Insights in Black Hole Thermodynamics Using Von Neumann Algebras

विद्या वाचस्पति की
उपाधि की अपेक्षाओं की आंशिक पूर्ति में प्रस्तुत शोध प्रबंध

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द्वारा / By

मोहम्मद अली / Mohd Ali

पंजीकरण सं. / Registration No.
20203749

शोध प्रबंध पर्यवेक्षक / Thesis Supervisor

सुनीता वरदराजन/ Suneeta Vardarajan

भारतीय विज्ञान शिक्षा एवं अनुसंधान संस्था न पुणे

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Dedicated to my parents and uncle, whose support made this possible.

Certificate

It is certified that the work contained in this thesis entitled **"Insights in Black Hole Thermodynamics Using Von Neumann Algebras"** by **Mohd Ali** has been carried out under my supervision and that it has not been submitted elsewhere for the award of any degree or diploma from any other university or institution.

Dr. Suneeta Vardarajan
(Supervisor)
Department of Physics
IISER Pune

Declaration

Name: **Mohd Ali**

Registration No.: **20203749**

Thesis Supervisor: **Dr. Suneeta Vardarajan**

Department: **Department of Physics**

Institute: **Indian Institute of Science Education and Research Pune**

Date of Joining Ph.D.: **1st September 2020**

Date of Synopsis: **22nd April 2025**

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To the best of my knowledge, it is an original work, both in terms of research content and narrative, and has not been submitted elsewhere, in part or in full, for a degree. Further, due credit has been attributed to the relevant source and collaborations with appropriate citations and acknowledgments, in line with established norms and practices. I hereby declare that I have adhered to the principles of academic honesty and integrity in my submission. I have not misrepresented, fabricated, or falsified any idea, data, fact, or source. I understand that any violation of these principles may result in disciplinary action by the Institute and may also lead to legal consequences from the original sources if proper citation or permission has not been obtained where required.

Md. Ali

Mohd Ali

Department of Physics

IISER Pune

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List of Publications

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Abstract

In recent years, algebraic quantum field theory and modular theory have provided significant insights into gravitational degrees of freedom. Notably, the works [1, 2, 3] demonstrate that the inclusion of gravitational degrees of freedom transforms the algebra of quantum fields on a curved spacetime from a type III to a type II crossed product algebra. Moreover, authors in [2, 3] have shown that the generalized entropy of the black hole exterior, evaluated at the bifurcation surface, equals the algebraic entropy of the associated type II crossed product algebra.
We have extended these results to arbitrary cuts on the horizon for black hole solutions in general relativity (GR) [4]. Specifically, we show that for QFT (including perturbative gravitons) in the static blackhole spacetime, the generalized entropy equals the algebraic entropy at any cut on the horizon, and the construction uses the Hartle-Hawking state. These results can also be extended to Kerr black holes under the assumption of a Hadamard stationary state. Furthermore, using the crossed product construction and modular theory, we present an algebraic proof of the local version of the generalized second law (GSL), where each step is manifestly finite—thanks to the Type II nature of the algebra. This finiteness provides a natural renormalization scheme, addressing a key assumption in Wall's proof of the GSL [5]. We have also studied deformations of modular operators and derived the averaged null energy condition (ANEC) for a class of spacetimes.

We further generalize the crossed product construction and the relation between generalized entropy and algebraic entropy beyond GR to arbitrary diffeomorphism-invariant theories [6]. In particular, we prove that the equality between generalized entropy and the entropy of the Type II crossed product algebra holds at the bifurcation surface in any such theory, and we provide a weaker form of the GSL in this broader context.

To study local algebras in higher curvature theories (HCT) and to better understand the nature of HCT itself, we investigate their causal structure. In particular, we analyze Generalized Quadratic Gravity (GQG) and Einsteinian Cubic Gravity (ECG). It is known that gravitons in higher curvature theories can propagate superluminally. This has important consequences for black holes: if the Killing horizon is not a characteristic surface for the fastest propagating mode, it cannot serve as a causal barrier. We show that GQG, which has genuine fourth-order equations of motion, admits only null characteristic surfaces, ensuring that the black hole horizon remains a causal barrier. We also perform a detailed characteristic analysis of ECG, finding that while all null surfaces are characteristic, not

all characteristic surfaces are null. Despite the presence of non-null characteristic surfaces, we establish that the black hole horizon in ECG remains a characteristic surface.

Chapter 1

Introduction

The quest for a quantum theory of gravity is one of the most profound challenges in theoretical physics. The length scale at which quantum gravity is important is the Planck scale $l_p = \sqrt{\frac{G_N \hbar}{c^3}}$. Nevertheless, we expect that at a length scale much larger than the Planck length, the quantum effects of gravity can be neglected. A full quantum theory of gravity must reduce to quantum field theory in the absence of gravitational degrees of freedom and to general relativity in the classical limit $\hbar \to 0$, with consistent higher curvature corrections. Yet, gravity is inherently nonlinear—it gravitates just like other matter fields, meaning that its quantum effects can, in principle, appear at all energy scales. This suggests that quantum aspects of gravity must be incorporated systematically, much like those of matter fields.

Although we do not yet know the full quantized theory of gravity, we can reasonably assume that if the characteristic de Broglie wavelength of a matter field is λ, satisfies $l_p \ll \lambda$, then quantum effects of matter fields and fluctuations of the gravitational background (gravitons) become important, while the full quantization of spacetime itself is not. This regime is known as the *semi-classical regime.*

The physics in this regime is governed by the small parameter $\frac{l_p}{\lambda} \ll 1$. The appropriate theoretical framework is *quantum field theory in a curved background*, incorporating graviton fluctuations.

This thesis explores various aspects of this semi-classical regime using both classical and contemporary tools from local quantum field theory and operator algebras. In this introduction part, we will give a broad overview of the algebraic quantum field theory and its usefulness in curved spacetime, especially in black hole spacetime. In Sec 1.1, we will briefly discuss the development of an algebraic approach to local quantum field theory.

1.1 Algebraic Quantum field theory

The framework of **local quantum field theory (LQFT)** is one of the most successful and well-tested frameworks. It elegantly describes the standard model of particle physics, which is a theory of all fundamental forces in our nature except gravity. The gravity can also be studied using the LQFT framework, but only as an effective field theory.

Several classical approaches to quantum field theory, such as canonical quantization and path integral quantization, have provided deep insights into quantum field theory and physical interactions. These methods have led to remarkable predictions, such as the value of the electron's gyromagnetic ratio to 11 decimal places. However, examining a system from multiple perspectives often deepens our understanding and may even prompt new questions that might not have naturally arisen within the conventional framework. There are other useful frameworks of LQFT, such as Algebraic quantum field theory (AQFT).

Algebraic Quantum Field Theory (AQFT) is one of the axiomatic approaches to Local Quantum Field Theory (LQFT), developed in the 1950s alongside the Wightman axiomatic framework by Haag, Kastler, Araki, Borchers, and others [7, 8, 9]. The primary goal of these axiomatic approaches was to establish a mathematically rigorous foundation for quantum field theory. However, as we will see, AQFT not only provides a precise formalism but also leads to many profound conceptual insights—some of which are difficult to even formulate within other approaches.

In AQFT, the fundamental objects of study are **local algebras of observables**, constructed from bounded functions of smeared field operators within bounded regions of spacetime. These algebras naturally exhibit the structure of **von Neumann algebras**, allowing us to analyze the general structure of the theory using von Neumann algebras and, more broadly, the powerful methods of **operator algebras**. All properties of quantum fields are now encoded in the **algebraic relations** among these observables. States are then defined as functionals on the algebra of observables. Further, we can obtain the traditional Hilbert space formulation using what is known as the GNS construction. This algebraic perspective is particularly effective for addressing questions related to local regions in spacetime and excels in deriving general theorems by focusing on the fundamental algebraic structure of a theory. For instance, Araki in [10] has shown that the local algebra of a subregion in any quantum field theory belongs to the **type** $\mathbf{III}_1$ class of von Neumann algebras. As we will discuss in later chapters, this is closely related to the divergence of **entanglement entropy** in quantum field theory.

From a physical standpoint, experimental measurements access **quantities such as momentum and energy**, rather than directly probing the quantum state of the system. The central idea of AQFT is to work directly with **observables**, making it a **natural framework for formulating and addressing observational questions** in quantum field theory. It also provides a natural setting for studying LQFT as an information-theoretic system.
The algebraic approach to QFT is a natural framework for studying QFT in curved spacetime. In Minkowski QFT, the primary objects of interest are scattering amplitudes, directly related to physical cross sections in scattering experiments. However, a well-defined S-matrix generally does not exist in curved spacetime. Additionally, in Minkowski space, the Poincaré group provides a distinguished vacuum state, leading to a Fock space construction over the Poincaré-invariant vacuum. In generic curved spacetimes, the absence of such symmetries implies no preferred vacuum and, consequently, no natural Hilbert space[11, 12]. This motivates an algebraic formulation, where the fundamental structure is the algebra of local observables, with states as secondary constructs.

As we have explained, quantum field theory can be studied through the lens of von Neumann algebras. These algebras were introduced by von Neumann in order to understand quantum system and later classified by Murray and von Neumann into three types: **Type I, Type II, and Type III**. This classification is based on key algebraic properties such as the existence of a trace, density matrices, and pure states.

Type	**Trace**	**Density matrix**	**Pure state**
Type I	Yes	Yes	Yes
Type II	Yes	Yes	No
Type III	No	No	No

More broadly, this can be viewed as a classification of quantum systems themselves. It is well known that any **finite-dimensional quantum system** is described by a **Type I von Neumann algebra**. As noted in the previous section, Araki proved that the local algebra of a subregion in quantum field theory is of Type III$_1$. The same algebraic structure was also obtained by Leutheusser and Liu in [13, 14] while studying $\mathcal{N} = 4$ super Yang-Mills theory. They showed that in the strict $N \to \infty$ limit, the algebra emerges as Type III$_1$. Now, what about Type II von Neumann algebras? Are they realized in any physical system?

1.2 Gravitational Algebra and Black hole thermodynamics

Recent developments have shown that the local algebra of observables in perturbative quantum gravity, in the limit where the gravitational coupling $G_N \to 0$, is of Type II. As we discussed earlier, in this limit, the effective description is given by quantum field theory in a curved background, with free gravitons included on the same footing as other matter fields. Adding dynamical gravity in the sense of graviton corrections requires implementing gravitational constraints in the theory. In gravity, diffeomorphisms that vanish at infinity are redundancies of the description. All observables in a theory of gravity must be invariant under such transformations, i.e., they must commute with the constraints associated with each diffeomorphism.

It is well known that in the full theory of quantum gravity, there are no local operators. This is because any local region can be mapped to any other by an appropriate diffeomorphism, preventing the definition of strictly local operators. However, in perturbative quantum gravity, these diffeomorphism constraints are implemented order by order. As a result, one can construct a gravitational algebra by dressing observables with respect to some heavy degrees of freedom that remain unaffected by linearized diffeomorphisms or by dressing them to the asymptotic boundary when available. Further, it turns out that implementing the gravitational constraints changes the local algebra of observables from type III to type II. In particular, the gravitational algebra has the structure of what is known as a Type II crossed product. This was first obtained by Witten [1] in the context of AdS black holes, where he showed that the algebra of observables in the exterior of a black hole in the limit $G_N \to 0$ is a type II crossed product. The fact that gravitational algebras are type II crossed product algebras has deep implications for the nature of local quantum degrees of freedom. Since type II algebras possess a trace, one can define a density matrix and an associated entropy. In black hole physics, it is already known that there exists Bekenstein's generalized entropy [15, 16], which is defined by

$$S_{\text{gen}} = \frac{A}{4G_N} + S_{\text{QFT}}, \tag{1.2.1}$$

where A is the black hole horizon area at an arbitrary cut of the event horizon and S_{QFT} is the entropy of the quantum field theory in the exterior of the black hole. When Bekenstein originally proposed this, he had in mind the thermodynamic entropy of matter outside the horizon. It was Sorkin [17] who proposed that if the matter was quantum, then S_{QFT} should be the entanglement entropy of the quantum fields in the exterior with the interior. Further, it was shown by Susskind and Uglum in [18] that if one interprets S_{QFT} as the entanglement entropy of the quantum field in the black hole exterior, then at one-loop in graviton, S_{gen} is UV finite and independent of the UV cutoff.

Note that S_{gen} is universal in the sense that for any black hole in general relativity, it can be defined by the same formula and exhibits the same UV behavior. This naturally raises several questions: Is there a deeper reason for this cancellation? Does this behavior suggest that the inclusion of gravity improves the UV properties? What exactly is S_{gen}? Is there a concrete interpretation of what this entropy measures, and who observes it?

As it turns out, these questions can be answered using the framework of von Neumann algebras and modular theory. It was shown by Chandrasekaran, Longo, Penington, and Witten in [2] and by Chandrasekaran, Penington, and Witten in [3] that S_{gen}, evaluated at the bifurcation surface of a black hole, is equal (up to a state-independent constant) to the algebraic entropy of a type II crossed product algebra associated with the black hole exterior. This result clarifies why S_{gen} is UV finite.
Now, if it is indeed true that the generalized entropy is equal to the algebraic entropy, then one must demonstrate this for all cuts on the horizon of a semi-classical black hole. As we will show, in general relativity, this can be proven for any static black hole in the Hartle–Hawking state and can further be extended to Kerr black holes.

In addition, there is a compelling proof of the generalized second law (GSL) by Aron Wall in [5], which assumes the existence of a suitable renormalization scheme for certain quantities like entanglement entropy. We will show that the type II crossed product construction provides such a scheme, and we will provide an algebraic version of the local generalized second law in which each step is manifestly finite [4].

It is also well known that any UV-complete theory of gravity will contain higher curvature corrections. One might therefore expect that it should still be possible to show that the generalized entropy equals the algebraic entropy up to a state-independent constant. We will demonstrate that this is indeed the case at the bifurcation surface [6].

A natural next question is whether the second law can be established for higher curvature theories. However, this requires us to first define local algebras in such theories. Since the causal structure in higher curvature gravity differs from that of general relativity, a prerequisite for this program is to understand the causal structure in these theories. This motivates our study of causal structures in higher curvature gravity [19].

Notably, understanding the causal structure in higher curvature gravity is also interesting in its own right. For example, it is a crucial first step towards addressing the problem of well-posedness.

1.3 Plan of thesis

In Chapter 2, we briefly describe the mathematics of operators used in the von Neumann algebras and algebraic quantum field theory. For completeness, we provide all the relevant definitions and theorems, along with illustrative examples from quantum mechanics.

In Chapter 3, we introduce von Neumann algebras and their classification, emphasizing their relevance in physics. For clarity, we provide the necessary definitions, theorems, and either their proofs or references to them. We also present useful examples from quantum mechanics and local quantum field theory. Additionally, we introduce generalized functions to facilitate a better understanding of the algebraic framework of quantum field theory.

In Chapter 4, we demonstrate how the techniques of von Neumann algebras and modular theory can be used to show that the generalized entropy is equal to the algebra entropy on arbitrary cuts of the horizon for any black hole. We also prove a local version of the Generalized Second Law (GSL) using the crossed product construction, in which each step is manifestly finite [4].

In Chapter 5, we extend the results of Chapter 4 to arbitrary diffeomorphism-invariant theories. In particular, we prove that the generalized entropy equals the algebraic entropy at the bifurcation surface. We also present a weaker version of the GSL in this broader context [6].

In Chapter 6, we study the causal structure of generalized quadratic gravity and Einsteinian cubic gravity [19]. We also discuss the implications of these structures for holography.

In Chapter 7, we provide a summary and synthesis of the main results from the three works presented in Chapters 4, 5, and 6.

Chapter 2

Mathematics of Operators for Quantum Theory

This section provides a concise yet rigorous introduction to the mathematical framework of Hilbert space operators and operator algebras, serving as a foundation for the exposition of von Neumann algebras. Our approach is to systematically present the essential definitions, lemmas, and theorems required for understanding von Neumann algebras. While some results will be proved in detail, others will be stated with references to sources where complete proofs can be found.

2.1 Some Mathematical Background

Definition 2.1. (Sequence)
A sequence ϕ in the set $\mathbf{M}$ is a map $\phi : \mathbb{N} \to \mathbf{M}$. We denote $\phi(n) = \phi_n$ for $n \in \mathbb{N}$ and the sequence by $\{\phi_n\}_{n=1}^{\infty}$ or $\{\phi_n\}$.

The simplest example of a sequence is $\{\frac{1}{n}\}$ in the set of real numbers.

Definition 2.2. (Vector Space)
A vector space $(\mathbf{V}, +, .)$ over a field $\mathbb{K}$ (which we will take to be either $\mathbb{R}$ or $\mathbb{C}$) is a set of vectors that comes with an addition $+ : V \times V \to V$ and scalar multiplication $. : \mathbb{K} \times V \to V$, and satisfies:

1. Under addition: associativity, commutativity, existence of an additive identity, and existence of additive inverses.
2. Under scalar multiplication: associativity, existence of a multiplicative identity, distributivity of scalar multiplication with respect to vectors, and distributivity of scalar multiplication with respect to field addition.

There are many interesting examples of vector spaces. For instance, $\mathbb{R}^d$ is a vector space over the field $\mathbb{R}$; the solution space of any linear differential operator forms a vector space; and $C^n(\mathbb{R})$, the space of n-times continuously differentiable functions on $\mathbb{R}$, is also a vector space over $\mathbb{R}$.
A vector space is finite-dimensional if every linearly independent subset in the vector space is a finite set; otherwise, it is an infinite-dimensional vector space. Once we have a vector space, we can define a normed space.

Definition 2.3. (Seminormed and normed space)
A complex vector space $(\mathbf{V}, +,)$ equipped with a map $\|.\| : \mathbf{V} \to \mathbb{R}^+$ and satisfy,

1. **Non negativity**: $||a|| \geq 0$.
2. **Homogenity**: $||za|| = |z|.||a||$.
3. **Triangle inequality**: $||a + b|| \leq ||a|| + ||b||$.

$\forall a, b \in \mathbf{V}$ and $z \in \mathbb{C}$ then $\mathbf{V}$ is a semi-normed space and $\|.\|$ is called a semi-norm. If the semi-norm also satisfies,

$$||a|| = 0 \implies a = 0 \tag{2.1.1}$$

then $\|.\|$ is a norm and $\mathbf{V}$ is the normed vector space.
If the space is normed, then we can use this norm to define the notion of distance via metric function $d(a, b) : \mathbf{V} \times \mathbf{V} \to \mathbb{R}^+$ as follows:

$$d(a, b) = ||a - b|| \; \forall a, b \in \mathbf{V} \tag{2.1.2}$$

Consider $(\mathbb{R}^d, +, .)$ vector space and consider the function $f_a : \mathbb{R}^d \to \mathbb{R}^d$ defined as,

$$f_a(X) = a.X \tag{2.1.3}$$

where . is a dot product in $\mathbb{R}^d$, $X \in \mathbb{R}^d$ and $a \in \mathbb{R}^d$ is a constant vector. Now consider a map $\|\|_{f_a} : \mathbb{R} \to \mathbb{R}$ that takes $X \to |f_a(X)|$, where $|f_a(X)|$ is the absolute value of $f_a(X)$. Then $\|\|_{f_a}$ is seminorm as $|f_a(X)| = 0$ does not imply $X = 0$. While its evident that for $X = (x_1, .., x_d) \in \mathbb{R}^d$ with $||X||_2 \equiv \sqrt{\sum_{i=1}^d x_i^2}$, $\|.\|_2$ is a norm on $\mathbb{R}^d$. As we have seen, a norm induces a notion of distance on a vector space, which allows us to rigorously define and analyze the convergence of sequences within the space.

Definition 2.4. (Convergent sequence)
Let $\{\phi_n\}$ be a sequence in normed space $(\mathbf{V}, \|.\|)$ and it is said to converge to $\phi \in \mathbf{V}$, if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : \|\phi_n - \phi\| < \varepsilon \tag{2.1.4}$$

Further, it can be easily shown using the triangle inequality that in a normed space, a convergent sequence converges to at most one element. In the above definition, convergence depends on a pre-assumed limit. However, one can define *Cauchy convergence*, which is intrinsic to a sequence and does not require, *a priori*, that the limit lie in the vector space.

Definition 2.5. (Cauchy sequence)
A sequence $\{\phi_n\}$ in a normed space$(\mathbf{V}, \|.\|)$ is Cauchy sequence if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n, m \geq N : \|\phi_n - \phi_m\| < \varepsilon \tag{2.1.5}$$

It is straightforward to prove that every convergent sequence is a Cauchy sequence; however, the converse is not always true. For example, in the normed space $(\mathbb{Q}, |\cdot|)$, where $|\cdot|$ denotes the absolute value, the sequence $\{x_n\}$ defined by $x_1 = \frac{3}{2}$ and

$$x_{n+1} = \frac{4 + 3x_n}{3 + 2x_n}$$

is a Cauchy sequence. Nevertheless, it converges to $\sqrt{2} \notin \mathbb{Q}$, showing that not all Cauchy sequences converge in $\mathbb{Q}$. Therefore, in the sense of Cauchy sequence $(\mathbb{Q}, |\cdot|)$ is not a complete space.

Definition 2.6. (Banach space)
A normed space is said to be complete if every Cauchy sequence in it is convergent. A complete normed vector space is called a Banach space.

The simplest example of Banach space is $(\mathbb{R}^d, \|.\|_2)$ or more generally, $(\mathbb{R}^d, \|.\|_p)$, where $\|X\|_p = (\sum_{i=1}^{d} x_i^p)^{\frac{1}{p}}$. Another Example is $\ell^p(\mathbb{N})$,

$$\ell^p(\mathbb{N}) = \{a = \{a_j\} : \|a\|_p < \infty\}$$

where,

$$\|a\|_p = (\sum_{i=1}^{\infty} a_i^p)^{\frac{1}{p}}, 1 \leq p < \infty$$

We can add more structure to the vector space and define the notion of the Hilbert space.

Definition 2.7. (Hilbert Space)
A complex vector space $\mathcal{H}$ equipped with a sesquilinear inner product

$$\langle \cdot \mid \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

is called a *pre-Hilbert space* if it satisfies the following properties:

1. **Hermitian property:** $\langle \varphi \mid \psi \rangle = \overline{\langle \psi \mid \varphi \rangle}$.
2. **Linearity in the second argument:** $\langle \varphi \mid a\psi_1 + b\psi_2 \rangle = a\langle \varphi \mid \psi_1 \rangle + b\langle \varphi \mid \psi_2 \rangle$.
3. **Positive-definiteness:** $\langle \psi \mid \psi \rangle \geq 0$, with equality if and only if $\psi = 0_{\mathcal{H}}$.

for all $\varphi, \psi, \psi_1, \psi_2 \in \mathcal{H}$ and $a, b \in \mathbb{C}$.

If $\mathcal{H}$ is also complete with respect to the norm defined by

$$\|\psi\| \equiv \sqrt{\langle \psi \mid \psi \rangle}$$

then it is called a *Hilbert space.*
Later, we will use the Dirac notation $|\psi\rangle$ to denote vectors in $\mathcal{H}$.

It must be evident from the above definition that every Hilbert space is a Banach space, but not every Banach space is a Hilbert space. This is because not every norm induces an inner product. The example is space $\ell^p(\mathbb{N})$, these are Banach spaces but not Hilbert spaces until $p = 2$. The necessary and sufficient condition for a Banach space $\mathcal{B}$ to be a

Hilbert space is that its norm must satisfy the parallelogram identity.

$$\|\phi_1 + \phi_2\|^2 + \|\phi_1 - \phi_2\|^2 = 2\|\phi_1\|^2 + 2\|\phi_2\|^2$$

for all $\phi_1, \phi_2 \in \mathcal{B}$. A Hilbert space is a **separable Hilbert space** if there exists a countable basis (a countable set of vectors such that any vector in the space can be written as a linear combination from the set; this set of vectors is called the basis, and the cardinality of the set is the dimension of the Hilbert space). Any finite-dimensional Hilbert space is isomorphic to $\mathbb{C}^N$ for some $N \in \mathbb{N}$, and any infinite-dimensional separable Hilbert space is isomorphic to $\ell^2(\mathbb{N})$.

In a quantum system, Hilbert space plays a crucial role, as it is used to define the state of the system. In particular, pure states correspond to rays in the Hilbert space and thus naturally belong to the projective Hilbert space. It is commonly thought that pure states correspond to vectors in the Hilbert space, but this is not entirely correct. For example, consider a unit vector $\phi_1 \in \mathcal{H}$. The vector $e^{i\alpha}\phi_1$, where α is a real number, is a different vector in the Hilbert space $\mathcal{H}$, yet in quantum mechanics, they represent the same physical state. Therefore, unit vectors in Hilbert space do not correspond directly to physical states; rather, pure states are described by equivalence classes of vectors differing by a global phase.

Definition 2.8. (Direct sum of Hilbert space)
Let $\mathcal{H}_i$ for $i = \{1, .., N\}$ be the Hilbert space, then the direct sum Hilbert space is denoted by $\oplus_{i=1}^N \mathcal{H}_i$ and defined as,

$$\oplus_{i=1}^N \mathcal{H}_i = \{(\phi_1, ..., \phi_N) : \phi_i \in \mathcal{H}_i\}$$

where the inner product over the direct sum Hilbert space is defined as,

$$\langle(\phi_1, ..., \phi_N)|(\psi_1, ..., \psi_N)\rangle_{\oplus_{i=1}^N \mathcal{H}} = \sum_{i=1}^N \langle\phi_i|\psi_i\rangle_{\mathcal{H}_i}$$

if $N \to \infty$, then we must also require that for each $(\phi_1, ..., \phi_N) \in \oplus_{i=1}^N \mathcal{H}_i$,

$$\sum_{i=0}^\infty \|\phi_i\|^2_{\mathcal{H}_i} < \infty$$

Further, if all the Hilbert space $\mathcal{H}_i = \mathcal{H}$, then we write $\oplus_{i=1}^N \mathcal{H}_i = \mathcal{H}^{\oplus N}$.

