S|=m-n\\i\in S,\, i-1\notin S}} \left(x_S f(\vec{g}) + x_{\mathrm{rep}_i S} f(\vec{g}), \mu_n(\vec{t}_S)\right) \\
&= \prod_{\substack{S\subset[m]\\|S|=m-n\\i\in S}} \left(x_{\mathrm{dsc}_i S} f\circ d_{i-1}(\vec{g}), \mu_n(\vec{t}_S)\right) \text{ (by (7.6))} \\
&= \prod_{\substack{S\subset[m-1]\\|S|=m-n-1}} \left(x_S f\circ d_{i-1}(\vec{g}), \mu_n(\vec{t}_{\mathrm{dsc}_i^{-1} S})\right) \\
&= \alpha_{m-1}^n f\left((g_1,t_1)\cdots(g_{i-2},t_{i-2})(g_{i-1}g_i,t_{i-1})(g_{i+1},t_{i+1})\cdots(g_m,t_m)\right). \qquad (7.14)
\end{aligned}$$

- If $t_m = 1$, then

$$\begin{aligned}
\alpha^n_m f\left((g_1,t_1)\cdots(g_m,t_m)\right) &= \prod_{\substack{S\subset[m]\\|S|=m-n\\m\in S}} \left(x_S f(\vec{g}), \mu_n(\vec{t}_S)\right)\\
&= \prod_{\substack{S\subset[m]\\|S|=m-n\\m\in S}} \left(x_S f(\vec{g}) + x_{\mathrm{rep}^{-1}_{m+1}S} f(\vec{g}), \mu_n(\vec{t}_S)\right) \text{ (by **Corollary 7.1.5**)}\\
&= \prod_{\substack{S\subset[m]\\|S|=m-n\\m\in S}} \left(x_{S'} f\circ d_m(\vec{g}), \mu_n(\vec{t}_S)\right) \text{ (by (7.6))}\\
&= \prod_{\substack{S\subset[m-1]\\|S|=m-n-1}} \left(x_S f\circ d_m(\vec{g}), \mu_n(\vec{t}_{S\cup\{m\}})\right)\\
&= \alpha^n_{m-1} f\left((g_2,t_2)\cdots(g_{m-1},t_{m-1})\right). \qquad (7.15)
\end{aligned}$$

- Calculations similar to but simpler than the above involving **Lemma 7.1.6** show that if $g_1 = 1_G$ for some i, then

$$\alpha^n_m f\left((g_1,t_1)\cdots(g_m,t_m)\right) = \alpha^n_{m-1} f\left((g_1,t_1)\cdots\widehat{(g_i,t_i)}\cdots(g_m,t_m)\right).$$

Remark 7.2.1. Observe that we only needed to use the fact that f is a cocycle in the $t_m = 1$ case (since **Corollary 7.1.5** relies on f being a cocycle). $\square$

In light of the above, the direct limit of $\alpha^n_m f$ over m yields a map $\alpha^n f : BG \to B^n A$. We have $\alpha^n f|_{B_n G} = \alpha^n_n f = \alpha_n f$ since $x_\emptyset f = f$. Furthermore, since $f \mapsto x_S f$ is a group homomorphism, so is

$$\alpha^n : Z^n_{\mathrm{c}}(G,A) \to H^n(BG,A).$$

It is not yet clear whether α^n factors through continuous cohomology; we show that it does in the next section. First, we show that α^n agrees with the maps $B : Z^1_{\mathrm{c}}(G,A) = H^1_{\mathrm{c}}(G,A) \to H^1(BG,A)$ and $\alpha : Z^2_{\mathrm{c}}(G,A) \to H^2(BG,A)$ for $n=1$ and $n=2$ respectively.

Proposition 7.2.2. *For $f : G \to A$ a continuous homomorphism (i.e., 1-cocycle), we have $\alpha^1 f = Bf$.*

Proof. For $m \geq 1$ and $i \in [m]$, let $S^m_i = [m] - \{i\}$ and $x^m_i = x_{S^m_i} f$. We wish to show that $x^m_i(g_1,\cdots,g_m) = f(g_i)$, which we will prove by induction on m. The base case of $m=1$ is immediate from definitions. For $m \geq 2$, we have

$$M(S_i^m) = \begin{cases} m & i \in [m-1] \\ m-1 & i = m \end{cases},$$

$$D(S_i^m) = \begin{cases} 1 & i \in [m-2] \\ 2 & i = m-1 \\ 1 & i = m \end{cases}, \text{ and}$$

$$(S_i^m)' = \begin{cases} S_i^{m-1} & i \in [m-1] \\ S_{m-1}^{m-1} & i = m \end{cases}.$$

Hence, (7.7) yields

$$x_i^m = \begin{cases} x_i^{m-1} \circ d_{m-1} & i \in [m-2] \\ x_{m-1}^{m-1} \circ d_{m-1} - x_{m-1}^{m-1} \circ d_{m-2} & i = m-1 \\ x_{m-1}^{m-1} \circ d_{m-2} & i = m \end{cases}.$$

The claim now follows using the induction hypothesis. □

Proposition 7.2.3. *The two maps $\alpha, \alpha^2 : Z_c^2(G, A) \to H^2(BG, A)$ are equal.*

Proof. The proof is essentially the same as that of **Theorem 5.2.1**, except with more moving parts. Fix a 2-cocycle $f \in Z_c^2(G, A)$ and let $\mathcal{E} = (E, \mu, p)$ be the corresponding extension (as in the proof of **Theorem 5.2.1**). For $m \geq 2$ and $1 \leq j < k \leq m$, let $S_{j,k}^m = [m] - \{j, k\}$ and $x_{j,k}^m = x_{S_{j,k}^m} f$. Recall the object $X_{\mathcal{E}}$ defined in Section 3.3, in particular that the inclusion $\iota_{BA} : BA \hookrightarrow X_{\mathcal{E}}$ is a homotopy equivalence (**Corollary 3.3.2**). Hence, $B\iota_{BA} : B^2A \hookrightarrow BX_{\mathcal{E}}$ is a weak homotopy equivalence. Consequently, it suffices to show that

$$(B\iota_{BA})_* \circ \alpha(f) = (B\iota_{BA})_* \circ \alpha^2(f).$$

By (5.6), this reduces to proving that

$$[B\iota_G] = (B\iota_{BA})_* \circ \alpha^2(f). \tag{7.16}$$

By (5.7), the restriction to B_mG ($m \geq 2$) of the left side in (7.16) is represented by

$$L_m = B\iota_G \circ \iota_m : B_mG \to BX_{\mathcal{E}}; (g_1, t_1) \cdots (g_m, t_m) \mapsto ((g_1, 1_A), 1, t_1) \cdots ((g_m, 1_A), 1, t_m).$$

Likewise, the restriction of the right side of (7.16) to B_mG admits the representative

$$R_m : B_mG \to BX_{\mathcal{E}}; (g_1, t_1) \cdots (g_m, t_m) \mapsto \prod_{1 \le j < k \le m} \left(\left(1_G, x^m_{j,k}(\vec{g})\right), \mu_2(t_j, t_k) \right),$$

where $\vec{g} := (g_1, \cdots, g_m)$. Let L and R be the respective direct limits, so that (7.16) reduces to showing that $L \approx R$. Equivalently, we may show that $L \cdot R^{-1}$ (interpreted in terms of the right-action of B^2A on $BX_{\mathcal{E}}$) is null-homotopic.

Analogous to the proof of **Theorem 5.2.1**, consider the homotopy $H^m_s : B_mG \to BX_{\mathcal{E}}$ $(s \in I)$ given by

$$H^m_s : (g_1, t_1) \cdots (g_m, t_m) \mapsto ((g_1, 1_A), s, t_1) \cdots ((g_m, 1_A), s, t_m) \cdot \prod_{1 \le j < k \le m} \left(\left(1_G, -x^m_{j,k}(\vec{g})\right), \frac{st_j}{t_k}, t_k \right). \tag{7.17}$$

Supposing for the moment that this is well-defined, it is immediate that $H^m_1 = L_m \cdot R^{-1}_m$ and H^m_0 is constant. We will now show that H^m_s is well-defined and $H^m_s|_{B_{m-1}G} = H^{m-1}_s$, so that the direct limit yields a null-homotopy of $L \cdot R^{-1}$.

For $m = 2$, it is clear that H^2_s is well-defined (this was proved while proving **Theorem 5.2.1**). For $m > 2$, we have the following. When $i, j, k \in [m]$ $(j \ne i \ne k)$ are fixed, we use the notation

$$j' := \begin{cases} j - 1 & i < j \\ j & i > j \end{cases} \text{ and}$$

$$k' := \begin{cases} k - 1 & i < k \\ k & i > k \end{cases}.$$

In particular, $\mathrm{dsc}_i S^m_{j,k} = S^{m-1}_{j',k'}$.

- Suppose $g_i = 1_G$ for some $i \in [m]$. Since $\alpha^2_m f|_{B_{m-1}G} = \alpha^2_{m-1} f$, we have

$$\prod_{1 \le j < k \le m} \left(\left(1_G, x^m_{j,k}(\vec{g})\right), \frac{t_j}{t_k}, t_k \right) = \prod_{\substack{1 \le j < k \le m \\ j \ne i \ne k}} \left(\left(1_G, x^{m-1}_{j',k'}(\vec{g}_i)\right), \frac{t_j}{t_k}, t_k \right).$$

The calculations used to prove this also yield

$$\prod_{1\le j<k\le m}\left(\left(1_G, x^m_{j,k}(\vec{g})\right), \frac{st_j}{t_k}, t_k\right) = \prod_{\substack{1\le j<k\le m\\ j\neq i\neq k}}\left(\left(1_G, x^{m-1}_{j',k'}(\vec{g}_i)\right), \frac{st_j}{t_k}, t_k\right).$$

This shows that

$$H^m_s\left((g_1,t_1)\cdots(g_m,t_m)\right) = H^{m-1}_s\left((g_1,t_1)\cdots\widehat{(g_i,t_i)}\cdots(g_m,t_m)\right).$$

- For the cases of $t_1 = 0$ and $t_m = 1$, the calculations done for (7.13) and (7.15) yield

$$H^m_s\left((g_1,t_1)\cdots(g_m,t_m)\right) = \begin{cases} H^{m-1}_s\left((g_2,t_2)\cdots(g_m,t_m)\right) & t_1 = 0 \\ H^{m-1}_s\left((g_1,t_1)\cdots(g_{m-1},t_{m-1})\right) & t_m = 1 \end{cases}.$$

- Suppose $t_{i-1} = t_i$ for some i with $1 < i \le m$. The calculation done for (7.14) yields

$$\prod_{1\le j<k\le m}\left(\left(1_G, x^m_{j,k}(\vec{g})\right), \frac{st_j}{t_k}, t_k\right) = \left(\left(1_G, x^m_{i-1,i}(\vec{g})\right), s, t_i\right)\cdot \prod_{\substack{1\le j<k\le m\\ j\neq i\neq k}}\left(\left(1_G, x^{m-1}_{j',k'}(\vec{g}_i)\right), \frac{st_j}{t_k}, t_k\right).$$

We also have

$$\begin{aligned}((g_1,1_A),s,t_1)\cdots((g_m,1_A),s,t_m) = ((g_1,1_A),s,t_1)\cdots((g_{i-2},1_A),s,t_{i-2})\\ \left((g_{i-1}g_i, f(g_{i-1},g_i)), s, t_i\right)((g_{i+1},1_A),s,t_{i+1})\cdots((g_m,1_A),s,t_m).\end{aligned}$$

Hence, to show that H^m_s is well-defined and $H^m_s|_{B_{m-1}G} = H^{m-1}_s$, it only remains to show that $x^m_{i-1,i}(\vec{g}) = f(g_{i-1},g_i)$ for $m \ge 2$ and $1 < i \le m$. This follows by inducting on m using (7.7) and considering the cases $i \in [n-2]$, $i = n-1$, and $i = n$ separately (this is similar to the proof of **Proposition 7.2.2**). $\square$

Example 7.2.4. We generalize the idea of **Example 3.1.3** to α^n. If $f \in Z^n_{\mathrm{c}}(G,A)$ is null-homotopic through cocycles, then f has a lift $\tilde{f} : G^{\wedge n} \to PA$ which is also a cocycle. By naturality of α^n, we have $(e_1)_* \circ \alpha^n_{G,PA}\tilde{f} = \alpha^n_{G,A} \circ (e_1)_*\tilde{f}$. The prior is 0 (since PA is contractible) and the latter is $\alpha^n_{G,A}f$, so $f \in \ker \alpha^n_{G,A}$.

The hypothesis on f is satisfied when G is discrete and $A = BA'$ for some k-group A'. This is because f must lift to a cocycle in EA', and EA' is contractible through group homomorphisms. $\square$

7.3 A topological counterpart to continuous cohomology's connecting morphism

Recall the evaluation map $e_1 : PA \to A^\circ$, which fits into a short exact sequence

$$1 \longrightarrow \Omega A \longrightarrow PA \xrightarrow{e_1} A^\circ \longrightarrow 1.$$

The connecting morphism $\delta^n : H^n_{\mathrm{c}}(G, PA, \Omega A) \to H^{n+1}_{\mathrm{c}}(G, \Omega A)$ from the corresponding long exact sequence of continuous cohomology is induced by the coboundary map

$$\delta^n : e_1^{-1}\left(Z^n_{\mathrm{c}}(G,A)\right) \to Z^{n+1}_{\mathrm{c}}(G, \Omega A).$$

Also, recall the weak homotopy equivalence $\theta_A : B\Omega A \to A^\circ; (\gamma, t) \mapsto \gamma(t)$ from Section 4.4.

