A torsion correspondence for non-compact arithmetic hyperbolic $3-$manifolds

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by

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Certificate

This is to certify that this dissertation entitled A torsion correspondence for non-compact arithmetic hyperbolic $3-$manifoldstowards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Kushtagi Sri Rama Chandraat Indian Institute of Science Education and Research under the supervision of Debargha Banerjee, Assistant Professor, Department of Mathematics, during the academic year 2017-2018.

Debargha Banerjee

Debargha Banerjee

Committee:
Debargha Banerjee
Diganta Borah

To

Mother.

Declaration

I hereby declare that the matter embodied in the report entitled A torsion correspondence for non-compact arithmetic hyperbolic 3−manifolds are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Debargha Banerjee and the same has not been submitted elsewhere for any other degree.

Kushtagi Sri Rama Chandra

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Abstract

The Cheeger-Müller theorem (formerly Ray-Singer conjecture) is one of the seminal results for closed orientable Riemannian manifolds. It implies that for a compact hyperbolic $3-$manifold, the analytic torsion and Reidemeister torsion coincide. An analogous result does not exist for non-compact hyperbolic $3-$manifolds. We explore a result that compares non-compact these torsions in arithmetic manifolds of a special kind.

Contents

Chapter 1

Introduction

We try to understand generalisations of classical modular forms, in particular, Bianchi modular forms. We follow [EGM] in order to do this and understand some interesting results emerging from such extension. For the rest of the work, our main reference is [CV]. In this chapter. we give the prerequisites for the forthcoming theory.

1.1 $3-$dimensional hyperbolic space

The upper half-space

$$\mathbb{H} : \{(x, y, r) \in \mathbb{R}^3 : r > 0\}$$

as a subspace of $\mathbb{R}^3$ models the 3-dimensional hyperbolic space. We shall often think of a point P in $\mathbb{H}$ as a quaternion

$$P = x + yi + rj = z + rj.$$

This viewpoint helps us make several observations, as shall be seen later. We equip $\mathbb{H}$ with the hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2}$$

which gives the hyperbolic Laplace-Beltrami operator as

$$\Delta = r^2\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial r^2}\right) - r\frac{\partial}{\partial r}.$$

Throughout this body of work, Laplacian shall mean the above operator.

1.1.1 Action on $\mathbb{H}$

The group $PSL_2(\mathbb{C})$ acts naturally on $\mathbb{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} P := \frac{aP+b}{cP+d} = \frac{(az+b)(\overline{cz+d}) + a\bar{c}r^2 + rj}{|cz+d|^2 + |c|^2r^2}.$$

In fact, it is the group of orientation preserving isometries on $\mathbb{H}$:

Theorem 1. *$Iso^+(\mathbb{H}) \cong PSL_2(\mathbb{C})$.*

Definition 1. *If $\gamma \in SL_2(\mathbb{C})$, $\gamma \neq \pm I$ we call γ*

$$\begin{array}{ccc} \textit{Parabolic} & \textit{if and only if} & |tr(\gamma)| = 2, tr(\gamma) \in \mathbb{R} \\ \textit{Hyperbolic} & \textit{if and only if} & |tr(\gamma)| > 2, tr(\gamma) \in \mathbb{R} \\ \textit{Elliptic} & \textit{if and only if} & |tr(\gamma)| < 2, tr(\gamma) \in \mathbb{R} \end{array}$$

and we say γ is loxodromic otherwise.

Definition 2 (cusp)**.** *A point $P \in \mathbb{P}^1(\mathbb{C})$ is called a cusp of $\Gamma \leq PSL_2(\mathbb{C})$ if Γ_P, the stabiliser subgroup of P in Γ, contains parabolic elements.*

Suppose $\Gamma < PSL_2(\mathbb{C})$ is discrete. We have

Theorem 2 (Poincaré)**.** *Γ is discontinuous if and only if it is discrete in $PSL_2(\mathbb{C})$*

Definition 3 (Kleinian group)**.** *A discrete subgroup Γ of $PSL_2(\mathbb{C})$ is called a Kleinian group.*

To understand $\Gamma\backslash\mathbb{H}$ when Γ is Kleinian, we define a fundamental domain.

1.1.2 Fundamental domain

Definition 4 (Fundamental domain)**.** *A fundamental domain of the discontinuous group $\Gamma < Iso^+(\mathbb{H})$ is a closed subset $\mathcal{F} \subset \mathbb{H}$ such that the following hold:*

1. *each $\Gamma-$orbit meets $\mathcal{F}$ at least once,*
2. *the interior of $\mathcal{F}$ meets each $\Gamma-$orbit at most once,*
3. *the Lebesgue measure of the boundary of $\mathcal{F}$ is zero.*

Every discrete $\Gamma < PSL_2(\mathbb{C})$ has a fundamental domain. One construction is given by the Dirichlet domain or the Poincaré normal polyhedron.

Definition 5 (Poincaré polyhedron)**.** *Let Γ be Kleinian. Let $Q \in \mathbb{H}$ be such that $\gamma Q \neq Q$ for every $\gamma \in \Gamma \backslash \{I\}$. Then the Poincaré normal polyhedron with center Q is defined as*

$$\mathcal{F}_Q(\Gamma) := \{P \in \mathbb{H} : d(P,Q) \leq d(\gamma P, Q)\ \forall \gamma \in \Gamma\}$$

Since Γ is countable, there always exists a $Q \in \mathbb{H}$ such that $\gamma Q \neq Q$ for every $\gamma \in \Gamma \backslash \{I\}$ (If not, there exists a surjection $\Gamma \to \mathbb{H}$, which can not happen).

Definition 6. *We say that a Kleinian group Γ has finite covolume, or that it is cofinite, if*

$$vol(\Gamma) := \int_{\mathcal{F}(\Gamma)} dv < \infty,$$

where dv is the hyperbolic volume element given by

$$dv = \frac{dxdydr}{r^3}.$$

Theorem 3. *For a hyperbolic manifold $M = \Gamma \backslash \mathbb{H}$ and a large constant T, each connected component of M_t for $t \leq T$ is isometric to one of the following*

1. *The quotient $\Gamma_1 \backslash H^T$ where $\Gamma_1 := \langle z \to z+1 \rangle$.*
2. *The quotient $\Gamma_2 \backslash H^T$ where $\Gamma_2 := \langle z \to z+1, z \mapsto z+\tau \rangle$, where $\Im\tau > 0, |\tau| \geq 1$.*

3. *The quotient $\langle\gamma\rangle\backslash U$, which is a torus, where γ is loxodromic, U is a tubular neighbourhood*

Definition 7. *A Kleinian group Γ is called geometrically finite if for some $Q \in \mathbb{H}$, $\mathcal{F}_Q(\Gamma)$ has finitely many faces.*

Theorem 4. *(Garland, Raghunathan) If Γ is a cofinite Kleinian group then it is geometrically finite*

In general a discontinuous group Γ need not be geometrically finite. Every geometrically finite group is finitely generated. However a finitely generated group need not be geometrically finite.

Theorem 5 (Kazhdan,Marghulis)**.** *Let Γ is a cofinite Kleinian group. Then, Γ is not cocompact if and only if it contains a parabolic element.*

Proposition 1. *A cofinite Kleinian group Γ has only finitely many $\Gamma-$classes of cusps.*

That is, the fundamental domain for $\Gamma\backslash\mathbb{H}$ has finitely many $\Gamma-$inequivalent cusps. Given a cofinite Kleinian group Γ, choose $A_1, \ldots, A_h \in PSL_2(\mathbb{C})$ so that

$$\eta_1 = A_1\infty, \ldots \eta_h = A_h\infty \in \mathbb{P}^1(\mathbb{C})$$

represent the $\Gamma-$classes of cusps. Also choose fundamental sets $\mathcal{P}_i$ for the action of $A_i^{-1}\Gamma_{\eta_i}A_i$ on $\mathbb{P}^1(\mathbb{C})\backslash\{\infty\} = \mathbb{C}$.

For $Y > 0$, define

$$\tilde{\mathcal{F}}_i(Y) := \{z + rj \in \mathbb{H} : z \in \mathcal{P}_i, r \geq Y\}.$$

Now let $Y_1, \ \ldots \ , Y_h \in \mathbb{R}$ be large enough so that $\mathcal{F}_i(Y_i) := A_i\tilde{\mathcal{F}}_i(Y_i)$ are contained in $A_i\mathbb{H}_{\lambda_i}$ for λ_i such that $A_iT^{\lambda_i}A_i^{-1} \in \Gamma$, where $\mathbb{H}_\lambda := \{z + rj \in \mathbb{H} : z \in \mathbb{C}, \ |r| > \lambda\}$.

Proposition 2. *With the above notations, for $i \neq j$ $A_i\mathbb{H}_{\lambda_i} \cap A_j\mathbb{H}_{\lambda_j} = \varnothing$.*

Thus $\mathcal{F}_\rangle(Y_i) \cap \mathcal{F}_j(Y_j) = \varnothing$. We call $\mathcal{F}_i(Y_i)$ the cusp end for the cusp η_i and write $\coprod \mathcal{F}_i(Y_i) =: \mathcal{F}_B$.

Theorem 6. *With above notations, there exists a compact set $\mathcal{F}_0$ in $\mathbb{H}$ such that*

$$\mathcal{F} := \mathcal{F}_0 \cup \mathcal{F}_B.$$

1.2 Bianchi modular forms

In this section we give the notion of an extension of the definition of classical modular forms.

Definition 8. *Let F be an imaginary quadratic field $\mathbb{Q}[\sqrt{-d}]$ and $\mathcal{O}_F$ be its ring of integers. A Kleinian group $PSL_2(\mathcal{O}_F)$ is called a Bianchi group.*

Definition 9. *Suppose $\mathfrak{i} \neq \mathfrak{o}$ is an ideal in $\mathcal{O}_F$.*

$$\Gamma(\mathfrak{i}) := \{\gamma \in PSL_2(\mathcal{O}_F) : \gamma \equiv 1 \mod \mathfrak{i}\}$$

is called the principal congruent subgroup of $PSL_2(\mathcal{O}_F)$ of level $\mathfrak{i}$.

A finite index subgroup of $PSL_2(\mathcal{O}_F)$ containing a principal congruent subgroup is called a congruent subgroup.

Given $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{C})$ and $P = z + rj = x + yi + rj \in \mathbb{H}$, we introduce the multiplier system

$$J(\gamma, P) := \begin{pmatrix} cz + d & -cr \\ \bar{c} & \overline{cz + d} \end{pmatrix}.$$

Given a function $f : \mathbb{H} \to \mathbb{C}^{k+1}$ and $\gamma \in PSL_2(\mathbb{C})$, we define the "slash operator"

$$(f|_k\gamma)(P) := Sym^k(J(\gamma, P)^{-1})f(\gamma P),$$

where Sym^k is the symmetric $k-$th power of the standard representation of $PSL_2(\mathbb{C})$ on $\mathbb{C}^2$.

Definition 10. *Let F be an imaginary quadratic field and $\mathcal{O}_F$ be its ring of integers. Let Γ be a congruence subgroup of the Bianchi group $PSL_2(\mathcal{O}_F)$. A Bianchi modular form for Γ with weight k is a real analytic function $f : \mathbb{H} \to C^{k+1}$ such that*

1. $f|_k\gamma = f \ \forall \gamma \in \Gamma$,
2. *f is harmonic,*
3. *f has, at worst a polynomial growth.*

Note that as there is no complex structure on $\mathbb{H}$ as the notion of holomorphicity is unavailable in this setting.

We shall see that as in the case of classical modular forms, the set $M(\Gamma, k)$ of Bianchi modular forms for Γ with weight k is a finite dimensional vector space.

For $k = 2$,

$$Sym^2 J(\gamma, P) = \frac{1}{|s|^2 + |t|^2} \begin{pmatrix} \bar{s}^2 & 2\bar{s}t & t^2 \\ -\bar{s}\bar{t} & |s|^2 - |t|^2 & st \\ \bar{t}^2 & -2s\bar{t} & s^2 \end{pmatrix},$$

where $s = cz + d$ and $t = cr$.

1.3 Eisenstein Series

The first examples in the classical case for modular forms were the Eisenstein series. In the same spirit, we define Eisenstein series for the 3−dimensional case. Before that, we define a Dirichlet series which shall be of technical convenience for us.

1.3.1 Poincaré series

Given a function $h(P)$ on $\mathbb{H}$, $(P = z + rj)$, a Poincaré series is defined as

$$f(P) = \sum_{\gamma \in \Gamma} h(\gamma P),$$

whenever the series converges absolutely. A simple case of this is when we take $h(P,Q) = \delta(P,Q)^{-1-s}$, $\delta := \cosh d$ where d is the hyperbolic distance function. δ is a point pair invariant under action of $PSL_2(\mathbb{C})$, meaning $\delta(P,Q) = \delta(\gamma P, \gamma Q)$ $\forall \gamma \in PSL_2(\mathbb{C})$. We define

$$H(P,Q;s) := \sum_{\gamma \in \Gamma} \delta(P, MQ)^{-1-s}. \tag{1.1}$$

This series will help us in the discussion of Eisenstein series that follows.

Proposition 3. *(1.1) converges absolutely and uniformly on compact subsets of* $\mathbb{H} \times \mathbb{H} \times \{s | \Re s > 1\}$.

Proof. Let $K \subset \mathbb{H} \times \mathbb{H} \times \{s | \Re s > 1\}$ be compact and $\sigma > 1$ be the minimum of real parts of s in K. For $(P,Q;s) \in K$, $\gamma \in \Gamma$

$$\delta(P,\gamma Q)^{-1-\Re s} \leq \delta(P,\gamma Q)^{-1-\sigma} \leq 4^{1+\sigma}\left(\frac{\delta(P,P_0)}{\delta(P_0,\gamma Q)}\right)^{1+\sigma} = 4^{1+\sigma}\left(\frac{\delta(P,P_0)}{\delta(\gamma^{-1}P_0,Q)}\right)^{1+\sigma}$$

$$\leq 4^{1+\sigma}\delta(P,P_0)4^{1+\sigma}\left(\frac{\delta(Q,Q_0)}{\delta(\gamma^{-1}P_0,Q_0)}\right)^{1+\sigma}$$

$$= 4^{2+2\sigma}\delta(P,P_0)\left(\frac{\delta(Q,Q_0)}{\delta(P_0,\gamma Q_0)}\right)^{1+\sigma} = 4^{2+2\sigma}\delta(P,P_0)^{1+\sigma}\delta(Q,Q_0)^{1+\sigma}\delta(P_0,\gamma Q_0)^{-1-\sigma}.$$

Thus, it suffices to show that $H(P_0, Q_0; \sigma)$ converges for some $P_0, Q_0 \in \mathbb{H}$. Suppose

$\mathcal{F}$ is a fundamental domain for Γ. Fix $Q_0 = j$ and $P_0 = P$

$$\int_{\mathcal{F}} H(P, j; \sigma) dv(P) = \int_{\mathcal{F}} \sum_{\gamma \in \Gamma} \delta(P, \gamma j)^{-1-\sigma} dv(P) = \sum_{\gamma \in \Gamma} \int_{\mathcal{F}} \delta(P, \gamma j)^{-1-\sigma} dv(P)$$
$$= \sum_{\gamma \in \Gamma} \int_{\mathcal{F}} \delta(\gamma P, j)^{-1-\sigma} dv(P)$$

$$= \int_{\cup_{\gamma \in \Gamma} \gamma \mathcal{F}} \delta(\gamma P, j)^{-1-\sigma} dv(P) = \int_{\mathbb{H}} \delta(P, j)^{-1-\sigma} dv(P).$$

Thus we have

$$\int_{\mathcal{F}} H(P, j; s) dv(P) = \int_{\mathbb{H}} \left(\frac{x^2 + y^2 + r^2 + 1}{2r} \right)^{-1-\sigma} \frac{dxdydr}{r^3} \stackrel{!}{=} \frac{2^{2+\sigma}\pi}{2\sigma} \int_0^\infty \frac{dr}{(1 + r^2)^{1+\sigma} r^{2-\sigma}}$$

As $\sigma > 1$, the right hand side is finite, implying that $H(P, j; \sigma)$ is finite for almost every $P \in \mathbb{H}$. Thus, $\exists P_0$ such that $H(P_0, j; s)$ converges. $\square$

The σ in the above proof is called the *abscissa of convergence* for Γ.

1.3.2 Eisenstein series

A point $\zeta \in \partial\mathbb{H}$ is a cusp if $A\zeta = \infty$ for some $A \in PSL_2(\mathbb{C})$ if $\exists$ a lattice $\Lambda \in \mathbb{C}$ so that

$$A\Gamma'_\zeta A^{-1} = \{T^\lambda | \lambda \in \Lambda\}.$$

Define

$$E^*_A(P, s) := \sum_{M \in A\Gamma'_\zeta A^{-1} \backslash A\Gamma A^{-1}} r(MP)^{1+s}.$$

If $M, T^\lambda M$ represent the same coset, $r(MP) = r(T^\lambda MP)$. Thus, $E^*_A(P, s)$ is independent of the choice of representatives M. If $S \in A\Gamma A^{-1}$, then writing $A\Gamma A^{-1} =: \tilde{\Gamma}$ and $A\Gamma'_\zeta A^{-1} =: \tilde{\Gamma}'_\zeta$, we have $\tilde{\Gamma} S = \tilde{\Gamma}$. Thus, with

$$\bigcup_M \tilde{\Gamma}'_\zeta MS = \tilde{\Gamma} S = \tilde{\Gamma},$$

if M runs over the said system of representatives, so does MS. Thus, so long as $E_A^*(P,s)$ converges absolutely,

$$E_A^*(P,s) = \sum_M r(MP)^{1+s} = \sum_{MS} r(MSP)^{1+s} = \sum_M r(MSP)^{1+s} = E_A^*(SP,s),$$

namely, $E_A^*(-,s)$ is $\tilde{\Gamma}-$invariant.

