Weighted equidistribution theorems in the theory of modular forms

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

Shreeya Behera

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

April, 2018

Supervisor: Kaneenika Sinha

Certificate

This is to certify that this dissertation entitled Weighted equidistribution theorems in the theory of modular forms towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Shreeya Behera at Indian Institute of Science Education and Research under the supervision of Kaneenika Sinha, Assistant Professor, Department of Mathematics , during the academic year 2018.

Kaneenika Sinha

Committee:

Kaneenika Sinha

Baskar Balasubramanyam

To my dear family.

Declaration

I hereby declare that the matter embodied in the report entitled Weighted equidistribution theorems in the theory of modular forms are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Kaneenika Sinha and the same has not been submitted elsewhere for any other degree.

Shreeya Behera

Shreeya Behera

Acknowledgments

I am indebted to my supervisor Dr. Kaneenika Sinha for the amount of time and resources she devoted towards me during my project. I have been fortunate enough to have a guide like her who always wanted me to do well in every aspect. I am thankful to her for the constant guidance as well as all the emotional and mental support.

Thanks are due to Dr. Baskar Balasubramanyam for his encouragement and guidance, and to Prof. A. Raghuram for playing a crucial role in shaping my career in mathematics and devoting his time whenever I needed.

This journey would not have ever been possible without my family, my parents and my sister who believed in me and always supported my decisions. Their constant support, encouragement, and love inspired me to keep going.

Maheshweta and Aditya, my dear friends, deserve my utmost thanks for always being on my side and motivating me to aim high and never to give up. I am grateful to each of my fellow mathematicians - batchmates, seniors, and juniors - for discussing new mathematical concepts or doubts that I had. Also, I would like to thank every friend of mine for supporting me through all my ups and downs and keeping me cheerful always.

I am grateful to all my course instructors at IISER Pune for giving me the opportunity to learn in such an enthusiastic environment and finally, the Mathematics department for providing me with an excellent research experience. Also, I would like to acknowledge the use of Bhaskara Lab for providing Latex typing facility to write this thesis.

Abstract

We give three results concerning the distribution of eigenvalues of Hecke operators acting on spaces of modular cusp forms of weight k with respect to $\Gamma_0(N)$ by attaching some weights to them. These results extend some classical results. In the 1960s, M. Sato and J. Tate made a conjecture regarding the distribution laws for the Fourier coefficients at primes of a fixed Hecke eigenform. In 1997, J-P Serre considered a vertical analogue of the Sato-Tate conjecture: he fixed a prime p and considered the set of p-th Fourier coefficients of all Hecke eigenforms of weight k with respect to $\Gamma_0(N)$. He then derived a distribution law for such families as $N + k \to \infty$. Serre's theorem was made effective by M. R. Murty and K. Sinha, who found explicit error terms in Serre's theorem. His theorem was also generalized by C. C. Li in 2004 to derive an equidistribution law for Serre's families by attaching some suitable weights to the elements. In our first theorem, we extend the work of Murty and Sinha and find the error term in Li's weighted equidistribution theorem.

In 2006, H. Nagoshi proved two theorems. In his first theorem, he showed that by varying the primes p and the weights k, the Sato-Tate distribution law holds and in his second theorem, he proves a type of central limit theorem for the Fourier coefficients at primes of Hecke eigenforms with respect to $\Gamma_0(1)$ and weights $k \to \infty$. Our second and third results are the weighted analogues of Nagoshi's first and second theorems respectively, with the weights as defined by Li.

Notations

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ denote the sets of natural number, integers, rational numbers and real numbers respectively.
- $\mathbb{C}$ denotes the set of complex numbers given by $\{z \in \mathbb{C} : z = x+iy, x, y \in \mathbb{R}, i = \sqrt{-1}\}$. For a complex number z, $Re(z)$ will denote the real part, that is, x, $Im(z)$ will denote the imaginary part, that is, y, $|z|$ its absolute value and $\overline{z}$ its complex conjugate.
- $\mathbb{H}$ denotes the upper-half complex plane.
- Let S be a finite set. $|S|$ or $\#S$ will denote the cardinality of S.
- Let $a, b \in \mathbb{Z}$, $a|b$ denotes that a is a divisor of b. The greatest common divisor of a and b is denoted by $gcd(a, b)$.
- $\pi(x)$ denotes the number of primes less than equal to x.
- $\pi_N(x)$ denots the number of primes coprime to N and less than equal to x, that is, $\pi_N(x) := |\{p \leq x : (p, N) = 1\}|$.
- $\mathrm{ord}_p r$ denotes the highest power of p which can divide r.
- Let f and g be real valued functions with $g(x) \neq 0$ for $|x| \geq a$.

  $$f \approx g$$

  denotes

  $$\lim_{x \to \infty} \frac{f(x)}{g(x)} \to 1.$$
- Let $g(x)$ be a positive function. We say

  $$f = \mathrm{O}_K(g) \text{ or } f \ll_K (g)$$

  if there exists a non-negative real number a and a positive constant $C = C(K)$, depending on some quantity K, such that, $|f(x)| \leq C(K)g(x)$ for all x such that $|x| \geq a$; if the constant $C(K)$ is absolute then we simply say

  $$f = \mathrm{O}(g) \text{ or } f(x) \ll g(x).$$

- We write

$$f = \mathrm{o}(g)$$

  if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \to 0.$$

- Let n be a positive integer. The Euler-ϕ function is given by

$$\phi(N) = N \prod_{p|N} \left(1 - \frac{1}{p}\right)$$

  where the product runs over primes p dividing N.

- Let n be a positive integer. Then

$$\psi(N) = N \prod_{p|N} \left(1 + \frac{1}{p}\right)$$

  where the product runs over primes p dividing N.

Organization of chapters

In the first chapter, we describe the concepts related to equidistribution. In the second chapter, we define fundamental notions in the theory of modular forms.

In Chapter 3, we state classical results about the distribution of families of Hecke eigenvalues. We also state the primary results of this thesis which constitute original research, namely Theorems 3.3.1, 3.3.2 and 3.3.3.

In Chapter 4, we describe the Kuznietsov trace formula which forms a primary tool in the proofs of the new theorems mentioned above.

In Chapter 5, we prove Theorems 3.3.1, 3.3.2 and 3.3.3. Lemmas 5.1.1, 5.1.2, 5.2.4 and 5.2.5 are subsidiary results that are required to prove the main theorems.

Contents

Chapter 1

Equidistribution

Let x be any real number. Let $\{x\}$ denotes the fractinal part of x, that is, $\{x\} = x - [x]$, where $[x]$ represents the greatest integer less than equal to x.

1.1 Uniform distribution modulo 1

1.1.1 Definitions

Definition 1.1.1. A sequence of real numbers $(x_n)_{n=1}^{\infty}$ is uniformly distributed (u.d.) mod 1 if, for every $a, b \in [0, 1]$ with $a < b$, we have

$$\lim_{N \to \infty} \frac{|\{n \leq N : \{x_n\} \in [a, b]\}|}{N} = b - a \tag{1.1}$$

(This condition tells us that the proportion of the fractional part of the sequences $\{x_n\}$ lying in the interval $[a, b]$ is asymptotic to the length of the interval $[a, b]$, $b - a$.

Remark 1.1.2. Without changing the above definition, $[a, b]$ could be replaced by $(a, b]$, $[a, b)$ or (a, b).

For convenience, we will assume that each term of the sequence $(x_n)_{n=1}^{\infty}$ lies between 0

and 1, that is, $0 \leq x_n < 1$.
Let $\chi_{[a,b]}$ be the characteristic function of an interval $[a,b] \subset [0,1)$.
Then, equation (1.1) can be written as:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[a,b]}(x_n) = \int_0^1 \chi_{[a,b]}(x)dx \tag{1.2}$$

Lemma 1.1.3. *Let the sequence of real numbers $(x_n)_{n=1}^{\infty}$ be uniformly distributed mod 1. Then, for any $a \in [0,1)$, we have*

$$\#\{n \leq N : \{x_n\} = a\} = o(N)$$

Proof. Let us take $b = a + \epsilon$ for $\epsilon > 0$.
Now,

$$|\{n \leq N : \{x_n\} \in [a,b]\}| \leq 2N(b-a) = 2N\epsilon \tag{1.3}$$

for all $N \geq N_0(\epsilon)$.
Thus,

$$\lim_{N\to\infty} \frac{|\{n \leq N : \{x_n\} \in [a,b]\}|}{N} \leq 2\epsilon$$

and as ϵ can be arbitrarily small, we get the desired result. □

We now let $\mathbb{T}$ denote the unit circle $\mathbb{R}/\mathbb{Z}$.

Theorem 1.1.4. *The following are equivalent:*

(a) The sequence of real numbers $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1.

(b) For any real valued, continuous functions $f : \mathbb{T} \to \mathbb{R}$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_{\mathbb{T}} f(x)dx \tag{1.4}$$

Proof. We first show that (a) implies (b).

Let us now consider a partition $0 \le a_0 < a_1 < ... < a_k < 1$ and define a step function

$$s(x) = \sum_{i=0}^{k-1} s_i \chi_{[a_i, a_{i+1}]}(x), s_i \in \mathbb{R} \tag{1.5}$$

By (a), we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} s(x_n) = \int_0^1 s(x)dx$$

Now, let us take some $\epsilon > 0$. We can always find step functions f_1 and f_2 such that $f_1(x) \le f(x) \le f_2(x)$ for all $x \in \mathbb{T}$ and $\int_0^1 (f_2(x) - f_1(x)) \le \epsilon$.
Thus,

$$\begin{aligned}
\int_0^1 f(x)dx - \epsilon &\le \int_0^1 f(x)dx - \int_0^1 (f_2(x) - f_1(x))dx \\
&= \int_0^1 ((f(x) - f_2(x)) + f_1(x))dx \\
&\le \int_0^1 f_1(x)dx \text{ (since } f(x) \le f_2(x) \text{ for all } x \in \mathbb{T})
\end{aligned}$$

But, f_1 is a step function, hence applying (1.5), we have

$$\begin{aligned}
\int_0^1 f_1(x)dx &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_1(x_n) \\
&\le \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) \\
&\le \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_2(x_n)
\end{aligned}$$

But, again f_2 is a step function, hence we can apply (1.5) and get,

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_2(x_n) &= \int_0^1 f_2(x)dx \\
&= \int_0^1 (f_2(x) - f(x))dx + \int_0^1 f(x)dx \\
&\le \int_0^1 f(x)dx + \epsilon
\end{aligned}$$

So, finally we have

$$\int_0^1 f(x)dx - \epsilon \leq \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) \leq \int_0^1 f(x)dx + \epsilon$$

Since this is true for any $\epsilon > 0$, we have,

$$\int_0^1 f(x)dx = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n)$$

Conversly, assuming (b) we need to show that the sequence (x_n) is uniformly distributed mod 1. Let us consider the interval $[a, b] \subset \mathbb{T}$. Let $\epsilon > 0$. Then there exists two continuous functions f_1 and f_2 on $\mathbb{T}$ such that $f_1(x) \leq \chi_{[a,b]}(x) \leq f_2(x)$ for $x \in \mathbb{T}$ and $\int_0^1 (f_2(x) - f_1(x))dx < \epsilon$.
Now, by (b),

$$\int_0^1 f_j(x)dx = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_j(x_n), j = 1, 2$$

Thus,

$$\begin{aligned} b - a - \epsilon &= \int_0^1 \chi_{[a,b]}(x)dx - \epsilon \\ &\leq \int_0^1 f_2(x)dx - \epsilon \leq \int_0^1 f_1(x)dx \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_1(x_n) \leq \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[a,b]}(x_n) \\ &\leq \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_2(x_n) = \int_0^1 f_2(x)dx \\ &\leq \int_0^1 f_1(x)dx + \epsilon \leq b - a + \epsilon \end{aligned}$$

As ϵ is arbitrary, we have

$$lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[a,b]}(x_n) = b - a$$

Therefore $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1. $\Box$

We can more generally consider a continuous, complex valued function $f : \mathbb{T} \to \mathbb{C}$. Applying Theorem 1.1.4 to real and imaginary parts of f, we deduce the following theorem.

Theorem 1.1.5. *The following are equivalent:*

1. *The sequence of real numbers $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1.*

2. *For any complex valued, continuous functions $f : \mathbb{T} \to \mathbb{C}$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_{\mathbb{T}} f(x)dx \tag{1.6}$$

Now, we would like to state Weyl's Criterion which allows us to reduce equidistribution questions to bounds on exponential sums. Before that, we would like to recall a theorem from analysis which will be required in proving Weyl's Criterion.

1.1.2 Weierstrass approximation theorem

Let us first recall what Fourier series are.
Let $f : \mathbb{R} \to \mathbb{T}$ be a continuous function. Then, the Fourier coefficient of f for any integer s is given by

$$\hat{f}(s) = \int_0^1 f(t)e^{-2\pi i s t}dt$$

The Fourier series of f is given by $\sum_s \hat{f}(s)e^{2\pi i s x}$.
Now, let us state and prove Fejér's Theorem which will eventually lead us to the Weierstrass approximation theorem.

Theorem 1.1.6. (Fejér) *If $f : \mathbb{T} \to \mathbb{C}$ is continuous, then*

$$\sigma_n(f,t) = \sum_{s=-n}^{n} \frac{n+1-s}{n+1} \hat{f}(s)e^{2\pi i s t} \to f(t)$$

uniformly as $n \to \infty$.

Before proceeding further, we first prove some properties of Fejér's kernel which would help us in proving Fejér's theorem.
Fejér's Kernel is expressed in either of the following two equivalent ways:

$$K_n(t) = \sum_{s=-n}^{n} \frac{n+1-s}{n+1} e^{2\pi ist} \text{ for any real t.} \tag{1.7}$$

$$K_n(t) = \frac{1}{n+1} \left(\frac{sin(n+1)\pi t}{sin(\pi t)} \right)^2 \text{ for } t \notin \mathbb{Z}. \tag{1.8}$$

Properties of Fejér's Kernel

i $\int_0^1 K_n(t)dt = 1$

ii $K_n(t) \geq 0$

iii For any fixed $0 < \delta < 1/2$,

$$\lim_{n \to \infty} \int_{|t|>\delta} K_n(t)dt = 0$$

i.e. $K_n(t) \to 0$ uniformly outside $[-\delta, \delta]$.

Proof. (i) Using 1.7 , we have

$$\begin{aligned}
\int_0^1 K_n(t)dt &= \int_0^1 \sum_{s=-n}^{n} \frac{n+1-s}{n+1} e^{2\pi ist} dt \\
&= \frac{1}{n+1} \sum_{s=-n}^{n} (n+1-s) \int_0^1 e^{2\pi ist} dt \\
&= \frac{1}{n+1} \sum_{s=0} (n+1-s)
\end{aligned}$$

The inner integral takes the value 1 only when $s = 0$ and is zero for rest all values. Thus we get the desire result.
(ii) It follows from equation 1.8 directly.
(iii) For $0 < t < 1$, $|sin(\pi t)| \geq 2t$. Thus

$$K_n(t) \leq \frac{1}{(n+1)(2t)^2} = \frac{1}{(n+1)(4t^2)}$$

Thus,

$$\int_{\delta}^{1/2} K_n(t)dt \leq \int_{\delta}^{1/2} \frac{1}{(n+1)(4t^2)}dt < \int_{\delta}^{\infty} \frac{1}{(n+1)(4t^2)}dt = \frac{1}{4(n+1)\delta}$$

Since, $K_n(t) = K_n(-t)$, we have,

$$\int_{-1/2}^{-\delta} K_n(t)dt < \frac{1}{4(n+1)\delta}.$$

Thus, $\int_{\delta<|t|<1/2} K_n(t)dt < \frac{1}{2(n+1)\delta} \Rightarrow \int_{\delta<|t|} K_n(t)dt < \frac{1}{2(n+1)\delta}$

So, as $n \to \infty$, $\int_{\delta<|t|} K_n(t)dt$ converges to 0 uniformly.

□

We observe that as $n \to \infty$,

$$\begin{aligned}
\sigma_n(f,t) &= \sum_{s=-n}^{n} \frac{n+1-s}{n+1} \hat{f}(s) e^{2\pi ist} \\
&= \sum_{s=-n}^{n} \frac{n+1-s}{n+1} e^{2\pi ist} \int_{-1/2}^{1/2} f(x)e^{-isx}dx (\text{ by definition of } \hat{f}(s)) \\
&= \int_{-1/2}^{1/2} f(x)K_n(t-x)dx \\
&= \int_{-1/2}^{1/2} f(t-x)K_n(x)dx \\
&\approx \int_{-\delta}^{\delta} f(t-x)K_n(x)dx \text{ (for large n and small } \delta \text{ and using property (iii))} \\
&\approx f(t) \int_{-\delta}^{\delta} K_n(x)dx \text{ (as f is continuous)} \\
&= f(t) \text{ (using property (i))}
\end{aligned}$$

Thus, $\sigma_n(f,t) \approx f(t)$ for large value of n.

Proof of Theorem 1.1.6: As f is a continuous and periodic function, it is bounded. Let us say $|f(x)| \leq M$ for all x. Now, for any $\epsilon > 0$ there exist $\delta > 0$ depending on ϵ such that $|f(x) - f(t)| \leq \epsilon/2$ whenever $|x - t| < \delta$. Also, there exist a positive N depending on δ by property (iii), such that for all $n \geq N$,

$$K_n(x) \leq \epsilon/4M \text{ for all } x \notin [-\delta, \delta].$$

Then, we have

$$\begin{aligned}
|\sigma_n(f,t) - f(t)| &= \left|\int_0^1 f(t-x)K_n(x)dx - f(t)\right| \\
&= \left|\int_0^1 f(t-x)K_n(x)dx - f(t)\int_0^1 K_n(x)dx\right| \\
&= \left|\int_0^1 f(t-x)K_n(x)dx - \int_0^1 f(t)K_n(x)dx\right| \\
&= \left|\int_0^1 (f(t-x) - f(t))K_n(x)dx\right| \\
&\leq \left|\int_{x\in[-\delta,\delta]} (f(t-x) - f(t))K_n(x)dx\right| + \left|\frac{1}{2\pi}\int_{x\notin[-\delta,\delta]} (f(t-x) - f(t))K_n(x)dx\right|
\end{aligned}$$

Now, we will use property (i) and the fact that $(f(t-x) - f(t)) \leq \epsilon/2$ so that the first integral is bounded by $\epsilon/2$. And, in the second integral, we bound $(f(t-x) - f(t))$ by $2M$ and the fact that $K_n(x) \leq \epsilon/4M$, the second integral too is bounded by $\epsilon/2$.
Thus $|\sigma_n(f,t) - f(t)| \leq \epsilon$ and our proof is complete.

Now, let us state and prove Weierstrass approximation theorem.

Theorem 1.1.7. Weierstrasss theorem :
Let $f : \mathbb{T} \to \mathbb{C}$ be continuous and periodic with period 1. Then, for every $\epsilon > 0$, there exists a trigonometric polynomial ϕ such that

$$\sup_{t\in[0,1]} |f(t) - \phi(t)| < \epsilon \tag{1.9}$$

Proof. We observe that $\sigma_n(f,t) = \int_{-1/2}^{1/2} f(t-x)K_n(x)dx = \int_0^1 f(t-x)K_n(x)dx$ (as f is a periodic function of period 1, we can change the limit from (-1/2 to 1/2) to (0 to 1)). $K_n(t)$ is a trigonometric polynomial. Hence, taking $\phi(t) = \sigma_n(f,t)$, we get the desired result. □

Thus, we see that, any periodic, continuous function $f : \mathbb{R} \to \mathbb{C}$ can be approximated by a sequence of trigonometric polynomials.