Proposition 7.3.1. *The following commutes up to a sign of* $(-1)^n$.

$$\begin{tikzcd}
e_1^{-1}\left(Z^n_c(G,A)\right) \arrow[r, "\delta^n"] \arrow[dd, "e_1"'] & Z^{n+1}_c(G, \Omega A) \arrow[d, "\alpha^{n+1}"] \\
 & H^{n+1}(BG, \Omega A) \arrow[d, "(\theta_A)_*"] \\
Z^n_c(G,A) \arrow[r, "\alpha^n"] & H^n(BG, A)
\end{tikzcd}$$

Proof. Fix a cocycle $f \in Z^n_{\mathrm{c}}(G,A)$ and a lift $\tilde{f} \in e_1^{-1}\left(Z^n_{\mathrm{c}}(G,A)\right)$, i.e., $\tilde{f}$ is a null-homotopy of f. We wish to show that the two maps

$$(-1)^n \alpha^n f, B^n\theta_A \circ \alpha^{n+1}(\delta \tilde{f}) : BG \to B^n A$$

are homotopic. We will explicitly construct a null-homotopy of the difference between these two maps using a technique similar to that used for **Proposition 7.2.3**. For $m \geq n+1$, consider the following homotopy.

$$H^m_s : (g_1, t_1)\cdots(g_m, t_m) \mapsto \prod_{\substack{S \subset [m] \\ |S| = m-n}} \left(x_S \tilde{f}(s)(\vec{g}), \mu_n(\vec{t_S})\right) \cdot \prod_{\substack{S \subset [m] \\ |S| = m-n-1}} \theta_A\left((-1)^{n+1} x_S(\delta\tilde{f})(\vec{g}), \mu_{n+1}(s\vec{t_S})\right). \tag{7.18}$$

Here, $s\vec{t_S}$ denotes coordinate-wise multiplication of s to all entries of t_S. Assuming that this homotopy is well-defined, it is clear that $H^m_1 = \alpha^n_m f + (-1)^{n+1} B^n\theta_A \circ \alpha^{n+1}_m(\delta\tilde{f})$ and H^m_0 is constant. Hence, the proposition will follow by taking direct limits if we show that H_s^m is well-defined with $H_s^m|_{B_{m-1}G} = H_s^{m-1}$. For this, we make the following checks.

- Suppose $g_i = 1_G$ for some $i \in [m]$. The corresponding calculation done in the beginning of Section 7.2 shows that

$$H_s^m\left((g_1, t_1) \cdots (g_m, t_m)\right) = H_s^{m-1}\Big((g_1, t_1) \cdots \widehat{(g_i, t_i)} \cdots (g_m, t_m)\Big).$$

Although those calculations were done in the context of cocycles (and $\tilde{f}$ is not a cocycle), this is not an issue by Remark 7.2.1.

- Suppose $t_1 = 0$ or $t_{i-1} = t_i$ for some i with $1 < i \leq m$. Appealing to Remark 7.2.1 once again, we use the calculations (7.13) and (7.14) to conclude that

$$H_s^m\left((g_1, t_1) \cdots (g_m, t_m)\right) = H_s^{m-1}\left((g_2, t_2) \cdots (g_m, t_m)\right)$$

if $t_1 = 0$ and

$$H_s^m\left((g_1, t_1) \cdots (g_m, t_m)\right) = H_s^{m-1}\left((g_1, t_1) \cdots (g_{i-2}, t_{i-2})(g_{i-1}g_i, t_i)(g_{i+1}, t_{i+1}) \cdots (g_m, t_m)\right)$$

if $t_{i-1} = t_i$.

- Suppose $t_m = 1$ (so Remark 7.2.1 no longer applies). The first product in the expression for H_s^m is

$$\begin{aligned}
\prod_{\substack{S \subset [m] \\ |S| = m-n}} \Big(x_S \tilde{f}(s)(\vec{g}), \mu_n(\vec{t}_S)\Big) &= \prod_{\substack{S \subset [m] \\ |S| = m-n \\ m \in S}} \Big(x_S \tilde{f}(s)(\vec{g}), \mu_n(\vec{t}_S)\Big) \\
&= \prod_{\substack{S \subset [m] \\ |S| = m-n \\ m \in S}} \Big(x_S \tilde{f}(s)(\vec{g}) + x_{\mathrm{rep}_{m+1}^{-1} S} \tilde{f}(\vec{g}), \mu_n(\vec{t}_S)\Big) \cdot \\
&\quad \prod_{\substack{S \subset [m] \\ |S| = m-n \\ m \in S}} \Big((-1)^n x_{S'} (\delta \tilde{f})(s)(\vec{g}), \mu_n(\vec{t}_S)\Big) \text{ (by **Lemma 7.1.4**)}.
\end{aligned}$$

The first product in the right side can be simplified just as in (7.15), so we obtain

$$\prod_{\substack{S\subset[m]\\|S|=m-n}} \left(x_S\tilde{f}(s)(\vec{g}),\mu_n(\vec{t}_S)\right) = \prod_{\substack{S\subset[m-1]\\|S|=m-n-1}} \left(x_S\tilde{f}(s)\circ d_m(\vec{g}),\mu_n(\vec{t}_{S\cup\{m\}})\right)\cdot \prod_{\substack{S\subset[m-1]\\|S|=m-n-1}} \left((-1)^n x_S(\delta\tilde{f})(s)(\vec{g}),\mu_n(\vec{t}_{S\cup\{m\}})\right). \tag{7.19}$$

Next, the second product in the expression for H_s^m is

$$\begin{aligned}
&\prod_{\substack{S\subset[m]\\|S|=m-n-1}} \theta_A\left((-1)^{n+1}x_S(\delta\tilde{f})(\vec{g}),\mu_{n+1}(s\vec{t}_S)\right)\\
&= \prod_{\substack{S\subset[m]\\|S|=m-n-1\\m\notin S}} \theta_A\left((-1)^{n+1}x_S(\delta\tilde{f})(\vec{g}),\mu_{n+1}(s\vec{t}_S)\right)\cdot \prod_{\substack{S\subset[m]\\|S|=m-n-1\\m\in S}} \theta_A\left((-1)^{n+1}x_S(\delta\tilde{f})(\vec{g}),\mu_{n+1}(s\vec{t}_S)\right)\\
&= \prod_{\substack{S\subset[m-1]\\|S|=m-n-1}} \theta_A\left((-1)^{n+1}x_S(\delta\tilde{f})(\vec{g}),\mu_{n+1}(s\vec{t}_S)\right)\cdot \prod_{\substack{S\subset[m]\\|S|=m-n-1\\m\in S}} \theta_A\left((-1)^{n+1}x_S(\delta\tilde{f})(\vec{g}),\mu_{n+1}(s\vec{t}_S)\right).
\end{aligned}$$

The first product in the right side of the above is the inverse of the second product in the right side of (7.19) (since $t_m = 1$ is the last coordinate of $\vec{t}_S$ when $m \notin S$). The second product in the right side of the above can be seen to be

$$\prod_{\substack{S\subset[m-1]\\|S|=m-n-1}} \theta_A\left((-1)^{n+1}x_S(\delta\tilde{f})\circ d_m(\vec{g}),\mu_{n+1}(s\vec{t}_{S\cup\{m\}})\right)$$

using (7.15) (since $\delta\tilde{f}$ is a cocycle). Combining these observations with (7.19) yields

$$H_s^m\left((g_1,t_1)\cdots(g_m,t_m)\right) = H_s^{m-1}\left((g_1,t_1)\cdots(g_{m-1},t_{m-1})\right),$$

as desired. □

Corollary 7.3.2. *The kernel of $\alpha^n : Z_c^n(G,A)\to H^n(BG,A)$ contains $B_c^n(G,A)$, i.e., α^n factors through a map $\alpha^n : H_c^n(G,A)\to H^n(BG,A)$.*

Proof. The claim is trivial for $n = 1$, since 0 is the only 1-coboundary. For $n > 1$, consider the following diagram.

$$\begin{tikzcd}
e_1^{-1}\left(Z_{\mathrm{c}}^{n-1}(G,BA)\right) \arrow[r, "\delta^{n-1}"] \arrow[dd, "e_1"'] & Z_{\mathrm{c}}^n(G,\Omega BA) \arrow[d, "\alpha^n"] & Z_{\mathrm{c}}^n(G,A) \arrow[l, "(\phi_A)_*"'] \arrow[d, "\alpha^n"] \\
 & H^n(BG,\Omega BA) \arrow[d, "(\theta_A)_*"] & H^n(BG,A) \arrow[l, "(\phi_A)_*"'] \arrow[dl, equal] \\
Z_{\mathrm{c}}^{n-1}(G,BA) \arrow[r, "\alpha^{n-1}"] & H^{n-1}(BG,BA) &
\end{tikzcd}$$

Here, $\phi_A : A \to \Omega BA$ is the weak homotopy equivalence (and group homomorphism) from **Lemma 2.8.1**. The top-right square commutes by naturality of α^n, and the lower-right triangle can be seen to commute using **Lemma 4.4.3**. Hence, the diagram commutes up to sign by **Proposition 7.3.1**.

Now fix a cochain $f \in C_{\mathrm{c}}^{n-1}(G,A)$ with coboundary $\delta f \in Z_{\mathrm{c}}^n(G,A)$ lying in the top right corner of the diagram. We wish to show that $\alpha^n\delta f = 0$. Let $f' = (\phi_A)_* f$, so that $f' \in C_{\mathrm{c}}^{n-1}(G,\Omega BA) \subset C_{\mathrm{c}}^{n-1}(G,PBA)$. Since $e_1 f' = 0$, we see that f' lies in the top left corner of the diagram. By definition of f', we have $\delta f' = (\phi_A)_*\delta f$. Commutativity of the diagram thus yields $\alpha^{n-1} \circ e_1 f' = \alpha^n \delta f$. The left side is 0 (since $e_1 f' = 0$), so the claim follows. $\square$

Corollary 7.3.3. *The following commutes up to a sign of $(-1)^n$.*

$$\begin{tikzcd}
H_c^n(G,PA,\Omega A) \arrow[r, "\delta^n"] \arrow[dd, "J_*"'] & H_c^{n+1}(G,\Omega A) \arrow[d, "\alpha^{n+1}"] \\
 & H^{n+1}(BG,\Omega A) \arrow[d, hook, "(\theta_A)_*"] \\
H_c^n(G,A) \arrow[r, "\alpha^n"] & H^n(BG,A)
\end{tikzcd}$$

7.4 An algorithmic description of $\ker\alpha^n$

If **Conjecture 6.3.1** holds, then **Theorem 6.3.2** and **Corollary 7.3.3** lend themselves to an algorithmic way of understanding $\ker\alpha^n$, which goes as follows. We start with a class $\zeta_0 \in H_{\mathrm{c}}^n(G,A)$ and ask whether it lies in $\ker\alpha^n$.

1. If ζ_0 does not have a null-homotopic representative in $Z_{\mathrm{c}}^n(G,A)$, then $\alpha_n\zeta_0 \neq 0$ (by **Theorem 6.3.2**). Hence, $\alpha^n\zeta_0 \neq 0$. [Algorithm terminates]

2. If ζ_0 has a null-homotopic representative, then it lies in im J_*. Let $\tilde{\zeta}_0 \in H^n_{\mathrm{c}}(G, PA, \Omega A)$ be a preimage and set $\zeta_1 = \delta^n \tilde{\zeta}_0 \in H^{n+1}_{\mathrm{c}}(G, \Omega A)$.

3. If ζ_1 does not have a null-homotopic representative in $Z^{n+1}_{\mathrm{c}}(G, \Omega A)$, then $\alpha_{n+1}\zeta_1 \neq 0$ (by **Theorem 6.3.2**). Hence, $\alpha^{n+1}\zeta_1 \neq 0$. Consequently, $\alpha^n \zeta_0 \neq 0$ (by **Corollary 7.3.3**). [Algorithm terminates]

4. If ζ_1 has a null-homotopic representative, then it lies in im J_*. Let $\tilde{\zeta}_1 \in H^{n+1}_{\mathrm{c}}(G, P\Omega A, \Omega^2 A)$ be a preimage and set $\zeta_2 = \delta^{n+1}\tilde{\zeta}_1 \in H^{n+2}_{\mathrm{c}}(G, \Omega^2 A)$.

5. If ζ_2 does not have a null-homotopic representative in $Z^{n+2}_{\mathrm{c}}(G, \Omega^2 A)$, then $\alpha_{n+2}\zeta_2 \neq 0$ (by **Theorem 6.3.2**). Hence, $\alpha^{n+2}\zeta_2 \neq 0$. Consequently, $\alpha^{n+1}\zeta_1 \neq 0$ (by **Corollary 7.3.3**) and $\alpha^n \zeta_0 \neq 0$ (by **Corollary 7.3.3**). [Algorithm terminates]

$$\vdots$$

If the algorithm never terminates, then $\alpha^n_m \zeta_0 = 0$ for all $m \geq n$. Hence, $\alpha^n \zeta_0 = 0$ in this case.

Remark 7.4.1. It is true unconditionally (i.e., without assuming **Conjecture 6.3.1**) that $\alpha^n \zeta_0 = 0$ if the algorithm never terminates.

If A is of finite type and **Conjecture 6.3.1** holds, then the algorithm determines whether $\zeta_0 \in \ker \alpha^n$ in only finitely many steps (since $\Omega^d A$ is weakly contractible for sufficiently large d). When G is connected and A is of finite type, we can say even more.

Proposition 7.4.2. *If G is d-connected ($d \geq 0$), then $G^{\wedge n}$ is $(n(d+1)-1)$-connected.*

Hence, if G is d-connected ($d \geq 0$), then the connectivity of $G^{\wedge(n+m)}$ grows linearly with m. If A of finite type, then the type of $\Omega^m A$ simultaneously falls linearly with m. In particular, if A has type t, then all maps $G^{\wedge(n+m)} \to \Omega^m A$ are null-homotopic when

$$(n+m)(d+1) - 1 \geq t - m,$$

i.e., $m \geq \frac{t+1-n(d+1)}{d+2}$. In particular, all cocycles in $Z^{n+m}_{\mathrm{c}}(G, \Omega^m A)$ are null-homotopic in this case.

The proof of **Proposition 7.4.2** requires some intermediate results.

Lemma 7.4.3. *Let*

- *X and Y be spaces with $f : X \to Y$ a homotopy equivalence,*

- *$\mathcal{D}$ be a disjoint union of disks (not necessarily of the same dimension), and*

- *$\phi : \partial\mathcal{D} \to X$ be a map, where $\partial\mathcal{D}$ denotes the union of the boundaries of all disks in $\mathcal{D}$.*

The map $\tilde{f} : X \sqcup_\phi \mathcal{D} \to Y \sqcup_{f\phi} \mathcal{D}$ induced by f is a homotopy equivalence.

Proof. This is a special case of [3, 7.5.7]. □

Lemma 7.4.4. *Let X be a CW complex and Y be a d-dimensional CW complex which is homotopy equivalent to $X^{(d)}$. There exists a CW complex X' which is homotopy equivalent to X and has d-skeleton Y.*

Proof. Let $Y_d = Y$ and $f_d : Y_d \to X^{(d)}$ be the given homotopy equivalence. By induction on $i > d$, we will produce an i-dimensional CW complex Y_i and a homotopy equivalence $f_i : Y_i \to X^{(i)}$ such that $Y_i^{(i-1)} = Y_{i-1}$ and $f_i|_{Y_{i-1}} = f_{i-1}$. Taking direct limits will yield a CW complex X' with $X'^{(i)} = Y_i$ and a map $f : X' \to X$ which restricts to homotopy equivalences $X'^{(i)} \to X^{(i)}$ for all i. Hence, f will be a homotopy equivalence by Whitehead's Theorem, proving the lemma.