Consider $E_A^*(AP,s) = \sum_M r(MAP)^{1+s}$. Suppose $M = ALA^{-1}$, then with $r(MAP) = r(ALP)$

$$E_A^*(AP,s) = \sum_{M\in\tilde{\Gamma}'_\zeta\backslash\tilde{\Gamma}} r(MAP)^{1+s} = \sum_{L\in\Gamma'_\zeta\backslash\Gamma} r(ALP)^{1+s} =: E_A(P,s).$$

As $E_A^*(-,s)$ is $\tilde{\Gamma}-$invariant, $E_A(-,s)$ is $\Gamma-$invariant

Suppose $S \in PSL_2(\mathbb{C})$ and $G = S^{-1}\Gamma S$. Then, $\eta = S^{-1}\zeta$ is a cusp of G and $AS\eta = \infty$. L runs through a system of representatives for right cosets of G'_η in G if and only if $M = SLS^{-1}$ runs correspondingly for Γ'_ζ in Γ. Thus,

$$E_{AS}(P,s) = E_A(SP,s)$$

where $E_{AS}(-,s)$ is the Eisenstein series for G,η. When $S \in \Gamma$, then, we have

$$E_{AS}(P,s) = E_A(P,s)$$

Proposition 4. *Suppose $\Gamma < PSL_2(\mathbb{C})$ is a Kleinian subgroup and $\zeta = A^{-1}\infty$ be a cusp of Γ for some $A \in PSL_2(\mathbb{C})$.*

1. *If $\Gamma = \Gamma_\zeta$, the Eisenstein series $E_A(P,s)$ is a finite sum and equals a constant multiple of $r(AP)^{1+s}$. The abscissa of convergence is $-\infty$.*
2. *If $\Gamma \neq \Gamma_\zeta$, the Eisenstein series $E_A(P,s)$ converges if and only if $H(P,Q;s)$ converges for some $Q \in \mathbb{H}$ (A necessary condition is $\Re s > 0$).*

Proof. (1.) is obvious since $[\Gamma_\zeta : \Gamma'_\zeta] < \infty$. For (2.) we may, without loss of generality, consider $\zeta = \infty$, $A = I$. Let $\Lambda \subset \mathbb{C}$ be the lattice corresponding to $\Gamma'_\infty \subset \Gamma_\infty$. For $s \in \mathbb{R}$,

$$H(P, j; s) = \sum_{\gamma \in \Gamma} \delta(\gamma P, j)^{-1-s} = \sum_{\gamma \in \Gamma} \left(\frac{2r(\gamma P)}{|z(\gamma P)|^2 + r(\gamma P)^2 + 1} \right)^{1+s} \tag{1.2}$$

$$=^! \; 2^{1+s} \sum_{\gamma \in \Gamma'_\infty \backslash \Gamma} r(\gamma P)^{1+s} \sum_{\lambda \in \Lambda} (|z(\gamma P) + \lambda|^2 + r(\gamma P)^2 + 1)^{-1-s}$$

The z in the inner sum may be seen as an element z^* of the fundamental parallelogram $\mathcal{P}$ of Λ. Suppose

$$\mathcal{P} = \{\alpha_1 \omega_1 + \alpha_2 \omega_2 \big| \; 2|\alpha_k| < 1, \; k = 1, 2\},$$

with minimal $|\omega_1|$ and $|\omega_2|$. It can be seen that for $z^*(\gamma P) \in \mathcal{P}$,

$$\frac{|\lambda|}{4} \leq |z^*(\gamma P) + \lambda| \qquad \forall \; 0 \neq \lambda \in \Lambda.$$

Note that there exists a C such that for a fixed P and for any $M \in \Gamma$ $r(MP) \leq C$, $|z^*(\gamma P)| \leq C$. We may thus write

$$(2C^2 + 1)^{-1-s} \leq \sum_{\lambda \in \Lambda} (|z^*(\gamma P) + \lambda|^2 + r(\gamma P)^2 + 1)^{-1-s} \tag{1.3}$$

$$\leq 1 + \sum_{\lambda \in \Lambda} (|z^*(\gamma P) + \lambda|^2)^{-1-s} \leq 1 + \sum_{\lambda \in \Lambda} \Big(\frac{|\lambda|^2}{16}\Big)^{-1-s}$$

Substituting (1.3) in (1.2), we see that if $H(P, j; s)$ converges for $s \in \mathbb{R}$, then $s > 0$ and the Eisenstein series

$$E(P, s) := \sum_{\gamma \in \Gamma'_\infty \backslash \Gamma} r(\gamma P)^{1+s} \tag{1.4}$$

converges. Now assume (1.4) converges. Notice that for $\gamma = \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix} \in \Gamma$, $\lambda \in \Lambda$

$$\begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix} T^{\lambda} = \begin{pmatrix} \cdot & \cdot \\ c & c\lambda + d \end{pmatrix} \in \Gamma.$$

Since $\Gamma \neq \Gamma_\infty$, $\exists M = \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix} \in \Gamma$ with $c \neq 0$. With $P = z + rj$, $E(P, s)$ is a convergent majorant to

$$\sum_{\lambda \in \Lambda} r(MT^{\lambda}P)^{1+s}.$$

The above series majorizes $\sum_{0 \neq \lambda \in \Lambda} |\lambda|^{-2-2s}$. As $E(P, s)$ converges, we have $s > 0$ and substituting (1.3) in (1.2), we see that if $H(P, j; s)$ converges. Since $H(P, j; s)$ converges if and only if $H(P, Q; s)$ converges for every $Q \in \mathbb{H}$, we are done. □

Proposition 5. *Suppose* $\Gamma < PSL_2(\mathbb{C})$ *is Kleinian and* $\zeta = A^{-1}\infty$, $\eta = B^{-1}\infty$ *be cusps of* Γ *for some* $A, B \in PSL_2(\mathbb{C})$. *Also assume* $\alpha > 0, \beta > \sigma_0$; σ_0 *being the abscissa of convergence. Then*

$$r(P)^{-1-s} E_A(P, s) = \sum_{\gamma \in \Gamma'_\zeta \backslash \Gamma} \left(\frac{r(A\gamma P)}{r(P)} \right)^{1+s} \tag{1.5}$$

converges uniformly for $(P, s) \in B^{-1}\{z + r'j \in \mathbb{H} | r' \geq \alpha\} \times \{s | \Re s \geq \beta\}$. *Define*

$$\delta_{\eta,\zeta} := \begin{cases} 1 & \eta \equiv \zeta \mod \Gamma \\ 0 & \eta \not\equiv \zeta \mod \Gamma \end{cases}$$

If $\eta \equiv \zeta \mod \Gamma$ *(i.e; η and ζ are $\Gamma-$equivalent cusps) choose* $L_0 \in \Gamma$ *so that* $\zeta = L_0\eta$. *Let* $AL_0B^{-1} = \begin{pmatrix} \cdot & \cdot \\ 0 & d_0 \end{pmatrix}$. *Then*

$$E_A(B^{-1}(z + rj), s) = (\delta_{\eta,\zeta}[\Gamma_\zeta : \Gamma'_\zeta]|d_0|^{-2-2s} + o(1))r^{1+s} \tag{1.6}$$

as $r \to \infty$, *uniformly in* $\{z + rj \in \mathbb{H} | r \geq \alpha\}, s \in \mathbb{C}, \Re s \geq \beta$.

Proof. Writing $P = z + rj$, $||cP + d||^2 = |cz + d|^2 + |c|^2 r^2$

$$r^{-1-s} E_A(P, s) = \sum_{\gamma \in \Gamma'_\zeta \backslash \Gamma} (||cP + d||)^{-1-s},$$

with $A\gamma = \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix}$. Suppose $P \in B^{-1}\{z + r'j | r' \geq \alpha\}$, $Q \in \{z + r'j | r' \geq \alpha\}$ so that $P = B^{-1}Q$. Then

$$E_A(P, s) = E_A(B^{-1}Q, s) = E_{AB^{-1}}(Q, s),$$

where $E_{AB^{-1}}$ is the Eisenstein series for $B\Gamma B^{-1}$ at cusp $B\zeta$. Thus we may just prove the first part of the proposition for $\zeta = \infty$ and $B = I$.

Suppose $K \subset \mathbb{C}$ is a compact subset, $P = z + rj$ with $z \in K$, $r \geq \alpha$. Fix $P_0 = z_0 + r_0 j \in K$. For $(c, d) \in \mathbb{C}^2 \backslash \{(0, 0)\}$ put

$$(\epsilon, \delta) := \sqrt{(|c|^2 + |d|^2)^{-1}}(c, d).$$

By compactness arguments

$$\frac{||cP_0 + d||^2}{||cP + d||^2} = \frac{||\epsilon P_0 + \delta||^2}{||\epsilon P + \delta||^2} \leq \frac{|\epsilon z_0 + \delta|^2 + |\epsilon|^2 r_0^2}{|\epsilon z + \delta|^2 + |\epsilon|^2 \alpha^2} < C$$

where C is independent of $(c, d), z \in K, r \geq \alpha$. Thus (1.5) is uniformly convergent for $(P, s) \in \{z + rj \in \mathbb{H} | z \in K, r \geq \alpha\} \times \{s | \Re s \geq \beta\}$. With $\eta = \infty$ and invariance of (1.5) under Γ'_∞, K can be replaced by $\mathbb{C}$.

For the second part we consider arbitrary B, η. By the above part,

$$\sum_{\gamma \in \Gamma'_\zeta \backslash \Gamma} ||c(z + rj) + d||^{-2-2s} \to 0$$

as $r \to \infty$ for $A\gamma B^{-1} = \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix}$ with $c \neq 0$. Now we want to get all elements so that

$A\gamma B^{-1} = \begin{pmatrix} \cdot & \cdot \\ 0 & d \end{pmatrix}$. But this happens if and only if $A\gamma B^{-1}$ fixes ∞ and thus ζ and η are $\Gamma-$equivalent. Choose L_0 as in the proposition. Then, for any $\gamma \in \Gamma$, $A\gamma B^{-1}$ looks like $\begin{pmatrix} \cdot & \cdot \\ 0 & d \end{pmatrix}$. Thus, $\gamma \in \Gamma_\zeta L_0$ and we have $[\Gamma_\zeta : \Gamma'_\zeta]$ choices for $\gamma \in \Gamma'_\zeta \backslash \Gamma$. For every such γ there is an $S = A^{-1} \begin{pmatrix} u^{-1} & b \\ 0 & u \end{pmatrix} A \in \Gamma_\zeta$ so that $\gamma = SL_0$ where u is a root of unity. Thus,

$$\begin{pmatrix} \cdot & \cdot \\ 0 & d \end{pmatrix} = A\gamma B^{-1} = \begin{pmatrix} u^{-1} & b \\ 0 & u \end{pmatrix} AL_0 B^{-1} = \begin{pmatrix} \cdot & \cdot \\ 0 & ud_0 \end{pmatrix}.$$

Thus, $|d| = |d_0|$ and the result (1.6) follows. $\square$

We have the following result from the Proposition[5]:

Corollary 1. *If $\Gamma < PSL_2(\mathbb{C})$ is Kleinian with cusps, then all Eisenstein series for Γ are of polynomial growth at all cusps of Γ.*

Also, for Laplace-Beltrami operator Δ, since $\Delta r^s = -(1-s^2)r^s$, we have $\Delta E_A(P,s) = -(1-s^2)E_A(P,s)$, whenever $E_A(P,s)$ is absolutely convergent. Thus, $E_A(-,s)$ is a modular function on $\mathbb{H}$. The following gives a Fourier expansion of the Eisenstein series.

Theorem 7. *Suppose Γ is a Kleinian subgroup of $PSL_2(\mathbb{C})$. Let $\zeta = A^{-1}\infty$ and $\eta = B^{-1}\infty$ for some $A, B \in PSL_2(\mathbb{C})$ be cusps of Γ. For $P \in \mathbb{H}$, $\Re s > \sigma_0$, $\Lambda-$invariant $E_A(B^{-1}P,s)$ has the Fourier expansion*

$$\begin{aligned} E_A(B^{-1}P,s) =& (\delta_{\eta,\zeta}[\Gamma_\zeta : \Gamma'_\zeta]|d_0|^{-2-2s})r^{1+s} + \frac{\pi}{|\Lambda|s}(\sum_{\mathcal{R}} |c|^{-2-2s})r^{1-s} \\ &+ \frac{2\pi^{1+s}}{|\Lambda|\Gamma(1+s)} \sum_{0\neq\mu\in\Lambda^v} |\mu|^s \left(\sum_{\mathcal{R}} \frac{e^{2\pi i\langle\mu,d/c\rangle}}{|c|^{2+2s}} \right) \end{aligned} \tag{1.7}$$

with $\begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix} \in \mathcal{R},\ c \neq 0,$ *where* $\mathcal{R}$ *is a double coset representative system*

$$A\Gamma'_\zeta A^{-1} \backslash A\Gamma B^{-1} / B\Gamma'_\eta B^{-1}.$$

Proof. Since E_A is $\Lambda-$invariant with polynomial growth and satisfies $\Delta E_A(P,s) = -\lambda E_A(P,s)$ we have

$$E_A(B^{-1}P, s) = \sum_{\mu \in \Lambda^v} a_\mu(r,s) e^{2\pi i \langle \mu, z \rangle}$$

where with $\mathcal{P}$ being the fundamental parallelogram for Λ:

$$a_\mu(r,s) = \frac{1}{|\Lambda|} \int_{\mathcal{P}} E_A(B^{-1}P, s) e^{-2\pi i \langle \mu, z \rangle} dxdy.$$

When E_A is uniformly convergent

$$a_\mu(r,s) = \frac{1}{|\Lambda|} \sum_{\gamma \in \Gamma'_\zeta \backslash \Gamma} \int_{\mathcal{P}} r(A\gamma B^{-1}P)^{1+s} e^{-2\pi i \langle \mu, z \rangle} dxdy.$$

We reduce $A\Gamma B^{-1}$ mod $B\Gamma'_\eta B^{-1}$ from the right. Notice that by Proposition[5] we have the coefficient for r^{1+s}. We compute the rest, that is, for $\mu \neq 0$, $c \neq 0$.

Note that for $\lambda \in \Lambda$ $\begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix} T^\lambda = \begin{pmatrix} \cdot & \cdot \\ c & c\lambda + d \end{pmatrix}$. By arguments of Proposition[5] different λ give different coset representatives for $A\Gamma'_\zeta A^{-1} \backslash A\Gamma B^{-1}$. Thus we have

$$\begin{aligned} a_\mu(r,s) =& \frac{1}{|\Lambda|} \sum_{\mathcal{R}} \sum_{\lambda \in \Lambda} \int_{\mathcal{P}} \left(\frac{r}{|cz + c\lambda|^2 + |c|^2 r^2} \right)^{1+s} e^{-2\pi i \langle \mu, z \rangle} dxdy \\ =& \frac{1}{|\Lambda|} \sum_{\mathcal{R}} \int_{\mathbb{C}} \left(\frac{r}{|cz + c\lambda|^2 + |c|^2 r^2} \right)^{1+s} e^{-2\pi i \langle \mu, z \rangle} dxdy \end{aligned}$$

The integral can be rewritten as

$$\begin{aligned}
&\int_{\mathbb{C}} \left(\frac{r}{|cz + c\lambda|^2 + |c|^2 r^2} \right)^{1+s} e^{-2\pi i \langle \mu, z \rangle} dxdy \\
&= \int_{\mathbb{C}} |c|^{-2-2s} e^{2\pi i \langle \mu, d/c \rangle} \left(\frac{r}{|z|^2 + 2r^2} \right)^{1+s} e^{-2\pi i \langle \mu, z \rangle} dxdy \\
&= |c|^{-2-2s} e^{2\pi i \langle \mu, d/c \rangle} \int_{\mathbb{C}} \left(\frac{r}{|z|^2 + 2r^2} \right)^{1+s} e^{-2\pi |\mu| x} dxdy
\end{aligned}$$

Straightforward evaluations alongside standard formulae for Γ function and K_s give us (1.7). □

1.3.3 Eisenstein series for $PSL_2(\mathcal{O}_F)$

Suppose $F = \mathbb{Q}(\sqrt{d})$ (with $d < 0$, discriminant of $F < 0$). $\mathfrak{M}$ be the set of all fractional ideals in F. Write $\mathcal{O}_F = \mathfrak{o}$ For every $\mathfrak{m} \in \mathfrak{M}$, there exist $\mathfrak{a}$, $\mathfrak{b} \in \mathfrak{o}$ so that $\mathfrak{m} = \mathfrak{a}\mathfrak{b}^{-1}$. Define norm of $\mathfrak{m}$:

$$N\mathfrak{m} = \frac{N\mathfrak{a}}{N\mathfrak{b}}.$$

We know the class group for F is $\mathcal{C}_F = \frac{\mathfrak{M}}{F^\times}$. For $\mathfrak{m} \in \mathfrak{M}$, we have the corresponding equivalence class $\mathfrak{m}^\# \in \mathcal{C}_F$.

Definition 11. *For $\mathfrak{m} \in \mathfrak{M}$, $P = z + rj \in \mathbb{H}$, $s \in \mathbb{C}$ $Res > 0$; we define*

$$E_{\mathfrak{m}}(P, s) := N\mathfrak{m}^{1+s} \sum_{\substack{c,d \in F \\ <c,d>=\mathfrak{m}}} \left(\frac{r}{||cP + d||^2} \right)^{1+s} \tag{1.8}$$

$$\hat{E}_{\mathfrak{m}}(P, s) := N\mathfrak{m}^{1+s} \sum_{c,d \in \mathfrak{m}} \left(\frac{r}{||cP + d||^2} \right)^{1+s} \tag{1.9}$$

One can verify that both $E_{\mathfrak{m}}(P, s)$ and $\hat{E}_{\mathfrak{m}}(P, s)$ depend only on $m^\#$. Later we give explicit relations between $E_{\mathfrak{m}}(P, s)$ and $\hat{E}_{\mathfrak{m}}(P, s)$ and $E_{\mathfrak{m}}(P, s)$ and $E_A(P, s)$ which establish that $E_{\mathfrak{m}}(P, s)$ and $\hat{E}_{\mathfrak{m}}(P, s)$ are also automorphic functions.

Definition 12. *For* $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$ *let*

$$\zeta(\mathfrak{m}, \mathfrak{n}, s) := (\mathfrak{m}\mathfrak{n}^{-1})^s \sum_{\lambda \in \mathfrak{m}\mathfrak{n}^{-1}} N\lambda^{-s} \tag{1.10}$$

and

$$\zeta(\mathfrak{m}^{\#}, s) := \sum_{\substack{\mathfrak{a} \in \mathfrak{m}^{\#} \\ \mathfrak{a} \subset \mathfrak{o}}} N\mathfrak{a}^{-s} \tag{1.11}$$

Lemma 1. *For* $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$*,*

$$\zeta(\mathfrak{m}, \mathfrak{n}, s) = |\mathfrak{o}^{\times}| \zeta((\mathfrak{n}\mathfrak{m}^{-1})^{\#}, s) \tag{1.12}$$

Proof. We see from definition that $\zeta(\mathfrak{m}, \mathfrak{n}, s) = \zeta(\mathfrak{m}\mathfrak{n}^{-1}, \mathfrak{o}, s)$. Thus, we may prove the statement for $\mathfrak{n} = \mathfrak{o}$. Since ζ depends only on the equivalence class of $\mathfrak{m}$ in $\mathcal{C}_F$, we may also assume that $\mathfrak{m} \subset \mathfrak{o}$. Thus all that is left to show is

$$\zeta(\mathfrak{m}, \mathfrak{o}, s) = |\mathfrak{o}^{\times}| \zeta((\mathfrak{m}^{-1})^{\#}, s),$$

which is a known result. See [Lan] pp 254. □

Proposition 6. *For* $\mathfrak{m} \in \mathfrak{M}$*,* $P \in \mathbb{H}$*,* $\Re s > 1$ *we have*

$$|\mathfrak{o}^{\times}| \hat{E}_{\mathfrak{m}}(P, s) = \sum_{\mathfrak{n} \in \mathcal{C}} \zeta(\mathfrak{m}, \mathfrak{n}, s+1) E_{\mathfrak{n}}(P, s) \tag{1.13}$$

Proof. Let $\mathfrak{n}$ run through a representative system $\mathcal{V}$ of $\mathcal{C}_F$. Suppose (γ, δ) generate some $\mathfrak{n} \in \mathcal{V}$ and for $\lambda \in \mathfrak{m}\mathfrak{n}^{-1}$ consider the map

$$(\lambda, (\gamma, \delta)) \mapsto (c, d) := (\lambda\gamma, \lambda\delta) \in \mathfrak{m} \oplus \mathfrak{m} \backslash \{(0, 0)\}.$$

It is easy to see that this map is surjective and thus every (c, d) has $|\mathfrak{o}^{\times}|$ preimages. Putting this in the right hand side of (1.13) gives the assertion. □

Now we show that $E_{\mathfrak{m}}(P, s)$ agrees with $E_A(P, s)$ (upto elementary factors). For this, we introduce a notation For $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSL_2(F)$ write the $\mathfrak{o}-$modules

$$\mathfrak{u}_A := \langle \gamma, \delta \rangle \in \mathfrak{M} \tag{1.14}$$

$$\mathfrak{v}_A := \langle \alpha, \beta \rangle \in \mathfrak{M} \tag{1.15}$$

The maps $A \mapsto \mathfrak{u}_A$ and $A \mapsto \mathfrak{v}_A$ are surjective.