Now, we are ready to state and prove Weyl's Criterion.

1.1.3 Weyl's Criterion

Theorem 1.1.8. **[Weyl, 1916]** *A sequence $(x_n)_{n=1}^{\infty}$ is uniformly distributed mod 1 if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i l x_n}=0,\ \textit{for all integers } l\neq 0 \tag{1.10}$$

Proof. Let $(x_n)_{n=1}^{\infty}$ be a sequence which is u.d. mod 1. Let us take $g(x)=e^{2\pi i l x}$, then using Theorem 1.1.5 we have,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i l x_n}=\int_0^1 e^{2\pi i l x}dx=0,\ \text{for } l\in\mathbb{Z} \text{ and } l\neq 0.$$

For the converse, let $(x_n)_{n=1}^{\infty}$ satisfy (1.10). Let $g:\mathbb{T}\to\mathbb{C}$ be a complex valued continuous function. We need to show that $\lim_{N\to\infty}\dfrac{1}{N}\sum_{n=1}^{N}g(x_n)=\int_0^1 g(x)dx$. Now let us take some arbitrary $\epsilon>0$. So, by Weierstrass approximation theorem, there exists a trignometric polynomial $\phi(x)$ that is, a finite linear combination of functions of the type $e^{2\pi i l x}, l\in\mathbb{Z}$, with complex coefficients, such that

$$\sup_{0\leq x\leq 1}|g(x)-\phi(x)|\leq\epsilon \tag{1.11}$$

Thus,

$$\begin{aligned}\left|\int_0^1 g(x)dx-\frac{1}{N}\sum_{n=1}^{N}g(x_n)\right| &\leq \left|\int_0^1 (g(x)-\phi(x))dx\right|+\left|\int_0^1\phi(x)dx-\frac{1}{N}\sum_{n=1}^{N}g(x_n)\right|\\ &\leq \left|\int_0^1 (g(x)-\phi(x))dx\right|+\left|\int_0^1\phi(x)dx-\frac{1}{N}\sum_{n=1}^{N}\phi(x_n)\right|\\ &+\left|\frac{1}{N}\sum_{n=1}^{N}(\phi(x_n)-g(x_n))\right|\end{aligned}$$

But by 1.11, the first term and the last term are $\leq\epsilon$.

Now, as we have $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i l x_n} = 0$, so if we take N large enough then we would have $\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i l x_n}\right|$ less than arbitrarily small number. By choosing this arbitrarily small number suitably, we can have $\frac{1}{N}\sum_{n=1}^{N} \phi(x_n)$ less than ϵ as ϕ is a finite linear combination of functions of the type $e^{2\pi i l x}, l \in \mathbb{Z}$, with complex coefficients. Therefore, the second term is less than ϵ. Therefore, by Theorem 1.1.5, $(x_n)_{n=1}^{\infty}$ is u.d. mod 1. $\square$

Now, we would like to introduce an important branch of trigonometric polynomials which provide a good approximation to the characteristic functions of intervals on $\mathbb{R}$, known as the Selberg-Beurling Polynomials.

1.2 Selberg-Beurling Polynomials

Selberg-Beurling Polynomials reduce the estimation of counting functions to evaluating finite exponential sums. Interested reader may look for detailed exposition by Montgomery (see [6], Chapter 1) or may look into the paper of Vaaler [16].
Let $I = [\alpha, \beta] \subset \left[\frac{-1}{2}, \frac{-1}{2}\right]$ and $M \geq 1$ be an integer. One can construct trigonometric polynomials $S_M^-(x)$ and $S_M^+(x)$ of degree less than or equal to M, respectively called the minorant and majorant Beurling-Selberg Polynomials for the interval I,

$$S_M^{\pm}(x) = \sum_{|m|\leq M} \hat{S}_M^{\pm}(m) e(mx),$$

such that

- For all $x \in \mathbb{R}, S_M^-(x) \leq \chi_I(x) \leq S_M^+(x)$
- $$\int_{-1/2}^{1/2} S_M^{\pm}(x)dx = \beta - \alpha \pm \frac{1}{M+1}.$$
- For $0 \leq |m| \leq M$,
$$|\hat{S}_M^{\pm}(m) - \hat{\chi}_I(m)| \leq \frac{1}{M+1} \tag{1.12}$$

Henceforth, we will use the following notation: for an interval $I = [A, B] \subset [-2, 2]$, we choose a subinterval

$$I_1 = [\alpha, \beta] \subset \left[0, \frac{1}{2}\right]$$

such that

$$\theta \in I_1 \iff 2cos(2\pi\theta) \in I$$

For $M \geq 1$, let

$$S^{\pm}_{M,1}(x) = \sum_{|m|\leq M} \hat{S}^{\pm}_{M,1}(m)e(mx)$$

denote the majorant and minorant Beurling-Selberg Polynomials for the interval I_1.
With view towards calculation in later sections, we denote, for $0 \leq |m| \leq M$,

$$\hat{\mathcal{S}}^{\pm}_{\mathrm{M}}(m) = \hat{S}^{\pm}_{M,1}(m) + \hat{S}^{\pm}_{M,1}(-m).$$

We have, for $1 \leq |m| \leq M$,

$$\hat{S}^{\pm}_{M,1}(m) = \hat{\chi_I}(m) + O(\frac{1}{M+1}) = \frac{e(-m\alpha) - e(-m\beta)}{2\pi i m} + O(\frac{1}{M+1})$$

Thus,

$$\hat{\mathcal{S}}^{\pm}_{\mathrm{M}}(m) = \hat{\chi_I}(m) + \hat{\chi_I}(-m) + O(\frac{1}{M+1}) = \frac{sin(2\pi m\beta) - sin(2\pi m\alpha)}{\pi m} + O(\frac{1}{M+1}) \quad (1.13)$$

1.3 van der Corput's Inequality

We now review an important inequality which is useful in the study of uniform distribution. It was introduced by Weyl and van der Corput.
In order to prove van der Corput's Inequality, we would need the following lemma:

Lemma 1.3.1. (Cauchy-Schwarz inequality) *Let $a_1, a_2, ..., a_n \in \mathbb{C}$ and $b_1, b_2, ..., b_n \in \mathbb{C}$, then,*

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 \leq \sum_{k=1}^{n} |a_k|^2 \sum_{i=1}^{n} |b_j|^2$$

Proof. Let us expand the term $\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j - a_jb_i)^2$. We get

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j - a_jb_i)^2 &= \sum_{i=1}^{n} a_i^2 \sum_{j=1}^{n} b_j^2 + \sum_{i=1}^{n} b_i^2 \sum_{j=1}^{n} a_j^2 - 2\sum_{i=1}^{n} a_ib_i \sum_{j=1}^{n} b_ja_j \\ &= 2\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) - 2\left(\sum_{i=1}^{n} a_ib_i\right)^2\end{aligned}$$

The left hand side of the above equation is greater than 0. Hence,

$$\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right).$$

□

Theorem 1.3.2. (van der Corput, 1931) *Let N be a positive integer and y_n be a complex number for $1 \leq n \leq N$ and let $y_n = 0$ if $n < 1$ or $n > N$. Let H be an integer with $1 \leq H \leq N$. Then*

$$\left|\sum_{n=1}^{N} y_n\right|^2 \leq \frac{(N+H)}{(H+1)}\sum_{n=1}^{N}|y_n|^2 + \frac{2(N+H)}{(H+1)}\sum_{h=1}^{H}\left(1 - \frac{h}{(H+1)}\right)\left|\sum_{n=1}^{N-h} y_{n+h}\overline{y_n}\right|$$

Proof. Let us try to expand the term $(H+1)^2\left|\sum_n y_n\right|^2$. Clearly,

$$(H+1)^2\left|\sum_n y_n\right|^2 = \left|\sum_{r=0}^{H}\sum_n y_n\right|^2 = \left|\sum_{r=0}^{H}\sum_n y_{n+r}\right|^2 = \left|\sum_n\sum_{r=0}^{H} y_{n+r}\right|^2$$

Since $0 \leq r \leq H$ and $y_n = 0$ if $n < 1$ or $n > N$, then when $r = 0, y_{n+r} = 0$ if $n < 1$ or $n > N$ and when $r = H, y_{n+r} = 0$ if $n < 1 - H$ or $n > N - H$. Thus the interval for n such that $y_{n+r} \neq 0$ for $0 \leq r \leq H$ is $n \in [-H+1, N]$. Thus, using Cauchy-Schwarz inequality by taking $a_i = 1$ and $b_i = \sum_{r=0}^{H} y_{i+r}$ we have

$$\left|\sum_{n=-H+1}^{N}\sum_{r=0}^{H} y_{n+r}\right|^2 = \sum_{k=-H+1}^{N} 1 \sum_{n=-H+1}^{N}\left|\sum_{r=0}^{H} y_{n+r}\right| = (N+H)\sum_{n=-H+1}^{N}\left|\sum_{r=0}^{H} y_{n+r}\right|$$

$$
\begin{aligned}
(H+1)^2\left|\sum_n y_n\right|^2 &\leq (N+H)\sum_n\left|\sum_{r=0}^{H} y_{n+r}\right|^2 = (N+H)\sum_n\left(\sum_{r=0}^{H} y_{n+r}\sum_{k=0}^{H}\overline{y_{n+k}}\right)\\
&= (N+H)\sum_n\sum_{r=0}^{H}\sum_{k=0}^{H} y_{n+r}\overline{y_{n+k}}\\
&= (N+H)\left[\sum_n\sum_{r=k=0}^{H}|y_{n+r}|^2+\sum_n\sum_{r\neq k} y_{n+r}\overline{y_{n+k}}\right]\\
&= (N+H)\left[(H+1)\sum_n|y_n|^2+\sum_n\sum_{r\neq k} y_{n+r}\overline{y_{n+k}}\right]
\end{aligned}
$$

Now, let us combine the terms corresponding to (r,k) and (k,r) to get second term in the inner sum as

$$
\sum_n\sum_{r\neq k} y_{n+r}\overline{y_{n+k}} = 2Re\left(\sum_n\sum_{r=0}^{H}\sum_{(r,k);k<r} y_{n+r}\overline{y_{n+k}}\right)
$$

Now, taking $m=n+k$, the above term can be written as

$$
2Re\left(\sum_m\sum_{r=0}^{H}\sum_{k<r} y_{m-k+r}\overline{y_m}\right) = 2Re\left(\sum_m\sum_{h=1}^{H} y_{m+h}\overline{y_m}\sum_{k<r;r-k=h} 1\right)
$$

But, the innermost sum is $H+1-h$. Hence,

$$
\begin{aligned}
(H+1)^2\left|\sum_n y_n\right|^2 &\leq (N+H)\left[(H+1)\sum_n|y_n|^2+2Re\left(\sum_m\sum_{h=1}^{H} y_{m+h}\overline{y_m}(H+1-h)\right)\right]\\
&= (N+H)\left[(H+1)\sum_n|y_n|^2+2\left(\sum_m\sum_{h=1}^{H} y_{m+h}\overline{y_m}(H+1-h)\right)\right]\\
\Rightarrow \left|\sum_n y_n\right|^2 &\leq \frac{(N+H)}{(H+1)^2}\left[(H+1)\sum_n|y_n|^2+2\left(\sum_m\sum_{h=1}^{H} y_{m+h}\overline{y_m}(H+1-h)\right)\right]\\
&= \frac{(N+H)}{(H+1)}\sum_n|y_n|^2+2\frac{(N+H)}{(H+1)}\sum_{h=1}^{H}\left(1-\frac{h}{H+1}\right)\sum_m y_{m+h}\overline{y_m}\\
&= \frac{(N+H)}{(H+1)}\sum_n|y_n|^2+2\frac{(N+H)}{(H+1)}\sum_{h=1}^{H}\left(1-\frac{h}{H+1}\right)\sum_n y_{n+h}\overline{y_n}
\end{aligned}
$$

Thus we get the desired result. □

Corollary 1.3.3. (van der Corput, 1931) *Let h be any positive integer. If the sequence $x_{n+r} - x_n$ is u.d. mod 1, the the sequence x_n is uniformly distributed mod 1.*

Proof. Let us take $y_n = e^{2\pi i m x_n}, m \neq 0$. By using Theorem 1.3.2, we have

$$\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i m x_n}\right|^2 \leq \frac{1+H/N}{H+1} + \frac{2(N+H)}{N^2(H+1)}\sum_{h=1}^{H}\left(1-\frac{h}{(H+1)}\right)\left|\sum_{n=1}^{N-h} e^{2\pi i m(x_{n+r}-x_n)}\right|$$

The second term in the above equation can be written as:

$$\frac{2(N+H)}{N^2(H+1)}\sum_{h=1}^{H}\left(1-\frac{h}{(H+1)}\right)\left|\sum_{n=1}^{N-h} e^{2\pi i m(x_{n+r}-x_n)}\right|$$
$$= \frac{2(1+H/N)}{(H+1)}\sum_{h=1}^{H}\left(1-\frac{h}{(H+1)}\right)\left|\frac{1}{N}\sum_{n=1}^{N-h} e^{2\pi i m(x_{n+r}-x_n)}\right|$$

We are given that $x_{n+r} - x_n$ is u.d. mod 1. Thus, the innermost term, that is, $\left|\frac{1}{N}\sum_{n=1}^{N-h} e^{2\pi i m(x_{n+r}-x_n)}\right| \to 0$ as $N \to \infty$. Hence $\frac{2(N+H)}{N^2(H+1)}\sum_{h=1}^{H}\left(1-\frac{h}{(H+1)}\right)\left|\sum_{n=1}^{N-h} e^{2\pi i m(x_{n+r}-x_n)}\right|$ vanishes as $N \to \infty$. Therefore, for sufficiently large N, we have,

$$\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i m x_n}\right|^2 << \frac{1}{H}$$

Thus, taking H large enough, we get the desired result. □

1.4 Examples of uniformly distributed sequences

Example 1.4.1. If θ is an irrational number, the the sequence $x_n = n\theta$ is u.d.
This can be seen easily using Weyl's criteria. We need to show that $\lim_{N\to\infty} \frac{1}{N}|\sum_{n=1}^{N} e^{2\pi i m n\theta}| \to 0$ for $m = 1, 2, 3, ...$ We can see that

$$\sum_{n=1}^{N} e^{2\pi i m n\theta} = \frac{e^{2\pi i m(N+1)\theta} - 1}{e^{2\pi i m\theta} - 1}$$

which is bounded by $2/|e^{2\pi i m\theta}|$. The denominator is nonzero as θ is irrational. Thus, $\lim_{N\to\infty} \frac{1}{N}|\sum_{n=1}^{N} e^{2\pi i m n\theta}| \to 0$

Example 1.4.2. If θ is a rational number, the the sequence $x_n = n\theta$ is not u.d.
Let $\theta = \dfrac{a}{b}$ with a, b coprime integers. Then for $m = b$, we have

$$\sum_{n=1}^{N} e^{2\pi i b(na/b)} = N$$

Thus Weyl's criterion fails in this case.

Example 1.4.3. If the sequence $(x_n)_{n=1}^{\infty}$ is u.d. mod 1, then the sequence $(mx_n)_{n=1}^{\infty}$ is u.d. mod 1 for a non-zero integer m.

This is a simple consequence of Weyl's Criterion. We have , for $l \neq 0$, and $m \neq 0$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i l m x_n} = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n} \text{ for } k = lm \neq 0$$

Since $(x_n)_{n=1}^{\infty}$ is u.d. mod 1, $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n} = 0$.
Thus, we are done.

Example 1.4.4. If the sequence $(x_n)_{n=1}^{\infty}$ is u.d. mod 1, then the sequence $(x_n + c)_{n=1}^{\infty}$ is u.d. mod 1 for some constant c.

This is again a consequence of Weyl's Criterion. We see that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i l(x_n+c)} = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i l x_n} C$$

where $C = e^{2\pi i l c}$.
As $(x_n)_{n=1}^{\infty}$ is u.d. mod 1, $C \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i l x_n} = 0$. Hence, $(x_n + c)_{n=1}^{\infty}$ is u.d. mod 1 for some constant c.

Example 1.4.5. If the sequence $(x_n)_{n=1}^{\infty}$ is u.d. mod 1 and $(y_n) \to c$ as $n \to \infty$, then $(x_n + y_n)_{n=1}^{\infty}$ is u.d. mod 1.

Let us assume that $c = 0$. As for any other case we can refer to example 1.4.4. Let $[a, b]$

be any interval. Let us now take $\epsilon > 0$ such that $2\epsilon < b - a$ and $|y_n| < \epsilon$ for all $n > N_0$. Then,

$$|n \leq N : (x_n + y_n) \in [a, b]| \geq |n \leq N : (x_n) \in [a + \epsilon, b - \epsilon]| - N_0$$

and

$$|n \leq N : (x_n + y_n) \in [a, b]| \leq |n \leq N : (x_n) \in [a + \epsilon, b - \epsilon]| + N_0$$

Since (x_n) is u.d. mod 1, $\lim_{N\to\infty} \frac{1}{N}|n \leq N : (x_n) \in [a + \epsilon, b - \epsilon]| = b - a - 2\epsilon$
Thus

$$b - a - 2\epsilon - \lim_{N\to\infty} \frac{N_0}{N} \leq \lim_{N\to\infty} \frac{1}{N}|n \leq N : (x_n + y_n) \in [a, b]| \leq b - a - 2\epsilon + \lim_{N\to\infty} \frac{N_0}{N}$$

$$\Rightarrow b - a - 2\epsilon \leq \lim_{N\to\infty} \frac{1}{N}|n \leq N : (x_n + y_n) \in [a, b]| \leq b - a - 2\epsilon$$

Therefore, $\lim_{N\to\infty} \frac{1}{N}|n \leq N : (x_n + y_n) \in [a, b]| = b - a$.
Hence, $(x_n + y_n)_{n=1}^{\infty}$ is u.d. mod 1 for $(y_n) \to c$.

Example 1.4.6. The sequence $(m^2\theta)_{m=1}^{\infty}$ is u.d. mod 1 for θ irrational.

Let us consider the sequence $(m + k)^2\theta - m^2\theta = 2km\theta + k^2\theta$. Here we see that the first term $2km\theta$ is mod 1 by example 1.4.3. Also, $k^2\theta$ is a constant term, hence by example 1.4.4, $2km\theta + k^2\theta$ is u.d. mod 1. Therefore, using corollary 1.3.3 we get the desire result.

1.5 Equidistribution

While many sequences are uniformly distributed with respect to the Lebesgue measure, we do come up with important sequences which are not distributed with respect to the Lebesgue measure but are distributed with respect to a different probablity measure $d\mu$.

Definition 1.5.1. Let X be a compact Hausdorff space with a measure $d\mu$. Let S_1, $S_2, \ldots$ be a sequence of finite nonempty subsets of X, such that each subset S_i has cardinality $|S_i|$. We say that $\{S_i\}$ is equidistributed in X with respect to $d\mu$ (or μ-equidistributed) if for any continuous complex-valued function f on X,

$$\lim_{i\to\infty} \frac{\sum_{x\in S_i} f(x)}{|S_i|} = \int_X f(x)d\mu(x). \tag{1.14}$$

Remark 1.5.2. If we take a sequence $\{x_n\}_{n=1}^{\infty}$ in X and denote S_i to be the set $\{x_1, x_2, \ldots x_i\}$, then Definition 1.5.1 gives us a notion of equidistribution for a sequence S_i in X.