Suppose f_{i-1} and Y_{i-1} have been constructed for some $i > d$. Let $\mathcal{D}$ be a disjoint union of i-disks and $\phi : \mathcal{D} \to X^{(i-1)}$ a map so that $X^{(i)} = X^{(i-1)} \sqcup_\phi \mathcal{D}$. Let $Y_i = Y_{i-1} \sqcup_{f_{i-1}\phi} \mathcal{D}$ and $f_i : Y_i \to X^{(i)}$ the map induced by f_{i-1}. By **Lemma 7.4.3**, Y_i and f_i are as desired. □

Corollary 7.4.5. *Suppose X is a d-connected CW complex, $d \geq 0$. There exists a CW complex X' which is homotopy equivalent to X and has d-skeleton a point.*

Proof. For $d = 0$, the corollary follows from [8, Propositions 0.17 & 1A.1]. Next, suppose $d > 0$ and X is $(d+1)$-dimensional. $H_{d+1}(X, \mathbb{Z})$ is free abelian, so let X' be a wedge of $(d+1)$-spheres indexed by a basis of $H_{d+1}(X, \mathbb{Z})$. Hurewicz's Theorem yields a map $X' \to X$ which induces isomorphism on $(d+1)$-st homology with integer coefficients. This map is a homotopy equivalence by [8, Corollary 4.33], so X' is as desired.

The general case now follows from **Lemma 7.4.4**. □

Corollary 7.4.6. *Let X_1 and X_2 be based CW complexes such that X_i is d_i-connected ($d_i \geq 0$). Then $X_1 \wedge X_2$ is $(d_1 + d_2 + 1)$-connected.*

Proof. The homotopy type of $X_1 \wedge X_2$ depends only on the homotopy types of X_1 and X_2, and X_i is homotopy equivalent to a CW complex with d_i-skeleton a point (by **Corollary 7.4.5**). Hence, we may assume, without loss of generality, that X_i has d_i-skeleton a point. Consequently, the $(d_1 + d_2 + 1)$-skeleton of $X_1 \times X_2$ is contained in $X_1 \vee X_2$. Thus, $X_1 \wedge X_2$ has $(d_1 + d_2 + 1)$-skeleton a point. □

Proof of ***Proposition 7.4.2****.* Apply **Corollary 7.4.6** $n - 1$ times. □

Although the algorithm requires **Conjecture 6.3.1** in general, some specific cases hold unconditionally in light of **Corollary 6.3.5** and **Theorem 6.4.2**.

Theorem 7.4.7. *1. The algorithm works unconditionally when G is discrete, $G = S^1$, or $G = S^3$.*

2. For $n = 1$, the algorithm works unconditionally up to step 3.

3. For $n = 2$, the algorithm works unconditionally up to step 1.

Proof. The case of G discrete or $G = S^1, S^3$ follows immediately from **Theorem 6.4.2**. The case of $n = 1, 2$ follows from Remark 6.3.3 and **Corollary 6.3.5**. □

7.5 Some examples

Example 7.5.1. Recall the set-up of **Example 6.5.3**. Since G is connected, $G^{\wedge 3}$ is 1-connected (by **Proposition 7.4.2**). Hence, all cocycles in $Z^3_{\mathrm{c}}(G, \Omega A)$ are null-homotopic. Since $G^{\wedge 4}$ is connected, all cocycles in $Z^4_{\mathrm{c}}(G, \Omega^2 A)$ are null-homotopic. Also, all cocycles in $Z^{n+2}_{\mathrm{c}}(G, \Omega^n A)$ are null-homotopic for $n > 2$ since $\Omega^n A$ is weakly contractible for $n > 2$. From Remark 7.4.1, it follows that $\ker \alpha = \ker \alpha^2 = \ker \alpha_2$. Hence, **Example 6.5.3** provides a complete description of the homotopy types of cocycles in $\ker \alpha$. □

Example 7.5.2. Recall the set-up of **Example 6.5.7**. We will use **Theorem 7.4.7** to show that $\zeta_0 := [f]$ is not in $\ker \alpha^1$. Since f is null-homotopic, we proceed to the second step in the algorithm and choose an explicit null-homotopy $\tilde{f} : G \to PA$ of f. We make the choice

$$\tilde{f}(x)(t) = e^{\frac{2\pi i x t}{m}},$$

where $x \in \{0, \cdots, m-1\}$. Hence,

$$\tilde{\zeta}_0 = [\tilde{f}] \in H^1_{\mathrm{c}}(G, PA, \Omega A) \text{ and}$$
$$\zeta_1 = \delta^1 \tilde{\zeta}_0 = [\delta \tilde{f}] \in H^2_{\mathrm{c}}(G, \Omega A).$$

For $x, y \in \{0, \cdots, m-1\}$, we have

$$\begin{aligned}\delta \tilde{f}(x, y)(t) &= \tilde{f}(x)(t) \cdot \tilde{f}(y)(t) \cdot \tilde{f}(x+y)(t)^{-1} \\ &= \begin{cases} 1 & x + y < m \\ e^{2\pi i t} & x + y \geq m. \end{cases}\end{aligned}$$

We will show that ζ_1 does not have a null-homotopic representative, so that the algorithm will terminate on the third step. It suffices to show that $q_* \zeta_1 \in H^2_{\mathrm{c}}(G, \pi_1(A))$ does not have a null-homotopic representative, where q is the projection $\Omega A \to \pi_1(A)$. Since $\pi_1(A)$ is discrete, it suffices to show that $q_* \zeta_1 \neq 0$.

Identifying $\pi_1(A) = \pi_1(S^1)$ with $\mathbb{Z}$ by choosing its generator to be the homotopy class of $t \mapsto e^{2\pi i t}$, the extension

$$1 \to \mathbb{Z} \to E \to G \to 1$$

induced by the 2-cocycle $q \circ \delta \tilde{f}$ is

$$1 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 1.$$

This is a non-trivial extension, so $q_* \zeta_1 \neq 0$ (by **Theorem 7.4.7**) as desired. □

Example 7.5.3. The preceding example can easily be generalized to cohomologies of higher degrees when $m = 2$. Let $G = \mathbb{Z}/2\mathbb{Z}$ and $A = S^1$, and fix odd $n \geq 1$. Consider the n-cocycle $f \in Z^n_{\mathrm{c}}(G, A)$ which takes $(1, \cdots, 1)$ (n-times) to $-1 \in A$.[1] This is indeed a cocycle, since

$$\begin{aligned}\delta f(1, \cdots, 1) &= f(1, \cdots, 1) \cdot \prod_{j=1}^{n} f(1, \cdots, 0, \cdots, 1)^{(-1)^j} \cdot f(1, \cdots, 1)^{(-1)^{n+1}} \\ &= f(1, \cdots, 1)^2 \text{ (since } n \text{ is odd)} \\ &= (-1)^2 = 1.\end{aligned}$$

[1]Note that $G^{\wedge n}$ has only two points, one of which is the base point $(0, \cdots, 0)$ and the other is $(1, \cdots, 1)$. Hence, defining f on $(1, \cdots, 1)$ determines it completely. Similar reasoning is used in the subsequent check that f is a cocycle.

In the indexed product, the j-th coordinate of the argument of f is 0 and all other coordinates are 1. Let $\zeta_0 = [f] \in H^n_{\mathrm{c}}(G, A)$. f is clearly null-homotopic, so we proceed to the second step of the algorithm and fix the null-homotopy

$$\tilde{f} : G^{\wedge n} \to PA; (1, \cdots, 1) \mapsto \left(t \mapsto e^{\pi i t}\right).$$

Hence,

$$\begin{aligned}\tilde{\zeta}_0 &= [\tilde{f}] \in H^n_{\mathrm{c}}(G, PA, \Omega A) \text{ and} \\ \zeta_1 &= \delta^n \tilde{\zeta}_0 = [\delta \tilde{f}] \in H^{n+1}_{\mathrm{c}}(G, \Omega A).\end{aligned}$$

We have

$$\begin{aligned}\delta \tilde{f}(1, \cdots, 1)(t) &= \tilde{f}(1, \cdots, 1)(t) \cdot \prod_{j=1}^{n} \tilde{f}(1, \cdots, 0, \cdots, 1)(t)^{(-1)^j} \cdot \tilde{f}(1, \cdots, 1)(t)^{(-1)^{n+1}} \\ &= \tilde{f}(1, \cdots, 1)(t)^2 \text{ (since } n \text{ is odd)} \\ &= e^{2\pi i t}.\end{aligned}$$

Hence, $q \circ \delta \tilde{f}(1, \cdots, 1) = 1$ (where q and the identification $\pi_1(A) \approx \mathbb{Z}$ are as in **Example 7.5.2**). It is not hard to see that $q_* \zeta_1 = [q \circ \delta \tilde{f}] \neq 0$, so ζ_1 does not have a null-homotopic representative. Hence, $\zeta_0 \notin \ker \alpha^n$ by the third step in the algorithm (since G is discrete, we use **Theorem 7.4.7**). □

Example 7.5.4. Suppose G is abelian and $A_\bullet$ is an abelian k-group. Let $h \in Z^2_{\mathrm{c}}(G, A_\bullet)$ be a cocycle which is not homotopic to a coboundary, so $[h] \notin \ker \alpha_{G, A_\bullet}$. Suppose h is symmetric, i.e., $h(g_1, g_2) = h(g_2, g_1)$ for $g_i \in G$. Let $\mathcal{E} = (E, \mu, p)$ be the extension of G by A induced by h. Note that E is abelian because h is symmetric.

Let $A = X_{\mathcal{E}}$, where $X_{\mathcal{E}}$ is as defined in Section 3.3. We will show that the inclusion $\iota_G : G \hookrightarrow X_{\mathcal{E}}; g \mapsto ((g, 1_A), 1)$ does not lie in $\ker \alpha^1_{G,A}$ using the algorithm and **Theorem 7.4.7**. First, observe that ι_G is null-homotopic — we have the null-homotopy

$$\tilde{\iota}_G : G \to PA; g \mapsto (t \mapsto ((g, 1_A), t)).$$

Hence, we proceed to the second step of the algorithm with

$$\zeta_0 = [\iota_G] \in H^1_{\mathrm{c}}(G, A) = \operatorname{Hom}(G, A) \text{ and}$$
$$\tilde{\zeta}_0 = [\tilde{\iota}_G] \in H^1_{\mathrm{c}}(G, PA, \Omega A).$$

A representative of $\zeta_1 = \delta^1\tilde{\zeta}_0$ is $\delta\tilde{\iota}_G$. We have

$$\begin{aligned}
\delta\tilde{\iota}_G(g_1, g_2)(t) &= \tilde{\iota}_G(g_1)(t) \cdot \tilde{\iota}_G(g_2)(t) \cdot \tilde{\iota}_G(g_1g_2)(t)^{-1} \\
&= ((g_1, 1_A) \cdot (g_2, 1_A), t)\,((g_1g_2, 1_A), t)^{-1} \\
&= ((1_G, h(g_1, g_2)), t) \\
&= \iota_{BA_\bullet}(h(g_1, g_2), t) \\
&= \iota_{BA_\bullet}\left(\phi_{A_\bullet} \circ h(g_1, g_2)(t)\right),
\end{aligned}$$

where $\iota_{BA_\bullet}$ is the inclusion $BA_\bullet \hookrightarrow X_{\mathcal{E}}; (a, t) \mapsto ((1_G, a), t)$ and $\phi_{A_\bullet}$ is the weak homotopy equivalence $A_\bullet \to \Omega BA; a \mapsto (t \mapsto (a, t))$ (see **Lemma 2.8.1**). Hence,

$$\delta\tilde{\iota}_G = \Omega\iota_{BA_\bullet} \circ \phi_{A_\bullet} \circ h.$$

We claim that this is not homotopic to a coboundary, so that the algorithm terminates on the third step and $\iota_G \notin \ker \alpha^1_{G,A}$. For the sake of contradiction, suppose $h' : G \to \Omega A$ were a 1-cocycle with $\delta h' \approx \delta\tilde{\iota}_G$. Since $\iota_{BA_\bullet}$ is a homotopy equivalence (by **Corollary 3.3.2**), $\Omega\iota_{BA_\bullet} \circ \phi_{A_\bullet}$ is a weak homotopy equivalence and there exists $h'' : G \to A_\bullet$ such that

$$\begin{tikzcd}
 & A_\bullet \arrow[d, "\Omega\iota_{BA_\bullet}\circ\phi_{A_\bullet}"] \\
G \arrow[ur, dashed, "h''"] \arrow[r, "h'"] & \Omega A
\end{tikzcd}$$

commutes up to homotopy. Hence, $\delta(\Omega\iota_{BA_\bullet} \circ \phi_{A_\bullet} \circ h'') = \Omega\iota_{BA_\bullet} \circ \phi_{A_\bullet} \circ \delta h''$ is homotopic to $\delta h' \approx \delta\tilde{\iota}_G = \Omega\iota_{BA_\bullet} \circ \phi_{A_\bullet} \circ h$. Since $\Omega\iota_{BA_\bullet} \circ \phi_{A_\bullet}$ is a weak homotopy equivalence, this implies that $h \approx \delta h''$. This contradicts the hypothesis on h. $\square$

7.6 $\ker \alpha^n$ and null-homotopy through cocycles

Example 7.2.4 shows that if $f \in Z^n_{\mathrm{c}}(G, A)$ is null-homotopic *through cocycles*, then $[f] \in \ker \alpha^n$. Morally, this says that if the information captured by $\zeta \in H^n_{\mathrm{c}}(G, A)$ is ‘strongly null-homotopic' in some sense, then $\zeta \in \ker \alpha^n$. The algorithm from Section 7.4 gives another way of interpreting 'strongly null-homotopic', and this interpretation completely characterizes $\ker \alpha^n$ if **Conjecture 6.3.1** holds. This naturally leads to the question of whether these two interpretations are equivalent. To be precise, does the algorithm terminate for a class in $H^n_c(G, A)$ if and only if it has no representative which is null-homotopic through cocycles?