Theorem 8. *If $\zeta \in \partial\mathbb{H}$ is a cusp for $\Gamma = PSL_2(\mathfrak{o})$ and $A\zeta = \infty$ for some $A \in PSL_2(F)$ then*

$$E_A(P, s) = \frac{1}{2}(N\mathfrak{u}_A)^{-1-s} E_{\mathfrak{u}_A}(P, s) \tag{1.16}$$

Proof. Let $L := \{(c, d) \in F^2 | \langle c, d \rangle = \mathfrak{u}_A\}$. From the above notation we see that for each $(c, d) \in L$ there exists an $M \in SL_2(\mathfrak{o})$ so that $M \begin{pmatrix} d \\ -c \end{pmatrix} = \begin{pmatrix} \delta \\ -\gamma \end{pmatrix}$. Thus we have a well defined map

$$\phi : L \to \Gamma'_\zeta \backslash \Gamma$$

where (c,d)$\mapsto \Gamma'_\zeta M. If(c,d)\in L$, $\phi(c, d) = \Gamma'_\zeta M (M \in \Gamma)$ then $AM = \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix}$. Conversely for $\Gamma'_\zeta M \in \Gamma'_\zeta \backslash \Gamma, AM = \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix}$ we have $\phi^{-1}(\Gamma'_\zeta M) = \{\pm(c, d)\}$. Putting these in the definitions of E_A and $E_{\mathfrak{m}}$ yields the assertion. □

Fourier expansion

Explicitly writing the Fourier expansion for E_m is difficult as we do not have a simple description for the representative system. For $\hat{E}_{\mathfrak{m}}$ we can give an explicit Fourier expansion as an analogue of Theorem[7] as shown in the below theorem

Theorem 9. *Let $\mathfrak{m} \in \mathfrak{M}$ and $\eta = B^{-1}\infty$ be a cusp of Γ. Let Λ be the corresponding* *lattice for η. Then, $\hat{E}_{\mathfrak{m}}(B^{-1}P, s)$ $(P \in \mathbb{H}, \Re s > 1)$ has Fourier expansion*

$$\hat{E}_{\mathfrak{m}}(B^{-1}P, s) = N\mathfrak{u}_B^{1+s}\zeta(\mathfrak{m}, \mathfrak{u}_B, s)r^{1+s} + \frac{\pi N\mathfrak{m}^{1+s}}{|\Lambda|s} \sum_{(c,d)\in\mathcal{R}_0} |c|^{-2-2s}\ r^{1-s}$$
$$+ \frac{2\pi^{1+s}N\mathfrak{m}^{1+s}}{|\Lambda|\Gamma(1+s)} \sum_{0\neq\mu\in\Lambda^v} |\mu|^s \left(\sum_{(c,d)\in\mathcal{R}_0} \frac{e^{2\pi i\langle\mu d/c\rangle}}{|c|^{2+2s}} \right) rK_s(2\pi|\mu|r)e^{2\pi i\langle\mu,z\rangle} \tag{1.17}$$

where $\mathcal{R}_0$ is a maximal set of representatives (c, d), $c \neq 0$ of $(\mathfrak{m} \oplus \mathfrak{m})B^{-1}/B\Gamma'_\eta B^{-1}$

Looking at the system of representatives in (1.17), writing $\mathfrak{u} := \mathfrak{u}_B$ and $\mathfrak{v} := \mathfrak{v}_B$ we define

$$\mathcal{L} := (\mathfrak{m} \oplus \mathfrak{m})B^{-1} \tag{1.18}$$

and for $0 \neq c_0 \in \mathfrak{m}\mathfrak{u}^{-1}$

$$\mathcal{L}(c_0) := \{(c_0, d) \in \mathcal{L}\}. \tag{1.19}$$

Note that $\mathcal{L} \subset \mathfrak{m}\mathfrak{u} \oplus \mathfrak{m}\mathfrak{v}$.

Lemma 2. *1. $\Lambda = \mathfrak{u}^{-2}$.*

2. If $(c, d) \in \mathcal{L}$ and $\omega \in \Lambda$, then $(c, c\omega + d) \in \mathcal{L}$.

3. If $0 \neq c_0 \in \mathfrak{m}\mathfrak{u}$, then $\mathcal{L}(c_0) \neq \emptyset$.

4. $\mathfrak{m}\mathfrak{u}^{-1} \subset \mathfrak{m}\mathfrak{v}$.

Thus the group $\Lambda = \mathfrak{u}^{-2}$ acts on $\mathcal{L}$ by

$$(c, d) \mapsto (c, c\omega + d)\ ((c, d) \in \mathcal{L}, \omega \in \Lambda). \tag{1.20}$$

The following lemma gives the number of orbits for this action restricted to $\mathcal{L}(c_0)$.

Lemma 3. *If $0 \neq c_0 \in \mathfrak{m}\mathfrak{u}$, then*

$$|\mathcal{L}(c_0)/\Lambda| = \frac{Nc_0}{N\mathfrak{m}N\mathfrak{u}}. \tag{1.21}$$

Proof. Consider the homomorphism of $\mathfrak{o}-$modules

$$\phi : \mathfrak{m}\mathfrak{v} \to (\mathfrak{m}\mathfrak{u}\mathfrak{v} \oplus \mathfrak{m}\mathfrak{u}\mathfrak{v})/\mathfrak{m} \oplus \mathfrak{m}$$

$$x \mapsto (\gamma x, \delta x) + \mathfrak{m} \oplus \mathfrak{m}.$$

By Lemma[2,d], we se that

$$\ker \phi = \mathfrak{m}\mathfrak{v} \cap \mathfrak{m}\mathfrak{u}^{-1} = \mathfrak{m}\mathfrak{u}^{-1},$$

hence, $c_0\mathfrak{u}^{-2} \subset \ker \phi$. Thus ϕ induces a homomorphism

$$\tilde{\phi} : \mathfrak{m}\mathfrak{v}/c_0\mathfrak{u}^{-2} \to (\mathfrak{m}\mathfrak{u}\mathfrak{v} \oplus \mathfrak{m}\mathfrak{u}\mathfrak{v})/\mathfrak{m} \oplus \mathfrak{m}.$$

Notice that

$$\lambda_0 := (c_0\alpha, c_0\beta) + \mathfrak{m} \oplus \mathfrak{m} \in (\mathfrak{m}\mathfrak{u}\mathfrak{v} \oplus \mathfrak{m}\mathfrak{u}\mathfrak{v})/\mathfrak{m} \oplus \mathfrak{m}$$

is an element of the range of $\tilde{\phi}$ and the map

$$\mathcal{L}(c_0)/\mathfrak{u}^{-2} \to \tilde{\phi}(\lambda_0)$$

$$\{(c_0, c_0\omega + d|\omega \in \mathfrak{u}^{-2}\} \mapsto -d + c_o\mathfrak{u}^{-2}$$

is a bijection. Thus

$$|\mathcal{L}(c_0)/\mathfrak{u}^{-2}| = |\tilde{\phi}^{-1}(\lambda_0) = |\ker \tilde{\phi}| = [\mathfrak{m}\mathfrak{u}^{-1} : c_0\mathfrak{u}^{-2}] = \frac{Nc_0}{N\mathfrak{m}N\mathfrak{u}}.$$

□

Lemma 4. *In Theorem[9], we have*

$$\frac{\pi N\mathfrak{m}^{1+s}}{|\Lambda|s} \sum_{(c,d)\in\mathcal{R}_0} |c|^{-2-2s} = \frac{2\pi}{\sqrt{|d_F|}s} N\mathfrak{u}_B^{1-s}\zeta(\mathfrak{m}, \mathfrak{u}_B^{-1}, s) \tag{1.22}$$

Proof. Let $\mathcal{R}_0$ be a maximal set of representatives $(c, d) \in \mathcal{L}, c \neq 0$ for the action

(1.20). For a fixed c_0, a first entry of an element of $\mathcal{R}_0$, the number of different d so that $(c_0, d) \in \mathcal{R}_0$ is given by Lemma[3]. Hence

$$\begin{aligned} N\mathfrak{m}^{1+s} \sum_{(c,d)\in\mathcal{R}_0} |c|^{-2-2s} &= N\mathfrak{m}^{1+s} \sum_{0\neq c\in\mathfrak{mu}} \frac{|\mathcal{L}(c)/\Lambda|}{Nc^{1+s}} \\ &= \frac{N\mathfrak{m}^s}{N\mathfrak{u}} \sum_{0\neq c\in\mathfrak{mu}} Nc^{-s} \\ &= N\mathfrak{u}^{-1-s}\zeta(\mathfrak{m}, \mathfrak{u}^{-1}; s). \end{aligned} \tag{1.23}$$

With the $\mathbb{Z}-$basis of $\mathfrak{o}$ as $\{1, \frac{d_F+\sqrt{d_F}}{2}\}$, the discriminant of $\mathfrak{o}$ is $\sqrt{|d_F|}/2$, thus

$$|\Lambda| = \frac{1}{2}\sqrt{|d_F|}N\mathfrak{u}^{-2}. \tag{1.24}$$

Equations (1.23) and (1.24) yield the desired result. □

Before giving explicit computation of higher Fourier coefficients, we state a few auxiliary results.

Definition 13. *For a fixed $0 \neq c_0 \in \mathfrak{mu}$, we define the following sum in the third term of right hand side of (1.17)*

$$S(\omega^v, c_0) := \sum_{(c_0,d)\in\mathcal{R}_0} e^{2\pi i\langle \omega^v, \frac{d}{c_0}\rangle} \quad (0 \neq \omega^v \in \Lambda^v). \tag{1.25}$$

Lemma 5. *If $0 \neq \omega^v \in \Lambda^v,\ 0 \neq c_0 \in \mathfrak{mu}$, then*

$$S(\omega^v, c_0) = 0 \quad if\ \frac{\omega^v}{\bar{c_0}} \notin (\mathfrak{mu}^{-1})^v. \tag{1.26}$$

Proof. Let $(c_0, d) \in \mathcal{L}$ and $x \in \mathfrak{mu}^{-1}$. Then $(c_0, d+x)B \in \mathfrak{m}\oplus\mathfrak{m}$ since $x\mathfrak{u} \subset \mathfrak{m}$. Thus, if (c_0, d) runs over a system of representatives for $\mathcal{L}(c_0)/\Lambda$, then so does $(c_0, x+d)$ for every fixed $x \in \mathfrak{mu}^{-1}$. Hence

$$S(\omega^v, c_0) = e^{2\pi i\langle \omega^v, \frac{x}{c_0}\rangle} S(\omega^v, c_0) \quad \forall x \in \mathfrak{mu}^{-1}.$$

Thus, $S(\omega^v, c_0) = 0$ or $e^{2\pi i \langle \omega^v, \frac{x}{c_0} \rangle} = 1$. The latter is true if and only if $\frac{\omega^v}{\bar{c}_0} \in (\mathfrak{m}\mathfrak{u}^{-1})^v$. □

Lemma 6. *Suppose* $0 \neq \omega^v \in \Lambda^v$, $0 \neq c_0 \in \mathfrak{m}\mathfrak{u}$, $\frac{\omega^v}{\bar{c}_0} \in (\mathfrak{m}\mathfrak{u}^{-1})^v$ *and* $(c_0, d_0) \in (\mathfrak{m} \oplus \mathfrak{m})B^{-1}$. *Then*

$$S(\omega^v, c_0) = \frac{Nc_0}{N\mathfrak{m}N\mathfrak{u}} e^{2\pi i \langle \omega^v, \frac{d_0}{c_0} \rangle}, \tag{1.27}$$

with the exponential being a root of unity.

Proof. If $(c_0, d) \in \mathcal{R}_0$, then, $((c_0, d_0) - (c_0, d))B \in \mathfrak{m} \oplus \mathfrak{m}$, that is $d_0 - d \in \mathfrak{m}\mathfrak{u}^{-1}$. Thus, all terms in the sum (1.25) are equal and Lemma[3] implies the result. □

Lemma 7. *If* $\mathfrak{n} \in \mathfrak{M}$, *then*

$$\mathfrak{n}^v = \frac{2}{\sqrt{|d_F|}} \bar{\mathfrak{n}}^{-1}, \tag{1.28}$$

in particular,

$$\Lambda^v = \frac{2}{\sqrt{|d_F|}} \bar{\mathfrak{u}}^{-1} \tag{1.29}$$

Proof. By definition

$$\mathfrak{n}^v = \{\lambda \in K | \langle \lambda, x \rangle = \frac{1}{2}(\lambda\bar{x} + \bar{\lambda}x) = \frac{1}{2} tr(\bar{\lambda}x) \in \mathbb{Z} \forall x \in \mathfrak{n}\}$$

Thus, $\mathfrak{n}^v = 2\mathfrak{n}^\perp$, where $\mathfrak{n}^\perp$ is the compliment of $\mathfrak{n}$ with respect to trace form. It is known that $\mathfrak{n}^\perp = \mathcal{D}^{-1}\mathfrak{n}^{-1}$ (see [Lan], p. 57), where $\mathcal{D}$ is the different of K, which is $\sqrt{|d_F|}\mathfrak{o}$ and rest follows. □

The choice of B in (1.17) is arbitrary; for example $T^\lambda B$ and B give the same lattice. We now define a normalisation condition on B to bypass this technicality. Recall that $\mathfrak{u}\mathfrak{v} \supset \mathfrak{o}$ (Lemma[2](d)).

Definition 14. *A matrix* $B \in PSL_2(F)$ *is called quasi integral if*

$$\mathfrak{u}_B\mathfrak{v}_B = \mathfrak{o}.$$

With $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, it is easy to see that B is quasi integral if and only if $\alpha\gamma, \alpha\delta, \beta\delta \in \mathfrak{o}$.

Lemma 8.

1. *If $\gamma^*, \delta^* \in F$, $(\gamma^*, \delta^*) \neq (0,0)$, then there exists a quasi integral matrix $B^* \in SL_2(K)$ such that $B^* = \begin{pmatrix} \cdot & \cdot \\ \gamma^* & \delta^* \end{pmatrix}$.*
2. *For every $\eta \in \mathbb{P}^1(F)$ there exists a quasi integral matrix $B \in PSL_2(F)$ such that $B\eta = \infty$.*
3. *For every $\mathfrak{n} \in \mathfrak{M}$ there exists a quasi integral matrix $B \in PSL_2(F)$ such that $\mathfrak{n} = \mathfrak{u}_B$.*

Proof. 1. The $\mathfrak{o}-$module $\mathfrak{q} := \langle \gamma^*, \delta^* \rangle$ is not $\{0\}$ by hypothesis. Hence, $\mathfrak{q}^h = \lambda\mathfrak{o}$ for some $0 \neq \lambda \in F$, with h being the class number of F. We choose a pair of generators α_0, β_0 for $\mathfrak{q}^{h-1}$. Then there exist $a, b, c, d \in \mathfrak{o}$ such that

$$a\alpha_0\gamma^* + b\alpha_0\delta^* + c\beta_0\gamma^* + d\beta_0\delta^* = \lambda.$$

Hence, with

$$\lambda\alpha^* := b\alpha_0 + d\beta_0,\ \lambda\beta^* = -a\alpha_0 - c\beta_0,$$

$B^* = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix}$ satisfies our requirements.

2. For $\eta = \infty$ take $B = I$ and for $\eta \in F$ there exists a quasi integral matrix $B \in PSL_2(F)$of the form $B = \begin{pmatrix} \alpha & \beta \\ 1 & -\eta \end{pmatrix}$
3. Immediate from 1.

□

Definition 15. *For $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}$, $s \in \mathbb{C}$ and $\omega \in F^\times$ let*

$$\sigma_s(\mathfrak{a}, \mathfrak{b}, \omega) := N\mathfrak{a}^{-s} \sum_{\substack{\lambda \in \mathfrak{ab} \\ \omega \in \lambda\mathfrak{a}^{-1}\mathfrak{b}}} N\lambda^s \tag{1.30}$$

Remark 1. *1. The sum (1.30) is finite. It is empty unless $\omega \in \mathfrak{b}^2$.*

2. If $\mu \in F^\times$, then

$$\sigma_s(\mu\mathfrak{a}, \mathfrak{b}, \omega) = \sigma_s(\mathfrak{a}, \mathfrak{b}, \omega) \tag{1.31}$$

3. The sum (1.30) satisfies the reciprocity formula

$$|\omega|^{-s}\sigma_s(\mathfrak{a}, \mathfrak{b}, \omega) = |\omega|^s \sigma_{-s}(\mathfrak{a}^{-1}, b, \omega). \tag{1.32}$$

Theorem 10. *Suppose $\mathfrak{m} \in \mathfrak{M}$ and $\eta \in \mathbb{P}^1(F)$ is a cusp of Γ. Choose a quasi integral matrix $B \in PSL_2(F), B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ such that $\eta = B^{-1}\infty$ and let $\mathfrak{u} := \langle \gamma, \delta \rangle$. Then $\hat{E}_{\mathfrak{m}}(B^{-1}P, s)$ where $P = z + rj \in \mathbb{H}, \Re s > 1$, has the Fourier expansion*

$$\begin{aligned} \hat{E}_{\mathfrak{m}}(B^{-1}P, s) &= N\mathfrak{u}^{1+s}\zeta(\mathfrak{m}, \mathfrak{u}, 1+s)r^{1+s} + \frac{2\pi}{\sqrt{|d_F|}s} N\mathfrak{u}^{1-s}\zeta(\mathfrak{m}, \mathfrak{u}^{-1}, s)r^{1-s} \\ &+ \frac{2^{2+s}\pi^{1+s}N\mathfrak{u}}{|d_F|^{\frac{s+1}{2}}\Gamma(1+s)} \sum_{0 \neq \omega \in \mathfrak{u}^2} |\omega|^s \sigma_{-s}(\mathfrak{m}, \mathfrak{u}, \omega) r K_s\left(\frac{4\pi|\omega|r}{\sqrt{|d_F|}}\right) e^{2\pi i \langle \frac{2\bar{\omega}}{\sqrt{d_F}}, z \rangle} \end{aligned} \tag{1.33}$$

Proof. From (1.17) and Lemma[4], we get the coefficients of r^{1+s} and r^{1-s}. We compute the higher coefficients. Let $0 \neq \omega^v \in \Lambda^v, 0 \neq c \in \mathfrak{m}\mathfrak{u}$. If $\frac{\omega^v}{\bar{c}} \notin (\mathfrak{m}\mathfrak{u}^{-1})^v$, by Lemma[5], $S(\omega, c) = 0$. Assume $\frac{\omega^v}{\bar{c}} \in (\mathfrak{m}\mathfrak{u}^{-1})^v$ and $(c, d) \in \mathcal{R}_0$. Then since $d \in \mathfrak{m}\mathfrak{v} = \mathfrak{m}\mathfrak{u}^{-1}$ (As B is quasi integral), $\langle \omega^v, \frac{d}{c} \rangle = \langle \frac{\omega^v}{\bar{c}}, d \rangle \in \mathbb{Z}$. Thus, all terms in the sum (1.25) are one and we have by Lemma[6]

$$S(\omega^v, c) = \frac{Nc}{N\mathfrak{m}N\mathfrak{u}}.$$

By (1.29), we can define a bijection between $\mathfrak{u}^2$ and Λ^v by

$$\mathfrak{u}^2 \ni \omega \mapsto \omega^v := \frac{2}{\sqrt{d_F}}\bar{\omega} \in \Lambda^v.$$

Putting $\frac{2}{\sqrt{d_F}}\bar{\omega}$ in place of ω^v, we have

$$\begin{aligned}
N\mathfrak{m}^{1+s}\sum_{(c,d)\in\mathcal{R}_0}\frac{e^{2\pi i\langle\omega^v,\frac{d}{c}\rangle}}{|c|^{2+2s}} &= N\mathfrak{m}^{1+s}\sum_{0\neq c\in\mathfrak{m}\mathfrak{u}}S(\omega^v,c)Nc^{-1-s}\\
&= \frac{N\mathfrak{m}^s}{N\mathfrak{u}}\sum_{\substack{c\in\mathfrak{m}\mathfrak{u}\\ \frac{\omega^v}{\bar{c}}=(\mathfrak{m}\mathfrak{u}^{-1})^v}} Nc^{-s}\\
&= \frac{N\mathfrak{m}^s}{N\mathfrak{u}}\sum_{\substack{c\in\mathfrak{m}\mathfrak{u}\\ \omega\in c\mathfrak{m}^{-1}\mathfrak{u}}} Nc^{-s}\\
&= \frac{1}{N\mathfrak{u}}\sigma_{-s}(\mathfrak{m},\mathfrak{u},\omega).
\end{aligned} \tag{1.34}$$

Inserting this and the result of Lemma[4] into (1.17) gives (1.33). □

Meromorphic continuation

We can give a meromorphic continuation of $\hat{E}_{\mathfrak{m}}(P,s)$ to the whole $s-$plane via (1.33).