Note: For the families of interest to us, we will take $X = [0, 1]$.

Let (x_n) be a sequence which is not uniformly distributed mod 1(that is not equidistributed with respect to the Lebesgue measure), then the Weyl's Criterion would fail for the given sequence. Therefore,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i m x_n} \neq 0 \text{ for some } m \in \mathbb{Z}, m \neq 0.$$

If $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i m x_n}$ exists, then let us denote it by c_m, known as the **Weyl limits**.

$$c_m = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i m x_n} \tag{1.15}$$

Let us suppose c_m exists for every integer m. Then, a theorem of Schoenberg and Wiener (see [4], Theorem 7.5) gives us a technique to construct a measure μ such that the sequence (x_n) is equidistributed with respect to μ. We have the following theorem which help us to find the measure μ.

Theorem 1.5.3. *If the limits c_m exist for every integer m and*

$$\lim_{M\to\infty} \sum_{m=1}^{M} |c_m|^2 = 0,$$

then the sequence (x_n) is equidistributed with respect to the measure

$$d\mu(x) = g(-x)dx, \text{ where } g(x) = \sum_{m\in\mathbb{Z}} c_m e^{2\pi i m x}.$$

1.6 Weighted equidistribution

In this section, we introduce the concept of weighted equidistribution. We attach some positive weights to each element in the sequence and see how the sequence is distributed.

1.6.1 Definition

Definition 1.6.1. Let X be a compact Hausdorff space with a measure $\mathrm{d}\mu$. Let S_1, $S_2, ...$ be a sequence of finite nonempty subsets of X, such that each subset S_i has cardinality $|S_i|$. Suppose each x in S_i has a real, positive weight ω_{ix} assigned to it. The sequence $\{S_i\}$ is ω-distributed with respect to a measure $d\mu$ if for any continuous complex valued function f on X,

$$\lim_{i\to\infty} \frac{\sum_{x\in S_i} \omega_{ix} f(x)}{\sum_{x\in S_i} \omega_{ix}} = \int_X f(x) d\mu(x). \tag{1.16}$$

By taking $\omega_{ix} = 1$ for each x, we recover Definition 1.5.1 without weights.

Proposition 1.6.2. *Let $X = [0, 1]$. For any interval $A \subseteq X$, let $\chi_A(x)$ denote the characteristic function of A. With the same notation as above, the sequence $\{S_i\}$ is ω-distributed with respect to the measure $d\mu$ if and only if for every interval $I = [a, b] \subseteq X$,*

$$\lim_{i\to\infty} \frac{\sum_{x\in S_i} \omega_{ix} \chi_I(x)}{\sum_{x\in S_i} \omega_{ix}} = \int_I d\mu(x). \tag{1.17}$$

Theorem 1.6.3. [Weighted version of Weyl's criterion] *For an integer m, let us define the* **weighted m-th Weyl limits**, *C_m as*

$$C_m := \lim_{i\to\infty} \frac{\sum_{x\in S_i} \omega_{ix} e(mx)}{\sum_{x\in S_i} \omega_{ix}}.$$

A sequence of finite nonempty subsets S_1, $S_2, \ldots$ in $[0, 1]$ is ω-distributed with respect to a measure $d\mu$ if and only if the limit C_m exists for all integers m and

$$C_m := \lim_{i\to\infty} \frac{\sum_{x\in S_i} \omega_{ix} e(mx)}{\sum_{x\in S_i} \omega_{ix}} = \int_0^1 e(mx) d\mu(x).$$

In this case, the sequence $\{S_i\}$ is ω-distributed with respect to the measure

$$d\mu(x) = g(-x)dx, \text{ where } g(x) = \sum_{m \in \mathbb{Z}} C_m e(mx).$$

Chapter 2

Modular forms

In this section, we review some basic properties of modular forms and Hecke operators. The family of sequences for which we study the distribution properties in this thesis arises using the concept of modular forms. Thus, these form the backbone of the problems in this thesis. This chapter is based on ([7], chapter 2-6).

2.1 Modular group

Definition 2.1.1. (General Linear Group) Let R be a commutative ring with 1. Then the set $\mathrm{GL}_2(R)$ denotes the ring of 2×2 matrices which are invertible.

Definition 2.1.2. (Special Linear Group) Let R be a commutative ring with 1. Then the set $\mathrm{SL}_2(R)$= $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in R; ad - bc = 1 \right\}$ forms a group with matrix multiplication called the special linear group.

There is an action of $\mathrm{SL}_2(R)$ on $\mathbb{C}$, given by

$$\gamma z = \frac{az + b}{cz + d}; \gamma \in \mathrm{SL}_2(R)$$

Definition 2.1.3. (Full Modular Group) Let us take $R = \mathbb{Z}$ in the *Special linear group.*

Then, the set

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}; ad - bc = 1 \right\}$$

forms a group with matrix multiplication called the full modular group.

Let us define two matrices S and T as $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

We have the following theorem:

Theorem 2.1.4. *The matrices S and T generates $SL_2(\mathbb{Z})$.*

Proof. We see that whenever S or T^n is multiplied to any matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we get the following:

$$S \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -c & -d \\ a & b \end{bmatrix}, \text{ the rows get interchanged with some sign change}$$

,

$$T^n \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+nc & b+nd \\ c & d \end{bmatrix} \text{ the first row gets added by } n \text{ times the second row}$$

Now, let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Let us assume $|a| \geq |c|$, if not then using S, we can interchange the row with a sign change. Now, using division formula, we have $a = cq + r$, where $0 \leq r < c$. So, we have,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} cq+r & b \\ c & d \end{bmatrix}.$$

Now multiplying T^{-q} to the above matrix we get the matrix $\begin{bmatrix} r & b-dq \\ c & d \end{bmatrix}$. Again using S, we interchange the rows and follow the above steps till we get the lowermost left entry as 0 and get a matrix of the form $\begin{bmatrix} p & q \\ 0 & r \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. So, we have $pq = 1 \Rightarrow p = q = \pm 1$. But, then $S^2 = 1$. Hence,

$$\begin{bmatrix} p & q \\ 0 & r \end{bmatrix} = \begin{bmatrix} \pm 1 & q \\ 0 & \pm 1 \end{bmatrix} = T^q \text{ or } S^2 T^{-q}$$

Thus, S and T generates $\mathrm{SL}_2(\mathbb{Z})$. □

2.1.1 Subgroups of the modular group

In this section, we define some important subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

Definition 2.1.5. (Principal congruence subgroup)
A principal congruence subgroup of level N is the group given by :

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\bmod N) \right\} \tag{2.1}$$

We see that it is a group of matrices in $\mathrm{SL}_2(\mathbb{Z})$ which are congruent to the identity matrix modulo N.

Definition 2.1.6. (Congruence Subgroup)
A subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is called a *congruence subgroup* if $\Gamma(N) \subset \Gamma$ for some N. Since $\Gamma(N)$ is of finite index in $\mathrm{SL}_2(\mathbb{Z})$, it follows that any congruence subgroup is also of finite index in $\mathrm{SL}_2(\mathbb{Z})$.

Definition 2.1.7. (Level of a Congruence Group)
Let N be the smallest positive integer such that $\Gamma(N) \subset \Gamma$. Then, Γ is said to be of level N.

Let us now define two special subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

Definition 2.1.8. (Hecke Subgroup)
Hecke subgroup is defined as a subgroup of $\mathrm{SL}_2(\mathbb{Z})$of the form

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod N \right\} \tag{2.2}$$

for some $N \geq 1$. It can be seen that, this is a group, and $\Gamma(N) \subset \Gamma_0(N)$, so these are congruence subgroups.

Another special subgroup of $\mathrm{SL}_2(\mathbb{Z})$is given by:

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod N \text{ and } d \equiv 1 \pmod N \right\} \tag{2.3}$$

We note that $\Gamma_1(N) \subset \Gamma_0(N)$. The function $\Gamma_0(N) \mapsto (\mathbb{Z}/N\mathbb{Z})^{\text{x}}$ sending $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto d \pmod N$ is a surjective group homomorphism with kernel $\Gamma_1(N)$. Therefore, $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$ and $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^{\text{x}}$.

Lemma 2.1.9. *For any given N, $\Gamma(N)$ is a normal subgroup of $\Gamma_1(N)$ and $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$. The following inclusion is satisfied:*

$$\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \ SL_2(\mathbb{Z})$$

We also have

- $[\Gamma_1(N) : \Gamma(N)] = N$
- $[\Gamma_0(N) : \Gamma_1(N)] = \phi(N)$ *where* $\phi(N) = N \prod_{p|N}(1 - \frac{1}{p})$ *is the Eulers totient function, and*
- $[SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right)$ *where the product is taken over all primes p dividing N.*

For proofs one can refer to ([7], Section 2.2 and 2.3)

Definition 2.1.10. (Upper half plane)
The upper half plane denoted by $\mathbb{H}$ is the set given by

$$\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$$

It is an open subset of $\mathbb{C}$ with usual topology.

The action of the group

$$GL_2^+(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, ad - bc => 0 \right\}$$

on $\mathbb{H}$ is given by

$$\begin{bmatrix} b & b \\ c & d \end{bmatrix} z = \frac{az+b}{cz+d}$$

Definition 2.1.11. (Extended Upper half plane)
The extended Upper half plane ($\mathbb{H}^*$) is given by

$$\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$$

That is, by adjoining all rational points and $\{i\infty\}$, known as the *cusps*, to the upper half plane we get the extended upper half plane.

Let Γ be a congruence subgroup. A cusp of Γ is a $\Gamma-$equivalence class of elements in $\mathbb{Q} \cup \{i\infty\}$ under the action of Γ. There is only one cusp of $\mathrm{SL}_2(\mathbb{Z})$as it acts transitively on $\mathbb{Q} \cup \{i\infty\}$. There are only finitely many cusps of Γ as every congruence subgroup has finite index.
The topology on $\mathbb{H}^*$ is given in the following way. For $z \in \mathbb{H}$, the usual fundamental system of neighborhoods is taken. For any cusp $y \neq i\infty$, all the sets given by $\{y\} \cup C^0$ is taken, where C^0 is the interior of the circle in $\mathbb{H}$, which is tangent to the real axis at y. Finally, for $y = i\infty$ the set $\{i\infty\} \cup \{z \in \mathbb{H} : Im(z) > c\}$ is taken as a fundamental open neighborhood of $\imath\infty$ for all $c > 0$.
If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{Q} \cup \{i\infty\}$ then $\gamma z \in \mathbb{Q} \cup \{i\infty\}$.

2.2 Fundamental Domain

Definition 2.2.1. Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathcal{F} \subset \mathbb{H}$ be a closed set with connected interior. Then, $\mathcal{F}$ is said to be a fundamental domain of Γ if

1. any $z \in \mathbb{H}$ is Γ-equivalent to a point in $\mathcal{F}$;
2. no two interior points of $\mathcal{F}$ are Γ-equivalent;
3. the boundary of $\mathcal{F}$ is a finite union of smooth curves.

Theorem 2.2.2. *The fundamental domain for the action of $SL_2(\mathbb{Z})$on $\mathbb{H}$ is given by*

$$\mathcal{F} = \left\{ z \in \mathbb{H} :\mid Re(z) \mid \leqslant \frac{1}{2}, \mid z \mid \geqslant 1 \right\}$$

Before proving the theorem, let us state and prove the following lemma which we will be required in proving above theorem.

Lemma 2.2.3. *The set of $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ such that $(a, b) \neq (0, 0)$ and for some $z \in \mathbb{H}$ if $|az + b| \leq 1$, then the set is finite and non empty.*

Proof. The second condition, i.e. the set is non empty can be seen easily by taking $(a, b) = (0, 1)$.
Now let us write $z = x + iy$. Then we have

$$|az + b| \leq 1 \Leftrightarrow (ax + b)^2 + a^2y^2 \leq 1 \Rightarrow |a| \leq 1/y$$

Hence a can only take finite values.
Again, for $|ax + b| \leq 1 \Rightarrow -1 \leq (ax + b) \leq 1 \Rightarrow -1 - ax \leq b \leq 1 + ax$. Hence, b can also take only finitely many values. □

Proof of Theorem 2.2.2 : In order to prove this, we need to show that the given $\mathcal{F}$ satisfies all the three condition which are there in the definition of fundamental domain.

(1) Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Then $\gamma z = \frac{az+b}{cz+d}$ and $Im(\gamma z) = \frac{Im(z)}{|cz+d|^2}$. From the above lemma we know that there are only finite values of (c, d) such that $|cz + d| \leq 1$. Hence, let us choose $\gamma \in \mathrm{SL}_2(\mathbb{Z})$such that $|cz + d|$ would attain a positive minimum value and thus, $Im(\gamma z) = \frac{Im(z)}{|cz+d|^2}$ would attain the maximal value.
We can adjust any $z \in \mathbb{H}$ by translationg by:

$$\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} z = z + n \text{ for } n \in \mathbb{Z}$$

Thus we can normalize z such that $|Re(z)| \leq 1/2$
Now, we will show that $\gamma z \in \mathcal{F}$.

If not, then we have $|\gamma z| < 1$ and $S(\gamma z) = \frac{-1}{\gamma z}$. Thus, $Im(S(\gamma z)) = \frac{Im(\gamma z)}{|\gamma z|^2} > Im(\gamma z)$, which contradicts the fact that $Im(\gamma z)$ was maximal. Hence $|\gamma z| \geq 1$. Therefore, every element of $\mathbb{Z}$ is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to some points in $\mathcal{F}$.

(2) Now we have to show that if $z, w \in$ interior of $\mathcal{F}$, then ther does not exist any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$such that $\gamma z = w$.

Let us prove this by the method of contradiction. So let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$such that $\gamma z = w$. We may assume that $Im(w) \geq Im(z)$. Therefore,

$$\begin{aligned} Im(w) \geq Im(z) &\Rightarrow \frac{Im(z)}{|cz+d|^2} \geq Im(z) \\ &\Rightarrow |cz+d| \leq 1 \\ &\Rightarrow Im(cz+d) = |c| Im(z) \leq 1 \end{aligned}$$

Now, since for any $z = x + iy \in \mathcal{F}, x$ can take maximum value of $|1/2|$ so $|y| \geq \sqrt{3}/2$. Hence $|c| \leq 2/\sqrt{3}$. Thus, $c = 0$ or ± 1. Without loss of generality let us assume $c = 1$. Now,

$$\begin{aligned} &|cz+d|^2 \leq 1 \\ &\Rightarrow (x+d)^2 + y^2 \leq 1 \\ &\Rightarrow (x+d)^2 + 3/4 \leq 1 \\ &\Rightarrow (x+d)^2 \leq 1/2 \end{aligned}$$

and as $|x| \leq 1/2$, we have $d = 0$. Hence $|cz+d| = |z| \leq 1$, which contradicts the fact that $z \in \mathcal{F}$. Therefore, no two interior points of $\mathcal{F}$ are $\Gamma-$equivalent.

(3) The boundary of $\mathcal{F}$ is clearly union of smooth curves.

2.3 Modular forms

Let us define few of the following concepts which will be later required in our section.

Definition 2.3.1. (Holomorphic and Meromorphic function):
Let T be open subset of $\mathbb{C}$, then a function $f : T \to \mathbb{C}$ is a holomorphic function if f is complex differentiable for all point $z \in T$, i.e. if

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists, and $h \to 0$ from any direction.
A function $f : T \to \mathbb{C} \cup \{\infty\}$ is a meromorphic function if f is holomorphic at all except (possibly) at a discrete set $D \subset T$, and at each $w \in D$ there is a positive integer n such that $(z-w)^n f(z)$ is holomorphic at w. We call n the *order* of f at w and is denoted by $v_w(f)$.
For example, $f(z) = e^z$ is a holomorphic function on $\mathbb{C}$ and $f(z) = \frac{1}{z-i}$ is not holomorphic but is meromorphic.

Definition 2.3.2. (j function):
Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(R)$and $z \in \mathbb{H}$, then

$$j(\gamma, z) = cz + d \tag{2.4}$$

Slash notation":
Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(R)$and for any holomorphic function $f \in \mathbb{H}$, we have

$$(f|\gamma)(z) := (det\gamma)^{k/2} j(\gamma, z)^{-k} f(\gamma z) \tag{2.5}$$

where k is the is related to the function f and we will get to know about it soon.

Definition 2.3.3. Weakly modular function:
It is is a holomorphic function $f(z) : \mathbb{H} \to \mathbb{C}$ such that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \text{ for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \tag{2.6}$$

We will use the topology on the extended upper half plane $\mathbb{H}^*$.
Fourier expansion:
Let $z = x + iy \in \mathbb{H}$ and ley y be fixed, then the function $e^{2\pi i z} = e^{-2\pi y} e^{2\pi i x}$ takes the line

$Im(z) = y$ in $\mathbb{H}$ to a circle centered at 0 of radius $e^{-2\pi y}$. We can extend this map by sending $i\infty$ to 0. Thus, we see that the fundamental neighborhoods of $i\infty$ are mapped to the fundamental neighborhoods of the origin.

We see that from definition 2.6 $f(z+1) = f\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} z\right) = f(z) \forall z \in \mathbb{H}$. Thus, there exists a well defined map f from the unit disc to $\mathbb{C}$ such that $e^{2\pi i z} \mapsto f(z)$ where $z \in \mathbb{H}$. Therefore, if $f(z) : \mathbb{H} \to \mathbb{C}$ is holomorphic, then $f(q)$ is holomorhic on punctured unit disc.

Therefore we obtain a Laurent series expansion $f(q) = \sum_{n=-\infty}^{\infty} a_n q^n$ where $a_n \in \mathbb{C}$. This is called the q-expansion, of f.

We can write the Fourier series at $i\infty$ as $f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i z n}$.

The function f is said to be holomorphic at $i\infty$ if $a_n = 0$ for all $n < 0$. And, we say f vanishes at $i\infty$ if $a_n = 0$ for all $n \leq 0$.

Definition 2.3.4. (Modular forms of weight k):
A modular form of weight $k \in \mathbb{Z}$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$is a holomorphic function $f(z) : \mathbb{H} \to \mathbb{C}$ such that

- $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z),$ for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.
- it is holomorphic at $\{i\infty\}$, that is, $v_{i\infty}(f) \geq 0$.

The set of all modular forms of weight k on $\mathrm{SL}_2(\mathbb{Z})$is denoted by $M_k(\mathrm{SL}_2(\mathbb{Z}))$.

Definition 2.3.5. (Cusp forms of weight k):
It is a modular form of weight $k \in \mathbb{Z}$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$such that the constant term a_0 is 0 in the Fourier expansion of the function at $\{i\infty\}$.
The set of all cusp forms of weight k on $\mathrm{SL}_2(\mathbb{Z})$is denoted by $S_k(\mathrm{SL}_2(\mathbb{Z}))$.

Proposition 2.3.6. *If $f : \mathbb{H} \mapsto \mathbb{C}$ is holomorphic which satisfies*

- $f(z+1) = f(z)$
- $f(-\frac{1}{z}) = z^k f(z)$

for all $z \in \mathbb{H}$, and is holomorphic at $i\infty$, then f is a modular form of weight k of $SL_2(\mathbb{Z})$.

