Aspects of field theories in the light-cone formalism

A thesis

submitted to the
Indian Institute of Science Education and Research, Pune
in partial fulfilment of the requirements for the
BS-MS Dual Degree Programme

by

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Certificate

This is to certify that this dissertation entitled "Aspects of field theories in the light-cone formalism" towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents work carried out by Nabha Shah at IISER Pune under the supervision of Dr. Sudarshan Ananth, Associate Professor, Physics during the academic year 2017-2018.

Nabha Shah

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Declaration

I hereby declare that the matter embodied in the report entitled "Aspects of field theories in the light-cone formalism" are the results of the work carried out by me at the Indian Institute of Science Education and Research, Pune under the supervision of Dr. Sudarshan Ananth and the same has not been submitted elsewhere for any other degree.

Nabha Shah

Dr. Sudarshan Ananth

Acknowledgements

I would like to thank my supervisor, Dr. Sudarshan Ananth, for all the guidance and support he has extended to me. His door has always been open, whether it comes to discussing my project or otherwise, and working on this thesis has been a wonderful discovery of the excitement of doing physics research. I am grateful to our collaborator, Prof. Lars Brink, and my TAC member, Prof. Anil D. Gangal, for their insights, help, and suggestions. I would also like to thank everyone involved in these projects, Aditya Kar, Sucheta Majumdar, and Mahendra Mali, without whom none of this work would exist.

I would like to acknowledge the resources and financial support provided by IISER Pune and the Department of Science and Technology through the DST - INSPIRE fellowship.

Lastly, the support I have received from my friends and family throughout my five years at IISER Pune has been so integral to my life here that it does not require a mention but most certainly deserves one.

Abstract

An ultraviolet finite theory of quantum gravity has been elusive. Gravity coupled to higher spin fields, described by the Vasiliev model, is one candidate for a finite theory of gravity but it has no known action principle. This thesis presents a light-cone method to determine interaction vertices, and hence Lagrangians, by demanding closure of the symmetry algebra of the background spacetime. In particular, we discuss the derivation of cubic interaction vertices for massless fields of arbitrary spin and quartic interaction vertices for spin-1 fields in four-dimensional Minkowski spacetime. The requirement of antisymmetric constants for odd spin fields is noted at the cubic level. It is observed that, at the quartic level, algebra closure forces the Jacobi identity onto these constants and dictates the existence of a gauge group. Some of the work being done to extend this method to higher spin fields in AdS_4 is included.

In addition, this thesis describes certain features of the pure gravity Hamiltonian that may point to hidden symmetries in the theory. We find that the Hamiltonian can be expressed as a quadratic form and determine its residual reparametrization invariances. The transformation properties of the quadratic form structure are studied.

Contents

Most of the work presented in this thesis is not original. This thesis is largely based on material from my publications listed below.

1. Deriving spin-1 quartic interaction vertices from closure of the Poincaré algebra,
   S. Ananth, A. Kar, S. Majumdar, N. Shah, *Nuclear Physics* B 926 (2018).

2. Gravitation and quadratic forms,
   S. Ananth, L. Brink, S. Majumdar, M. Mali, N. Shah, *Journal of High Energy Physics* (JHEP) 1703 (2017).

Chapter 1

Introduction

The Standard Model of particle physics classifies all known elementary particles and provides a quantum description of their interactions through three of the four fundamental forces: the electromagnetic, weak, and strong forces. On the other hand, the classical theory of general relativity, described by the Einstein-Hilbert action, successfully explains all gravitational phenomena. However, reconciling the quantum nature of matter with this classical description of spacetime geometry, and obtaining a quantum theory of gravity runs into several problems. We briefly discuss the divergence issues in quantizing the Einstein-Hilbert action to motivate work done as part of this thesis.

Quantum gravity

With an accurate description of gravity provided by the general theory of relativity and no observed quantum gravitational effects, one may ask why we need a theory of quantum gravity at all. While several arguments motivate the need for a quantum theory such as the inconsistency of having singularities inside black-holes (see [1] and the references therein), a simple way to understand the requirement [2] is through the source term in the gravitational equations of motion,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = T_{\mu\nu}.^{1} \tag{1.1}$$

The R.H.S. depends on the dynamical variable of the theory, the spacetime metric $g_{\mu\nu}$, while the equations are sourced on the L.H.S. by the stress-energy tensor, $T_{\mu\nu}$, which depends on other fields that are present, and the metric itself. Since the stress-energy tensor has a quantum nature and the gravitational field has a dependence on this quantity, we need to be able to quantize gravity.

Scattering amplitudes in quantum field theory are evaluated from quantizing a perturbative expansion of the action. One of the issues with quantum general relativity are the divergences produced in perturbatively quantizing the Einstein-Hilbert action. For example, the inclusion of

[1] For details of the notation in this equation, see Chapter 4.

matter fields at the one-loop level introduces non-renormalizable divergences in quantizing the gravity Lagrangian [3, 4, 5, 6, 7, 8]. Also, quantum effects from gravitons contribute at higher orders [9, 10, 11]. The removal of divergences that occur in perturbative quantum general relativity by renormalization requires the introduction of additional terms in the equations of motion [12] but these extra degrees of freedom produce an unstable universe. Non-perturbative approaches to quantizing gravity have also not been successful (for more details, see the reviews [1, 2] and references therein).

Our approach to the problem involves searching for new symmetries that can help curb divergences in the theory. A possible solution is the introduction of higher spin fields in a theory of gravity.

Higher spin fields

Currently, all observed elementary particles have spin values of zero, half or one. The electromagnetic, weak, and strong forces are mediated by spin-1 particles that are described by quantum Yang-Mills theory. The successful quantization of gravity would produce spin-2 gravitons as mediators. Though they are not known to occur in nature, several forays have been made into studying theories with higher integer spin fields (spin ≥ 3). In flat spacetimes, no-go theorems forbid the consistent formulation of interacting higher spin theories [13, 14, 15], but this is not true on curved spacetime backgrounds. In particular, self-consistent and gauge invariant interacting equations of motion for massless higher spin fields in four-dimensional dS/AdS spacetime exist as solutions to Vasiliev's equations [16, 17, 18]. A property of these higher spin systems is the existence of a higher spin symmetry algebra: consistent equations of motion for a spin-3 field require the introduction of a spin-4 field and so on, creating an infinite higher spin tower. With this enhanced symmetry, the theory is expected to have interesting divergence properties. Coupling these spins to a spin-2 field may even provide an example of an ultraviolet finite theory of gravity. This makes it worthwhile to study configurations that involve fields with spin ≥ 2, even though no such fundamental particles have been observed thus far. However, there is no known action principle Vasiliev's equations, which prevents study of the quantum properties of such systems.

We propose finding interaction vertices for higher spins in an AdS_4 background using a method in the light-cone formalism which is illustrated in this thesis. The Lagrangian for a theory reflects the isometries of the spacetime in which it resides and we impose closure of the light-cone isometry algebra of the background as a first principles approach to constructing interacting theories. The method was introduced in [19], where cubic interaction vertices have been derived for arbitrary integer spin fields in four-dimensional flat spacetime. An extension of their work to higher orders could lead to a Lagrangian for such fields that bypasses the covariant no-go theorems and whether this is possible is discussed in [20, 21, 22]. Since the existence of such theories is uncertain, we choose to focus on higher spins in dS/AdS spacetimes, where the Vasiliev model provides consistent equations of motion. As proof of concept, we have derived spin-1 quartic interaction vertices in flat spacetime, thereby filling a gap in the literature. The derivation also demonstrates the strength of the method in forcing certain properties onto the Lagrangian. For related discussions about no-go theorems and higher-spin fields in the light-cone formalism, see [23, 24, 25, 26].

Chapter 2 provides a brief overview of required light-cone concepts. The algebra closure method is explained in Chapter 3, by reviewing the derivation of cubic interaction vertices in flat spacetimes from [19], and is then applied to determine quartic interaction vertices for spin-1 fields. Chapter 4 details ongoing work in AdS_4 spacetimes.

In Chapter 5, we move away from the isometry algebra of the background spacetime and higher spins, and focus once again on pure gravity. It was shown in [27, 28] that the light-cone Hamiltonians for pure Yang-Mills theory and $N = 4$ superYang-Mills theory can be expressed as quadratic forms. $N = 8$ supergravity is also known to have this structure to quartic order [29]. We show that this is also a feature of the pure gravity Hamiltonian in four dimensions up to quartic order in its perturbative expansion. Loop amplitude calculations of $N = 8$ supergravity have shown that there are unexpected cancellations in divergences that render the theory finite up to four loops in four dimensions [30]. Not all these enhanced cancellations can be explained by supersymmetry alone and it is possible that they occur due to additional symmetries at the pure gravity level itself. Special mathematical structures can be signatures to hidden symmetries and we hope that the quadratic form structure is an indicator of such a symmetry. We also determine residual reparametrization invariances of the Hamiltonian to order one in the coupling constant.

Chapter 2

The light-cone formalism

The light-cone form of relativistic dynamics is based on a choice of coordinates that sets the initial surface from which a system evolves to be a surface tangential to the light-cone. It was first proposed by Dirac in 1949 [31] as an equivalent form of studying relativistic quantum mechanics in which dynamical variables are expressed in terms of their values on the light-front. Apart from light-cone field theory, literature that uses this form of dynamics also uses the terminology light-front field theory, field theory in the infinite momentum frame, and null plane field theory.

There are several advantages to working in the light-cone formalism. For our purposes, the two most important aspects are that it allows one to work with only the physical degrees of freedom of a field (absence of auxiliary fields) and leads to a reduction in the number of symmetry generators that are dependent on interactions. For example, in four-dimensional flat spacetime, only three of the ten light-cone Poincaré generators are dynamical. These properties are illustrated in this chapter which provides an introduction to notation, conventions and other information about the light-cone gauge.

2.1 Coordinates

The light-cone formulation of field dynamics requires the introduction of light-cone coordinates. For four-dimensional Minkowski spacetime, with metric signature $(-,+,+,+)$, these coordinates are given by

$$x^{\pm} = \frac{x^0 \pm x^3}{\sqrt{2}} \,, \qquad x = \frac{x^1 + ix^2}{\sqrt{2}} \,, \qquad \text{and} \qquad \bar{x} = \frac{x^1 - ix^2}{\sqrt{2}} \,, \tag{2.1}$$

with the corresponding derivatives

$$\partial^{\mp} = \frac{\partial^0 \mp \partial^3}{\sqrt{2}} \,, \qquad \bar{\partial} = \frac{\partial^1 - i\partial^2}{\sqrt{2}} \,, \qquad \text{and} \qquad \partial = \frac{\partial^1 + i\partial^2}{\sqrt{2}} \,. \tag{2.2}$$

In this convention, the invariant spacetime interval

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -(dx^0)^2 + \sum_{i=1}^{3} (dx^i)^2, \tag{2.3}$$

becomes

$$ds^2 = -2\,x^+ x^- + 2\,x\,\bar{x} \tag{2.4}$$

and

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.5}$$

Thus, the scalar product of two vectors has the form

$$A^\mu B_\mu = \eta_{\mu\nu} A^\mu B^\nu = -A^+ B^- - A^- B^+ + A\,\bar{B} + \bar{A}\,B, \tag{2.6}$$

where A^μ and B^μ are redefined in the same way as the coordinates.

Either of the two null coordinates, x^+ or x^-, can be chosen as the time coordinate and the convention we follow is to take x^+ as the evolution parameter. The remaining coordinates represent the three spatial directions, with x and $\bar{x}$ termed as transverse directions. Redefining all vectors and tensors in this fashion allow us the make the light-cone gauge choice as demonstrated for electromagnetism in the next section.

2.2 Maxwell theory in the light-cone gauge

The Lagrangian density for Maxwell theory is

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \qquad \text{with} \qquad \mu, \nu = 0, 1, 2, 3, \tag{2.7}$$

where, in terms of the electromagnetic four potential, A_μ,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.8}$$

The expression (2.7) is invariant under the transformation,

$$A_\mu \to A_\mu + \partial_\mu \Lambda, \tag{2.9}$$

where Λ is a scalar. Thus, the theory allows for one gauge choice. We make the light-cone gauge choice,

$$A_- = -\,A^+ = 0, \tag{2.10}$$

and impose (2.10) in the equations of motion,

$$\partial_\mu F^{\mu\nu} = 0, \tag{2.11}$$

which results in two kinds of equations. Dynamical equations contain the time derivative, ∂^-, and carry information about the evolution of the system, while equations that do not involve ∂^- give constraints on components of the field. In particular, when $\nu = -$,

$$A^- = \frac{\bar{\partial}}{\partial^+}A + \frac{\partial}{\partial^+}\bar{A}, \tag{2.12}$$

where the operator $\frac{1}{\partial^+} = -\frac{1}{\partial_-}$ is defined following [32, 33] as

$$\frac{1}{\partial_-}f(x^-) = \int_{-\infty}^{\infty} f(y^-)\varepsilon(x^- - y^-)dy^-, \tag{2.13}$$

ε being the Heaviside step function.