In this section, we produce a counter-example for the 'if' part of the above question. The main tool for this is a k-group analogue of the standard mapping cone construction for spaces. Recall that, given a based map $\phi : X \to Y$ between based spaces, the mapping cone C_ϕ is the pushout of $X \hookrightarrow CX$ and ϕ.

Now, consider a continuous homomorphism $f : A \to G$ whose image is contained in the center $Z(G)$ of G. Let K be the subgroup

$$K := \{(a, f(a)^{-1}) \mid a \in A\} \leq EA \times G.$$

This is indeed a subgroup since A is abelian, and it is normal in $EA \times G$ since $\operatorname{im} f \subset Z(G)$. Define

$$E_f := \frac{EA \times G}{K}.$$

Observe that the image of $CA \times G$ in E_f is C_f, and C_f generates E_f as a group (since CA generates EA). Also, the projection $E_f \to BA$ is a G-bundle — it is the pushforward of $p_A : EA \to BA$ along f.[2]

Proposition 7.6.1. *If A is a CW group, both A and G have cellular multiplication, and $f : A \to G$ is cellular, then E_f has a cell structure which renders $G \subset E_f$ a subcomplex and the projection $EA \times G \to E_f$ cellular.*

The proof is relegated to the end of this section. Now, consider the case of $A = B\mathbb{Z}$, $G = S^1$, and f generated by $(1, t) \mapsto e^{2\pi i t}$. The hypotheses of **Proposition 7.6.1** are satisfied, so E_f is a k-group and α^1_{G,E_f} makes sense. We claim that the algorithm does not terminate on the inclusion $G \hookrightarrow E_f$, but it is not null-homotopic through 1-cocycles (group homomorphisms). The first part follows from the fact that E_f is weakly contractible, which can be seen as follows. $\pi_1(f)$ is an isomorphism, which yields that the connecting morphism $\pi_2(BA) \to \pi_1(G)$ from the long exact sequence for the bundle $E_f \to BA$ is also an isomorphism. The same long exact sequence now yields that E_f is weakly contractible.[3]

[2]The projection $E_f \to BA$ is the k-group analogue of the map $C_\phi \to \Sigma X$ which collapses Y.

[3]In fact E_f is contractible, since it is a CW complex by **Proposition 7.6.1**.

To see that $G \hookrightarrow E_f$ is not null-homotopic through 1-cocycles, assume the contrary. Composing a given null-homotopy through 1-cocycles with the projection $E_f \to BA$ yields a continuous homomorphism $S^1 \to \Omega BA$ which induces an isomorphism on fundamental groups. In particular, we have obtained a non-trivial continuous homomorphism $S^1 \to \Omega BA$. This is a contradiction because every open set in S^1 contains a torsion element and ΩBA has no torsion. This concludes the counter-example.

Proof of ***Proposition 7.6.1***. Recall from Section 2.2 that, under the present hypotheses, EA is a CW complex with subcomplexes D_nA and E_nA. Write p for the projection $EA \times G \to E_f$. By inducting on n, we will produce a cell structure for $X_n := p(E_nA \times G)$ so that the restriction $p : E_nA \times G \to X_n$ is cellular and X_{n-1} is a subcomplex of X_n. Note that we also have $X_n = p(D_nA \times G)$. Since $D_0 = \{1_{EA}\}$ is a point, the cell structure on X_0 is the same as that for G and $p : D_0 \times G \to X_0$ is a homeomorphism. Since f and the multiplication $G \times G \to G$ are cellular, so is $p : E_0 \times G = A \times G \to X_0; (a, g) \mapsto (1_{EA}, f(a)g)$.

Now, suppose $n \geq 1$ and the cell structure on X_{n-1} is given. For $\mathcal{I}$ an indexing set, let

$$\{e_j : D^{d_j} \to D_nA \times G \mid j \in \mathcal{I}\}$$

be the characteristic maps of the cells of $(D_nA - E_{n-1}A) \times G$. Observe that

- $p \circ e_j$ restricts to a homeomorphism from the open disk $\operatorname{int} D^{d_j}$ to its image, and
- the union over $\mathcal{I}$ of the images of these open disks is $(D_nA - E_{n-1}A) \times G$.

Hence, we obtain a cell structure on $X_n = p(D_nA \times G)$ by adding the cells $p \circ e_j$ to X_{n-1}. Cellularity of $p : E_{n-1}A \times G \to X_{n-1}$ ensures that the intersection of X_{n-1} with the image of a gluing map $p \circ \partial e_j$ is contained in $X_{n-1}^{(d_j-1)}$. Clearly, $p : D_nA \times G \to X_n$ is cellular; it remains to show that $p : E_nA \times G \to X_n$ is also cellular.

Write λ for the A-action $D_nA \times A \to E_nA$. Since f and the multiplication $G \times G \to G$ are cellular, so is the composition

$$D_nA \times A \times G \xrightarrow{\lambda \times \mathrm{id}_G} E_nA \times G \xrightarrow{p} X_n \ ,$$

which takes $(x, a, g) \in D_nA \times A \times G$ to $p(x, f(a)g)$. Also, λ is cellular and restricts to a homeomorphism from $(D_nA - E_{n-1}A) \times (A - 1_A)$ to $E_nA - D_nA$. Combining the above observations, we see that $p : E_nA \times G \to X_n$ is cellular. $\square$

Remark 7.6.2. One can show that, under the hypotheses of **Proposition 7.6.1**, the multiplication on E_f is also cellular. This allows for iterating the construction by taking a sequence of cellular homomorphisms $f_1 : A_1 \to G$, $f_2 : A_2 \to E_{f_1}, \cdots, f_n : A_n \to E_{f_{n-1}}$ (with $\mathrm{im}\, f_i \subset Z(E_{f_{i-1}})$) and obtaining a CW group E_{f_n}. This is analogous to how CW (and pCW) complexes are obtained by iteratively gluing disks — gluing an n-disk to a space X amounts to taking the mapping cone of a map $S^{n-1} \to X$. $\square$

Chapter 8

Surjectivity of α for discrete A

Throughout this chapter, A is assumed to be discrete. Write π_0 for $\pi_0(G)$. Joshi–Spallone proved that α is injective under this hypothesis, and in Remark 1.1.3 we saw that our results yield this too. On the topic of surjectivity, they prove

Theorem 8.0.1. *If A has prime order and $G = G^\circ \rtimes \pi_0(G)$, then α is an isomorphism.*

The hypothesis of A having prime order came about due to the same hypothesis appearing in [1, Lemma 1.12], which Joshi–Spallone used to conclude that $\alpha_{\pi_0,A}$ is an isomorphism. **Theorem 5.3.1** yields bijectivity of $\alpha_{\pi_0,A}$ without such a hypothesis, so we have

Theorem 8.0.2. *If A is discrete and $G = G^\circ \rtimes \pi_0(G)$, then α is an isomorphism.*

In order to include this improved result within the context of their manuscript [9], Joshi–Spallone decided, in consultation with the author, to add the requisite material from Chapter 5 to a new version of their manuscript. At the time of writing, this updated version (Jain–Joshi–Spallone) is in preparation.

In discussions with the author, Joshi–Spallone proposed the following approach to proving that $\alpha_{G,A}$ is an isomorphism (in the absence of hypotheses on G). In the proof of [9, Theorem 10.4], they produce the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{E}(\pi_0, A) \arrow[r, "♀^*"] \arrow[d, "\alpha"] & \mathbb{E}(G, A) \arrow[r, "\iota^*"] \arrow[d, "\alpha"] & \mathbb{E}(G^\circ, A)^{\pi_0} \arrow[d, "\alpha"] \\
0 \arrow[r] & H^2(B\pi_0, A) \arrow[r, "B♀^*"] & H^2(BG, A) \arrow[r, "B\iota^*"] & H^2(BG^\circ, A)^{\pi_0}
\end{tikzcd}, \tag{8.1}$$

where $♀ : G \to \pi_0$ is the projection and $\iota : G^\circ \hookrightarrow G$ is the inclusion. They show that both rows are exact (and the squares commute by naturality of α). Suppose one could produce maps $\delta : \mathbb{E}(G^\circ, A) \to H^3_{\mathrm{gp}}(\pi_0, A)$ and $\delta : H^2(BG^\circ, A) \to H^3(B\pi_0, A)$ so that the extended diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{E}(\pi_0, A) \arrow[r, "\female^*"] \arrow[d, "\alpha"] & \mathbb{E}(G, A) \arrow[r, "\iota^*"] \arrow[d, "\alpha"] & \mathbb{E}(G^\circ, A)^{\pi_0} \arrow[r, "\delta"] \arrow[d, "\alpha"] & H^3_{\mathrm{gp}}(\pi_0, A) \arrow[d, "\alpha^3"] \\
0 \arrow[r] & H^2(B\pi_0, A) \arrow[r, "B\female^*"] & H^2(BG, A) \arrow[r, "B\iota^*"] & H^2(BG^\circ, A)^{\pi_0} \arrow[r, "\delta"] & H^3(B\pi_0, A)
\end{tikzcd}$$

also commutes and has exact rows.[1] All vertical arrows apart from $\alpha_{G,A}$ are known to be isomorphisms, so it would follow that $\alpha_{G,A}$ is also an isomorphism (by the Five Lemma). We propose the following modified strategy: produce a map $\delta : \mathbb{E}(G^\circ, A) \to H^3_{\mathrm{gp}}(\pi_0, A)$ so that

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{E}(\pi_0, A) \arrow[r, "\female^*"] \arrow[dd, "\alpha"] & \mathbb{E}(G, A) \arrow[r, "\iota^*"] \arrow[dd, "\alpha"] & \mathbb{E}(G^\circ, A)^{\pi_0} \arrow[r, "\delta"] \arrow[dd, "\alpha"] & H^3_{\mathrm{gp}}(\pi_0, A) \arrow[dd, dashed] \arrow[dr, "\alpha^3"] & \\
& & & & & H^3(B\pi_0, A) \arrow[dl, "\overline{B\female}^*"] \\
0 \arrow[r] & H^2(B\pi_0, A) \arrow[r, "B\female^*"] & H^2(BG, A) \arrow[r, "B\iota^*"] & H^2(BG^\circ, A)^{\pi_0} \arrow[r, "\delta^2"] & H^3(BG/BG^\circ, A) &
\end{tikzcd} \tag{8.2}$$

commutes and has an exact top row, where

- the dotted arrow is defined to make the triangle commute,
- δ^2 is (the appropriate restriction of) the connecting morphism for the pCW pair (BG, BG°), and
- $\overline{B\female} : BG/BG^\circ \to B\pi_0$ is obtained by factoring $B\female : BG \to B\pi_0$ through BG/BG°.

The definition of δ^2 automatically ensures exactness of the bottom row, and we now show that $\overline{B\female}^*$ is injective. Hence, the dotted arrow is also injective and the Five Lemma would still yields that $\alpha_{G,A}$ is an isomorphism.

Remark 8.0.3. The existence of such δ would imply that $\operatorname{im} \delta^2 \subset \operatorname{im} \overline{B\female}^*$, so ultimately one would obtain a map $H^2(BG^\circ, A)^{\pi_0} \to H^3(B\pi_0, A)$ (as in the strategy proposed by Joshi–Spallone) anyway. $\square$

Proposition 8.0.4. $\overline{B\female}^* : H^3(B\pi_0, A) \to H^3(BG/BG^\circ, A)$ *is injective.*

The proof requires the following lemma.

[1] Actually, their proposal involved the classical isomorphism in place of $\alpha^3_{\pi_0,A}$. We noted in the beginning of Chapter 7 that this agrees with $\alpha^3_{\pi_0,A}$.

Lemma 8.0.5. *$\pi_1(\overline{B♀})$ is an isomorphism and $\pi_2(BG/BG^\circ)$ is trivial.*

Proof. By [9, Corollary 6.7.2], $\pi_1(B♀)$ is an isomorphism. Also, BG° is 1-connected (since G° is connected), so van Kampen's Theorem implies that the quotient map $BG \to BG/BG^\circ$ induces an isomorphism on fundamental groups. Hence, $\pi_1(\overline{B♀})$ is an isomorphism.

Again by [9, Corollary 6.7.2], $\pi_2(B\iota)$ is an isomorphism. Hence, $\pi_2(BG, BG^\circ)$ is trivial. The second part of the lemma now follows from **Theorem 2.6.12** with $r = 0$ and $s = 1$. ☐

*Proof of **Proposition 8.0.4**.* Let X be the space obtained by gluing cells of dimension 4 and above to BG/BG° so that $\pi_n(X)$ is trivial for $n > 1$ (this can be done in light of the second part of **Lemma 8.0.5**).[2] Hence, $\overline{B♀}$ extends to a map $\tilde{\overline{B♀}} : X \to B\pi_0$ which induces isomorphism on fundamental groups (by the first part of **Lemma 8.0.5**). Since X and $B\pi_0$ are both $K(\pi_0, 1)$-spaces, $\tilde{\overline{B♀}}$ is in fact a weak homotopy equivalence. In particular,

$$\tilde{\overline{B♀}}^* : H^3(B\pi_0, A) \to H^3(X, A)$$

is an isomorphism. Next, we observe that the map

$$H^3(X, A) \to H^3(BG/BG^\circ, A)$$

induced by $BG/BG^\circ \hookrightarrow X$ is injective, since X was constructed by gluing cells of dimensions 4 and above to BG/BG°. Hence, taking the third cohomology of

$$\begin{tikzcd} X \arrow[dr, "\tilde{\overline{B♀}}"] & \\ BG/BG^\circ \arrow[u, hook] \arrow[r, "\overline{B♀}"'] & B\pi_0 \end{tikzcd}$$

proves the proposition. ☐

To prove that $\alpha_{G,A}$ is an isomorphism, it now suffices to produce a map δ which fits into (8.2), rendering the top row exact and the rightmost square commutative. In the following section, we produce a candidate for δ and show that it makes the top row of (8.2) exact. However, we are unable to prove the commutativity of the square. Nonetheless, the Five Lemma yields the following partial result.

Theorem 8.0.6. *If A is discrete and $H^3_{\mathrm{gp}}(\pi_0(G), A)$ is trivial, then $\alpha_{G,A}$ is an isomorphism.*

[2]This is the standard construction of the first term in the Postnikov tower of BG/BG°.