Lemma 2.3.7. *For $k \in \mathbb{Z}$ and odd $M_k(SL_2(\mathbb{Z})) = \{0\}$.*

Proof. By using equation 2.6, we have

$$f(z) = f\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} z\right) = (-1)^k f(z) = -f(z)$$

Hence, we get $M_k(\mathrm{SL}_2(\mathbb{Z})) = \{0\}$ for k odd. □

2.3.1 Examples of modular forms

Definition 2.3.8. Eisenstein series of weight $k \geq 4$:
The Eisenstein series of weight $k > 2$ and k even integer is a function on $\mathbb{H}$ as

$$G_k(z) = \sum_{m,n}{}^{'} \frac{1}{(mz+n)^k}, z \in \mathbb{H} \tag{2.7}$$

where summation is over all $(m, n) \in \mathbb{Z}^2$, with $(m, n) \neq (0, 0)$.

Lemma 2.3.9. *Let $B(m, n) = am^2 + 2bmn + cn^2; a, b, c \in \mathbb{R}$ be a binary quadractic form which is positive definite i.e. $a > 0$ and $ac-b^2 > 0$. Then $B(m, n) \geq \mu(m^2+n^2)$ for $\mu \in \mathbb{R}^+$.*

Proof. We see that $B(m, n) = (m, n) \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix}$ and we can diagonalize $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ as it is a real symmetric matrix. Thus there exist a matrix P such that

$$P \begin{bmatrix} a & b \\ b & c \end{bmatrix} P^t = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$$

Let us put $\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} m \\ n \end{bmatrix}$. So $B(m, n) = \mu_1 x^2 + \mu_2 y^2$ and μ_1, μ_2 are real and positive as B is positive definite.
Let $\mu = \min(\mu_1, \mu_2) > 0$. Then, $B(m, n) \geq \mu(x^2 + y^2)$. But $(x^2 + y^2) = (m^2 + n^2)$ since P is orhtonormal. Hence, $B(m, n) \geq \mu(m^2 + n^2)$. □

Lemma 2.3.10. *For any $z \in \mathbb{H}$, equation 2.7 is absolutely convergent and converges uniformly on compact subsets of $\mathbb{H}$. Therefore, G_k is a holomorphic function on $\mathbb{H}$.*

Proof. Let $z = x + iy \Rightarrow |mz + n|^2 = |m(x + iy) + n|^2 = (x^2 + y^2)m^2 + 2xmn + n^2$. We see that this is a binary quadractic form with respect to the matrix $\begin{bmatrix} (x^2+y^2) & x \\ x & 1 \end{bmatrix}$ and since $z \in \mathbb{H}, y > 0$, so this is a positive definite. Thus, using above lemma we have $|mz + n|^2 \geq \mu(m^2 + n^2)$ for some positive $\mu \in \mathbb{R}$. Therefore,

$$\sum_{m,n}{}' \frac{1}{(mz+n)^k} \leq \sum_{m,n}{}' \frac{1}{(m^2+n^2)^k/2} = \sum_{s=1}^{\infty} \frac{r_2(s)}{s^{k/2}}$$

where $r_2(s)$ counts the number of ways s can be written as sum of two squares. Thus, $r_2(s) = \mathrm{O}(s^{\epsilon})$ for $\epsilon > 0$. Thus the series converges. Therefore, by Weierstrass M-test the series i.e. $G_k(z)$ is a holomorphic function. □

Lemma 2.3.11. *The holomorphic function G_k satisfies Proposition 2.3.6.*

Proof. We need to prove that $G_k(z + 1) = G_k(z)$ and $G_k(-1/z) = z^k G_k(z)$. Also, $G_k(z)$ is holomorphic at $i\infty$.
Now,

$$G_k(z+1) = \sum_{m,n}{}' \frac{1}{(m(z+1)+n)^k} = \sum_{m,n}{}' \frac{1}{(mz+(m+n))^k}$$

and since $(m, n) \in \mathbb{Z}^2$, with $(m, n) \neq (0, 0)$, so $(m, m+n) \neq (0, 0)$. Hence $G_k(z+1) = G_k(z)$.
Now again,

$$G_k(-1/z) = \sum_{m,n}{}' \frac{1}{(-m/z+n)^k} = z^k \sum_{m,n}{}' \frac{1}{(-m+nz)^k} = z^k G_k(z)$$

as $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\} \Rightarrow (n, -m) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$.
Now, the behavior of $G_k(z)$ at $z = i\infty$ come from the term when $m = 0$. So,

$$G_k(i\infty) = \sum_{n \in \mathbb{Z} \setminus 0} \frac{1}{n^k}.$$

Thus, for all odd values of $k, G_k(i\infty) = 0$ as we can pair up the summands corresponding to

(m,n) with $(-m,-n)$. And, for k even we have $G_k(i\infty) = 2\zeta(k)$, where ζ is the Riemann zeta function. $\Box$

Thus from lemma 2.3.10 and lemma 2.3.11, we see that $G_k(z)$ is a modular form of weight k. Hence it will have a Fourier expansion, which is given by the following proposition.

Proposition 2.3.12. *For every even $k \geq 4$, the Fourier expansion for $G_k(z)$ is given by*

$$G_k(z) = 2\zeta(k) + \frac{2(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \tag{2.8}$$

where $q = e^{2\pi iz}$ and $\sigma_s(n) = \sum_{d|n} d^s$.

Before going into the details of the proof let us first state and prove **Lipschitz formula.**

For $k \geq 1$ and $z \in \mathbb{H}$, we have

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi inz} \tag{2.9}$$

The above formula can be derived by using the definition of $\pi\text{cot}\pi z$. We know that $\pi\text{cot}\pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{(z+n)} + \frac{1}{(z-n)} \right) = \sum_{n\in\mathbb{Z}} \frac{1}{(z+n)}$. Also,

$$\pi\text{cot}\pi z = \pi \frac{\text{cos}\pi z}{\text{sin}\pi z} = \pi i \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} = \pi i \frac{e^{2\pi iz} + 1}{e^{2\pi iz} - 1} \tag{2.10}$$

$$= \pi i \frac{e^{2\pi iz} - 1 + 2}{e^{2\pi iz} - 1} = \pi i - \frac{2\pi i}{1 - e^{2\pi iz}} = \pi i - 2\pi i \sum_{n=0}^{\infty} e^{2\pi inz} \tag{2.11}$$

Thus, $\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)} = \pi i - 2\pi i \sum_{n=0}^{\infty} e^{2\pi inz}$ and the successive differentiation of this equation gives us the **Lipschitz formula.**

Proof of Proposition 2.3.12:

We have

$$G_k(z) = \sum_{m,n}{}^{'} \frac{1}{(mz+n)^k} = 2\zeta(k) + \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k} = 2\zeta(k) + 2\sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k}$$

We use Lipschitz formula in the inner sum to get

$$G_k(z) = 2\zeta(k) + 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-2\pi i)^k}{(k-1)!}n^{k-1}e^{2\pi imnz} = 2\zeta(k) + 2\frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}n^{k-1}q^{mn}$$

Now, let us collect the term such that $mn = s$. Thus, the coefficient of q^s is $\sigma_{k-1}(s)$. Therefore,

$$G_k(z) = 2\zeta(k) + \frac{2(-2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

Now, to make the coefficient of $G_k(z)$ equals to 1, we need to divide $G_k(z)$ by $2\zeta(k)$ and we get

$$E_k(z) = \frac{G_k(z)}{2\zeta(k)} = 1 + \frac{2(-2\pi i)^k}{2\zeta(k)(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n = 1 + \frac{2k(-2\pi i)^k}{2\zeta(k)k!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

But, $2\zeta(k) = -\frac{(-2\pi i)^k B_k}{k!}$, where B_k is the *Bernoulli number*. Therefore,

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

which is a modular form of weight k for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Also, $E_k(i\infty) = 1$.

We see that,

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n$$

$$E_6(z) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n$$

Now, we prove the following lemma.

Lemma 2.3.13. *For $k \geq 4$, $M_k(SL_2(\mathbb{Z})) = \mathbb{C}E_k \oplus S_k(SL_2(\mathbb{Z}))$. Therfore, dim $M_k(SL_2(\mathbb{Z})) =$ 1+ dim $S_k(SL_2(\mathbb{Z}))$.*

Proof. Let $g \in dim M_k(\mathrm{SL}_2(\mathbb{Z}))$ and $\mu = g(i\infty)$ be the constant term in the Fourier expansion of g.

Now, $g = \mu E_k - \mu E_k + g = \mu E_k + (g - \mu E_k)$. As the $g(i\infty) = \mu$ and constant term of $\mu E_k = \mu$, $(g - \mu E_k) \in S_k(\mathrm{SL}_2(\mathbb{Z}))$.
Therefore, $M_k(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}E_k \oplus S_k(\mathrm{SL}_2(\mathbb{Z}))$ and dim $M_k(\mathrm{SL}_2(\mathbb{Z})) = 1+$ dim $S_k(\mathrm{SL}_2(\mathbb{Z}))$. $\square$

Definition 2.3.14. Tha Ramanujan delta function is given by

$$\Delta(z) = \frac{E_4^3(z) - E_6^2(z)}{1728} = q - 24q^2 + 252q^3 + ... = \sum_{n=1}^{\infty} \tau(n)q^n \tag{2.12}$$

$\Delta(z)$ is a non trivial cusp form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$.

2.3.2 Eisenstein series of weight $k = 2$

For $k = 2, G_k(z)$ does not satisfy Proposition 2.3.12 since $\sum'_{m,n} \frac{1}{(m(z+1)+n)^2}$ fails to converge. But for $k = 2$, we have the following Proposition.

Proposition 2.3.15. *The function G_2 and E_2 satisfy the following:*

$$G_2(-1/z) = z^2 G_2(z) - 2\pi i z \tag{2.13}$$

and

$$E_2(-1/z) = z^2 E_2(z) + \frac{6z}{\pi i} \tag{2.14}$$

To prove the proposition, we need the following lemma.

Lemma 2.3.16. *For all $z \in \mathbb{H}$,*

$$\sum_{n=-\infty}^{\infty} \sum_{m \neq 0} \left(\frac{1}{mz+n} - \frac{1}{mz+n+1} \right) = \frac{-2\pi i}{z} \tag{2.15}$$

and

$$\sum_{m \neq 0} \sum_{n=-\infty}^{\infty} \left(\frac{1}{mz+n} - \frac{1}{mz+n+1} \right) = 0 \tag{2.16}$$

Proof. Since, $\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{(z+n)} + \frac{1}{(z-n)}\right)$,

$$\sum_{n=-\infty}^{\infty}\sum_{m\neq 0}\left(\frac{1}{mz+n} - \frac{1}{mz+n+1}\right) = \frac{1}{z}\sum_{\substack{n=-\infty\\ n\neq 0,1}}^{\infty}\left(\pi\cot\left(\frac{\pi(n-1)}{z}\right) - \frac{z}{n-1} - \pi\cot\left(\frac{\pi n}{z}\right) + \frac{z}{n}\right)$$
$$+\sum_{m\neq 0}\left(\frac{1}{mz-1} - \frac{1}{mz}\right) + \sum_{m\neq 0}\left(\frac{1}{mz} - \frac{1}{mz+1}\right)$$

We have

$$\begin{aligned}\sum_{m\neq 0}\left(\frac{1}{mz-1} - \frac{1}{mz}\right) + \sum_{m\neq 0}\left(\frac{1}{mz} - \frac{1}{mz+1}\right) &= \sum_{m\neq 0}\left(\frac{1}{mz-1} - \frac{1}{mz+1}\right)\\ &= -\frac{1}{z}\sum_{m\neq 0}\left(\frac{1}{(1/z)+m} - \frac{1}{(1/z)-m}\right)\\ &= -\frac{2}{z}\left(\pi\cot\left(\frac{\pi}{z}\right) - z\right) = -\frac{2\pi}{z}\cot\left(\frac{\pi}{z}\right) + 2\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{z}\sum_{\substack{n=-\infty\\ n\neq 0,1}}^{\infty}\left(\pi\cot\left(\frac{\pi(n-1)}{z}\right) - \frac{z}{n-1} - \pi\cot\left(\frac{\pi n}{z}\right) + \frac{z}{n}\right)\\ &= \frac{1}{z}\lim_{N\to\infty}\sum_{\substack{n=-N\\ n\neq 0,1}}^{N}\left(\pi\cot\left(\frac{\pi(n-1)}{z}\right) - \pi\cot\left(\frac{\pi n}{z}\right) + \frac{z}{n} - \frac{z}{n-1}\right)\\ &= \frac{1}{z}\lim_{N\to\infty}\left\{2\pi\cot\left(\frac{\pi}{z}\right) - \pi\cot\left(\frac{\pi(N+1)}{z}\right) - \pi\cot\left(\frac{\pi N}{z}\right) + \frac{z}{N}\frac{z}{N+1} - 2z\right\}\\ &= \frac{2\pi}{z}\cot\left(\frac{\pi}{z}\right) - 2 - \frac{2\pi}{z}\cot\left(\frac{\pi N}{z}\right)\end{aligned}$$

As, $z \in \mathbb{H}$, by (2.10) we have

$$\lim_{N\to\infty}\cot\left(\frac{\pi}{z}\right) = i\lim_{N\to\infty}\frac{e^{2\pi i(N/z)}+1}{e^{2\pi i(N/z)}-1} = 2i\lim_{N\to\infty}\frac{1}{e^{2\pi i(N/z)}-1} = i$$

Putting everything together, we have

$$\sum_{n=-\infty}^{\infty}\sum_{m\neq 0}\left(\frac{1}{mz+n} - \frac{1}{mz+n+1}\right) = \frac{-2\pi i}{z}$$

Now,

$$\sum_{m\neq 0}\sum_{n=-\infty}^{\infty}\left(\frac{1}{mz+n}-\frac{1}{mz+n+1}\right)=\sum_{m\neq 0}\lim_{N\to\infty}\sum_{n=-N}^{N}\left(\frac{1}{mz+n}-\frac{1}{mz+n+1}\right)$$
$$=\sum_{m\neq 0}0=0$$

□

Proof.of Proposition 2.3.15 Let us first observe that,

$$\frac{1}{(mz+n)^2}-\left(\frac{1}{mz+n}-\frac{1}{mz+n+1}\right)=\frac{-1}{(mz+n)^2(mz+n-1)}$$

The series,

$$\sum_{(m,n)\neq}\sum_{(0,0),(0,1)}\frac{-1}{(mz+n)^2(mz+n-1)}$$

converges absolutely. Therefore,

$$G_2(z)=2\zeta(2)+\sum_{m\neq 0}\sum_{n=-\infty}^{\infty}\frac{1}{(mz+n)^2}$$
$$=2\zeta(2)+\sum_{m\neq 0}\sum_{n=-\infty}^{\infty}\left\{\frac{1}{(mz+n)^2}-\left(\frac{1}{mz+n}-\frac{1}{mz+n+1}\right)+\left(\frac{1}{mz+n}-\frac{1}{mz+n+1}\right)\right\}$$

Using (2.16), we have,

$$G_2(z)=2\zeta(2)+\sum_{m\neq 0}\sum_{n=-\infty}^{\infty}\left\{\frac{1}{(mz+n)^2}-\left(\frac{1}{mz+n}-\frac{1}{mz+n+1}\right)\right\}$$

We can interchange the double summation as it is absolutely convergent. Thus we get,

$$G_2(z)=2\zeta(2)+\sum_{n=-\infty}^{\infty}\sum_{m\neq 0}\left\{\frac{1}{(mz+n)^2}-\left(\frac{1}{mz+n}-\frac{1}{mz+n+1}\right)\right\} \tag{2.17}$$

Now,

$$\begin{aligned} G_2(-1/z) &= 2\zeta(2) + \sum_{m\neq 0}\sum_{n=-\infty}^{\infty} \frac{z^2}{(nz-m)^2} \\ &= 2\zeta(2)(1+z^2) + \sum_{m\neq 0}\sum_{n=\neq 0} \frac{z^2}{(nz-m)^2} \\ &= 2\zeta(2)z^2 + \sum_{m=-\infty}^{\infty}\sum_{n\neq 0} \frac{z^2}{(nz-m)^2} \\ &= 2\zeta(2)z^2 + \sum_{n=-\infty}^{\infty}\sum_{m\neq 0} \frac{z^2}{(nz-m)^2} \end{aligned}$$

By (2.17), we have

$$G_2(z) = 2\zeta(2) + \left(G_2(-1/z) - 2\zeta(2)z^2\right) z^{-2} - \sum_{n=-\infty}^{\infty}\sum_{m\neq 0}\left(\frac{1}{mz+n} - \frac{1}{mz+n+1}\right)$$

Using (2.15), we get $G_2(-1/z) = z^2 G_2(z) - 2\pi i z$
($G_2(z)$ is not a modular form, but ia an example of quasi-modular form.)
By definition of $E_2(z)$ and by equation (2.13), we get the desired value for $E_2(1/z)$. □

Theorem 2.3.17. *The cusp form* $\Delta = \frac{E_4^3 - E_6^2}{1728} \in S_{12}(SL_2(\mathbb{Z}))$ *is Ramanujan's infinite product*

$$\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

For proof one can refer to ([7], Theorem 5.1.4).

The Fourier coefficients of this series are denoted by $\tau(n)$, so $\Delta = \sum_{n=1}^{\infty} \tau(n) q^n$.

Remark 2.3.18. Ramanujan in 1916 conjuctured the following properties of τ

- $\tau(mn) = \tau(m)\tau(n)$ for $(m, n) = 1$ and $m, n \in \mathbb{Z}^+$
- if p is prime, then $\tau(p^{\alpha+1}) = \tau(p)\tau(p^{\alpha}) - p^{11}\tau(p^{\alpha-1}), \alpha \in \mathbb{N}$
- if p is a prime, then $|\tau(p)| \leq 2p^{11/2}$

The first two properties were already proven by Mordell in 1917 and the last by Deligne in 1974.

2.4 Valence Formula

Let f be a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$. Then for any $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $v_w(f) = v_{\gamma w}(f)$, since $f(z) = (cz+d)^{-k} f\left(\frac{(az+b)}{cz+d}\right)$, where $v_w(f)$ is the order of f at w.
We note that $v_w(f)$ depends only on the orbit of w under $\mathrm{SL}_2(\mathbb{Z})$, so we need to study order of $f, v_z(f)$ only for z in fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$.

Theorem 2.4.1. *Let $0 \neq f$ be a modular function of weight k for the full modular group $SL_2(\mathbb{Z})$. Then,*

$$v_{i\infty}(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\rho(f) + \sum_{\substack{w \in \mathfrak{F} \\ w \neq i, \rho}} v_w(f) = \frac{k}{12}$$

where $\rho = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$

For proof one can refer to ([7], Section 4.3).

2.5 Dimension Formula

It is an immediate application of valence formula.

Theorem 2.5.1. *For, $k \geq 0$,*

$$dim M_k(SL_2(\mathbb{Z})) = \begin{cases} \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \text{ mod } 12 \\ \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 2 \text{ mod } 12 \end{cases}$$

To prove this, we need the following lemma.