Theorem 11. *With notation as in Theorem[10] and $B = I$, the Eisenstein series $\hat{E}_{\mathfrak{m}}(P,s)$ satisfies the following functional equation*

$$\Big(\frac{2\pi}{\sqrt{|d_F|}}\Big)^{-(1+s)}\Gamma(1+s)\hat{E}_{\mathfrak{m}}(P,s) = \Big(\frac{2\pi}{\sqrt{|d_F|}}\Big)^{-(1-s)}\Gamma(1-s)\hat{E}_{\mathfrak{m}^{-1}}(P,-s) \quad (s\in\mathbb{C}), \tag{1.35}$$

and has a meromorphic continuation to the whole $s-$plane. The Eisenstein series is holomorphic everywhere except at $s = 1$, where it has a simple pole with residue

$$res_{s=1}\hat{E}_{\mathfrak{m}}(P,s) = \frac{4\pi^2}{|d_F|} \tag{1.36}$$

and

$$\hat{E}_{\mathfrak{m}}(P,-n) = 0 \quad \forall n\geq 2, \tag{1.37}$$

whereas

$$\hat{E}_{\mathfrak{m}}(P,-1) = -1. \tag{1.38}$$

Proof. Define

$$Z(\mathfrak{m},\mathfrak{u},s) := (\frac{2\pi}{\sqrt{|d_F|}})^{-s}\Gamma(s)\zeta(\mathfrak{m},\mathfrak{u},s) \tag{1.39}$$

and rewrite (1.33) as

$$(\frac{2\pi}{\sqrt{|d_F|}})^{-(1+s)}\Gamma(1+s)\hat{E}_{\mathfrak{m}}(B^{-1}P,s) = Z(\mathfrak{m},\mathfrak{u},1+s)N\mathfrak{u}^{1+s}r^{1+s} + Z(\mathfrak{m},\mathfrak{u}^{-1},s)N\mathfrak{u}^{1-s}r^{1-s}$$
$$+ 2N\mathfrak{u}\sum_{0\neq\omega\in\mathfrak{u}^2}|\omega|^s\sigma_{-s}(\mathfrak{m},\mathfrak{u},\omega)rK_s(\frac{4\pi|\omega|r}{\sqrt{|d_F|}})e^{2\pi i\langle\frac{2\bar{\omega}}{\sqrt{d_F}},z\rangle} \tag{1.40}$$

It is known that (1.39) has a meromorphic continuation to the whole $s-$plane and satisfies the functional equation

$$Z(\mathfrak{m},\mathfrak{u},1-s) = Z(\mathfrak{u},\mathfrak{m},s) \tag{1.41}$$

Also, by (1.12) we have

$$Z(\mathfrak{m}^{-1},\mathfrak{u}^{-1},s) = Z(\mathfrak{u},\mathfrak{m},s). \tag{1.42}$$

Thus, the zeroth coefficients in (1.40) satisfy (1.35). Now, since $K_s(t)$ is an even entire function of $s\in\mathbb{C}$, (1.32) implies that the infinite series in (1.40) term-wise satisfies (1.35) and (1.35) follows for the Eisenstein series.

The points of interest are $s = 0, 1$ since otherwise, $\hat{E}_{\mathfrak{m}}(P,s)$ is holomorphic. The factor of r^{1-s} in (1.33) has a simple pole at $s = 1$ while the other terms are holomorphic at $s = 1$. Thus, by (1.12) we have

$$res_{s=1}\hat{E}_{\mathfrak{m}}(P,s) = \frac{2\pi}{\sqrt{|d_F|}}|\mathfrak{o}^\times|res_{s=1}\zeta((\mathfrak{m}^{-1})^\#,s).$$

From [Lan], p. 259 we have

$$res_{s=1}\zeta((\mathfrak{m}^{-1})^\#,s) = \frac{2\pi}{|\mathfrak{o}^\times|\sqrt{|d_F|}}$$

and (1.36) follows.

At $s = 0$, since $\zeta((\mathfrak{m}^{-1})^{\#}, 0) = -\frac{1}{|\mathfrak{o}|^{\times}}$, the poles of the factors r^{1+s} and r^{1-s} cancel and thus, the Eisenstein series has no pole at $s = 0$. Now (1.37) and (1.38) are directly computed from (1.35) and (1.36). □

1.3.4 Kronecker limit formula

As the points of interest in the above theorem were $s = 0$ and $s = 1$ (as they are poles for the zeta function) and the formulae (1.37) and (1.38) do not deal with these points, we give the following theorems. The first one gives us an explicit computation of $\hat{E}_{\mathfrak{m}}(B^{-1}P, 0)$ and the second is a limit formula for $\hat{E}_{\mathfrak{m}}(B^{-1}P, s)$ as $s \to 1$.

Theorem 12. *Let the notation be as in Theorem[10] and for $\mathfrak{a} \in \mathfrak{M}$ define*

$$g(\mathfrak{a}) := (2\pi)^{-12} N\mathfrak{a}^6 |\Delta(\mathfrak{a})| = (2\pi)^{-12} N(\langle 1, \tau \rangle)^6 |\Delta(\tau)|,$$

where $\mathfrak{a} = \lambda\langle 1, \tau\rangle$ with $\lambda \in F^{\times}$ and $\tau \in \mathbb{C}, \Im\tau > 0$ and $\Delta = g_2^3 - 27g_3^2$ is the discriminant (from the theory of elliptic functions). Then

$$\hat{E}_{\mathfrak{m}}(B^{-1}P, 0) = \frac{4\pi N\mathfrak{u}}{\sqrt{|d_F|}} r \Bigg(\log(rN\mathfrak{u}) + \gamma - \log 2\pi - \log |d_F|^{\frac{1}{2}} - \frac{1}{12} \log(g(\mathfrak{m}\mathfrak{u}^{-1}) g(\mathfrak{m}^{-1}\mathfrak{u}^{-1}) \\ + \sum_{0 \neq \omega \in \mathfrak{u}^2} \sigma_0(\mathfrak{m}, \mathfrak{u}, \omega) K_0\big(\frac{4\pi|\omega| r}{\sqrt{|d_F|}} e^{2\pi i \langle \frac{2\bar{\omega}}{\sqrt{d_F}}, z\rangle} \Bigg) \tag{1.43}$$

where γ is Euler's constant.

Proof. It is known that

$$\zeta(\mathfrak{m}, \mathfrak{u}, 1+s) = \frac{2\pi}{\sqrt{|d_F|}} \Big(\frac{1}{s} + 2\gamma - \log |d_F| - \frac{1}{6} \log g(\mathfrak{m}\mathfrak{u}^{-1}) + O(s) \Big) \quad \text{as } s \to 0 \tag{1.44}$$

Thus

$$N\mathfrak{u}^{1+s} \zeta(\mathfrak{m}, \mathfrak{u}, 1+s) r^{1+s} = \frac{2\pi N\mathfrak{u}}{\sqrt{|d_F|}} r \Big(\log(rN\mathfrak{u}) + \frac{1}{s} + 2\gamma - \log |d_F| - \frac{1}{6} \log g(\mathfrak{m}\mathfrak{u}^{-1} + O(s) \Big) \quad \text{as } s \to 0.$$

Also, we have a functional equation for the zeta function from which we have

$$\frac{2\pi}{\sqrt{|d_F|}s}\zeta(\mathfrak{m},\mathfrak{u}^{-1},s) = \big(\frac{2\pi}{\sqrt{|d_F|}s}\big)^{2s}\frac{\Gamma(1-s)}{\Gamma(1+s)}\zeta(\mathfrak{u}^{-1},\mathfrak{m},1-s).$$

Putting (1.44) with $-s$, expanding $\frac{2\pi}{\sqrt{|d_F|}s}$ as an exponential and expanding the Gamma functions we have

$$\begin{aligned}\frac{2\pi}{\sqrt{|d_F|}s}\zeta(\mathfrak{m},\mathfrak{u}^{-1},s) &= \big(1+s\log\frac{4\pi^2}{|d_F|}+\dots\big)\frac{1+\gamma s+\dots}{1-\gamma s+\dots}\frac{2\pi}{\sqrt{|d_F|}}.\\ &\quad\big(-\frac{1}{s}+2\gamma-\log|d_F|-\frac{1}{6}\log g(\mathfrak{m}^{-1}\mathfrak{u}^{-1})+O(s)\big)\\ &= \frac{2\pi}{\sqrt{|d_F|}}\big(-\frac{1}{s}+2\log 2\pi-\frac{1}{6}\log g(\mathfrak{m}^{-1}\mathfrak{u}^{-1})+O(s)\big)\ as\ s\to 0\end{aligned}$$

Thus

$$\frac{2\pi}{\sqrt{|d_F|}s}N\mathfrak{u}^{1-s}\zeta(\mathfrak{m},\mathfrak{u}^{-1},s)r^{1-s} = \frac{2\pi N\mathfrak{u}}{\sqrt{|d_F|}}r\big(\log(rN\mathfrak{u})-\frac{1}{s}+2\log 2\pi-\frac{1}{6}\log g(\mathfrak{m}^{-1}\mathfrak{u}^{-1})+O(s)\big)$$
$$as\ s\to 0.$$

Putting these results in (1.33), the assertion follows. □

Theorem 13. *With notation as in Theorem[12]*

$$\begin{aligned}\lim_{s\to 1}\big(\hat{E}_{\mathfrak{m}}(B^{-1}P,s)-\frac{4\pi^2}{|d_F|}\frac{1}{s-1}\big) &= |\mathfrak{o}^\times|\zeta((\mathfrak{m}^{-1}\mathfrak{u})^{\#},2)N\mathfrak{u}^2r^2\\ &\quad+\frac{4\pi^2}{|d_F|}\Bigg(2\gamma-1-\log|d_F|-\log(rN\mathfrak{u})-\frac{1}{6}\log g(\mathfrak{m}\mathfrak{u})\\ &\quad+2N\mathfrak{u}\sum_{0\neq\omega\in\mathfrak{u}^2}|\omega|\sigma_{-1}(\mathfrak{m},\mathfrak{u},\omega)rK_1(\frac{4\pi|\omega|r}{\sqrt{|d_F|}})e^{2\pi i\langle\frac{2\bar{\omega}}{\sqrt{d_F}},z\rangle}\Bigg).\end{aligned}\qquad(1.45)$$

Proof. From proof of Theorem[12]

$$\begin{aligned}
&\frac{2\pi}{\sqrt{|d_F|}s} N\mathfrak{u}^{1-s}\zeta(\mathfrak{m},\mathfrak{u}^{-1},s)r^{1-s} - \frac{4\pi^2}{|d_F|}\frac{1}{s-1} \\
&= \frac{4\pi^2}{|d_F|}\Bigg((1-(s-1)+\dots)(1-(s-1)\log(rN\mathfrak{u})+\dots) \\
&.\left(\frac{1}{s-1} + 2\gamma - \log|d_F| - \frac{1}{6}\log g(\mathfrak{m}\mathfrak{u}) + O(s-1)\right) - \frac{1}{s-1}\Bigg) \\
&= \frac{4\pi^2}{|d_F|}\left(2\gamma - 1 - \log|d_F| - \log(rN\mathfrak{u}) - \frac{1}{6}\log g(\mathfrak{m}\mathfrak{u})\right) + O(s-1) \ as\ s \to 1.
\end{aligned}$$

The assertion follows. □

Theorem[13] is an analogue of Kronecker's first limit formula.

1.4 Noncompact hyperbolic $3-$manifolds

Suppose M is a hyperbolic 3−manifold of finite volume with cusps $\mathcal{C}_1, \dots, \mathcal{C}_h$. For a cusp $\mathcal{C}$, we have an isometry

$$\sigma : \mathcal{C}_P \equiv \{x + yi + rj \in \mathbb{H} : r \geq 1\}/\Gamma_{\mathcal{C}}.$$

σ is not unique. We choose a σ. We call a cusp $\mathcal{C}$ relevant if Γ_P acts on $x + yi$ by translation. We are only concerned with such singularities. The nonrelevant cusps are singularities from the orbifold structure (they arise for orbifold primes, which in our case are primes above 2 and 3.

We may assume that the cusps are disjoint.

We define the height of a point as a measure of how high in a cusp.

$$Ht(P) := \begin{cases} 1 & P \notin \cup\mathcal{C}_i \\ r(\sigma(P) & P \in \mathcal{C}_i. \end{cases}$$

We drop σ for notational convenience. A choice of σ is assumed. We also define

$$M_B = \amalg_i \mathbb{H}/\Gamma_i.$$

M_B models the geometry of M at its cusps. Define the height truncation M_T of M as

$$M_T = \{P \in M : r(P) \leq T\}$$

and set its boundary ∂M_T for sufficiently large T as ∂M.

1.4.1 Eigenfunctions and eigenforms

For a function f on a hyperbolic $3-$manifold M, we define its constant term f_N via

$$f_N \circ \sigma^{-1}(P) = \frac{1}{ar(\mathcal{C}_i)} \int_{(x,y)\in\mathbb{C}/\Gamma_{\mathcal{C}_i}} f \circ \sigma^{-1}(P) dxdy$$

where σ_i are isometries of cusps. The constant term f_N is a function on M_B and we say f is cuspidal if $f_N \equiv 0$.

Lemma 9. *If f is an eigenfunction of the Laplacian with eigenvalue $-\lambda^2$, then there exist constants A, b_0 such that*

$$||f(P) - f_N(P)|| \leq ||f||_{L^2(M_T)} e^{-b_0 Ht(P)}, \quad A(1+\lambda^2) \leq Ht(P) \leq T/2.$$

Denote the space of functions that are multiples of r^{1+s} on connected components of M_B by $C^\infty(s)$. The spaces of $1-$forms that are multiples of $r^{\pm s}(dx \pm idy)$ by $\Omega^{\pm}(s)$ respectively.

From the theory of Eisenstein series, we see that for $f \in C^\infty(0)$ and $t \neq 0$, there is a unique eigenfunction of the Laplacian E such that

$$E(f,s)\ f.y^s + g.y^{-s}$$

where $g = \Psi(t)f$, $\Psi(t) : C^\infty(0) \to C^\infty(0)$ being the scattering matrix.

The Maass-Selberg relations for $s, t \in i\mathbb{R}$ imply, for $f \in C^\infty(s)$ and $f' \in C^\infty(t)$

$$\begin{aligned}\langle \wedge^T E(f,s), \wedge^T E(f',t)\rangle &= \langle f, f'\rangle \frac{T^{s-t}}{s-t} + \langle \Psi(s)f, \Psi(t)f'\rangle \frac{T^{t-s}}{t-s} \\ &\quad + \langle \Psi(s)f, f'\rangle \frac{T^{-s-t}}{-s-t} + \langle f, \Psi(t)f'\rangle \frac{T^{t+s}}{t+s}.\end{aligned}$$

As $s \to t$, this reduces to

$$\begin{aligned}|| \wedge^T E(f,s)||^2 &= 2\log T\langle f, f'\rangle - \langle -\Psi(s)^{-1}\Psi'(s)f, f\rangle \\ &\quad + \langle \Psi(s)f, f\rangle \frac{T^{2s}}{2s} - \langle f, \Psi(s)f\rangle \frac{T^{-2s}}{2s},\end{aligned} \tag{1.46}$$

where $\Phi'(s) = d\Phi(s)/ds$.

A similar result holds for $1-$forms in $\Omega^+(0)$ considering

$$E(\omega, s)\ r^s\omega + r^{-\omega}\Phi^+(s)\omega$$

where $\Phi^+(s)$ is the scattering matrix on $\Omega^+(0)$.

By Maass-Selberg relations, as above we get

$$\sum_\omega ||E(\omega, s)||^2 = 2h_{rel}\log T - Tr(\Phi^+(s)^{-1}\Phi^+(s)') \tag{1.47}$$

where ω runs over an orthonormal basis for $\Omega^+(0)$. Similar result holds for $\Omega^-(0)$.

For 2 and $3-$ forms, one can apply the Hodge star to the above forms and deduce similar results.

1.5 Noncompact arithmetic manifolds

We work with arithmetic manifolds of a particular kind.

Let $\mathbb{G}$ be an algebraic group over $\mathbb{Q}$ given by

$$\mathbb{G} = Res_{F/\mathbb{Q}}GL_1(D)/\mathbb{G}_m$$

where D is a quaternion algebra over F ramified at a set S that contains all real places. Define

$$Y(K_\Sigma) := \mathbb{G}(F)\backslash\mathbb{H} \times \mathbb{G}(\mathbb{A}_f)/K_\Sigma, \tag{1.48}$$

with $K_\Sigma = \Pi_v K_v$, where under the identification of $D_v^\times$ with $GL_2(F_v)$ and $\mathbb{G}(F_v)$ with $PGL_2(F_v)$:

$$K_v = \begin{cases} PGL_2(\mathcal{O}_v) & v \notin \Sigma \\ \Gamma_0(\pi_v) & v \in \Sigma - S \\ Im(\mathcal{B}_v^\times \to \mathbb{G}(F_v)) & else \end{cases}$$

where $\mathcal{B}_v$ is a maximal order in D_v.

We denote by $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ a Borel subgroup and by $N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ its unipotent radical. $A(F)$ denotes the diagonal torus.

We define the height $Ht(x)$ of $x \in Y(K)$ as the maximal height of a lift $g_x \in \mathbb{G}(\mathbb{A})$. Define Y_B as

$$Y_B(K_\Sigma) = B(F)\backslash\mathbb{H} \times \mathbb{G}(\mathbb{A}_f)/K_\Sigma.$$

This models $Y(K_\Sigma)$ at its cusps.

Proposition 7. *If $\square$ is a finite set of finite places disjoint from Σ and $K' = K_\Sigma \cap K_\square$, then*

$$Y_B(K') \cong \{1,2\}^{|\square|} \times Y(K_\Sigma). \tag{1.49}$$

1.6 Some homology

In subsequent chapters, a lot of theory involves homology. This section gives a quick introduction to the relevant topics.

Definition 16 (simplex)**.** *The $n-$simplex, Δ^n, is the simplest geometric figure determined by a collection of $n+1$ points in Euclidean space $\mathbb{R}^n$.*

Geometrically, it can be thought of as the complete graph on $(n+1)$ vertices, which is solid in n dimensions

Definition 17 (singular simplex)**.** *Given a topological space X that can be seen as a subset of $\mathbb{R}^n$, a singular $n-$simplex in X is a continuous map $\sigma : \Delta^n \to X$.*

Definition 18 ($n-$chain)**.** *Let $C_n(X)$ be the free abelian group with the set of singular $n-$simplices of X as basis. Elements of $C_n(X)$ are called singular $n-$chains and are finite formal sums: $P_i = \sum g_i\sigma_i$, where $g_i \in \mathbb{Z}$.*

Definition 19 (boundary map)**.** *The map $\delta_n : C_n(X) \to C_{n-1}(X)$ defined as*

$$\delta(\sigma) = \sum_i (-1)^n \sigma|_{[v_0,\ldots,v_{i-1},v_{i+1}\ldots,v_n]}$$

where v_i are $0-$simplices is called the boundary map.

We usually drop the subscript of δ_n.