Remark 8.0.7. The hypothesis of **Theorem 8.0.6** is satisfied if, for instance, A and $\pi_0(G)$ are finite and have coprime orders. $\square$

8.1 A candidate for δ

8.1.1 Loops and paths

For paths $\gamma, \gamma_1, \gamma_2 : I \to G$, write $\gamma_1 \cdot \gamma_2$ for their pointwise product. If $\gamma_1(1) = \gamma_2(0)$, then $\gamma_1 * \gamma_2$ is their concatenation. γ^{-1} is the pointwise inverse of γ and γ^{*-1} is the reverse path $t \mapsto \gamma(1-t)$. For $0 \leq a \leq b \leq 1$, write $\gamma|_{[a,b]}$ for the path

$$t \mapsto \gamma(a + (b-a)t).$$

If γ_1 and γ_2 have the same end-points, write $\gamma_1 \approx \gamma_2$ when there is an end-point fixing homotopy between γ_1 and γ_2.

Recall that if λ is a loop based at 1_G, then

$$\lambda^{-1} \approx \lambda^{*-1}. \tag{8.3}$$

Also,

$$\lambda \cdot \gamma \approx \gamma * (\lambda \cdot \gamma(1)) \approx \gamma \cdot \gamma(1)^{-1} \cdot \lambda \cdot \gamma(1). \tag{8.4}$$

In particular, if $\gamma(1) = 1_G$ then

$$\gamma \cdot \lambda \approx \gamma * \lambda \approx \lambda \cdot \gamma. \tag{8.5}$$

If $\gamma_1(1) = 1_G$, then

$$\gamma_1 \cdot \gamma_2 = (\gamma_1 \cdot \gamma_2(0)) * \gamma_2. \tag{8.6}$$

8.1.2 A motivating construction

The following theorem is a restatement of [9, Theorem 4.9], but we spell out its proof in a way which motivates the upcoming techniques.

Theorem 8.1.1. *If G is connected, then every A-cover $p : E \to G$ with a choice base point in $p^{-1}(1_G)$ has a unique group structure which makes it a central extension of G by A.*

Proof. Write 1_E for the chosen base point of E, and identify A as a subspace of E as $a \mapsto 1_E \cdot a$. Given $e_1, e_2 \in E$, we will define their product $e_1 e_2$ such that

- E is a k-group with this product, and
- the maps p and $A \hookrightarrow E$ are group homomorphisms,
- E a central extension of G by A with these maps, and
- the action of A on E coming from the A-cover structure is the same as that coming from the group structure on E.

Each component of E meets A, so pick $a_1, a_2 \in A$ and paths $\gamma_1, \gamma_2 : I \to E$ with $\gamma_i(0) = a_i$ and $\gamma_i(1) = e_i$ for $i = 1, 2$. Let γ_{12} be the unique lift of $p\gamma_1 \cdot p\gamma_2$ with $\gamma_{12}(0) = a_1 a_2$ (where '$\cdot$' denotes the pointwise product of paths in G) and define $e_1 e_2 := \gamma_{12}(1)$. Standard lifting arguments can be used to check that

- this definition is independent of the choice of a_i and γ_i,
- 1_E is the identity with respect to this product,
- this product is associative and has inverses,
- p and $A \hookrightarrow E$ are group homomorphisms, and
- the two actions of A on E agree.

This product can also be realized as a lift of the composition

$$E \times E \overset{p\times p}{\to} G \times G \to G,$$

where the second map is the product on G. Continuity and uniqueness can now be checked using standard lifting arguments. $\square$

8.1.3 Extending the construction

For $g \in G$, write $\bar{g}$ for the class of g in π_0. We now attempt to apply this procedure when G is not connected; the obstruction in doing so will give the required map δ. Hence, fix an A-bundle $p : E \to G$. It will be convenient to regard E as the pullback of $EA \to BA$ along some map $\phi : G \to BA$. Hence,

$$E = \{(g, x) \in G \times EA \mid \phi(g) = p_A(x)\}$$

and p is the first projection. Let $\tau : \pi_0 \to G$ be a choice of coset representatives, with $\tau(\overline{1_G}) = 1_G$.

A simple necessary condition for E to have a central extension structure is that the homotopy class of ϕ should be fixed under the two-sided translation action of G on the domain, so we assume at the outset that this condition is satisfied. Hence, we may assume, by adjusting ϕ up to homotopy if necessary, that

$$\phi(\tau(\bar{g})g') = \phi(g') \text{ for } \overline{g} \in \pi_0, g' \in G^\circ. \tag{8.7}$$

Also, the homotopy class of ϕ is fixed under the conjugation action of G on itself.

Since G is well-pointed, we may further assume that $\phi(1_G) = 1_{BA}$, and consequently $\phi(\tau(\bar{g})) = 1_{BA}$ for $\bar{g} \in \pi_0$. In other words, the fiber $p^{-1}(\tau(\bar{g}))$ is given by

$$p^{-1}(\tau(\bar{g})) = \{(\tau(\bar{g}), a) \mid a \in A\}.$$

Now, consider the following attempt at mimicking the procedure from the proof of **Theorem 8.1.1**, with $1_E = (1_G, 1_A)$. Let $e_1, e_2 \in E$. Each component of E contains a point of the form $(\tau(\bar{g}), a)$ for some $\bar{g} \in \pi_0$ and $a \in A$, so pick $\overline{g_1}, \overline{g_2} \in \pi_0$, $a_1, a_2 \in A$, and paths $\gamma_1, \gamma_2 : I \to E$ satisfying $\gamma_i(0) = (\tau(\overline{g_i}), a_i)$ and $\gamma_i(1) = e_i$ for $i = 1, 2$.

The next step would be to let γ_{12} be the unique lift of $p\gamma_1 \cdot p\gamma_2$ starting at some cleverly chosen point in the fiber of $\tau(\overline{g_1})\tau(\overline{g_2})$. A naïve choice would be the point $(\tau(\overline{g_1})\tau(\overline{g_2}), a_1a_2)$, which does not work since it need not be a point in E. This can be somewhat remedied as follows. Choose, as part of the the information about G in the set-up, paths $\varepsilon_{\overline{g_1}, \overline{g_2}} : I \to G^\circ$ with

$$\begin{aligned} \varepsilon_{\overline{g_1}, \overline{g_2}}(0) &= (\tau(\overline{g_1})\tau(\overline{g_2}))^{-1} \tau(\overline{g_1 g_2}), \\ \varepsilon_{\overline{g_1}, \overline{g_2}}(1) &= 1_G, \text{ and} \\ \varepsilon_{\overline{g}, \overline{1_G}} = \varepsilon_{\overline{1_G}, \overline{g}} &\equiv 1_G. \end{aligned} \tag{8.8}$$

It will be convenient to view ε as a map $\pi_0^2 \times I \to G^\circ$. Now, let γ_{12} be the unique lift of $p\gamma_1 \cdot p\gamma_2 \cdot \varepsilon_{\overline{g_1}, \overline{g_2}}$ starting at $(\tau(\overline{g_1 g_2}), a_1a_2)$. Define $e_1e_2 := \gamma_{12}(1)$. Once again, one checks using standard lifting arguments that

- this definition is independent of the choice of a_1, a_2 and γ_1, γ_2,

- this product is continuous,

- $(g, x)(1_G, a) = (g, xa) = (1_G, a)(g, x)$ for $(g, x) \in E$ and $a \in A$, and
- $p(e_1 e_2) = p(e_1)p(e_2)$.

This last property is satisfied precisely because of how $\varepsilon_{\overline{g_1}, \overline{g_2}}$ was chosen. However, we refrain from saying that p and $A \hookrightarrow E$ are homomorphisms for now — we do not yet know whether the product on E satisfies the group axioms.

Assuming for the moment that the product thus far defined is associative, it is easy to construct inverses. Hence, this product makes $A \to E \to G$ into a central extension if and only if it is associative. To examine associativity, let $e_1, e_2, e_3 \in E$ and $\overline{g_i}, a_i, \gamma_i$ be as before. Define

$$\begin{aligned}\gamma'_{12,3} &:= (p\gamma_1 \cdot p\gamma_2 \cdot \varepsilon_{\overline{g_1}, \overline{g_2}}) \cdot p\gamma_3 \cdot \varepsilon_{\overline{g_1 g_2}, \overline{g_3}} \text{ and} \\ \gamma'_{1,23} &:= p\gamma_1 \cdot (p\gamma_2 \cdot p\gamma_3 \cdot \varepsilon_{\overline{g_2}, \overline{g_3}}) \cdot \varepsilon_{\overline{g_1}, \overline{g_2 g_3}}.\end{aligned}$$

Observe that $\gamma'_\bullet(0) = 1_G$ and $\gamma'_\bullet(1) = g_1 g_2 g_3$, where $\bullet$ can be substituted by '$12, 3$' or '$1, 23$'. Let $\gamma_{12,3}$ and $\gamma_{1,23}$ be the respective lifts to E starting at $(\tau(\overline{g_1 g_2 g_3}), a_1 a_2 a_3)$. Associativity holds if and only if $\gamma_{12,3}(1) = \gamma_{1,23}(1)$.

Note that $(\gamma'_{12,3})^{-1} \cdot \gamma'_{1,23}$ is a loop at 1_G, so associativity holds if and only if the homotopy class of $(\gamma'_{12,3})^{-1} \cdot \gamma'_{1,23}$ lies in the kernel of $\pi_1(\phi)$. Writing out the definitions, we obtain

$$\left(\gamma'_{12,3}\right)^{-1} \gamma'_{1,23} = (\varepsilon_{\overline{g_1 g_2}, \overline{g_3}})^{-1} \cdot (p\gamma_3)^{-1} \cdot (\varepsilon_{\overline{g_1}, \overline{g_2}})^{-1} \cdot p\gamma_3 \cdot \varepsilon_{\overline{g_2}, \overline{g_3}} \cdot \varepsilon_{\overline{g_1}, \overline{g_2 g_3}}. \tag{8.9}$$

We also have a homotopy of loops

$$t \mapsto (\varepsilon_{\overline{g_1 g_2}, \overline{g_3}})^{-1} \cdot p\gamma_3|_{[0,t]}^{-1} \cdot (\varepsilon_{\overline{g_1}, \overline{g_2}})^{-1} \cdot p\gamma_3|_{[0,t]} \cdot \varepsilon_{\overline{g_2}, \overline{g_3}} \cdot \varepsilon_{\overline{g_1}, \overline{g_2 g_3}}$$

from

$$(\varepsilon_{\overline{g_1 g_2}, \overline{g_3}})^{-1} \cdot \tau(\overline{g_3})^{-1} \cdot (\varepsilon_{\overline{g_1}, \overline{g_2}})^{-1} \cdot \tau(\overline{g_3}) \cdot \varepsilon_{\overline{g_2}, \overline{g_3}} \cdot \varepsilon_{\overline{g_1}, \overline{g_2 g_3}}$$

to (8.9). To summarize,

Proposition 8.1.2. *Let $\tau : G \to \pi_0$ be a choice of coset representatives with $\tau(\overline{1_G}) = 1_G$. Let $\phi : G \to BA$ be a based map satisfying (8.7) whose homotopy class is fixed under the conjugation action of G on itself, and let $E = \phi^* EA$ the corresponding A-cover of G. Given a choice of ε satisfying (8.8), the above-defined product on E makes $A \to E \to G$ a central* *extension if and only if the homotopy class of the loop*

$$\epsilon_{\overline{g_1},\overline{g_2},\overline{g_3}} := \left(\varepsilon_{\overline{g_1 g_2},\,\overline{g_3}}\right)^{-1} \cdot \tau(\overline{g_3})^{-1} \cdot \left(\varepsilon_{\overline{g_1},\,\overline{g_2}}\right)^{-1} \cdot \tau(\overline{g_3}) \cdot \varepsilon_{\overline{g_2},\,\overline{g_3}} \cdot \varepsilon_{\overline{g_1},\,\overline{g_2 g_3}},$$

based at 1_G*, lies in* $\ker(\pi_1(\phi))$ *for* $\overline{g_1}$, $\overline{g_2}$, $\overline{g_3} \in \pi_0$.

Remark 8.1.3. Every central extension of G by A can be obtained in the above way — if ϕ and a product $\times : E \times E \to E$ are given (such that $A \to E \to G$ is a central extension), then ε can be chosen as follows. For $\overline{g_1}, \overline{g_2} \in \pi_0$, let $\tilde{\varepsilon}_{\overline{g_1},\,\overline{g_2}}$ be any path in E from a point in

$$p^{-1}\left((\tau(\overline{g_1})\tau(\overline{g_2}))^{-1}\tau(\overline{g_1 g_2})\right)$$

to a point in $p^{-1}(1_{BA})$. Define $\varepsilon_{\overline{g_1},\,\overline{g_2}} := p\tilde{\varepsilon}_{\overline{g_1},\,\overline{g_2}}$. One checks that this choice of ε satisfies the hypothesis of **Proposition 8.1.2** and $e_1 e_2 = e_1 \times e_2$ for $e_1, e_2 \in E$. $\square$

Remark 8.1.4. When $G = G^\circ \rtimes \pi_0$, we may choose τ to be a group homomorphism and ε to be constant. Hence, **Proposition 8.1.2** yields that ι^* from (8.1) is surjective. Thus, we recover **Theorem 8.0.2**. $\square$

We now give another perspective on **Proposition 8.1.2**. Let τ be as in the proposition and $\bar{\phi} : G^\circ \to BA$ be a based map whose homotopy class is fixed under the conjugation action of G on G°. We can extend $\bar{\phi}$ to $\phi : G \to BA$ as

$$\phi(\tau(\bar{g})g') := \bar{\phi}(g') \text{ for } \bar{g} \in \pi_0, g' \in G^\circ,$$

so that ϕ satisfies the hypothesis of the proposition. Of course, this construction can be reversed by simply restricting ϕ to get $\bar{\phi}$, and this gives an isomorphism

$$[G, BA]^G \approx [G^\circ, BA]^G.$$

Here, $[G, BA]^G$ denotes the fixed points of the two-sided action of G on $[G, BA]$, and $[G^\circ, BA]^G$ denotes the fixed points of the conjugation action of G on $[G^\circ, BA]$. Both of these actions factor through π_0, so we can also write

$$[G, BA]^{\pi_0} \approx [G^\circ, BA]^{\pi_0}. \tag{8.10}$$

Next, let $E = \phi^* EA$ as before and pick arbitrary ε satisfying (8.8). We have a map $f_{\bar{\phi}} : \pi_0^3 \to \pi_1(BA) \approx A$ given by

$$(\overline{g_1}, \overline{g_2}, \overline{g_3}) \mapsto \bar{\phi}_* \left[\epsilon_{\overline{g_1},\overline{g_1},\overline{g_3}}\right],$$

thought of as an inhomogeneous 3-cochain in $C^3_{\mathrm{gp}}(\pi_0, A)$ (with π_0 acting trivially on A). It is natural to ask whether this cochain is a cocycle, and how it depends on the choice of ε. We answer these questions in the next two lemmas.