In the direct sum of Hilbert spaces, the vectors of two different Hilbert spaces are orthogonal. Therefore, the dimension of the direct sum Hilbert space is the sum of the dimensions of each Hilbert space in the direct sum.

Definition 2.9. (Direct product of Hilbert space)
Let $\mathcal{H}_i$ for $i = \{1, .., N\}$ be the Hilbert space, then the direct product Hilbert space is denoted by $\otimes_{i=1}^{N}\mathcal{H}_i$ and defined as a completion,

$$\otimes_{i=1}^{N}\mathcal{H}_i = \{(\phi_1, ..., \phi_N) : \phi_i \in \mathcal{H}_i\}$$

With respect to the inner product over the direct product of Hilbert spaces,

$$\langle(\phi_1, ..., \phi_N)|(\psi_1, ..., \psi_N)\rangle_{\otimes_{i=1}^{N}\mathcal{H}} = \prod_{i=1}^{N}\langle\phi_i|\psi_i\rangle_{\mathcal{H}_i}$$

Further, if all the Hilbert space $\mathcal{H}_i = \mathcal{H}$, then we write $\otimes_{i=1}^{N}\mathcal{H}_i = \mathcal{H}^{\otimes N}$.

The direct product of Hilbert spaces naturally arises in the quantum mechanics of multiparticle systems. An important application of direct sums and direct product Hilbert spaces is that they allow us to define the Fock space.

Definition 2.10. (Fock space)
Let $\mathcal{H}$ be the Hilbert space, then we can define the Fock space associated with $\mathcal{H}$ as a Hilbert space,

$$\mathcal{F}(\mathcal{H}) = \oplus_{n=0}^{\infty}\ \mathcal{H}^{\otimes n}$$

where $\otimes^0\mathcal{H} = \mathbb{C}$.

The Fock space is used as the state space in quantum field theory, which describes systems with an indefinite number of particles. Having established the essential structure for understanding the state space of a quantum system, we now move toward the study of operators in quantum systems. Let us introduce a few more concepts and definitions that will be needed for the discussion of von Neumann algebras.

Definition 2.11. (Banach Algebra)
A *Banach algebra* $\mathcal{B}$ is a Banach space equipped with a bilinear map

$$\mu : \mathcal{B} \times \mathcal{B} \to \mathcal{B}, \quad (f, g) \mapsto fg,$$

such that the multiplication is associative and satisfies the submultiplicative norm condition:

$$\|fg\| \leq \|f\| \, \|g\|, \quad \forall f, g \in \mathcal{B}.$$

Banach algebras play a crucial role in quantum mechanics by providing a robust mathematical framework for analyzing operators on quantum systems. They facilitate the study of spectral properties and allow physical observables to be represented as elements of the algebra, enabling us to study the quantum dynamics.

Definition 2.12. (Banach * algebra)
A Banach *-algebra $\mathcal{A}$ is a Banach algebra over complex numbers equipped with a map

$$* : \mathcal{A} \to \mathcal{A}$$

satisfying the following properties:

1. $(A^*)^* = A, \forall A \in \mathcal{A}$
2. $(AB)^* = B^* A^*, \forall A, B \in \mathcal{A}$
3. $(A + B)^* = A^* + B^* \; \forall A, B \in \mathcal{A}$
4. $(zA)^* = \bar{z} A^*$, for all $z \in \mathbb{C}$, for every $A \in \mathcal{A}$.
5. $\|A^*\| = \|A\|, \forall A \in \mathcal{A}$

If there exists a unit element $I \in \mathcal{A}$ such that $AI = IA = A$ for all $A \in \mathcal{A}$, then $\mathcal{A}$ is a unital Banach *-algebra. We can define an important structure called $C*$ algebra using this Banach* algebra.

Definition 2.13. (C * algebra)
A C^*- algebra is a special type of Banach*-algebra where the norm satisfies the following property.

$$\|A^* A\| = \|A^*\| \|A\| = \|A\|^2$$

In quantum mechanics, observables are represented as self-adjoint operators on a Hilbert space. C* algebras provide a natural framework for studying these operators, especially in the algebraic formulation of quantum mechanics. In algebraic quantum field theory (AQFT), local algebras of observables are modeled using C*-algebras and von Neumann algebras.

Definition 2.14. (Linear Operator)
A linear operator is a linear map $\mathbf{A} : \mathcal{D}_A \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space and $\mathcal{D}_A \subseteq \mathcal{H}$ is the domain of the operator, satisfying:

$$\mathbf{A}(\alpha|\psi\rangle + \beta|\varphi\rangle) = \alpha\mathbf{A}|\psi\rangle + \beta\mathbf{A}|\varphi\rangle$$

for all $|\psi\rangle, |\varphi\rangle \in \mathcal{D}_A$ and $\alpha, \beta \in \mathbb{C}$.

As we know, observables in quantum mechanics are self-adjoint linear operators. While we have defined linear operators, we still need to define self-adjoint operators. To do so, we must first define the adjoint of a linear operator. However, before that, we need to introduce the concept of densely defined operators, as only densely defined operators have adjoints.

Definition 2.15. (Densely Defined Operator)
A linear operator is said to be *densely defined* if its domain $\mathcal{D}_A$ is dense in the Hilbert space $\mathcal{H}$.
More precisely, $\mathcal{D}_A$ is dense in $\mathcal{H}$ if for every $|\psi\rangle \in \mathcal{H}$ and every $\varepsilon > 0$, there exists $|\varphi\rangle \in \mathcal{D}_A$ such that

$$\||\varphi\rangle - |\psi\rangle\| < \varepsilon.$$

In simple terms, a dense subspace of a Hilbert space is a subspace such that any vector in the Hilbert space can be approximated arbitrarily well by vectors from it. More precisely, for any vector $|\psi\rangle \in \mathcal{H}$, there exists a sequence $\{|\varphi_n\rangle\} \subset \mathcal{D}_A$ that converges to $|\psi\rangle$. This convergence is guaranteed by the completeness of the Hilbert space. This property is particularly useful in physics: to prove a statement about an operator, it is often sufficient to establish it on a dense subspace of the Hilbert space.

Definition 2.16. (Adjoint of an Operator)
Let $\mathbf{A}$ be densely defined linear operator, $\mathbf{A} : \mathcal{D}_A \to \mathcal{H}$, the adjoint of the operator is a map $\mathbf{A}^\dagger : \mathcal{D}_{A^\dagger} \to \mathcal{H}$, such that,

$$\mathcal{D}_{A^\dagger} = \{|\psi\rangle \in \mathcal{H} : \forall\ |\varphi\rangle \in \mathcal{D}_A, \exists\ |\chi\rangle \in \mathcal{H} : \langle\psi|\mathbf{A}|\varphi\rangle = \langle\chi|\varphi\rangle\}$$

and

$$A^\dagger|\psi\rangle = |\chi\rangle$$

It can easily be shown by the properties of the inner product that the adjoint is well defined, i.e., there exists a unique $|\chi\rangle$ in the above definition. Further, we would like to emphasize that to define an operator, we need to give its domain and its action on each element in the domain. Therefore, if we have two operators $\mathbf{A}$ and $\mathbf{B}$, they are equal iff $\mathcal{D}_A = \mathcal{D}_B$ and $\mathbf{A}|\psi\rangle = \mathbf{B}|\psi\rangle$ for all $|\psi\rangle \in \mathcal{H}$. Now, we can define a self-adjoint operator,

Definition 2.17. (Self adjoint Operator)
A densely defined operator $\mathbf{A} : \mathcal{D}_A \to \mathcal{H}$, is self adjoint if it equals to its adjoint $\mathbf{A}^\dagger : \mathcal{D}_{A^\dagger} \to \mathcal{H}$, that is

1. The domain must coincide: $\mathcal{D}_A = \mathcal{D}_{A^\dagger}$.
2. They should have same action $\forall|\varphi\rangle \in \mathcal{D}_A$: $\mathbf{A}|\varphi\rangle = \mathbf{A}^\dagger|\varphi\rangle$

In quantum theory, self-adjoint operators are of prime importance. The observables in quantum theory, such as the momentum operator in the quantum harmonic oscillator, are self-adjoint operators. As we can see from the definition, self-adjointness requires the domains of the operator and its adjoint to coincide, which is a highly non-trivial condition. Therefore, nothing guarantees *a priori* that an operator is self-adjoint. Sometimes, a symmetric operator is mistaken for a self-adjoint operator.

First, let us define a symmetric operator and then examine whether the momentum operator in quantum mechanics is always self-adjoint.

Definition 2.18. (Symmetric opertor)
A densely defined operator $\mathbf{A} : \mathcal{D}_A \to \mathcal{H}$, is symmetric if

$$\forall\ |\psi\rangle, |\varphi\rangle \in \mathcal{D}_A; \langle\psi|\mathbf{A}|\varphi\rangle = \langle\mathbf{A}\psi|\varphi\rangle$$

As should be clear from the definition, every self-adjoint operator is symmetric, but not every symmetric operator is self-adjoint.

Now let us consider the momentum operator, which is defined by

$$P : \mathcal{D}_P \to \mathcal{L}^2([0,1]),$$

where $\mathcal{L}^2([0,1])$ is the space of square-integrable functions on the interval $[0,1]$.

We choose the domain $\mathcal{D}_P := C^1([0,1]) \subset \mathcal{L}^2([0,1])$ such that

$$\psi(0) = \psi(1) = 0, \quad \psi(x) \in C^1([0,1]).$$

Although it is easy to show that P is symmetric using integration by parts and the boundary conditions, it is generally difficult to determine whether $\mathcal{D}_{P^*} = \mathcal{D}_P$. It can be proven (a proof can be found in [20]) that the domain of the adjoint, $\mathcal{D}_{P^*}$, is larger than the domain $\mathcal{D}_P$. This is a very general property of symmetric operators, as we will see. Additionally, one would like to determine whether a self-adjoint extension exists. But first, we need to define the extension of an operator.

Definition 2.19. (Extension of an operator)
Let $\mathbf{A} : \mathcal{D}_A \to \mathcal{H}$ and $\mathbf{B} : \mathcal{D}_B \to \mathcal{H}$ be are linear operators; we say $\mathbf{B}$ is an extension of $\mathbf{A}$ and denote it by $\mathbf{A} \subseteq \mathbf{B}$, if

1. Domain of $\mathbf{A}$ is contained in domain of $\mathbf{B}$: $\mathcal{D}_A \subseteq \mathcal{D}_B$.
2. They should have the same action on the intersection of their domain: $\forall |\psi\rangle \in \mathcal{D}_A, A|\psi\rangle = B|\psi\rangle$.

Proposition 2.20. *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are densely defined and $\boldsymbol{A} \subseteq \boldsymbol{B}$, then $\boldsymbol{B}^\dagger \subseteq \boldsymbol{A}^\dagger$.*

Proof. Let $|\psi\rangle \in \mathcal{D}_{B^\dagger}$,then there exists $|\varphi\rangle \in \mathcal{H}$, such that

$$\forall\ |\chi\rangle \in \mathcal{D}_B : \langle\psi|\mathbf{B}|\chi\rangle = \langle\varphi|\chi\rangle$$

Since $\mathbf{A} \subseteq \mathbf{B}$, we have $\mathcal{D}_A \subseteq \mathcal{D}_B$ and therefore

$$\forall\ |\eta\rangle \in \mathcal{D}_A \subseteq \mathcal{D}_B, \langle\psi|\mathbf{B}|\eta\rangle = \langle\psi|\mathbf{A}|\eta\rangle = \langle\varphi|\eta\rangle$$

Implies, $|\psi\rangle \in \mathcal{D}_{A^\dagger}$ and hence, $\mathcal{D}_{B^\dagger} \subseteq \mathcal{D}_{A^\dagger}$. Further, $\mathbf{B}^\dagger|\psi\rangle = \mathbf{A}^\dagger|\psi\rangle = |\varphi\rangle$.

$$\implies \mathbf{B}^\dagger \subseteq \mathbf{A}^\dagger$$

□

Proposition 2.21. *If $\boldsymbol{A}$ is symmetric operator, then $\boldsymbol{A} \subseteq \boldsymbol{A}^\dagger$.*

Proof. Let $|\psi\rangle \in \mathcal{D}_A$ and $|\eta\rangle = \mathbf{A}|\psi\rangle$ some element in $\mathcal{H}$, then by symmetric property,

$$\forall|\varphi\rangle \in \mathcal{D}_A, \langle\psi|\mathbf{A}|\varphi\rangle = \langle\eta|\varphi\rangle$$

implies $|\psi\rangle \in \mathcal{D}_{A^\dagger}$, therefore $\mathcal{D}_A \subseteq \mathcal{D}_{A^\dagger}$. Further, $\forall\psi \in \mathcal{D}_A, \mathbf{A}^\dagger|\psi\rangle = \mathbf{A}|\psi\rangle$.

$$\implies \mathbf{A} \subseteq \mathbf{A}^\dagger$$

□

The above proposition is very important, as it tells us that a self-adjoint operator does not have any self-adjoint extension. Let $\mathbf{A}$ and $\mathbf{B}$ are self adjoint operators and $\mathbf{A} \subseteq \mathbf{B}$. Therefore, $\mathbf{B} = \mathbf{B}^\dagger \subseteq \mathbf{A}^\dagger = \mathbf{A} \subseteq \mathbf{B}$, implies $\mathbf{B} = \mathbf{A}$.
In quantum mechanics, we have a Hamiltonian, which is constructed by adding and multiplying different operators. Therefore, it is natural to ask whether we can add and multiply any two operators. In general, the answer is no. If we have two operators $\mathbf{A}$ and $\mathbf{B}$ with domains $\mathcal{D}_A$ and $\mathcal{D}_B$, respectively, then

1. The domain of $\mathbf{A} + \mathbf{B}$ is $\mathcal{D}_{A+B} = \mathcal{D}_A \cap \mathcal{D}_B$.
2. The domain of $\mathbf{A}.\mathbf{B}$ is $\mathcal{D}_{AB} = \{|\psi\rangle \in \mathcal{D}_B : \mathbf{B}|\psi\rangle \in \mathcal{D}_A\}$.
3. The domain of $\lambda\mathbf{A}$, where $\lambda \in \mathbb{C}$ is $\mathcal{D}_{\lambda A} = \mathcal{D}_A$
4. The domain of inverse operator $\mathbf{A}^{-1}$ is $\mathcal{D}_{A^{-1}} = \mathbf{A}\mathcal{D}_A$
5. If $\mathbf{A}^{-1}$ is bounded then $\mathbf{A}^{-1}\mathbf{A} \subset \mathbf{A}\mathbf{A}^{-}1$

I want to emphasize that, up to this point, all the discussion has been about general linear operators. We would also like to introduce the concept of the graph of an operator, which will be useful later in understanding the closed operator.

Definition 2.22. (Graph of an Operator)
The graph $\mathcal{G}(\mathbf{A})$ of a linear operator $\mathbf{A} : \mathcal{D}_A \subseteq \mathcal{H} \to \mathcal{H}$ is subspace of $\mathcal{H} \oplus \mathcal{H}$ defined as,

$$\mathcal{G}(\mathbf{A}) = \{(|\psi\rangle, \mathbf{A}|\psi\rangle) \in \mathcal{H} \oplus \mathcal{H} \mid |\psi\rangle \in \mathcal{D}_A\}$$

The above definition is simply a generalization of the definition of the graph of a function. For any function $f : (a, b) \subset \mathbb{R} \to \mathbb{R}$, the graph is a subset of $\mathbb{R} \oplus \mathbb{R}$ (or the Cartesian product $\mathbb{R} \times \mathbb{R}$), where each element is represented by an ordered pair $(x, f(x))$ with $x \in (a, b)$. It must be clear that the graph of an operator nicely captures all the information about the action of an operator in its domain. It is particularly helpful in studying the extension of an operator in an infinite-dimensional Hilbert space.

2.1.1 Bounded Operators on Hilbert space

In the previous subsection, we were very careful with the domain of the operators and consistently considered it to be a subspace of the Hilbert space. The reason is simple: not all operators have a well-defined action on every vector in the Hilbert space. Operators that do are called bounded operators, as we will see.

Definition 2.23. (Bounded Operator)
A linear operator $\mathbf{A} : \mathcal{H} \to \mathcal{H}$ is bounded iff $\exists\ C > 0$ such that for all $|\psi\rangle \in \mathcal{H}$, we have,

$$\|\mathbf{A}|\psi\rangle\| \leq C\||\psi\rangle\|$$

From the above definition for a bounded operator, $\|\mathbf{A}|\psi\rangle\| \leq C\||\psi\rangle\|$ or equivalently $\frac{\|\mathbf{A}|\psi\rangle\|}{\||\psi\rangle\|} \leq C$, here $|\psi\rangle$ non-zero vector in $\mathcal{H}$ as for a zero vector, the inequality is trivial. Further, using the linearity of the norm, the inequality is the same as $\|\mathbf{A}|\psi\rangle\| \leq C$ for all unit vectors $|\psi\rangle \in \mathcal{H}$. Furthermore, it tells us that $C = \sup_{\||\psi\rangle\|=1} \|\mathbf{A}|\psi\rangle\|$, where sup is the superimum or least upper bound. This motivates the definition of operator norm.

Definition 2.24. (Operator norm)
The norm of a linear operator $\mathbf{A} : \mathcal{H} \to \mathcal{H}$ is denoted by $\|\mathbf{A}\|$ and defined as

$$\|\mathbf{A}\| = \sup_{\||\psi\rangle\|=1} \|\mathbf{A}|\psi\rangle\|$$

for $|\psi\rangle \in \mathcal{H}$.

therefore, for a bounded operator $\|\mathbf{A}\| < \infty$. One thing that you might be thinking about is why I have taken the domain of a bounded operator as full $\mathcal{H}$. The reason is,

Theorem 2.1.1. **Hahn Banach Theorem**
Let $\mathbf{A} : \mathcal{D}_A \subset \mathcal{H} \to \mathcal{H}$ is linear bounded operator densely define on $\mathcal{H}$ with an operator norm $\|\mathbf{A}\| = C$, then there exists a continuous extension $\hat{\mathbf{A}} : \mathcal{H} \to \mathcal{H}$, such that $\hat{\mathbf{A}}|_{\mathcal{D}_A} = \mathbf{A}$ and $\|\hat{\mathbf{A}}\| = C$.

We will not prove the theorem here, but it can be found in [21]. The theorem states that every bounded operator defined on a dense subspace of $\mathcal{H}$ can be uniquely extended to an operator defined on the entire $\mathcal{H}$ while preserving the same operator norm. Therefore, when working with a bounded operator, we can take its domain to be $\mathcal{H}$.
The bounded operators have many nice properties. Let $\mathcal{B}(\mathcal{H})$ be the collection of all bounded operators on the Hilbert space $\mathcal{H}$. Then

1. Adjoint of a bounded operator is bounded, i.e., $\|\mathbf{A}\| < \infty \implies \|\mathbf{A}^\dagger\| < \infty$.
2. Using (1) and the fact that bounded operators are well-defined on full Hilbert space,
   - $(\mathbf{A} + \mathbf{B})^\dagger = \mathbf{A}^\dagger + \mathbf{B}^\dagger$
   - $(\lambda\mathbf{A})^\dagger = \bar{\lambda}\mathbf{A}^\dagger$
   - $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger$
   - $\mathbf{A}^{\dagger\dagger} = \mathbf{A}$

   for $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\mathcal{H})$.
3. Further using properties of the norm,
   - $\|\alpha\mathbf{A}\| = |\alpha|\|\mathbf{A}\|$
   - $\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$
   - $\|\mathbf{AB}\| \leq \|\mathbf{A}\|\|\mathbf{B}\|$

   for $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\mathcal{H})$ and $\alpha \in \mathbb{C}$
4. $\mathbf{A}^\dagger\mathbf{A}$ for $\mathbf{A} \in \mathcal{B}(\mathcal{H})$ is a positive operator,i.e $\forall|\psi\rangle \in \mathcal{H}, \langle\psi|\mathbf{A}^\dagger\mathbf{A}|\psi\rangle \geq 0$
5. $\mathcal{B}(\mathcal{H})$ is a Banach * algebra.
6. Every bounded symmetric operator is self-adjoint.

Above, we have listed some important and useful properties of bounded operators. The first property is straightforward: since $\mathbf{A}$ is bounded, it is defined on the entire Hilbert space (if initially defined on a subspace, the **Hahn-Banach theorem** guarantees the existence of a unique extension to the full Hilbert space $\mathcal{H}$). Therefore, $\mathcal{D}_A$ is dense and $\mathbf{A}^\dagger$ exists. It follows from the Reisz representation theorem (linear isomorphism of H and its dual) that $\mathcal{D}_{A^\dagger} = \mathcal{H}$.

Next, it can be easily shown that $\|\mathbf{A}^\dagger\| = \|\mathbf{A}\|$, which implies that $\mathbf{A}^\dagger$ is bounded. Properties (2) and (3) can be proven straightforwardly. The fourth property follows from the fact that the norm of any state is positive when applied to state $|\chi\rangle = \mathbf{A}|\psi\rangle$. The proof of the fifth property can be found in any standard text on functional analysis; for example, see Rudin [21]. The sixth property follows from (1) and the symmetry of the operator. We would also like to mention that bounded operators are continuous, i.e, for any sequence $\{|\psi_n\rangle\} \to |\psi\rangle$, the sequence $\{\mathbf{A}|\psi_n\rangle\} \to \mathbf{A}|\psi\rangle$. This follows from the fact that

$$\|\mathbf{A}|\psi_n\rangle - \mathbf{A}|\psi\rangle\| = \|\mathbf{A}(|\psi_n\rangle - |\psi\rangle)\| \leq C\||\psi_n\rangle - |\psi\rangle\| \tag{2.1.6}$$

and therefore $\{|\psi_n\rangle\} \to |\psi\rangle$ implies $\{\mathbf{A}|\psi_n\rangle\} \to \mathbf{A}|\psi\rangle$. Now, what will happen if the operator is not a bounded operator?

2.1.2 Unbounded operator on Hilbert space

All operators that are not bounded are called unbounded operators. This means that for an unbounded operator, there always exists an element in $\mathcal{H}$ that is not mapped back to $\mathcal{H}$ by the operator. Hence, an unbounded operator can only be defined on the subspace of the Hilbert space.

Definition 2.25. (Unbounded Operator)
A linear operator $\mathbf{A} : \mathcal{D}_A \to \mathcal{H}$ is unbounded if it is not bounded or Equivalently $\forall C > 0$ there exists a $|\psi\rangle \in \mathcal{D}_A$, such that,

$$\|\mathbf{A}|\psi\rangle\| \geq C\||\psi\rangle\|$$

The above definition tells us that for an unbounded operator, $\|\mathbf{A}|\psi\rangle\| \leq C\||\psi\rangle\|$ for some vector $|\psi\rangle$, but there exists at least one element $|\psi\rangle$ in its domain such that $\|\mathbf{A}|\psi\rangle\| \geq C\||\psi\rangle\|$ for all $C > 0$.

We want to emphasize that, in quantum mechanics, we often work with unbounded operators. For example, the momentum operator and the position operator in quantum mechanics are unbounded, or at least one of them must be unbounded.

Let us prove this statement. We know that

$$[\mathbf{q}, \mathbf{p}] = i\hbar$$

and, more generally, for $n \in \mathbb{Z}^+$,

$$[\mathbf{q}, \mathbf{p}^n] = i\hbar n \mathbf{p}^{n-1}.$$

Now, taking the norm on both sides and using the fact that

$$\|n\hbar \mathbf{P}^{n-1}\| = \|[\mathbf{q}, \mathbf{p}^n]\| \leq 2\|\mathbf{q}\|\|\mathbf{p}\|\|\mathbf{p}^{n-1}\|,$$

we obtain

$$\|\mathbf{p}^{n-1}\|\|\mathbf{q}\|\|\mathbf{p}\| \geq \frac{n\hbar}{2}\|\mathbf{p}^{n-1}\|$$

Since n can be any integer, this inequality implies that both $\mathbf{q}$ and $\mathbf{p}$ cannot be bounded operators.

Another example of an unbounded operator is the number operator $\mathcal{N} = \mathbf{a}^\dagger \mathbf{a}$ in the quantum harmonic oscillator. Let $|n\rangle$ be the nth occupation state in Fock space, where the number operator acts as

$$\mathcal{N}|n\rangle = n|n\rangle.$$

Hence,

$$\|\mathcal{N}|n\rangle\| = n^2.$$

Since for any constant $C > 0$, there exists an $n \in \mathbb{Z}^+$ such that $n^2 > C$, it follows that $\mathcal{N}$ is unbounded.

Let us go one step further and determine the domain of $\mathcal{N}$. The first thing to notice is that the domain of $\mathcal{N}$ is not all of $\ell^2(\mathbb{C})$. For any state $|\psi\rangle \in \ell^2(\mathbb{C})$, we can write

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle,$$

where the coefficients satisfy

$$\sum_{n=0}^{\infty} |c_n|^2 < \infty.$$

However, this condition does not necessarily imply that

$$\sum_{n=0}^{\infty} n^2 |c_n|^2 < \infty.$$

For example, choosing $c_n = \frac{1}{n}$, we see that $\sum_{n=0}^{\infty} |c_n|^2 < \infty$, but

$$\sum_{n=0}^{\infty} n^2 |c_n|^2 = \sum_{n=0}^{\infty} \frac{n^2}{n^2} = \sum_{n=0}^{\infty} 1 = \infty.$$

Therefore, the domain of $\mathcal{N}$ is not $\ell^2(\mathbb{C})$ but rather

$$\mathcal{D}_N = \left\{ |\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \in \ell^2(\mathbb{C}) : \sum_{n=0}^{\infty} n^2 |c_n|^2 < \infty \right\}.$$

Therefore, $\mathcal{D}_N$ is a subset of $\ell^2(\mathbb{C})$. Furthermore, $\mathcal{D}_N$ is dense in $\ell^2(\mathbb{C})$, since $|n\rangle \in \mathcal{D}_N$ for any n, and there is no nontrivial vector in $\ell^2(\mathbb{C})$ that is orthogonal to $\mathcal{D}_N$, i.e $\mathcal{D}_N^{\perp} = \{0_\ell\}$. Therefore, $\overline{\mathcal{D}_N} = \ell^2(\mathbb{C})$.
Having established that quantum mechanics necessarily involves unbounded operators, let us now list some important properties of unbounded operators.