Substituting for A^+ and A^- in (2.7) using (2.10) and (2.12) reduces the Lagrangian to an expression with only two independent components, A and $\bar{A}$, with

$$\mathscr{L} = \frac{1}{2}\bar{A}\Box A, \tag{2.14}$$

where $\Box = 2\,(\partial\bar{\partial} - \partial^+\partial^-)$. A photon has only two physical degrees of freedom and these are captured in A and $\bar{A}$. The absence of auxiliary fields is one of the advantages of working in the light-cone formulation of dynamics. In general, any massless field of arbitrary integer spin, λ, has two physical degrees of freedom in four dimensions. Using a complex field, ϕ, to define the particle, the representation is chosen such that ϕ and $\bar{\phi}$ are eigenstates of the helicity operator with the eigenvalues, $+\lambda$ and $-\lambda$ respectively.

The following table lists the helicity and length-dimension of some of the commonly occurring variables and operators since these values will be of use later.

Quantity	Helicity	Dim $[L]$
x	$+1$	$+1$
$\bar{x}$	-1	$+1$
∂	$+1$	-1
$\bar{\partial}$	-1	-1
A	$+1$	-1
$\bar{A}$	-1	-1
∂^+	0	-1

2.3 Light-cone Poincaré algebra

A relativistic quantum field theory in Minkowski spacetime has to have a Poincaré invariant action. Generators of Poincaré symmetry satisfy the algebra,

$$\begin{aligned}
&[P^\mu, P^\nu] = 0 \, , \\
&[P^\rho, J^{\mu\nu}] = i \, (\eta^{\rho\mu} P^\nu - \eta^{\rho\nu} P^\mu) \, , \\
&[J^{\mu\nu}, J^{\rho\sigma}] = -i \, (\eta^{\mu\rho} J^{\nu\sigma} - \eta^{\mu\sigma} J^{\nu\rho} - \eta^{\nu\rho} J^{\mu\sigma} + \eta^{\nu\sigma} J^{\mu\rho}) \, .
\end{aligned} \tag{2.15}$$

A required ingredient in what follows are explicit expressions for generators of the Poincaré algebra in light-cone coordinates. As differential operators acting on a free field with spin, λ, we find that the generators,

$$P \; = -i\partial \, , \tag{2.16}$$

$$\bar{P} \; = -i\bar{\partial} \, , \tag{2.17}$$

$$P^+ = -i\partial^+, \tag{2.18}$$

$$P^- = -i\partial^-, \tag{2.19}$$

$$J = i \, (x\bar{\partial} - \bar{x}\partial - \hat{\lambda}), \tag{2.20}$$

$$J^{+-} = i \, (x^- \partial^+ - x^+ \partial^-), \tag{2.21}$$

$$J^- = i \, (x\partial^- - x^- \partial - \hat{\lambda} \frac{\partial}{\partial^+}), \tag{2.22}$$

$$J^+ = i \, (x\partial^+ - x^+ \partial), \tag{2.23}$$

$$\bar{J}^+ = i \, (\bar{x}\partial^+ - x^+ \bar{\partial}), \tag{2.24}$$

$$\bar{J}^- = i \, (\bar{x}\partial^- - x^- \bar{\partial} - \hat{\lambda} \frac{\bar{\partial}}{\partial^+}). \tag{2.25}$$

satisfy the light-cone equivalent of (2.15) (see Appendix A) provided

$$\partial^- = \frac{\partial\bar{\partial}}{\partial^+}, \tag{2.26}$$

which is the on-shell condition for a free field. The spin part of the Lorentz algebra, which appears in the expressions for J, J^-, and $\bar{J}^-$ with the operator $\hat{\lambda}$, can be determined in two different ways [19, 34]. One option is to use the covariant formula for the variation of a field under a Lorentz transformation, impose the light-cone gauge condition, and work out the compensating gauge transformation required to maintain the gauge. For example, for a spin-1 fields, A^μ, we would set $A^+ = 0$ and work out the gauge transformation required to keep this component zero after a rotation. The other method is to note that the spin part of J is an operator that measures the helicity of the field (has egienvalue $+\lambda$ for ϕ and $-\lambda$ for $\bar{\phi}$), and use algebra closure to derive the spin parts of the other rotation generators.

Once (2.26) is substituted wherever ∂^- appears, we can choose to work on the surface $x^+ = 0$ since it simplifies calculations. Generators get classified as kinematical generators,

$$P^+ \ , \ P \ , \ \bar{P} \ , \ J \ , \ J^+ \ , \ \bar{J}^+, \ \text{and} \ J^{+-}, \tag{2.27}$$

which do not involve time derivatives and dynamical generators,

$$P^- \ , \ J^- \ , \ \text{and} \ \bar{J}^-, \tag{2.28}$$

that do involve a substitution for ∂^- and pick up corrections in interacting theories. The fact that there are only three dynamical generators is another advantage of the light-cone formalism.

Chapter 3

Adding interactions

Results in this chapter are reproduced from the publication: S. Ananth, A. Kar, S. Majumdar, N. Shah, Nuclear Physics B 926 (2018).

With knowledge of light-cone coordinates, gauge choice, and Poincaré symmetry algebra, we proceed to demonstrate how self-interaction vertices can be derived by requiring algebra closure order-by-order in the coupling constant. The method and results that are discussed in this chapter are based on the work in [19] and the publication: S. Ananth, A. Kar, S. Majumdar, N. Shah, *Nucl. Phys.* B 926 (2018).

3.1 The method

As seen in section 2.2, the Lagrangian for a free massless field of arbitrary spin is given by

$$L = \frac{1}{2}\int d^3x\, \bar{\phi}\, \Box\, \phi = \int d^3x\, (\bar{\phi}\, \partial\, \bar{\partial}\, \phi - \bar{\phi}\, \partial^+ \partial^- \phi), \tag{3.1}$$

with the corresponding Hamiltonian being

$$H = \int d^3x\, (\bar{\phi}\, \partial\, \bar{\partial}\, \phi). \tag{3.2}$$

In terms of the infinitesimal variation caused by the Hamiltonian operator on a field,

$$H = \int d^3x \partial^+ \bar{\phi}\, \delta_H \phi. \tag{3.3}$$

The variation δ_H is defined as

$$\delta_H \equiv \partial^- \phi = i\delta_{P^-}\phi = \{\phi,\, H\}.^1 \tag{3.4}$$

[1]When writing infinitesimal variations, we leave out the infinitesimal parameter

where

$$\{A,B\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial B}{\partial q}\frac{\partial A}{\partial p} \,, \tag{3.5}$$

is the Poisson bracket and in our case, the dynamical variable q is the field ϕ and the corresponding momentum, $p = \frac{\partial L}{\partial \dot{q}}$, is $\partial^+\phi$. Therefore, the dynamical information of the system is carried in the expression for $\partial^-\phi$. For a free theory, the algebra closed provided (2.26), which comes from the equation of motion for a free field,

$$\Box\, \phi = 2\,(\,\partial\,\bar{\partial} - \partial^+\partial^-)\,\phi = 0. \tag{3.6}$$

A theory with self-interactions will have a Lagrangian density of the form

$$\mathscr{L} = \frac{1}{2}\bar{\phi}\Box\phi + f(\phi,\bar{\phi}), \tag{3.7}$$

leading to corrections to the expression for ∂^-. Our goal is to determine these corrections by demanding algebra closure order-by-order in the coupling constant of the theory, α, just as (2.26) was found as an output of requiring consistent commutation relations at the free level. In order to do so, we note that the generators that pick up corrections at higher orders, apart from P^-, are the other dynamical generators with

$$\delta_{J^-}\phi = \delta^0_{J^-}\phi + ix\delta^\alpha_H + \delta^\alpha_{S^-} + \mathrm{O}(\alpha^2)\,, \tag{3.8}$$

and

$$\delta_{\bar{J}^-}\phi = \delta^0_{\bar{J}^-}\phi + i\bar{x}\delta^\alpha_H + \delta^\alpha_{\bar{S}^-} + \mathrm{O}(\alpha^2)\,. \tag{3.9}$$

δ^0 represents the free level expressions of the generators and δ^α represents corrections at the next order in the coupling constant. The procedure to determine these corrections involves making an ansatz for their general structure and using commutation relations among the various generators to fix the ambiguities.

3.2 Cubic interaction vertices

We expect the Lagrangian density for a field with self-interactions to have the structure,

$$\mathscr{L} = \frac{1}{2}\bar{\phi}\Box\phi + \alpha\bar{\phi}\phi\phi + \alpha\phi\bar{\phi}\bar{\phi} + \mathrm{O}(\alpha^2), \tag{3.10}$$

where α is the coupling constant and derivatives have not yet been inserted. The Hamiltonian variation will therefore have the form,

$$\delta_H\phi = \frac{\partial\bar{\partial}}{\partial^+}\phi + \alpha\phi\phi + \alpha\bar{\phi}\phi + \mathrm{O}(\alpha^2). \tag{3.11}$$

In this section, we review the derivation of cubic self-interaction terms for massless fields of arbitrary integer spin, λ [19]. Beginning with an ansatz for the order α terms in the Hamiltonian variation, we will demand closure of the Poincaré algebra. The restrictions imposed by these commutation relations fix the form of the ansatz. Since the two types of terms that appear, $\bar{\phi}\phi$ and $\phi\phi$, do not mix, they can each be worked out separately. The ansatz for the latter kind is a sum of terms of the form,

$$\delta_H^\alpha \phi = \alpha K\, \partial^{+\mu} \left[\bar{\partial}^B \partial^C \partial^{+\rho} \phi \; \bar{\partial}^D \partial^E \partial^{+\sigma} \phi \right], \tag{3.12}$$

where μ, ρ, σ B,C,D,E are integers, and K is a constant to be fixed by the algebra. Initially, we obtain constraints on the values of these integers using the commutation relations of the Hamiltonian variation with kinematical generators. The commutator $[P^-, J^{+-}] = -iP^-$, when acted on fields at order α,

$$[\delta_H \phi, \delta_{J^{+-}} \phi]^\alpha = [\delta_H^0 \phi, \delta_{J^{+-}}^\alpha \phi] + [\delta_H^\alpha \phi, \delta_{J^{+-}}^0 \phi] = -i\delta_H^\alpha \phi\ , \tag{3.13}$$

yields [19]

$$\mu + \rho + \sigma = -1. \tag{3.14}$$

Only the second commutator in (3.13) contributes since $\delta_{J^{+-}}^\alpha \phi = 0$. From the commutator of the ansatz with δ_J, we find that

$$B + D - C - E = \lambda\ . \tag{3.15}$$

Using (3.14), dimensional analysis and helicity, we further determine that

$$B + D = \lambda \qquad \text{and} \qquad C = E = 0\ . \tag{3.16}$$

The commutator of δ_H with $\delta_{\bar{J}^+}$ is trivial but that with δ_{J^+} mixes terms with different derivative structures and leads to a more complex constraint than the simple relations that are obtained above. The constraint [19],

$$\begin{aligned}[]
[\delta_{J^+}, \delta_H]^\alpha \phi = \alpha\, K\, [\, i\, B\, \partial^{+\mu} (\partial^{+(\rho+1)} \bar{\partial}^{(B-1)} \phi \partial^{+\sigma} \bar{\partial}^D \phi) \\
+\ i\, D\, \partial^{+\mu} (\partial^{+\rho} \bar{\partial}^B \phi \partial^{+(\sigma+1)} \bar{\partial}^{(D-1)} \phi)\,] = 0\ ,
\end{aligned} \tag{3.17}$$

restricts the values of the coefficient, K, in the sum of terms. Taking the commutator with δ_{J^-} or $\delta_{\bar{J}^-}$ requires expressions for the spin corrections of (3.8) and (3.9). We follow [19] in making a guess that

$$\delta_{S^-}^\alpha \phi = \alpha \phi \phi \qquad \text{and} \qquad \delta_{\bar{S}^-}^\alpha = \alpha \phi \bar{\phi}, \tag{3.18}$$

where derivatives have not yet been inserted. Since we are only looking at terms that have the structure, $\phi\phi$, we can take the commutator of δ_H with the orbital part of $\delta_{\bar{J}^-}$ to get [19]

$$
\begin{aligned}
[\delta_{\bar{J}^-}, \delta_H]^\alpha = i\,(\mu+1-\lambda)\,\frac{\bar{\partial}}{\partial^+}\,\alpha\,K\,\partial^{+\mu}[\bar{\partial}^B\partial^{+\rho}\phi\;\bar{\partial}^D\partial^{+\sigma}\phi] \\
+\;\alpha\,K\,\{\;i\,(\rho+\lambda)\,\partial^{+\mu}[\bar{\partial}^{(B+1)}\partial^{+(\rho-1)}\phi\;\bar{\partial}^D\partial^{+\sigma}\phi] \\
+\;i\,(\sigma+\lambda)\,\partial^{+\mu}[\bar{\partial}^B\partial^{+\rho}\phi\;\bar{\partial}^{(D+1)}\partial^{+(\sigma-1)}\phi]\;\} = 0\,.
\end{aligned}
\tag{3.19}
$$