Notice that

$$\begin{aligned}\delta_{n-1}\delta_n =& \sum_{i<j}(-1)^i(-1)^j\sigma|_{[v_0,\ldots,v_{i-1},v_{i+1},\ldots,v_{j-1},v_{j+1},\ldots,v_n]}\\ &+\sum_{i>j}(-1)^i(-1)^{j-1}\sigma|_{[v_0,\ldots v_{j-1},v_{j+1},\ldots,v_{i-1},v_{i+1}\ldots,v_n]}\end{aligned}$$

and the right hand side is zero. Thus we have

Proposition 8. $\delta_{n-1}\delta_n = 0$

Definition 20 (singular homology)**.** *The $n-th$ homology group for a topological space X is defined as $H_n(X) = \ker(\delta_n)/Im(\delta_{n+1})$.*

If the coefficients of C_n are from an abelian ring R instead of $\mathbb{Z}$, we write $H_n(X,R)$ for the $n-$th homology group.

Definition 21 (exact sequence)**.** *A sequence of homomorphisms d_n of abelian groups C_n*

$$\ldots \longrightarrow C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \longrightarrow \ldots \longrightarrow 0$$

with $\ker(d_n) = Im(d_{n+1})$ for each n is called an exact sequence.

Definition 22 (chain complex). *A sequence of homomorphisms (called boundaries) δ_n of abelian groups C_n*

$$\dots \longrightarrow C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \longrightarrow \dots \longrightarrow 0$$

with $\delta_{n+1}\delta_n = 0$ for each n is called a chain complex. We denote it by (C, δ_C)

Homology thus measures how off a chain complex is from being exact.

Definition 23 (chain map). *A chain map between two complexes (A, δ_A) and (B, δ_B) is a collection of maps $f_n : A_n \to B_n$ such that f commutes with δ_A and δ_B as follows:*

$$\begin{tikzcd} \dots \arrow[r] & A_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\ \dots \arrow[r] & B_{n+1} \arrow[r, "\delta_{n+1}"] & B_n \arrow[r, "\delta_n"] & B_{n-1} \arrow[r] & \dots \end{tikzcd} \tag{1.50}$$

Theorem 14. *A chain map between two chain complexes induces homomorphisms between homology groups.*

Proof. From the commutative diagram (1.50) $f\delta_A = \delta_B f$. Thus f maps cycles to cycles and boundaries to boundaries and hence induces a homomorphism $f_* : H_n(A) \to H_n(B)$. □

Definition 24 (relative homology). *Given a space X and a subspace $A \subset X$, define $C_n(X, A) := C_n(X)/C_n(A)$. The natural boundary map $\delta : C_n(X, A) \to C_{n-1}(X, A)$ gives the chain complex*

$$\dots \longrightarrow C_{n+1}(X, A) \xrightarrow{\delta_{n+1}} C_n(X, A) \xrightarrow{\delta_n} C_{n-1}(X, A) \longrightarrow \dots \longrightarrow 0$$

Homology of this chain complex is called relative homology.

Relative homology fits in the exact sequence

$$\dots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \longrightarrow \dots \longrightarrow H_0(X,A) \longrightarrow 0$$

Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & C_n(A) \arrow[r, "i"] \arrow[d, "\delta"] & C_n(X) \arrow[r, "j"] \arrow[d, "\delta"] & C_n(X,A) \arrow[r] \arrow[d, "\delta"] & 0 \\ 0 \arrow[r] & C_{n-1}(A) \arrow[r, "i"] & C_{n-1}(X) \arrow[r, "j"] & C_{n-1}(X,A) \arrow[r] & 0 \end{tikzcd} \tag{1.51}$$

where i and j are inclusion and quotient maps respectively.

Write $C_n(A) = A_n$, $C_n(X) = B_n$, $C_n(X,A) = C_n$. Then, from (1.50), we see that i and j are chain maps. Thus they induce homomorphisms i_* and j_* as in Theorem[14]. As j is surjective for a cycle $c \in C_n$ we have $c = j(b)$ for some $b \in B_n$. For $\delta b \in B_{n-1}$, $j(\delta b) = \delta(j(b)) = \delta(c) = 0$ since c is a cycle. Thus $\delta b \in \ker(j)$. As the rows are exact, $\ker(j) = Im(i)$. Thus $\delta b = i(a)$ for some $a \in A_{n-1}$. Commutativity gives $i(\delta(a)) = \delta(i(a)) = \delta(\delta(b)) = 0$, which implies $\delta(a) = 0$. Thus, a is a cycle representing $[a] \in H_{n-1}(A)$. Define

$$\delta : H_n(C) \to H_{n-1}(A)$$

$$[c] \to [a].$$

Now,

- Since i is injective, a is uniquely determined by δb.
- For a c, suppose we choose b' instead of b. Then, $j(b) = j(b')$ implies that $j(b) - j(b') = 0$. That is, $j(b - b') = 0$, namely $b - b' \in \ker(j)$. So, $b - b' = i(a')$ and $\delta(b + i(a')) = \delta(b) + \delta(i(a')) = i(a) + i\delta(a') = i(a + \delta a')$ for some $a' \in A_n$. Now, a' is a cycle ($\delta a' \cong 0$) and thus $a + \delta a' \cong a$

- Choosing a c_* from the coset of c, we have $c_* = c + \delta c'$, $c' = j(b')$ for some $b' \in B_n$. So, $c + \delta c' = j(b) + \delta(j(b')) = j(b + \delta b')$. Changing c is the same as changing b to a homologous element, and hence does not affect a.

The map $H_n(C) \to H_{n-1}(A)$ is a homomorphism.

Theorem 15. *The sequence*

$$\ldots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} \ldots$$

is exact.

Proof. We have the following inclusions from left to right:

1. $Im(i_*) \subset \ker(j_*) : \; ji = 0 \; \Rightarrow \; j_* i_* = 0$.
2. $Im(j_*) \subset \ker(\delta) : \; \delta j = 0 \; \Rightarrow \; \delta j_* = 0$.
3. $Im(\delta) \subset \ker(i_*) : i_* \delta[c] = [\delta b] = 0 \; \Rightarrow i_* \delta = 0$.
4. $\ker(j_*) \subset Im(i_*)$: A cycle $b \in B_n$ represents a homology class in $\ker(j_*)$ such that $j(b) = \delta(c')$ for some $c' \in C_{n+1}$. j is surjective implies $c' = j(b')$ for some $b' \in B_{n+1}$. Now, $j(b) = \delta(c') = \delta j(b')$ implies $j(b - \delta b') = 0$ and $b - \delta b' = i(a)$. Also, $i\delta(a) = \delta i(a) = \delta(b - \delta(b')) = \delta b = 0$ since i is injective and b is a cycle. Thus, $i_*[a] = [b]$.
5. $\ker(\delta) \subset Im(j_*)$: Suppose c represents a homology class in $\ker(\delta)$. Then, with $a = \delta a'$ for some $a' \in A_n$, $\delta(b - i(a')) = \delta(b) - \delta(i(a')) = \delta(b) - i(a) = 0$. Thus, $b - i(a')$ is a cycle. Also, $j(b - i(a')) = j(b) - ji(a') = j(b) = c$. Thus $\ker(\delta) \subset Im(j_*)$
6. $\ker(i_*) \subset Im(\delta)$: Consider a cycle $a \in A_{n-1}$ such that $i(a) = \delta(b)$ for some $b \in B_n$. Now $\delta(j(b)) = j(\delta(b)) = ji(a) = 0$. Thus, jb is a cycle and $\delta[jb] = [a]$.

The above six inclusions prove the theorem. $\square$

One might wonder what happens if we reverse the arrows in a chain complex. This gives us cohomology.

Definition 25 (cochain complex). *A cochain complex is a sequence* (C, d_C)*:*

$$0 \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C1 \longrightarrow \dots \longrightarrow C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n++1}} \dots$$

where C^n *are abelian groups and the maps* d^n *are homomorphisms such that* $d^{n-1}d^n = 0$*.* C^n *are called cochains and* d^n *are called coboundary maps.*

Definition 26 (cohomology). *In the above sequence, we define* $H^n(C) := \ker(d^{n+1})/Im(d^n)$ *as the cohomology group for the cochain complex* (C, d_C)*.*

We call a chain map $f : C \to C'$ between two chain complexes a cochain map. Like chain maps for homology, cochain maps induce homomorphisms of cohomology $f^* : H^n(C) \to H^n(C')$.

Given a topological space with chain complex $(C_i, \delta_i$, fix an abelian group G, and replace each group C_i with its dual group $C^i := cC_i^* = \hom(C_i, G)$ and δ_i with $d^i : C^{i-1} \to C^i$. The $n-$th cohomology of this cochain complex (C^i, d^i) is called the $n-$th cohomology of X, $H^n(X, G)$.

Two homotopic maps from X to Y induce the same homomorphism on cohomology (just as on homology).

Given a topological space X, its subspace A a group G, dualize the short exact sequence

$$0 \longrightarrow C_n(A) \longrightarrow C_n(X) \xrightarrow{i} C_n(X, A) \xrightarrow{j} 0$$

via group G to get

$$0 \longleftarrow C^n(A, G) \xleftarrow[i^*]{} C^n(X, G) \xleftarrow[j^*]{} C^n(x, A) \longleftarrow 0 \ .$$

This sequence is exact.

Definition 27 (relative cohomology). *Define the relative coboundary map, $d : C^{n-1}(X, A, G) \to C^n(X, A, G)$ as the map induced by the restrictions of the maps $d : C^{n-1}(X, G) \to C^n(XG)$. This defines the relative cohomology groups $H^n(X, A, G)$. When $G = \mathbb{Z}$, we write $H^n(X, A)$.*

As in homology, we have the following long exact sequence for cohomology:

$$\dots \longrightarrow H^n(X, A, G) \xrightarrow{j^*_*} H^n(X, G) \xrightarrow{\delta} H^n(A, G) \xrightarrow{i^*_*} \dots \tag{1.52}$$

Definition 28 (compactly supported cohomology). *We define compactly supported cohomology as the direct limit*

$$H^n_c(X) := \varinjlim_{K \in I} H^n(X, X \backslash K)$$

where K denote compact subsets of X.

If X is compact, $H^n_c(X) = H^n(X)$ since X has a unique maximal compact subset (itself).

Compactly supported cohomology H_c is not homotopy invariant.

Consider the long exact sequences

$$\dots \to H^j_c(M, \mathbb{Z}) \to H^j(M, \mathbb{Z}) \to H^{j+1}_c(M, \mathbb{Z}) \to \dots.$$

We define cuspidal cohomology as

$$H^j_!(M, \mathbb{Z}) = Im(H^j_c(M, \mathbb{Z}) \to H^j(M, \mathbb{Z})).$$

Theorem 16 (Poincaré duality). *Let M be an orientable manifold of dimension n. Then, $H^i_c(M) \cong H_{n-i}(M)$.*

Consider the groupoid whose objects are given by $\mathbb{G}(\mathbb{A}_f)/K$ with morphisms coming from left multiplication by $\bar{\mathbb{G}}$. Define the space classifying this as $Y(K)^\wedge$.

Definition 29 (congruence homology)**.** *For an abelian group A, we define $H_{1,cong}(Y(K), A)$ as the quotient of $H_1(Y(K), A)$ under*

$$H_1(Y(K), A) \to H_1(Y(K)^\wedge, A).$$

Dualizing this, we define $H^1_{cong}(Y(K), A)$ as the image of

$$H_1(Y(K)^\wedge, A) \hookrightarrow H_1(Y(K), A).$$

We write $h_{lif}(\Sigma)$ for the order of the cokernel of

$$H_1(Y(\Sigma))_{tors} \to H_1(Y(\Sigma))_{cong}.$$

For a prime ideal $\mathfrak{q}$ with norm $N(\mathfrak{q}) > 3$, we define $h_{lif}(\Sigma; \mathfrak{q})$ as the order of cokernel of the quotient of $H_1(Y(\Sigma), \mathbb{Z})_{cong}$ by span of $[\mathfrak{q}]^i H_1(Y(\Sigma), \mathbb{Z})_{tors}$, where $[\mathfrak{q}]$ is an automorphism of $H_1(Y(\Sigma), \mathbb{Z})_{cong}$ (via left multiplication).

Chapter 2

Borel-Moore homology

We introduce the notion on Borel-Moore homology in this chapter. In the later chapters, it is a powerful tool that reduces results nicely.

There are a few equivalent definitions of Borel-Moore homology.

Definition 30 (locally finite chain). *Let $\sigma = \sum_i n_i \sigma_i$ be an element of the complex $C'_n(X)$ of a topological space X. We say σ is locally finite if for every $x \in X$, there exists a neighbourhood of x, $U \subseteq X$ such that*

$$\{\sigma_i | n_i \neq 0, \sigma_i(\Delta^n) \cap U \neq \emptyset\}$$

is finite.

Definition 31 (Borel-Moore homology).

1. *Consider the chain complex*

$$\dots \longrightarrow C'_{n+1}(X) \xrightarrow{\delta_{n+1}} C'_n(X) \xrightarrow{\delta_n} C'_{n-1}(X) \xrightarrow{\delta_{n-1}} \dots \tag{2.1}$$

   The $n-th$ Borel-Moore homology group is $H_n^{BM} := \ker \delta_n / Im \delta_{n+1}$.

2. *Suppose $\hat{X} = X \cup \{\infty\}$ is the one point compactification of X (if it exists). Then $H_n^{BM} := H_n(\hat{X}, \infty)$.*

3. *(Borel-Moore duality) Suppose M is an orientable manifold of dimension n. Then $H_i^{BM}(M,\mathbb{Z})^{n-i}(M,\mathbb{Z})$.*

It is clear by one or more of the above definitions that $H_n^{BM}(X) = H_n(X)$ if X is compact. The definitions also suggest the embedding $H_n(X) \hookrightarrow H_n^{BM}(X)$.

2.1 Homology and Borel-Moore homology of hyperbolic $3-$manifolds

Suppose M is a hyperbolic $3-$manifold. We have a long exact sequence

$$\dots H_{i+1}(M,\mathbb{Z}) \leftarrow H_i(\partial M,\mathbb{Z}) \leftarrow H_i^{BM}(M,Z) \leftarrow H_i(M,\mathbb{Z})\dots$$

via the following isomorphisms.

$$H_c^n(M,\mathbb{Z}) \equiv H_n(M,\mathbb{Z})$$

$$H^{3-n}(M,\mathbb{Z}) \equiv H_n^{BM}(M,\mathbb{Z})$$

$$\mathbb{H}^{2-n}(\partial M,\mathbb{Z}) \equiv H_n(\partial M,\mathbb{Z}).$$

Definition 32 (cuspidal homology)**.** *We define $n-th$ cuspidal homology as*

$$H_{j,!}(M,\mathbb{Z}) = Im(H_j(M,\mathbb{Z}) \to H_j^{BM}(M,\mathbb{Z})).$$

If p is an orbifold prime, as we are concerned with only relevant cusps, we localise $\mathbb{Z}$ away from p.

Suppose $Y(K)$ is a noncompact arithmetic manifold as in (1.48). Consider the long exact sequence

$$\dots \to H^{i-1}(\partial Y(K),-) \to H_c^i(Y(K),-) \to H^i(Y(K),-) \to H^i(\partial Y(K),-)\dots,$$

where $\partial Y = \partial Y_T$ for a chosen T. Thus, for homology we have corresponding exact sequence

$$\cdots \to H_2(\partial Y(K), -) \to H_1(\partial Y(K), -) \to H_1(Y(K), -) \to H_1^{BM}(Y(K), -) \ldots$$

In particular, we have cuspidal cohomology $H_{1,!}(Y(K), \mathbb{Z})$ as the quotient of $H_1(\partial Y(K), \mathbb{Z})$ by the image of $H_1(\partial Y(K), \mathbb{Z})$.

Chapter 3

Modular Symbols

The methods used here allow us to have a bound on the Eisenstein torsion. These are not necessary for main discussion of the report but are included for the interesting segues into several other topics they lead to.

In this chapter, we work with an arbitrary open compact subgroup $K \subset \mathbb{G}(\mathbb{A}_f)$. Let $\bar{\mathbb{Z}}$ denote the ring of algebraic integers.

Theorem 17. *Let $s \in H^1(\partial Y(K), \bar{\mathbb{Q}})$ be a Hecke eigenclass that lies in the image of $H^1(Y(K), \bar{\mathbb{Q}})$. If s is integral, namely, it lies in the image of $H^1(\partial Y(K), \bar{\mathbb{Z}})$, then s is in the image of a Hecke eigenclass $\tilde{s} \in H^1(Y(K), \bar{\mathbb{Z}}[1/e])$.*

We define e later.

3.0.1 Defining e

For a Grossencharacter $\chi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$, let

$$I(\chi) := \{f : \mathbb{G}(\mathbb{A}) \to \mathbb{C} \mid f(bg) = \chi(b)||b||_{\mathbb{A}}^2 f(g) \ for \ b \in B(\mathbb{A})\}.$$

If α is algebraic, define $num(\alpha)$ as the product of primes dividing the numerator of α. Let S be the finite set of places where K is not maximal, that is, the set of places

v such that $PGL_2(\mathcal{O}_v) \not\subset K$. Set $e_1 = 30h_F \prod_{v\in S} q_v(q_v - 1)$. Now e is defined as

$$e = e_1 \prod_{\chi} num(L^{alg,S}(1, \chi^2)),$$

where $L^{alg,S}$ is the algebraic part of $L-$function omitting places in S and χ is a Grossencharacter in $X(K) := \{\chi | I(\chi)^K \neq 0\}$.

3.1 Modular Symbols as Borel-Moore homology

Let $\alpha,\ \beta \in \mathbb{P}^1(F)$ and $g \in \mathbb{G}(\mathbb{A}_f)$. The geodesic from α to β (with the points seen as elements of $\mathbb{P}^1(\mathbb{C})$ translated by g gives a class in $H_1^{BM}(Y(K))$, denoted by $\langle \alpha, \beta; g\rangle$.

It is easy to see that modular symbols satisfy

$$\langle \alpha, \beta; g\rangle + \langle \beta, \gamma; g\rangle + \langle \gamma, \alpha; g\rangle = 0.$$

We say the triple $\langle \alpha, \beta; g\rangle$ is admissible if there exists an $\epsilon \in \mathbb{G}(F)$ such that $\epsilon\alpha = \beta$, $\epsilon g K = gK$. The image of $\langle \alpha, \beta; K\rangle$ under $H_1^{BM}(Y(K)) \to H_0(\partial Y(K))$. Thus, $\langle \alpha, \beta; K\rangle$ can be lifted to a class $[\alpha, \beta; K]$ in $H_1(Y(K)), \mathbb{Z})$.

Lemma 10. *The classes $[\alpha, \beta; K]$ for admissible triples along with $H_1(\partial Y(K), \mathbb{Z})$ generate $H_1(Y(K), \mathbb{Z})$*

Proof. For $\Gamma = \mathbb{G}(F) \cup gKg^{-1}$, the connected component of $Y(K)$ containing $1 \times g$ is isomorphic to $M := \Gamma\backslash\mathbb{H}$. It is sufficient to show the lemma for M.

Fix $z_0 \in \mathbb{H}$. Then $H_1(M, \mathbb{Z})$ is generated by the geodesic between z_0 and γz_0, projected onto M.

Let C be a horoball around ∞ and $D = \gamma C$. Let $\mathcal{G}$ be the geodesic between ∞ and $\gamma\infty$. Let $z_0 \in C$ such that $\gamma z_0 \in D$. Let $w_0 \in \mathcal{G} \cup C$ and $w_1 \in \mathcal{G} \cup D$.

Let P_1 be a path from z_0 to w_0 entirely in C. Let P_2 be the geodesic from w_0 to w_1 along $\mathcal{G}$. Let P_3 be a path from w_1 to γz_0 entirely in D. The path $P_1 + P_2 + P_3$ gives a path between $z_0$and γz_0, whose projection to M represents $[\gamma] \in H_1(M, \mathbb{Z}) \equiv \Gamma^{ab}$.

But P_1, P_3 lie entirely in cusps of M and P_2 differs from $\mathcal{G}$ by segments in these cusps. Thus, both $P_1 + P_2 + P_3$ and $\langle\infty, \gamma\infty; g\rangle$ represent the same BM-homology class.