Let $\mathbf{A}$ be some densely defined unbounded operator and $\mathbf{B}$ is some bounded operator, then

- $(\mathbf{A} + \mathbf{B})^\dagger = \mathbf{A}^\dagger + \mathbf{B}^\dagger$
- $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger \mathbf{A}^\dagger$

If $\mathbf{A}$ and $\mathbf{B}$ both are densely defined unbounded operators, then it can be verified that,

1. $(\mathbf{A} + \mathbf{B})^\dagger \supset \mathbf{A}^\dagger + \mathbf{B}^\dagger$
2. $(\mathbf{AB})^\dagger \supset \mathbf{B}^\dagger \mathbf{A}^\dagger$

The way to verify the above statements is to use the definition of the adjoint and then prove that the domain of the operator on the right-hand side is contained in the domain of the operator on the left-hand side.
Unlike bounded operators, unbounded operators do not form an algebra or even a linear vector space. This is solely because each operator comes with its own domain. Furthermore, unbounded operators are not continuous. In fact, one can prove that an operator is continuous if and only if it is bounded. We have already proved that bounded operators

are continuous (2.1.6). Although we will not prove the reverse statement, it can be easily shown using linearity and the fact that continuity requires every open set in an image to have an inverse image as an open set in the preimage; for a full proof, see Rudin [21]. However, there is a generalization of continuity to a class of unbounded operators called closed operators.

Definition 2.26. (Closed and Preclosed Operator)
A linear unbounded operator $\mathbf{A} : \mathcal{D}_A \subset \mathcal{H} \to \mathcal{H}$ is **closed** if its graph $\mathcal{G}(\mathbf{A})$ is closed under the norm (graph norm) induced by $\mathcal{H} \oplus \mathcal{H}$, i.e., if $\mathcal{G}(\mathbf{A})$ is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$.
Furthermore, an operator is **preclosed** if it admits an extension to a closed operator. If $\mathbf{A}$ is the preclosed operator, then we will denote its closure by $\bar{\mathbf{A}}$.

Let us explain what the above definition means. It essentially states that operator $\mathbf{A}$ is closed, if $\{|\varphi_n\rangle\}$ is a sequence in $\mathcal{D}_A$ such that $|\varphi_n\rangle \to |\varphi\rangle$ and $\mathbf{A}|\varphi_n\rangle \to |\psi\rangle$, then it follows that $|\psi\rangle = \mathbf{A}|\varphi\rangle$. It is exactly the definition of closed subspace applied to the graph.

We emphasize that this does not imply that for every convergent sequence $\{|\varphi_n\rangle\}$, the sequence $\{\mathbf{A}|\varphi_n\rangle\}$ necessarily converges. Instead, it tells us that if $\{\mathbf{A}|\varphi_n\rangle\}$ does converge, then it does so consistently, meaning that it satisfies $|\psi\rangle = \mathbf{A}|\varphi\rangle$. Let us list some useful properties related to closed and preclosed operators:

1. If $\mathbf{A}$ is densely defined operator then $\mathbf{A}^\dagger$ is closed.
2. $\mathbf{A}$ is densely defined preclosed operator iff $\mathbf{A}^\dagger$ is densely defined. In that case the closure $\bar{\mathbf{A}} = \mathbf{A}^{\dagger\dagger}$.
3. Every self-adjoint operator is closed.
4. If $\mathbf{A}$ is symmetric operator then, $\mathbf{A} \subseteq \bar{\mathbf{A}} \subseteq \mathbf{A}^\dagger$.

The proof of the above statements can be found in [chapter 13] of [21].

2.1.3 Spectral properties of an Operator

In this subsection, we delve into the concept of the spectrum of operators, a fundamental notion in quantum mechanics, where measurable quantities correspond to spectral values of operators. To build a rigorous foundation, we first introduce the inverse of an operator, which serves as a key tool in formally defining the spectrum.

Definition 2.27. (Invertible operator)
For every linear operator $\mathbf{A} : \mathcal{D}_A \to \mathcal{H}$, we can define kernel $ker(\mathbf{A})$ and range $ran(\mathbf{A})$ as,

$$ker(\mathbf{A}) = \{|\psi\rangle \in \mathcal{D}_A : \mathbf{A}|\psi\rangle = 0\}, \quad ran(\mathbf{A}) = \{\mathbf{A}|\psi\rangle : |\psi\rangle \in \mathcal{D}_A\}$$

$\mathbf{A}$ is said to be invertible iff,

$$ker(\mathbf{A}) = \{0\}, \quad \overline{ran(\mathbf{A})} = \mathcal{H}$$

Further we can define inverse operator $\mathbf{A}^{-1} : \mathbf{A}\mathcal{D}_A \to \mathcal{D}_A$ such that,

$$\mathbf{A}\mathbf{A}^{-1} = 1_{\mathbf{A}\mathcal{D}_A} \quad \mathbf{A}^{-1}\mathbf{A} = 1_{\mathcal{D}_A}$$

The above notion of invertibility is precisely a generalization of the invertibility of functions to operators. The reason we focus on the inverse of an operator is that, in the case of any matrix M (which acts on finite dimensional vector space), eigenvalues are determined by studying the kernel of $M - \lambda I$. We will now generalize it to operators on infinite dimensional Hilbert space.
Consider a densely defined linear operator $\mathbf{A} : \mathcal{D}_A \subset \mathcal{H} \to \mathcal{H}$, we can define a family of operators

$$\mathbf{A}_\lambda = \mathbf{A} - 1\lambda$$

for every $\lambda \in \mathbb{C}$. If $\mathbf{A}_\lambda$ is invertible then we can define a resolvent operator $\mathbf{R}_\lambda(\mathbf{A})$ as,

$$\mathbf{R}_\lambda(\mathbf{A}) = (\mathbf{A}_\lambda)^{-1} = (\mathbf{A} - 1\lambda)^{-1}$$

Further, we can define resolvent set $\rho(\mathbf{A})$ as set of all $\lambda \in \mathbb{C}$ such that,

1. $\mathbf{R}_\lambda(\mathbf{A})$ exists.
2. $\mathbf{R}_\lambda(\mathbf{A})$ is bounded.
3. $\mathbf{R}_\lambda(\mathbf{A})$ is densely defined in $\mathcal{H}$.

or equivalently,

$$\rho(\mathbf{A}) = \{\lambda \in \mathbb{C} : \mathbf{R}_\lambda(\mathbf{A}) \in \mathcal{B}(\mathcal{H})\}$$

Now, we can define the spectrum of an operator,

Definition 2.28. (Spectrum of an Operator)
The spectrum $\sigma(\mathbf{A})$ of an operator $\mathbf{A}$ is defined as the complement of resolvent set in $\mathbb{C}$,

$$\sigma(\mathbf{A}) = \mathbb{C} \setminus \rho(\mathbf{A})$$

We can further classify the spectrum into three disjoint set:

1. **Point spectrum $\sigma_p(\mathbf{A})$:** The point spectrum $\sigma_p(\mathbf{A}) \subset \sigma(\mathbf{A})$ such that $\forall\lambda \in \sigma_p(\mathbf{A})$, $\mathbf{R}_\lambda(\mathbf{A})$ does not exist. That is, there exists a non-trivial element $|\psi\rangle \in \mathcal{D}_A$ such that $(\mathbf{A} - 1\lambda)|\psi\rangle = 0$. This implies that λ is an eigenvalue and $|\psi\rangle$ is an eigenvector.
2. **Continous Spectrum $\sigma_c(\mathbf{A})$:** The continuous spectrum $\sigma_c(\mathbf{A}) \subset \sigma(\mathbf{A})$ such that $\forall\lambda \in \sigma_c(\mathbf{A})$, $\mathbf{R}_\lambda(\mathbf{A})$ exists and defined densely on $\mathcal{H}$, but is unbounded.
3. **Residual spectrum $\sigma_r(\mathbf{A})$:** It is set in $\sigma(\mathbf{A})$ such that $\mathbf{R}_\lambda(\mathbf{A})$ exist as an bounded or unbounded operator but $\mathbf{R}_\lambda(\mathbf{A})$ is not densely defined.

As it must be clear that

$$\sigma_p(\mathbf{A}) \cup \sigma_c(\mathbf{A}) \cup \sigma_r(\mathbf{A}) = \sigma(\mathbf{A})$$

and

$$\sigma_i(\mathbf{A}) \cap \sigma_j(\mathbf{A}) = \delta_{ij}\sigma_i(\mathbf{A}),$$

where $i, j \in \{p, c, r\}$. It is possible that some of these subsets of the spectrum are empty. For example, consider any linear operator $\mathbf{T} : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a finite-dimensional Hilbert space. If $\mathbf{T}$ is injective, then it is also bijective, implying that such operators have only a **point spectrum**, i.e.,

$$\sigma_p(\mathbf{A}) = \sigma(\mathbf{A})$$

Therefore, the usual intuition that the spectrum of an operator consists solely of its eigenvalues holds only in finite-dimensional spaces. In infinite-dimensional spaces, the spectrum may include not only eigenvalues (point spectrum) but also the continuous and residual spectrum.

Now, let us consider a **particle in a box** of length unity. In this case, the position operator

$$\hat{\mathbf{q}} : L^2([0,1]) \to L^2([0,1])$$

acts as

$$\hat{\mathbf{q}}\psi(q) = q\psi(q), \quad \text{for } \psi(q) \in L^2([0,1]).$$

Notice that in this case position operator is bounded, $\|\hat{\mathbf{q}}\psi(q)\| = \|q\psi(q)\| = |q|\|\psi\| < 1$, where $\psi(q)$ is unit vector. To determine the point spectrum, we look for solutions to the eigenvalue equation:

$$\hat{\mathbf{q}}\psi(q) = q\psi(q) = \lambda\psi(q).$$

Clearly, the equation $q\psi(q) = \lambda\psi(q)$ cannot be satisfied for all $q \in [0,1]$ unless $\psi(q)$ is identically zero. This implies that the position operator has no point spectrum. It also confirms that the position operator exists. Furthermore, it is evident that the range of the operator is the entire space $L^2([0,1])$. However, $\mathbf{R}_\lambda(\hat{\mathbf{q}})$ is not bounded everywhere, since for any vector, $\mathbf{R}_\lambda(\hat{\mathbf{q}})$ diverges at $q = \lambda$ if $\lambda \in [0,1]$. Therefore, the position operator has a continuous spectrum given by

$$\sigma(\hat{\mathbf{q}}) = [0,1]$$

Now, we would like to list here a few spectral properties of operators.

1. The spectrum $\sigma(\mathbf{A})$ of any bounded operator $\mathbf{A}$, is closed, bounded, non empty set of $\mathbb{C}$. Further, it is bounded by the $\|A\|$.

2. For the bounded operators, the spectrum is real iff it is self-adjoint.

3. The point spectrum of a self-adjoint operator is the eigenvalues.

4. The spectrum of any self-adjoint operator is a closed subset of $\mathbf{R}$.

5. For any self adjoint operator (bounded or unbounded) $\mathbf{A} : \mathcal{D}_A \to \mathcal{H}$, $\lambda \in \mathbb{R}$ belongs to the spectrum of $\mathbf{A}$ iff, $\exists$ a sequence $\{|\psi_n\rangle\}$ in $\mathcal{D}_A$, such that,

$$\lim_{n\to\infty} \frac{\|(\mathbf{A} - \lambda\mathbf{1})|\psi_n\rangle\|}{\||\psi_n\rangle\|} = 0$$

We will not prove these properties here, but the proof can be found in [20]. The first property follows directly from the definition of a bounded operator. Since the norm of a bounded operator exists and is given by the supremum over all expectation values of the operator, the spectrum must be bounded by the norm of the bounded operator.

The second and third properties are generalizations of what we already know for Hermitian finite-dimensional matrices. However, the second property has an interesting implication: if an operator has a real spectrum but is not self-adjoint, then it must be unbounded.

The fourth property states that even for an unbounded self-adjoint operator, the spectrum remains real and closed. Definitely it won't be bounded, because if it is bounded, then the operator is bounded.

The fifth property is particularly interesting as it introduces the notion of approximate eigenvectors and eigenvalues. It tells us that every element in the spectrum of a self-adjoint operator can approximately be considered an eigenvalue. This means there exists a sequence in the domain such that the sequential limit of the transformed sequence under the operator behaves as an eigenvalue equation in the limit. This can be seen in the above example of the position operator. We know that the Dirac delta function $\delta(q - \lambda)$ can act as an eigenfunction for the position operator. However, the Dirac delta function does not belong to the space of square-integrable functions. Instead, we can replace the Dirac delta function with a sequence of functions in $L^2([0, 1])$ that converges to the Dirac delta function. This allows us to interpret the continuous spectrum as an approximate eigenvalue spectrum.

Chapter 3

von Neumann Algebras for Physicist

This chapter aims to make the mathematical ideas of von Neumann algebras and operator algebras in general accessible to a broad physics audience without sacrificing much rigor. As outlined in the introduction, operator algebras—particularly von Neumann algebras—serve as powerful tools for exploring quantum systems, including quantum fields and gravity. Given that von Neumann originally developed these concepts to formalize quantum mechanics, it is natural that they have found extensive applications in physics. In recent years, von Neumann algebras have led to significant developments in quantum field theory and gravity. To ensure that the presentation is self-contained, we provide essential definitions, propositions, and theorems. Furthermore, this section serves as a reference for the mathematical concepts related to von Neumann algebras that will be used in later chapters.

3.1 Operator topologies

The point of this subsection is to introduce the definition of commonly used operator topologies. As we will see, the choice of topology plays a crucial role in defining different classes of operator algebras, with closure under various topologies leading to distinct algebraic structures. We will briefly discuss these topologies here, but the interested reader can look at [22, 23].

Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the space of bounded operators on $\mathcal{H}$. One may induce many interesting $\mathcal{B}(\mathcal{H})$ topologies. The most commonly used topologies are norm, ultrastrong, ultraweak, strong, and weak. The topologies of our interest are the norm, strong, and weak topologies.

Let $\{\mathbf{T}_\alpha\} \in \mathcal{B}(\mathcal{H})$ be the sequence of a bounded operator or more precesily net [1] of operators in $\mathcal{B}(\mathcal{H})$. Now we can define the operator topologies as,

- **Operator Norm(ON):** We say $\{\mathbf{T}_\alpha\} \to \mathbf{T}$ in norm topology iff,

$$\|\mathbf{T}_\alpha - \mathbf{T}\| \to 0. \tag{3.1.1}$$

  where $\|.\|$ is an operator norm defined in (2.24). In norm topology, an operator is small if its supremum norm is small.

- **Strong Operator(So):** We say $\{\mathbf{T}_\alpha\} \to \mathbf{T}$ in the strong topology if, for every $|\psi\rangle \in \mathcal{H}$, we have $(\mathbf{T} - \mathbf{T}_\alpha)|\psi\rangle \to 0$. In **So**, an operator is said to be small if its action on any state is small.

- **Weak Operator(Wo):** We say $\{\mathbf{T}_\alpha\} \to \mathbf{T}$ in the weak topology if, for every $|\psi\rangle \in \mathcal{H}$, we have $\langle\psi|(\mathbf{T} - \mathbf{T}_\alpha)|\psi\rangle \to 0$. In **Wo**, an operator is said to be small if its expectation value in any state is small.

- **Ultrastrong Operator(USo):** We say $\{\mathbf{T}_\alpha\} \to \mathbf{T}$ in the ultrastrong topology if, for every positive trace-class operator ρ on $\mathcal{H}$, we have $(\mathbf{T} - \mathbf{T}_\alpha)\rho(\mathbf{T} - \mathbf{T}_\alpha)^\dagger \to 0$ in the trace norm.[2] In **USo**, an operator is said to be small if the expectation value positive operator $(\mathbf{T} - \mathbf{T}_\alpha)^\dagger(\mathbf{T} - \mathbf{T}_\alpha)$ in any density matrix is small.

- **Ultraweak Operator(UWo):** We say $\{\mathbf{T}_\alpha\} \to \mathbf{T}$ in the ultraweak topology if, for every positive trace-class operator $\rho \in \mathcal{H}$, we have $\mathrm{tr}(\rho(\mathbf{T} - \mathbf{T}_\alpha)) \to 0$. In **UWo**, an operator is said to be small if its expectation value in any density matrix is small.

Notice that the convergence is measured with respect to a density matrix whenever we use "Ultra" topologies. In the case of non-ultra topologies, the convergence is measured in pure states. These operator topologies have interesting properties,

[1] A net is a generalization of a sequence; roughly speaking, a net can be thought of as a sequence where indexing is done by a directed set (a set which is preordered such that for any two elements there is a greater element under the preorder). Moreover, the reason to introduce net is that in general topological spaces, the convergence of sequences is not enough to completely determine the topology.

[2] Note that this is different from the condition $\mathbf{T}_\alpha\rho\mathbf{T}_\alpha^\dagger \to \mathbf{T}\rho\mathbf{T}^\dagger$. That condition, since it is not a function of the difference $\mathbf{T} - \mathbf{T}_\alpha$, is not compatible with the vector space structure of $\mathcal{B}(\mathcal{H})$.

1. If $\{\mathbf{T}_\alpha\} \to \mathbf{T}$ in the norm topology then $\mathbf{T}_\alpha \to \mathbf{T}$ in strong topology. If $\mathbf{T}_\alpha \to \mathbf{T}$ in strong topology, then $T_\alpha \to T$ in weak topology.

2. The convergence of net $\{\mathbf{T}_\alpha\} \to \mathbf{T}$ under different topologies follows the ordering,

$$\text{norm} \to \text{ultrastrong} \to \text{strong} \to \text{weak},$$

$$\text{norm} \to \text{ultrastrong} \to \text{ultraweak} \to \text{weak},$$

The first property follows directly from the fact that the operator norm is given by the supremum over $\|(\mathbf{T}_\alpha - \mathbf{T})|\psi\rangle\|$. This immediately implies that $\|\mathbf{T}_\alpha - \mathbf{T}\| \geq \|(\mathbf{T}_\alpha - \mathbf{T})|\psi\rangle\|$. Moreover, applying the Cauchy–Schwarz inequality, we obtain $|\langle\psi|(\mathbf{T}_\alpha - \mathbf{T})|\psi\rangle| \leq \|(\mathbf{T}_\alpha - \mathbf{T})|\psi\rangle\|$. A similar argument can be used to verify the second property. The ordering in operator topologies, as described above, implies that if $\{\mathbf{T}_\alpha\} \to \mathbf{T}$ in the norm topology, then the sequence also converges in all weaker topologies. However, the converse does not hold; convergence in a weaker topology does not necessarily imply convergence in the norm topology.
Notational convention: Just for notational convenience we will not write the operator in bold from here.

3.2 von Neumann algebra

Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a unital $*$-algebra (2.12). As defined earlier, if $\mathcal{A}$ is closed in the norm topology, it is called a C^*-algebra. However, if $\mathcal{A}$ is closed in the weak topology, it is called a von Neumann algebra.

Definition 3.1. (**von Neumann algebra**):
The algebra of operators $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra, if $\mathcal{A}$ is a unital $*$-algebra and is closed under the Wo topology.

From the above ordering of convergence of a net in different operator topologies, we observe that the norm topology is stronger than the weak operator topology (WOT), which means it has more open sets. Since a closed set is the complement of an open set, the closure of a set in the norm topology is smaller (or finer) than its closure in the weak operator topology. Therefore, the closure in a weaker topology (like WOT) is larger than the closure in a stronger topology (like norm topology). Therfore

$$\text{C* algebra} \supset \text{von Neumann algebra}$$

The above definition of von Neumann algebra is very abstract, but it turns out that one can give a simpler, more useful, and equivalent definition of von Neumann algebra using the notion of commutant of the algebra. In addition, it will give us an easier way to think about von Neumann algebras in physical systems. So, let us introduce the commutant of an algebra,

Definition 3.2. (**Commutant of algebra**):
Let $\mathcal{A}$ be any subset of bounded operators i.e $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, then commutant of $\mathcal{A}$ is denoted by $\mathcal{A}'$ and defined as,

$$\mathcal{A}' = \Big\{ a \in \mathcal{B}(\mathcal{H}) : \forall b \in \mathcal{A} \quad [a, b] = 0 \Big\} \tag{3.2.1}$$

Remark: $\mathcal{A}$ is any subset, need not be a von Neumann algebra.

Since $\mathcal{A}' \subset \mathcal{B}(\mathcal{H})$, we can also define the commutant of $\mathcal{A}'$, which we will call a double commutant and denote it as $\mathcal{A}''$.

$$\mathcal{A}'' = \Big\{ c \in \mathcal{B}(\mathcal{H}) : \forall d \in \mathcal{A}' \quad [c, d] = 0 \Big\} \tag{3.2.2}$$

From the above definition, it is easy to see that

$$\mathcal{A} \subset \mathcal{A}'' \tag{3.2.3}$$

Let $a \in \mathcal{A}$ be any bounded operator, then $\forall\ b \in \mathcal{A}'$ we have $[a, b] = 0$. From the above definition, $\mathcal{A}''$ is a collection of bounded operators that commute with all the elements of $\mathcal{A}'$, which implies $a \in \mathcal{A}''$. Thus, $\mathcal{A} \subset \mathcal{A}''$. Similarly, we can define higher commutants. We will denote $n+1$-th commutant of $\mathcal{A}$ as $\mathcal{A}^{(n+1)}$ and define it as,

$$\mathcal{A}^{(n+1)} = \Big\{ c \in \mathcal{B}(\mathcal{H}) : \forall d \in \mathcal{A}^{(n)} \quad [c, d] = 0 \Big\} \tag{3.2.4}$$

Now we want to prove that $\mathcal{A}' = \mathcal{A}'''$. Notice that (3.2.3) holds for any subset of $\mathcal{B}(\mathcal{H})$, which implies that $\mathcal{A}' \subset \mathcal{A}'''$.

Now, let $a \in \mathcal{A}'''$. By definition,

$$[a, \mathcal{A}''] = 0.$$

but, since $\mathcal{A} \subset \mathcal{A}''$

$$[a, \mathcal{A}] = 0.$$

This implies that $a \in \mathcal{A}'$, and hence, $\mathcal{A}''' \subset \mathcal{A}'$.

Since we have both inclusions, it follows that

$$\mathcal{A}' = \mathcal{A}'''. \tag{3.2.5}$$

We can again use the fact that $\mathcal{A}$ can be any set in $\mathcal{B}(\mathcal{H})$ and the above equation to show that,

- For any $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, $\mathcal{A}^{(2n+1)} = \mathcal{A}'$ $\forall n \in \mathbb{N}$.
- For any $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, $\mathcal{A}^{(2n+2)} = \mathcal{A}''$ $\forall n \in \mathbb{N}$.

Now our aim is to show that there is an equivalent definition of von Neumann algebra in terms of commutants. To set up an equivalent definition, we need to introduce the concepts of projection.

Definition 3.3. (Projection)
An operator, $\mathbf{P} \in \mathcal{B}(\mathcal{H})$, is said to be a projection iff

- $P = P^\dagger$ Adjointness
- $P^2 = P$

The *rank* of a projection P on a Hilbert space $\mathcal{H}$ is defined as the dimension of its range:

$$\mathrm{rank}(P) := \dim(\mathrm{ran}(P))$$

The set of all projections in $\mathcal{B}(\mathcal{H})$ is denoted as $\mathcal{P}_{\mathcal{B}(\mathcal{H})}$. It is clear that if P is projection then,

1. $1 - P$ is also a projection.

$$(1-P)^2 = 1 - P - P + P^2 = 1 - P$$

2. $P\mathcal{H}$ is a closed subspace of $\mathcal{H}$.

Proof. Let $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$. Then,

$$\alpha P|\psi\rangle + \beta P|\phi\rangle = P(\alpha|\psi\rangle + \beta|\phi\rangle) \in P\mathcal{H}.$$

So, $P\mathcal{H}$ is closed under linear combinations and hence is a subspace.

Now, let us show that $P\mathcal{H}$ is closed.

Let $\{P|\psi_n\rangle\} \subset P\mathcal{H}$ be a sequence such that

$$P|\psi_n\rangle \to |\xi\rangle \in \mathcal{H}.$$

We want to show that $|\xi\rangle \in P\mathcal{H}$, i.e., that there exists some $|\phi\rangle \in \mathcal{H}$ such that $|\xi\rangle = P|\phi\rangle$.

Since P is bounded and linear, we can write:

$$P(P|\psi_n\rangle) = P^2|\psi_n\rangle = P|\psi_n\rangle \to |\xi\rangle.$$

Also, by continuity of P, it follows that

$$P(P|\psi_n\rangle) \to P|\xi\rangle.$$

Therefore,

$$P|\xi\rangle = \lim_{n\to\infty} P(P|\psi_n\rangle) = \lim_{n\to\infty} P|\psi_n\rangle = |\xi\rangle.$$

Thus, $|\xi\rangle = P|\xi\rangle$, which shows that $|\xi\rangle \in \mathrm{Im}(P) = P\mathcal{H}$.