Lemma 2.5.2. *The following statements are true.*

(i) $M_k(SL_2(\mathbb{Z})) =$ *0 for all* $k < 0$ *and* $k = 2$*;*

(ii) $M_k(SL_2(\mathbb{Z}))$ *is one-dimensional for* $k = 0, 4, 6, 8,$ *and* 10 *and is spanned by* $1, E_4, E_5, E_8$ *and* E_{10} *respectively;*

(iii) Multiplication by Δ *gives an isomorphism of* $M_{k-12}(SL_2(\mathbb{Z}))$ *onto* $S_k(SL_2(\mathbb{Z}))$ *for all* $k \in \mathbb{Z}$*.*

Proof.

(i) By applying valence formula, we get that for f whose weight is $k < 0$ and $k = 2$, is identically zero.

(ii) We see that for $k \leq 10$ the right hand side of valence formula is less than 1. Thus, if there was a non trivial cusp form of weight $k \leq 10$, then there would have been a contribution 1 from the zero of $i\infty$, which is not possible as right hand side is less than 1. Therefore, $S_k(\mathrm{SL}_2(\mathbb{Z})) = 0$ for $k \leq 10$. Now by using lemma 2.3.13, we have $M_k(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}E_k$ for $k \leq 10$.

(iii) Let $f \in E_{k-12}(\mathrm{SL}_2(\mathbb{Z}))$, then $f\Delta \in S_k(\mathrm{SL}_2(\mathbb{Z}))$. Now for the converse, let $g \in S_k(\mathrm{SL}_2(\mathbb{Z}))$. Since, $\Delta(z)$ is a non trivial cusp form of weight 12 which does not vanishes on the upper half plane, we can define an analytic function $f(z) = g(z)/\Delta(z)$ which is of weight $k - 12$. Therefore, this proves (iii).

□

Proof.of Theorem 2.5.1: The dimension formula is true for $k \leq 10$ by (i) and (ii) of Lemma 2.5.2. By part (iii) of Lemma 2.5.2 and Lemma 2.3.13, for $k \geq 12$ we have

$$\begin{aligned}\mathrm{dim} M_k(\mathrm{SL}_2(\mathbb{Z})) &= 1 + \mathrm{dim} S_k(\mathrm{SL}_2(\mathbb{Z})) \\ &= 1 + \mathrm{dim} M_{k-12}(\mathrm{SL}_2(\mathbb{Z}))\end{aligned}$$

Thus, by induction, we get the dimension formula. □

Corollary 2.5.3. *For* $k \geq 4$*, we have*

$$dim S_k(SL_2(\mathbb{Z})) = \begin{cases} \lfloor \frac{k}{12} \rfloor - 1 & \text{if } k \equiv 2 \text{ mod } 12 \\ \lfloor \frac{k}{12} \rfloor & \text{if } k \not\equiv 2 \text{ mod } 12 \end{cases}$$

2.6 Hecke operators of level 1

A family of operators mapping each space M_k of modular forms onto itself are called Hecke operators. We define the $m-th$ Hecke operator T_m for $f \in M_k(SL_2(\mathbb{Z}))$ as

$$T_m(f) := m^{k/2-1} \sum_{\substack{ad=m \\ d>0}} \sum_{b (mod\, d)} f| \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \tag{2.18}$$

An eigenform is a modular form which is an eigen vector for all the Hecke operators.
Properties of Hecke operator:

(i) Let m be any positive integer. Let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$, then $T_m(f) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$. Similarly for $g \in S_k(\mathrm{SL}_2(\mathbb{Z}))$, then $T_m(g) \in S_k(\mathrm{SL}_2(\mathbb{Z}))$.

(ii) Let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ has a Fourier expansion as

$$f(z) = \sum_{n=0}^{\infty} \mu(n) e^{2\pi i n z}$$

at $i\infty$, then the Fourier expansion of $T_m(f)$ at $i\infty$ is given by

$$T_m(f(z)) = \sum_{n=0}^{\infty} \left(\sum_{d|m,n} d^{k-1} \mu(\frac{mn}{d^2}) \right) e^{2\pi i n z}$$

(iii) For any prime p and $r \in \mathbb{N}$,

$$T_p T_{p^r} = T_{p^{r+1}} + p^{k-1} T_{p^{r-1}}$$

and

$$T_m T_n = T_{mn} \text{ whenever (m,n)} = 1$$

In particular, we note that all Hecke Operator commute.

Remark 2.6.1. If f is a normalized eigenform we have :

$$T_n(f) = \lambda_n f \; for \; some \; \lambda_n$$

but

$$a_n(f) = a_1(T_n(f)) = a_1(\lambda_n f) = \lambda_n a_1(f) = \lambda_n$$

since $a_1(f) = 1$.

Hence we can recover the Fourier coefficients of a normalized eigenform by considering the associated eigenvalues of Hecke Operators : $\lambda_n = a_n(f)$. Thus, we see that for our previous example,

$$T_m(\Delta) = \tau(m)\Delta$$

2.7 Modular forms of level N

So far we have discussed modular forms of level 1. Now in this section, we will generalize the concept of modular forms to a higher level.
For more detail, the reader can look at ([7], Chapter 6-8).
For the purposes of this thesis, we will be concerned with Hecke congruence subgroup ($\Gamma_0(N)$) of level N, so our definition would be with respect to $\Gamma_0(N)$.

Definition 2.7.1. (Modular forms of level N)
A modular form of weight k with respect to $\Gamma_0(N)$ is a function $f : \mathbb{H} \to \mathbb{C}$, such that

- f is holomorphic on $\mathbb{H}$.
- $f|\gamma = f$ for all $\gamma \in \Gamma_0(N)$.
- f is holomorphic at the cusps, that is if a is a cusp then $v_a(f) \geq 0$.

f is called a cusp form of weight k with respect to $\Gamma_0(N)$, if f vanishes at the cusps. The space of modular forms of weight k and level N is denoted by $M(N, k)$ and the space of modular forms of weight k and level N is denoted by $S(N, k)$.
We now define the **Petersson Inner Product** for $f, g \in S(N, k)$.

Definition 2.7.2. (Petersson Inner Product)
Let $f, g \in S(N, k)$. The Petersson inner product of f and g is given by

$$(f, g) = \int\int_{\mathfrak{F}} y^k f(z)\overline{g(z)}\frac{dxdy}{y^2}$$

where $z = x + iy$.

2.8 Hecke operators of level N

Hecke operators are family of linear operators that preserve the spaces $M(N,k)$ and $S(N,k)$ for each weight k and level N. We will study the distribution of eigenvalues of these operators. Let us define,

$$\Delta^n(N) = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in GL_2(\mathbb{Z}) : a, b, d \in \mathbb{Z}, 0 \leq b < dad = n, gcd(a, N) = 1 \right\}.$$

Now, let us define the $n-th$ Hecke operator for level N.

Definition 2.8.1. Let $f \in M(N,k)$ and n be a positive integer. The, the $n-th$ **Hecke operator** T_n is given by:

$$T_n(f) = n^{\frac{k}{2}-1} \sum_{\gamma \in \Delta^n(N)} f|\gamma$$

Definition 2.8.2. Let $f \in M(N,k)$. We say f to be a **Hecke eigenform** if, for each n such that $\gcd(n, N) = 1$, there exists a complex number λ_n, such that

$$T_n(f) = \lambda_n.f$$

Chapter 3

History and Motivation of the problem

3.1 Introduction

Let $S(N,k)$ be the space of cusp forms of weight k ($k \geq 2$ is an even integer) with respect to $\Gamma_0(N)$ and for any integer $n \geq 1$, let $T_n(N,k)$ be the n-th Hecke operator acting on $S(N,k)$. Let $s(N,k)$ be the dimension of $S(N,k)$. Let p be a prime such that $(p,N) = 1$. Let $E(N,k)$ denote a basis of Hecke eigenforms of $S(N,k)$. An eigenform $h \in E(N,k)$ will have a Fourier expansion of the form

$$h(z) = \sum_{n=1}^{\infty} a_n(h) e(nz).$$

where $e(x) = e^{2\pi i x}$.
Let T'_n denote the normalised Hecke operator

$$T'_n = \frac{T_n}{n^{(k-1)/2}}$$

and $\lambda_h(n)$, an eigenvalue of $h(z)$ corresponding to T'_n.
Let r be a fixed positive integer. Let p be a fixed prime and $r_p = \operatorname{ord}_p r$. For $h(z) \in E(N,k)$, we define the weight

$$\omega_h^r := \frac{|a_r(h)|^2 e^{-4\pi r}}{||h||^2},$$

where $||h||$ denotes the Petersson norm of h.

Let p be a prime number such that gcd $(p, N) = 1$ By a theorem of Deligne [2], the eigenvalues $\lambda_h(p)$ lie in $[-2, 2]$. We consider the following families formed by these Hecke eigenvalues:

(a) (Sato-Tate family) Let N and k be fixed and $h \in E(N, k)$. We consider the sequence $\{\lambda_h(p)\}$ as $p \to \infty$.

(b) (Vertical Sato-Tate family) For a fixed prime p, we consider the families $\{\lambda_h(p) : h \in E(N, k)\}$ as $N \to \infty$.

(c) (Average Sato-Tate family) We consider the families $\{\lambda_h(p) : p \leq x, h \in E(N, k)\}$ as $N \to \infty, x \to \infty$.

We can also study the distribution of these family by attaching weights ω_h^r.

3.2 Equidistribution theorems for Hecke eigenvalues

The Sato-Tate conjecture (now proved by the work of Richard Taylor et al) states that for $h(z) \in E(N, k)$, the sequence $\{\lambda_h(p)\}_{\substack{p\to\infty \\ (p,N)=1}}$ is equidistributed in $[-2, 2]$ with respect to the measure

$$d\mu_\infty(x) = \begin{cases} \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}dx & \text{if } x \in [-2, 2] \\ 0 & \text{otherwise.} \end{cases}$$

That is, by Definition 1.5.1, for any continuous function $f : [-2, 2] \to \mathbb{R}$,

$$\lim_{x\to\infty} \frac{1}{\pi_N(x)} \sum_{\substack{p\leq x \\ (p,N)=1}} f(\lambda_h(p)) = \int_{-2}^{2} f(x)d\mu_\infty(x).$$

Equivalently, for any interval $I = [A, B] \subset [-2, 2]$,

$$\lim_{x\to\infty} \frac{1}{\pi_N(x)} \sum_{\substack{p\leq x \\ (p,N)=1}} \chi_I(\lambda_h(p)) = \int_{A}^{B} d\mu_\infty(x).$$

Here, $\pi_N(x)$ denotes the number of primes p less than or equal to x such that $(p, N) = 1$.

In 1997, Serre [14] proved a vertical analogue of the Sato-Tate conjecture by fixing a prime p and varying the Hecke eigenforms. Let p be a fixed prime. He showed that as $N + k \to \infty$ with the restrictions that $(p, N) = 1$ and $k \geq 2$ is an even integer, the sequence of multisets

$$S_N := \{\lambda_h(p),\ h \in E(N, k)\}$$

is equidistributed with respect to the measure

$$d\mu_p(x) = \frac{p+1}{(p^{1/2} + p^{-1/2})^2 - x^2} d\mu_\infty(x).$$

That is, for any continuous function $f : [-2, 2] \to \mathbb{R}$,

$$\lim_{\substack{N+k\to\infty \\ (p,N)=1 \\ k \text{ even}}} \frac{1}{s(N,k)} \sum_{h\in E(N,k)} f(\lambda_h(p)) = \int_{-2}^{2} f(x) d\mu_p(x).$$

Equivalently, for any interval $I = [A, B] \subset [-2, 2]$,

$$\lim_{\substack{N+k\to\infty \\ (p,N)=1 \\ k \text{ even}}} \frac{1}{s(N,k)} \sum_{h\in E(N,k)} \chi_I(\lambda_h(p)) = \int_{A}^{B} d\mu_p(x).$$

In 2004, Charles Li [5] obtained an interesting generalisation of Serre's equidistribution theorem. He observed that by attaching suitable weights to each element in the multiset S_N, one can derive a weighted distribution measure for the sequence $\{S_N\}$. He proved the following theorem:

Theorem 3.2.1. *Let $k \geq 3$ be an even integer. Let r be a fixed positive integer. Let p be a fixed prime and $r_p = ord_p\, r$. Define, for $h(z) \in E(N, k)$, the weight*

$$\omega_h^r := \frac{|a_r(h)|^2 e^{-4\pi r}}{||h||^2},$$

where $||h||$ denotes the Petersson norm of h. The family of sets $\{S_N : (p, N) = 1\}$ is ω_h^r-distributed with respect to the measure

$$\sum_{i=0}^{r_p} X_{2i}(x) d\mu_\infty(x), \ \text{as } N \to \infty$$

where $X_n(x)$ is the n-the Chebychev polynomial defined by

$$X_n(2\cos\phi) = \frac{\sin(n+1)\phi}{\sin\phi}.$$

That is, for a continuous function $f : [-2,2] \to \mathbb{R}$,

$$\lim_{\substack{N\to\infty \\ (p,N)=1}} \frac{1}{s(N,k)} \sum_{h\in E(N,k)} \omega_h^r f(\lambda_h(p)) = \int_{-2}^{2} f(x) \left(\sum_{i=0}^{r_p} X_{2i}(x) \right) d\mu_\infty(x).$$

Equivalently, for any interval $I = [A,B] \subset [-2,2]$,

$$\lim_{\substack{N\to\infty \\ (p,N)=1}} \frac{1}{s(N,k)} \sum_{h\in E(N,k)} \omega_h^r \chi_I(\lambda_h(p)) = \int_{A}^{B} \left(\sum_{i=0}^{r_p} X_{2i}(x) \right) d\mu_\infty(x).$$

Remark 3.2.2. We note that in Serre's theorem, we may vary N as well as k. However, in the above theorem, the weight k is fixed and the levels N vary.

This gives us the corollary:

Corollary 3.2.3. *If p does not divide r, the family of sets $\{S_N : (p,N) = 1\}$ is ω_h^r-distributed with respect to the Sato-Tate measure $d\mu_\infty(x)$, as $N \to \infty$.*

In 2009, the error terms in Serre's theorem were obtained by M. R. Murty and Sinha ([9]). They proved the following theorem which describes the rate of convergence to the measure $d\mu_p(x)$ effectively:

Theorem 3.2.4. *Let N be a positive integer, k be a positive even integer and p be a prime number coprime to N. For an interval $[a,b] \subset [-2,2]$,*

$$\frac{1}{s(N,k)} \# \{h \in E(N,k) : \lambda_h(p) \in [\alpha,\beta]\} = \int_a^b d\mu_p(x) + \mathrm{O}\left(\frac{\log p}{\log kN} \right),$$

where the implied constant is effectively computable.

In 2006 Nagoshi [10] investigated the following and proved two theorems which tell us the following:

Theorem 3.2.5. *Let $k = k(x)$ such that $\frac{log(k)}{log(x)} \to \infty$ as $x \to \infty$. Then for any continuous real valued function g on $[-2, 2]$, we have*

$$\frac{1}{\pi(x)s(1,k)} \sum_{\substack{p \le x \\ h \in E(1,k)}} g(\lambda_h(p)) \to \frac{1}{2\pi} \int_{-2}^{2} g(t)\sqrt{4-t^2}dt$$

as $x \to \infty$.

The second main result is the following:

Theorem 3.2.6. *Let $k = k(x)$ such that $\frac{log(k)}{log(x)} \to \infty$ as $x \to \infty$. Then for any bounded continuous real valued function g on $\mathbb{R}$, we have*

$$\frac{1}{s(1,k)} \sum_{h \in E(1,k)} g\left(\frac{\sum_{p \le x} \lambda_h(p)}{\sqrt{\pi(x)}}\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) e^{\frac{-t^2}{2}} dt$$

as $x \to \infty$.

Cho and Kim in their paper ([1], Theorem 4.1) generalized Nagoshi's theorem for higher level N and proved the following theorem.

Theorem 3.2.7. *Suppose that $\frac{log(N)}{log(x)} \to \infty$ as $x \to \infty$. For a continuous real valued function g on $\mathbb{R}$,*

$$\frac{1}{s(N,k)} \sum_{h \in E(N,k)} g\left(\frac{\sum_{\substack{p \le x \\ (p,N)=1}} \lambda_h(p)}{\sqrt{\pi_N(x)}}\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) e^{\frac{-t^2}{2}} dt$$

as $x \to \infty$.

3.3 Overview of new results.

We now state the new results proved in this thesis.

3.3.1 Weighted analogue of Murty and Sinha's theorem

Let $W_r(N,k) = \sum_{h \in E(N,k)} \omega_h^r$.
We prove the following theorem which is the weighted analog of Theorem 3.2.4.

Theorem 3.3.1. *Let $k \geq 3$ be a fixed even integer. Let p be a fixed prime. Then for any interval $I = [a, b] \subseteq [-2, 2]$ and with notations as defined above, we have,*

$$\frac{1}{W_r(N,k)} \sum_h \omega_h^r \chi_I(\lambda_h(p)) = \int_I \sum_{i=0}^{r_p} X_{2i}(x)\mu_\infty(x)dx + \mathrm{O}\left(\frac{\log p}{\log N}\right)$$

3.3.2 Weighted analogue of Nagoshi's Theorems

We also prove the following two theorems which are the weighted analogues of Nagoshi's theorems 3.2.5 and 3.2.6.

Theorem 3.3.2. *Let $N = N(x)$ such that $\frac{\log N}{\log x} \to \infty$ as $x \to \infty$. Let p be prime such that $(p, N) = 1$. Then for any continuous real valued function g on $[-2, 2]$, we have*

$$\frac{1}{\pi_N(x)W_r(N,k)} \sum_{\substack{p \leq x \\ h \in E(N,k)}} \omega_h^r g(\lambda_h(p)) \to \frac{1}{2\pi} \int_{-2}^{2} g(t)\sqrt{4-t^2}dt$$

as $x \to \infty$.

The second theorem is the following:

Theorem 3.3.3. *Let $N = N(x)$ such that $\frac{\log N}{\log x} \to \infty$ as $x \to \infty$. Then for any bounded continuous real valued function g on $\mathbb{R}$, we have*

$$\frac{1}{W_r(N,k)} \sum_{h \in E(N,k)} \omega_h^r g\left(\frac{\sum_{p \leq x} \lambda_h(p)}{\sqrt{\pi_N(x)}}\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t)e^{\frac{-t^2}{2}}dt$$

as $x \to \infty$.

Chapter 4

Trace Formula

4.1 Introduction

The Hecke eigenvalues carry a lot of important and interesting arithmetical information. However, it is difficult to get these eigenvalues directly. Thus, we study the trace of Hecke operators. In this chapter, we study about two important class of Trace formula mainly the Eichler-Selberg trace formula and Kuznietsov trace formula.

4.2 The Eichler-Selberg Trace Formula

Eichler-Selberg trace formula gives the formula for the trace of n^{th} Hecke operator T_n in terms of class numbers of binary quadratic forms. For level $N = 1$, it was discovered by Selberg in the year 1956 on the trace formula for $SL_2(\mathbb{Z})$. In the same year, Eichler obtained a formula for $k = 2$ and square free level. Hijikata gave the trace formula for T_n for the level N, such that $gcd(n, N) = 1$. Oesterlé in his thesis [11], gave a more generalized formula for the space $S(N, k)$ and Nebentypus χ, where χ is a Dirichlet character mod (N) valid for all n and N. This formula is known as the Eichler-Selberg trace formula. When we take χ as the trivial character in this formula, we get a formula for the trace of T_n on $S(N, k)$.