Now, admissible triples generate a subgroup $L < H_1(M, \mathbb{Z})$ so that

$$Im(L \to H_1^{BM}(M, Z)) \supset Im(H_1(M, \mathbb{Z}) \to H_1^{BM}(M, Z)).$$

That is, $H_1(M, \mathbb{Z})$ is generated by L along with $H_1(\partial M, \mathbb{Z})$. $\square$

Thus modular symbols generate the Borel-Moore homology for compact manifolds.

3.2 Denominator avoidance

We define "denominator of a modular symbol", a notion that corresponds to bad primes in Eisenstein integral over it.

Before that we introduce the notion of Bruhat-Tits buildings.

$G_p := SL_2(\mathbb{Q}_p)$, for prime p.

Definition 33 (Bruhat-Tits building)**.** *The simplicial complex X_p, called the Bruhat-Tits building of G_p, is such that:*

(1) there is a continuous action of G_p on X_p, by simplicial automorphisms and this action is proper,

(2) X_p is contractible,

(3) X_p is finite-dimensional: $\dim X_p = n - 1 = rank_{\mathbb{Q}_p} G_p$

(4) X_p is locally finite.

We say a finite place v is in the denominator of $\langle\alpha, \beta; g\rangle$ if the geodesic between $\alpha_v, \beta_v \in \mathbb{P}^1(F_v)$ inside the Bruhat-Tits building of $\mathbb{G}(F_v)$ does not pass through $gPGL_2(\mathcal{O}_v)$.

For $\gamma \in \mathbb{G}(F)$, v is in the denominator of $\alpha_v, \beta_v \in \mathbb{P}^1(F_v)$ if and only if it is in the denominator of $\gamma\alpha_v, \beta_v \in \mathbb{P}^1(F_v)$. Thus, "denominator" is invariant under $\mathbb{G}(F)$.

Now, since $\mathbb{G}(F)$ acts 2−transitively on $\mathbb{P}^1(F)$, we have $\gamma \in \mathbb{G}(F)$ such that $\gamma(\alpha, \beta) = (0, \infty)$. Thus, a modular form is equivalent to $\langle 0, \infty; g\rangle$ for suitable g.

It is easy to see that v is not in the denominator if and only if $g \in \mathbb{A}(F_v).PGL_2(\mathcal{O}_v)$.

Theorem 18. *Suppose $p \nmid e$. The Eisenstein series, integrated over modular forms falls in, $\bar{\mathbb{Z}}_p$, the set of all the algebraic numbers that are integral at all primes above p.*

The following lemma helps in establishing Theorem[18].

Lemma 11. *For prime $p > 5$, $\langle \alpha, \beta; g\rangle$ can be written as a sum of modular symbols with denominators not containing any place v such that p does not divide q_v or $q_v - 1$ That is, for any integer q, $p \nmid q_v$ or $p \nmid q_v - 1$, at all places v.*

Chapter 4

Analytic torsion and Reidemeister torsion

In this chapter, we give a result comparing analytic and Reidemeister torsions for noncompact arithmetic manifolds.

4.1 Analytic torsion

4.1.1 Regularized trace

For a noncompact manifold M and the Laplacian operator Δ, $e^{-t\Delta}$ and its analogues for $i-$forms on M are generally not trace class. but there is a fairly natural way of regularizing trace. in fact we know that its "regularization"

$$e^{-\Delta t} - e^{-\tilde{\Delta} t}\mathfrak{p}$$

is of trace class where $\tilde{\Delta}$ is the essentially self adjoint and

$$e^{-\tilde{\Delta} t}\mathfrak{p}[f] = \int_{M_B} K(., P, t) f(p) dP$$

where K is the heat kernel. $\mathfrak{p}[f](r) = \frac{1}{\mathcal{P}} \int_{\mathcal{P}} f(x, y, r)dxdy$, $\mathcal{P}$ being a fundamental parallelogram.

The regularized trace $tr^*(e^{-t\Delta})$ is characterized by the constant term of the unique linear function of $\log T$ such that

$$\int_M \wedge^T K(x, x; t)dx \sim k_0 \log T + tr^*(e^{-t\Delta}) \tag{4.1}$$

where K is the integral kernel of $e^{-t\Delta}$. (The notation $A \sim B$ means $A(T) - B(T) \to 0$ as $T \to \infty$).

Then, an application of Selberg trace formula gives

$$tr^*(e^{-t\Delta}) \sim at^{-3/2} + bt^{-1/2} + ct^{-1/2} \log t + d + O(t^{1/2}).$$

We define the regularized determinant as

$$\overset{*}{\det}(\Delta) := \exp\left(\frac{d}{ds}\bigg|_{s=0} \left(\Gamma^{-1}(s) \int_0^\infty (\lim_{t\to\infty} tr^*(e^{-t\Delta}) - tr^*(e^{-t\Delta}))t^s \frac{dt}{t}\right)\right).$$

Definition 34 (analytic torsion)**.** *The regularized analytic torsion is defined as*

$$\tau_{an}(M) = \exp(\frac{1}{2} \sum_j (-1)^{j+1} j \log \overset{*}{\det}(\Delta_j)),$$

with Δ_j being the regularized determinants on the $j-$forms.

4.1.2 Comparing regularized traces

Consider two hyperbolic $3-$manifolds M and M'. Then, one can compute the difference between their regularized trace.Suppose $\mathcal{C}$ and $\mathcal{C}'$ are cusps of M, M' respectively such there is an isometry $\sigma : \mathcal{C} \to \mathcal{C}'$. Then

$$\begin{aligned} tr^*(e^{-t\Delta_M}) - tr^*(e^{-t\Delta_{M'}}) = & \int_{M-\mathcal{C}} K(x,x;t) - \int_{M-\mathcal{C}'} K'(x',x';t) \\ & + \int_{\mathcal{C}} K(x,x;t) - K'(\sigma(x),\sigma(x);t). \end{aligned} \tag{4.2}$$

This is immediate if we write

$$tr^*(e^{-t\Delta}) = \lim_{T\to\infty} \int_{M-\mathcal{C}} K(x,x;t) + \int_{\mathcal{C}_T} K(x,x;t)) - k_0(t)\log T$$

where $C_T = M_T \cup \mathcal{C}$ and $k_0(t)$ is as in (4.1). It is clear that (4.2) leads to a comparision of analytic torsions of manifolds M, M'.

4.2 Reidemeister torsion

Reidemeister torsion was the first 3-manifold invariant able to distinguish between manifolds which are homotopy equivalent but not homeomorphic. The Reidemeister torsion of an exact complex (homology) generalizes the volume of a linear transform.

4.2.1 Regulator

We define a particular subgroup $\mathcal{H}$ of harmonic forms and via homomorphism between homology and these forms we define regulators.

Let M be a hyperbolic $3-$manifold of finite volume, with a choice of height Ht. We define inner product on forms on a particular space $\mathcal{H}^i$ of harmonic forms (i=0,1,2) so that

$$\mathcal{H}^i \to H^i(M,\mathbb{C}) \tag{4.3}$$

is an isomorphism.

- $\mathcal{H}^0$ contains constant functions.
- $\mathcal{H}^1$ contains cuspidal harmonic $1-$forms alongside $Eis(\omega)$ for $\omega \in \Omega^+(0)$.

- $\mathcal{H}^2$ contains cuspidal harmonic 2−forms alongside forms of type $*E(f,1)$ ($*$ being the Hodge star) for $f \in C^\infty(0)$ is such that f lies in the kernel of residue of $\Psi(s)$ at $s = 1$.

It is easy to see (direct computation) that (4.3) is an isomorphism. For $i = 1, 2$, we have

$$\mathcal{H} = \Omega^i_{cusp}(M) \oplus \Omega^i_{Eis}(M),$$

that is, H^i decomposes into space of cuspidal j−forms $\Omega^i_{cusp}(M)(\omega_N = 0)$ and its orthogonal compliment under standard L^2 inner product.

$\Omega^i_{cusp}(M)$ maps to $H^i_!(M)$ isomorphically under (4.3).

We define an inner product via the norm

$$||\omega||^2 = \lim_{T\to\infty} \int_{M_T} \langle \omega, \omega \rangle / \log T$$

on $\Omega^1_{Eis}(M)$, where $\langle ., . \rangle$ comes from Riemannian structure. We define an inner product on $\mathcal{H}^1$ as direct sum of the two inner products. Similarly, we define an inner product on $\mathcal{H}^2$, using the following norm on $\Omega^2_{Eis}(M)$:

$$||\omega||^2 = \lim_{T\to\infty} \int_{M_T} \langle \omega, \omega \rangle / T^2.$$

(The choice of norms is more or less arbitrary for $\Omega^j_{Eis}(M)$).

Definition 35 (regulator)**.** *We define the regulator of a homology $H_{i,?}$ of a hyperbolic 3−manifold M (? marks BM−homology and cuspidal homology) as*

$$reg(H_{i,?}(M)) = \left| \det \int_{\gamma_i} \omega_j \right|$$

where γ_i and ω_j are chosen as follows:

For $i = 0$ or 1, $\gamma_i \in H_i(M, \mathbb{Z})$ projects to a basis of $H_i(M, \mathbb{Z})_{tf}$. ω_j form an orthonormal basis for $\mathcal{H}^i$.

For $i = 2$, $\gamma_i \in H_i(M, \mathbb{Z})$ projects to a basis of the torsion free quotient of

$H_{i,!}(M, \mathbb{Z})$. ω_j *form an orthonormal basis for* $\Omega^2_{cusp}(M)$.

For $i = 3$, $\gamma_i \in H_i(M, \mathbb{Z})$ *projects to a basis of the torsion free quotient of* $H_i^{BM}(M, \mathbb{Z})$. ω_j *form an orthonormal basis for the space of harmonic* $3-$*forms (which are precisely multiples of the volume form on each component)*

We define the regulator of M *as*

$$reg(M) = \frac{reg(H_1)reg(H_3^{BM})}{reg(H_0)reg(H_{2,!})}$$

Definition 36 (Reidemeister torsion)**.** *We define Reidemeister torsion as*

$$\tau_R := |H_1(M, \mathbb{Z})_{tors}|^{-1} reg(M).$$

We now give a relation between the regulators of $H_2(M)$ and $H_{2,!}(M)$. Consider the sequence

$$H_3^{BM}(M, \mathbb{Z}) \to H_2(\partial M, \mathbb{Z}) \to H_2(M, \mathbb{Z}) \to H_{2,!}(M, \mathbb{Z}).$$

All these groups are torsion free (away from orbifold singularities). Suppose there are no orbifold singularities. Let $\delta_1, \ldots, \delta_h$ be the generators of

$$H_2(\partial M, \mathbb{Z})/Im(H_3^{BM} \to H_2(\partial M, \mathbb{Z}))$$

and fix $\gamma_1, \ldots, \gamma_r \in H_2(M, \mathbb{Z})$ such that their images span $H_{2,!}$. Then,

$$H_2(M, \mathbb{Z}) = \bigoplus_j \mathbb{Z}\gamma_j \oplus \bigoplus_i \mathbb{Z}\delta_i.$$

Now, fix orthonormal bases $\omega_1, \ldots, \omega_r$ for $\Omega^2(M)_{cusp}$ and $\eta_1, \ldots, \eta_h$ for $\Omega^2(M)_{Eis}$.

Notice that $\int_{\delta_i} \omega_j = 0$ since cuspidal forms vanish at cusps. We thus have

$$\begin{aligned} reg(H_2) &= |\det(\int_{\gamma_i} \omega_j). \det(\int_{\delta_i} \eta_j)| \\ &= reg(H_{2,!}).|\det(\int_{\delta_i} \eta_j)|. \end{aligned}$$

4.2.2 Comparision of regulators

Comparing regulators of two manifolds leads to a comparision of Reidemeister torsions of the manifolds. We have the following result which helps in the proof of the main theorem.

Proposition 9. *Suppose $\mathfrak{q}$ is a prime in the level set Σ. Also suppose that*

$$H_1(Y(\Sigma - \mathfrak{q}, \mathbb{C}) \equiv H_1(Y(\Sigma, \mathbb{C}).$$

Then away from orbifold singularities:

$$\frac{reg(H_1(Y(\Sigma - \mathfrak{q}) \times \{1,2\})}{reg(H_1(Y(\Sigma))} = \frac{\sqrt{D}}{|h_{lif}(\Sigma - \mathfrak{q}; \mathfrak{q})|}$$

$$\frac{reg(H_{2,!}(Y(\Sigma - \mathfrak{q}) \times \{1,2\})}{reg(H_{2,!}(Y(\Sigma))} = \frac{|h_{lif}(\Sigma - \mathfrak{q}; \mathfrak{q})|}{\sqrt{D}}$$

$$\frac{reg(H_0(Y(\Sigma - \mathfrak{q}) \times \{1,2\})}{reg(H_0(Y(\Sigma))} = \frac{\sqrt{vol(Y(\Sigma)}}{vol(Y(\Sigma - \mathfrak{q}))}$$

$$\frac{reg(H_3^{BM}(Y(\Sigma - \mathfrak{q}) \times \{1,2\})}{reg(H_3^{BM}(Y(\Sigma))} = \frac{vol(Y(\Sigma - \mathfrak{q})}{\sqrt{vol(Y(\Sigma))}},$$

where D is $\det(T_{\mathfrak{q}}^2 - (1 + N(\mathfrak{q}))^2|_{H_1(Y(\Sigma - \mathfrak{q}), \mathbb{C})})$, $T_{\mathfrak{q}}$ being the Hecke operator.

4.3 Main theorem

The following result was conjectured by Ray and Singer [RS] and was proved by Cheeger [Ch1] and Müller [M].

Theorem 19. *For a closed orientable Riemannian manifold M*

$$\tau_{an}(M) = \tau_R(M).$$

This is the Cheeger-Müller theorem. The main theorem of this report is a result that considers

$$\alpha(M) = \frac{\tau_{an}(M)}{\tau_R(M)},$$

(which is 1 if M is compact) for a noncompact hyperbolic $3-$manifold M and compares this quantity for different manifolds arising from level raising.

For level sets Σ, $\Sigma \cup \{\mathfrak{p}\}$, $\Sigma \cup \{\mathfrak{q}\}$ and $\Sigma \cup \{\mathfrak{p}, \mathfrak{q}\}$ set

$$Y = Y(K_\Sigma) \times \{1,2\}^2, Y_\mathfrak{q} = Y(K_\Sigma \cap K_{\{\mathfrak{q}\}}) \times \{1,2\};$$

$$Y_\mathfrak{q} = Y(K_\Sigma \cap K_{\{\mathfrak{q}\}}) \times \{1,2\}, Y_{\mathfrak{pq}} = Y(K_\Sigma \cap K_{\{\mathfrak{p},\mathfrak{q}\}}).$$

Given a choice of height functions on $Y(K_{\mathfrak{pq}})$ and $Y(K_\mathfrak{q})$, since we have isometries

$$Y(K_\mathfrak{q})_B \equiv Y(K)_B \times \{1,2\}$$

$$Y(K_{\mathfrak{pq}})_B \equiv Y(K_\mathfrak{p})_B \times \{1,2\},$$

fix height functions on Y and $Y_\mathfrak{p}$ so that they preserve these isometries at cusps.

Theorem 20 (main theorem)**.** *With the above notations and height functions, upto orbifold primes*

$$\alpha(Y_{\mathfrak{pq}})\alpha(Y) = \alpha(Y_\mathfrak{p})\alpha(Y_\mathfrak{q})$$

In the following section we list a few results that enable us in proving Theorem[20]

4.3.1 Proof of main theorem

Theorem 21 (truncation invariance)**.** *Suppose M and M' are two hyperbolic manifolds (of form Y(K)) such that M_B and M'_B are isometric. Then*

$$\lim_{T\to\infty} \log\alpha(M) + \log\alpha(M') - \log\alpha(M_B) - \log\alpha(M'_B) = 0$$

Suppose M and M' are true manifolds (they have no orbifold singularities). With some modifications, the following lines (due to Cheeger [Ch2]) hold true for the orbifolds.

$\alpha(M)$ depends only on germ of metric on M near the boundary. As the metric germs of Y and $Y_{\mathfrak{q}}$ are the same and so are the metric germs of $Y_{\mathfrak{p}}$ and $Y_{\mathfrak{pq}}$, we have $\alpha(Y_T) = \alpha(Y_{\mathfrak{q},T})$ and $\alpha(Y_{\mathfrak{pq}}, T) = \alpha(Y_{\mathfrak{p}}, T)$.

Now Theorem[21] implies, that by taking limits we get Theorem[20].

Theorem 22. *Put*

$$f(s) = \det(Id - Y^{-4s}\Phi^-(-s)\Phi^+(s)).$$

Let $0 \le a_1 \le a_2 \le \dots$ be the non-negative real roots of $t \to f(it)$. Set $\bar\lambda_j = a_j^2$. Let $\mathcal{E}$ be the set of eigenvalues a cuspidal co-closed $1-$form can take. Consider the set $\{t \in \mathbb{R}^+ | t^2 \in \mathcal{E}\}$ with multiplicity. Let $\mathcal{E}_T$ be the set of eigenvalues of Laplacian on co-closed $1-$ forms on M_T.

Set $T_{max} := (\log T)^{100}$. There exists $a > 0$ such that, with $\delta = e^{-aT}$, for sufficiently large T

1. *If $b = \dim H^1(M, \mathbb{C})$, then $\lambda_1 = \dots = \lambda_b = 0$ and $\lambda_{b+1} > 0$. The same holds for $\bar\lambda_i$.*
2. *For any j, with $\sqrt{|\lambda_j|} \le T_{max}$ we have $|\lambda_j - \bar\lambda_j| \le \delta$.*
3. *Assertions $(1), (2)$ hold for eigenvalues of Laplacian on (co-closed) $0-$forms replacing $f(s)$ by*
$$g(s) = \det(Id - \frac{1-s}{1+s}Y^{-2s}\Psi^s),$$

taking only strictly positive roots of $t \to g(it)$, $\bar{\lambda}_j = 1 - a_j^2$ *and replacing* b *by* $\dim H^0(M, \mathbb{C})$.

4. *Assertions* (1), (2) *hold for eigenvalues of Laplacian on (co-closed)* 2*−forms (equivalently* 1*−forms) replacing* $f(s)$ *by*

$$g'(s) = \det(Id + T^{-2s}\Psi(s)),$$

replacing b *by* $\dim H^2(M, \mathbb{C})$ *(equivalently* $\dim H^1(M, \mathbb{C})$*) and taking* $\bar{\lambda}_j$ *as follows:*

Let $0 < u_1 \leq u_1 \leq \cdots \leq u_h < 1$ *be the roots of* $g'(t)$ *(with multiplicity) for* $t \in (0, 1]$, *with parameters of cusp forms*

$$\{t \in [0, 1] | 1 - t^2 \ is\ an\ eigenvalue\ of\ a\ cuspidal\ 3 - form\ on\ M\}$$

and let $a_j = iu_j$ *for* $1 \leq j \leq h$. *Let* $0 < a_{h+1} \leq a_{h+2} \leq \ldots$ *be the positive real roots of* $t \to g'(it)$ *that are parameters of cusp forms. Put* $\bar{\lambda}_j = 1 + a_j^2$.

Write $A(s) = \Phi^-(-s)\Phi^+(s) : \Omega^+(s) :\to \Omega^+(s)$. We have the following result regarding the solutions of the form $s = it$ $(t \in \mathbb{R})$ to

$$f(it) := \det(1 - T^{-4it}A(it)) = 0 \tag{4.4}$$

Let $\nu_j : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be such that $e^{i\nu_j(t)}$ is an eigenvalue for $A(it)$ for $1 \leq j \leq h$.