Hence, $P\mathcal{H}$ is closed. □

3. $\|P\| = 1 \implies P$ is a positive operator.

Further, Let $P_\alpha \in \mathcal{B}(\mathcal{H})$ be the set projections. We say they are orthogonal projection if $P_\alpha P_\beta = 0$ whenever $\alpha \neq \beta$. One simple example of orthogonal projection is P and $1 - P$. It can also be shown that for any closed subspace of $\mathcal{H}$ there exists a unique projection $P \in \mathcal{B}(\mathcal{H})$. It is also easy to show that

1. If $P_1, P_2 \in \mathcal{P}_{\mathcal{B}(\mathcal{H})}$, then $P_1 + P_2 \in \mathcal{P}_{\mathcal{B}(\mathcal{H})} \iff P_1P_2 = 0$.
2. If $P_1, P_2 \in \mathcal{P}_{\mathcal{B}(\mathcal{H})}$, then $P_1P_2 \in \mathcal{P}_{\mathcal{B}(\mathcal{H})} \iff P_1P_2 = P_2P_1$.
3. If $P_1, P_2 \in \mathcal{P}_{\mathcal{B}(\mathcal{H})}$ then $P_1 \leq P_2 \iff P_1P_2 = P_1 \iff P_1\mathcal{H} \subset P_2\mathcal{H}$

Let us prove the first property. Let $P_1 + P_2$ be a projection, then,

$$(P_1 + P_2) = (P_1 + P_2)^2 = P_1 + P_2 + (P_1P_2 + P_2P_1) \tag{3.2.6}$$

Implies, $P_1P_2 + P_2P_1 = 0$. But then

$$P_2P_1P_2 = -P_2P_1 = -P_1P_2$$

Implies, $P_2P_1 = P_1P_2 = 0$. Conversely, we know that $(P_1 + P_2)^\dagger = P_1 + P_2$ and if $P_2P_1 = P_1P_2 = 0$ then $(P_1 + P_2)^2 = P_1 + P_2 + (P_1P_2 + P_2P_1) = P_1 + P_2$.
To prove the second property, notice that if P_1P_2 is a projection, then $P_1P_2 = (P_1P_2)^\dagger = P_2P_1$. Conversely, if $P_2P_1 = P_1P_2$ then $(P_1P_2)^\dagger = P_2P_1 = P_1P_2$ and $(P_1P_2)^2 = P_1P_2P_1P_2 = P_1^2P_2^2 = P_1P_2$.
Now, let us prove the third property.

Assume that $P_1P_2 = P_1$. Since we know that $0 \leq P_1 \leq 1$, it follows that

$$P_2P_1P_2 \leq P_2$$

From this, we can deduce that $P_1 \leq P_2$.

Conversely, assume that $P_1 \leq P_2$. Then,

$$0 \leq (1 - P_2)P_1(1 - P_2) \leq (1 - P_2)P_2(1 - P_2) = 0$$

This implies that

$$(1 - P_2)P_1(1 - P_2) = 0,$$

which further leads to

$$P_1(1 - P_2) = 0$$

To prove that $P_1P_2 = P_1$ if and only if $P_1\mathcal{H} \subset P_2\mathcal{H}$, note that

$$P_1\mathcal{H} \subset P_2\mathcal{H} \iff (1 - P_2)\mathcal{H} \subset (1 - P_1)\mathcal{H} \iff P_1(1 - P_2)\mathcal{H} = 0 \iff P_1(1 - P_2) = 0.$$

This completes the proof of the third property.
We can also define projections onto the closure of the union and intersection of projections.

Let $\{P_i\}_{i=1}^N$ be any family of projection on $\mathcal{B}(\mathcal{H})$. Then, we define:

$$\bigvee_{i=1}^{N} P_i \equiv \text{the projection on the subspace} \overline{\sum_{i=1} P_i\mathcal{H}} \tag{3.2.7}$$

$$\bigwedge_{i=1}^{N} P_i \equiv \text{the projection on the subspace} \cap_{i=1} P_i\mathcal{H} \tag{3.2.8}$$

Now, we will use the properties of projections to prove von Neumann's bicommutant theorem.

Theorem 3.2.1. **(von-Neumann's bicommutant theorem):**
Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be the unital $*$-algebra of operators. Then the So- closure of $\mathcal{A}$ or equivalently Wo- closure of $\mathcal{A}$ coincides with the bicommutant $\mathcal{A}''$ of $\mathcal{A}$.

Proof. Let $a \in \mathcal{A}''$, we want to show that $a \in \bar{\mathcal{A}}$[3], i.e. for each $|\psi\rangle \in \mathcal{H}$ and any $\varepsilon > 0$, there exists $b \in \mathcal{A}$ with $\|(a-b)\,|\psi\rangle\| < \varepsilon$. This will imply that $\mathcal{A}'' \subset \bar{\mathcal{A}}$.
Let $|\psi\rangle \in \mathcal{H}$ be the fixed element, we can define the subspace $\mathcal{A}\,|\psi\rangle = \Big\{c\,|\psi\rangle : c \in \mathcal{A}\Big\}$. It is invariant under the action of any element of $\mathcal{A}$. Its closure $\overline{\mathcal{A}|\psi\rangle}$ in the norm of $\mathcal{H}$ is a closed linear subspace. We can associate the orthogonal projection $P : \mathcal{H} \to \overline{\mathcal{A}|\psi\rangle}$ onto the subspace. P is a bounded linear operator, and it can easily be shown that $P \in \mathcal{A}'$. Which we will prove as the next lemma.
Since $P \in \mathcal{A}'$, $bP = Pb$ for all $b \in \mathcal{A}$. Also, from the definition of the bicommutant, we must have $aP = Pa$. Since $\mathcal{A}$ is unital, $|\psi\rangle \in \mathcal{A}\,|\psi\rangle$, and so $|\psi\rangle = P\,|\psi\rangle$.
Hence $a\,|\psi\rangle = aP\,|\psi\rangle = Pa\,|\psi\rangle \in \overline{\mathcal{A}|\psi\rangle}$. So for each $\varepsilon > 0$, there exists b in $\mathcal{A}$ with $\|(a-b)\,|\psi\rangle\| < \varepsilon$. Therefore $\mathcal{A}'' \subset \bar{\mathcal{A}}$.
Now we want to show that $\bar{\mathcal{A}} \subset \mathcal{A}''$. We have already proved that $\mathcal{A} \subset \mathcal{A}''$.

Next, we note that the strong operator topology (So) is stronger than the weak operator topology (Wo). Consequently, closure in Wo is larger than closure in So. Therefore, it suffices to show that the weak closure of $\mathcal{A}$ is contained in $\mathcal{A}''$.

This follows from the fact that, under the weak operator topology, continuity ensures that for all $a \in \bar{\mathcal{A}}$, we have

$$[a, \mathcal{A}'] = 0.$$

This, in turn, implies that $\bar{\mathcal{A}} \subset \mathcal{A}''$. Therefore, we conclude that

$$\mathcal{A}'' = \bar{\mathcal{A}}.$$

[3] $\bar{\mathcal{A}}$ is a closure of $\mathcal{A}$

□

To complete the proof, we just have to prove that,

Lemma 3.4. *The orthogonal projector $P \in \mathcal{A}'$.*

Proof. Let $|\psi\rangle \in \mathcal{H}$ be the fixed element, Then $P|\psi\rangle \in \overline{\mathcal{AH}}$. Since $\overline{\mathcal{AH}}$ is closed subspace, there must exist the limiting sequence $\{A_n|\psi\rangle\}_n$ with $a_n \in \mathcal{A}$ which limits to $P|\psi\rangle$. Let $b \in \mathcal{A}$ then $ba_n|\psi\rangle \in \mathcal{AH}$, and therefore $bP|\psi\rangle \in \overline{\mathcal{AH}}$. Which implies $PbP|\psi\rangle = bP|\psi\rangle$. Since it holds for all $|\psi\rangle$, $PbP = bP$ for all $b \in \mathcal{A}$. Since the orthogonal projectors are self-adjoint operator, for any $|\psi\rangle, |\phi\rangle \in \mathcal{H}$,

$$\langle\phi|bP\psi\rangle = \langle\phi|PbP|\psi\rangle = \left\langle (PbP)^\dagger\phi \middle| \psi \right\rangle = \left\langle Pb^\dagger P\phi \middle| \psi \right\rangle = \left\langle b^\dagger P\phi \middle| \psi \right\rangle \tag{3.2.9}$$

$$\left\langle b^\dagger P\phi \middle| \psi \right\rangle = \langle P\phi|b\psi\rangle = \langle\phi|Pb\psi\rangle$$

Thus, $Pb = bP$ for $b \in \mathcal{A}$. Similarly, it can be shown that the orthogonal projector M on $\overline{\mathcal{A}'|\psi\rangle}$ for any $|\psi\rangle \in \mathcal{H}$ belong to $\mathcal{A}$, i.e $M \in \mathcal{A}$. □

This completes the proof of the von Neumann bicommutant theorem. So from von Neumann's bicommutant theorem,(the closure of $\mathcal{A}$ in WO)= $\mathcal{A}''$. This implies that we can replace the Wo closure condition with the equality condition of the algebra with its bicommutant in the definition of von Neumann algebra.

Definition 3.5. (von-Neumann algebra):
The algebra of operators $cA \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra. If $\mathcal{A}$ is a unital $*$-algebra, and it is equal to its bicommutant.

$$\mathcal{A} = \mathcal{A}'' \tag{3.2.10}$$

This is an extremely useful definition because, to check whether an operator belongs to a von Neumann algebra $\mathcal{A}$, it suffices to verify that it commutes with every operator in $\mathcal{A}'$. It is also useful in constructing von Neumann algebra, for example, if we have a unital *-algebra, we can get a von Neumann algebra by taking its double commutant. From now on, we will use the above definition of a von Neumann algebra.

Consider $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, which is not necessarily a von Neumann algebra. We previously established that for any subset $\mathcal{A}$,

$$\mathcal{A}' = \mathcal{A}'''.$$

This implies that for any $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, the commutant $\mathcal{A}'$ is always a von Neumann algebra. As we know, in physics, commutators are intimately connected with microcausality in quantum field theory. Therefore, it is natural to realize that von Neumann algebras arise as appropriate mathematical structures to describe the set of local observatories associated with the causal closure of a local subregion of a physical system [24, 25]. We will discuss this in detail in a later section.

Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{A}'$ its commutant. It is not necessarily the case that they are disjoint. This leads to the concept of the center of an algebra.

Definition 3.6. (Centre:)
For a von Neumann algebra $\mathcal{A}$, the centre Z of the algebra is defined as,

$$Z = \mathcal{A} \cap \mathcal{A}' \tag{3.2.11}$$

These are the collection of all the elements that are common in both $\mathcal{A}$ and $\mathcal{A}'$.

The center of an algebra plays a crucial role in the classification of von Neumann algebras. Operators in the center are used to define superselection rules in charged systems and gauge theories [26]. The centre will play a very important role in the classification of von Neumann algebras, as we will see. Using the centre of algebra, we can define factor algebras,

Definition 3.7. (Factor)
The von Neumann algebra is a factor if the center Z is trivial.

$$Z = CI_{\mathcal{A}}, \text{ where } c \in \mathbb{C}.$$

Factors are a fundamental concept in the theory of von Neumann algebras. A remarkable theorem by von Neumann states that any von Neumann algebra that is not a factor can always be decomposed into a direct sum or direct integral of factors. Thus, a factor von Neumann algebra is an irreducible von Neumann algebra, meaning that studying factors is sufficient to understand the structure of von Neumann algebras. There is also a strong physical motivation for the importance of these factors. In quantum field theory, the local algebra associated with a given spacetime region is typically a factor. Now, we want to move to a composite system.

It is natural to ask: if two systems are disjoint, and each has its own algebra of observables, what is the algebra describing the total system? The answer is that the algebra of the total system is given by the tensor product of the algebras associated with each system.

Thus, let us now define the tensor product of von Neumann algebras.

Definition 3.8. (Tensor product of two von Neuman algebras):
Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces, and $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the *tensor product Hilbert space* of $\mathcal{H}_1$ and $\mathcal{H}_2$. Let $B(\mathcal{H}_1)$ and $B(\mathcal{H}_2)$ be the algebra of bounded linear operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then for any $a_1 \otimes a_2 \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, it is uniquely determined by $(a_1 \otimes a_2) |\psi_1 \otimes \psi_2)\rangle = a_1 |\psi_1\rangle \otimes a_2 |\psi_2\rangle$ for all $|\psi_i\rangle \in \mathcal{H}_i$. For von Neumann algebras $\mathcal{A}_i \subset B(\mathcal{H}_i)$, the von Neumann algebra generated by $\{a_1 \otimes a_2 : a_1 \in \mathcal{A}_1, a_2 \in \mathcal{A}_2\}$ is denoted by $\mathcal{A}_1 \otimes \mathcal{A}_2$ and called the *tensor product* of $\mathcal{A}_1, \mathcal{A}_2$.

Using the tensor product of von Neumann algebras, we can show

Theorem 3.2.2. **(Commutation Theorem for Tensor Products):**
The commutation theorem for tensor products states that for von Neumann algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, the commutant of their tensor product is the tensor product of their commutants. Specifically:

$$(\mathcal{A}_1 \otimes \mathcal{A}_2)' = \mathcal{A}_1' \otimes \mathcal{A}_2'$$

Proof. To prove this, we need to show two inclusions:

1. $(\mathcal{A}_1 \otimes \mathcal{A}_2)' \supseteq \mathcal{A}_1' \otimes \mathcal{A}_2'$
2. $(\mathcal{A}_1 \otimes \mathcal{A}_2)' \subseteq \mathcal{A}_1' \otimes \mathcal{A}_2'$

1. $(\mathcal{A}_1 \otimes \mathcal{A}_2)' \supseteq \mathcal{A}_1' \otimes \mathcal{A}_2'$

Let $a \in \mathcal{A}_1'$ and $b \in \mathcal{A}_2'$. We need to show that $a \otimes b$ commutes with every element of $\mathcal{A}_1 \otimes \mathcal{A}_2$.
Take any $c \in \mathcal{A}_1$ and $d \in \mathcal{A}_2$. Then:

$$(a \otimes b)(c \otimes d) = ac \otimes bd$$

Since $a \in \mathcal{A}_1'$, $ac = ca$. Similarly, since $b \in \mathcal{A}_2'$, $bd = db$. Thus:

$$ac \otimes bd = ca \otimes db = (c \otimes d)(a \otimes b)$$

This shows that $a \otimes b$ commutes with $c \otimes d$. Therefore,

$$a \otimes b \in (\mathcal{A}_1 \otimes \mathcal{A}_2)'$$

This proves $\mathcal{A}_1' \otimes \mathcal{A}_2' \subseteq (\mathcal{A}_1 \otimes \mathcal{A}_2)'$.

2. $(\mathcal{A}_1 \otimes \mathcal{A}_2)' \subseteq \mathcal{A}_1' \otimes \mathcal{A}_2'$

Let $e \in (\mathcal{A}_1 \otimes \mathcal{A}_2)'$. We need to show that $e \in \mathcal{A}_1' \otimes \mathcal{A}_2'$. Since e commutes with every element of $\mathcal{A}_1 \otimes \mathcal{A}_2$, it commutes with elements of the form $c \otimes I_{\mathcal{A}_2}$ for $c \in \mathcal{A}_1$ and $I_{\mathcal{A}_1} \otimes d$ for $d \in \mathcal{A}_2$, where $I_{\mathcal{A}_1}$ and $I_{\mathcal{A}_2}$ are the identity operators on $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. This implies:

$$e(c \otimes I_{\mathcal{A}_2}) = (c \otimes I_{\mathcal{A}_2})e \quad \text{for all } c \in \mathcal{A}_1$$

$$e(I_{\mathcal{A}_1} \otimes d) = (I_{\mathcal{A}_1} \otimes d)e \quad \text{for all } d \in \mathcal{A}_2$$

These commutation relations imply that e can be decomposed as $e = a \otimes b$, where $a \in \mathcal{A}_1'$ and $b \in \mathcal{A}_2'$. This is a consequence of the structure of von Neumann algebras and the fact that e commutes with the generators of $\mathcal{A}_1 \otimes \mathcal{A}_2$.
Therefore, $e \in \mathcal{A}_1' \otimes \mathcal{A}_2'$, which proves $(\mathcal{A}_1 \otimes \mathcal{A}_2)' \subseteq \mathcal{A}_1' \otimes \mathcal{A}_2'$.

Combining the two inclusions, we have:

$$(\mathcal{A}_1 \otimes \mathcal{A}_2)' = \mathcal{A}_1' \otimes \mathcal{A}_2'$$

This completes the proof of the commutation theorem for tensor products. □

The above theorem is helpful in thinking about the algebra of a composite system and the algebra of multiple subregion in local quantum field theory. For completeness and the fact that any von Neumann algebra can be written as the direct sum of the factor algebras, we would like to introduce the direct sum of von Neuman algebra.

Definition 3.9. (Direct sum of von Neumann algebra:)
Let $\mathcal{A}_1$ and $\mathcal{A}_2$ are von Neumann algebras, then their direct sum is defined as:

$$\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2.$$

This means that every element $a \in \mathcal{A}$ can be written as a pair $a = (a_1, a_2)$, where $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$ and satisfty,

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2),$$

$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1, a_2 b_2),$$

$$(a_1, a_2)^\dagger = (a_1^\dagger, a_2^\dagger).$$

Further, following the similar steps as in the tensor product of von Neumann algebra, it can be shown that $(\mathcal{A}_1 \oplus \mathcal{A}_2)' = \mathcal{A}_1' \oplus \mathcal{A}_2'$.
Now, let us see some examples of von Neumann algebras. Let us start with a finite-dimensional Hilbert space.

Example 3.1. *(Algebra of $n \times n$ matrix)*
*Let $M_n(\mathbb{C})$ denote the algebra of $n \times n$ complex matrices acting on the Hilbert space $\mathcal{H} = \mathbb{C}^n$. This is a unital *-algebra, since it is closed under addition, multiplication, scalar multiplication, and taking adjoints (i.e., conjugate transpose).*

The commutant of $M_n(\mathbb{C})$ is

$$M_n(\mathbb{C})' = \{\lambda I \mid \lambda \in \mathbb{C}\} = \mathbb{C} \cdot I,$$

since the only matrices that commute with all matrices in $M_n(\mathbb{C})$ are scalar multiples of the identity matrix.

Taking the commutant again, we find

$$M_n(\mathbb{C})'' = (M_n(\mathbb{C})')' = M_n(\mathbb{C}),$$

so $M_n(\mathbb{C})$ is equal to its double commutant and is therefore a von Neumann algebra by the double commutant theorem.

Example 3.2. *(Finite Dimensional Interacting System)*
Consider an n-dimensional quantum system interacting with another n-dimensional quantum system. The algebra of observables for these systems is given by:

$$\mathcal{A}_1 = span\{a \otimes I_n \,|\, a \in M_n(\mathbb{C})\}\,, \quad \mathcal{A}_2 = span\{I_n \otimes b \,|\, b \in M_n(\mathbb{C})\}\,,$$

acting on the Hilbert space $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^n$.

*Clearly, $\mathcal{A}_1$ and $\mathcal{A}_2$ are *-algebras, and we have:*

$$\mathcal{A}_1' = \mathcal{A}_2, \qquad \mathcal{A}_2' = \mathcal{A}_1$$

Hence, each is the commutant of the other, and both are von Neumann algebras.

Example 3.3. *(Weyl Algebra in Quantum Mechanics)*
Let $\hat{Q}$ and $\hat{P}$ be the position and momentum operators in quantum mechanics acting on the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$. These operators satisfy the canonical commutation relation:

$$[\hat{Q}, \hat{P}] = i\hbar.$$

Define the Weyl operators as

$$W(q,p) \equiv e^{i(q\hat{Q} - p\hat{P})}, \quad \textit{where } (q,p) \in \mathbb{R}^2.$$

These operators satisfy the Weyl form of the canonical commutation relations:

$$W(q_1,p_1)W(q_2,p_2) = e^{-\frac{i\hbar}{2}(q_1p_2 - q_2p_1)}W(q_1+q_2, p_1+p_2).$$

The Weyl algebra is defined as the von Neumann algebra generated by all Weyl operators:

$$\mathcal{A}_{Weyl} = \{W(q,p) : (q,p) \in \mathbb{R}^2\}''.$$

Since this algebra is closed under adjoint, weak operator limits, and contains the identity and linear combinations of Weyl operators, it is indeed a von Neumann algebra.

3.3 The Local Algebras in Quantum Field Theory

In this subsection, we study the operator aspects of quantum field theory. The goal is to construct a local algebra of observables in QFT or, equivalently, an algebra of subregions in QFT. If quantum fields are fields of operators, constructing such an algebra is straightforward: restrict the field to the subregion, take all bounded operators constructed from the localized field, and complete the set under the weak topology. However, quantum fields are not operator-valued fields but rather operator-valued distributions. Thus, the first step is to construct well-defined operators from the quantum field that act on a dense subset of the Hilbert space. Let's take a small digression and understand a little bit about generalized functions, also known as distributions.

3.4 Introduction to Generalized functions

Generalized functions, or distributions, extend the classical notion of functions to accommodate singularities, non-smooth behavior, and idealized physical quantities such as point charges and mass distributions. Their necessity was recognized in both physics and mathematics, with Dirac's introduction of the delta function in quantum mechanics and Schwartz's rigorous formulation of distribution theory. This revolutionary framework, pioneered by Laurent Schwartz in the 1940s [27, 28], extends classical calculus to handle singularities, jumps, and infinite values with mathematical precision. At its core lies the Dirac delta function—a mathematical phantom, zero everywhere except at a single point where it is "infinitely tall," yet integrates to unity. This function is essential for modeling point sources in classical mechanics, defining canonical commutation relations, and normalizing eigenfunctions of operators with continuous spectra (e.g., position and momentum operators in quantum mechanics), all of which require this broader framework.

Beyond these foundational aspects, generalized functions play a crucial role in the analysis of differential equations, particularly in defining Green's functions (or propagators) in quantum field theory and facilitating non-smooth solutions to PDEs. They provide a rigorous framework for handling singularities in physical models and offer deep insights into the structure of quantum fields.

Generalized functions or distributions are defined as continuous linear functionals on spaces of test functions. The primary test function spaces include $\mathcal{D}(\mathbb{R}^d)$, the space of compactly supported smooth functions, and the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, consisting of rapidly decaying smooth functions. So, let us start by introducing the space of the test function.

3.4.1 Test Functions

Let $f : \mathbb{R}^d \to \mathbb{C}$ be a smooth function (infinitely differentiable function) denoted by $C^\infty(\mathbb{R}^d)$ and multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\alpha_j \in \mathbb{N}$. We can define order $|\alpha| = \sum_{j\leq d} \alpha_j$ and

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}.$$

For $x \in \mathbb{R}^d$, we denote

$$x^\alpha = x_1^{\alpha_1} .. x_d^{\alpha_d}, \qquad |\mathbf{x}|^2 = \sum_{j\leq d} |x_j|^2.$$

For each integer k, we define the seminorm[4]

$$\|f\|_k = \sup_{\mathbf{x}\in\mathbb{R}^d} (1 + |\mathbf{x}|^2)^k \sum_{|\alpha|\leq k} |D^\alpha f(\mathbf{x})|. \tag{3.4.1}$$

Then,

Definition 3.10. (Schwartz Functions) $\mathcal{S}(\mathbb{R}^d)$:
The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ consists of all functions $f \in C^\infty(\mathbb{R}^d)$ such that

$$\|f\|_k < \infty \quad \forall k \in \mathbb{Z}$$

In other words, a function belongs to $\mathcal{S}(\mathbb{R}^n)$ if and only if f together with all its derivatives falls off as $|x| \to \infty$ faster than any power of $|x|^{-1}$.

Notice that,

1. $\mathcal{S}(\mathbb{R}^d)$ is a vector space, that is

$$f, g \in \mathcal{S}(\mathbb{R}^d), \alpha, \beta \in \mathbb{C} \implies \alpha f + \beta g \in \mathcal{S}(\mathbb{R}^d)$$

[4] It is easy to check that satisfy condition for the seminorm given in 2.3

2. For $f, g \in \mathcal{S}(\mathbb{R}^d)$, the product $fg \in \mathcal{S}(\mathbb{R}^d)$.
3. The seminorm in (3.4.1) is a norm if and only if $k = 0$.
4. If $f(\mathbf{x}) \in \mathcal{S}(\mathbb{R}^d)$, then an arbitrary derivative of $f(\mathbf{x})$ also belongs to $\mathcal{S}(\mathbb{R}^d)$.
5. If $h(\mathbf{x})$ is a smooth function such that both $h(\mathbf{x})$ and all its derivatives are polynomially bounded, i.e., $\forall \alpha \in \mathbb{N}^d$ there exist $n \in \mathbb{N}$ and $C_n > 0$ such that

$$|h^{(\alpha)}(\mathbf{x})| \leq C_n(1 + |\mathbf{x}|^2)^n, \quad \forall \alpha \in \mathbb{N}^d,$$

then for any $f(\mathbf{x}) \in \mathcal{S}(\mathbb{R}^d)$, the product $h(\mathbf{x})f(\mathbf{x})$ also belongs to $\mathcal{S}(\mathbb{R}^d)$. Such functions are called ***multipliers***. It can easily be shown that the collection of all multipliers forms a vector space.

It can also be shown that for a function $h(\mathbf{x})$ to be a multiplier, it is necessary and sufficient that it be smooth and that both $h(\mathbf{x})$ and its derivatives grow at infinity no faster than some polynomial.