Lemma 8.1.5. *$f_{\bar\phi}$ depends on the choice of ε only up to a 3-coboundary.*

Proof. Let ε' be another map $\pi_0^2 \times I \to G^\circ$ satisfying (8.8), and let ϵ' and $f'_{\bar\phi}$ be the corresponding analogues of ϵ and $f_{\bar\phi}$ respectively. It is clear that $f_{\bar\phi} = f'_{\bar\phi}$ if $\varepsilon_{\overline{g_1},\overline{g_2}} \approx \varepsilon'_{\overline{g_1},\overline{g_2}}$ for all $\overline{g_1}$, $\overline{g_2} \in \pi_0$. Hence, we may assume that, for all $\overline{g_1}$, $\overline{g_2} \in \pi_0$, the path $\varepsilon'_{\overline{g_1},\overline{g_2}}$ is given by $\varepsilon_{\overline{g_1},\overline{g_2}} * \lambda_{\overline{g_1},\overline{g_2}}$ for some loop $\lambda_{\overline{g_1},\overline{g_2}}$ at 1_G.

Several applications of (8.5) show that there is an end-point fixing homotopy

$$\epsilon'_{\overline{g_1},\overline{g_1},\overline{g_3}} \approx \epsilon_{\overline{g_1},\overline{g_1},\overline{g_3}} * (\lambda_{\overline{g_1 g_2},\overline{g_3}})^{-1} * \left(\tau(\overline{g_3})^{-1} \cdot (\lambda_{\overline{g_1},\overline{g_2}})^{-1} \cdot \tau(\overline{g_3})\right) * \lambda_{\overline{g_2},\overline{g_3}} * \lambda_{\overline{g_1},\overline{g_2 g_3}}$$

for all $(\overline{g_1},\overline{g_2},\overline{g_3}) \in \pi_0^3$. Taking homotopy classes and applying $\bar\phi_*$ on both sides yields the following, where $\pi_1(BA)$ is written additively.

$$\begin{aligned}
f'_{\bar\phi}(\overline{g_1},\overline{g_2},\overline{g_3}) &= f_{\bar\phi}(\overline{g_1},\overline{g_2},\overline{g_3}) + \bar\phi_*\left[(\lambda_{\overline{g_1 g_2},\overline{g_3}})^{-1}\right] + \bar\phi_*\left[\tau(\overline{g_3})^{-1} \cdot (\lambda_{\overline{g_1},\overline{g_2}})^{-1} \cdot \tau(\overline{g_3})\right] \\
&\quad + \bar\phi_*\left[\lambda_{\overline{g_2},\overline{g_3}}\right] + \bar\phi_*\left[\lambda_{\overline{g_1},\overline{g_2 g_3}}\right] \\
&= f_{\bar\phi}(\overline{g_1},\overline{g_2},\overline{g_3}) + \bar\phi_*\left[(\lambda_{\overline{g_1 g_2},\overline{g_3}})^{-1}\right] + \bar\phi_*\left[(\lambda_{\overline{g_1},\overline{g_2}})^{-1}\right] + \bar\phi_*\left[\lambda_{\overline{g_2},\overline{g_3}}\right] + \bar\phi_*\left[\lambda_{\overline{g_1},\overline{g_2 g_3}}\right] \\
&= f_{\bar\phi}(\overline{g_1},\overline{g_2},\overline{g_3}) - \bar\phi_*\left[\lambda_{\overline{g_1 g_2},\overline{g_3}}\right] - \bar\phi_*\left[\lambda_{\overline{g_1},\overline{g_2}}\right] + \bar\phi_*\left[\lambda_{\overline{g_2},\overline{g_3}}\right] + \bar\phi_*\left[\lambda_{\overline{g_1},\overline{g_2 g_3}}\right].
\end{aligned}$$

The second equality follows from the hypothesis on $\bar\phi$, and the third follows from (8.3). The lemma now follows. $\square$

Lemma 8.1.6. *$f_{\bar\phi}$ is a 3-cocycle.*

Proof. The proof is essentially a long calculation which involves going back and forth between products and concatenations of paths in order to get cancellations. First, we establish some notation. 1_G will sometimes be used to denote the constant path at the identity. Fix $\overline{g_i} \in \pi_0$ ($1 \le i \le 4$), and write g_i for $\tau(\overline{g_i})$, g_{ij} for $\tau(\overline{g_i g_j})$, and so on. Write $\varepsilon_{i,j}$ for $\varepsilon_{\overline{g_i},\overline{g_j}}$, likewise $\varepsilon_{ij,k}$ for $\varepsilon_{\overline{g_i g_j},\overline{g_k}}$, and so on. Hence, for instance,

$$\varepsilon_{12,34} := \varepsilon_{\overline{g_1 g_2},\overline{g_3 g_4}}.$$

Similar notation is used for ϵ. For loops λ_1, λ_2 with a common base point, write

$$\lambda_1 \overset{\bar\phi}{\approx} \lambda_2$$

if $\bar{\phi} \circ \lambda_1 \approx \bar{\phi} \circ \lambda_2$. In particular, recall that

$$g\lambda g^{-1} \overset{\bar{\phi}}{\approx} \lambda \tag{8.11}$$

for all $g \in G$ and loops λ based at 1_G, since the homotopy class of $\bar{\phi}$ is fixed under the conjugation action of G. To prove the lemma, we need to show that

$$\epsilon_{2,3,4} * (\epsilon_{12,3,4})^{*-1} * \epsilon_{1,23,4} * (\epsilon_{1,2,34})^{*-1} * \epsilon_{1,2,3} \overset{\bar{\phi}}{\approx} 1_G.$$

By (8.5) and (8.3), it suffices to show that

$$\epsilon_{1,2,3} \cdot (\epsilon_{12,3,4})^{-1} \cdot \epsilon_{1,23,4} \cdot (\epsilon_{1,2,34})^{-1} \cdot \epsilon_{2,3,4} \overset{\bar{\phi}}{\approx} 1_G.$$

We now write out the left hand side, omitting '$\cdot$' and using angular brackets $\langle\rangle$ to enclose products which are loops at 1_G. Terms which are important for the next step are highlighted.

$$\begin{aligned}
&\epsilon_{1,2,3} \, (\epsilon_{12,3,4})^{-1} \, \epsilon_{1,23,4} \, (\epsilon_{1,2,34})^{-1} \, \epsilon_{2,3,4} \\
&= \left\langle (\varepsilon_{12,3})^{-1} \, g_3^{-1} \, (\varepsilon_{1,2})^{-1} \, g_3 \, \varepsilon_{2,3} \, \varepsilon_{1,23} \right\rangle \left\langle (\varepsilon_{12,34})^{-1} \, (\varepsilon_{3,4})^{-1} \, g_4^{-1} \, \varepsilon_{12,3} \, g_4 \, \varepsilon_{123,4} \right\rangle \\
&\quad \left\langle (\varepsilon_{123,4})^{-1} \, g_4^{-1} \, (\varepsilon_{1,23})^{-1} \, g_4 \, \varepsilon_{23,4} \, \varepsilon_{1,234} \right\rangle \left\langle (\varepsilon_{1,234})^{-1} \, (\varepsilon_{2,34})^{-1} \, g_{34}^{-1} \, \varepsilon_{1,2} \, g_{34} \, \varepsilon_{12,34} \right\rangle \\
&\quad \left\langle (\varepsilon_{23,4})^{-1} \, g_4^{-1} \, (\varepsilon_{2,3})^{-1} \, g_4 \, \varepsilon_{3,4} \, \varepsilon_{2,34} \right\rangle \\
&= \left\langle (\varepsilon_{12,3})^{-1} \, g_3^{-1} \, (\varepsilon_{1,2})^{-1} \, g_3 \, \varepsilon_{2,3} \, \varepsilon_{1,23} \right\rangle \left((\varepsilon_{12,34})^{-1} \, (\varepsilon_{3,4})^{-1} \, g_4^{-1} \, \varepsilon_{12,3} \right) \left((\varepsilon_{1,23})^{-1} \, g_4 \, \varepsilon_{23,4} \right) \\
&\quad \left((\varepsilon_{2,34})^{-1} \, g_{34}^{-1} \, \varepsilon_{1,2} \, g_{34} \, \varepsilon_{12,34} \right) \left\langle (\varepsilon_{23,4})^{-1} \, g_4^{-1} \, (\varepsilon_{2,3})^{-1} \, g_4 \, \varepsilon_{3,4} \, \varepsilon_{2,34} \right\rangle
\end{aligned} \tag{8.12}$$

An application of (8.5) allows for more cancellations. We continue from (8.12).

$$\begin{aligned}
&\approx \left\langle (\varepsilon_{12,3})^{-1} \, g_3^{-1} \, (\varepsilon_{1,2})^{-1} \, g_3 \, \varepsilon_{2,3} \, \varepsilon_{1,23} \right\rangle \left((\varepsilon_{12,34})^{-1} \, (\varepsilon_{3,4})^{-1} \, g_4^{-1} \, \varepsilon_{12,3} \right) \left((\varepsilon_{1,23})^{-1} \, g_4 \, \varepsilon_{23,4} \right) \\
&\quad \left\langle (\varepsilon_{23,4})^{-1} \, g_4^{-1} \, (\varepsilon_{2,3})^{-1} \, g_4 \, \varepsilon_{3,4} \, \varepsilon_{2,34} \right\rangle \left((\varepsilon_{2,34})^{-1} \, g_{34}^{-1} \, \varepsilon_{1,2} \, g_{34} \, \varepsilon_{12,34} \right) \\
&= \left\langle (\varepsilon_{12,3})^{-1} \, g_3^{-1} \, (\varepsilon_{1,2})^{-1} \, g_3 \, \varepsilon_{2,3} \, \varepsilon_{1,23} \right\rangle \left((\varepsilon_{12,34})^{-1} \, (\varepsilon_{3,4})^{-1} \, g_4^{-1} \, \varepsilon_{12,3} \right) (\varepsilon_{1,23})^{-1} \\
&\quad \left((\varepsilon_{2,3})^{-1} \, g_4 \, \varepsilon_{3,4} \right) \left(g_{34}^{-1} \, \varepsilon_{1,2} \, g_{34} \, \varepsilon_{12,34} \right)
\end{aligned} \tag{8.13}$$

Interchanging the highlighted factors would also lead to several cancellations, but this cannot be done since (8.5) is not applicable. Hence, we instead use (8.4) and (8.11) to continue

(8.13):

$$\begin{aligned}
&\approx \Big(\big((\varepsilon_{12,34})^{-1}\,(\varepsilon_{3,4})^{-1}\,g_4^{-1}\,\varepsilon_{12,3}\big) * \big((\varepsilon_{12,3})^{-1}\,g_3^{-1}\,(\varepsilon_{1,2})^{-1}\,g_3\,\varepsilon_{2,3}\,\varepsilon_{1,23}\,g_4^{-1}\big)\Big)\\
&\quad (\varepsilon_{1,23})^{-1}\,\big((\varepsilon_{2,3})^{-1}\,g_4\,\varepsilon_{3,4}\big)\,\big(g_{34}^{-1}\,\varepsilon_{1,2}\,g_{34}\,\varepsilon_{12,34}\big)\ \text{(by (8.4))}\\
&\overset{\bar\phi}{\approx} \boxed{g_4}\,\Big(\big((\varepsilon_{12,34})^{-1}\,(\varepsilon_{3,4})^{-1}\,g_4^{-1}\,\varepsilon_{12,3}\big) * \big((\varepsilon_{12,3})^{-1}\,g_3^{-1}\,(\varepsilon_{1,2})^{-1}\,g_3\,\varepsilon_{2,3}\,\varepsilon_{1,23}\,g_4^{-1}\big)\Big)\\
&\quad (\varepsilon_{1,23})^{-1}\,\big((\varepsilon_{2,3})^{-1}\,g_4\,\varepsilon_{3,4}\big)\,\big(g_{34}^{-1}\,\varepsilon_{1,2}\,g_{34}\,\varepsilon_{12,34}\big)\,\boxed{g_4^{-1}}\ \text{(by (8.11))}\\
&= \Big(\big(g_4\,(\varepsilon_{12,34})^{-1}\,(\varepsilon_{3,4})^{-1}\,g_4^{-1}\,\varepsilon_{12,3}\big) * \big\langle g_4\,(\varepsilon_{12,3})^{-1}\,g_3^{-1}\,(\varepsilon_{1,2})^{-1}\,g_3\,\varepsilon_{2,3}\,\varepsilon_{1,23}\,g_4^{-1}\big\rangle\Big)\\
&\quad \boxed{(\varepsilon_{1,23})^{-1}\,\big((\varepsilon_{2,3})^{-1}\,g_4\,\varepsilon_{3,4}\big)\,\big(g_{34}^{-1}\,\varepsilon_{1,2}\,g_{34}\,\varepsilon_{12,34}\big)\,g_4^{-1}} \qquad (8.14)
\end{aligned}$$