For a negative integer Δ congruent to 0 or 1 (mod 4), let

$$B(\Delta) = \{ax^2 + bxy + cy^2 : a, b, c \in \mathbb{Z}, a > 0, b^2 - 4ac = \Delta\}.$$

Let $b(\Delta)$ denotes the set of primitive forms, that is,

$$b(\Delta) = \{f(x, y) \in B(\Delta) : gcd(a, b, c) = 1\}.$$

The right action of the group $\mathrm{SL}_2(\mathbb{Z})$on $B(\Delta)$ is given by,

$$f(x, y) \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} := f(\alpha x + \beta y, \gamma x + \delta y), \text{ for } f(x, y) \in B(\Delta).$$

It is known fact that this action has only finitely many orbits. Let the number of orbits of $b(\Delta)$ be given by $\mathfrak{h}(\Delta)$. Let h_w be defined as follows :

$$h_w(3) = 1/3,$$

$$h_w(4) = 1/2,$$

$$h_w(\Delta) = \mathfrak{h}(\Delta) \text{ for } \Delta < -4.$$

Theorem 4.2.1. ***(Eichler-Selberg Trace Formula)***
Let n be a positive integer coprime to N. The trace Tr of T_n acting on $S(N, k)$ is given by

$$Tr\ T_n = \sum_{i=1}^{4} A_i(n),$$

where $A_i(n)'s$ are as follows:

$$A_1(n) = \frac{k-1}{12}\psi(N) \begin{cases} n^{(k/2-1)}, & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise} \end{cases}$$

$$A_2(n) = \frac{-1}{2} \sum_{t \in \mathbb{Z}, t^2 < 4n} \frac{\mathcal{Q}^{k-1} - \overline{\mathcal{Q}}^{k-1}}{\mathcal{Q} - \overline{\mathcal{Q}}} \sum_f h_w\left(\frac{t^2 - 4n}{f^2}\right) \mu(t, f, n)$$

$$A_3(n) = - \sum_{d|n, 0<d\leq\sqrt{n}} d^{k-1} \sum_{c|N} \phi\left(gcd\left(c, \frac{c}{N}\right)\right)$$

and

$$A_4(n) = \begin{cases} \sum_{t|n, t>0} t & \text{if } k = 2, \\ 0 & \text{otherwise} \end{cases}$$

In the above terms,

- $\mathcal{Q}$ *and* $\overline{\mathcal{Q}}$ *are the complex zeroes of the polynomial* $x^2 - tx + n$.
- *The inner sum for* $A_2(n)$ *runs over all positive divisors* f *of* $t^2 - 4n$ *such that* $\frac{(t^2-4n)}{f^2} \in \mathbb{Z}$ *is congruent to* 0 *or* 1 *(mod* 4*).*
- *We have*
$$\mu(t, f, n) = \frac{\psi(N)}{\psi(N/N_f)} M(t, n, NN_f),$$
where $N_f = gcd(N, f)$ *and* $M(t, n, NN_f)$ *denotes the number of elements of* $(\mathbb{Z}/N\mathbb{Z})^*$ *which lift to solutions of* $x^2 - tx + n \equiv 0 \bmod NN_f$.
- *In* $A_3(n)$*, in the first summation, if there is a contribution from the term* $d = \sqrt{n}$*, it should be multiplied by* $1/2$*. In the inner sum, we also need the condition that* $gcd(c, N/c)$ *divides* $gcd(N, n/d - d)$.

Now we state some results involving estimates of the trace formula.

For any general N, Serre [14] proved the following:

Proposition 4.2.2. *If* n *is a square,*

$$\left| Tr\ T_n - \frac{k-1}{12} n^{k/2-1} \psi(N) \right| \ll_n n^{k/2} N^{1/2} d(N)$$

where $d(N)$ *is the number of positive divisors of* N.

Corollary 4.2.3. *The Eichler-Selberg trace formula to the case* $n = 1$ *gives us a formula for the dimension of* $s(N, k)$*. Thus,*

$$s(N, k) = \frac{k-1}{12} \psi(N) + \mathrm{O}(N^{1/2} d(N)),$$

where $d(N)$ is the number of divisor of N.

In order to prove Theorem 3.2.1, the primary tool used was the Kuznietsov trace formula. We now state a consequence of this formula as derived by Li ([5], Thm. 4.8).

4.3 Kuznietsov Trace Formula

Bessel functions are defined as solutions $y(x)$ of the differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\alpha^2)y=0$$

for a complex number α. Bessel functions of the first kind $J_\alpha(x)$ have the following series expansion around $x=0$:

$$J_\alpha(x):=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha}.$$

For positive integers r, n and u, the Kloosterman sums are defined as follows:

$$\mathrm{Kl}_u(n,r):=\sum_{\substack{s_1,s_2\in\mathbb{Z}/u\mathbb{Z}\\ s_1s_2\equiv n \bmod \mathrm{u}}} e^{\frac{2\pi i}{u}(rs_1+rs_2)}.$$

Using the above defined notation, we are now ready to state a consequence of the Kuznietsov trace formula as derived by Li ([5], Theorem 4.8):

Theorem 4.3.1. *Let k be an even number ≥ 3. Let n, N, r be positive integers such that $(n,N)=1$. Suppose n can be factorized as $n=\prod_{p|n}p^{n_p}$. With the above notation,*

$$\sum_{h\in E(N,k)}\left(\prod_{p|n}X_{n_p}(\lambda_h(p))\right)\frac{|a_r(h)|^2e^{-4\pi r}}{||h||^2}$$

$$=\begin{cases}\frac{e^{-4\pi r}(4\pi r)^{k-1}}{(k-2)!}\psi(N) & \textit{if } n \textit{ is a square and } n^{1/2}|r\\ 0 & \textit{otherwise.}\end{cases}$$

$$+\frac{e^{-4\pi r}(4\pi i)^k r^{k-1}}{2(k-2)!}\psi(N)\sum_{v=1}^{\infty}\frac{1}{Nv}J_{k-1}\left(\frac{4\pi n^{1/2}r}{Nv}\right)Kl_{vN}(n,r).$$

For $\lambda_h(p) \in S_N$, we choose $\theta_p^h \in [0,\pi]$ so that $2\cos\theta_p^h = \lambda_h(p)$. We consider the families

$$F_N := \left\{ \pm\frac{\theta_p^h}{2\pi} \pmod 1 : \mathrm{h} \in \mathrm{S_N} \right\}.$$

In order to calculate the weighted m-th Weyl limit with the weight ω_h^r, we first observe that for $m \geq 2$,

$$\begin{aligned}
\sum_{t\in F_N} e(mt) = \sum_{h\in E(N,k)} 2\cos(m\theta_p^h) & \\
&= \sum_{h\in E(N,k)}\left(\frac{\sin(m+1)\theta_p^h}{\sin\theta_p^h} - \frac{\sin(m-1)\theta_p^h}{\sin\theta_p^h}\right)\\
&= \sum_{h\in E(N,k)} X_m(\lambda_h(p)) - X_{m-2}(\lambda_h(p))\\
&= \sum_{h\in E(N,k)} \lambda_h(p^m) - \lambda_h(p^{m-2}).
\end{aligned}$$

From this, we deduce that for $m \in \mathbb{Z}$,

$$\frac{\sum_{h\in E(N,k)}\omega_h^r(2\cos(m\theta_p^h))}{\sum_{h\in E(N,k)}\omega_h^r}$$

$$= \begin{cases} 1, & \text{if } m = 0,\\ \frac{\sum_{h\in E(N,k)}\omega_h^r\lambda_h(p)}{\sum_{h\in E(N,k)}\omega_h^r}, & \text{if } |m| = 1,\\ \frac{\sum_{h\in E(N,k)}\omega_h^r(\lambda_h(p^m)-\lambda_h(p^{m-2}))}{\sum_{h\in E(N,k)}\omega_h^r}, & \text{if } |m| \geq 2.\end{cases}$$

To simplify our expressions, henceforth, we denote

$$W_r(N,k) := \sum_{h\in E(N,k)}\omega_h^r.$$

From Theorem 4.3.1, we deduce that for a prime power p^m with $m \geq 2$,

$$\sum_{h\in E(N,k)}(X_m(\lambda_h(p)) - X_{m-2}(\lambda_h(p)))\frac{|a_r(h)|^2e^{-4\pi r}}{||h||^2} = \frac{e^{-4\pi r}(4\pi r)^{k-1}}{(k-2)!}\psi(N)(A_1(m)+A_2(m)),$$

where

$$A_1(m) = \begin{cases} -1 & \text{if m is even, } p^{\frac{m-2}{2}} || r, \\ 0, & \text{otherwise} \end{cases}$$

and

$$A_2(m) = 2\pi \sum_{v=1}^{\infty} \frac{1}{Nv} \left\{ J_{k-1}\left(\frac{4\pi p^{m/2} r}{Nv}\right) \mathrm{Kl}_{vN}(p^m, r) - J_{k-1}\left(\frac{4\pi p^{m/2-1} r}{Nv}\right) \mathrm{Kl}_{vN}(p^{m-2}, r) \right\}.$$

Now, we state the following lemmas which would be required later in our calculation.

Lemma 4.3.2. *From ([3], equation (5.16)), we have*

$$J_k(x) \leq min\{x^k, x^{-1/2}\}$$

where $J_k(x)$ is the Bessel function of first kind.

Lemma 4.3.3. *([5], Lemma 5.1)*

$$Kl_u(n, r) \leq un$$

Chapter 5

Proofs of the main theorems

In this chapter we prove Theorems 3.3.1, 3.3.2 and 3.3.3.

5.1 Proof of Theorem 3.3.1

Now, we use the technique used by Murty and Sinha in their paper ([9]) to derive similar results by attaching weights.
For $h \in E(N,k)$ and $(p,N) = 1$ we choose $\theta_p^h \in [0,\pi]$ so that $2\cos\theta_p^h = \lambda_h(p)$. For a positive even integer $k \geq 3$ and for $N \geq 1$, we define the family

$$F_{N,k} := \left\{ \pm\frac{\theta_p^h}{2\pi} \,(\mathrm{mod}\ 1) : \mathrm{h} \in \mathrm{E(N,k)} \right\}.$$

Let $I = [a,b]$ be a fixed subinterval contained in $[-2,2]$. We choose a subinterval

$$I_1 = [\alpha,\beta] \subseteq \left[0, \frac{1}{2}\right]$$

so that

$$\frac{\theta_p^h}{2\pi} \in I_1 \iff 2\cos\theta_p^h \in I.$$

We also denote $I_2 = (\alpha,\beta]$.

We define

$$\mathcal{N}_I(N,k) = \sum_{h \in E(N,k)} \omega_h^r \chi_I(\lambda_h(p)).$$

We observe that, for $h \in E(N,k)$,

$$\omega_h^r \chi_I(\lambda_h(p)) = \omega_h^r \left[\chi_{I_1}\left(\frac{\theta_p^h}{2\pi}\right) + \chi_{I_2}\left(\frac{-\theta_p^h}{2\pi}\right)\right].$$

Let $S_{M,1}^{\pm}(x)$ be the Selberg-Beurling approximating polynomials for $\chi_{I_1}(x)$. We have

$$\begin{aligned}\sum_{h\in E(N,k)} \omega_h^r \left[S_{M,1}^{-}\left(\frac{\theta_p^h}{2\pi}\right) + S_{M,1}^{-}\left(-\frac{\theta_p^h}{2\pi}\right)\right] &\le \sum_{h\in E(N,k)} \omega_h^r \left[\chi_{I_1}\left(\frac{\theta_p^h}{2\pi}\right) + \chi_{I_2}\left(\frac{-\theta_p^h}{2\pi}\right)\right] \\ &\le \sum_{h\in E(N,k)} \omega_h^r \left[S_{M,1}^{+}\left(\frac{\theta_p^h}{2\pi}\right) + S_{M,1}^{+}\left(-\frac{\theta_p^h}{2\pi}\right)\right].\end{aligned} \tag{5.1}$$

We observe, for any $M \ge 1$,

$$\begin{aligned}\mathcal{N}_I(N,k) &\le \sum_{h\in E(N,k)} \omega_h^r \left[S_{M,1}^{+}\left(\frac{\theta_p^h}{2\pi}\right) + S_{M,1}^{+}\left(-\frac{\theta_p^h}{2\pi}\right)\right] \\ &= \sum_{h\in E(N,k)} \omega_h^r \sum_{|m|\le M} \hat{S}_{M,1}^{+}(m) \left(e\left(m\,\frac{\theta_p^h}{2\pi}\right) + e\left(-m\,\frac{\theta_p^h}{2\pi}\right)\right) \\ &= \sum_{h\in E(N,k)} \omega_h^r \sum_{|m|\le M} \hat{S}_{M,1}^{+}(m)(2\cos m\,\theta_p^h) \\ &= \hat{\mathcal{S}}_{\mathrm{M}}^{+}(0) \sum_{h\in E(N,k)} \omega_h^r + \sum_{m=1}^{M} \hat{\mathcal{S}}_{\mathrm{M}}^{+}(m) \sum_{h\in E(N,k)} \omega_h^r (2\cos m\,\theta_p^h).\end{aligned} \tag{5.2}$$

By considering the lower bound for $\mathcal{N}_I(N,k)$, we have

$$\begin{aligned}&\hat{\mathcal{S}}_{\mathrm{M}}^{-}(0) \sum_{h\in E(N,k)} \omega_h^r + \sum_{m=1}^{M} \hat{\mathcal{S}}_{\mathrm{M}}^{-}(m) \sum_{h\in E(N,k)} \omega_h^r (2\cos m\,\theta_p^h) \le \sum_{h\in E(N,k)} \omega_h^r \chi_I(\lambda_h(p)) \\ &\le \hat{S}_{M,1}^{+}(0) \sum_{h\in E(N,k)} \omega_h^r + \sum_{m=1}^{M} \hat{\mathcal{S}}_{\mathrm{M}}^{+}(m) \sum_{h\in E(N,k)} \omega_h^r (2\cos m\,\theta_p^h).\end{aligned} \tag{5.3}$$

We observe that for $h \in E(N,k)$, if $m = 1$, then

$$2\cos(m\theta_p^h) = \lambda_h(p),$$

and if $m \geq 2$,

$$\begin{aligned} 2\cos(m\theta_p^h) &= \left(\frac{\sin(m+1)\theta_p^h}{\sin\theta_p^h} - \frac{\sin(m-1)\theta_p^h}{\sin\theta_p^h} \right) \\ &= X_m(\lambda_h(p)) - X_{m-2}(\lambda_h(p)) = \lambda_h(p^m) - \lambda_h(p^{m-2}). \end{aligned} \tag{5.4}$$

From Theorem 4.3.1, we deduce that for a prime power p^m with $m \geq 2$,

$$\begin{aligned} &\sum_{h \in E(N,k)} \omega_h^r \left(\lambda_h(p^m) - \lambda_h(p^{m-2}) \right) \\ &= \frac{e^{-4\pi r}(4\pi r)^{k-1}}{(k-2)!}\psi(N)A_1(m) + \frac{e^{-4\pi r}(4\pi i)^k r^{k-1}}{2(k-2)!}\psi(N)A_2(m), \end{aligned} \tag{5.5}$$

where

$$A_1(m) = \begin{cases} -1 & \text{if m is even, } p^{\frac{m-2}{2}} || r, \\ 0, & \text{otherwise} \end{cases}$$

and

$$A_2(m) = 2\pi \sum_{v=1}^{\infty} \frac{1}{Nv} \left\{ J_{k-1}\left(\frac{4\pi p^{m/2} r}{Nv} \right) \mathrm{Kl}_{vN}(p^m, r) - J_{k-1}\left(\frac{4\pi p^{m/2-1} r}{Nv} \right) \mathrm{Kl}_{vN}(p^{m-2}, r) \right\}.$$

Let $m \geq 2$ be an even integer. In equation (5.5), we observe that there is a non-zero contribution from the first term on the right hand side if and only if

$$\frac{m}{2} = r_p + 1.$$

This contribution is

$$\frac{e^{-4\pi r}(4\pi r)^{k-1}}{(k-2)!}\psi(N).$$

We now choose a positive integer $M \geq 2(r_p+1)$. Thus, from equation (5.2), we deduce

$$\begin{aligned}\mathcal{N}_I(N,k) \leq \hat{\mathcal{S}}^+_{\mathrm{M}}(0)W_r(N,k) - \hat{\mathcal{S}}^+_{\mathrm{M}}(2r_p+2)\frac{e^{-4\pi r}(4\pi r)^{k-1}}{(k-2)!}\psi(N) \\ + \psi(N)\frac{e^{-4\pi r}(4\pi i)^k r^{k-1}}{2(k-2)!}\sum_{m=2}^{M}\hat{\mathcal{S}}^+_{\mathrm{M}}(m)A_2(m) + \hat{\mathcal{S}}^+_{\mathrm{M}}(1)\sum_{h\in E(N,k)}\omega_h^r\lambda_h(p).\end{aligned} \tag{5.6}$$

By ([5], Corollary 5.3), we have

$$\frac{e^{-4\pi r}(4\pi r)^{k-1}}{(k-2)!}\psi(N) = \sum_{h\in E(N,k)}\omega_h^r + \mathrm{O}\left(\frac{\psi(N)}{N^{k-1}}\right). \tag{5.7}$$

Substituting equation (5.7) in (5.6), we have

$$\begin{aligned}\mathcal{N}_I(N,k) \leq \hat{S}^+_{M,1}(0)W_r(N,k) - \hat{\mathcal{S}}^+_{\mathrm{M}}(2r_p+2)\left[\sum_{h\in E(N,k)}\omega_h^r + \mathrm{O}\left(\frac{\psi(N)}{N^{k-1}}\right)\right] \\ + \psi(N)\frac{e^{-4\pi r}(4\pi i)^k r^{k-1}}{2(k-2)!}\sum_{m=1}^{M}\hat{\mathcal{S}}^+_{\mathrm{M}}(m)A_2(m).\end{aligned} \tag{5.8}$$

We recall, from Section 1.2,

$$\hat{\mathcal{S}}^{\pm}_{\mathrm{M}}(0) = 2(\beta-\alpha) \pm \frac{2}{M+1}$$

and for $m \geq 1$,

$$\hat{\mathcal{S}}^{\pm}_{\mathrm{M}}(m) = \frac{\sin(2\pi m\beta) - \sin(2\pi m\alpha)}{m\pi} + O\left(\frac{1}{M}\right).$$

Thus, by (5.8),

$$\begin{aligned}&\sum_h \omega_h^r\chi_I(\lambda_h(p)) - \left[2(\beta-\alpha) - \frac{\sin(2\pi(2r_p+2)\beta) - \sin(2\pi(2r_p+2)\alpha)}{(2r_p+2)\pi}\right] \\ &= \mathrm{O}\left(\frac{|W_r(N,k)|}{M}\right) + \mathrm{O}\left(\frac{|\,\hat{\mathcal{S}}^{\pm}_{\mathrm{M}}(2r_p+2)|\psi(N)}{N^{k-1}}\right) + \mathrm{O}\left(\frac{1}{M}\left(W_r(N,k) + \frac{\psi(N)}{N^{k-1}}\right)\right) \\ &+ \mathrm{O}\left(\psi(N)\frac{e^{-4\pi r}(4\pi)^k r^{k-1}}{2(k-2)!}\sum_{m=1}^{M}|\,\hat{\mathcal{S}}^{\pm}_{\mathrm{M}}(m)A_2(m)|\right).\end{aligned} \tag{5.9}$$