Further , we say the sequence $\{f_n\} \in \mathcal{S}(\mathbb{R}^d)$ convergence to f in $\mathcal{S}(\mathbb{R}^d)$, if

$$D^\alpha f_n(\mathbf{x}) \overset{\mathbf{x} \in \mathbb{R}^d}{\Longrightarrow} D^\alpha f(\mathbf{x}) \tag{3.4.2}$$

where $x^\beta = x_1^{\beta_1}..x_d^{\beta_d}$.
We call all functions in $\mathcal{S}(\mathbb{R}^d)$ **test function**. Now, let us look at some examples of Schwartz functions or test functions.
Example 1:
Consider $f(x) = P(x)e^{-|x|}$, where $P(x)$ is a polynomial of $x \in \mathbb{R}$ of order m, it is a Schwartz function.

Proof. Note that any derivative of $f(x)$ will always take the form $g(x)e^{-|x|}$, where $g(x)$ is some polynomial. Let us denote the α-th derivative as $g_\alpha(x)e^{-|x|}$, where $g_\alpha(x)$ is some polynomial and it must be clear that,

$$g_\alpha(x) < C(1 + |x|^2)^{m+\alpha}$$

where $C = |g(0)| + 1$. In addition, for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$

$$e^{-|x|} \leq \frac{1}{1 + |x| + ... + |x|^n}$$

therefore, there must exist a $n_k \in \mathbb{N}$ for each $k \in \mathbb{N}$ such that

$$\sup_{x\in\mathbb{R}}(1+|x|^2)^k \sum_{|\alpha|\leq k} |D^\alpha f(x)| < \sup_{x\in\mathbb{R}} \frac{Ck(1+|x|^2)^{2k+m}}{1+|x|+....+|x|^{n_k}} < \infty$$

Therefore all $f(x) = P(x)e^{-|x|}$, where $P(x)$ is a polynomial of x, is a Schwarzt function. □

Since $e^{-x^2} \leq e^{-|x|}$, any function of the form $f(x) = P(x)e^{-x^2}$ where $P(x)$ is polynomial is a Schwartz function. More generally, any $f(x) = P(x)e^{-|x|^n}$ for n being integer is a Schwartz function.

Example 2

Any smooth function with compact support is a Schwartz function.

Proof. Any smooth function with compact support is bounded [21]. Furthermore, any polynomial is also bounded in a compact region of $\mathbb{R}^d$. This implies that (3.4.1) is finite. In other words, a smooth function with compact support is nonzero only in a compact region and hence decays faster than any power of $|\mathbf{x}|^{-1}$. Therefore, it belongs to $\mathcal{S}(\mathbb{R}^d)$. An example of such a function is a *bump function*.

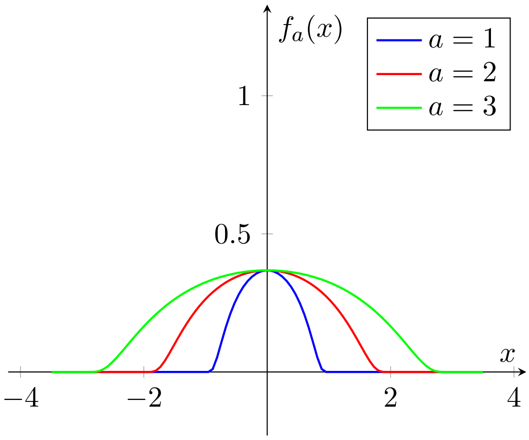
A **bump function** is a smooth function (infinitely differentiable) that has compact support, meaning it is nonzero only in a finite region and smoothly vanishes outside that region. A common example is given by:

$$f_a^b(\mathbf{x}) = \begin{cases} e^{-\frac{a^2}{a^2-\mathbf{x}^2}}, & \text{if } |\mathbf{x}| \leq a, \\ 0, & \text{if } |\mathbf{x}| > a. \end{cases}$$

where $a > 0$ and $\mathbf{x} \in \mathbb{R}^d$.

Graph of the Bump Function for Different Values of a

Plot of $f_a^b(\mathbf{x})$ for $a = 1, 2, 3$



$\square$

It can be easily checked that $f_a(x)$ is smooth but not real analytic. The nonanalytic behavior of $f_a(\mathbf{x})$ follows from the fact that all derivatives of $f_a(\mathbf{x})$ are zero at $|\mathbf{x}| = a$. Hence, the Taylor expansion does not exist at $|\mathbf{x}| = a$. These compactly supported smooth functions are closed in themselves.

Definition 3.11. (Test function with compact support) $\mathcal{D}(\mathbb{R}^d)$:
The collection of all smooth functions with compact support forms a vector space denoted by $\mathcal{D}(\mathbb{R}^d)$.

It must be clear from the above definition and Example 2 that $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$. Further , we say the sequence $\{f_n\} \in \mathcal{D}(\mathbb{R}^d)$ convergence to f in $\mathcal{D}(\mathbb{R}^d)$, if

$$D^\alpha f_n(\mathbf{x}) \overset{\mathbf{x}\in\mathbb{R}^d}{\Longrightarrow} D^\alpha f(\mathbf{x}) \tag{3.4.3}$$

It follows that any sequence convergent in $\mathcal{D}(\mathbb{R}^d)$ is also convergent in $\mathcal{S}(\mathbb{R}^d)$. Furthermore, for any smooth function f, the product fg with $g \in \mathcal{D}(\mathbb{R}^d)$ remains in $\mathcal{D}$. Therefore, multiplication by a smooth function is an automorphism in $\mathcal{D}(\mathbb{R}^d)$. $f_a(x)$ is a good example of the function which belongs $\mathcal{D}(\mathbb{R}^d)$.

Some properties of the test functions:

1. The Schwartz functions are bounded functions. It clearly follows from the definition just take $k = 0$.

2. Schwartz functions are absolutely integrable, i.e., $\mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$.

   Proof. Let $f(\mathbf{x}) \in \mathcal{S}(\mathbb{R}^d)$ and consider a ball $B(r_0)$ of radius r_0 centered at the origin. Then,
$$\int_{\mathbb{R}^d} |f(\mathbf{x})| d^d x = \int_{B(r_0)} |f(\mathbf{x})| d^d x + \int_{\mathbb{R}^d \setminus B(r_0)} |f(\mathbf{x})| d^d x.$$
   Since $f(\mathbf{x})$ is bounded in $B(r_0)$, there exists $M > 0$ such that
$$\int_{B(r_0)} |f(\mathbf{x})| d^d x \leq M.$$
   For $|\mathbf{x}| > r_0$, the rapid decay of $f(\mathbf{x})$ ensures that there exists $C_k > 0$ such that
$$|f(\mathbf{x})| \leq \frac{C_k}{(1 + |\mathbf{x}|^2)^k}, \quad \forall k > 0.$$
   Choosing $k > d/2$, we obtain
$$\int_{\mathbb{R}^d \setminus B(r_0)} |f(\mathbf{x})| d^d x < \infty.$$
   Thus,
$$\int_{\mathbb{R}^d} |f(\mathbf{x})| d^d x < \infty,$$
   which proves that $f(\mathbf{x}) \in L^1(\mathbb{R}^d)$. □

3. The Fourier transform of any $f \in \mathcal{S}(\mathbb{R}^d)$ is well defined, that is
$$|\tilde{f}(\mathbf{p})| = \Big| \int_{\mathbb{R}^d} e^{i\mathbf{p}.\mathbf{x}} f(\mathbf{x}) d^d x \Big| \leq \int_{\mathbb{R}^d} |f(\mathbf{x})| d^d x < \infty$$
   where $\tilde{f}(\mathbf{p})$ is Fourier transform of $f(\mathbf{x})$.

4. For $f \in \mathcal{S}(\mathbb{R}^d)$ the Fouier transform $\tilde{f}(\mathbf{p})$ also belongs to $\mathcal{S}(\mathbb{R}^d)$. It follows from the fact for $f \in \mathcal{S}(\mathbb{R}^d)$, $|\mathbf{x}|^n f \in \mathcal{S}(\mathbb{R}^d)$ and all derivatives of f also belongs to $\mathcal{S}(\mathbb{R}^d)$.

5. If $f \in \mathcal{D}(\mathbb{R})$, then $\tilde{f}$ is an entire analytic function, and there exist a $r_0 > 0$ and $B_n > 0$ such that $|p^n \tilde{f}(p)| \leq B_n e^{r_0 Im(p)}$. This is also true in d dimensions.

   Proof. Let *supp*(f) be contained in the region $B(r_0)$ of radius r_0 around the origin. then
$$\tilde{f}(p) = \int_{\mathbb{R}^d} e^{ipx} f(x) dx = \int_{B(r_o)} e^{ipx} f(x) dx \tag{3.4.4}$$

Now let p be a complex variable, then,

$$\frac{1}{2\pi i}\oint_{\gamma_z}\frac{\tilde{f}(p)}{p-z}dp = \int_{B(r_o)}\frac{1}{2\pi i}\Big[\oint_{\gamma_z}\frac{e^{ipx}}{p-z}dp\Big]f(x)dx = \int_{B(r_o)}e^{izx}f(x)dx = \tilde{f}(z)$$

where γ_z is any contour in the complex plane that encloses z. Therefore, it $\tilde{f}(z)$ is an analytic function on the complex plane. Since it does not have any poles, it is entire.

$$\begin{aligned}|p^n\tilde{f}(p)| &= |\int_{\mathbb{R}^d}(-i)^n\frac{\partial^n}{\partial x^n}e^{-ipx}f(x)dx| \\ &= |\int_{\mathbb{R}^d}e^{-ipx}f^n(x)dx| \\ &\leq \int_{\mathbb{R}^d}e^{xIm(p)}|f^n(x)|dx \\ &\leq e^{r_0 Im(p)}\int_{B(r_0)}|f^n(x)|dx\end{aligned}$$

where r_0 is the minimum radius of an open ball $B(r_0)$ containing the support of $f^n(x)$. Since $f^n(x)$ is bounded there must exist $B_n > 0$ such that $\int_{B(r_0)}|f^n(x)|dx = B_n$. Implies,

$$|p^n\tilde{f}(p)| \leq B_n e^{r_0 Im(p)} \tag{3.4.5}$$

It can easily be checked that the above inequality is true even if we replace $\tilde{f}(x)$ by any of its derivatives. This implies when $p \in \mathbb{R}$ then $\tilde{f} \in \mathcal{S}(\mathbb{R})$. It is straightforward to extend this to d dimensions. This basically follows from the fact that $e^{iz.x}$ is an entire analytic function for any complex vector z. □

3.4.2 Generalized Functions or Tempered Distribution

Generalized functions are continuous linear functionals defined on the space of test functions. When the space of test functions is $\mathcal{S}(\mathbb{R}^d)$, these functionals are called tempered distributions. The space of tempered distributions is the dual space of $\mathcal{S}(\mathbb{R}^d)$, i.e., $T : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$, and it is denoted by $\mathcal{S}'(\mathbb{R}^d)$. Let $f \in \mathcal{S}(\mathbb{R}^d)$ and $T \in \mathcal{S}'(\mathbb{R}^d)$, then we denote action of T on f as,

$$T(f) = (T, f) \tag{3.4.6}$$

and it satisfies the following properties,

1. **Linearity:** $(T, \alpha f_1 + \beta f_2) = \alpha(T, f_1) + \beta(T, f_2)$.

2. **Continuity:** There exist $k \in \mathbb{N}$ and $C > 0$, such that

$$|(T, f_1)| \leq C\|f_1\|_k$$

where $f_1, f_2 \in \mathcal{S}(\mathbb{R}^d)$ and $\alpha, \beta \in \mathbb{C}$. These are the necessary and sufficient conditions for T to belong to $\mathcal{S}(\mathbb{R}^d)$[29]. We can also write it formally as,

$$T(f) = \int_{\mathbb{R}^d} \overline{T(\mathbf{x})} f(\mathbf{x}) d^d x \tag{3.4.7}$$

The above integral representation is formal because it assumes that $T(x)$ is defined at each point. In general, a distribution may not be defined point-wise, but has a well-defined action on test functions. However, there are **regular distributions**, which are essentially functions that are locally integrable and have polynomial growth (tempered), for which the above integral representation is well defined. This is because the product of a Schwartz function with any polynomially bounded function is again a Schwartz function and is integrable. For example, The Lebesgue measure dx is a regular tempered distribution. It is defined as,

$$(dx, f) = \int_{\mathbb{R}} f(x) d^d x \tag{3.4.8}$$

Furthermore, any continuous measure of type $\rho(x)dx$, where $\rho(x)$ is a polynomially bounded smooth function, is a regular distribution. Let us see an example where the integral representation (3.4.6) is formal. But before that, let us define the support of a distribution.

Definition 3.12. (Support of a distribution):
A closed subset Q of $\mathbb{R}^d$ is called the support of T if, for any $f \in \mathcal{S}(\mathbb{R}^d)$ with support $R = \mathbb{R}^d \setminus Q$, it holds that

$$(T, f) = 0.$$

Delta distribution δ_a:

The delta distribution is defined by

$$(\delta_{\mathbf{a}}, f) = f(\mathbf{a}), \tag{3.4.9}$$

where f is any test function in $\mathcal{S}(\mathbb{R}^d)$.

The first thing to notice is that $\delta_{\mathbf{a}}$ is linear, and continuity follows from the fact that

$$|(\delta_{\mathbf{a}}, f)| = |f(\mathbf{a})| \leq \|f\|_{k=0}.$$

Hence, it is a tempered distribution.

Let g be any smooth function with compact support such that $g(\mathbf{a}) = 0$. Then,

$$(\delta_{\mathbf{a}}, g) = 0.$$

This implies that the support of $\delta_{\mathbf{a}}$ is

$$\mathrm{supp}(\delta_{\mathbf{a}}) = \{\mathbf{a}\}.$$

Therefore, if we modify the test function away from $\mathbf{a}$, the delta distribution will not detect the change. Now, assume that $\delta_{\mathbf{a}}$ has an integral representation:

$$f(\mathbf{a}) = \int_{\mathbb{R}^d} \delta_{\mathbf{a}}(\mathbf{x}) f(\mathbf{x})\, d^d x. \tag{3.4.10}$$

In the theory of Riemann integration, any function supported at a single point has a vanishing integral. Even the integral of the product of an integrable function with such a function is zero. This implies that if $\delta_{\mathbf{a}}$ were Riemann integrable, then the right-hand side of the expression

$$\int_{\mathbb{R}^d} \delta_{\mathbf{a}}(\mathbf{x}) f(\mathbf{x})\, d^d x$$

would be zero for all f, which contradicts the defining property of the delta distribution. Hence, $\delta_{\mathbf{a}}$ is not Riemann integrable. To see it more explicitly, take $d = 1$ and $a = 0$. Consider a sequence of bump functions $f^b_{n^{-1}}(x)$ defined in (3.4.1). Clearly,

$$\lim_{n\to\infty} f^b_{n^{-1}}(x) = \begin{cases} e^{-1}, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

By the assumed integral representation, we obtain

$$e^{-1} = \lim_{n\to\infty} \int_{\mathbb{R}} \delta_0(x) f^b_{n^{-1}}(x)\, dx. \tag{3.4.11}$$

since $f^b_{n^{-1}}(x)$ is only non zero in $[-\frac{1}{n}, \frac{1}{n}]$,

$$e^{-1} = \lim_{n\to\infty} \int_{-\frac{1}{n}}^{\frac{1}{n}} \delta_0(x) f^b_{n^{-1}}(x)\, dx. \leq \lim_{n\to\infty} \int_{-\frac{1}{n}}^{\frac{1}{n}} sup(\delta_0(x)) dx$$

if $\delta_0(x)$ is bounded as required for Riemann integrable functions. Then there exists $C > 0$,

$$e^{-1} \leq \lim_{n\to\infty} \frac{2C}{n} = 0$$

This is a clear contradiction. It tells us that there is no Riemann-integrable function $\delta_a(x)$ that satisfies (3.4.10). Moreover, it can also be shown that it is not Lebesgue integrable. Therefore, the integral representation is formal and does not exist in the actual mathematical sense.

3.4.3 Derivative of Tempered Distribution:

As we have already mentioned, the purpose of defining generalized functions is to extend the concept of functions beyond point-wise definitions. Consequently, we must also generalize the definitions of derivatives, products, and the Fourier transform appropriately. Let $T(x)$ be a polynomially bounded function and suppose that it has a derivative; then

$$(T', f) = \int_{\mathbb{R}^d} \overline{T(\mathbf{x})}' f(\mathbf{x}) d^d x = -\int_{\mathbb{R}^d} \overline{T(\mathbf{x})} f'(\mathbf{x}) d^d x = -(T, f')$$

where the derivative can be with respect to any component of $\mathbf{x}$. Further, the extra term that we get after biparts vanishes because the Schwartz function vanishes at $|\mathbf{x}| \to \infty$ faster than any polynomial. Now, we will state this as the definition of the derivative of a tempered distribution. This is a natural extension of the definition of derivatives from functions to generalized functions.

Definition 3.13. (Derivative of a distribution):
Let T be a tempered distribution. The n-th derivative $T^{(n)}$ of the tempered distribution T is defined by the equation,

$$(T^{(n)}, f) = (-1)^n (T, f^{(n)}) \tag{3.4.12}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$ and $f^{(n)}$ is the n-th derivative of f.

The above definition allows us to define the n-derivative of the delta distributions.

$$(\delta_a^{(n)}, f) = (-1)^n f^{(n)}(a) \tag{3.4.13}$$

where f is any Schwartz function and $f^{(n)}$ is its derivatives. Using the fact that locally integrable functions can be thought of as distributions, we can define the delta distribution as the derivative of some discontinuous locally integrable function

$$\Theta(x-a) = \begin{cases} 1, & \text{if } x > a, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4.14}$$

Now it follows that, for $f \in \mathcal{S}(\mathbb{R})$,

$$(\frac{\partial \Theta(x-a)}{\partial x}, f) = -\int_{\mathbb{R}} \Theta(x-a) f'(x) = f(a) = (\delta_a, f) \tag{3.4.15}$$

implies that $\frac{\partial \Theta(x-a)}{\partial x} = \delta_a$ in the sense of distributions. Another example of such distribution that is derivative of some locally integrable function is Cauchy's principal value distribution. Consider the derivatives of $\log|x|$.

$$(\frac{\partial \log|x|}{\partial x}, f) = -\int_{-\infty}^{\infty} \log|x| f' dx = \lim_{\epsilon \to 0} \{ \int_{-\infty}^{-\epsilon} \frac{f(x)}{x} dx + \int_{\epsilon}^{\infty} \frac{f(x)}{x} dx \} \tag{3.4.16}$$

where $f \in \mathcal{S}(\mathbb{R})$,

$$(\frac{\partial \log|x|}{\partial x}, f) = \text{p.v} \int_{\mathbb{R}} \frac{f(x)}{x} dx \equiv (\frac{1}{x}, f) \tag{3.4.17}$$

where p.v denotes the principal value of the integrals. In addition, we want to make the notation that whenever we write $\frac{1}{x}$, it should be thought of as a distribution and defined through the principal value integral over some test function.

The above equation can easily be generalized; it can easily be shown that,

$$(\frac{(-1)^{n-1}}{n-1!} \frac{\partial^n \log|x|}{\partial x^n}, f) = (\frac{1}{x^n}, f)$$

We can further consider a locally integrable function defined as,

$$\log(x \pm i0_+) \equiv \lim_{\epsilon \to 0_+} \log(x \pm i\epsilon) \tag{3.4.18}$$

$$= \log|x| + i \lim_{\epsilon \to 0_+} arg(x \pm i\epsilon) \tag{3.4.19}$$

$$= \log|x| + i\pi\Theta(\mp x) \tag{3.4.20}$$

using (3.4.15) and (3.4.17), it follows

$$\frac{1}{x \pm i0_+} \equiv \frac{\partial}{\partial x} \log(x \pm i0_+) = \text{p.v}\frac{1}{x} \mp i\pi\delta_0 \tag{3.4.21}$$

in the sense of the distributions. It also follows that,

$$\frac{1}{(x \pm i0_+)^n} \equiv \frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^n}{\partial x^n}\log(x \pm i0_+) = \text{p.v}\frac{1}{x^n} \pm \frac{(-1)^n}{(n-1)!}i\pi\delta_0^{(n-1)} \tag{3.4.22}$$

where $\delta_0^{(n)}$ is the n-th derivative of the delta distribution.

Notation: From now on, we will formally write δ_0 as $\delta(x)$, and we will always keep in mind that it's not a function; its a tempered distribution.

3.4.4 Multiplication of Tempered Distributions:

The product of two tempered distributions is not defined in general, since one wants the associative law of multiplication to hold, which does not hold in general for distributions. Let $Q, T \in \mathcal{S}^*$ be any pair of distributions and $f \in \mathcal{S}$, then

$$(QT, f) \text{ is not defined.} \tag{3.4.23}$$

since, in general, $\bar{T}f \notin \mathcal{S}$. For example, take $T = \Theta(x)$ and $Q = \delta(x)$. Then $\delta(x)\Theta(x)$ is not defined because $\Theta(x)f(x) \notin \mathcal{S}$ for all $f \in \mathcal{S}$. For another example, consider the following equation in the sense of distributions:

$$x\delta(x) = 0 \qquad\qquad \frac{1}{x}x = 1 \tag{3.4.24}$$

Now,

$$\frac{1}{x}(x\delta(x)) = \frac{1}{x}0 = 0$$
$$(\frac{1}{x}x)\delta(x) = \delta(x)$$

implies that multiplication is not associative. Although multiplication is not defined in general, there is a wider class $\mathcal{S}^*$ for which the multiplication is well defined. Take T to be an element in the space of multipliers; then for any $Q \in \mathcal{S}^*$, the product,

$$(QT, f) = (Q, \bar{T}f) \tag{3.4.25}$$

This is well-defined because T is a polynomially bounded smooth function and $Tf \in \mathcal{S}$. Also, it follows from the definition that $QT = TQ$. In some cases, it is also possible to define the singular product of distributions where neither of the distributions in the product belongs to the space of multipliers; however, we will not discuss that here. Knowing the fact that the product of a multiplier and a distribution is a distribution, we can ask if

$$fQ = T \tag{3.4.26}$$

where $T \in \mathcal{S}^*$ and f is a multiplier, then can we find Q in the space of distributions. If f^{-1} exists for all x and is a multiplier, then $Q = f^{-1}T$. However, if $f(x)$ has zeros (n distinct zeroes), then the problem becomes somewhat non-trivial. Let $u \in \mathcal{S}$ and $v = \bar{f}u$, then,

$$(fQ, u) = (Q, \bar{f}u) = (Q, v) = (T, u)$$

The above equation uniquely specifies Q in the subspace of $\mathcal{S}$, which contains all test functions that vanish at the zeroes of f. Now, to completely specify Q, it is sufficient to specify it for $u_0^i \in \mathcal{S}$ such that $u_0^i(z_i) = 1$, where z_i is the i-th zero of f. Any arbitrary $u \in \mathcal{S}$ can written as,

$$u = \sum_{i=1}^{n} u(z_i)u_0^i + v$$

where $v = fu_1$, $u_1 \in \mathcal{S}$ and satisfy $v(z_i) = 0$. Now

$$(Q, u) = \sum_{i=1}^{n} u(z_i)(Q, u_0^i) + (T, u_1)$$

which can also be written as,

$$(Q, u) = \sum_{i=1}^{n} u(z_i)C_i + (T, u_1) \tag{3.4.27}$$

where C_i are some arbitrary functions. The above equation completely specifies Q, which satisfies $fQ = T$. If T is a multiplier, then we can write the general solution as,

$$Q = \sum_{i=1}^{n} C_i\delta(z_i) + p.v\frac{1}{f}T$$

if f has finitely many zeroes and f^{-1} is a good multiplier away from those zeroes. Here f^{-1} should be thought of in terms of the Cauchy principal value. Moreover, the first term of the above equation is the solution to the homogeneous part of the equation (3.4.26), and the second term is the solution to the particular part of (3.4.26). One straightforward application of the above result is,

$$xQ = 0 \tag{3.4.28}$$

implies, $Q = C\delta(x)$, where C is some constant. Another interesting example is

$$(p^2 - m^2)G_p = 1 \tag{3.4.29}$$

The solution of the above equation is,

$$G_p = (p^2 - m^2)^{-1} + (\Theta(p_0)f_1(\vec{p}) + \Theta(-p_0)f_2(\vec{p}))\delta^d(p^2 - m^2) \tag{3.4.30}$$

where $(p^2 - m^2)$ is defined as a Cauchy principal value and f_1 and f_2 are multiples. We can further generalize (3.4.27) for the case when f has degenerate zeroes. Let m_i be the degeneracy of zeroes at z_i, then the solution of $fQ = T$ is,

$$Q = \sum_{i=1}^{n} \sum_{l=0}^{l=m_i-1} \frac{C_i^l}{l!}\delta^{(l)}(z_i) + p.v\frac{1}{f}T \tag{3.4.31}$$

This basically followed from the fact that,

$$(x-a)^n \frac{1}{m!}\delta^{(m)}(x-a) = \begin{cases} 0, & \text{if } n > m, \\ \frac{(-1)^n}{(m-n)!}\delta^{(m-n)}(x-a), & \text{otherwise} \end{cases} \tag{3.4.32}$$

It can also be proved by induction using the above fact. It follows from equation (3.4.31) that any distribution that has point support can always be written in terms of a delta distribution and its derivatives.