It turns out that (3.17) and (3.19), along with (3.14) and (3.16), are sufficient to determine all unkowns in (3.12). The general answer for arbitrary spin [19] was determined by identifying the pattern across specific values of λ. For $\lambda = 0$, equations (3.17) and (3.19) are satisfied by the solution $\mu = -1$, $\rho = 0$, and $\sigma = 0$, resulting in the interaction vertex of the ϕ^3 theory,

$$\delta_H^\alpha \phi = \alpha \frac{1}{\partial^+}(\phi\phi). \tag{3.20}$$

$\lambda = 1$ admits a solution for $\mu = 0$ but it is non-trivial only if antisymmetric structure constants, f^{abc}, are introduced such that

$$\delta_H^\alpha \phi^a = \alpha f^{abc}\left[\frac{\bar{\partial}}{\partial^+}\phi^b\phi^c - \phi^b\frac{\bar{\partial}}{\partial^+}\phi^c\right]\,. \tag{3.21}$$

The need for these constants is an indicator of the gauge group that is present for fields with odd spins. When $\lambda = 2$, there exists a solution with $\mu = 1$,

$$\delta_H^\alpha \phi = \alpha\partial^+\left[\frac{\bar{\partial}^2}{\partial^{+2}}\phi\phi - 2\frac{\bar{\partial}}{\partial^+}\phi\frac{\bar{\partial}}{\partial^+}\phi + \phi\frac{\bar{\partial}^2}{\partial^{+2}}\phi\right]\,. \tag{3.22}$$

For $\lambda = 3$, a solution exists with $\mu = 2$ and antisymmetric structure constants and the pattern continues as λ is increased further. There is also pattern in the derivative structure of the terms, and their coefficients follow Pascal's triangle. A general solution can be written down for an arbitrary λ value. For even spin [19],

$$\delta_H^\alpha \phi = \alpha\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\partial^{+(\lambda-1)}\left[\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\,\phi\,\frac{\bar{\partial}^n}{\partial^{+n}}\,\phi\right]\,, \tag{3.23}$$

and for odd spin [19]

$$\delta_H^\alpha \phi^a = \alpha f^{abc}\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\partial^{+(\lambda-1)}\left[\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\,\phi^b\,\frac{\bar{\partial}^n}{\partial^{+n}}\,\phi^c\right]\,. \tag{3.24}$$

Equations (3.23) and (3.24) do not complete the process of determining interaction vertices through algebra closure. Two steps remain: working out the $\bar{\phi}\phi$ pieces of the Hamiltonian variation and determining expressions δ_S^α and $\delta_{\bar{S}}^\alpha$ for which the algebra is consistent. Proceeding in a way that is similar to that followed for the $\phi\phi$ terms, the $\bar{\phi}\phi$ terms can be found to be [19]

$$\delta_H^\alpha \phi = 2\alpha \sum_{n=0}^{\lambda} (-1)^n \binom{\lambda}{n} \frac{1}{\partial^{+(\lambda+1)}} \left[\frac{\partial^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\, \bar{\phi}\, \partial^n\, \partial^{+(2\lambda-n)}\, \phi \right] , \tag{3.25}$$

with f^{abc} inserted for odd spin. The spin correction, $\delta_{S^-}^\alpha$ is determined by the relation,

$$[\delta_{J^-}, \delta_H]^\alpha\, \phi = 0. \tag{3.26}$$

The unknown piece in (3.26) is $\delta_{S^-}^\alpha$, and demanding that the equation hold gives [19]

$$\delta_{S^-}^\alpha \phi = -2\, i\, \alpha\, \lambda \sum_{n=0}^{\lambda-1} (-1)^n \binom{\lambda-1}{n}\, \partial^{+(\lambda-1)} \left[\frac{\bar{\partial}^{(\lambda-1-n)}}{\partial^{+(\lambda-n)}}\, \phi \frac{\bar{\partial}^n}{\partial^{+n}}\, \phi \right]. \tag{3.27}$$

The expression for $\delta_{\bar{S}^-}^\alpha$ is obtained similarly from the commutator

$$[\delta_{\bar{J}^-}, \delta_H]^\alpha\, \phi = 0\,, \tag{3.28}$$

giving [19]

$$\begin{aligned} \delta_{\bar{S}^-}^\alpha \phi = -2\, i\, \alpha\, \lambda \sum_{n=0}^{\lambda-1} (-1)^n \binom{\lambda-1}{n} \frac{1}{\partial^{+(\lambda+1)}} \Bigg[&\frac{\partial^{(\lambda-1-n)}}{\partial^{+(\lambda-n)}}\, \bar{\phi}\, \partial^n\, \partial^{+(2\lambda-n)}\, \phi \\ &+ 3 \frac{\partial^{(\lambda-1-n)}}{\partial^{+(\lambda-1-n)}}\, \bar{\phi}\, \partial^n\, \partial^{+(2\lambda-n-1)}\, \phi \Bigg]. \end{aligned} \tag{3.29}$$

Again, for odd spins, these expressions will involve f^{abc}. A consistency check on (3.27) and (3.29) comes from requiring

$$[\delta_{J^-}, \delta_{\bar{J}^-}]^\alpha\, \phi = 0, \tag{3.30}$$

which does hold true.

Now that we have the expressions for δ_H^α, we can write down the action upto this order. For even spin [19],

$$S = \int d^4x \left(\frac{1}{2} \bar{\phi} \Box \phi + \alpha \sum_{n=0}^{\lambda} (-1)^n \binom{\lambda}{n} \bar{\phi}\, \partial^{+\lambda} \left[\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}} \phi \frac{\bar{\partial}^n}{\partial^{+n}} \phi \right] + c.c. \right), \tag{3.31}$$

and for odd spin [19],

$$S = \int d^4x \left(\frac{1}{2} \bar{\phi}^a \Box \phi^a + \alpha f^{abc} \sum_{n=0}^{\lambda} (-1)^n \binom{\lambda}{n} \bar{\phi}^a\, \partial^{+\lambda} \left[\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}} \phi^b \frac{\bar{\partial}^n}{\partial^{+n}} \phi^c \right] + c.c. \right). \tag{3.32}$$

The actions for $\lambda = 1$ and $\lambda = 2$ match those obtained from light-cone gauge-fixing the covariant Yang-Mills [35] and Einstein-Hilbert actions [36] respectively.

3.3 Quartic interaction vertices for spin-1 fields

The method outlined in the previous section can be extended to higher orders in the coupling constant. This section describes the derivation of the quartic interaction vertex for spin-1 fields in a four-dimensional Minkowski spacetime. When $\lambda = 1$, the states ϕ and $\bar{\phi}$ have helicity eigenvalues $+1$ and -1 respectively and we denote them as the vector components, A and $\bar{A}$. 2α is identified with the dimensionless coupling constant, g, of Yang-Mills theory. We make an ansatz for $\delta_H^{g^2} A$ as was done at order g. A generic structure is provided by a sum of terms of the form,

$$\delta_H^{g^2} A^a = g^2\, K\, f^{abc}\, f^{cde}\, \partial^{+\mu} \left[\bar{\partial}^B\, \partial^C\, \partial^{+\rho}\, \phi^b\ \partial^{+\sigma} \left(\bar{\partial}^D\, \partial^E\, \partial^{+\eta} A^d\ \bar{\partial}^F\, \partial^G\, \partial^{+\delta}\, \bar{A}^e \right) \right] , \tag{3.33}$$

where the constant, K, and integers, μ, ρ, σ, η, δ, B, C, D, E, F, and G, will be determined by the algebra. Notice the need for two antisymmetric constants to create a non-zero term that contains three fields. The requirement that the Hamiltonian have zero helicity disqualifies terms that have three A fields or three $\bar{A}$ fields. Other combinations with different positions of the derivatives may be possible but these structures can be generated starting from (3.33).

The commutation relation , $[\delta_J, \delta_H]^{g^2} A^a = 0$, yields

$$B + D + F = C + E + G = \lambda - 1\ . \tag{3.34}$$

For $\lambda = 1$, (3.34) imposes that there will be no transverse derivatives. Therefore, (3.33) simplifies to

$$\delta_H^{g^2} A^a = +g^2\, K\, f^{abc}\, f^{cde}\, \partial^{+\mu} \left[\partial^{+\rho} A^b\ \partial^{+\sigma} \left(\partial^{+\eta} A^d\ \partial^{+\delta}\, \bar{A}^e \right) \right] . \tag{3.35}$$

We commute the ansatz with $\delta_{J^{+-}}$ to get

$$\mu + \rho + \sigma + \eta + \delta = -1\ . \tag{3.36}$$

Once condtions (3.34) and (3.36) have been imposed, the commutation relation that fixes $\delta_H^{g^2}$ is

$$[\delta_{J^-}\, ,\, \delta_H]\, A^a = 0\ . \tag{3.37}$$

At order g^2, this commutator has three parts

$$[\delta_{J^-}^0\, ,\, \delta_H^{g^2}] A^a + [\delta_{J^-}^g\, ,\, \delta_H^g] A^a + [\delta_{J^-}^{g^2}\, ,\, \delta_H^0] A^a = 0\ . \tag{3.38}$$

Two kinds of field structures, AAA and $AA\bar{A}$, are present in $[\delta_{J^-}^g\, ,\, \delta_H^g] A^a$, which is composed of the orbital piece,

$$[\delta^{g}_{L^-}\,,\,\delta^{g}_{H}]A^a\ , \tag{3.39}$$

and the spin piece,

$$[\delta^{g}_{S^-}\,,\,\delta^{g}_{H}]A^a. \tag{3.40}$$

Thus, we require the result,

$$\delta^{g}_{H}A^a = +g\,f^{abc}\left\{-A^c\,\frac{\bar\partial}{\partial^+}A^b+\frac{1}{\partial^{+2}}(\partial^{+2}A^b\,\frac{\partial}{\partial^+}\bar A^c)-\frac{1}{\partial^{+2}}(\partial\partial^+A^b\,\bar A^c)\right\} \tag{3.41}$$

from the previous section, along with the spin generators at order g,

$$\delta^{g}_{\bar S^-}A^a = -g\,f^{abc}\,\frac{1}{\partial^{+2}}\left(\frac{1}{\partial^+}\bar A^c\,\partial^{+2}A^b+3\bar A^c\partial^+A^b\right)\ , \tag{3.42}$$

and

$$\delta^{g}_{S^-}A^a = +g\,f^{abc}\,\frac{1}{\partial^+}A^b\,A^c\ . \tag{3.43}$$

The following computation details the result for terms of the form $AA\bar A$ which, for $[\delta^{g}_{J^-}\,,\,\delta^{g}_{H}]A^a$, are given by

$$[\delta^{g}_{L^-}\,,\,\delta_H{}^g]A^a = -g\,f^{abc}\,A^c\,\frac{1}{\partial^+}(\delta_H{}^g A^b)\ , \tag{3.44}$$

and

$$\begin{aligned}[\delta^{g}_{S^-}\,,\,\delta^{g}_{H}]A^a =\ & +g^2\,f^{abc}\left\{f^{bde}\frac{1}{\partial^{+2}}\left(\partial^{+2}(\frac{1}{\partial^+}A^d A^e)\frac{\partial}{\partial^+}\bar A^c\right)-f^{bde}\frac{1}{\partial^{+2}}\left(\partial\partial^+(\frac{1}{\partial^+}A^d A^e)\bar A^c\right)\right.\\ & -f^{cde}\,\frac{1}{\partial^{+2}}\left(\partial^{+2}A^b\,\frac{\partial}{\partial^{+3}}(\frac{1}{\partial^+}A^e\,\partial^{+2}\bar A^d+3A^e\partial^+\bar A^d)\right)\\ & \left.+f^{cde}\,\frac{1}{\partial^{+2}}\left(\partial\partial^+A^b\,\frac{1}{\partial^{+2}}(\frac{1}{\partial^+}A^e\,\partial^{+2}\bar A^d+3A^e\partial^+\bar A^d)\right)\right\}\\ & -g\,f^{abc}\,\delta^{g}_{H}A^c\,\frac{1}{\partial^+}A^b-g\,f^{abc}\,A^c\,\frac{1}{\partial^+}(\delta^{g}_{H}A^b)\ .\end{aligned} \tag{3.45}$$

In order to satisfy (3.38),

$$[\delta^{0}_{J^-}\,,\,\delta^{g^2}_{H}]A^a+[\delta^{g^2}_{J^-}\,,\,\delta^{0}_{H}]A^a\ , \tag{3.46}$$

must produce these terms with a negative sign. Since (3.46) cannot produce terms with the structure AAA, the ones obtained from $[\delta^{g}_{J^-}\,,\,\delta^{g}_{H}]A^a$ have to vanish. It can be checked that they indeed do so.