Lemma 12. *There exists* T_0 *such that for* $T \geq T_0$*:*

1. *The number of solutions to (4.4) in the interval* $0 < t \leq B$ *is* $\left[\frac{4Blog(T) - \nu_j(T)}{2\pi}\right]$. *(*$[\cdot]$ *is the greatest integer function.)*
2. *If* $|f(it)| < \epsilon$ *and* $|t| \leq T_{max}$, *there exists* $t' \in \mathbb{R}$ *with* $f(it') = 0$ *and* $|t' - t| \ll \epsilon^{1/h_{rel}}$.
3. *Suppose* $t_0 \in \mathbb{R}$ *with* $|t_0| \leq T_{max}$. *Let* $[x, y]$ *be the positive definite inner product*

on $\Omega^+(0)$ defined as

$$[x, y]_{it_0} := \langle 4 \log T - \frac{A'(it_0)}{A(it_0)}, y \rangle$$

where $A'(s) = dA(s)/ds$. Let $t_1 \neq t_2$ be such that $|t_j - t_0| < \epsilon$ for $j = 1, 2$ and v_1, v_2 be such that $T^{-4it_j} A(it_j) v_j = v_j$ for $j = 1, 2$. Then,

$$[v_j, w_j]_{it_0} = O(\epsilon ||v_j|| ||w_j|| (\log T)^2)).$$

4. *Suppose $|t| \leq T_{max}$, and v with $||v|| = 1$ are such that $||T^{-4it} A(it) v - v|| < \epsilon$. Then there exist pairs $(t_j \in \mathbb{R}, w_j \Omega^+(0))$ for $1 \leq j \leq m$ for some $m \leq h_{rel}$ and an absolute constant M such that*
   - $T^{-4it_j} A(it_j) w_j = w_j$;
   - $||t - t_i|| \ll \epsilon^{1/M}$;
   - $||v - \sum w_j|| \leq T_{max} \epsilon^{1/M}$;
   - $||w_j|| \leq T_{max}$

It is easy to see that $reg(H_0(M_T))$ approaches $reg(H_0(M))$ as $T \to \infty$. We also have

Proposition 10.

$$reg(H_0(M_T)) \sim reg(H_0(M)),$$

$$reg(H_1(M_T)) \sim reg(H_1(M))(\log T)^{-h_{rel}},$$

$$reg(H_2(M_T)) \sim reg(H_2(M))(\log T)^{-2h},$$

$$reg(H_3(M_T)) \sim reg(H_3^{BM}(M)) \prod_N \frac{1}{\sqrt{vol(N)}},$$

where $A \sim B$ means that the ratio A/B approaches 1 as $T \to \infty$ and N runs over connected components of M.

Taking suitable ratios in Proposition[10] for manifolds M and M' yields

$$\frac{reg(M).reg(M'_T)}{reg(M_T).reg(M')} \sim \left(\frac{\det'(2R'/T^2))}{\det'(2R/T^2)}\right)^{1/2} \tag{4.5}$$

where R is the residue of Ψ at $s=1$ and $\det'(A)$ is the product of nonzero eigenvalues of A.

4.3.2 Proof of Theorem[21]

Fix two truncation parameters T, T' $(0 < T' < T)$. Let $K(x,y;t)$ be the heat kernel of $j-$forms on M and $k_t(x) := K(x,x;t)$. Similarly define $K_T, k_{t,T}$ for M_T. Set $k_\infty(x) = \lim_{t\to\infty} k_t(x)$. Define $k_{\infty,T}$ as for $k_{t,T}$. Similarly define K', k', k'_∞ for M' and $K'_T, k'_{t,T}, k'_{\infty,T}$ for M'_T.

Set $\delta_t(x) = (k_t(x) - k_{t,T}(x))$ for $x \in M_T$, with M_T seen as a subset of M. Similarly define δ'_t.

Set

$$I_j(T,t) := (tr^* e^{-t\Delta_M} - tr\ e^{-t\Delta_{M_T}}) - (tr^* e^{-t\Delta_{M'}} - tr\ e^{-t\Delta_{M'_T}}),$$

where Laplacians are taken on $j-$forms and $I_j(T,\infty) := \lim_{t\to\infty} I_j(T,t)$

Writing $I_*(T,t) := \frac{1}{2}\sum_j j(-1)^{j+1} I_j(T,t)$, one can see that the part concerning the analytic torsion in Theorem[21] is

$$-\frac{d}{ds}|_{s=0}\Gamma(s)^{-1}\int_0^\infty (I_*(T,t) - I_*(T,\infty))t^s\frac{dt}{t}.$$

For some j, write $I_j(T,t) = I(T,t)$. Then the discussion in section 4.1.2 gives

$$\begin{aligned} I(T,t) = &\int_{M_{T'}} \delta_t(x) - \int_{M'_{T'}} \delta'_t(x) \\ &+ \int_{M_{[T',T]}} (k_t(x) - k'_t(x)) - (k_{t,T}(x) - k'_{t,T}(x))) \\ &+ \int_{\mathcal{C}_T} (k_t(x) - k'_t(x)). \end{aligned} \tag{4.6}$$

We get $I(T,\infty)$ by putting k_∞ for k_t everywhere.

$$\lim_{t\to\infty} tr^* e^{-t\Delta_M} = \begin{cases} b_0 = \dim H_0(M,\mathbb{C}) & j = 0,3 \\ \dim H_{j,!}(M,\mathbb{C}) & j = 1,2 \end{cases}$$

and $\lim_{t\to\infty} tr\ e^{-t\Delta_{M_T}} = \dim H_j(M_T, \mathbb{C}))$. Similar results hold for M', M'_T. Now,

$$\dim H_j(M,\mathbb{C}) - \dim H_{j,!}(M,\mathbb{C}) = \begin{cases} \frac{1}{2}\dim H_1(\partial M) & j=1 \\ \dim H_2(\partial M) - b_0(M) & j=2. \end{cases}$$

Now boundaries ∂M and ∂M_T are homotopy equivalent. Thus $I(T,\infty) = 0$ for $j \neq 2,3$. We get

$$\frac{d}{ds}|_{s=0}\Gamma(s)^{-1}\int_0^B I(T,\infty) = \begin{cases} (\log T + \gamma)(b_0(M) - b_0(M')) & j=2,3 \\ 0 & j=0,1. \end{cases}$$

The functions ν_i are real analytic away from a discrete set of points. Also, by (1.46) $-\nu_i' = -dv_i/dt$ is bounded below whenever differentiable. If v_i is an eigenvalue corresponding to $\lambda_i = e^{iv_it}$, then

$$\lambda_i' = i\frac{\langle A'v_i, v_i\rangle}{\langle v_i, v_i\rangle}$$

where $\lambda_i' = d\lambda_i/dt$. Now $i\nu_i' = \lambda_i'/\lambda_i = i\frac{\langle (A'/A)v_i, v_i\rangle}{\langle v_i, v_i\rangle}$ implies that $-\nu_i' = \langle -(A'/A)v_i, v_i\rangle/||v_i||^2$. This is bounded below (By (1.46)). Note that

$$\sum -\nu_i' = tr(-A'/A) = -(\det A)'/\det A.$$

We claim that for any fixed $B > 0$,

$$\lim_{T\to\infty}\int_0^B I(T,t)\frac{dt}{t} = 0.$$

Proving this involves seeing that each of the the integrands in (4.6) is uniformly approaching zero as $T \to \infty$.

By (4.5), we see that

$$\begin{aligned}(\log \tau_R(M) - \log_R(M_T)) - (\log \tau_R(M') - \log \tau_R(M'_T)) \\ + \frac{1}{2}(\log \overset{\prime}{\det}(T^{-2}R) - \log \overset{\prime}{\det}(T^{-2}R')) \to 0,\end{aligned} \tag{4.7}$$

where R, R' are residues of scattering matrices Ψ and Ψ' for M, M' respectively.

For $t \in [B, \infty)$, thus, showing that

$$\begin{aligned}-\frac{d}{ds}|_{s=0}\Gamma(s)^{-1}\int_B^\infty (I_*(T,t) - I_*(T,\infty))t^s\frac{dt}{t} \\ + \frac{1}{2}(\log \overset{\prime}{\det}(T^{-2}R) - \log \overset{\prime}{\det}(T^{-2}R')) \to -\frac{1}{2}(\log B + \gamma)(b_0(M) - b_0(M'))\end{aligned} \tag{4.8}$$

as $T \to \infty$ proves Theorem[21].

Now each term of $I_j(T,t)$ can be written as a summation of $e^{-\lambda t}$ over eigenvalues of $j-$forms on one of M, M_T, M', and M'_T. We define I_j^c to be the restriction of these summations to co-closed forms. This is helpful as via

$$I_j(T,t) - I_j(T,\infty) = (I_j^c(T,t) - I_j^c(T,\infty)) + (I_{j-1}^c(T,t) - I_{j-1}^c(T,\infty))$$

one can reduce the analysis of (4.8) to co-compact case.

Suppose $N(x)$ is the number of eigenvalues on co-closed $1-$forms on M_T in $(0, x^2]$. Suppose the eigenvalues are $0 < \lambda_1 \leq \lambda_2 \ldots$. Define N_{cusp} as the number of eigenvalues of co-closed cusplidal $1-$forms on M.

Set an "error term" as

$$E(x) = N(x) - N_{cusp}(x) - \frac{4xh_{rel}\log T - \sum_{i=1}^h \nu_i(x)}{2\pi}. \tag{4.9}$$

Writing $\omega(s) := \det(\Phi^-(-s)\Phi^+(s))$, it is easy to see that $\omega(s)\omega(-s) = 1$ and thus, ω'/ω is an even function.

By Lemma[12](1) and Theorem[22], $E(x)$ is bounded when $x \leq T_{max}(T)$, T_0 for some T_0. Indeed

$$\int_X^{X+1} \left|E(x) + \sum_i \{\frac{4x\log T - \nu_i}{2\pi}\}\right| \leq ae^{-bT}) \tag{4.10}$$

for some absolute constants a, b whenever $X \leq T_{max}$. Here $\{\cdot\}$ is the fractional part function.

$$\begin{aligned}
\sum_{\lambda_i \neq 0} e^{-\lambda_i t} &= \int_0^\infty e^{-x^2 t} dN(x) \\
&=^{(a)} \int_0^\infty N(x).2xte^{-x^2 t} dx \\
&=^{(b)} \sum \frac{-\nu_i(0)}{2\pi} + \frac{1}{2\pi}\int_0^\infty e^{-x^2 t}(4h_{rel}\log T - \frac{\omega'}{\omega}(ix) + 2\pi\frac{dN_{cusp}}{dx})dx \\
&+ \int_0^\infty E(x).2xte^{-x^2 t} \\
&=^{(c)} \frac{1}{2\pi}\int_{-\infty}^\infty e^{-x^2 t}(2h_{rel}\log T - \frac{\omega'}{\omega}(ix)) + \sum_{\lambda \neq 0} e^{-\lambda t} \\
&+ \int_0^\infty E(x).2xte^{-x^2 t}.
\end{aligned} \tag{4.11}$$

In (4.11), equality (a) is integration by parts. Equality (b) is by expanding $N(x)$ via (4.9). Equality (c) uses the facts that $_i(0) = 0$ for every i (follows from (1.46) and (1.47)) and that ω'/ω is even.

To proceed further, we need the following lemma

Lemma 13. *Suppose I is an open interval in $\mathbb{R}$ and m be a monotonically increasing piecewise differentiable function $I \to (-1/2, 1/2)$ so that $m' \in [A, B]$ where $B - A \geq 1$. Suppose $f : I \to \mathbb{R}$ is smooth and $|f| + |f'| \leq M$ for some constant M. Then*

$$\int f(x).m(x)dx \ll M\frac{B-A}{A^2}.$$

In view of (4.11) and its analogue for M_T', with $E'(x)$ as the error term for M_T', we may write $\int_B^\infty (I_i^c(T,t) - I_1^c(T,\infty))dt/t$ as

$$\int_B^\infty (\int_0^\infty E(x) - E'(x))2xe^{-x^2 t}dxdt = 2\int_0^\infty (E(x) - E'(x))e^{-x^2 B}dx/x. \tag{4.12}$$

Fix $0 < \epsilon < 1$. We estimate the right hand side of (4.12) by splitting the integral into

$$[0, \frac{1}{100\log T}) \cup [\frac{1}{100\log T}, \epsilon) \cup [\epsilon, T_{max}) \cup [T_{max}, \infty).$$

- The integral $\int_{T_{max}}^\infty$: It is easy to see by the definition that $E(x) - E'(x)$ is bounded by a polynomial in x, with coefficients that have at worst, polynomial growth in $\log T$. Now, $\int_{T_{max}}^\infty x^N e^{-x^2 B}$ decays faster than any polynomial in $\log T$.
- The integral in $[\epsilon, T_{max}]$, under the approximation for $E(x)$(equivalently $E'(x)$) given by (4.10) looks like

  $$\int_\epsilon^{T_{max}} e^{-x^2 T}\frac{dx}{x}\sum_j\{\frac{4x\log T - \nu_i}{2\pi}\}$$

  upto an exponentially small error.

  Split the integral as a union$cup(a_j, a_{j+1})$so that $\{\frac{4x\log T - \nu_i}{2\pi}\}$ lies in $(-1/2, 1/2)$ as x goes from a_i to a_{i+1}). The number of such intervals in $O(T_{max}\log T)$. An application of Lemma[13] gives that the integral is $O(()^{-1/2})$, depending on ϵ and B.
- For $\int_0^{\frac{1}{100\log T}}$, note that, for $x \leq \frac{1}{100\log T}$

  $$|E(x) - E'(x)| \leq \sum_i |\nu_i| + |\nu_i'|$$

  from definition of $E(x)$ and the fact that M has no cuspidal eigenvalues and M_T has no eigenvalues x^2 for this range of x. Now, in this interval, $\nu_i(x)/x$ is bounded and independent of T. Thus, the integral is bounded by $O((\log T)^{-1/2})$.

– We are left with $\int_{\frac{1}{100\log T}}^{\epsilon}$. Due to (4.10) it suffices to bound

$$\int_{\frac{1}{100\log T}}^{\epsilon}\frac{dx}{x}\Big|\sum_j\{\frac{4x\log T-\nu_i}{2\pi}\}-\sum_j\{\frac{4x\log T-\nu_i'}{2\pi}\}\Big|.$$

To bound this, it is enough to consider the greatest integer function instead of the fractional part function in the above integral. This is bounded upto constants by $|\nu_i(x)/x|$. The integral of $|\nu_i(x)/x|$ is bounded by $O(\epsilon)$.

The above bounds show that

$$\lim_{T\to\infty}\Big|\int_B^{\infty}I_1^c(T,t)-I_1^c(T,\infty))\frac{dt}{t}\Big|=0$$

- (0−forms) Here $\Phi^-(-it)\Phi^+(it)$ is replaced by $\Psi(it)$.
- (2−forms) Here, the analysis is a bit different owing to the formulation in Theorem[4.11](4). The same analysis give above shows that

$$\begin{aligned}\int_T^{\infty}\frac{dt}{t}(I_2^c(T,t)-I_2^c(T,\infty)) &= o(1)+\log\det{}'(2T^{-2}R)-\log\det{}'(2T^{-2}R')\\ &+(b_0(M)-b_0(M'))(\log T+\gamma)\end{aligned}$$

  as $T\to\infty$.
- (3−forms) The coclosed 3−forms are exactly the multiples of the volume form and have all eigenvalues zero. Hence their contribution is zero.

This proves Theorem[21] and hence the main theorem.

Appendix A

Spectral theory of the Laplacian operator

Here we analyse the Laplace-Beltrami operator Δ, which in (x, y, r) coordinates is

$$\Delta = r^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2}) - r\frac{\partial}{\partial r},$$

in its natural domain within the Hilbert space $L^2(\Gamma\backslash\mathbb{H})$.

A.0.1 Essential self adjointness of Δ

Let $\Gamma < PSL_2(\mathbb{C})$ be discrete. We know that $L^2(\Gamma\backslash\mathbb{H})$ is the set of all Borel measurable functions $f : \mathbb{H} \to \mathbb{C}$ that are $\Gamma-$ invariant satisfying

$$\int_{\mathcal{F}} |f|^2 dv < \infty,$$

where $\mathcal{F}$ is a fundamental domain of Γ. For $f, g \in L^2(\Gamma\backslash\mathbb{H})$ we have an inner product

$$\langle f, g\rangle := \int_{\mathcal{F}} f\bar{g} dv \tag{A.1}$$

We want to define Δ on an appropriate domain $\mathcal{D} \subset L^2(\Gamma\backslash\mathbb{H})$ so as to make it self adjoint. We know that for a C^2-function f, $(\Delta f)\circ M = \Delta(f\circ M) = \Delta f$. Thus Δf is automatically $\Gamma-$ invariant (but not necessarily in $L^2(\Gamma\backslash\mathbb{H})$) and we are motivated to define the following domains.

Definition 37. *Let $\Gamma < PSL_2(\mathbb{C})$ be discrete. Define*

$$\mathcal{D} := \{f \in L^2(\Gamma\backslash\mathbb{H}) \cap C^2(\mathbb{H}) | \Delta f \in L^2(\Gamma\backslash\mathbb{H})\}, \tag{A.2}$$

$$\mathcal{D}^\infty := \{f \in L^2(\Gamma\backslash\mathbb{H}) \cap C^\infty(\mathbb{H}) | \pi_\Gamma(supp(f)) \text{ is compact in} \Gamma\backslash\mathbb{H}\}. \tag{A.3}$$

Here $\pi_\Gamma : \mathbb{H} \to \Gamma\backslash\mathbb{H}$ is the natural projection map and $supp(f)$ is the support of f.

Note that $\mathcal{D}^\infty \subset \mathcal{D}$ and that $\mathcal{D}^\infty$, $\mathcal{D}$ are dense in $L^2(\Gamma\backslash\mathbb{H})$. We prove the following version of partition of unity lemma.

Lemma 14. *Let $\Gamma < PSL_2(\mathbb{C})$ be discrete. Then there exist $C^\infty-$functions with compact support $h_\nu : \mathbb{H} \to [0,1]$ for $\nu \in \mathbb{N}$ such that $0 \le h_\nu \le 1$. Furthermore there are relatively compact open neighbourhoods U_ν with $supp(h_\nu) \subset U_\nu$ such that the sets MU_ν for $M \in \Gamma, \nu \in \mathbb{N}$ form a locally finite covering of $\mathbb{H}$ and*

$$1 = \sum_{\substack{M\in\Gamma \\ \nu\in\mathbb{N}}} h_\nu \circ M.$$

That is, there exists a $\Gamma-$invariant $C^\infty-$partition of unity on $\mathbb{H}$.

Proof. Suppose $(U_\nu)_{\nu\in\mathbb{N}}$ is a family of relatively compact open subsets of $\mathbb{H}$ such that $(MU_\nu)_{(M,\nu)\in\Gamma\times\mathbb{N}}$ is a locally finite open covering of $\mathbb{H}$. Also assume that for every U_ν, there exists an open subset V_ν with $\bar{V}_\nu \subset U_\nu$ such that $(MU_\nu)_{(M,\nu)\in\Gamma\times\mathbb{N}}$ is a covering for $\mathbb{H}$. For every $\nu \in \mathbb{N}$ choose a $C^\infty-$function $g_\nu \ge 0$ such that $g_\nu(x) > 0$ for all $x \in \bar{V}_\nu$ and $supp(g_\nu) \subset U_\nu$. It is known that such functions exist. Then

$$g := \sum_{M\in\Gamma,\nu\in\mathbb{N}} g_\nu \circ M$$

is strictly positive, $\Gamma-$invariant $C^{\infty}-$function on $\mathbb{H}$. Hence $h_{\nu} := g_{\nu}/g$ has the required properties.