3.4.5 Fourier Transform of the distributions:

To define the Fourier transform $\mathcal{F}$ of a distribution, we will adapt the same strategy used in defining the derivative of a distribution. First, we will examine the properties of functions whose Fourier transform exists, viewed from the perspective of functionals on the space of test functions. Then, we will elevate these properties to define the Fourier transform of distributions. Let f be an integrable function and $\tilde{f}$ be its Fourier transform. Consider

$g \in \mathcal{S}(\mathbb{R}^d)$ defined in the momentum space, then

$$(\tilde{f}, g(p)) = \int \overline{\tilde{f}(p)} g(p) d^d p = \int \overline{f(x)} \tilde{g}(x) d^d x = (f, \tilde{g}(x)) \tag{3.4.33}$$

Now we will declare the above equation as a definition of the Fourier transform for any distribution. Notice that it is a consistent definition since $g \in \mathcal{S}(\mathbb{R}^d)$, the Fourier transform of g exists, unique and belongs to $\mathcal{S}(\mathbb{R}^d)$.

Definition 3.14. (Fourier transform of a distribution):
Let $T \in \mathcal{S}^*$ be a tempered distribution and let $f \in \mathcal{S}$. The Fourier transform $\mathcal{F}(T) = \tilde{T} \in \mathcal{S}^*$ is defined by

$$(\tilde{T}, f(p)) := (T, \tilde{f}(x)), \tag{3.4.34}$$

where p and x denote the Fourier and spacetime variables, respectively.

Since for any $f \in \mathcal{S}$, the Fourier transform $\tilde{f} \in \mathcal{S}$ is uniquely defined, it follows that the Fourier transform is an isomorphism from $\mathcal{S}^*$ onto itself. Therefore, from the above definition, for all $T \in \mathcal{S}^*$, the Fourier transform $\tilde{T} \in \mathcal{S}^*$. We can also define the inverse Fourier transform,

$$(\mathcal{F}^{-1}(T), f(x)) = (T, \mathcal{F}^{-1}(f)(p)) \tag{3.4.35}$$

It follows that,

$$\mathcal{F}^{-1}(\mathcal{F}(T)) = T \tag{3.4.36}$$

Let us list some nice properties of the Fourier transform of $T \in \mathcal{S}^*$,

1. Fourier transform of derivative of distribution, $\mathcal{F}(\partial^n_{x^\alpha} T) = (ip_\alpha)^n \mathcal{F}(T)$.
2. Fourier transform of polynomial times distribution, $\mathcal{F}(x^n T) = (-i)^n \partial^n_p \mathcal{F}(T)$.
3. Fourier transform of the Fourier transform of distribution, $\mathcal{F}(\mathcal{F}(T))(x) = T(-x)$.

Now, let us see some examples,

1. Fourier transform of the Delta distribution :

$$(\tilde{\delta}(p), f(p)) = (\delta(x), \tilde{f}(x)) = \tilde{f}(0) = \int \Big(\frac{1}{(2\pi)^{\frac{d}{2}}}\Big) f(p) d^d p \tag{3.4.37}$$

Implies, $\tilde{\delta}(p) = \frac{1}{(2\pi)^{\frac{d}{2}}}$. Now consider,

$$(\delta_a(x), f(x)) = f(a) = \int \Big(\frac{e^{iap}}{(2\pi)^{\frac{d}{2}}}\Big)\tilde{f}(p)d^dp = (\frac{e^{iap}}{(2\pi)^{\frac{d}{2}}}, \tilde{f}(p)) \tag{3.4.38}$$

Implies, $F(\delta_a)(p) = F(\delta(x-a))(p) = \frac{e^{iap}}{(2\pi)^{\frac{d}{2}}}$

2. Fourier transform of step function Θ:

$$(\tilde{\Theta}(p), f(p)) = (\Theta(x), \tilde{f}(x))) = \int_0^\infty \Theta(x)\tilde{f}(x)dx = \int_{\mathbb{R}} f(p)\Big(\int_0^\infty e^{-ip.x}dx\Big)dp \tag{3.4.39}$$

Implies,

$$\tilde{\Theta}(p) = \frac{i}{p + i0_+} \tag{3.4.40}$$

There are many other interesting properties of generalized functions, but we will conclude our discussion here, as the material covered is sufficient for subsequent developments. For further details, see [29]. Now, let us present a concrete application of the generalized (tempered distribution).

We utilize our understanding of distributions to demonstrate that quantum fields are, in fact, operator-valued distributions.

Proposition 3.15. ***(Quantum fields are operator valued distribution):***
Let $\phi(x)$ be the scalar quantum field (not necessarily free) in d-dimensional Minkowski spacetime; then the covariance under the Poincaré group implies that the quantum fields are not local operators but rather operator-valued distributions.

Proof. Let $|\Omega\rangle$ be a Poincaré invariant vacuum state. Consider 2-pt Wightman function:

$$\langle\Omega|\phi^\dagger(x)\phi(y)|\Omega\rangle$$

Then from translation invariance

$$\langle\Omega|\phi^\dagger(x+a)\phi(y+a)|\Omega\rangle = \langle\Omega|\phi^\dagger(x)\phi(y)|\Omega\rangle$$

where $a \in \mathbb{R}^d$. Implies there must exist a continuous function F on $\mathbb{R}^d$ such that,

$$\langle\Omega|\phi^\dagger(x)\phi(y)|\Omega\rangle = F(x-y)$$

Further, let $\{x_i\}_{i=1}^n$ be set of points in $\mathbb{R}^{(1,d-1)}$ and $\{z_i\}_{i=1}^n$ be a complex number, then

$$\sum_{1\leq i,j\leq n} F(x_i - x_j)\bar{z}_i z_j = \langle\Omega|\sum_{i=1}^{n}\phi^\dagger(x_i)\bar{z}_i\sum_{j=1}^{n}\phi(x_j)z_j|\Omega\rangle = \|\sum_{j=1}^{n}\phi(x_j)z_j|\Omega\rangle\| \geq 0$$

Implies $F(x)$ is a positive function; for definition, see [30]. Now, from the theorem of Bochner, which states that a complex-valued function $F \in \mathbb{R}^{d-1,1}$ is positive iff it is a Fourier transform of a nonnegative measure μ on $\mathbb{R}^{d-1,1}$,

$$F(x-y) = \langle\Omega|\phi^\dagger(x)\phi(y)|\Omega\rangle = \frac{1}{\sqrt{(2\pi)^d}}\int_{\mathbb{R}^{d-1,1}} e^{-ip.(x-y)}d\mu(p) \tag{3.4.41}$$

$d\mu(p) \geq 0$ and Lorentz invariant, i.e. $\mu(\Lambda p) = \mu(p)$, where Λ is a restricted Lorentz transformation. Now if we assume that $\phi(x)$ is an operator and $|\Omega\rangle$ is in its domain then,

$$\|\phi(x)|\Omega\rangle\|^2 = \int_{\mathbb{R}^{d-1,1}} d\mu(p) < \infty$$

Implies that the measure is integrable. The most general measure invariant under restricted Lorentz transformation is [31, 32],

$$d\mu(p) = \big(C\delta^d(p) + \Theta(p_0)\int_{-\infty}^{\infty}\delta(p^2 - m^2)d\rho(m^2)\big)d^dp \tag{3.4.42}$$

where C is a positive constant and $\rho(m^2)$ is non decreasing function with a polynomial growth. Note that the first term in the above equation is the only integrable part. Therefore,

$$\langle\Omega|\phi^\dagger(x)\phi(y)|\Omega\rangle = C$$

So we have learned that if $\phi(x)$ is a local operator (defined at a point), then the 2-pt function is just a constant. Hence, $\phi(x)$ with a nontrivial 2-pt function cannot be an operator. Let $f(x)$ and $g(x)$ be a real smooth function with compact support, then

$$\langle\Omega|\phi^\dagger[f]\phi[g]|\Omega\rangle = \sqrt{(2\pi)^d}\int_{\mathbb{R}^{d-1,1}}\tilde{f}^*(p)\tilde{g}(p)d\mu(p) < \infty \tag{3.4.43}$$

where,

$$\phi[f] := \int_{\mathbb{R}^{d-1,1}}\phi(x)f(x)d^dx \tag{3.4.44}$$

and $\tilde{f}$ and $\tilde{g}$ are the Fourier transform of f and g respectively. The finiteness of the equation (3.4.43) follows from the fact that the Fourier transform of a compactly supported function is a Schwartz function. Hence, the quantum fields can be thought of as operator-valued distributions. □

From the above proposition, we have learned that the homogeneity and isotropy of spacetime (or equivalently, Poincaré invariance) imply that fields are not defined as an operator at any specific point in spacetime, but rather as an operator-valued distribution. The second term in equation (3.4.42) exhibits a UV divergence in the limit $\mathbf{p}^2 \to \infty$(large momentum), which is regulated using a smooth function with compact support.

We emphasize two important points. First, similar results are expected to hold for higher-spin fields as well. Second, we anticipate that the above result remains valid in curved spacetime. Since any curved spacetime locally resembles Minkowski space at sufficiently small length scales, the UV divergence must be the same.

To better understand the distributional nature of the field and the precise meaning of $\phi[f]$, we now consider a free scalar field.

3.5 Example of Free Scalar field

Let us consider the free scalar field in d-dimensional Minkowski spacetime. Our approach here will be to start with the well-known canonical quantization, examine some characteristics of the scalar field, and then demonstrate that the smeared field operator is indeed a densely defined unbounded operator.

To begin, let us write the field expansion in terms of creation and annihilation operators.

$$\phi(x) = \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1}} \frac{1}{2\omega_{\mathbf{p}}} \left(e^{-ip.x} a_{\mathbf{p}} + e^{ip.x} a^{\dagger}_{\mathbf{p}}\right) \tag{3.5.1}$$

where $\omega_p = (\mathbf{p}^2 + m^2)^{\frac{1}{2}}$ and $p.x = \omega_p x^0 - \mathbf{p}.\vec{x}$. In the above equation, the creation and the annihilation operator are evaluated at $x^0 = 0$, and they satisfy equal time commutator,

$$[a_{\mathbf{p}}, a^{\dagger}_{\mathbf{p}'}] = 2\omega_{\mathbf{p}}(2\pi)^{d-1}\delta^{d-1}(\mathbf{p} - \mathbf{p}') \tag{3.5.2}$$

All other commutators are zero. The above equation tells us that $a_{\mathbf{p}}$ and $a^{\dagger}_{\mathbf{p}}$ are not really operator as their commutator results in a delta function (distribution). It will be helpful to write the (3.5.2) as,

$$\phi(x) = \psi(x) + \psi^{\dagger}(x) \tag{3.5.3}$$

where $\psi(x)$ and $\psi^\dagger(x)$ contain the $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$, respectively. It follows from (3.5.2),

$$[\psi(x), \psi^\dagger(y)] = \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1}} \frac{1}{2\omega_{\mathbf{p}}} e^{-ip(x-y)} = \Delta_+(x-y) \tag{3.5.4}$$

$\Delta_+(x-y)$ is clearly a distribution. It is not difficult to show that [31],

$$\begin{aligned}\Delta_+(x-y) = -i\frac{\pi\Theta(x^0-y^0)}{(2\pi)^{d-2}}\delta((x-y)^2) + \frac{\Theta((x-y)^2)}{2(2\pi)^{d-2}\sqrt{(x-y)^2}}\big[N_1(m\sqrt{(x-y)^2}))\\ + i\Theta(x^0-y^0)J_1(m\sqrt{(x-y)^2})\big] + \frac{m\pi\Theta(-(x-y)^2)}{(2\pi)^{d-2}\sqrt{-(x-y)^2}}K_1(m\sqrt{-(x-y)^2})\end{aligned} \tag{3.5.5}$$

where N_1 is Neumann function, J_1 is Bessel function and K_1 is modified Bessel function. It can easily be seen that the above integral diverges when $(x-y)^2 = 0$, if $x^\mu \to y^\mu$ or they are null separated. Further, let $|\Omega\rangle$ be Poincare invariant vacuum, such that $a_{\mathbf{p}}|\Omega\rangle = 0$. Then 2pt function is,

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \langle\Omega|[\psi(x), \psi^\dagger(y)]|\Omega\rangle = \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1}} \frac{1}{2\omega_{\mathbf{p}}} e^{-ip(x-y)} = \Delta_+(x-y) \tag{3.5.6}$$

As we have already seen, that $\Delta_+(x-y)$ diverges as $x^\mu \to y^\mu$ implies, $\|\phi(x)|\Omega\rangle\| \to \infty$. This divergence is Universal in the sense that a 2-pt function in any state at high energy must have the same behavior. This is due to the fact that any state at high energy looks like a vacuum state, and therefore, the 2-pt function should have the same UV behavior. Implies $\|\phi(x)|\Psi\rangle\| \to \infty$, for any state. This tells us that the quantum field $\phi(x)$ is too singular to be thought of as any operator-valued function, as the norm of the state that one gets after the action of the field on any arbitrary state is divergent (field operator does not map any state in Hilbert space back to Hilbert space). Now, notice in the equation (3.5.5) that the singular behavior at the coincidence point is coming from the delta distribution, and we know that we can deal with such a distribution by smearing over a suitable test function (Schwartz function). Now consider the smeared 2-pt function,

$$\langle\Omega|\phi[f]\phi[g]|\Omega\rangle = \int_{\mathbb{R}} f(x)\Delta_+(x-y)g(y)d^dxd^dy \tag{3.5.7}$$

where $\phi[f] := \int_{\mathbb{R}^d} \phi(x)f(x)d^dx$ and $f, g \in \mathcal{S}(\mathbb{R}^d)$ are real functions. For future ease, we want to write,

$$\int_{\mathbb{R}} f(x)\Delta_+(x-y)g(y)d^dxd^dy := (f|g) \tag{3.5.8}$$

Now, it is evident from equation (3.5.4) and (3.5.5) that equation (3.5.7) is finite everywhere. Hence, $\phi(x)$ is an operator-valued distribution. We again want to emphasize that the above analysis is easily generalized to curved spacetime, at least in Hadamard states. This generalization works because, at a very small length scale—much smaller than the scale of spacetime curvature—the behavior of fields is the same as in flat space. Hence, the UV divergence of the two-point function in curved spacetime can also be handled by smearing the fields. The above analysis is just an explicit demonstration of the general proposition (3.15) that we have proved above.

Now, if $\phi[f]$ is truly an operator, is it bounded or unbounded? What is the domain of the operator? To investigate this, we need to define the action of the operator on some subspace of the Hilbert space. Let us start by defining the n-particle state of identical bosons in the momentum basis using the creation operator $a_{\mathbf{p}}$.

$$\prod_{i=1}^{n} a^{\dagger}_{\mathbf{p}_i}|\Omega\rangle = |\mathbf{p}_1, ..., \mathbf{p}_n\rangle \tag{3.5.9}$$

In the above equation, we assumed that all $\mathbf{p}_i$ are distinct. However, if some $\mathbf{p}_i$ appear $n_{\mathbf{p}_i}$ times, then the above expression must be divided by $\sqrt{n_{\mathbf{p}_i}}$ for each such $\mathbf{p}_i$. These states are not normalizable, since $\langle \mathbf{p}'|\mathbf{p}\rangle = (2\pi)^{d-1} 2\omega_{\mathbf{p}} \delta^{d-1}(\mathbf{p} - \mathbf{p}')$ (eigenmodes of momentum operator) and therefore does not belongs to the Hilbert space. This is exactly because $a_{\mathbf{p}}$ was not an operator. The presence of a delta distribution in $\langle \mathbf{p}'|\mathbf{p}\rangle$ tells us that we must smear $|\mathbf{p}\rangle$ with some test function, that is, we should work with the states of type

$$\int_{\mathbb{R}^{d-1}} f_1(\mathbf{p})|\mathbf{p}\rangle \frac{1}{2\omega_{\mathbf{p}}} \frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1}} \tag{3.5.10}$$

where $f_1(p) \in \mathcal{S}(\mathbb{R}^d)$. As we already know, the Schwartz functions are square-integrable; therefore, these state belongs to the one particle Hilbert space $\mathcal{H}$. It can easily be shown,

$$\Psi^{\dagger}[f_1]|\Omega\rangle := \int_{\mathbb{R}^d} \Psi^{\dagger}(x) f_1(x) d^d x |\Omega\rangle = \int_{\mathbb{R}^{d-1}} f_1(\mathbf{p})|\mathbf{p}\rangle \frac{1}{2\omega_{\mathbf{p}}} \frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1}} \tag{3.5.11}$$

Now following the same trick, we can define the general state in the Fock space,

$$|\Phi\rangle = f_0|\Omega\rangle + \int_{\mathbb{R}^{d-1}} f_1(\mathbf{p}_1)|\mathbf{p}_1\rangle \frac{1}{2\omega_{\mathbf{p}_1}} \frac{d^{d-1}\mathbf{p}_1}{(2\pi)^{d-1}} + ... + \frac{1}{\sqrt{n!}} \int_{\mathbb{R}^{d-1}} \prod_{i=1}^{n} g_i(\mathbf{p}_i)|\mathbf{p}_1, ..\mathbf{p}_n\rangle \frac{1}{2\omega_{\mathbf{p}_i}} \frac{d^{d-1}\mathbf{p}_i}{(2\pi)^{d-1}} + ... \tag{3.5.12}$$

where $g_i(\mathbf{p}_i) \in \mathcal{S}(\mathbb{R}^d)$ and $f_0 \in \mathbb{C}$. The n-th term depicted above is clearly a n particle state, which belongs to the Hilbert space $\mathcal{H}^{\otimes_s n}$ (symmetrized tensor product) and therefore $|\Phi\rangle \in \mathcal{F}_{\mathcal{B}}(\mathcal{H})$, where $\mathcal{F}_{\mathcal{B}}(\mathcal{H})$ is a bosonic Fock Space . Also, notice that the above expression can also be written as,

$$|\Phi\rangle = f_0|\Omega\rangle + \Psi^\dagger[f_1]|\Omega\rangle + ... + \frac{1}{\sqrt{n!}}\prod_{i=1}^{n}\Psi^\dagger[g_i]|\Omega\rangle + ... \tag{3.5.13}$$

or equivalently,

$$|\Phi\rangle = f_0|\Omega\rangle + \phi[f_1]|\Omega\rangle + ... + \frac{1}{\sqrt{n!}} : \prod_{i=1}^{n}\phi[g_i] : |\Omega\rangle + ... \tag{3.5.14}$$

From the above equation, we have learned that the n-particles state $|n; \{g_1, .., g_n\}\rangle$ can be written as,

$$|n; \{g_1, .., g_n\}\rangle = \frac{1}{\sqrt{n!}} : \prod_{i=1}^{n}\phi[g_i] : |\Omega\rangle = \frac{1}{\sqrt{n!}}\prod_{i=1}^{n}\Psi^\dagger[g_i]|\Omega\rangle \tag{3.5.15}$$

Further, using (3.5.4) or the wick contraction, it can be shown that,

$$\langle m; \{f_1, ..., f_m\}|n; \{g_1, .., g_n\}\rangle = \delta_{nm}\frac{1}{n!}\sum_{\sigma\in\mathcal{S}^n}\prod_{i=1}^{n}(f_i|g_{\sigma(i)}) \tag{3.5.16}$$

where $\mathcal{S}^n$ is symmetric group on n objects. The above equation tells us that the state with different particles has no overlap. Further, there is no $(f|f)$ or $(g|g)$ because of the normal order. It can easily be shown that

$$\prod_{i=1}^{n}\phi[f_i]|\Omega\rangle = \text{Sum over normal ordered field} \tag{3.5.17}$$

Now, there is a beautiful theorem in QFT known as the Reeh-Schlieder theorem, which says that

$$\mathcal{H}_{RS} = \sum_{\oplus n}\mathcal{H}_n \tag{3.5.18}$$

where

$$\mathcal{H}_n = \{\sum_n \alpha_n|\psi_n\rangle : |\psi_n\rangle = \prod_{i=1}^{n}\phi[f_i]|\Omega\rangle, n\in\mathbb{Z}_+, \alpha\in\mathbb{C}\}$$

$\mathcal{H}_{RS}$ is dense in $\mathcal{F}_\mathcal{B}(\mathcal{H})$, ie $\overline{\mathcal{H}_{RS}} = \mathcal{F}_\mathcal{B}(\mathcal{H})$. Using (3.5.17), we can write,

$$\mathcal{H}_n = \{\sum_n \alpha_n|\psi_n\rangle : |\psi_n\rangle = \prod_{i=1}^{n} : \phi[f_i] : |\Omega\rangle, n\in\mathbb{Z}_+, \alpha\in\mathbb{C}\} \tag{3.5.19}$$

therefore, any state in Bosonic Fock space can be approximated with the state $|\Phi\rangle$ that we have constructed in (3.5.14). Now to find the domain of the operator $\phi[f]$, we first check the action of $\phi[f]$ on the state $|n;\{g_1,..,g_n\}\rangle$. It can easily be shown using (3.5.4) that,

$$\phi[f]|n;\{g_1,..,g_n\}\rangle = \sqrt{(n+1)}|n{+}1;\{f,g_1,..,g_n\}\rangle + \frac{1}{\sqrt{n}}\sum_{i=0}^{n}(f|g_i)|n{-}1;\{g_1,..,g_{i-1},g_{i+1}.,g_n\}\rangle \tag{3.5.20}$$

therefore $\phi[f]$ has good action on this states. Further, it can be shown by using the above equation, or equivalently, wick contraction that $\phi[f]$ has bounded action on such states,

$$\|\phi[f]|n;\{g_1,..,g_n\}\rangle\| \leq \sqrt{n}\sqrt{(g|g)}\||n;\{g_1,..,g_n\}\rangle\| \tag{3.5.21}$$

The above inequality can be further saturated. To show this, let us consider the case when $g_i(x) = f(x)$. Then, from (3.5.20),

$$\phi[f]|n;\{f,..,f\}\rangle = \sqrt{(n+1)}|n+1;\{f,f,..,f\}\rangle + \sqrt{n}(f|f)|n-1;\{f,..,f\}\rangle \tag{3.5.22}$$

therefore,

$$\|\phi[f]|n;\{f,..,f\}\rangle\| = \sqrt{(2n+1)}\sqrt{(f|f)}\||n;\{f,..,f\}\rangle\| \tag{3.5.23}$$

In order to get the above expression, we used the relation in (3.5.16). It implies that for every $C > 0$ there exists a $n \in \mathbb{Z}_+$ such that

$$\frac{\|\phi[f]|n;\{f,..,f\}\rangle\|}{\sqrt{\langle f|f\rangle}\||n;\{f,..,f\}\rangle\|} = \sqrt{(2n+1)} \geq C \tag{3.5.24}$$

Hence, $\phi[f]$ is an unbounded operator. Furthermore, it follows from the above equation that the domain of $\phi[f]$ is $\mathcal{H}_{RS}$. Since $\mathcal{H}_{RS}$ is dense in $\mathcal{F}_{\mathcal{B}}(\mathcal{H})$, the unbounded operator $\phi[f]$ is densely defined. It can easily be shown using (3.5.20) that $\phi[f]$ is a symmetric operator. Now, since $\phi[f]$ is densely defined and symmetric, it follows that $\phi[f]$ exists and $\phi[f] \subset \phi[f]^\dagger$, which implies that $\phi[f]^\dagger$ is dense. Therefore, $\phi[f]$ is a closed operator with closure $\overline{\phi[f]} = \phi[f]^{\dagger\dagger}$. It can further be shown that $\overline{\phi[f]}$ is self-adjoint [33], and it satisfies the equation of motion in the sense of distribution, that is, for all $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\phi[(\Box^2 + m^2)f] = 0 \tag{3.5.25}$$

3.5.0.1 Algebra of observables

Let M be the global hyperbolic spacetime and $\Sigma \subset M$ be the Cauchy surface. Let $V \subset \Sigma$ be an open set, small enough so that its closure $\bar{V}$ is not all of Σ [5]. Since $\bar{V}$ is not all of

[5] We are interested in the local algebra of observables.

Σ, the complement V' of $\bar{V}$ will also be a non-empty open set, disjoint from V. Clearly, V and V' are spacelike separated and we can contain them in the open sets $\mathcal{U}_V \subset M$ and $\mathcal{U}_{V'} \subset M$ respectively, which are also spacelike separated and are open sets in spacetime. There can be multiple choices for $\mathcal{U}_V$ and $\mathcal{U}_{V'}$, and one chooses them as large as they want as long as they are spacelike separated.

Definition 3.16. (Local algebra of the region $\mathcal{U}$):
Let $\mathcal{U}$ be the open set in spacetime. Then, the local algebra $\mathcal{A}_{\mathcal{U}}$ of the region $\mathcal{U}$ is defined as the von Neumann algebra of the bounded operators supported in $\mathcal{U}$.

Consider the quantum field theory in D dimension spacetime (M, g), where g denotes the metric on the spacetime M. For example, the scalar field $\phi(x)$ in Minkowski spacetime. We want to emphasize that the field $\phi(x)$ is not an operator in Hilbert space. Rather, it's an operator-valued distribution. Its distribution nature is apparent through the delta function appearing in the canonical commutation relation or through the universal singularity in its two-point function. To make it an unbounded operator, we need to smear it. Let $f(x)$ be the smooth function supported only in $\mathcal{U}$, then we can define smear field operator as

$$\phi[f] = \int_{\mathcal{U}} \sqrt{-g} d^D x f(x) \phi(x) \tag{3.5.26}$$

Now, it's an unbounded operator, as we have discussed in the previous subsection. We can construct a bounded operator from $\phi[f]$ by considering $F(\phi[f])$, where F is a complex-valued bounded function. For example, $e^{i\phi[f]}$ is a bounded operator. We can construct a more general bounded operator by considering multivariable bounded functions of these smeared field operators. The $*-$local algebra is then constructed by taking these simple operators and the operators that one can construct from them by taking their sums, products, and hermitian conjugates. Now, the von Neumann algebra associated with the local algebra is defined by closing the $*-$ local algebra under weak operator topology or equally by taking the double commutant.