The commutator of $\delta^{g^2}_{J^-}$ with δ^{0}_{H} has a spin part and an orbital part,

$$[\delta^{g^2}_{L^-},\delta^{0}_{H}]A^a+[\delta^{g^2}_{S^-},\delta^{0}_{H}]A^a\ . \tag{3.47}$$

It turns out that the spin correction to J^- at order g^2 vanishes as will be explained in the next section.

Equation (3.38) is satisfied by the solution which has two terms with

$$(\mu=-1\,;\rho=+1\,;\sigma=-2\,;\eta=0\,;\delta=+1)+(\mu=0\,;\rho=0\,;\sigma=-2\,;\eta=+1\,;\delta=0)\ , \tag{3.48}$$

and the explicit computation of (3.46) for these values gives

$$\begin{aligned}
f^{abc}f^{cde}[&-\frac{1}{\partial^{+2}}(\partial^+\partial A^b\frac{1}{\partial^{+2}}(\bar{A}^e\partial^+A^d))+\frac{1}{\partial^{+2}}(\partial^+\partial A^b\frac{1}{\partial^{+2}}(\partial^+\bar{A}^eA^d))\\
&+\frac{1}{\partial^{+2}}(\frac{\partial}{\partial^+}A^b\bar{A}^e\partial^+A^d)+2\frac{1}{\partial^{+2}}(\partial^+A^b\frac{1}{\partial^{+2}}(\bar{A}^e\partial\partial^+A^d))\\
&-\frac{1}{\partial^{+2}}(\partial^+A^b\frac{1}{\partial^+}(\partial^+\bar{A}^e\frac{\partial}{\partial^+}A^d))-2\frac{1}{\partial^{+2}}(\partial^+A^b\frac{1}{\partial^+}(\frac{\partial}{\partial^+}\bar{A}^e\partial^+A^d))\\
&+\frac{1}{\partial^{+2}}(\partial^+A^b\frac{1}{\partial^{+2}}(\partial^+\partial\bar{A}^eA^d))-2\frac{1}{\partial^{+2}}(\partial^{+2}A^b\frac{1}{\partial^{+3}}(\partial^+\bar{A}^e\partial A^d))\\
&+4\frac{1}{\partial^{+2}}(\partial^{+2}A^b\frac{1}{\partial^{+3}}(\partial\bar{A}^e\partial^+A^d))+2\frac{1}{\partial^{+2}}(\partial^{+2}A^b\frac{1}{\partial^{+3}}(\bar{A}^e\partial\partial^+A^d))\\
&-\frac{1}{\partial^{+2}}(\partial^{+2}A^b\frac{1}{\partial^{+2}}(\partial^+\bar{A}^e\frac{\partial}{\partial^+}A^d))-\frac{1}{\partial^{+2}}(\partial^{+2}A^b\frac{1}{\partial^{+2}}(\frac{\partial}{\partial^+}\bar{A}^e\partial^+A^d))\\
&-\frac{1}{\partial^{+2}}(A^b\frac{\partial}{\partial^+}\bar{A}^e\partial^+A^d)-\frac{1}{\partial^{+2}}(\partial^+A^b\frac{1}{\partial^{+2}}(\partial^+\bar{A}^e\partial A^d))\\
&+4\frac{1}{\partial^{+2}}(\partial^+A^b\frac{1}{\partial^{+2}}(\partial\bar{A}^e\partial^+A^d))]\ .
\end{aligned} \tag{3.49}$$

We show these terms explicitly to point out that the calculation is not as straightforward as simply crossing off terms across (3.44), (3.45) and (3.49). For (3.38) to hold true, the antisymmetric constants, f^{abc}, that were introduced at order g have to satisfy the Jacobi identity,

$$f^{abc}f^{bde}+f^{abd}f^{bec}+f^{abe}f^{bcd}\;=0\ . \tag{3.50}$$

With this result, we now see that f^{abc} are structure constants of a gauge group. The AAA terms that appear in (3.40) also vanish if and only if (3.50) holds. Thus,

$$\delta_H^{g^2}A^a=g^2f^{abc}f^{cde}\left[\frac{1}{\partial^+}\left(\partial^+A^b\,\frac{1}{\partial^{+2}}(\partial^+\bar{A}^eA^d)\right)-A^b\,\frac{1}{\partial^{+2}}(\bar{A}^e\,\partial^+A^d)\right]\ . \tag{3.51}$$

Though it has not been definitively shown that this solution is unique, it seems non-trivial to identify a different set of values for the integer powers of the derivatives that will also satisfy all commutators. It can be verified that (3.51) leads to the same quartic interaction vertex as that obtained by light-cone gauge-fixing the covariant Yang-Mills Lagrangian [35].

The spin generator at order g^2

We determine that corrections to the spin generator at order g^2 cannot exist for spin-1 fields based on helicity and dimensions. The free field spin generator,

$$\delta^0_{S^-} A^a = -\frac{\partial}{\partial^+} A^a \ , \tag{3.52}$$

has length-dimension, -1, and helicity, +2 (see section 2.2). Therefore, at order g^2, the generator should have the form,

$$\delta^{g^2}_{S^-} A \sim g^2 A A \bar{A}\, \partial \, \frac{1}{\partial^{+3}} \ , \tag{3.53}$$

with the derivatives sprinkled on the fields. However, the commutator,

$$[\delta_{J^{+-}}, \delta_{J^-}]^{g^2} A^a = -i \delta^{g^2}_{J^-} A^a \ , \tag{3.54}$$

works only if the number of ∂^+ in the denominator is one greater than in the numerator, similar to the condition (3.36) for the Hamiltonian variation. There is no combination of fields and derivatives that allows for the correct helicity and dimension values. Therefore, it is not possible to construct a valid expression for $\delta^{g^2}_{S^-} A^a$. An expression for $\delta^{g^2}_{\bar{S}^-} A^a$ is also ruled out by the same argument. The fact the spin generator at this order vanishes is not surprising since the commutator $[\bar{P}, J^-] = -iP^-$ implies that the spin generator has one less transverse derivative than the Hamiltonian variation, and the spin one Hamiltonian at order g^2 has zero transverse derivatives.

Obtaining the order g^2 pieces of the Hamiltonian and spin generators completes the construction of the entire light-cone Poincaré algebra for spin-1 fields in four-dimensional Minkowski spacetime since the Lagrangian for Yang-Mills theory terminates at this order.

3.4 Structure of vertices for $\lambda \geq 2$

Using helicity and dimensional arguments like in the previous section, we can put limits on the structure of δ_H for spins greater than 1. For $\lambda = 2$, the dynamical variables are denoted h and $\bar{h}$ and

$$\delta^0_H h = \frac{\partial \bar{\partial}}{\partial^+} h \ , \tag{3.55}$$

has a length-dimension of -2 and a helicity of $+2$. Therefore, at order κ^2, where κ is the coupling constant for gravity, we expect

$$\delta^{\kappa^2}_H h \sim \kappa^2 h h \bar{h} \ , \tag{3.56}$$

since, for a spin two field, κ has a length-dimension of 1. (3.56) has the correct helicity but the wrong dimension. Using constraints like (3.36) and dimensional analysis,

$$\delta_H^{\kappa^2} h \sim \kappa^2\, hh\bar{h}\; (\partial\bar{\partial})\, \frac{1}{\partial^+}\; . \tag{3.57}$$

The actual expression for $\delta_H^{\kappa^2}$ will be a sum of terms of the form (3.57) with equal numbers of ∂^+ and $\frac{1}{\partial^+}$ fixed in by algebra closure to act on the various fields. This is in agreement with the light-cone gauge-fixed gravity Lagrangian [36]. At order κ^3, helicity and dimensions dictate that

$$\delta_H^{\kappa^3} h \sim \kappa^3\, hh\bar{h}\bar{h}\, \partial^2\, \frac{1}{\partial^+} + \kappa^3\, hhh\bar{h}\, \bar{\partial}^2\, \frac{1}{\partial^+}\; , \tag{3.58}$$

which matches in structure with [37]. In principle, the exact expressions should be derivable by our method but, due to the large number of terms involved at orders κ^2 and κ^3, algebraic computations are tedious for spins ≥ 2.

In general, for spin λ,

$$\delta_H^{\alpha^2} \phi \sim \alpha^2\, \phi\phi\bar{\phi}\; (\partial\bar{\partial})^{\lambda-1}\, \frac{1}{\partial^+}\; , \tag{3.59}$$

where α has length-dimension of $\lambda - 1$.

Cubic interaction vertices for higher spin fields were found in [19] but this does not guarantee the existence of consistent Lagrangians for higher spins in flat spacetime. The first step in checking whether they do would involve determining the existence of higher order interaction vertices. Dynamical commutators would be essential in checking whether vertices with the structure (3.59) are valid.

Just like the analysis for δ_H, the analysis for the spin generator at this order yields

$$\delta_{S^-}^{\alpha^2} \phi \sim \alpha^2\, \phi\, \phi\, \bar{\phi}\, \partial\, (\partial\bar{\partial})^{\lambda-2}\, \frac{1}{\partial^+}\; , \tag{3.60}$$

which has one less transverse derivative than the Hamiltonian.

Chapter 4

Curved spacetimes: dS/AdS

Cubic interaction vertices for spins ≥ 3 were found in flat-spacetime. This does not imply the existence of a consistent Lagrangian for these fields since the algebra may not close at higher orders. However equations of motion describing higher spin fields (spin ≥ 3) in curved spacetimes are known to exist [16, 17, 18]. We therefore propose using the method illustrated in the previous chapter to determine interaction vertices for higher spins in such spacetimes. Since our universe has positive curvature, we would like to study theories in dS spacetimes. However, AdS spacetimes are easier to work with and we therefore begin with such spacetimes to understand modifications that occur when curvature come into play. This chapter lays out the basics of AdS spacetimes and the light-cone generators that satisfy the corresponding isometry algebra.