At 0, the highlighted path evaluates to $g := g_{123}^{-1}\,g_{1234}\,g_4^{-1} \in G^\circ$. Hence, we can continue (8.14) as follows using (8.6):

$$\begin{aligned}
&\approx \big(g_4\,(\varepsilon_{12,34})^{-1}\,(\varepsilon_{3,4})^{-1}\,g_4^{-1}\,\varepsilon_{12,3}\,g\big) * \boxed{\big(g_4\,(\varepsilon_{12,3})^{-1}\,g_3^{-1}\,(\varepsilon_{1,2})^{-1}\,g_3\,\varepsilon_{2,3}\,\varepsilon_{1,23}\,g_4^{-1}\,g\big)} *\\
&\quad \big((\varepsilon_{1,23})^{-1}\,\big((\varepsilon_{2,3})^{-1}\,g_4\,\varepsilon_{3,4}\big)\,\big(g_{34}^{-1}\,\varepsilon_{1,2}\,g_{34}\,\varepsilon_{12,34}\big)\,g_4^{-1}\big)\\
&\overset{\bar\phi}{\approx} \Big(g_4\,(\varepsilon_{12,34})^{-1}\,(\varepsilon_{3,4})^{-1}\,g_4^{-1}\,\varepsilon_{12,3}\,\boxed{g}\Big) * \Big((\varepsilon_{12,3})^{-1}\,g_3^{-1}\,(\varepsilon_{1,2})^{-1}\,g_3\,\varepsilon_{2,3}\,\varepsilon_{1,23}\,\boxed{g}\Big)\,\boxed{*}\\
&\quad \big((\varepsilon_{1,23})^{-1}\,\big((\varepsilon_{2,3})^{-1}\,g_4\,\varepsilon_{3,4}\big)\,\big(g_{34}^{-1}\,\varepsilon_{1,2}\,g_{34}\,\varepsilon_{12,34}\big)\,g_4^{-1}\big)\ \text{(by (8.11))}\\
&\approx \boxed{\Big(\big(g_4\,(\varepsilon_{12,34})^{-1}\,(\varepsilon_{3,4})^{-1}\,g_4^{-1}\,\varepsilon_{12,3}\big) * \big\langle(\varepsilon_{12,3})^{-1}\,g_3^{-1}\,(\varepsilon_{1,2})^{-1}\,g_3\,\varepsilon_{2,3}\,\varepsilon_{1,23}\big\rangle\Big)}\\
&\quad (\varepsilon_{1,23})^{-1}\,\big((\varepsilon_{2,3})^{-1}\,g_4\,\varepsilon_{3,4}\big)\,\big(g_{34}^{-1}\,\varepsilon_{1,2}\,g_{34}\,\varepsilon_{12,34}\big)\,g_4^{-1}\ \text{(using (8.6))} \qquad (8.15)
\end{aligned}$$

(8.5) now applies on the highlighted concatenation, so (8.15) can be continued as

$$\begin{aligned}
&\approx \Big(g_4\,(\varepsilon_{12,34})^{-1}\,(\varepsilon_{3,4})^{-1}\,g_4^{-1}\,\boxed{\varepsilon_{12,3}}\Big)\,\Big\langle\boxed{(\varepsilon_{12,3})^{-1}}\,g_3^{-1}\,(\varepsilon_{1,2})^{-1}\,g_3\,\boxed{\varepsilon_{2,3}\,\varepsilon_{1,23}}\Big\rangle\\
&\quad \boxed{(\varepsilon_{1,23})^{-1}}\,\Big(\boxed{(\varepsilon_{2,3})^{-1}}\,g_4\,\varepsilon_{3,4}\Big)\,\big(g_{34}^{-1}\,\varepsilon_{1,2}\,g_{34}\,\varepsilon_{12,34}\big)\,g_4^{-1}\\
&= \big(g_4\,(\varepsilon_{12,34})^{-1}\,(\varepsilon_{3,4})^{-1}\,g_4^{-1}\big)\,\big(g_3^{-1}\,(\varepsilon_{1,2})^{-1}\,g_3\big)\,\big(g_4\,\varepsilon_{3,4}\big)\,\big(g_{34}^{-1}\,\varepsilon_{1,2}\,g_{34}\,\varepsilon_{12,34}\big)\,g_4^{-1}\\
&= g_4\,(\varepsilon_{12,34})^{-1}\,\big\langle(\varepsilon_{3,4})^{-1}\,g_4^{-1}\,g_3^{-1}\,(\varepsilon_{1,2})^{-1}\,g_3\,g_4\,\varepsilon_{3,4}\,g_{34}^{-1}\,\varepsilon_{1,2}\,g_{34}\big\rangle\,\varepsilon_{12,34}\,g_4^{-1} \qquad (8.16)
\end{aligned}$$

The last equality simply involves some re-bracketing to highlight that a new loop, based at 1_G, has been obtained (it is left to the reader to verify that it is indeed a loop at 1_G). The reader should also verify that the following is a homotopy of loops based at 1_G:

$$t \mapsto (\varepsilon_{3,4})^{-1}\,g_4^{-1}\,g_3^{-1}\,\varepsilon_{1,2}\big|_{[t,1]}^{-1}\,g_3\,g_4\,\varepsilon_{3,4}\,g_{34}^{-1}\,\varepsilon_{1,2}\big|_{[t,1]}\,g_{34}.$$

Hence, (8.16) may be continued as

$$\begin{aligned} &\approx g_4\,(\varepsilon_{12,34})^{-1}\left\langle(\varepsilon_{3,4})^{-1}\,g_4^{-1}\,g_3^{-1}\,g_3\,g_4\,\varepsilon_{3,4}\,g_{34}^{-1}\,g_{34}\right\rangle\varepsilon_{12,34}\,g_4 \\ &= 1_G. \end{aligned}$$

□

Putting the last two lemmas together, we see that $\bar{\phi} \to f_{\bar{\phi}}$ induces a well-defined map

$$[G^\circ, BA]^{\pi_0} \to H^3_{\mathrm{gp}}(\pi_0, A).$$

Composing with the isomorphism $\mathbb{E}(G^\circ, A)^{\pi_0} \approx [G^\circ, BA]^{\pi_0}$ (which comes from **Theorem 8.1.1**) yields the desired map $\delta : \mathbb{E}(G^\circ, A)^{\pi_0} \to H^3_{\mathrm{gp}}(\pi_0, A)$. In terms of this map, **Proposition 8.1.2** and Remark 8.1.3 can be restated as

Theorem 8.1.7. *The sequence*

$$\mathbb{E}(G, A) \xrightarrow{\iota^*} \mathbb{E}(G^\circ, A)^{\pi_0} \xrightarrow{\delta} H^3_{\mathrm{gp}}(\pi_0, A)$$

is exact.

Theorem 8.0.6 now follows.

Chapter 9

Miscellaneous

9.1 Homotopy and singular (co)homology groups of good pCW complexes

As a general principle, the n-skeleton of a CW complex determines all of its elementary homotopy invariants (homotopy, singular homology, and singular cohomology groups) with dimension at most $n-1$. We will now prove analogous results for good pCW complexes. Throughout this section, X is a good pCW complex (with pCW structure as in Section 2.6) and A is discrete.

Lemma 9.1.1. *Every compact subset of X meets only finitely many sets of the form $X_m - X_{m-1}$.*

Proof. Suppose $K \subset X$ with $K \cap (X_{m_i} - X_{m_i - 1}) \neq \emptyset$ for $m_1 < m_2 < \cdots$. Pick $x_i \in K \cap (X_{m_i} - X_{m_i-1})$, and let $U_i = X - (\{x_1, x_2, \cdots\} - \{x_i\})$. For all i and m, $U_i \cap X_m$ is the complement of finitely many points in X_m. Hence, U_i is open in X (here we used that pCW complexes are Hausdorff). The open cover $\{U_1, U_2, \cdots\}$ of K has no finite subcover, so K is not compact. $\square$

Corollary 9.1.2. *The inclusions $X_m \hookrightarrow X$ induce isomorphisms*

$$\varinjlim_m H_n(X_m, A) \xrightarrow{\sim} H_n(X, A) \text{ and}$$
$$\varinjlim_m \pi_n(X_m) \xrightarrow{\sim} \pi_n(X).$$

Proof. The corollary essentially follows from **Lemma 9.1.1**, since the images of all maps $\Delta^n \to X$ and $S^n \to X$ are compact. $\square$

Lemma 9.1.3. *Suppose X is a good pCW complex. For $i \geq M(n)$, the maps*

$$H_j(X_i, A) \to H_j(X, A) \text{ and}$$
$$\pi_j(X_i) \to \pi_j(X),$$

induced by $X_i \hookrightarrow X$, are isomorphisms for $j < n$ and surjections for $j = n$. Likewise, for $i \geq M(n)$,

$$H^j(X, A) \to H^j(X_i, A)$$

is an isomorphism for $j < n$ and an injection for $j = n$.

Proof. It suffices to prove the claim regarding the homology and homotopy groups, since the claim regarding cohomology groups would follow using the Universal Coefficients Theorem. By **Corollary 9.1.2**, it suffices to show that, for $i \geq M(n)$,

$$H_j(X_i, A) \to H_j(X_{i+1}, A) \text{ and}$$
$$\pi_j(X_i) \to \pi_j(X_{i+1})$$

are isomorphisms for $j < n$ and surjections for $j = n$. Since $Y_{i+1}/Z_{i+1} \cong X_{i+1}/X_i$, the space X_{i+1} ($i \geq M(n)$) is obtained by attaching cells of dimension $n+1$ and above to X_i. The claim now follows using standard cellularity results. $\square$

Corollary 9.1.4. *For $i \geq n$, the maps*

$$H_j(B_iG, A) \to H_j(BG, A) \text{ and}$$
$$\pi_j(B_iG) \to \pi_j(BG),$$

induced by $B_iG \hookrightarrow BG$, are isomorphisms for $j < n$ and surjections for $j = n$. Likewise, for $i \geq n$,

$$H^j(BG, A) \to H^j(B_iG, A)$$

is an isomorphism for $j < n$ and an injection for $j = n$.

Proof. Use **Proposition 2.6.4** and **Lemma 9.1.3**. $\square$

Corollary 9.1.5. *E_nG is $(n-1)$-connected. Consequently, the restriction $p_G : E_nG \to B_nG$ is a universal G-bundle with respect to CW complexes of dimension at most $n-1$.*

Proof. E_nG is clearly connected, so it remains to show that $\pi_i(E_nG) = 0$ for $1 \leq i \leq n$. For this we use the diagram

$$\begin{array}{ccccccccccc}
\pi_n(B_nG) & \longrightarrow & \pi_{n-1}(G) & \longrightarrow & \pi_{n-1}(E_nG) & \longrightarrow & \pi_{n-1}(B_nG) & \longrightarrow & \dots & \longrightarrow & \pi_0(G) \\
\downarrow & & \| & & \downarrow & & \downarrow & & & & \| \\
\pi_n(BG) & \longrightarrow & \pi_{n-1}(G) & \longrightarrow & \pi_{n-1}(EG) & \longrightarrow & \pi_{n-1}(BG) & \longrightarrow & \dots & \longrightarrow & \pi_0(G)
\end{array},$$

which commutes and has exact rows. By **Corollary 9.1.4**, $\pi_i(B_nG) \to \pi_i(BG)$ is an isomorphism for $1 \leq i < n$ and surjection for $i = n$. Also, EG is contractible so $\pi_i(EG) = 0$ for $1 \leq i \leq n$. Hence, the Five Lemma proves the corollary. □

9.2 Classifying spaces and suspensions

Commutativity of (5.1) shows that the suspension map

$$\Sigma : [X, G]_* \to [\Sigma X, \Sigma G]_*$$

is injective, for X a based pCW complex. This reasoning works even when G is an arbitrary k-group (not necessarily a CW group). This immediately provides a necessary condition for a space K to have the based homotopy type of a k-group:

Proposition 9.2.1. *If a based space K has the based homotopy type of a k-group, then the suspension map*

$$\Sigma : [X, K]_* \to [\Sigma X, \Sigma K]_*$$

is injective for all pCW complexes X.

The classical necessary condition that $\pi_1(K)$ must be abelian follows from the special case $X = S^1$ of the above. Also, taking $G = S^3$ (the group of unit quaternions) and $X = S^n$ shows that the suspension map

$$\pi_n(S^3) \to \pi_{n+1}(S^4)$$

is injective for all $n \geq 1$.

9.3 Some interesting k-rings

In this section, all abelian groups are written additively (including groups of the form BA' for A' an abelian k-group).

A *k-ring* is a CG space R with a ring structure, so that addition makes it an abelian k-group and multiplication makes it a k-monoid (in particular R must have a multiplicative identity). The most ubiquitous examples of non-discrete k-rings are the matrix rings $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ and their subrings. However, being contractible, these are somewhat trivial from a homotopy-theoretic perspective. In this section, we exhibit some k-rings which do not have weakly contractible components.

Remark 9.3.1. If R is a k-ring whose additive and multiplicative identities lie in the same component (for instance, if R is connected), then R must be contractible. Indeed, a path $\gamma : I \to R$ with $\gamma(0) = 0_R$ and $\gamma(1) = 1_R$ yields a null-homotopy $t \mapsto (r \mapsto r \cdot \gamma(t))$. In fact, this shows that the underlying additive k-group of R is contractible through group homomorphisms. □

Let $C_m = \mathbb{Z}/m\mathbb{Z}$, and consider the ring $R = \mathrm{End}(BC_m)$ of continuous group endomorphisms of BC_m. R is a subset of $\mathrm{Map}(BC_m, BC_m)$, so it can be topologized as a subspace. This renders addition and composition of maps continuous. Hence, R is a k-ring if it is well-pointed. We show something much stronger:

Proposition 9.3.2. *Composition with $S^1 \hookrightarrow BC_m; t \mapsto (1, t)$ defines a homeomorphism*

$$\xi_m : R \to \Omega BC_m.$$

In particular, R is well-pointed, and hence a k-ring.

Proof. $\{1\} \times I$ generates BC_m, so it is clear that ξ_m is injective. To check surjectivity, we will show that, given a based loop $\gamma \in \Omega BC_m$, the map

$$BC_m \to BC_m; (g_1, t_1) + \ldots + (g_n, t_n) \mapsto g_1\gamma(t_1) + \ldots + g_n\gamma(t_n) \tag{9.1}$$

is continuous. Here, gx denotes $x + x + \ldots + x$ (g times) for $g \in C_m$ and $x \in BC_m$. First, observe that the image of γ is contained in B_NC_m for some $N \in \mathbb{N}$ (since S^1 is compact). Hence, (9.1) takes B_nC_m to $B_{nN}C_m$ for all $n \in \mathbb{N}$. Continuity of the restriction of (9.1) to B_nC_m is now easy to show, so (9.1) is continuous. Hence, ξ_m is a continuous bijection.

Now, we check the continuity of ξ_m^{-1}. It is enough to prove continuity when R is given the compact-open topology (instead of the k-ification of the compact-open topology). Hence, consider a compact set $K \subset BC_m$ and an open set $U \subset BC_m$. Let $S(K,U) = \{f \in R \mid f(K) \subset U\}$. We wish to show that $\xi_m(S(K,U))$ is open in ΩBC_m. Fix $f_0 \in S(K,U)$, and we will produce a neighborhood of $\xi_m(f_0)$ which is contained in $\xi_m(S(K,U))$.