From the above construction of the local algebra, it is not difficult to see that the algebra $\mathcal{A}_{\mathcal{U}}$ in the region $\mathcal{U}$ will be the same as the algebra $\mathcal{A}_{\hat{\mathcal{U}}}$ of causal completion $\hat{\mathcal{U}}$ of $\mathcal{U}$. It follows from the fact that the operator in the causal diamond can be written in terms of the operator on the Cauchy slice using the evolution equation of the fields.

Let us now understand more about the association of algebras to local regions, and the properties we expect such an association to satisfy. One of the most basic properties is the following: if $\mathcal{U}_1 \subset \mathcal{U}_2$, then the associated algebras should satisfy

$$\mathcal{A}_{\mathcal{U}_1} \subset \mathcal{A}_{\mathcal{U}_2} \quad \text{(Isotony)} \tag{3.5.27}$$

We expect that the algebra of a smaller region is contained within the algebra of a larger region is known as the **isotony** property of local algebras. Isotony guarantees that extending the region that we probe in real experiments should at least retain the measurements we could already make in a smaller subregion. Further, if we denote the causal completion of the region containing two causally complete region $\hat{\mathcal{U}}_1$ $\hat{\mathcal{U}}_2$ by $\hat{\mathcal{U}}_1 \vee \hat{\mathcal{U}}_2 = (\hat{\mathcal{U}}_1 \cup \hat{\mathcal{U}}_2)''$ and the von Neumann algebra associated with the $\hat{\mathcal{U}}_1 \vee \hat{\mathcal{U}}_2$ by $\mathcal{A}_{\hat{\mathcal{U}}_1} \vee \mathcal{A}_{\hat{\mathcal{U}}_2} = (\mathcal{A}_{\hat{\mathcal{U}}_1} \cup \mathcal{A}_{\hat{\mathcal{U}}_2})''$. Then we also expect the local algebra to follow,

$$\mathcal{A}_{\hat{\mathcal{U}}_1 \vee \hat{\mathcal{U}}_2} = \mathcal{A}_{\hat{\mathcal{U}}_1} \vee \mathcal{A}_{\hat{\mathcal{U}}_2} \quad \text{(Additivity)} \tag{3.5.28}$$

The additivity property (as expressed in the equation above) states that the algebra associated with a larger region can be generated from the algebras of smaller regions contained within it. Together with isotony, additivity provides an algebraic formulation of the idea that degrees of freedom are local. Specifically, it reflects the fact that one cannot reconstruct the algebra of a larger region from the algebra of its smaller subregions alone, since there exist local degrees of freedom in the larger region that are not accessible within the smaller one.

If $\hat{\mathcal{U}}'$ be the causal complement of $\hat{\mathcal{U}}$ (a maximally open set which is spacelike seperated from $\mathcal{U}$), then the algebra $\mathcal{A}_{\hat{\mathcal{U}}'}$ associated with $\hat{\mathcal{U}}'$ will commute with the algebra $\mathcal{A}_{\hat{\mathcal{U}}}$.

$$[\mathcal{A}_{\hat{\mathcal{U}}}, \mathcal{A}_{\hat{\mathcal{U}}'}] = 0 \tag{3.5.29}$$

The above equation can equivalently be written as $\mathcal{A}_{\hat{\mathcal{U}}} \subseteq (\mathcal{A}_{\hat{\mathcal{U}}'})'$. Therefore,

$$\text{Causality} \implies \mathcal{A}_{\hat{\mathcal{U}}} \subseteq (\mathcal{A}_{\hat{\mathcal{U}}'})' \tag{3.5.30}$$

In many interesting quantum field theories, and at least for a wide class of open regions such as a single open ball, it can be shown that $\mathcal{A}_{\hat{\mathcal{U}}'} = \mathcal{A}'_{\hat{\mathcal{U}}}$. This property was proposed by Haag and is known as **Haag duality** [34]. An important example where Haag duality holds is in Rindler space, as proven by Bisognano and Wichmann [35]. Haag duality can be interpreted as the requirement that the algebra associated with a region is *maximal*, in the sense that it includes all operators compatible with causality. For more details on local algebras, see [24].

However, Haag duality can be violated in various cases. For example, in gauge theories, it may fail for non-simply connected open regions in spacetime [36]. Moreover, there are theories with global symmetries where charged operators can be locally constructed. In such cases, there exist superselection sectors known as **DHR** (Doplicher, Haag, and Roberts) sectors. In theories admitting DHR sectors, regions with nontrivial topology generally violate Haag duality. Furthermore, as shown in [37], attempts to restore duality often lead to the failure of additivity of algebras; the two properties cannot be simultaneously preserved. Haag duality has several nontrivial implications. Notably, Haag duality in the vacuum sector is equivalent to the absence of spontaneously broken gauge symmetries [38, 39, 40].

As we know, the field operator does not commute at time-like and light-like separations. $\phi[f]$ and $\phi[g]$ will not commute if the support of f is time-like or light-like separated from the support of g. Therefore, $\mathcal{A}_{\hat{\mathcal{U}}}$ can have a non-trivial centre only if one can construct an operator supported on the bifurcation surface $B = \hat{\mathcal{U}} \cap \hat{\mathcal{U}}'$ (intersection right and left Rindler wedges), which commutes with $\mathcal{A}'_{\hat{\mathcal{U}}}$ and $\mathcal{A}_{\hat{\mathcal{U}}}$. However, we should not expect that such an operator will always exist. Because there is a region (the Cauchy horizon of $\hat{\mathcal{U}}$) in $\hat{\mathcal{U}}$ that is light-like separated from B, and the operator in B has to commute with the operator on the horizon. This will not happen unless there is some symmetry for which charges can be localized at the bifurcation surface. For example, if we have only one scalar field, then we know that null-separated fields do not commute, and we don't have any such symmetry. Thus, generically, the local algebra of the quantum field is a factor.

3.6 States of the quantum system and classification of von Neumann algebra

In this section, we present a physicist's perspective on the classification of von Neumann algebras, following the lecture notes of Sorce [25]. We begin by exploring the relationship between the existence of a trace on the algebra and the density matrix. This will lead us to a classification scheme based on whether a trace can be defined, either directly or through some renormalization procedure. In the next subsection, we will introduce the classification of von Neumann algebras using the theory of projections. Let us begin with the definition of the trace.

Definition 3.17. (Trace on von Neumann algebra):
A trace on the von Neumann algebra $\mathcal{A}$ is a map $\tau : \mathcal{A}_+ \to [0, \infty]$, where $\mathcal{A}_+$ are collection of all positive operators in $\mathcal{A}$, satisfying:

1. Homogenity: $\tau(\lambda a) = \lambda \tau(a)$ for all $a \in \mathcal{A}_+$ and all $\lambda \geq 0$
2. Additivity: $\tau(a + b) = \tau(a) + \tau(b)$ for all $a, b \in \mathcal{A}_+$.
3. Unitarity Equivalence: $\tau(UaU^\dagger) = \tau(a)$ for all $a \in \mathcal{A}_+$ and all unitary $U \in \mathcal{A}$.

The above definition arises from identifying the key properties of the Hilbert space trace and elevating them to define the abstract trace. The first two properties ensure that the trace is a linear map, while the third property generalizes the notion that the Hilbert space trace is independent of the choice of basis to an abstract setting. We want to emphasize that all the elements need not have finite trace. For example, the identity operator has infinite trace with respect to the Hilbert space trace in infinite dimensions. One might wonder why the trace is defined only on positive operators, given that a von Neumann algebra contains many more elements. However, this restriction is sufficient, and we can extend the trace using an interesting theorem to a bigger subset of $\mathcal{A}$.

Theorem 3.6.1. Every operator $a \in \mathcal{A}$ can always be written as,

$$a = a_+^R - a_-^R + ia_+^I - ia_-^I \tag{3.6.1}$$

where $a_\pm^{R/L} \in \mathcal{A}_+$.

Proof. For any bounded operator a, there exists a bounded adjoint $a^\dagger$. We can always decompose any bounded operator into a sum of two self-adjoint operators A_1 and A_2:

$$a = A_1 + iA_2$$

where

$$A_1 = \frac{1}{2}(a + a^\dagger), \qquad A_2 = \frac{1}{2i}(a - a^\dagger)$$

Since A_1 and A_2 are self-adjoint, they each have a real spectrum. Using the spectral decomposition, we can further decompose them into the difference of two positive operators:

$$A_1 = a_+^R - a_-^R, \qquad A_2 = a_+^I - a_-^I$$

where

$$a_{\pm}^{R} = A_1\,\Theta(\pm A_1), \qquad a_{\pm}^{I} = A_2\,\Theta(\pm A_2)$$

and Θ denotes the Heaviside step function. a^R and a^I are the restrictions of a to the real and imaginary parts of the spectrum. It is evident from the above expressions that $a_+^R, a_-^R, a_+^I, a_-^I$ are all positive operators. Therefore, we can express any bounded operator a as a linear combination of positive operators:

$$a = a_+^R - a_-^R + i a_+^I - i a_-^I$$

□

Further, if $a, b \in \mathcal{A}_+$ and $a \leq b$, then $b = a + (b - a)$, where $b - a$ is again a positive operator. This implies $\tau(a) \leq \tau(b)$.

Now, let $a \in \mathcal{A}$, and define $|a| \equiv \sqrt{a^\dagger a}$. By the spectral theorem, we have $|a| \geq a_{\pm}^{R/I}$, which means that $|a|$ dominates both the positive and negative parts of the real and imaginary components of a and $\tau(|a|) \geq \tau(a_{\pm}^{R/I})$. Implies if $\tau(|a|) < \infty$, then the sum $\tau(a_+^R) - \tau(a_-^R) + i\,\tau(a_+^I) - i\,\tau(a_-^I)$ is well defined as each term in sum is finite.

Now, using the above theorem and observation, we can extend the trace to all trace class operators denoted by $\mathcal{A}_T$.

Definition 3.18. (Extended trace):
Let $\mathcal{A}_\tau$ be the collection of all operators $T \in \mathcal{A}$ such that $\tau(|T|) < \infty$. The set $\mathcal{A}_\tau$ is called the space of *trace-class operators*. We can define an *extended trace* τ_{ext} on $\mathcal{A}_\tau$ by

$$\tau_{\text{ext}}(a) \equiv \tau(a_+^R) - \tau(a_-^R) + i\,\tau(a_+^I) - i\,\tau(a_-^I), \tag{3.6.2}$$

where a_+^R, a_-^R, a_+^I, and a_-^I denote the positive and negative parts of the real and imaginary components of the a, respectively.

The above definition by construction is well-defined. It can be easily shown that it satisfies all three properties of trace. And for the positive operator $\tau_{ext} = \tau$. Furthermore, the above trace has many nice properties:

1. If $a \in \mathcal{A}_\tau$ and U is some unitary in $\mathcal{A}$, then $\tau_{ext}(aU) < \infty$. This follows from the fact that $|a|$ dominates aU.

2. Using the first property and the unitary equivalence of τ, it follows that $\tau_{ext}(aU) = \tau_{ext}(Ua)$.

3. It follows from the second property and the fact that any operator can be written as a sum of two Hermitian operators, and that any Hermitian operator can be written as a linear combination of two unitary operators, that

$$\tau_{ext}(ab) = \tau_{ext}(ba) \tag{3.6.3}$$

for any $a \in \mathcal{A}_\tau$ and $b \in \mathcal{A}$. This is called the Cyclicity of trace.

4. Using the third property and the fact that when $a \in \mathcal{A}$ and $\tau(aa^\dagger)$ is infinite, $\tau(a^\dagger a)$ is infinite. We get

$$\tau(aa^\dagger) = \tau(a^\dagger a) \quad \forall a \in \mathcal{A} \tag{3.6.4}$$

Using the trace, we can define the concepts of state or density matrix.

Definition 3.19. (State or Density matrix):
A *state* or *density matrix* ρ of a system is a positive trace-class operator with respect to a trace τ, and satisfies

$$\tau(\rho) = 1 \qquad \rho \in \mathcal{A}_+ \cap \mathcal{A}_\tau$$

The state is called *pure* if and only if

$$\rho^2 = \rho$$

From the above definition, it is evident that the notion of a state is fundamentally linked to the existence of a trace, as the trace is used to define the state itself. Also, in practice, we are interested in finding the expectation value of an operator in some state described by the density matrix ρ, that we can assign by the following relation,

$$\langle a \rangle_\rho \equiv \tau_{ext}(\rho a) \tag{3.6.5}$$

It follows from the third property that $\tau_{ext}(\rho a)$ is well defined for all $a \in \mathcal{A}$. Therefore, the trace that we have defined is sufficient for our purpose.

From now on, we will omit the subscript *ext* and simply write τ to denote the extended trace τ_{ext}. Unless stated otherwise, all traces will be understood to refer to τ_{ext}.

We also emphasize that the above definition of the trace does not uniquely fix the trace; in

general, multiple traces may exist in a given von Neumann algebra. One might also wonder why we need to consider such general notions of trace, given that we always have the usual Hilbert space trace at our disposal. The reason is that the Hilbert space trace may not be well defined, and therefore, to capture the relevant physical or algebraic structure, we need a more general trace. Let us demonstrate this through an example.
Suppose we have a quantum system with a separable Hilbert space $\mathcal{H}$. In an actual experiment, we only have access to bounded operators, since we always operate with finite energy in experiments, and our measurement devices have finite resolution. Therefore, we have access to all bounded operators on $\mathcal{H}$, i.e., $\mathcal{B}(\mathcal{H})$.

But do we also have access to any density matrices?

Indeed, there are positive trace-class operators in $\mathcal{B}(\mathcal{H})$. For example, given any orthonormal sequence $|\psi_n\rangle \in \mathcal{H}$ and any sequence $p_n \in [0,1]$ with $\sum_n p_n = 1$, the operator

$$\rho = \sum_n p_n \, |\psi_n\rangle\langle\psi_n|$$

is a positive trace-class operator with respect to the Hilbert space trace, which is defined as

$$\mathrm{Tr}_{\mathcal{H}}(a) = \sum_n \langle\psi_n|a|\psi_n\rangle, \tag{3.6.6}$$

where a is some operator. It is also clear that ρ is a bounded operator with $\mathrm{Tr}_{\mathcal{H}}(\rho) = 1$. Therefore, ρ is a density matrix.

Now, suppose there is another quantum system in the Andromeda galaxy with Hilbert space $\mathcal{H}'$. It is natural to ask, what operators do we have access to?

Now that we are aware of the presence of another system, we must extend our Hilbert space to incorporate the degrees of freedom of the system in the Andromeda galaxy. Correspondingly, the algebra of operators must also be extended. We now have access to $\mathcal{B}(\mathcal{H}) \otimes 1_{\mathcal{H}'}$: operators that act as a bounded operator on $\mathcal{H}$ and as the identity on $\mathcal{H}'$.

Let us ask again: Do we have access to any density matrices?

To answer this, we return to the definition of a density matrix: it must be a trace-class operator with unit trace. The requirement that the trace be equal to one implies that the answer depends on the dimension of $\mathcal{H}'$. Let the dimension of $\mathcal{H}'$ be d, so that $1_{\mathcal{H}'}$ has trace d. Then, for any density matrix $\rho \in \mathcal{B}(\mathcal{H})$, we have access to the operator

$$\frac{\rho}{d} \otimes 1_{\mathcal{H}'} = \rho \otimes \frac{1_{\mathcal{H}'}}{d}, \tag{3.6.7}$$

which is itself a density matrix on the extended system.

However, if $\mathcal{H}'$ is infinite-dimensional, then $\frac{\rho}{d} \otimes 1_{\mathcal{H}'}$ is not a density matrix, infact the algebra $\mathcal{B}(\mathcal{H}) \otimes 1_{\mathcal{H}'}$ does not contains any density matrices with respect to Hilbert space trace, because there is no way to normalize $\rho \otimes 1_{\mathcal{H}'}$ to obtain unit trace on the combined system.

Therefore, we learn the following:

When $\mathcal{H}'$ is infinite-dimensional, there is no density matrix in $\mathcal{B}(\mathcal{H}) \otimes 1_{\mathcal{H}'}$ with respect to the Hilbert space trace of the combined system, because there is no way to normalize $\rho \otimes 1_{\mathcal{H}'}$ to have unit trace.
It might seem bizarre that knowing that there is another system out there that we do not have access to, can make the algebra of observables $\mathcal{B}(\mathcal{H}) \otimes 1_{\mathcal{H}'}$ not have a density matrix. We want the reader to notice that this is not a density matrix with respect to the Hilbert space trace. But since we can only access algebra of observable $\mathcal{B}(\mathcal{H}) \otimes 1_{\mathcal{H}'}$. It is legitimate to ask whether we can think of ρ as an effective density matrix for our system, which is a subsystem of the combined system. Let us rephrase this question as "Whether there is any consistent way to define a new trace for the algebra of observables of our system such that we can compute expectation values $\langle a \otimes 1_{\mathcal{H}'} \rangle_{\rho \otimes 1_{\mathcal{H}'}}$ of any operator a and that gives $\rho \otimes 1_{\mathcal{H}'}$ all the properties of a quantum state". This question is the central theme of the type classification of von Neumann algebras. The answer, in our case, is yes.
For any bounded operator $a \otimes 1_{\mathcal{H}'}$, we define its *effective* expectation value in the "state" $\rho \otimes 1_{\mathcal{H}'}$ in the obvious way:

$$\langle A \otimes 1_{\mathcal{H}'} \rangle_{\rho \otimes 1_{\mathcal{H}'}} \equiv \operatorname{Tr}_{\mathcal{H}}(\rho A). \tag{3.6.8}$$

here $\operatorname{Tr}_{\mathcal{H}}$ can be thought of as a renormalized trace for $\mathcal{B}(\mathcal{H}) \otimes 1_{\mathcal{H}'}$. So we have learned that

The operator $\rho \otimes 1_{\mathcal{H}'}$ is not a true density matrix with respect to the Hilbert space trace on the combined system. Nevertheless, it is a density matrix with respect to the algebra $\mathcal{B}(\mathcal{H}) \otimes 1_{\mathcal{H}'}$, relative to a certain *renormalized* trace. This allows us to consistently assign expectation values to $\rho \otimes 1_{\mathcal{H}'}$, thereby endowing it with the properties of a quantum state.

This renormalization procedure is not specific to the particular operator $\rho \otimes 1_{\mathcal{H}'}$; it can be applied analogously to any positive trace-class operator. This makes it clear that we need to define a more general notion of trace in order to meaningfully assign expectation values to the observables we have access to.

Let us take another step. Suppose we are given access to some general von Neumann algebra $\mathcal{A}$—for example, the von Neumann algebra associated with a subregion in the vacuum sector of a quantum field theory (3.5.0.1). It is important to realize that all actual experiments involve operators localized in the region where the experiment is performed. Hence, we never truly access the global state defined on the full algebra. What matters for us is the state that is associated with, or induced on, the local subregion. Therefore, the relevant notion of a state should be the density matrix associated with that subregion.

Within any von Neumann algebra $\mathcal{A}$, we have the set of positive operators by $\mathcal{A}_+$. Are any of these operators genuine density matrices? Perhaps not in the traditional sense. However, as in the previous discussion, some operators in $\mathcal{A}_+$ may act as effective density matrices for observables in $\mathcal{A}$. This motivates a natural question: does there exist a consistent renormalization scheme on $\mathcal{A}$ that promotes certain operators in $\mathcal{A}_+$ to well-defined quantum states?

The answer depends on the type of algebra of observables, which we will introduce in the next subsection. In particular, we will see that for a von Neumann factor, if the trace exists, then there exists only one consistent renormalized trace (i.e., renormalization scheme), up to an overall normalization, that acts on every operator in a well-defined way.

With respect to this renormalized trace, the type classification of von Neumann factors can be understood in the following terms:

- A factor $\mathcal{A}$ is said to be of **type I** if the renormalized trace allows some (possibly all) operators in $\mathcal{A}_+$ to be interpreted as pure states, and others as mixed states (density matrices that are not pure). That is, $\mathcal{A}$ contains *renormalizable pure and mixed states.*
- A factor $\mathcal{A}$ is of **type II** if the renormalized trace turns some (possibly all) operators in $\mathcal{A}_+$ into mixed states, but none into pure states. That is, $\mathcal{A}$ contains *renormalizable mixed states but no renormalizable pure states.*
- A factor $\mathcal{A}$ is of **type III** if, even after applying the renormalized trace, there are no operators in $\mathcal{A}_+$ that qualify as density operators. That is, $\mathcal{A}$ contains *no renormalizable states.*

Further,

- A factor $\mathcal{A}$ is said to be **finite** if the renormalized trace assigns a finite value to every operator in $\mathcal{A}_+$. That is, every positive operator in $\mathcal{A}$ becomes a renormalizable state.

- A factor $\mathcal{A}$ is **infinite** if the renormalized trace fails to normalize at least one operator in $\mathcal{A}_+$. That is, there exists at least one positive operator in $\mathcal{A}$ that is not a renormalizable state.

Note that based on the above definitions, every type III factor is infinite, while type I and II factors can be either finite or infinite. Let us conclude this section with some terminology and examples.

- A finite factor of type II is called **type $\mathbf{II_1}$**.
- An infinite factor of type II is called **type $\mathbf{II_\infty}$**.
- A finite factor of type I is called **type $\mathbf{I_n}$**, where n is a positive integer that encodes additional information about the factor (e.g., the dimension of the Hilbert space on which it acts).
- An infinite factor of type I is called **type $\mathbf{I_\infty}$**.

Examples:

- **Type $\mathbf{I_n}$**: $n \times n$ matrix algebra; system of n qubits.
- **Type $\mathbf{I_\infty}$**: Quantum harmonic oscillator.
- **Type $\mathbf{II_1}$**: The thermodynamic limit of two spin chains maximally entangled with each other; algebra of observables accessible to a static observer in de Sitter spacetime.
- **Type $\mathbf{II_\infty}$**: Algebra of observables in the exterior of a black hole in perturbative quantum gravity.
- **Type $\mathbf{III_1}$**: Local algebra of observables in quantum field theory.

More details can be found in [24, 3, 6, 41]. To define the renormalized trace, we need an effective way to discuss all positive operators in a von Neumann algebra. As is well known from spectral theory, any positive operator can be expressed in terms of its spectral projections. Therefore, projections can be viewed as the building blocks of positive operators.

By studying the projections associated with a von Neumann algebra and defining the renormalized trace on them, we can extend this definition to all positive operators. Thus, in the next section, we will study the theory of projections, their classification, and how this induces an algebraic classification of von Neumann algebras.

We will then define the renormalized trace on these projections, which in turn allows us to construct density matrices associated with the algebra and define the expectation values of positive operators in renormalized states—quantities of physical interest.

3.6.1 The theory of projections and Murray von Neumann classification

In this section, we will briefly talk about the theory of projectors, and then we will use it to outline the Murray-von Neumann classification of von Neumann algebras.

Definition 3.20. (Isometry):
A bounded operator $V \in \mathcal{B}(\mathcal{H})$ is said to be an isometry if,

$$\|V|\psi\rangle\| = \||\psi\rangle\| \quad \forall|\psi\rangle \in \mathcal{H}$$

It is evident from the above definition that the norm of isometry is unity. If V is an isometry, then the following interesting properties hold.

1. $\langle V\psi|V\phi\rangle = \langle\psi|\phi\rangle$ for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}$.

   Proof.

$$\begin{aligned}
\||\psi\rangle\|^2 + 2Re(\langle\psi|\phi\rangle) + \||\phi\rangle\|^2 &= \langle\psi+\phi|\psi+\phi\rangle \\
&= \||\psi+\phi\rangle\| \\
&= \|V(|\psi\rangle + |\phi\rangle)\|^2 \\
&= \langle V(\psi+\phi)|V(\psi+\phi)\rangle \\
&= \|V|\psi\rangle\|^2 + 2Re(\langle V\psi|V\phi\rangle) + \||\phi\rangle\|^2
\end{aligned}$$

   Implies, $Re(\langle\psi|\phi\rangle) = Re(\langle V\psi|V\phi\rangle)$ for any $|\phi\rangle$. So, let us take $i|\phi\rangle$ instead of $|\phi\rangle$. Then, we will get

$$Im(\langle\psi|\phi\rangle) = Re(-i\langle\psi|\phi\rangle) = Re(-i\langle V\psi|V\phi\rangle) = Im(\langle V\psi|V\phi\rangle)$$

   □

   Hence,$\langle V\psi|V\phi\rangle = \langle\psi|\phi\rangle$.

2. $V^\dagger V = I_{\mathcal{H}}$ and converse is also true.

Proof. This direction $\implies$ follows from property 2. For any $|\psi\rangle, |\phi\rangle \in \mathcal{H}$,

$$\langle\psi|\phi\rangle = \langle V\psi|V\phi\rangle = \langle\psi|V^\dagger V\phi\rangle \implies \langle\psi|V^\dagger V - I_{\mathcal{H}}|\phi\rangle = 0$$

which implies,

$$V^\dagger V - I_{\mathcal{H}} = 0$$

Now, let us prove the converse. We will assume that $V^\dagger V = I_{\mathcal{H}}$, then

$$\|V|\psi\rangle\|^2 = \langle V\psi|V\psi\rangle = \langle V^\dagger V\psi|\psi\rangle = \||\psi\rangle\|^2$$

Implies, $\|V\psi\rangle\| = \||\psi\rangle\|$ for all $|\psi\rangle \in \mathcal{H}$. Hence, V is an isometry. □

An isometry is an injective map but need not be a surjective one. We emphasize that $VV^\dagger$ is not necessarily the identity operator. It becomes the identity if and only if V is unitary.