4.1 AdS spacetimes

The simplest way of moving from flat to curved spacetimes is by introducing a constant curvature. AdS spacetimes are maximally symmetric solutions to Einstein's equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 0, \tag{4.1}$$

where the introduction of a negative cosmological constant, Λ, results in spacetimes with a constant negative scalar curvature. The metric, $g_{\mu\nu}$, is the dynamical variable of the theory with the Ricci tensor, $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$, and the Ricci scalar, $R = g^{\mu\nu} R_{\mu\nu}$. The Riemann tensor, R itself is defined in terms of the metric by the relations,

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu} \Gamma^{\rho}_{\nu\sigma} - \partial_{\nu} \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma} \ , \tag{4.2}$$

where

$$\Gamma^{\rho}_{\nu\sigma} = \frac{1}{2} g^{\rho\mu} \left(\partial_{\nu} g_{\mu\sigma} + \partial_{\sigma} g_{\nu\mu} - \partial_{\mu} g_{\nu\sigma} \right) . \tag{4.3}$$

AdS spacetimes can be thought of as being embedded in a higher dimensional flat spacetime that has two timelike directions. In particular, AdS_4 can be embedded in a $(3,2)$ dimensional flat spacetime. Given the metric of this spacetime,

$$ds^2 = -X_0^2 + \sum_{i=1}^{3} X_i^2 - X_4^2 \ , \tag{4.4}$$

the AdS_4 spacetime is the surface that satisfies the constraint,

$$-X_0^2 + \sum_{i=1}^{3} X_i^2 - X_4^2 = -L^2 \ , \tag{4.5}$$

where L is the AdS radius. Each value of L corresponds to a particular value of the cosmological constant through the relation,

$$\Lambda = -\frac{3}{L^2} \ . \tag{4.6}$$

In Poincare coordinates, the invariant interval for AdS_4 is

$$ds^2 = \frac{1}{z^2}\left\{-dt^2 + dx_1^2 + dx_2^2 + dz^2\right\} \ , \qquad \text{with} \qquad z > 0 \ , \tag{4.7}$$

where the cosmological constant has been set to unity. As in the case of flat spacetimes, light-cone coordinates are introduced by defining

$$x^{\pm} = \frac{t \pm x^2}{\sqrt{2}}, \qquad x = x^1, \qquad \text{and} \qquad z = z \ . \tag{4.8}$$

4.2 Isometry algebra and generators

AdS_4 inherits the $SO(3,2)$ symmetry of the higher dimensional flat spacetime in which it is embedded. The commutation relations satisfied by the by the translation and rotation generators of the AdS_4 spacetime are not only equivalent to the rotations of this parent spacetime but also equivalent to those of the three dimensional conformal algebra. We choose to work in this conformal basis of generators in which the algebra is

$$\begin{aligned}
&[D, P^\mu] = P^\mu, \qquad [D, K^\mu] = -K^\mu, \qquad [P^\mu, P^\nu] = 0 \ , \qquad [K^\mu, K^\nu] = 0 \ , \\
&[P^\mu, J^{\rho\sigma}] = -\eta^{\mu\rho} P^\sigma + \eta^{\mu\sigma} P^\rho, \qquad [K^\mu, J^{\rho\sigma}] = -\eta^{\mu\rho} K^\sigma + \eta^{\mu\sigma} K^\rho, \\
&[P^\mu, K^\nu] = -\eta^{\mu\nu} D + J^{\mu\nu}, \qquad [J^{\mu\nu}, J^{\rho\sigma}] = -\eta^{\nu\rho} J^{\mu\sigma} - 3 \text{ terms} \ ,
\end{aligned} \tag{4.9}$$

with $\mu, \nu, \rho, \sigma = 0, 1, 2$. For the light-cone form of these relations see Appendix B. The dynamical generators are

$$P^-, \ J^-, \ \text{and} \ K^-. \tag{4.10}$$

In order to derive explicit expressions for the light-cone generators, we begin by defining the translation generators

$$P = \partial \ , \tag{4.11}$$

$$P^+ = \partial^+, \tag{4.12}$$

$$P^- = \partial^-. \tag{4.13}$$

We then assume the form of the generators J^{+-}, J^-, D, and K^+, to be given by

$$J^{+-} = x^+\partial^- - x^-\partial^+, \tag{4.14}$$

$$J^- = a\,(x^-\partial - x\partial^-) + B\,\frac{\partial_z}{\partial^+}, \tag{4.15}$$

$$D = e\,(-x^+\partial^- - x^-\partial^+ + x\partial + z\partial_z + 1), \tag{4.16}$$

$$K^+ = -\frac{b}{2}\,(x^2 + z^2 - 2x^+x^-)\partial^+ + cx^+D\ . \tag{4.17}$$

The piece in J^- with the coefficient B is the spin part of the generator. The constants a, b, c, and e are fixed using commutation relations such as

$$[K^+, P^-] = J^{+-} - D, \quad [J^{+-}, J^-] = -J^-, \quad \text{and} \quad [D, P^-] = P^-. \tag{4.18}$$

We then proceed to derive the remaining generators. The expression for K is determined using

$$[J^-, K^+] = -K\ , \tag{4.19}$$

and then used in the commutator

$$[J^-, K] = -K^-, \tag{4.20}$$

to obtain K^-. The relation

$$[K, P^+] = -J^+ \tag{4.21}$$

determines J^+.

At the end of this process, we find that the complete set of generators given by

$$P = \partial \,, \tag{4.22}$$

$$P^+ = \partial^+, \tag{4.23}$$

$$P^- = \partial^-, \tag{4.24}$$

$$J^{+-} = x^+\partial^- - x^-\partial^+, \tag{4.25}$$

$$J^- = x^-\partial - x\partial^- + B\,\frac{\partial_z}{\partial^+}, \tag{4.26}$$

$$J^+ = x^+\partial - x\partial^+, \tag{4.27}$$

$$D = -x^+\partial^- - x^-\partial^+ + x\partial + z\partial_z + 1, \tag{4.28}$$

$$K^+ = -\frac{1}{2}(x^2 + z^2 - 2x^+x^-)\partial^+ + x^+D \tag{4.29}$$

$$K = -\frac{1}{2}(x^2 + z^2 - 2x^+x^-)\partial + xD - Bz + x^+B\frac{\partial_z}{\partial^+}, \tag{4.30}$$

$$K^- = -\frac{1}{2}(x^2 + z^2 - 2x^+x^-)\partial^- + x^-D + xB\frac{\partial_z}{\partial^+} - zB\frac{\partial}{\partial^+} - B^2\frac{1}{\partial^+}. \tag{4.31}$$

satisfies the AdS algebra at the free level provided

$$\partial^- = \frac{1}{2}\left(\frac{\partial^2}{\partial^+} + \frac{\partial_z^2}{\partial^+}\right). \tag{4.32}$$

The strucutre of these generators is in keeping with expressions derived in [38].

With these explicit forms, we are currently working on determining the interaction vertices for a spin-2 field in this spacetime. The results will be compared against the known light-cone Lagrangian [39] as a test of the generators. The unknown in these generators, B, depends on the spin of the field and will be fixed in the same calculation. Once we verify these expressions, they will be used to determine higher spin interaction vertices in AdS_4.

Chapter 5

An aside: hints of additional symmetries in the gravity Hamiltonian

The work discussed in this chapter is part of the publication: S. Ananth, L. Brink, S. Majumdar, M. Mali, N. Shah, Journal of High Energy Physics (JHEP) 1703 (2017).

Chapter 3 details the derivation of interaction vertices for fields with arbitrary spin using Poincaré invariance. In this chapter, we move away from the symmetry algebra of the space-time and study some properties of the gravity Hamiltonian, which is composed of spin-2 fields. We demonstrate that the Hamiltonian has certain residual reparametrization invariances and an interesting quadratic form structure that may be pointers to deeper symmetries.

5.1 Gravity in the light-cone gauge

In the absence of a cosmological constant, the Einstein-Hilbert action that describes gravitation in four dimensions is given by

$$S_{EH} = \int d^4x\, \mathscr{L} = \frac{1}{2\,\kappa^2} \int d^4x\, \sqrt{-g}\, R\ , \tag{5.1}$$

with the corresponding equations of motion,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0\ . \tag{5.2}$$

where $g = det\ (g_{\mu\nu})$. The dimensionful coupling constant of the theory, κ, is related to Newton's constant by the relation $\kappa^2 = 8\pi G$. The Lagrangian has general coordinate invariance and we are allowed four gauge choices. In the light-cone formulation, we initially make the three choices [36, 40],

$$g_{--} = g_{-i} = 0\ , \qquad \text{with} \qquad i = 1,2\ , \tag{5.3}$$

motivated by values of the Minkowski metric, $\eta_{--} = \eta_{-i} = 0$. We further parametrize the remaining components of the metric as

$$g_{+-} = -e^{\phi} \ , \qquad \text{and} \qquad g_{ij} = e^{\psi} \gamma_{ij} \ , \tag{5.4}$$

where ϕ and ψ are real parameters, and γ_{ij} is a two-by-two, symmetric matrix with determinant equal to one. Like in the case of Maxwell and Yang-Mills theories, substituting using (5.3) divides the equations of motion into dynamical equations and constraints. Out of the four constraints, the one with $\mu = \nu = -$,

$$2\,\partial_-\phi\,\partial_-\psi - 2\,\partial_-^2\psi - (\partial_-\psi)^2 + \frac{1}{2}\partial_-\gamma^{ij}\,\partial_-\gamma_{ij} = 0 \ , \tag{5.5}$$

can be solved by making the allowed fourth gauge choice [36],

$$\phi = \frac{\psi}{2}, \tag{5.6}$$

to give

$$\psi = \frac{1}{4}\frac{1}{\partial_-^2}\,(\partial_-\gamma^{ij}\,\partial_-\gamma_{ij}) \ . \tag{5.7}$$

The other constraints eliminate more degrees of freedom. For example, (5.2) with $\mu = i$, and $\nu = -$, results in

$$\begin{aligned} g^{-i} \; &= \; \mathrm{e}^{-\phi}\,\frac{1}{\partial_-}\Big[\gamma^{ij}\,\mathrm{e}^{\phi-2\psi}\,\frac{1}{\partial_-}\Big\{\,\mathrm{e}^{\psi}\Big(\frac{1}{2}\partial_-\gamma^{kl}\,\partial_j\,\gamma_{kl} - \partial_-\,\partial_j\,\phi \\ &\quad - \partial_-\,\partial_j\,\psi + \partial_j\phi\,\partial_-\,\psi\Big) + \partial_l\Big(\mathrm{e}^{\psi}\,\gamma^{kl}\,\partial_-\,\gamma_{jk}\Big)\Big\}\Big] \ . \end{aligned} \tag{5.8}$$

With these constraints implemented, the action reads as [36]

$$\begin{aligned} S \; = \; & \frac{1}{2\kappa^2}\int d^4x\, e^{\psi}\left(2\,\partial_+\partial_-\phi + \partial_+\partial_-\psi - \frac{1}{2}\partial_+\gamma^{ij}\partial_-\gamma_{ij}\right) \\ & -e^{\phi}\gamma^{ij}\left(\partial_i\partial_j\phi + \frac{1}{2}\partial_i\phi\,\partial_j\phi - \partial_i\phi\,\partial_j\psi - \frac{1}{4}\partial_i\gamma^{kl}\partial_j\gamma_{kl} + \frac{1}{2}\partial_i\gamma^{kl}\partial_k\gamma_{jl}\right) \\ & -\frac{1}{2}e^{\phi-2\psi}\gamma^{ij}\frac{1}{\partial_-}R_i\frac{1}{\partial_-}R_j \ , \end{aligned} \tag{5.9}$$

where

$$R_i \equiv e^{\psi}\left(\frac{1}{2}\partial_-\gamma^{jk}\partial_i\gamma_{jk} - \partial_-\partial_i\phi - \partial_-\partial_i\psi + \partial_i\phi\,\partial_-\psi\right) + \partial_k(e^{\psi}\gamma^{jk}\partial_-\gamma_{ij}) \ .$$