Let $N_K \in \mathbb{N}$ so that $K \subset B_{N_K}C_m$. Let $V \subset BC_m$ be a neighborhood of the identity so that $f_0(V+K) = \{f_0(x+y) \mid x \in V, y \in K\} \subset U$ (here, we use compactness of K and the fact that BC_m is a τ-group (by Remark 2.1.5)). Let $W \subset BC_m$ be a neighborhood of the identity so that $mN_KW = W + W + \ldots + W$ (mN_K times) is contained in V (once again, we have used that BC_m is a τ-group). Hence, the neighborhood

$$\{\gamma \in \Omega BC_m \mid \gamma(t) - f_0(1,t) \in W\} \subset \Omega BC_m$$

of $\xi_m(f_0)$ is contained in $\xi_m(S(K,U))$. □

Clearly, ξ_m is also a homomorphism of additive groups. Hence, transfer of structure yields a multiplication on ΩBC_m which makes it a k-ring! However, this is still not very interesting from a homotopy-theoretic perspective — ΩBC_m has m components, all of which are weakly contractible (see **Lemma 2.8.1**). Nonetheless, this is a step in the right direction. The proof of **Proposition 9.3.2** generalizes easily to prove

Proposition 9.3.3. *Suppose $m_1, m_2 \in \mathbb{N}$ such that m_2 divides m_1. Composition with $I^{n_1}/\partial I^{n_1} \hookrightarrow B^{n_1}C_{m_1}; x \mapsto (1,x)$ yields a homeomorphism*

$$\mathrm{Hom}(B^{n_1}C_{m_1}, B^{n_2}C_{m_2}) \to \Omega^{n_1}B^{n_2}C_{m_2}.$$

Remark 9.3.4. The condition of m_2 dividing m_1 is needed so that the analogue of (9.1) is well-defined. □

Now, consider the k-ring

$$\begin{aligned}\mathrm{End}(BC_m \times B^2C_m) &\cong \mathrm{End}(BC_m) \times \mathrm{Hom}(BC_m, B^2C_m) \times \mathrm{Hom}(B^2C_m, BC_m) \times \mathrm{End}(B^2C_m)\\ &\cong \Omega BC_m \times \Omega B^2C_m \times \Omega^2 BC_m \times \Omega^2 B^2 C_m \text{ (by \textbf{Proposition 9.3.3})}.\end{aligned}$$

By **Lemma 2.8.1**, this has the weak homotopy type of $C_m \times BC_m \times C_m$. In particular, its components are not weakly contractible. Similar examples include

$$\mathrm{End}\left(B^{n_1}C_m \times B^{n_2}C_m \times \ldots \times B^{n_\ell}C_m\right).$$

Chapter 10

Further questions

10.1 Conjecture 6.3.1 and its equivalents

The importance of **Conjecture 6.3.1** for the study of continuous cohomology and central extensions is apparent through our work. It first came up in Chapter 6, where we saw that the study of $\ker \alpha_n$ is intimately linked with the conjecture. In Chapter 7, we gave a complete characterization of $\ker \alpha^n$ (in particular, $\ker \alpha$) by assuming **Conjecture 6.3.1**.

In this section, we wish to convince the reader of the importance of this conjecture in the broader context of homotopy theory, and provide a perspective that might aid an eventual proof. This calls for the conjecture to be placed in a framework that is interesting from a homotopy-theoretic perspective, i.e., all the objects in the conjecture must be defined using ideas that are ubiquitous in homotopy theory.

For this, we turn to the equivalent formulation **Conjecture 6.4.3**. The objects of interest are singular cohomology, the classifying space BG, and the Milgram–Steenrod filtration $B_1G \subset B_2G \subset \cdots$. Singular cohomology needs no introduction, and BG can be understood as either a classifying space for G-bundles or a delooping of G (see **Lemma 2.8.1**). What is so interesting, from a homotopy-theoretic perspective, about the Milgram–Steenrod filtration? An answer is provided by Stasheff [13, 14] through the framework of A_n-spaces, whose implications in our context can be summarized as saying that the cohomology of B_nG carries information regarding A_n-maps out of G (viewed as an A_n-space in a trivial way).

To make use of this framework, however, we must show that the Milgram–Steenrod filtration is the same as that of Milnor (up to homotopy). This is needed since Milnor's construction is older and more popular than that of Milgram–Steenrod, and the literature generally uses the prior when stating results.[1]

10.1.1 Revisiting Milnor versus Milgram–Steenrod

In Section 2.3, we produced a homotopy equivalence $\bar{\Psi} : \bar{B}G \to BG$ in the case of G discrete. In fact, the same construction also works in the general case of G a CW group. Furthermore, the restriction $\bar{\Psi}_n := \bar{\Psi}|_{\bar{B}_n G} : \bar{B}_n G \to B_n G$ is also a homotopy equivalence, where $\bar{B}_n G$ is the image of $\bar{E}_n G$ (the $(n+1)$-fold join of G) in $\bar{B}G$.

Theorem 10.1.1. *$\bar{\Psi}_n$ and $\bar{\Psi}$ are homotopy equivalences for G a CW group.*

In particular, **Conjecture 6.4.3** is equivalent to

Conjecture 10.1.2. *For A a discrete abelian group, the restriction maps $H^d(\bar{B}_{n-1}G, A) \to H^d(\bar{B}_{n-2}G, A)$ and $H^d(\bar{B}G, A) \to H^d(\bar{B}_{n-2}G, A)$ have the same image.*

Sketch of proof of ***Theorem 10.1.1****.* First, one observes, in similar fashion as (6.1), that

$$E_n G / D_n G \cong \Sigma^n G^{\wedge(n+1)}.$$

Since $D_n G$ is contractible and $D_n G \hookrightarrow E_n G$ is a cofibration, we thus obtain a homotopy equivalence

$$E_n G \to \Sigma^n G^{\wedge(n+1)}. \tag{10.1}$$

Next, for $1 \le i \le n$, let $X_i^n \subset \bar{E}_n G$ be the subspace consisting of points

$$[g_0, s_0, \cdots, g_n, s_n]$$

with $g_{i-1} = g_i$ (here, we used notation from Section 2.3). Let $X_{n+1}^n \subset \bar{E}_n G$ be the subspace consisting of points

$$[g_0, s_0, \cdots, g_n, s_n]$$

with $g_n = 1_G$. It is easy to see that, for each $S \subset [n+1]$, the space

$$\bigcap_{i \in S} X_i^n$$

[1] In fact, the paper [10] which first developed the Milgram–Steenrod construction came four years after [13, 14].

is contractible. Hence,

$$X^n := \bigcup_{i \in [n+1]} X_i^n$$

is weakly contractible. Now,

- $\bar{E}_n G / X^n \cong \Sigma^n G^{\wedge(n+1)}$ and
- $(\bar{E}_n G, X^n)$ is a pCW pair,

so we obtain a homotopy equivalence

$$\bar{E}_n G \to \Sigma^n G^{\wedge(n+1)} \tag{10.2}$$

(by **Theorem 2.6.8**). (10.1) and (10.2) fit into the commuting triangle

$$\begin{tikzcd} \bar{E}_n G \arrow[r, "\Psi_n"] \arrow[d] & E_n G \arrow[dl] \\ \Sigma^n G^{\wedge(n+1)} & \end{tikzcd},$$

so Ψ_n is a homotopy equivalence. By the Five Lemma and the long exact sequences of homotopy groups for the bundles $E_n G \to B_n G$ and $\bar{E}_n G \to \bar{B}_n G$, we now see that $\bar{\Psi}_n$ is a weak homotopy equivalence, and hence a homotopy equivalence (by **Theorem 2.6.8**, since $B_n G$ and $\bar{B}_n G$ are pCW complexes).

$\bar{B}G$ is a good pCW complex with n-skeleton $\bar{B}_n G$, so $\bar{\Psi}$ is also a weak homotopy equivalence (see **Corollary 9.1.2**), and hence a homotopy equivalence. □

10.2 The images of α and α^n

We studied the image of α only in the case of A discrete, in which case the following conjecture implies that α is an isomorphism.

Conjecture 10.2.1. *The rightmost square in* (8.2) *commutes.*

Next, we consider $\operatorname{im} \alpha^n$. It is clear that

$$\operatorname{im} \alpha^n \subset \operatorname{im} \left(H^n(BG/B_{n-1}G, A) \to H^n(BG, A) \right)$$

(by definition of $\alpha_n = \iota_n^* \circ \alpha^n$). When G and A are discrete, this is an equality. Hence, one might conjecture that this is also an equality in general. Here is an example to show that this does not hold. Let $G = \mathbb{Z}/2\mathbb{Z}$, $A_1 = B\mathbb{Z}$, and $A_2 = S^1$. By **Example 7.2.4**, $\alpha^n_{G,A_1} = 0$. Also, $\alpha^n_{G,A_2} \neq 0$ for odd n by **Example 7.5.3**. However, A_1 and A_2 are both $K(\mathbb{Z}, 1)$-spaces, and hence weakly homotopy equivalent.[2] In particular,

$$\operatorname{im}\left(H^n(BG/B_{n-1}G, A_1) \to H^n(BG, A_1)\right) \text{ and}$$
$$\operatorname{im}\left(H^n(BG/B_{n-1}G, A_2) \to H^n(BG, A_2)\right)$$

are isomorphic but $\operatorname{im}\alpha^n_{G,A_1}$ and $\operatorname{im}\alpha^n_{G,A_2}$ are not. In fact, this example shows that $\operatorname{im}\alpha^n_{G,A}$ depends on something more than just the homotopy-theoretic data about BG, its filtration $B_1G \subset B_2G \subset \cdots$, and the Ω-spectrum $\mathbb{A}$. The same is true for $\operatorname{im}\alpha = \alpha(\mathbb{E}(G, A))$, since $\mathbb{E}(G, A_i) = H^2_{\mathrm{c}}(G, A_i)$ $(i = 1, 2)$ in the preceding example.

10.3 α^n and Yoneda extensions

The story of α started with central extensions, and we later narrowed our attention to second continuous cohomology (which corresponds with 'topologically trivial' extensions) since our main goal was to understand $\ker\alpha$. Conversely, one could view α as an 'extension-based' definition of α^2 which reduces to the original definition of α^2 when only 'topologically trivial' extensions are considered. This begs the question:

Question 10.3.1. Is there an 'extension-based' definition of α^n which reduces to our definition when only 'topologically trivial' extensions are considered?

It is well-known (see, for instance, [19, Vista 3.4.6]) that for discrete groups G and A, group cohomology $H^n_{\mathrm{gp}}(G, A)$ is isomorphic to the group (under Baer sums) of Yoneda extensions of length n modulo equivalences. In this context, a Yoneda extension of length n is an exact sequence

$$\mathcal{E} : 0 \longrightarrow E_0 = A \xrightarrow{f_0} E_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} E_n \xrightarrow{f_n} E_{n+1} = \mathbb{Z} \longrightarrow 0$$

in the category of G-modules (with G acting trivially on A and $\mathbb{Z}$). For G a CW group and A an abelian k-group, [15, §3] describes how the above can be generalized to give a

[2] Something much stronger is true. $(1, t) \mapsto e^{2\pi i t}$ generates a continuous homomorphism $f : A_1 \to A_2$ which is also a weak homotopy equivalence. Since A_1 and A_2 are CW complexes, f is a homotopy equivalence. Furthermore, $B^n f : B^n A_1 \to B^n A_2$ is also a group homomorphism and a homotopy equivalence by the same reasoning. In particular, the homotopy equivalence between $B^n A_1$ and $B^n A_2$ captures some group-theoretic data, and yet $\operatorname{im}\alpha^n_{G,A_1} \neq \operatorname{im}\alpha^n_{G,A_2}$ (cf. Section 7.6).

correspondence between continuous cohomology $H^n_{\mathrm{c}}(G,A)$ and Yoneda extensions of length n which are 'topologically trivial'. To make this precise, we need some definitions.

For G a CW group, a G-module is an abelian k-group E with a continuous G-action $G \times E \to E$ through group automorphisms. Morphisms of G-modules are continuous and G-equivariant group homomorphisms. For A a fixed abelian k-group, a Yoneda extension of length n is an exact sequence

$$\mathcal{E} : 0 \longrightarrow E_0 = A \xrightarrow{f_0} E_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} E_n \xrightarrow{f_n} E_{n+1} = \mathbb{Z} \longrightarrow 0$$

in the category of G-modules (with G acting trivially on A and $\mathbb{Z}$) such that, for $1 \le i \le n$,

- $\mathrm{im}\, f_{i-1} = \ker f_i$ is a k-group,
- the map $\bar{f}_i : \mathrm{coker} f_{i-1} \to \mathrm{im}\, f_i$ induced by f_i is a homeomorphism, and
- the projection $E_i \to \mathrm{coker} f_{i-1}$ is a numerable $(\mathrm{im}\, f_{i-1})$-bundle.

Equivalence of Yoneda extensions is defined in the same way as in the discrete case, and $\mathbb{YE}^n(G,A)$ is the group (under Baer sums) of equivalence classes of length n Yoneda extensions. The Yoneda extension $\mathcal{E}$ is said to be topologically trivial if the bundles $E_i \to \mathrm{coker} f_{i-1}$ are trivial, and $\mathbb{YE}^n_\bullet(G,A)$ is the group of equivalence classes of topologically trivial length n Yoneda extensions.

The precise statement of [15, §3] (alluded to previously) is that there is a natural isomorphism

$$\mathbb{YE}^n_\bullet(G,A) \approx H^n_{\mathrm{c}}(G,A).$$

Hence, we may state **Question 10.3.1** more precisely as

Question 10.3.2. Does there exist a natural map $\beta^n : \mathbb{YE}^n(G,A) \to H^n(BG,A)$ whose restriction to $\mathbb{YE}^n_\bullet(G,A) \approx H^n_{\mathrm{c}}(G,A)$ is α^n?

Remark 10.3.3. [20, Theorem 4] gives an isomorphism between $\mathbb{YE}^*(G,A)$ and the sheaf cohomology of BG when A is discrete and G is finite dimensional and has countably many cells (i.e., G is a Lie group). □

In the $n = 2$ case, it would be nice to have agreement with α:

Question 10.3.4. Is there a natural isomorphism $\mathbb{YE}^2(G,A) \approx \mathbb{E}(G,A)$?

Question 10.3.5. If the answers to both the preceding questions are affirmative, then does the map $\mathbb{E}(G,A) \approx \mathbb{YE}^2(G,A) \to H^2(BG,A)$ agree with α?

Since we have already studied $\ker \alpha^n$, it would be nice if $\ker \alpha^n = \ker \beta^n$:

Question 10.3.6. If the answer to **Question 10.3.2** is affirmative, is there an analogue of **Theorem 3.1.1** for β^n? In particular, do we have $\ker \beta^n = \ker \alpha^n$ (with appropriate identifications)?

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