But,

$$\begin{aligned}
&2(\beta-\alpha) - \frac{\sin(2\pi(2r_p+2)\beta) - \sin(2\pi(2r_p+2)\alpha)}{(2r_p+2)\pi} \\
&= 2\int_\alpha^\beta (1-\cos 2\pi(2r_p+2)t)dt.
\end{aligned} \tag{5.10}$$

Also, with the substitution $x = 2\cos 2\pi t$,

$$\begin{aligned}
&\sum_{i=0}^{r_p} X_{2i}(x)\mu_\infty(x) \\
&= 4\sum_{i=0}^{r_p} \frac{\sin(2i+1)(2\pi t)}{\sin 2\pi t}\sin^2 2\pi t = 2(1-\cos 2\pi(2r_p+2)t).
\end{aligned} \tag{5.11}$$

We deduce

$$\begin{aligned}
&\sum_h \omega_h^r \chi_I(\lambda_h(p)) - \int_I \sum_{i=0}^{r_p} X_{2i}(x)\mu_\infty(x)dx \\
&= \mathrm{O}\left(\frac{|W_r(N,k)|}{M}\right) + \mathrm{O}\left(\frac{|\,\hat{\mathcal{S}}_{\mathrm{M}}^{\pm}(2r_p+2)|\psi(N)}{N^{k-1}}\right) + \mathrm{O}\left(\frac{1}{M}\left(W_r(N,k) + \frac{\psi(N)}{N^{k-1}}\right)\right) \\
&+ \mathrm{O}\left(\psi(N)\frac{e^{-4\pi r}(4\pi)^k r^{k-1}}{2(k-2)!}\sum_{m=1}^{M} |\,\hat{\mathcal{S}}_{\mathrm{M}}^{\pm}(m)A_2(m)|\right).
\end{aligned} \tag{5.12}$$

We now state the following two lemmas:

Lemma 5.1.1. *With the notations as above, we have*

$$|A_2(m)| \ll \frac{p^{(k+1)(\frac{m}{2})}}{N^{k+1}}$$

Proof. Using Lemma 4.3.2 and 4.3.3 we have,

$$\begin{aligned}
A_2(m) &= 2\pi \sum_{v=1}^{\infty} \frac{1}{Nv} \left\{ \left(\frac{4\pi p^{m/2} r}{Nv} \right)^{k-1} vNp^m - \left(\frac{4\pi p^{m/2-1} r}{Nv} \right)^{k-1} vNp^{m-2} \right\} \\
&= 2\pi \left(\frac{4\pi r}{N} \right)^{k-1} \sum_{v=1}^{\infty} \left\{ \frac{p^{\frac{m}{2}(k+1)} - p^{(\frac{m}{2}-1)(k+1)}}{v^{k-1}} \right\} \\
&\ll \left(\frac{p^{\frac{m}{2}(k+1)}}{N^{k-1}} \right)
\end{aligned}$$

Thus $A_2(m) = \mathrm{O}\left(\frac{p^{\frac{m}{2}(k+1)}}{N^{k-1}} \right)$ □

Lemma 5.1.2. *With the same notations as used above, we have*

$$\psi(N) \sum_{m=1}^{M} |\, \hat{\mathcal{S}}_{\mathrm{M}}^{\pm}(m) A_2(m)| = \mathrm{O}\left(\frac{\psi(N)}{N^{k-1}} (p^{k+1})^{\frac{M}{2}} \right)$$

Proof. Using the fact that $\hat{\mathcal{S}}_{\mathrm{M}}^{\pm}(m) \leq \dfrac{1}{m}$, we have

$$\psi(N) \sum_{m=1}^{M} |\, \hat{\mathcal{S}}_{\mathrm{M}}^{\pm}(m) A_2(m)| \ll \psi(N) \sum_{m=1}^{M} \left| \frac{1}{m} \left(\frac{p^{\frac{m}{2}(k+1)}}{N^{k-1}} \right) \right| < \psi(N) \sum_{m=1}^{M} \left| \frac{p^{\frac{M}{2}(k+1)}}{N^{k-1}} \right|$$

Thus, $\psi(N) \sum_{m=1}^{M} |\, \hat{\mathcal{S}}_{\mathrm{M}}^{\pm}(m) A_2(m)| = \mathrm{O}\left(\dfrac{\psi(N)}{N^{k-1}} (p^{k+1})^{\frac{M}{2}} \right)$ □

Thus, by using Lemma 5.1.2 and (5.12) , we are now ready to prove Theorem 3.3.1.

Proof. Using (5.12) we have,

$$\begin{aligned}
&\frac{1}{|W_r(N,k)|} \left[\sum_h \omega_h^r \chi_I(\lambda_h(p)) - \int_I \sum_{i=0}^{r_p} X_{2i}(x) \mu_\infty(x) dx \right] \\
&= \mathrm{O}\left(\frac{1}{M} \right) + \mathrm{O}\left(\frac{(p^{k+1})^{\frac{M}{2}}}{N^{k-1}} \right)
\end{aligned} \tag{5.13}$$

Now taking $p^{(k+1)(\frac{M}{2})} \approx N^{\frac{k-1}{2}}$, we have $\frac{k+1}{2} M \log p \approx \frac{k-1}{2} \log N$.
Thus $M \approx \frac{\frac{k-1}{2} \log N}{\frac{k+1}{2} M \log p}$.

Therefore, taking $M = \left\lfloor \frac{\frac{k-1}{2}\log N}{\frac{k+1}{2}M\log p} \right\rfloor$, we get the desired result. $\square$

Thus, we found a bound for the weighted analog of Murty and Sinha's work ([9], Theorem 2).

5.2 Proof of Theorem 3.3.2 and 3.3.3

Before going into the proof of Theorems 3.3.2 and 3.3.3, we state the following lemmas.

Lemma 5.2.1. *(Corollary 5.3, [5])*
Let r be a fixed integer and ω_h^r be the weight as defined before. Let $k \geq 3$ and $h \in E(N,k)$. Then

$$W_r(N,k) = \sum_{f \in E(N,k)} \omega_h^r = \psi(N)\frac{e^{-4\pi r}(4\pi r)^{k-1}}{(k-2)!} + \mathrm{O}\left(\frac{\psi(N)}{N^{k-1}}\right).$$

Lemma 5.2.2. *Let $n = p_1^{j_1}p_2^{j_2}...p_u^{j_u}$, r be a fixed integer and ω_h^r be the weight as defined before. Let $h \in E(N,k)$, then*

$$\sum_{h \in E(N,k)} \omega_h^r \lambda_h(p_1^{j_1}p_2^{j_2}...p_u^{j_u}) = True\left(j_i's \text{ are even}, \frac{j_i}{2} \leq r_{p_i}\right) W_r(N,k) + \mathrm{O}\left(\frac{\psi(N)}{N^{k-1}} n^{\frac{k+1}{2}}\right)$$

Lemma 5.2.3. *(Lemma 2, [10])*
Suppose p is prime. For any $n \geq 1$, we have

$$(\lambda_h(p))^n = \sum_{j=0}^{n} b_n(j)\lambda_h(p^j)$$

where,

$$b_n(j) := \frac{2^{n+1}}{\pi}\int_0^{\pi} \cos^n\theta \sin(j+1)\theta \sin\theta d\theta \tag{5.14}$$

Also, $b_n(j) = 0$ if n is odd and j is even or if n is even and j is odd.

5.2.1 Proof of Theorem 3.3.2

We use the technique used by Nagoshi ([10], Section 3).
By Weierstrass approximation theorem there exists polynomial function $p(t)$ such that for a given continuous real valued function $g(t)$ on compact set $[-2,2]$ and for $\epsilon > 0$, we have $|g(t)-p(t)| < \epsilon$. So, it will be sufficient to prove Theorem 3.3.2 for $g = X_n$, that is, the n^{th} Chebychev polynomial. Then, $X_n(\lambda_h(p)) = \lambda_h(p^n)$. Thus,

$$\frac{1}{\pi_N(x)W_r(N,k)} \sum_{\substack{p\leq x\\ h\in E(N,k)}} \omega_h^r X_n(\lambda_h(p)) = \frac{1}{\pi_N(x)W_r(N,k)} \sum_{\substack{p\leq x\\ h\in E(N,k)}} \omega_h^r(\lambda_h(p^n)).$$

We note that in the summation we take all primes p less than equal to x such that $(p,N)=1$.

Now, we need to show

$$\frac{1}{\pi_N(x)W_r(N,k)} \sum_{\substack{p\leq x\\ h\in E(N,k)}} \omega_h^r(\lambda_h(p^n)) \to \frac{1}{2\pi}\int_{-2}^{2} X_n(t)\sqrt{4-t^2}dt$$

as $x\to\infty$.
Let us change the variable $t\mapsto 2\cos(2\pi x)$, then, for $n\geq 1$.

$$\frac{1}{2\pi}\int_{-2}^{2} X_n(t)\sqrt{4-t^2}dt = \int_0^{1/2} \frac{\sin((n+1)2\pi x)}{\sin(2\pi x)} 2\sin^2(2\pi x)dx = 0 \tag{5.15}$$

Now, using Lemma 5.2.2, we have, for $n\geq 1$,

$$\begin{aligned}
\frac{1}{\pi_N(x)W_r(N,k)} \sum_{\substack{p\leq x\\ h\in E(N,k)}} \omega_h^r(\lambda_h(p^n)) &= \frac{1}{\pi_N(x)W_r(N,k)} \sum_{p\leq x} \left(\text{True}\left(n \text{ is even}, \frac{n}{2}\leq r_p\right) W_r(N,k)\right) \\
&\quad + \mathrm{O}\left(\frac{1}{\pi_N(x)W_r(N,k)} \sum_{p\leq x} \frac{\psi(N)}{N^{k-1}} p^{n\frac{k+1}{2}}\right) \\
&< \frac{1}{\pi_N(x)} \sum_{p\leq x} \left(\text{True}\left(n \text{ is even}, \frac{n}{2}\leq r_p\right)\right) + \mathrm{O}\left(\frac{x^{n\frac{k+1}{2}}}{N^{k-1}}\right)
\end{aligned}$$

r_p is equals to the highest power of p dividing the fixed integer r. So, as p increases, after a certain point r_p would be zero for all p greater than some particular value. So, as $x \to \infty$, the first term would be zero in the above equation.
Now, for the second term to tend to zero as $x \to \infty$, we need to have $N > x^{n\frac{(k+1)}{2(k-1)}}$.
Thus, for $\frac{\log N}{\log x} \to \infty$,

$$\frac{1}{\pi_N(x)W_r(N,k)} \sum_{\substack{p \leq x \\ h \in E(N,k)}} \omega_h^r X_n(\lambda_h(p)) \to 0 \tag{5.16}$$

as $x \to \infty$.

Now, for $h \in E(N,k)$, we choose $\theta_p^h \in [0,\pi]$ so that $2\cos\theta_p^h = \lambda_h(p)$. For a positive even integer $k \geq 3$ and for $N \geq 1$, we define the family

$$F_{N,k} := \left\{ \pm\frac{\theta_p^h}{2\pi} \pmod 1 : \text{h} \in \text{E(N, k)} \right\}$$

as in Section 5.1.
Now, we calculate the weighted Weyl's limit for $m \in \mathbb{Z}$, that is,

$$C_m = \lim_{x\to\infty} \frac{1}{2\pi_N(x)W_r(N,k)} \sum_{\substack{p \leq x \\ h \in E(N,k)}} \omega_h^r 2\cos(m\theta_p^h)$$

Thus, for $m = 0$,

$$C_0 = \lim_{x\to\infty} \frac{1}{2\pi_N(x)W_r(N,k)} \sum_{\substack{p \leq x \\ h \in E(N,k)}} 2\omega_h^r = 1.$$

For $|m| = 1$,

$$C_m = \lim_{x\to\infty} \frac{1}{2\pi_N(x)W_r(N,k)} \sum_{\substack{p \leq x \\ h \in E(N,k)}} \omega_h^r \lambda_h(p) \to 0$$

For $|m| = 2$,

$$C_m = \lim_{x\to\infty} \frac{1}{2\pi_N(x)W_r(N,k)} \sum_{\substack{p\le x \\ h\in E(N,k)}} \omega_h^r(\lambda_h(p^2) - 1) = -1/2$$

ALso, for $|m| > 2$, $C_m = 0$. Thus, by Theorem 1.6.3, we obtain the following measure:

$$\mu(x) = \sum_{m\in\mathbb{Z}} C_m e(mx) = 1 - \frac{1}{2}(e(2x) + e(-2x)) = 2\sin^2(2\pi x)$$

Thus, by using Equation (5.15) and (5.16), we have the following:

$$\frac{1}{\pi_N(x)W_r(N,k)} \sum_{\substack{p\le x \\ h\in E(N,k)}} \omega_h^r X_n(\lambda_h(p)) \to \frac{1}{2\pi}\int_{-2}^{2} X_n(t)\sqrt{4-t^2}dt$$

as $x \to \infty$.
Thus, Theorem 3.3.2 is proved, that is, for any continuous real valued function g on $[-2, 2]$, $N = N(x)$ such that $\frac{\log N}{\log x} \to \infty$ and for primes p such that $(p, N) = 1$, we have

$$\frac{1}{\pi_N(x)W_r(N,k)} \sum_{\substack{p\le x \\ h\in E(N,k)}} \omega_h^r g(\lambda_h(p)) \to \frac{1}{2\pi}\int_{-2}^{2} g(t)\sqrt{4-t^2}dt$$

as $x \to \infty$.

5.2.2 Proof of Theorem 3.3.3

We use the technique used by Nagoshi ([10], Section 4).
By Weierstrass approximation theorem there exists polynomial function $p(t)$ such that for a given continuous real valued function $g(t)$ on compact set $[-2, 2]$ and for $\epsilon > 0$, we have $|g(t)-p(t)| < \epsilon$. So, it is sufficient for getting Theorem 3.3.3 to prove for each positive integer $l \ge 0$,

$$\frac{1}{W_r(N,k)} \sum_{h\in E(N,k)} \omega_h^r \left(\frac{\sum_{p\le x}\lambda_h(p)}{\sqrt{\pi_N(x)}}\right)^l \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^l e^{\frac{-t^2}{2}} dt \text{ as } x \to \infty. \quad (5.17)$$

We note that in the above equation and in what follows, the sum $\sum_{p\le x}$ runs over ptimes p which are coprime to N.

By the multinomial formula,

$$w_h^r\left(\sum_{p\le x}\lambda_h(p)\right)^l = w_h^r\sum_{u=1}^{l}\sum_{(l_1,l_2,...,l_u)}^{(1)}\frac{l!}{l_1!l_2!...l_u!}\frac{1}{u!}\sum_{(p_1,p_2,...,p_3)}^{(2)}\lambda_h(p_1)^{l_1}\lambda_h(p_2)^{l_2}...\lambda_h(p_u)^{l_u},$$

where $\sum_{(l_1,l_2,...,l_u)}^{(1)}$ means sum over u-tuples $(l_1,l_2,...,l_u)$ of positive integer satisfying $l_1+l_2+...+l_u=l$, and $\sum_{(p_1,p_2,...,p_u)}^{(2)}$ is the sum over u-tuples $(p_1,p_2,...,p_3)$ of distinct primes which are not greater than x and $(p_i,N)=1$ for $i=1,2,...,u$.

Since, $\lambda_h(mn)=\sum_{d|(m,n)}\lambda_h(\frac{mn}{d^2})$ and by using Lemma 5.2.3, we have,

$$\sum_{h\in E(N,k)}\omega_h^r\lambda_h(p_1)^{l_1}\lambda_h(p_2)^{l_2}...\lambda_h(p_u)^{l_u} = \sum_{h\in E(N,k)}\omega_h^r\left[\sum_{j_1=0}^{l_1}b_{l_1}(j_1)\lambda_h(p_1^{j_1})\right]...\left[\sum_{j_u=0}^{l_u}b_{l_u}(j_u)\lambda_h(p_u^{j_u})\right] \tag{5.18}$$

$$= \sum_{h\in E(N,k)}\omega_h^r\sum_{\substack{0\le j_1\le l_1\\ 0\le j_u\le l_u}}b_{l_1}(j_1)...b_{l_u}(j_u)\lambda_h(p_1^{j_1}...p_u^{j_u}) \tag{5.19}$$

$$= \sum_{\substack{0\le j_1\le l_1\\ 0\le j_u\le l_u}}b_{l_1}(j_1)...b_{l_u}(j_u)\sum_{h\in E(N,k)}\omega_h^r\lambda_h(p_1^{j_1}...p_u^{j_u}) \tag{5.20}$$

We have the following lemmas under this setting.

Lemma 5.2.4. *Let k be fixed and $N\ge\left(x^{\frac{l}{2}+l^2\frac{k+1}{2}}(\log x)^{\frac{l}{2}}\right)^{\frac{1}{k-1}}$. Assume the u-tuple $(l_1,l_2,...,l_u)$ such that l_m is odd for some m. Then*

$$\frac{1}{\pi_N(x)^{\frac{l}{2}}}\frac{1}{W_r(N,k)}\sum_{(p_1,p_2,...,p_3)}^{(2)}\sum_{h\in E(N,k)}\omega_h^r\lambda_h(p_1)^{l_1}\lambda_h(p_2)^{l_2}...\lambda_h(p_u)^{l_u}\mapsto 0$$

as $x\mapsto\infty$.

Proof. Let $\sum_{(j_1...j_u)}^{(3)}$ be the sum of u-tuples $(j_1,...,j_u)$ of integers satisfying j_i's even and $0\le j_i\le 2r_{p_i}$ for each $0\le i\le u$. Let $n=p_1^{l_1}...p_u^{l_u}$.