Since everything is now in terms of γ_{ij}, this light-cone action expresses gravity with only its physical degrees of freedom. We now expand this action in orders of the coupling constant, κ, by further rewriting γ_{ij} as

$$\gamma_{ij} = (e^M)_{ij} \,, \tag{5.10}$$

where

$$M = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & -h_{11} \end{pmatrix} \,, \tag{5.11}$$

is a traceless matrix since $det(\gamma_{ij}) = 1$. We write the physical degrees of freedom as helicity eigenstates,

$$h = \frac{(h_{11} + i h_{12})}{\sqrt{2}}, \qquad \bar{h} = \frac{(h_{11} - i h_{12})}{\sqrt{2}} \,, \tag{5.12}$$

with helicity being +2 for h and -2 for $\bar{h}$. After the rescaling,

$$h \to \frac{h}{\kappa} \,, \tag{5.13}$$

the Lagrangian density to order κ is given by [36]

$$\mathscr{L} = \frac{1}{2}\bar{h}\Box h + 2\kappa\left\{\bar{h}\partial_-^2\left[-h\frac{\bar{\partial}^2}{\partial_-^2}h + \frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h\right] + \text{complex conjugate}\right\}, \tag{5.14}$$

which matches the expression derived by algebra closure. We run into a problem at order κ^2 where time derivatives appear in the expression for the Lagrangian. This is handled by a field redefinition which replaces the $\partial_+ = -\partial^-$ with $\frac{\partial\bar{\partial}}{\partial_-}$. Using the relation

$$\mathscr{H} = \partial^+ h\, \partial^- h \,-\, \mathscr{L} \,, \tag{5.15}$$

we get the Hamiltonian density up to order κ^2 [36],

$$\begin{aligned}
\mathscr{H} \;=\;& \partial\bar{h}\bar{\partial}h - 2\kappa\bar{h}\partial_-^2\left\{-h\frac{\bar{\partial}^2}{\partial_-^2}\,h + \frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h\right\} - 2\kappa h\partial_-^2\left\{-\bar{h}\frac{\partial^2}{\partial_-^2}\,\bar{h} + \frac{\partial}{\partial_-}\bar{h}\frac{\partial}{\partial_-}\bar{h}\right\} \\
& -4\kappa^2\left\{-2\frac{1}{\partial_-^2}\left(\frac{\bar{\partial}}{\partial_-}h\partial_-^3\bar{h} - h\partial_-^2\bar{\partial}\bar{h}\right)\frac{1}{\partial_-^2}\left(\frac{\partial}{\partial_-}\bar{h}\partial_-^3 h - \bar{h}\partial_-^2\partial h\right)\right. \\
& +\frac{1}{\partial_-^2}(\bar{\partial}h\partial_-^2\bar{h} - \partial_- h\partial_-\bar{\partial}\bar{h})\frac{1}{\partial_-^2}(\partial\bar{h}\partial_-^2 h - \partial_-\bar{h}\partial_-\partial h) - 3\frac{1}{\partial_-}(\bar{\partial}h\partial_-\bar{h})\frac{1}{\partial_-}(\partial_- h\partial\bar{h}) \\
& +\frac{1}{\partial_-}(\bar{\partial}h\partial_-\bar{h} - \partial_- h\bar{\partial}\bar{h})\frac{1}{\partial_-}(\partial\bar{h}\partial_- h - \partial_-\bar{h}\partial h) + 3\frac{1}{\partial_-}(\partial_- h\partial_-\bar{h})\frac{1}{\partial_-}(\bar{\partial}h\partial\bar{h}) \\
& \left.+\left[\frac{1}{\partial_-^2}(\partial_- h\partial_-\bar{h}) - h\bar{h}\right](\bar{\partial}h\partial\bar{h} + \partial h\bar{\partial}\bar{h} - \partial_- h\frac{\partial\bar{\partial}}{\partial_-}\bar{h} - \partial_-\bar{h}\frac{\partial\bar{\partial}}{\partial_-}h)\right\}.
\end{aligned} \tag{5.16}$$

5.2 Residual reparametrization invariances

Choosing to work in a particular gauge removes the gauge redundancy in a theory. However, we find that certain transformations of the field still leave the gravity Hamiltonian invariant. Under the coordinate transformation,

$$x \to x + \xi(\bar{x}) \,, \qquad \bar{x} \to \bar{x} + \bar{\xi}(x) \,, \tag{5.17}$$

the field transforms as [41]

$$\delta h = \frac{1}{2\kappa}\partial\xi + \xi\bar{\partial}h + \bar{\xi}\partial h \,, \tag{5.18}$$

up to order κ^0, where ξ satisfies [41]

$$\partial_-\xi = 0 \qquad \text{and} \qquad \bar{\partial}\,\xi = 0 \,. \tag{5.19}$$

Since, at order κ^{-1},

$$\partial_-(\delta h) = 0 \,, \qquad \bar{\partial}(\delta h) = 0 \,, \tag{5.20}$$

the variation in the Hamiltonian to order κ^{-1} is zero. At order κ^0,

$$\delta H^{(\kappa^0)} = \delta\,(\partial\bar{h}\bar{\partial}h) + 2\kappa\,\delta^{\kappa^{-1}}\left\{\bar{h}\,\partial_-{}^2\left(h\frac{\bar{\partial}^2}{\partial_-{}^2}h - \frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h\right) + \text{c.c.}\right\} . \tag{5.21}$$

On substituting the appropriate variation for the fields, (5.21) turns out to be zero. Since δh has terms of order κ^{-1}, studying the variation of the Hamiltonian at order κ would involve the quartic interaction vertex. The relevant terms are

$$\delta H^{(\kappa)}_{c,q} = \delta^{\kappa^0}(\text{cubic terms}) + \delta^{\kappa^{-1}}(\text{quartic terms}) \,, \tag{5.22}$$

which give a non-zero net contribution of

$$\delta H^{(\kappa)}_{c,q} = \left(+2\kappa\,\bar{\partial}\bar{\xi}\;h\,\partial h\,\bar{\partial}\bar{h} - 2\kappa h\,\bar{\partial}\bar{\xi}\;\partial_-\bar{h}\,\frac{\partial\bar{\partial}}{\partial_-}h\right) + \text{c.c.} \tag{5.23}$$

The details of this calculation are provided in Appendix C. At this stage, the Hamiltonian is clearly not invariant under δh. However, there is another possible source of order κ terms,

$$\delta H^{(\kappa)}_{k} = \delta^{\kappa}\,(\text{kinetic terms}). \tag{5.24}$$

We determine new terms at order κ which, when added to (5.18), leave the Hamiltonian invariant.

$$\delta h = \frac{1}{2\kappa}\partial\xi + \xi\bar{\partial}h + \bar{\xi}\partial h - \kappa\,\bar{\partial}\bar{\xi}\;h\,h + 2\kappa\,\partial\xi\,\frac{1}{\partial_-}(\bar{h}\,\partial_-h) \,, \tag{5.25}$$

and

$$\delta\bar{h} \;=\; \frac{1}{2\kappa}\partial\bar{\xi} \;+\; \xi\bar{\partial}\bar{h} \;+\; \bar{\xi}\partial\bar{h} \;-\; \kappa\,\partial\xi\;\bar{h}\,\bar{h} \;+\; 2\kappa\,\bar{\partial}\bar{\xi}\;\frac{1}{\partial_-}(h\,\partial_-\bar{h})\,, \tag{5.26}$$

result in

$$\delta H^{(\kappa)} \;=\; 0\,, \tag{5.27}$$

proving invariance of the light-cone Hamiltonian, to order κ^2, under the residual reparametrizations (5.25) and (5.26).

5.3 Quadratic form of the gravity Hamiltonian to order κ^2

Apart from the residual reparametrization invariance, we find that another interesting property of the gravity Hamiltonian is that it can be expressed in the form,

$$H = \int d^3x\;\mathscr{D}\bar{h}\,\bar{\mathscr{D}}h\;, \tag{5.28}$$

up to order κ^2, with the derivative,

$$\begin{aligned}
\mathscr{D}\bar{h} &= \partial\bar{h} + 2\kappa\,\frac{1}{\partial_-^2}\,\Big(\frac{\bar{\partial}}{\partial_-}h\,\partial_-^3\bar{h} - h\,\partial_-^2\bar{\partial}\bar{h}\Big) \\
&+2\,\kappa^2\,\frac{1}{\partial_-}\{\partial_-^2\bar{h}\,\frac{1}{\partial_-^3}(\partial_-^3 h\frac{\partial}{\partial_-}\bar{h} - \partial_-^2\partial h\bar{h})\} + 2\,\kappa^2\,\frac{1}{\partial_-}\{\,\frac{\partial}{\partial_-^4}(\bar{h}\partial_-^2 h)\partial_-^3\bar{h}\} \\
&-2\,\kappa^2\,\partial_-^2\bar{h}\,\frac{1}{\partial_-^4}(\partial_-^2 h\partial\bar{h} - 2\partial_-\partial h\partial_-\bar{h}) + 2\,\kappa^2\,\partial_-\bar{h}\frac{1}{\partial_-^2}(\partial_- h\partial\bar{h} - 2\partial h\partial_-\bar{h}) \\
&+6\,\kappa^2\,\frac{1}{\partial_-^2}(\partial_- h\partial_-\bar{h})\,\partial\bar{h} - 6\,\kappa^2\,\partial_-\bar{h}\,\frac{1}{\partial_-^2}(\partial_- h\partial\bar{h}) - 2\,\kappa^2\,\frac{1}{\partial_-^2}(\partial_- h\partial_-\bar{h})\partial\bar{h} \\
&+4\kappa^2\,h\bar{h}\,\partial\bar{h} + 4\kappa^2\,\frac{\partial}{\partial_-}\{\partial_-\bar{h}\,(\frac{1}{\partial_-^2}(\partial_- h\partial_-\bar{h}) - h\bar{h})\} + 2\kappa^2\,\partial_-^2\bar{h}\frac{1}{\partial_-^4}(\partial_-^2\partial h\bar{h}) \\
&-2\kappa^2\,\partial_-\{\partial_-\bar{h}\frac{1}{\partial_-^2}(\bar{h}\partial h)\} - 2\kappa^2\,\partial\{\bar{h}\frac{1}{\partial_-^2}(\partial_-\bar{h}\partial_- h)\} - 2\kappa^2\,\partial_-^2\bar{h}\frac{1}{\partial_-^3}(\partial_-\partial h\bar{h}) \\
&+2\kappa^2\partial_-\partial\{\bar{h}\frac{1}{\partial_-^3}(h\partial_-^2\bar{h})\} + 2\kappa^2\partial\{\partial_-\bar{h}\frac{1}{\partial_-^3}(\bar{h}\partial_-^2 h)\} + 2\kappa^2\partial_-^2\{\bar{h}\frac{1}{\partial_-^3}(\partial_-\bar{h}\partial h)\},
\end{aligned} \tag{5.29}$$

and $\bar{\mathscr{D}}h$ being the complex conjugate. For details of the calculation see Appendix D. Thus, the pure gravity Hamiltonian to this order can be expressed as a quadratic form. Up to order κ^0, $\mathscr{D}\bar{h}$ transforms just like the field with

$$\delta(\mathscr{D}\bar{h}) = (\xi\bar{\partial} + \bar{\xi}\partial)\,\mathscr{D}\bar{h}\,, \tag{5.30}$$

and is hence said to transform 'covariantly'. This is in keeping with the properties of Yang-Mills theory. However, the property does not carry to order κ where

$$
\begin{aligned}
\delta\,(\mathscr{D}\bar{h})^{\kappa} \;\; &= \;\; +\kappa\,\partial\xi\,\partial\{\,\partial_-\bar{h}\frac{1}{\partial_-}\bar{h}\} \\
&\quad +2\kappa\,\bar{\partial}\bar{\xi}\,h\,\partial\,\bar{h} + \kappa\,\bar{\partial}\,\bar{\xi}\,\partial\,\partial_-\,\bar{h}\,\frac{1}{\partial_-}\,h - \kappa\,\bar{\partial}\,\bar{\xi}\,\frac{1}{\partial_-}\{\,\partial_-\,\partial\,\bar{h}\,h\}\,, \qquad (5.31)
\end{aligned}
$$

does not transform like the field, h. At this point, it is unclear if the factorization of the Hamiltonian into $\bar{\mathscr{D}}h$ and $\mathscr{D}\bar{h}$ is unique at order κ^2. We therefore study the transformation properties of $\mathscr{D}\bar{h}$ by making a general ansatz for the structure of $\delta(\mathscr{D}\bar{h})$ based on helicity, (5.25) and (5.26), and dimensional analysis. Thus, we have for $\delta(\mathscr{D}\bar{h})$,

$$
\delta(\mathscr{D}\bar{h}) \;= 0 + (\,\xi\bar{\partial} + \bar{\xi}\partial\,)\,\mathscr{D}\bar{h} - \kappa\partial\xi\,\sum_i \alpha_i\bar{\hat{A}}_i\,(\bar{\hat{B}}_i\bar{h}\bar{\hat{C}}_i\bar{h}) + \;2\kappa\,\bar{\partial}\bar{\xi}\,\sum_j \beta_j\,\bar{\hat{P}}_j(\bar{\hat{Q}}_j h\bar{\hat{R}}_j\bar{h})\,. \qquad (5.32)
$$

and for $\delta(\bar{\mathscr{D}}h)$

$$
\delta(\bar{\mathscr{D}}h) = 0 + \,(\,\xi\bar{\partial} + \bar{\xi}\partial\,)\,\bar{\mathscr{D}}h - \kappa\bar{\partial}\bar{\xi}\,\sum_i \alpha_i\hat{A}_i\,(\hat{B}_i h\,\hat{C}_i h) + 2\kappa\partial\xi\sum_j \beta_j\,\hat{P}_j(\hat{Q}_j\bar{h}\hat{R}_j h)\,, \qquad (5.33)
$$

where α_i and β_j are constants and $\hat{A}_i$, etc. are operators to be determined later. At this point we note that, if the operators take the values,

$$
\begin{aligned}
&\alpha = 1\,,\,\hat{A} \;=\; \bar{\partial}\;,\,\hat{B} \;=\; \hat{C} \;=\; 1\;, \\
&\beta = 1\,,\,\hat{P} \;=\; \frac{1}{\partial_-}\;,\,\hat{Q} \;=\; 1\;,\,\hat{R} \;=\; \partial_-\bar{\partial}\;, \qquad (5.34)
\end{aligned}
$$

the derivative transforms covariantly. We know that the Hamiltonian is invariant under the field transformations (5.25) and (5.26) and thus

$$
\delta H = 0 \;\Longrightarrow\; \int d^3x\,[\,\delta(\mathscr{D}\bar{h})\,\bar{\mathscr{D}}h \;+\; \mathscr{D}\bar{h}\,\delta(\bar{\mathscr{D}}h)] = 0\,. \qquad (5.35)
$$