To complete the proof, we construct the families (U_{ν}) and (V_{ν}). Consider a fixed Poincaré polyhedron $\mathcal{F}$ and write

$$\mathcal{F}_n := \mathcal{F} \cap B(\bar{j}, n) \ \ (n \geq 1),$$

where $B(j, n)$ is the hyperbolic ball centred at j with radius n. Cover $\mathcal{F}_1$ by finitely many open hyperbolic unit balls, call them $V_1, \ldots, V_{k_1 - 1}$. Proceed inductively as follows: Suppose for $n \in \mathbb{N}$ with $n > 1$, the hyperbolic unit balls $V_1, \ldots, V_{k_n - 1}$ are such that they cover $\mathcal{F}_n$. If $\mathcal{F}_{n+1} - \mathcal{F}_n$ is empty, set

$$V_{k_n} := \emptyset, \quad k_{n+1} = k_n + 1.$$

If $\mathcal{F}_{n+1} - \mathcal{F}_n$ is non empty, cover $\mathcal{F}_{n+1} - \mathcal{F}_n$ by finitely many hyperbolic unit balls $V_{k_n}, \ldots, V_{k_{n+1}-1}$ such that their centres are in $\mathcal{F}_{n+1} - \mathcal{F}_n$. Let U_{ν} be the hyperbolic ball of radius 2 concentric with V_{ν}. Put $U_{\nu} := \emptyset$ if $V_{\nu} = \emptyset$. Then U_{ν} and V_{ν} are as required. $\square$

The elements of $\mathcal{D}^{\infty}$ can be represented as below.

Lemma 15. *$\mathcal{D}^{\infty}$ is the set of functions $g : \mathbb{H} \to \mathbb{C}$ such that*

$$g = \sum_{M \in \Gamma} h \circ M \tag{A.4}$$

where $h \in C_c^{\infty}(\mathbb{H})$.

Proof. It is easy to see that any g of the form (A.4) is in $\mathcal{D}^{\infty}$. We prove the other direction. Let $g \in \mathcal{D}^{\infty}$ be given, h_{ν} ($\nu \in \mathbb{N}$ be as in Lemma[14]. Since $\pi_{\Gamma}(supp(g))$ is compact in $\Gamma\backslash\mathbb{H}$, there exists a finite $F \subset \mathbb{N}$ such that

$$supp(g) \cap M^{-1} supp(h_{\nu}) = \emptyset$$

for all $\nu \in \mathbb{N} - F$, $M \in \Gamma$. Then $g.(h_n u \circ M) = 0$ for all $\nu \in \mathbb{N} - F$, $M \in \Gamma$. Thus,

$$g = \sum_{M \in \Gamma, \nu \in F} g.(h_n u \circ M) = \sum_{M \in \Gamma} h \circ M,$$

where

$$h := \sum_{\nu \in F} g.h_\nu \in C_c^\infty(\mathbb{H}).$$

□

We give the following result (without proof)

Proposition 11. *Let $A : \mathcal{D}_A \to \boldsymbol{H}$ be an operator defined on a dense subspace $\mathcal{D}_A$ of the complex Hilbert space $\boldsymbol{H}$. The following are equivalent.*

1. *A is essentially self adjoint.*
2. *A is symmetric, and $(A + i)\mathcal{D}_A$ and $(A - i)\mathcal{D}_A$ are dense in $\boldsymbol{H}$.*

We want to show that $-\Delta : \mathcal{D} \to L^2(\Gamma\backslash\mathbb{H})$ and $-\Delta : \mathcal{D}^\infty \to L^2(\Gamma\backslash\mathbb{H})$ are essentially self adjoint. We give the following results first.

Lemma 16. *Let $f, g \in C^1(\mathbb{H})$ and put*

$$\boldsymbol{Gr}(f, g) := r^2(f_x\bar{g}_x + f_y\bar{g}_y + f_r\bar{g}_r), \tag{A.5}$$

where f_x, f_y, f_r are partial derivatives for f in (x, y, r) coordinates. Then

$$\boldsymbol{Gr}(f, g) \circ T = \boldsymbol{Gr}(f \circ T, g \circ T) \tag{A.6}$$

for all isometries T on $\mathbb{H}$.

Lemma 17. *Let $T \in PSL_2(\mathbb{C})$ and let $h \in C^1(\mathbb{H})$ be $T-$invariant. Then*

$$\omega := \frac{1}{r}\frac{\partial h}{\partial x} dy \wedge dr + \frac{1}{r}\frac{\partial h}{\partial y} dr \wedge dx + \frac{1}{r}\frac{\partial h}{\partial r} dx \wedge dy \tag{A.7}$$

is $T-$invariant.

Theorem 23. *Let $\Gamma < PSL_2(\mathbb{C})$ be discrete and let $\mathcal{F}$ be a fundamental domain for Γ. Then for all $f, g \in \mathcal{D}$ $\boldsymbol{Gr}(f,g)$ is $\Gamma-$invariant and $\nu-$integrable over $\mathcal{F}$ and satisfies*

$$\langle -\Delta f, g\rangle = \int_{\mathcal{F}} \boldsymbol{Gr}(f,g)dv. \tag{A.8}$$

In particular, $-\Delta : \mathcal{D} \to L^2(\Gamma\backslash\mathbb{H})$ is symmetric and positive, that is, $\langle -\Delta f, g\rangle = \langle f, -\Delta g\rangle$, $\langle -\Delta f, f\rangle \geq 0$ for every $f, g \in \mathcal{D}$.

We prove this result only for groups of finite covolume.

Proof. Firstly, as the statement is independent of choice of the fundamental domain, choose it to be a Poincaré normal polyhedron if Γ is cocompact and $\mathcal{F}_0 \cup M_B$ in case Γ is of finite covolume but not cocompact. where M_B is the union of all cusp ends and $\mathcal{F}_0$ is a compact set (here, a polyhedron). Such choice exists. Denote the fundamental domain by $\mathcal{F}$. By Lemma[A.5], $\mathbf{Gr}(f,g)$ is $\Gamma-$invariant for $f, g \in \mathcal{D}$. We are to show that $\mathbf{Gr}(f,g)$ is $v-$integrable and (A.8) holds. It suffices to show this for $f = g$ as applying polarisation (alongside Cauchy-Schwarz inequality) gives the desired result.

Let $f \in \mathcal{D}$. Consider the $\Gamma-$invariant differential form

$$\omega := \frac{1}{r} f_x \bar{f} dy \wedge dr + \frac{1}{r} f_y \bar{f} dr \wedge dx + \frac{1}{r} f_r \bar{f} dx \wedge dy. \tag{A.9}$$

We have

$$d\omega = (\Delta f)\bar{f}dv + \mathbf{Gr}(f,f)dv. \tag{A.10}$$

Now, we have, for sufficiently large R

$$J(R) := \int_{\mathcal{F}_R} ((\Delta f)\bar{f} + \mathbf{Gr}(f,f))dv = \int_{\mathcal{F}_R} d\omega = \int_{\partial\mathcal{F}_R} \omega \tag{A.11}$$

where $\mathcal{F}_R$ is $\mathcal{F}$ truncated at height $r = R$.

Now, on $\partial\mathcal{F}_R$, the contributions from $\Gamma-$equivalent pairs are in opposite orientations and hence cancel out. We are left with the contributions of cross sections at cusps. Writing $\mathcal{Q}_\nu(R) := \{z + Rj | z \in \mathcal{P}_v\}$, where $\mathcal{P}_\nu$ is a fundamental domain in $\mathbb{C}$

for the action of $B_\nu \Gamma_{\eta_\nu} B_\nu^{-1}$ on $\mathbb{P}^1\mathbb{C} - \infty = \mathbb{C}$, ($\eta_\nu$ being a cusp for Γ in $\mathbb{H}$) we get

$$J(R) = \sum_{nu=1}^{h} \int_{B_\nu^{-1}\mathcal{Q}_\nu} \omega$$

Writing $P = (x, y, r)$ and $BP = P' = (x', y,' r')$ and dropping index ν we have

$$\begin{aligned}
&|\int_Y^R (\int_{B^{-1}\mathcal{Q}(t)} \omega)\frac{dt}{t}| = |\int_Y^R (\int_{B^{-1}\mathcal{Q}(t)} \omega')\frac{dt}{t}| \\
&= |\int_Y^R (\int_{\mathcal{Q}t} (\frac{\partial}{\partial r'} f(P'))f(\bar{P'})\frac{dx'dy'}{r})\Big|_{r'=t} \frac{dt}{t}| \\
&\leq \left(\int_{\mathcal{P}\times[Y,R]} |r' f_{r'}(P')|^2 dv\right)^{1/2} \left(\int_{\mathcal{P}\times[Y,R]} |f(P')|^2 dv\right)^{1/2} \qquad \text{(A.12)} \\
&\leq \left(\int_{\mathcal{P}\times[Y,R]} \mathbf{Gr}(f,f)\circ B dv\right)^{1/2} \left(\int_{\mathcal{P}\times[Y,R]} |f\circ B|^2 dv\right)^{1/2} \\
&= \left(\int_{\mathcal{F}[Y,R]} \mathbf{Gr}(f,f) dv\right)^{1/2} \left(\int_{\mathcal{F}[Y,R]} |f|^2 dv\right)^{1/2},
\end{aligned}$$

where $\mathcal{F}_\nu[Y,R] := B_\nu^{-1}(\mathcal{P}_\nu \times [Y,R])$. Now from Cauchy-Schwarz inequality, we have

$$|\int_Y^R J(r)\frac{dr}{r}| \leq h\left(\int_{\mathcal{F}_R} \mathbf{Gr}(f,f)dv\right)^{1/2} \|f\| \qquad \text{(A.13)}$$

If we show $\lim_{R\to\infty} J(R) = 0$, we are done. This follows if we have

$$\int_{\mathcal{F}} \mathbf{Gr}(f,f)dv < \infty, \qquad \text{(A.14)}$$

Since we'll have that $\lim_{R\to\infty} J(R)$ exists and is finite ($f \in \mathcal{D}$) and if this limit is nonzero, the left hand side of (A.13) is infinite while the right hand side is finite, a contradiction.

Now, we show (A.14) . Assume the contrary

$$\int_{\mathcal{F}} \mathbf{Gr}(f,f)dv \to +\infty, \quad R \to \infty.$$

Define

$$\phi(R) := \Re \int_{R_0}^{R} J(r)\frac{dr}{r}.$$

Then $\phi(R) \to \infty$ as $R \to \infty$ (From (A.11)).

We have $R\phi'(R) = \Re J(R)$. In (A.11), first term of the right hand side is bounded, hence, there exist constants $c > 0$, $R_1 \geq Y$ such that

$$R\phi'(R) = \Re J(R) > c \int_{\mathcal{F}_R} \mathbf{Gr}(f, f)dv > 0$$

for all $R \geq R_1$. Thus, $\phi'(R) > 0$ for all $R \geq R_1$. From (A.13) we can infer that there is a constant $C > 0$ such that $0 < \phi(R) \leq C\sqrt{R\phi'(R)}$ for all $R \geq R_1$. Hence s

$$\frac{1}{R}C^2\frac{\phi'(R)}{\phi(R)^2} \qquad \forall R \geq R_1.$$

This implies that

$$\log R - \log R_1 \leq C^2\big(\frac{1}{\phi(R_1)} - \frac{1}{\phi(R)}\big) \qquad \forall R \geq R_1,$$

a contradiction to $\lim_{R\to\infty} \phi(R)$ is infinite. □

Theorem 24. *Let $\Gamma \in PSL_2(\mathbb{C})$ be discrete. Then the operators*

$$-\Delta : \mathcal{D}^\infty \to L^2(\Gamma\backslash\mathbb{H}), \qquad -\Delta : \mathcal{D} \to L^2(\Gamma\backslash\mathbb{H})$$

are essentially self adjoint and have the same self adjoint extension.

Proof. An essentially self adjoint operator in a Hilbert space has a unique self adjoint extension. Thus, Δ with domain $\mathcal{D}^\infty$ and Δ with domain $\mathcal{D}$ have the same self adjoint extension if both the operators are essentially self adjoint. As Δ is symmetric in both domains,and $\mathcal{D}^\infty \subset \mathcal{D}$, we only have to show that $(\Delta + i)\mathcal{D}^\infty$ and $(\Delta - i)\mathcal{D}^\infty$ are dense subspaces of $L^2(\Gamma\backslash\mathbb{H})$. But Δ has only real coefficients and $\mathcal{D}^\infty$ is conjugation invariant. Hence it suffices to show that $(\Delta + i)\mathcal{D}^\infty$ is dense in $L^2(\Gamma\backslash\mathbb{H})$. Let $u \in L^2(\Gamma\backslash\mathbb{H})$ and assume $\langle u, \Delta f + if\rangle = 0$ for all $f \in \mathcal{D}^\infty$. If we show that $u = 0$, it will imply that the closure of $(\Delta + i)\mathcal{D}^\infty$ is $L^2(\Gamma\backslash\mathbb{H})$.

Using Lemma[15] any $f \in \mathcal{D}^\infty$ can be written as

$$f = \sum_{M\in\Gamma} h \circ M \tag{A.15}$$

for some $h \in C_c^\infty(\mathbb{H})$. Conversely, for any $h \in C_c^\infty(\mathbb{H})$, (A.15) defines an $f \in \mathcal{D}^\infty$. The sum in (A.15) is finite if the variables range over a compact set. Thus,

$$\Delta f = \sum_{M\in\Gamma} (\Delta h) \circ M.$$

By dominated convergence theorem and $\Gamma-$invariance of u, we write

$$0 = \sum_{M\in\Gamma} \int_{\mathcal{F}} u(P)\overline{(\Delta h + ih) \circ M(P)}dv(P) = \int_{\mathbb{H}} u\overline{(\Delta + i)h}dv. \tag{A.16}$$

By assumption this is true for all $h \in C_c^\infty(\mathbb{H})$. By Weyl's lemma, u is almost everywhere equal to a C^2-function. Applying Green's formula and Stokes' theorem to a ball B such that $\bar{B} \subset \mathbb{H}$

$$\int_B (-\Delta u\bar{h} - u\overline{\Delta h})dv = \int_{\partial B} \sum_{x,y,r} (u\bar{h}_x - u_x\bar{h})\frac{dy \wedge dr}{r}. \tag{A.17}$$

Choosing B such that $supp(h) \subset B$, the right hand side of (A.17) vanishes and we are left with

$$\int_{\mathbb{H}} u\overline{\Delta h}dv = \int_{\mathbb{H}} (\Delta u)\bar{h}dv.$$

Thus, from (A.16)

$$\int_{\mathbb{H}} ((\Delta - i)u)\bar{h}dv = 0 \quad \forall h \in C_c^\infty(\mathbb{H}).$$

Hence, $\Delta u = iu$. But by Theorem[23] $\Delta : \mathcal{D} \to L^2(\Gamma\backslash\mathbb{H})$ has only real eigenvalues $(\langle \Delta f, f\rangle = \langle \lambda f, f\rangle \leq 0$, thus λ is real.) and thus $u = 0$. $\square$

Definition 38. *Define*

$$\tilde{\Delta} : \tilde{\mathcal{D}} \to L^2(\Gamma\backslash\mathbb{H}) \tag{A.18}$$

to be the unique self adjoint extension of $-\Delta : \mathcal{D}^\infty \to L^2(\Gamma\backslash\mathbb{H})$ *and of* $-\Delta : \mathcal{D} \to L^2(\Gamma\backslash\mathbb{H})$.

An element $g \in L^2(\Gamma\backslash\mathbb{H})$ belongs to $\tilde{\mathcal{D}}$ iff there exists a sequence $(f_n)_{n\geq 1}$ in $\mathcal{D}^\infty$ (correspondingly in $\mathcal{D}$) converging to g such that $(\Delta f_n)_{n\geq 1}$ converges in $L^2(\Gamma\backslash\mathbb{H})$. In this case,

$$\tilde{\Delta} g := \lim_{n\to\infty} \Delta f_n.$$

Lemma 18. *If* $f \in \tilde{\mathcal{D}}$ *is twice continuously differentiable, then* f *is in* $\mathcal{D}$.

Proof. Take an arbitrary $g \in \mathcal{D}^\infty$ and write g as in Lemma[15] with $h \in C_c^\infty(\mathbb{H})$. Let $\mathcal{F}$ be a fundamental domain of Γ. Then,

$$\begin{aligned} \int_{\mathcal{F}} (\tilde{\Delta} f)\bar{g}\,dv &= \langle \tilde{\Delta} f, g\rangle = \langle f, \tilde{\Delta} g\rangle = \langle f, \Delta g\rangle \\ &= \int_{\mathbb{H}} f(\overline{\Delta h})\,dv = \int_{\mathbb{H}} (\Delta f)\bar{h}\,dv = \int_{\mathcal{F}} (\Delta f)\bar{g}\,dv. \end{aligned} \tag{A.19}$$

Since $g \in \mathcal{D}^\infty$ is arbitrary this implies $\tilde{\Delta} f = \Delta f \in L^2(\Gamma\backslash\mathbb{H})$. $\square$

Appendix B

Some special values of Eisenstein series

We choose $\mathfrak{m} = \mathfrak{o}$ and restrict ourselves to the case $P = rj \in \mathbb{H}$, $r > 0$. Then,

$$||cP + d||^2 = |d|^2 + r^2|c|^2 \quad (c, d \in \mathfrak{o}). \tag{B.1}$$

If $r^2 \in \mathbb{N}$, then $\hat{E}_{\mathfrak{o}}(rj, s)$ is the following summation

$$\hat{E}_{\mathfrak{o}}(rj, s) = r^{1+s} \sum_{n=1}^{\infty} \frac{a_n(r^2)}{n^{1+s}}, \tag{B.2}$$

where $a_n(r^2)$ is the number of ways of writing n as the sum $m_1^2 + m_2^2 + r^2(m_3^2 + m_4^2)$ where $m_1, ..., m_4$ are integers. We define for $n, k \in \mathbb{N}$

$$\sigma_1(n, k) := \sum_{0<d|n, k\nmid n} d. \tag{B.3}$$

The Dirichlet series associated with (B.3) is

$$\sum_{n=1}^{\infty} \frac{\sigma_1(n, k)}{n^{1+s}} = \big(1 - \frac{1}{k^s}\big)\zeta(s)\zeta(1+s), \tag{B.4}$$

$\zeta(s)$ being the Riemann zeta function. For the rest of the section we take $K = \mathbb{Q}[i]$, and thus $\mathfrak{o} = \mathbb{Z}[i]$

Example 1.

$$\hat{E}_{\mathfrak{o}}(j,s) = 8(1-2^{-2s})\zeta(s)\zeta(1+s). \tag{B.5}$$

By Jacobi's four square formula, we have

$$a_n(1) = 8\sigma_1(n,4)$$

and (B.4) implies the result.

Example 2.

$$\hat{E}_{\mathfrak{o}}(\sqrt{2}j,s) = 2^{\frac{s+3}{2}}\Big(2(1-2^{-3s}) - (2^{-s}-2^{-2s})\Big)\zeta(s)\zeta(1+s). \tag{B.6}$$

The following result is due to Liouville and Pepin:

$$a_n(2) = \begin{cases} 4\sigma_1(n) & n \textit{ is odd} \\ 8\sigma_1(m) & m = \frac{n}{2} \textit{ is odd} \\ 24\sigma_1(m) & m = \frac{n}{2^{-\nu}} \textit{ is odd}, \ \nu \geq 2 \end{cases} \tag{B.7}$$

One can see

$$\hat{E}_{\mathfrak{o}}(\sqrt{2}j,s) = 4.2^{\frac{1+s}{2}}\frac{(1-2^{-3s}) - 2^{-1}(2^{-s}-2^{-3s})}{(1-2^{-s})(1-2^{-1-s})}\sum_{k=0}^{\infty}\frac{\sigma_1(2k+1)}{(2k+1)^{1+s}}$$

and

$$\zeta(s)\zeta(1+s) = \sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n^{1+s}} = (1-2^{-s})^{-1}(1-2^{-1-s})^{-1}\sum_{k=0}^{\infty}\frac{\sigma_1(2k+1)}{(2k+1)^{1+s}}.$$

Here $\sigma_1(n)$ is the usual divisor sum. The result follows.

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