Using Lemma 5.2.2 in equation (5.20), we have

$$\sum_{h\in E(N,k)} \omega_h^r \lambda_h(p_1)^{l_1}\lambda_h(p_2)^{l_2}...\lambda_h(p_u)^{l_u} = \sum_{(j_1...j_u)}^{(3)} \left[b_{l_1}(j_1)...b_{l_u}(j_u) W_r(N,k) + \mathrm{O}\left(\frac{W_r(N,k)}{N^{k-1}} n^{(\frac{k+1}{2})}\right)\right]$$

The first term in the left hand side is zero since, each $j_i's$ are even and l_m is odd for some m. Thus,

$$\sum_{h\in E(N,k)} \omega_h^r \lambda_h(p_1)^{l_1}\lambda_h(p_2)^{l_2}...\lambda_h(p_u)^{l_u} = \sum_{(j_1...j_u)}^{(3)} \mathrm{O}\left(\frac{W_r(N,k)}{N^{k-1}} n^{(\frac{k+1}{2})}\right)$$

Now,

$$\begin{aligned}
\sum_{(p_1,p_2,...,p_3)}^{(2)} \sum_{h\in E(N,k)} \omega_h^r \lambda_h(p_1)^{l_1}\lambda_h(p_2)^{l_2}...\lambda_h(p_u)^{l_u} &= \sum_{(p_1,p_2,...,p_3)}^{(2)} \sum_{(j_1...j_u)}^{(3)} \mathrm{O}\left(\frac{W_r(N,k)}{N^{k-1}} n^{(\frac{k+1}{2})}\right) \\
&= \frac{W_r(N,k)}{N^{k-1}} \sum_{(p_1,p_2,...,p_3)}^{(2)} \mathrm{O}_l(x^{ul\frac{k+1}{2}}) \\
&\leq \frac{W_r(N,k)}{N^{k-1}} \pi_N(x)^u \, \mathrm{O}_l(x^{ul\frac{k+1}{2}}) \\
&\ll_l \frac{W_r(N,k)}{N^{k-1}} \pi_N(x)^l x^{l^2\frac{k+1}{2}} \ll \frac{W_r(N,k)}{N^{k-1}} x^{l+l^2\frac{k+1}{2}}
\end{aligned}$$

Thus,

$$\frac{1}{\pi_N(x)^{\frac{l}{2}}} \frac{1}{W_r(N,k)} \frac{W_r(N,k)}{N^{k-1}} x^{l+l^2\frac{k+1}{2}} = \frac{1}{\pi_N(x)^{\frac{l}{2}}} \frac{1}{N^{k-1}} x^{l+l^2\frac{k+1}{2}}$$

Therefore, for $N \geq \left(x^{\frac{l}{2}+l^2\frac{k+1}{2}}(\log x)^{\frac{l}{2}}\right)^{\frac{1}{k-1}}$,

$$\frac{1}{\pi_N(x)^{\frac{l}{2}}} \frac{1}{W_r(N,k)} \sum_{(p_1,p_2,...,p_3)}^{(2)} \sum_{h\in E(N,k)} \omega_h^r \lambda_h(p_1)^{l_1}\lambda_h(p_2)^{l_2}...\lambda_h(p_u)^{l_u} \mapsto 0$$

as $x \mapsto \infty$. □

Lemma 5.2.5. *Let k be fixed. Let $N = N(x)$. Assume that an u-tuple $(l_1, l_2, ..., l_u)$ such that l_m is even for each $0 \leq m \leq u$. If $l_m = 2$ for all m, then we have,*

$$\frac{1}{\pi_N(x)^{\frac{l}{2}}}\frac{1}{W_r(N,k)}\sum_{(p_1,p_2,...,p_3)}^{(2)}\sum_{h\in E(N,k)}\omega_h^r\lambda_h(p_1)^{l_1}\lambda_h(p_2)^{l_2}...\lambda_h(p_u)^{l_u}\mapsto 1$$

as $x\mapsto\infty$ *and* $N\geq x^{\frac{l^2(k+1)}{4(k-1)}}$*;*
if not then, we have,

$$\frac{1}{\pi_N(x)^{\frac{l}{2}}}\frac{1}{W_r(N,k)}\sum_{(p_1,p_2,...,p_3)}^{(2)}\sum_{h\in E(N,k)}\omega_h^r\lambda_h(p_1)^{l_1}\lambda_h(p_2)^{l_2}...\lambda_h(p_u)^{l_u}\mapsto 0$$

as $x\mapsto\infty$ *and* $N\geq\left(\log x\ x^{(\frac{l}{2})l(\frac{k+1}{2})-1}\right)^{\frac{1}{k-1}}$.

Proof. Let us first consider the case when $l_m=2$ for all m. Since $l_1+...+l_u=l$, we have, $u=\dfrac{l}{2}$.
As $b_2(0)=b_2(2)=1$, it follows that

$$\sum_{h\in E(N,k)}\sum_{\substack{j_i=0,2\\0\leq i\leq u}}\omega_h^r b_{l_1}(j_1)...b_{l_u}(j_u)\lambda_h(p_1^{j_1}...p_u^{j_u})=\sum_{h\in E(N,k)}\sum_{\substack{j_i=0,2\\0\leq i\leq u}}\omega_h^r\lambda_h(p_1^{j_1}...p_u^{j_u})$$

$$=\sum_{h\in E(N,k)}\omega_h^r\lambda_h(1)+\sum_{(j_1,...,j_u)}^{(4)}\sum_{h\in E(N,k)}\omega_h^r\lambda_h(p_1^{j_1}...p_u^{j_u})$$

$$=W_r(N,k)+\sum_{(j_1,...,j_u)}^{(4)}\left[\mathrm{True}(\frac{j_i}{2}\leq r_{p_i})W_r(N,k))+\mathrm{O}\left(\frac{W_r(N,k)}{N^{k-1}}n^{\frac{k+1}{2}}\right)\right]$$

where $n=p_1^{j_1}...p_u^{j_u}$ and $\sum_{(j_1,...,j_u)}^{(4)}$ is the sum over u-tuple $(j_1,...,j_u)$ satisfying $j_i=0$ or 2 and that $(j_1,...,j_u)\neq(0,...,0)$.

Now,using ([10], equation (25)), we have

$$\sum_{(p_1,p_2,...,p_3)}^{(2)} \left(W_r(N,k) + \sum_{(j_1,...,j_u)}^{(4)} \left[\mathrm{True}(\frac{j_i}{2} \leq r_{p_i}) W_r(N,k) + \mathrm{O}(\frac{W_r(N,k)}{N^{k-1}} n^{\frac{k+1}{2}}) \right] \right)$$

$$= W_r(N,k) \left(\pi_N(x)^{\frac{l}{2}} + \mathrm{O}_l(\pi_N(x)^{\frac{l}{2}-1}) + \sum_{(p_1,p_2,...,p_3)}^{(2)} \sum_{(j_1,...,j_u)}^{(4)} \mathrm{True}(\frac{j_i}{2} \leq r_{p_i}) \right)$$

$$+ \sum_{(p_1,p_2,...,p_3)}^{(2)} \sum_{(j_1,...,j_u)}^{(4)} \mathrm{O}\left(\frac{W_r(N,k)}{N^{k-1}} (p_1^{j_1}...p_u^{j_u})^{\frac{k+1}{2}} \right)$$

$$= W_r(N,k) \left(\pi_N(x)^{\frac{l}{2}} + \mathrm{O}_l(\pi_N(x)^{\frac{l}{2}-1}) + \sum_{(p_1,p_2,...,p_3)}^{(2)} \sum_{(j_1,...,j_u)}^{(4)} \mathrm{True}(\frac{j_i}{2} \leq r_{p_i}) \right) + \mathrm{O}_l \left(\frac{W_r(N,k)}{N^{k-1}} \pi_N(x)^{\frac{l}{2}} x^{\frac{l^2}{2}\frac{k+1}{2}} \right)$$

Now,

$$\frac{1}{W_r(N,k)\pi_N(x)^{\frac{l}{2}}} \left[W_r(N,k) \left(\pi_N(x)^{\frac{l}{2}} + \mathrm{O}_l(\pi_N(x)^{\frac{l}{2}-1}) + \sum_{(p_1,p_2,...,p_3)}^{(2)} \sum_{(j_1,...,j_u)}^{(4)} \mathrm{True}(\frac{j_i}{2} \leq r_{p_i}) \right) \right]$$

$$+ \frac{1}{W_r(N,k)\pi_N(x)^{\frac{l}{2}}} \left[\mathrm{O}_l \left(\frac{W_r(N,k)}{N^{k-1}} \pi_N(x)^{\frac{l}{2}} x^{\frac{l^2}{2}\frac{k+1}{2}} \right) \right]$$

$$= \frac{1}{\pi_N(x)^{\frac{l}{2}}} \left[\pi_N(x)^{\frac{l}{2}} + \mathrm{O}_l(\pi_N(x)^{\frac{l}{2}-1}) + \sum_{(p_1,p_2,...,p_3)}^{(2)} \sum_{(j_1,...,j_u)}^{(4)} \mathrm{True}(\frac{j_i}{2} \leq r_{p_i}) + \mathrm{O}_l \left(\frac{\pi_N(x)^{\frac{l}{2}} x^{\frac{l^2}{2}\frac{k+1}{2}}}{N^{k-1}} \right) \right]$$

$$= 1 + \mathrm{O}_l \left(\frac{x^{\frac{l^2}{2}\frac{k+1}{2}}}{N^{k-1}} \right)$$

Thus, for $N \geq x^{\frac{l^2(k+1)}{4(k-1)}}$,
$\frac{1}{\pi_N(x)^{\frac{l}{2}}} \frac{1}{W_r(N,k)} \sum_{(p_1,p_2,...,p_3)}^{(2)} \sum_{h \in E(N,k)} \omega_h^r \lambda_h(p_1)^{l_1} \lambda_h(p_2)^{l_2} ... \lambda_h(p_u)^{l_u} \mapsto 1$ as $x \mapsto \infty$.

Now coming to second case $(l_1, l_2, ..., l_u) \neq (2, 2, ..., 2)$. We have

$$u \leq \frac{l}{2} - 1.$$

Now,

$$\sum_{h\in E(N,k)} \omega_h^r \lambda_h(p_1)^{l_1}\lambda_h(p_2)^{l_2}...\lambda_h(p_u)^{l_u} = \sum_{(j_1,...,j_u)}^{(5)} b_{l_1}(j_1)...b_{l_u}(j_u) \sum_{h\in E(N,k)} \omega_h^r \lambda_h(p_1^{j_1}...p_u^{j_u})$$
$$= \sum_{(j_1,...,j_u)}^{(5)} b_{l_1}(j_1)...b_{l_u}(j_u) \left[\text{True}(\frac{j_i}{2} \leq r_{p_i})W_r(N,k)) + \text{O}(\frac{W_r(N,k)}{N^{k-1}} n^{\frac{k+1}{2}})\right]$$

where $\sum_{(j_1,...,j_u)}^{(5)}$ denotes the sum over the $u-$tuple $(j_1,...,j_u)$ of even integers satisfying $0 \leq j_i \leq l_i$ for each $i = 1,...,u$. Hence,

$$\sum_{(p_1,p_2,...,p_3)}^{(2)} \sum_{h\in E(N,k)} \omega_h^r \lambda_h(p_1)^{l_1}\lambda_h(p_2)^{l_2}...\lambda_h(p_u)^{l_u} = \sum_{(p_1,p_2,...,p_3)}^{(2)} \sum_{(j_1,...,j_u)}^{(5)} b_{l_1}(j_1)...b_{l_u}(j_u)(\text{True}(\frac{j_i}{2} \leq r_{p_i})W_r(N,k))$$
$$+ \sum_{(p_1,p_2,...,p_3)}^{(2)} \sum_{(j_1,...,j_u)}^{(5)} b_{l_1}(j_1)...b_{l_u}(j_u)\, \text{O}\left(\frac{W_r(N,k)}{N^{k-1}} n^{\frac{k+1}{2}}\right)$$

Here the first part will vanish as $x \mapsto \infty$. Therefore we need to consider only the second part, that is, $\sum_{(p_1,p_2,...,p_3)}^{(2)} \sum_{(j_1,...,j_u)}^{(5)} b_{l_1}(j_1)...b_{l_u}(j_u)\, \text{O}\left(\frac{W_r(N,k)}{N^{k-1}} n^{\frac{k+1}{2}}\right)$.
Using steps from ([10], first equation in page number 305), we have,

$$\sum_{(p_1,p_2,...,p_3)}^{(2)} \sum_{(j_1,...,j_u)}^{(5)} b_{l_1}(j_1)...b_{l_u}(j_u)\, \text{O}\left(\frac{W_r(N,k)}{N^{k-1}} n^{\frac{k+1}{2}}\right) <<_l \pi_N(x)^u \frac{W_r(N,k)}{N^{k-1}} x^{(\frac{l}{2}-1)l\frac{k+1}{2}}$$

Therefore,

$$\frac{1}{\pi_N(x)^{\frac{l}{2}}} \frac{1}{W_r(N,k)} \pi_N(x)^u \frac{W_r(N,k)}{N^{k-1}} x^{(\frac{l}{2}-1)l\frac{k+1}{2}} \leq \frac{1}{\pi_N(x)} \frac{1}{N^{k-1}} x^{(\frac{l}{2}-1)l\frac{k+1}{2}}$$

Thus for $N \geq \left(\log x\ x^{(\frac{l}{2})l(\frac{k+1}{2})-1}\right)^{\frac{1}{k-1}}$,

$$\frac{1}{\pi_N(x)^{\frac{l}{2}}} \frac{1}{W_r(N,k)} \sum_{(p_1,p_2,...,p_3)}^{(2)} \sum_{h\in E(N,k)} \omega_h^r \lambda_h(p_1)^{l_1}\lambda_h(p_2)^{l_2}...\lambda_h(p_u)^{l_u} \mapsto 0$$

as $x \mapsto \infty$.

□

From Lemma 5.2.4 and 5.2.5, for an $u-$tuple $(l_1, l_2, ..., l_u)$, we have

$$\frac{1}{\pi_N(x)^{\frac{l}{2}}}\frac{1}{W_r(N,k)}\sum_{(p_1,p_2,...,p_3)}^{(2)}\sum_{h\in E(N,k)}\omega_h^r\lambda_h(p_1)^{l_1}\lambda_h(p_2)^{l_2}...\lambda_h(p_u)^{l_u}\to\begin{cases}1 & \text{if } (l_1,...,l_u)=(2,...,2)\\ 0 & \text{otherwise,}\end{cases} \tag{5.21}$$

for $\frac{\log N}{\log x}\to\infty$ as $x\to\infty$.

Let x be a positive integer and l be odd. Then by $l_1 + l_2 + ... + l_3 = l$ and (5.21) all the terms of

$$\frac{1}{\pi_N(x)^{l/2}W_r(N,k)}\sum_{h\in E(N,k)}w_h^r\left(\sum_{p\le x}\lambda_h(p)\right)^l$$
$$=\sum_{u=1}^{l}\sum_{(l_1,l_2,...,l_u)}^{(1)}\frac{l!}{l_1!l_2!...l_u!}\frac{1}{u!}\frac{1}{\pi_N(x)^{l/2}W_r(N,k)}\sum_{(p_1,p_2,...,p_3)}^{(2)}\sum_{h\in E(N,k)}w_h^r\lambda_h(p_1)^{l_1}\lambda_h(p_2)^{l_2}...\lambda_h(p_u)^{l_u}$$

goes to zero as $x\to\infty$.
Now, let l be even. Then by (5.21), as $x\to\infty$, all terms of

$$\frac{1}{\pi_N(x)^{l/2}W_r(N,k)}\sum_{h\in E(N,k)}w_h^r\left(\sum_{p\le x}\lambda_h(p)\right)^l$$

go to 0 except for $u=\frac{l}{2}$ and $l_1=l_2=...=l_u=2$, which goes to $l!/(l_1!...l_u!u!)=\frac{l!}{2^{\frac{l}{2}}(l/2)!}$. Therefore, $\frac{1}{\pi_N(x)^{l/2}W_r(N,k)}\sum_{h\in E(N,k)}w_h^r\left(\sum_{p\le x}\lambda_h(p)\right)^l$ goes to $\frac{l!}{2^{\frac{l}{2}}(l/2)!}$.

But then it is known that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}t^le^{\frac{-t^2}{2}}dt=\begin{cases}\frac{l!}{2^{\frac{l}{2}}(l/2)!}, & \text{if } l \text{ is even,}\\ 0, & \text{otherwise}\end{cases} \tag{5.22}$$

Thus Theorem 3.3.3 is proved.

Chapter 6

Conclusion

In this thesis, we proved the weighted analog of Murty and Sinha's work [9] as well as Nagoshi's work [10].

6.1 Future Goals

In future, we would like to investigate the following related questions. Following some recent results of Prabhu and Sinha [12], we would like to answer the following questions in the context of Li's weighted equidistribution theorem. Let I be a fixed subinterval of $[-2, 2]$. For such an I,

- What does the asymptotic variance

$$\frac{1}{|W_r(N,k)|} \sum_{h \in E(N,k)} \left[\omega_h^r \chi_I(\lambda_h(p)) - \int_I \sum_{i=0}^{r_p} X_{2i}(s) d\mu_\infty(x) \right]^2$$

  converge to as $N + k \to \infty$?

- If the asymptotic variance exists, then let it be denoted by $Var(I)$. Our next goal would be to find how does the following expression

$$\frac{1}{|W^r(N)|}\sum_{h\in E(N,k)} g\left(\frac{w_h^r\chi_I(x_p^h) - \int_I \sum_{i=0}^{r_p} X_{2i}(s)d\mu_\infty(x)}{\sqrt{Var(I)}}\right)$$

behave for any bounded, continuous, real valued function g on $\mathbb{R}$. That is, does there exist a distribution measure $\mu(t)$ such that

$$\frac{1}{|W^r(N)|}\sum_{h\in E(N,k)} g\left(\frac{w_h^r\chi_I(x_p^h) - \int_I \sum_{i=0}^{r_p} X_{2i}(s)d\mu_\infty(x)}{\sqrt{Var(I)}}\right) \mapsto \int_{\mathbb{R}} g(t)d\mu(t)?$$

Bibliography

[1] Peter J. Cho and Henry H. Kim,“Central limit theorem for Artin L-functions ”, International Journal of Number Theory, Vol 13 No. 1 (2017) 1-14.

[2] Deligne P., “La conjecture de Weil. I.”, Inst. Hautes Études Sci. Publ. Math., (43), (1974),273-307.

[3] Iwaniec H. , “Topics in classical automorphic forms”, Graduate studies in Mathematics, Vol. 17, American Mathematical Society, 1991.

[4] Kuipers L. and Niederreiter H. , “Uniform distribution of sequences”, John Wiley and Sons, 1974.

[5] Charles C.C. Li, “Kuznietsov trace formula and weighted distribution of Hecke eigenvalues”, Journal of Number Theory 104 (2004) 177-192.

[6] Montgomery H. L., “Ten lectures on the interface between analytic number theory and harmonic analysis”, CBMS Regional Conference Series in Mathematics 84, Washington, DC, Conference Board of the Mathematical Sciences, 1994.

[7] Murty M. R., Michael Dewar and Hester Graves “Problems in the Theory of Modular Forms ”,(Hindustan Book Agency).

[8] Murty M. R., “Problems in Analytic Number Theory”, Second Edition, Graduate Texts in Mathematics 206, Springer-Verlag, 2008.

[9] Murty M. R. and Sinha K., “Effective equidistribution of eigenvalues of Hecke operators”, J. Number Theory, 129 (3), (2009), 681-714.

[10] Nagoshi H., “Distribution of Hecke Eigenvalues”, Proceedings of the American Mathematical Society 134 (2006), 3097-3106.

[11] Oesterlé, “Sur la trace des opérateurs de Hecke”, Thése de 3é cycle(1977) Orsay.

[12] Prabhu N. and Sinha K., “Fluctuations in the distribution of Hecke eigenvalues about the Sato-Tate measure”, to appear in International Mathematics Research Notices.

[13] Prabhu N., “Fluctuations in the distribution of Hecke eigenvalues”, PhD thesis, Indian Institute of Science Education and Research, Pune,2017.

[14] Serre P., “Répartition asymptotique des valeurs propres de l’opérateur de Hecke T_p. J. Amer. Math. Soc., 10(1),(1997),75-102.

[15] Sinha K., “Effective Equidistribution of eigenvalues of Hecke Operators”, PhD thesis, Queen’s University at Kingston , 2006.

[16] Vaaler J. D., “Some extremal functions in Fourier analysis”, Bulletin of the American Mathematical Society , 12 (1985) 183-216