At order κ^0,

$$
\begin{aligned}
\delta H = \;\; &\int d^3x\,[\,(\delta(\mathscr{D}\bar{h}))^{\kappa^0}\,\bar{\partial}h \;+\partial\bar{h}(\delta(\bar{\mathscr{D}}h))^{\kappa^0}\,]\,, \qquad (5.36) \\
= \;\; &\int d^3x\,[\,\bar{\xi}\partial^2\bar{h}\bar{\partial}h \;+\; \partial\bar{h}\bar{\xi}\partial\bar{\partial}h]\,. \qquad (5.37)
\end{aligned}
$$

and integrating a ∂ from the $\bar{h}$ in the first term yields $(\delta H)^{\kappa^0} \;=\; 0$.

At order κ,

$$
\begin{aligned}
(\delta H)^{\kappa} &= \int d^3x\,[(\delta(\mathscr{D}\bar{h}))^{\kappa}\bar{\partial}h + (\delta(\mathscr{D}\bar{h}))^{\kappa^0}(\bar{\mathscr{D}}h)^{\kappa} + (\mathscr{D}\bar{h})^{\kappa}(\delta(\bar{\mathscr{D}}h))^{\kappa^0} + \partial\bar{h}(\delta(\bar{\mathscr{D}}h))^{\kappa}]\,, \\
&= \kappa\int d^3x\,\{[\bar{\xi}\partial(\mathscr{D}\bar{h})^{\kappa} + 2\bar{\partial}\bar{\xi}\sum_j \beta_j\bar{\hat{P}}_j(\bar{\hat{Q}}_j h\bar{\hat{R}}_j\bar{h})]\bar{\partial}h + \bar{\xi}\partial^2\bar{h}(\bar{\mathscr{D}}h)^{\kappa} \qquad (5.38) \\
&\quad + [\bar{\xi}\partial(\bar{\mathscr{D}}h)^{\kappa} - \bar{\partial}\bar{\xi}\sum_i \alpha_i\hat{A}_i\,(\hat{B}_i h\,\hat{C}_i h)]\,\partial\bar{h} + (\mathscr{D}\bar{h})^{\kappa}\bar{\xi}\partial\bar{\partial}h\}\,. \qquad (5.39)
\end{aligned}
$$

Integrating a ∂ from $\bar{h}$ in the last term of (5.38) leads to a cancellation against the first term in (5.39). The last term of (5.39) is cancelled against the first term of (5.38) by integrating a ∂, and we are left with the expression,

$$
(\delta H)^{\kappa} = \kappa\int d^3x\,[2\bar{\partial}\bar{\xi}\sum_j \beta_j\bar{\hat{P}}_j(\bar{\hat{Q}}_j h\bar{\hat{R}}_j\bar{h})\bar{\partial}h - \bar{\partial}\bar{\xi}\sum_i \alpha_i\hat{A}_i\,(\hat{B}_i h\,\hat{C}_i h)\,\partial\bar{h}]\,. \qquad (5.40)
$$

Substituting (5.34) into (5.40) , leaves us with

$$
(\delta H)^{\kappa} = +2\,\kappa\int d^3x\;\bar{\partial}\bar{\xi}\,\frac{1}{\partial_-}(\,\partial_- h\,\partial\bar{h}\,)\,\bar{\partial}h + \text{c.c.} \neq 0 \qquad (5.41)
$$

Thus, derivative $\bar{\mathscr{D}}h$ and its complex conjugate cannot be covariant. The exact form of $\mathscr{D}\bar{h}$, at order κ^2, is irrelevant to this analysis and shows that the derivative cannot be covariant even if the factorization allows for a different expression at this order.

Chapter 6

Conclusion and future work

Theories constructed in the light-cone gauge are not manifestly invariant under the symmetries of the spacetime in which they reside, and have to be checked for this property. By using the requirement of algebra closure as a method to determine interaction vertices, we invert the process to obtain a first principles approach to construct Lagrangians. Chapter 3 outlines the use of this method to derive cubic self-interaction vertices for massless fields of arbitrary spin and quartic interaction vertices for spin-1 fields. The method has also been used to derive the Hamiltonian for $N = 4$ superYang-Mills [28] and to obtain cubic interaction vertices for $N = 1$ supergravity [34], fields with three different spin values [42], fermions and bosons [43], and all maximally extended supersymmetry multiplets where the maximum helicity is an integer [44]. For fields with spin values of one and two, the process serves as a proof of concept since the solutions were already known from gauge-fixing the covariant Lagrangians. However, even at this stage, there are some important aspects to be noted. The Yang-Mills Lagrangian is completely fixed by the method using only spacetime symmetries, with the gauge group emerging as a necessity of Poincaré invariance. Cubic order algebra closure introduces antisymmetric constants that are forced to satisfy the Jacobi identity at quartic order. (Similar results for the structure constants are obtained using symmetry arguments in a covariant approach [45].)

With a constructive method for determining Lagrangians, the goal for us now is to apply it in the context of higher spins in curved spacetimes. Having fixed the light-cone generators of the AdS_4 isometry algebra, we hope to work out cubic interaction vertices in this spacetime. The benefit of doing so is twofold: there is no known Lagrangian for higher spins in AdS, and the structure of the higher spin algebra is unknown. Like the introduction of the gauge group of Yang-Mills theory, algebra closure could provide indications of this structure. It would also be interesting to see whether a generalized series could be found for the cubic interaction vertex of arbitrary integer spin fields similar to that in flat space.

However, the method has its limitations. It yields solutions for the Hamiltonian perturbatively in the coupling constant of the theory with the process getting progressively more complicated. In cases where an infinite expansion exists, the method is not ideal for determining the entire

Lagrangian, nor is it possible to determine whether the Lagrangian will be consistent without the full closed-form expression. At the same time, it still has its usefulness in determining possible interaction vertices that can then be studied for their quantum properties.

The quadratic form structure of the gravity Hamiltonian and its residual reparametrization invariances, though unrelated to the algebra closure method, were studied with the same goal of examining possible sources of enhanced symmetries in theories of gravity. It is interesting to note that the quadratic form structure occurs only for pure and maximally supersymmetric Yang-Mills and gravity. The reason for this property and its implications have to be studied further. The residual reparametrization invariance calculation showed us that order κ corrections to δh are necessary to maintain invariance of the Hamiltonian. A question this raises is whether there is an infinite series in all orders that may be integrated to obtain a finite symmetry. The cause of these symmetry structures that we have found are as yet unclear, but we hope that they will point to hidden properties of the pure gravity Lagrangian.

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Appendix A

Light-cone Poincaré algebra in four dimensions

In terms of light-cone generators, the non-vanishing commutation relations of the four-dimensional Poincaré algebra are

$$\begin{aligned}
&[P^-,J^{+-}] \;=\; -iP^- \;, \qquad [P^-,J^+] = -iP \;, \qquad [P^-,\bar{J}^+] \;=\; -i\bar{P} \\
&[P^+,J^{+-}] \;=\; iP^+ \;, \qquad [P^+,J^-] \;=\; -iP \;, \quad [P^+,\bar{J}^-] \;=\; -i\bar{P} \\
&[P,\bar{J}^-] \;=\; -iP^- \;, \qquad [P,\bar{J}^+] \;=\; -iP^+ \;, \quad [P,J] \;=\; P \\
&[\bar{P},J^-] \;=\; -iP^- \;, \qquad [\bar{P},J^+] \;=\; -iP^+ \;, \quad [\bar{P},J] \;=\; -\bar{P} \\
&[J^-,J^{+-}] \;=\; -iJ^- \;, \qquad [J^-,\bar{J}^+] \;=\; iJ^{+-}+J \;, \quad [J^-,J] \;=\; J^- \\
&[\bar{J}^-,J^{+-}] \;=\; -i\bar{J}^- \;, \qquad [\bar{J}^-,J^+] \;=\; iJ^{+-}-J \;, \quad [\bar{J}^-,J] \;=\; -\bar{J}^- \\
&[J^{+-},J^+] \;=\; -iJ^+ \;, \qquad [J^{+-},\bar{J}^+] \;=\; -i\bar{J}^+ \;, \\
&[J^+,J] \;=\; J^+ \;, \qquad [\bar{J}^+,J] \;=\; -\bar{J}^+ \;. \qquad \text{(A.1)}
\end{aligned}$$

Appendix B

Light-cone AdS$_4$ algebra

In terms of light-cone generators, the non-vanishing commutation relations of the three-dimensional conformal algebra are

$$[J^{+-},J^{+}]=J^{+},\quad [J^{+-},J^{-}]=-J^{-},\quad [J^{+},J^{-}]=J^{+-},\quad [J^{+-},P^{+}]=P^{+},$$

$$[J^{+-},P^{-}]=-P^{-},\quad [J^{+-},K^{+}]=K^{+},\quad [J^{+-},K^{-}]=-K^{-},\quad [J^{+},P]=-P^{+},$$

$$[J^{+},P^{-}]=-P,\quad [J^{+},K]=-K^{+},\quad [J^{+},K^{-}]=-K,\quad [J^{-},P^{+}]=-P,$$

$$[J^{-},P]=-P^{-},\quad [J^{-},K^{+}]=-K,\quad [J^{-},K]=-K^{-},\quad [K^{+},P]=J^{+},$$

$$[K^{+},P^{-}]=J^{+-}-D,\quad [K^{-},P]=J^{-},\quad [K^{-},P^{+}]=-J^{+-}-D,$$

$$[K,P^{+}]=-J^{+},\quad [K,P^{-}]=-J^{-},\quad [K,P]=-D,\quad [D,P^{+}]=P^{+},$$

$$[D,P]=P,\quad [D,P^{-}]=P^{-},\quad [D,K^{+}]=-K^{+},\quad [D,K]=-K,\quad [D,K^{-}]=-K^{-}.$$

Appendix C

Detailed computation of $\delta H^{(\kappa)}_{c,q}$

In this appendix, we provide the details of computing $\delta H^{(\kappa)}_{c,q}$. We begin by varying the first cubic term of the Hamiltonian,

$$\delta^{\kappa^0}(\text{first cubic term}) \;=\; 2\kappa(\bar{\xi}\partial\bar{h} \,+\, \xi\bar{\partial}\bar{h}){\partial_-}^2\left(h\frac{\bar{\partial}^2}{{\partial_-}^2}h \,-\, \frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h\right)$$

$$+\; 2\kappa\bar{h}\,{\partial_-}^2\left((\xi\bar{\partial}h \,+\, \bar{\xi}\partial h)\frac{\bar{\partial}^2}{{\partial_-}^2}h \,+\, h\frac{\bar{\partial}^2}{{\partial_-}^2}(\xi\bar{\partial}h \,+\, \bar{\xi}\partial h) \,-\, 2\,\frac{\bar{\partial}}{\partial_-}(\xi\bar{\partial}h \,+\, \bar{\xi}\partial h)\frac{\bar{\partial}}{\partial_-}h\right)\;,$$

$$=\; 2\kappa\,\bar{\xi}\;\partial\bar{h}\,{\partial_-}^2\left(h\frac{\bar{\partial}^2}{{\partial_-}^2}h \,-\, \frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h\right)$$

$$+\; 2\kappa\bar{h}\,{\partial_-}^2\left(\bar{\xi}\partial h\frac{\bar{\partial}^2}{{\partial_-}^2}h \,+\, h\frac{\bar{\partial}^2}{{\partial_-}^2}(\bar{\xi}\partial h) \,-\, 2\,\frac{\bar{\partial}}{\partial_-}(\bar{\xi}\partial h)\frac{\bar{\partial}}{\partial_-}h\right)\; + W(\xi)\;,$$

$$=\; \mathscr{X} \,+\, \mathscr{Y} \,+\, W(\xi)\;, \tag{C.1}$$

where

$$\begin{aligned} W &= 2\kappa\,\xi\bar{\partial}\bar{h}\,\partial_-^2\left(h\frac{\bar{\partial}^2}{\partial_-^2}h \,-\, \frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h\right) \\ &\quad +\, 2\kappa\,\bar{h}\,\partial_-^2\left(\xi\bar{\partial}h\frac{\bar{\partial}^2}{\partial_-^2}h \,+\, h\frac{\bar{\partial}^2}{\partial_-^2}(\xi\bar{\partial}h) \,-2\frac{\bar{\partial}}{\partial_-}(\xi\bar{\partial}h)\frac{\bar{\partial}}{\partial_-}h\right)\;, \\ &= 0\;, \end{aligned} \tag{C.2}$$

by partial integrations. Thus, ξ-dependent terms vanish. In a similar fashion, there will be no $\bar{\xi}$-dependent terms in the variation of the second cubic term (complex conjugate). We simplify

$\mathscr{X}$ and $\mathscr{Y}$ further using partial integration to give

$$\mathscr{X} \;=\; -2\kappa\,\xi\,\bar{h}\,\partial_-{}^2\partial\left(h\frac{\bar{\partial}^2}{\partial_-{}^2}h\right)+2\kappa\,\xi\,\bar{h}\,\partial_-{}^2\partial\left(\frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h\right)\,, \tag{C.3}$$

and

$$\begin{aligned}\mathscr{Y} \;=\; & 2\kappa\bar{h}\,\bar{\xi}\partial_-{}^2\,\partial\left(h\frac{\bar{\partial}^2}{\partial_-{}^2}h\right)\;-\;2\kappa\bar{h}\bar{\xi}\partial_-{}^2\,\partial\left(\frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h\right) \\ & -\;4\kappa\,\bar{\partial}\bar{\xi}\;\frac{\partial}{\partial_-}h\;\frac{\bar{\partial}}{\partial_-}h\,\partial_-^2\bar{h}\;+\;2\kappa\,\bar{\partial}^2\bar{\xi}\;\frac{\partial}{\partial_-^2}h\,h\,\partial_-^2\bar{h}\;+\;4\kappa\,\bar{\partial}\bar{\xi}\;\frac{\partial\bar{\partial}}{\partial_-^2}h\,h\,\partial_-^2\bar{h}\,.\end{aligned} \tag{C.4}$$

The first two terms in (C.4) cancel against (C.3) and thus,

$$\delta^{\kappa^0}(\text{cubic terms})\;=\;-4\kappa\,\bar{\partial}\bar{\xi}\;\frac{\partial}{\partial_-}h\;\frac{\bar{\partial}}{\partial_-}h\,\partial_-^2\bar{h}\;+\;2\kappa\,\bar{\partial}^2\bar{\xi}\;\frac{\partial}{\partial_-^2}h\,h\,\partial_-^2\bar{h}\;+\;4\kappa\,\bar{\partial}\bar{\xi}\;\frac{\partial\bar{\partial}}{\partial_-^2}h\,h\,\partial_-^2\bar{h}\,. \tag{C.5}$$

In the κ^{-1} variation of the quartic vertex, we focus on the $\bar{\xi}$ terms since the ξ-dependent terms are simply complex conjugates.

$$\delta^{\kappa^{-1}}(\text{quartic terms})\;=\mathscr{A}\;+\;\mathscr{B}\;+\;\mathscr{C}\;+\;\mathscr{D}\,, \tag{C.6}$$

where

$$\begin{aligned}\mathscr{A}\;&=\;-\,4\kappa\,\bar{\partial}\bar{\xi}\;\partial h\;\frac{1}{\partial_-{}^2}\;\left(\frac{\bar{\partial}}{\partial_-}h\,\partial_-{}^3\,\bar{h}\;-\;h\,\partial_-{}^2\bar{\partial}\bar{h}\right) \\ &=\;4\kappa\,\bar{\partial}\bar{\xi}\;\frac{\partial}{\partial_-}h\;\frac{\bar{\partial}}{\partial_-}h\,\partial_-^2\bar{h}\;-\;4\kappa\,\bar{\partial}^2\bar{\xi}\;\frac{\partial}{\partial_-^2}h\,h\,\partial_-^2\bar{h}\;-\;4\kappa\,\bar{\partial}\bar{\xi}\;\frac{\partial\bar{\partial}}{\partial_-^2}h\,h\,\partial_-^2\bar{h}\,,\end{aligned} \tag{C.7}$$

$$\begin{aligned}\mathscr{D}\;&=\;-\,2\kappa^2\,\partial h\,\bar{\partial}^2\bar{\xi}\;\left(\frac{1}{\partial_-{}^2}\,(\partial_-h\,\partial_-\bar{h})\;-\;h\,\bar{h}\right) \\ &=\;-\,2\kappa\,\bar{\partial}^2\bar{\xi}\;\frac{\partial}{\partial_-}h\,\partial_-h\,\bar{h}\;+\;2\kappa\,\bar{\partial}^2\bar{\xi}\;\frac{\partial}{\partial_-^2}h\,h\,\partial_-^2\bar{h}\,.\end{aligned} \tag{C.8}$$

Taken together, the terms in (C.7) and the second term in (C.8) cancel the entire contribution from the cubic vertex. We find that the term,

$$\mathscr{B}=+\,2\kappa\,\bar{\partial}^2\bar{\xi}\,h\frac{1}{\partial_-}\,(\partial\bar{h}\,\partial_-h\;-\;\partial_-\bar{h}\,\partial h)\,, \tag{C.9}$$

and the first term in (C.8) give

$$\mathscr{B}-\;2\kappa\,\bar{\partial}^2\bar{\xi}\;\frac{\partial}{\partial_-}h\,\partial_-h\,\bar{h}=+\kappa\bar{\partial}^2\bar{\xi}\,hh\partial\bar{h}\,. \tag{C.10}$$

the third term has the form,

$$\begin{aligned}\mathscr{C} &= +2\kappa h\,\bar{\partial}\bar{\xi}\,\left(\,\bar{\partial}h\,\partial\bar{h} + \partial h\,\bar{\partial}\bar{h} - \partial_{-}\bar{h}\,\frac{\partial\bar{\partial}}{\partial_{-}}h - \partial_{-}h\,\frac{\partial\bar{\partial}}{\partial_{-}}\,\bar{h}\right) \\ &= +2\,\kappa\,\bar{\partial}\bar{\xi}\,h\partial h\bar{\partial}\bar{h} - \kappa\,\bar{\partial}^{2}\bar{\xi}\,hh\partial\bar{h} - 2\kappa h\,\bar{\partial}\bar{\xi}\,\partial_{-}\bar{h}\,\frac{\partial\bar{\partial}}{\partial_{-}}h\,.\end{aligned} \tag{C.11}$$

Thus,

$$\delta H^{(\kappa)}_{c,q} = \left(+2\kappa\,\bar{\partial}\bar{\xi}\,h\,\partial h\,\bar{\partial}\bar{h} - 2\kappa h\,\bar{\partial}\bar{\xi}\,\partial_{-}\bar{h}\,\frac{\partial\bar{\partial}}{\partial_{-}}h\right) + \text{c.c.} \tag{C.12}$$

Appendix D

Detailed computation of $\mathscr{D}\bar{h}$

At order κ^0, it is trivial to note that

$$\mathscr{D}\bar{h} = \partial\bar{h} \qquad \text{and} \qquad \bar{\mathscr{D}}h = \bar{\partial}h \, . \tag{D.1}$$

The order κ term of the Hamiltonian will come from

$$\mathscr{H} = \mathscr{D}\bar{h}(\kappa)\,\bar{\partial}h + \partial\bar{h}\,\bar{\mathscr{D}}h(\kappa) \, , \tag{D.2}$$

and it is easy to see that

$$(\mathscr{D}\bar{h})^{\kappa} = +2\kappa\,\frac{1}{\partial_-^2}\,\Big(\frac{\bar{\partial}}{\partial_-}h\,\partial_-^3\bar{h} - h\,\partial_-^2\bar{\partial}\bar{h}\Big) \, , \tag{D.3}$$

with the complex conjugate giving $(\bar{\mathscr{D}}h)^{\kappa}$. We now need to determine the contribution to $\mathscr{D}\bar{h}$ at order κ^2 from the terms in the Hamiltonian density, (5.16). The product, $\mathscr{D}\bar{h}(\kappa)\ \bar{\mathscr{D}}h(\kappa)$, reproduces one-half of the first term of the Hamiltonian density at order κ^2 (second line in (5.16)). The remaining half of the second line and all the other terms at order κ^2 have to come from

$$\mathscr{D}\bar{h}(\kappa^2)\,\bar{\partial}h + \partial\bar{h}\,\bar{\mathscr{D}}h(\kappa^2) \, . \tag{D.4}$$

The contribution to $\mathscr{D}\bar{h}$ from line 2 of (5.16) is

$$+2\,\kappa^2\,\frac{1}{\partial_-}\{\partial_-^2\bar{h}\frac{1}{\partial_-^3}(\partial_-^3 h\frac{\partial}{\partial_-}\bar{h} - \partial_-^2\partial h\bar{h})\} \tag{D.5}$$

$$+2\,\kappa^2\,\frac{1}{\partial_-}\{\,\frac{\partial}{\partial_-^4}(\bar{h}\partial_-^2 h)\partial_-^3\bar{h}\}, \tag{D.6}$$

with terms that cannot immediately be written in the form $X\bar{\partial}h$ or $Y\partial\bar{h}$ (remaining terms) being

$$-2\,\kappa^2\,h\partial_-^2\bar{h}\,\frac{\bar{\partial}}{\partial_-^4}(\partial_-^2\partial h\bar{h}) + \text{c.c.} \tag{D.7}$$

From line 3, $\mathscr{D}\bar{h}$ gets

$$-2\,\kappa^2\,\partial_-^2\bar{h}\,\frac{1}{\partial_-^4}(\partial_-^2 h\partial\bar{h}-2\partial_-\partial h\partial_-\bar{h})\;, \tag{D.8}$$

and the remaining term is

$$-4\,\kappa^2\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{\partial}\bar{h})\,\frac{1}{\partial_-^2}(\partial_-\partial h\partial_-\bar{h})\;. \tag{D.9}$$

From line 4, the contribution to $\mathscr{D}\bar{h}$ is

$$+2\,\kappa^2\,\partial_-\bar{h}\frac{1}{\partial_-^2}(\partial_-h\partial\bar{h}-2\partial h\partial_-\bar{h})\;, \tag{D.10}$$

and the remaining term is

$$-4\,\kappa^2\,\frac{1}{\partial_-}(\partial_-h\bar{\partial}\bar{h})\,\frac{1}{\partial_-}(\partial h\partial_-\bar{h})\;. \tag{D.11}$$

From line 5 - I, the contribution to $\mathscr{D}\bar{h}$ is

$$+6\,\kappa^2\,\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})\,\partial\bar{h}\;, \tag{D.12}$$

and from line 5-II,

$$-6\,\kappa^2\,\partial_-\bar{h}\,\frac{1}{\partial_-^2}(\partial_-h\partial\bar{h})\;. \tag{D.13}$$

Line 6 again has a contribution to $\mathscr{D}\bar{h}$,

$$\begin{aligned}&-2\,\kappa^2\,\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})\partial\bar{h}\\&+4\kappa^2\,h\bar{h}\,\partial\bar{h}\\&+4\kappa^2\,\frac{\partial}{\partial_-}\{\partial_-\bar{h}\,(\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})-h\bar{h})\}\end{aligned} \tag{D.14}$$

and the remaining term,

$$-4\kappa^2\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})\partial h\bar{\partial}\bar{h}\;. \tag{D.15}$$

When taken together, all the remaining terms can again be written in the form $X\bar{\partial}h$ or $Y\partial\bar{h}$ and add factors of X or Y to $\mathscr{D}\bar{h}$ or $\bar{\mathscr{D}}h$.