Definition 3.21. (Partial isometry):
An operator $V \in \mathcal{B}(\mathcal{H})$ is said to be a partial isometry if there exists a closed subspace $\mathcal{M} \subset \mathcal{H}$, such that

$$\|V\,|\psi\rangle\| = \||\psi\rangle\|, \forall\,|\psi\rangle \in \mathcal{M}$$
$$V\,|\psi\rangle = 0, \text{for any } \;|\psi\rangle \in \mathcal{M}^\perp$$

Notice that in the above definition, $\mathcal{M}^\perp = \ker(V)$, and therefore a partial isometry is an isometry on $\mathcal{M} = \ker(V)^\perp$. We can further show that these are equivalent,

1. V is a partial isometry.
2. $V^\dagger$ is a partial isometry.
3. $V^\dagger V$ is a projection.
4. $VV^\dagger$ is a projection.
5. $V^\dagger VV^\dagger = V^\dagger$.
6. $VV^\dagger V = V$.

Proof. Let us start by showing that 1) $\implies$ 5). Assume that V is an isometry. We want to show that for any $|\psi\rangle, |\phi\rangle \in \mathcal{H}$,

$$\langle V^\dagger V V^\dagger \psi|\phi\rangle = \langle V^\dagger \psi|\phi\rangle.$$

First, suppose $|\phi\rangle \in \ker(V)$. Then,

$$\langle V^\dagger V V^\dagger \psi|\phi\rangle = \langle V V^\dagger \psi|V\phi\rangle = 0 = \langle \psi|V\phi\rangle = \langle V^\dagger \psi|\phi\rangle.$$

Now suppose $|\phi\rangle \in \ker(V)^\perp$. Then,

$$\langle V^\dagger V V^\dagger \psi|\phi\rangle = \langle V(V^\dagger \psi)|V\phi\rangle.$$

Since V is an isometry on $\ker(V)^\perp$ and satisfies property 1),

$$\langle V^\dagger V V^\dagger \psi|\phi\rangle = \langle V(V^\dagger \psi)|V\phi\rangle = \langle V^\dagger \psi|\phi\rangle.$$

Thus, we conclude that $V^\dagger V V^\dagger = V^\dagger$.

Now, it is easy to show that 5) $\implies$ 6):

$$V = (V^\dagger)^\dagger = (V^\dagger V V^\dagger)^\dagger = V V^\dagger V.$$

Similarly, 6) $\implies$ 5):

$$V^\dagger = (V)^\dagger = (V V^\dagger V)^\dagger = V^\dagger V V^\dagger.$$

Therefore, 5) $\iff$ 6).

It is also straightforward to see that 5) $\implies$ 3). First, note that $V^\dagger V$ is self-adjoint, and

$$(V^\dagger V)^2 = V^\dagger V V^\dagger V = (V^\dagger V V^\dagger)V = V^\dagger V,$$

which implies that $V^\dagger V$ is a projection. Similarly, by analogous steps, 5) $\implies$ 4).

Now we show 4) $\implies$ 1). Assume $VV^\dagger$ is a projection. Let $|\phi\rangle \in \ker(V)^\perp = \overline{\text{ran}(V^\dagger)}$. Then, there exists a sequence $\{|\phi_n\rangle\} \subset \text{ran}(V^\dagger)$ such that $\lim_{n\to\infty} |\phi_n\rangle = |\phi\rangle$. Then,

$$\begin{aligned}\|V|\phi\rangle\|^2 &= \lim_{n\to\infty} \|V|\phi_n\rangle\|^2 \\ &= \lim_{n\to\infty} \langle VV^\dagger\phi_n|VV^\dagger\phi_n\rangle \\ &= \lim_{n\to\infty} \langle (VV^\dagger)^2\phi_n|\phi_n\rangle \\ &= \lim_{n\to\infty} \|V^\dagger\phi_n\|^2 = \||\phi\rangle\|^2.\end{aligned}$$

Hence, V is a partial isometry.

By similar reasoning, it can be shown that 3) $\implies$ 2). It is also straightforward to show that 2) $\implies$ 6). Hence, all the statements are equivalent. □

Theorem 3.22. *If V is a partial isometry, then* $\text{ran}(V)$ *is closed subspace of $\mathcal{H}$, further $VV^\dagger$ is the projection on* $\text{ran}(V)$ *and $V^\dagger V$ is projection on* $\text{ran}(V^\dagger)$.

Proof. To show that $\text{ran}(V)$ is closed, suppose that $|\phi\rangle \in \overline{\text{ran}(V)}$ then there exists a sequence $\{|\phi_n\rangle\} \in \mathcal{H}$ such that $|\phi\rangle = \lim_{n\to\infty} V|\phi_n\rangle$. Then, using the continuity of bounded operators,

$$V(V^\dagger|\phi\rangle) = \lim_{n\to\infty} V(V^\dagger V|\phi_n\rangle) = \lim_{n\to\infty} V|\phi\rangle = |\phi\rangle$$

implies $|\phi\rangle \in \text{ran}(V)$, $\overline{\text{ran}(V)} \subset \text{ran}(V)$. Therefore, $\text{ran}(V) = \overline{\text{ran}(V)}$ is closed.
Let $|\phi\rangle \in \text{ran}(V)$, then $\exists\ |\psi\rangle \in \mathcal{H}$ such that $|\phi\rangle = V|\psi\rangle$, then

$$VV^\dagger|\phi\rangle = VV^\dagger V|\psi\rangle = V|\psi\rangle = |\phi\rangle$$

Further if $|\phi\rangle \in \text{ran}(V)_\perp = \ker(V^\dagger)$, then $VV^\dagger|\phi\rangle = 0$. Hence $VV^\dagger$ is a projection on $\text{ran}(V)$. Similarly, $V^\dagger V$ is projection on $\text{ran}(V^\dagger)$. □

Now, we would like to give some examples of isometry and partial isometry.

Example 3.4. ***(Isometry):***

$$S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \quad S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$$

- *$S^\dagger S = I$, so S is an isometry.*
- *$SS^\dagger \neq I$, so S is not unitary.*

- *S preserves norms but is not surjective.*

Example 3.5. ***(Partial Isometry):***

$$S^\dagger : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \quad S^\dagger(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$$

- *$S^\dagger$ is a partial isometry.*
- $\ker(S^\dagger) = span\{(1,0,0,\dots)\}$
- *$S^\dagger$ is an isometry on* $\ker(S^\dagger)^\perp$

Note that $S = \frac{a^\dagger}{\sqrt{(1+a^\dagger a)}}$ and $(x_1, x_2, x_3, ..) = \sum_{n=0}^{\infty} x_{n+1}|n\rangle$ in the quantum harmonic oscillator. Another outcome of partial isometry is the polar decomposition theorem.

Theorem 3.23. ***Polar decomposition:***
For any operator $x \in \mathcal{B}(\mathcal{H})$, there exists a unique positive operator $a \in \mathcal{B}(\mathcal{H})$ and a unique partial isometry $V \in \mathcal{B}(\mathcal{H})$, such that

$$x = Va \tag{3.6.9}$$

$$V^\dagger V = s(a) \tag{3.6.10}$$

where $s(a)$ is projection on $\overline{\mathrm{ran}(a)}$.

Proof. Let us define a positive operator $a = (x^\dagger x)^{\frac{1}{2}}$ and define v_0 on $a\mathcal{H}$ by,

$$v_0 a|\psi\rangle = x|\psi\rangle \quad \forall|\psi\rangle \in \mathcal{H}$$

Clearly,

$$\|v_0 a|\psi\rangle\|^2 = \|x|\psi\rangle\|^2 = \langle x^\dagger x\psi|\psi\rangle = \langle a^2\psi|\psi\rangle = \|a|\psi\rangle\|^2$$

From the continuity of v_0, it is an isometry on $\overline{a\mathcal{H}}$. We can easily extend it as a partial isometry on full $\mathcal{H}$ as follows,

$$V|\psi\rangle = \begin{cases} V_0|\psi\rangle & \text{for} \quad |\psi\rangle \in \overline{a\mathcal{H}} \\ 0 & \text{for} \quad \text{otherwise,} \end{cases}$$

From the properties of partial isometry $s(a) = V^\dagger V$ is a projection on $\overline{a\mathcal{H}}$. Now, it can easily be checked,

$$x = Va$$
$$V^\dagger V = s(a)$$

Uniqueness follows from the fact that a is unique for the given $x \in \mathcal{B}(\mathcal{H})$ and V is unique for a. It follows from the above proof that,

$$x^\dagger = bV^\dagger \tag{3.6.11}$$
$$V^\dagger V = s(b) \tag{3.6.12}$$

where $b = (xx^\dagger)^{\frac{1}{2}}$. When x is self-adjoint then $a = b = |x|$ and $s(a) = s(b) = s(|x|)$. □

Definition 3.24. Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. An operator $V \in \mathcal{A}$ is a partial isometry if $V^\dagger V$ is a projection.

From the properties of the partial isometry, it follows that $V^\dagger$ is also a partial isometry. Moreover, for $E, F \in \mathcal{P}_\mathcal{A}$ and defined as $E = V^\dagger V$ and $F = VV^\dagger$,

$$VE = V \qquad FV = V \tag{3.6.13}$$
$$VEV^\dagger = F \qquad V^\dagger FV = E \tag{3.6.14}$$

It follows from (3.6.13) that,

$$V(E\mathcal{H}) = V(V^\dagger V\mathcal{H}) = VV^\dagger V\mathcal{H} = V\mathcal{H} \subseteq F\mathcal{H} \tag{3.6.15}$$

Now for any $|\phi\rangle \in \mathcal{H}$, we have $EV^\dagger |\phi\rangle = V^\dagger |\phi\rangle \in E\mathcal{H}$, and thus

$$F|\phi\rangle = VV^\dagger |\phi\rangle = V(V^\dagger |\phi\rangle) \in V(E\mathcal{H}), \tag{3.6.16}$$

so $F\mathcal{H} \subseteq V(E\mathcal{H})$. Therefore,

$$V(E\mathcal{H}) = F\mathcal{H} \tag{3.6.17}$$

Therefore, V maps the subspace $E\mathcal{H}$ to $F\mathcal{H}$. Since V is an isometry on $E\mathcal{H}$, it implies that $\dim(E\mathcal{H}) = \dim(F\mathcal{H})$. This shows that V is a map between projections of the same rank. Furthermore, if two projections have the same rank, then there is a partial isometry that connects them. This is because any two Hilbert spaces of the same dimension are isomorphic. Thus, partial isometries are the appropriate algebraic objects to define equivalence between projections, as they compare only the "relevant parts" (the images of the projections, or the rank of the projections) rather than the entire space.

Definition 3.25. (Equivalent projection):
Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then $E, F \in \mathcal{P}_{\mathcal{A}}$ are said to be equivalent with respect to $\mathcal{A}$ if there exists a partial isometry $V \in \mathcal{A}$, such that $E = V^\dagger V$ and $F = VV^\dagger$. We will denote it as $E \sim F$.

Another way of putting the above definition is that two projections in a von Neumann algebra are equivalent if their ranks are the same relative to the von Neumann algebra. We can easily show that $\sim$ is an equivalence relation. Let $P, Q, R \in \mathcal{P}_{\mathcal{A}}$.

1. **Reflexivity:** Let $V = P$. Then $V^\dagger V = VV^\dagger = P$, so $P \sim P$.
2. **Symmetry:** If $P \sim Q$ via partial isometry V, then $V^\dagger V = P$, $VV^\dagger = Q$. Let $W = V^\dagger$, then $W^\dagger W = Q$, $WW^\dagger = P$, so $Q \sim P$
3. **Transitivity:** If $P \sim Q$ via partial isometry V, and $Q \sim R$ via partial isometry W, define $U = WV$. Then:

$$U^\dagger U = V^\dagger W^\dagger WV = V^\dagger QV = V^\dagger V = P,$$

$$UU^\dagger = WVV^\dagger W^\dagger = WQW^\dagger = WW^\dagger = R.$$

So $P \sim R$

Therefore, $\sim$ is an equivalence relation.

Definition 3.26. (Central projection):
A projection in the von Neumann algebra $\mathcal{A}$ is called a central projection if it belongs to the centre $Z_{\mathcal{A}}$ of $\mathcal{A}$.

Central projection has a very nice property that it preserves the equivence of projections. That is,

$$E \sim F, \;\; P \in \mathcal{P}_{Z_{\mathcal{A}}} \implies EP \sim FP \tag{3.6.18}$$

This follows from the fact that there exists a partial isometry V such that $E = V^\dagger V$ and $F = VV^\dagger$. Now we can define a partial isometry $W = VP$[6]. Since $W^\dagger W \sim WW^\dagger \implies$

[6] It follows properties of partial isometry that we have proved and the fact that $W^\dagger W$ is a projection.

$EP \sim FP$.

Another interesting thing we can define using central projection is the concept of central support of the element of algebra. Let $a \in \mathcal{A}$, we define

$$Z(a) = \bigwedge\{E \in \mathcal{P}_{Z_\mathcal{A}}; Ea = a\} \tag{3.6.19}$$

clearly, $Z(a) \in \mathcal{P}_{Z_\mathcal{A}}$. So it is the smallest central projection, and $Z(a)a = aZ(a) = a$. $Z(a)$ is also known as central support of a. Further, it is easy to show that the central support of two equivalent projections is equal. That is,

$$E \sim F \implies \quad Z(E) = Z(F) \tag{3.6.20}$$

It follows from the fact that if V is a partial isometry then $Z(VV^\dagger) = Z(V)$ and $Z(V^\dagger V) = Z(V)$, implies $Z(E) = Z(F)$.

Proposition 3.27. *Let $\mathcal{A}$ be a von Neumann algebra. Let $\{E_i\}_{i\in I}$,$\{F_i\}_{i\in I} \in \mathcal{P}_\mathcal{A}$, where E_i are mutually orthogonal projections and F_i are also mutually orthogonal projections such that $E_i \sim F_i$ for all $i \in I$. Then*

$$\bigvee_{i\in I} E_i \sim \bigvee_{i\in I} F_i$$

Proof. Since $E_i \sim F_i$ for each $i \in I$, there exists a partial isometry $V_i \in \mathcal{A}$ such that

$$V_i^\dagger V_i = E_i \quad \text{and} \quad V_i V_i^\dagger = F_i$$

Because the $\{E_i\}_{i\in I}$ are mutually orthogonal, and likewise the $\{F_i\}_{i\in I}$, the ranges of the V_i are mutually orthogonal, and their initial spaces are orthogonal as well. Thus, for each $|\psi\rangle \in \mathcal{H}$, only countably many $V_i|\psi\rangle$ are non-zero, and their sum converges in norm. Hence, the series

$$V := \sum_{i\in I} V_i$$

converges in the strong operator topology in $\mathcal{A}$, since $\mathcal{A}$ is a von Neumann algebra (closed under strong limits), and the sum of strongly orthogonal partial isometries converges strongly.

Now compute:

$$V^\dagger V = \sum_{i,j} V_i^\dagger V_j = \sum_i V_i^\dagger V_i = \sum_i E_i = \bigvee_{i\in I} E_i,$$

and similarly,

$$VV^\dagger = \sum_{i,j} V_i V_j^\dagger = \sum_i V_i V_i^\dagger = \sum_i F_i = \bigvee_{i\in I} F_i.$$

Thus, V is a partial isometry in $\mathcal{A}$ such that

$$V^\dagger V = \bigvee_{i\in I} E_i, \quad VV^\dagger = \bigvee_{i\in I} F_i,$$

so we conclude that

$$\bigvee_{i\in I} E_i \sim \bigvee_{i\in I} F_i \tag{3.6.21}$$

□

The above result tells us that the equivalence of projections is compatible with orthogonal sums.

Now, we have the notion of the equivalence of projections, but we want to go one step further and define the notion of sub-projection that is compatible with the equivalence of projections.

Definition 3.28. (Sub equivalence):
Let $\mathcal{A}$ be the von Neumann algebra, and let $E, F \in \mathcal{P}_\mathcal{A}$. We call E subequivalent to F, and denote it as $E \preceq F$ (and read it as F dominates E), if $\exists$ a partial isometry $V \in \mathcal{A}$ such that $E = V^\dagger V$ and $VV^\dagger \equiv Q \leq F$. E is equivalent to some subprojection of F.

The relation $\preceq$ on the projections is a preorder relation (reflexive and transitive). Let $E, F, G \in \mathcal{P}_\mathcal{A}$.

1. **Reflexivity:** Take $V = E$, then $V^\dagger V = E$, $VV^\dagger = E \leq E$, so $E \preceq E$.
2. **Transitivity:** Suppose $E \preceq F$ via partial isometry $V \in \mathcal{A}$, so $V^\dagger V = E$, $VV^\dagger \leq F$; and $F \preceq G$ via partial isometry $W \in \mathcal{A}$, so $W^\dagger W = F$, $WW^\dagger \leq G$. Define $U = WV \in \mathcal{A}$. Then:

$$U^\dagger U = V^\dagger W^\dagger W V = V^\dagger F V = V^\dagger V = E,$$

$$UU^\dagger = WVV^\dagger W^\dagger \leq WFW^\dagger = WW^\dagger \leq G.$$

Hence, $E \preceq G$.

Thus, $\preceq$ is reflexive and transitive, hence a preorder. Given the above definitions, it is natural to ask whether any two projections can be compared using the preorder relation. The answer is, in general, no. In finite-dimensional Hilbert spaces, any two projections are comparable via dimension: for instance,

$$\mathrm{rank}(P) \leq \mathrm{rank}(Q) \quad \Rightarrow \quad P \preceq Q.$$

However, in infinite-dimensional case, it may happen that they are not directly comparable. We can prove following theroem.

Theorem 3.29. *If $E, F \in \mathcal{P}_{\mathcal{A}}$, then following statements are equivalent:*

1. *E and F are centrally orthogonal,i.e $Z(E)Z(F) = 0$.*
2. *$E\mathcal{A}F = \{0\}$*
3. *For all nonzer projection $E_1 \leq E$ and $F_1 \leq F$, E_1 and F_1 are inequivalent.*

Proof. We start with 1) $\implies$ 2). Let $a \in \mathcal{A}$. Then

$$EaF = Z(E)EaZ(F)F = EaFZ(E)Z(F) = 0, \tag{3.6.22}$$

which implies that $E\mathcal{A}F = \{0\}$.

Now, to show 2) $\implies$ 1), $E\mathcal{A}F = \{0\}$. Then $EZ(F) = 0$, and therefore $Z(E)Z(F) = 0$.

For 2) $\implies$ 3), suppose that statement 3) is not true. Then, there exist $E_1 = VV^\dagger$ and $F_1 = V^\dagger V$, satisfying $E_1 \leq E$ and $F_1 \leq F$, we have

$$EVF = EE_1VF_1F = E_1VF_1 = V \tag{3.6.23}$$

by 2), V must vanish. Implies, statement 3) must be true if 2) is true.

Now, to prove that 3) $\implies$ 2), assume that 2) is not true. Then there exists $0 \neq a \in E\mathcal{A}F$. But since $a = EaF = Ea = Fa$, it follows that $s(a) \leq E$ and $s(a^\dagger) \leq F$, which contradicts 3).

Therefore, 3) implies 2). $\square$

The above theorem shows that there exist projections E and F for which neither $E \preceq F$ nor $F \preceq E$ holds. However, there is a weaker notion of comparison for the projections.

Theorem 3.6.2. **(The Comparison theorem)**:
For any $E, F \in \mathcal{P}_{\mathcal{A}}$, there exists a $P \in \mathcal{P}_{Z_{\mathcal{A}}}$ where $Z_{\mathcal{A}}$ is the center of $\mathcal{A}$, such that

$$EP \preceq FP,$$
$$F(1-P) \preceq E(1-P)$$

Proof. Let $\{(E_i, F_i)\}$ be a maximal pair of families of mutually orthogonal projections with $E_i \leq E$, $F_i \leq F$, and $E_i \sim F_i$. Let

$$E_0 = \bigvee_i E_i, \quad F_0 = \bigvee_i F_i.$$

Then from the Proposition (3.27), $E_0 \sim F_0$. We can define the orthogonal complements $E' = E - E_0$, $F' = F - F_0$. Since E_0 and F_0 are maximal, there cannot be any equivalent subprojection of E' and F'. It follows from the previous theorem that $Z(E')Z(F') = 0$ or equivalently $E'Z(F') = 0$.
Let P be the central support of F', i.e $P = Z(F)$. Then $P \in \mathcal{P}_{Z_{\mathcal{A}}}$, and we have:

$$EP = E_0P \sim F_0P \leq FP, \quad \text{and} \quad F(1-P) = F_0(1-P) \sim E_0(1-p) \leq E(1-P).$$

Hence,

$$EP \preceq FP, \quad \text{and} \quad F(1-P) \preceq E(1-P),$$

as claimed. □

The above theorem tells us that even if two projections cannot be directly compared, we can still compare them on subspaces determined by central projections P. Moreover, it tells us that when $\mathcal{A}$ is a factor, any two projections are directly comparable.
Classification of Projections and von Neumann algebras:
Let E be a projection in a von Neumann algebra $\mathcal{A}$. Then E is said to be:

1. **Abelian:** If $E\mathcal{A}E$ is abelian.

2. **Finite:** If for $F \in \mathcal{P}_{\mathcal{A}}$, $E \sim F \leq E \implies E = F$. That is, E is not equivalent to any proper subprojection of itself.

3. **Infinite:** If it is not finite. That is, E can be equivalent to some proper subprojection of itself.

4. **Properly infinite:** If for every central projection F, EF is either infinite or zero. There is no central subspace on which F is non-trivial and finite.

5. **Purely infinite:** If for every projection $F \leq E$, F is either infinite or zero. That is, E should not have any non-trivial finite subprojection.

6. **Minimal:** If for any $F \in \mathcal{P}_{\mathcal{A}}$, $F \leq E \implies E = F$.

Further, we can prove some very useful theorems. As we will see, they play an important role in the decomposition of von Neumann algebras.

Theorem 3.30. *A nonzero projection $E \in \mathcal{P}_{\mathcal{A}}$ is* abelian *if, for every $F \in \mathcal{P}_{\mathcal{A}}$ with $F \leq E$, we have*

$$F = EZ(F),$$

where $Z(F)$ denotes the central support of F.

Furthermore, if $\mathcal{A}$ is a factor, then a projection E is abelian if and only if it is minimal.

Proof. Let $E, F \in \mathcal{P}_{\mathcal{A}}$ and $F \leq E$, then we want to show that if E is abelian, then $F = E$. Since $E\mathcal{A}E$ is commutative, for any $a \in \mathcal{A}$, we have

$$Fa(E - F) = F(EaE)(E - F) = F(E - F)(EaE) = 0 \tag{3.6.24}$$

hence $F\mathcal{A}(E - F) = \{0\}$, implies $F = EZ(F)$. If $\mathcal{A}$ is a factor, then if $F = E$. Conversely, if E is a minimal then $F \leq F$, implies $F = E = EZ(F)$. □

Another very useful theorem will be useful.

Theorem 3.31. *Let $\{E_i\}$ be a family of finite (respectively abelian) projections, centrally orthogonal support. Then $E = \bigvee_{i \in I} E_i$ is a finite (respectively abelian) projection.*

Proof. Let F is some projection,such that $F \leq E$, $F \sim E$. Using the fact that $Z(E_i)$ are mutually orthogonal,

$$(E - E_i) \bigvee_{n \neq i} Z(E_n) = E - E_i \implies E_i = EZ(E_i) \tag{3.6.25}$$

since E_i are finite,

$$E_i = EZ(E_i) = FZ(E_i) \leq F \quad \forall i \in I \tag{3.6.26}$$

Implies $E \leq F$[7]. Therefore $E = F$ and E is finite.
Now, let $\{E_i\}$ be a family of abelian projections with centrally orthogonal supports in a von Neumann algebra. Let F be a projection such that $F \leq E := \bigvee_{i \in I} E_i$.

Then we can define

$$F = \bigvee_{i \in I} F_i, \quad \text{where} \quad F_i := E_i F E_i.$$

Since the central supports $Z(F_i)$ and $Z(F_j)$ are centrally orthogonal for $i \neq j$, it follows that

$$Z(F_i)Z(F_j) = \delta_{ij} Z(F_j).$$

Thus, the family $\{Z(F_i)\}$ consists of mutually orthogonal central projections.

Furthermore, since $F_i E_i = F_i$, it follows that $F_i \leq E_i$. Because E_i is abelian, any subprojection is of the form $E_i Z$ for some central projection Z (relative to the center of $\mathcal{A}$ restricted to E_i). Hence,

$$F_i = E_i Z(F_i).$$

Therefore,

$$F = \bigvee_{i \in I} F_i = \bigvee_{i \in I} E_i Z(F_i) = E \left(\bigvee_{i \in I} Z(F_i) \right).$$

Since $\bigvee_{i \in I} Z(F_i)$ is a central projection, it follows that

$$F = EZ(F),$$

where $Z(F)$ is the central support of F.

Thus, for every projection $F \leq E$, we have $F = EZ(F)$. This proves that E is an abelian projection. E is abelian[8]. $\square$

The above classification of projections provides a useful way to analyze von Neumann algebras. Since von Neumann algebras are unital, and identity I dominates every projection. Therefore, we can naturally extend the notion of finite and infinite to von Neumann algebras as follows. Let $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ be the von Neumann algebra. Then $\mathcal{A}$ is said to be:

[7] The Finitness is used in the second equality. That is $E \sim F \implies Z(E_i)E \sim Z(E_i)F$ and since $Z(E_i)E$ is finite. Implies $Z(E_i)E = Z(E_i)F$.

[8] See theorem (3.30)

- **Finite, infinite, properly infinite, and purely infinite**, if I is a finite, infinite, properly infinite, and purely infinite projection, respectively.
- **Semifinite**, if any non-zero central projection contains a non-zero finite projection.

Murray von Neumann Classification:
Let $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ be the von Neumann algebra. Then $\mathcal{A}$ is said to be:

1. **Type I:** If for every non zero central projection E, there exists a non-zero abelian projection in $\mathcal{A}$. Further
   - It is Type I_{fin} if it is finite.
   - It is Type I_{∞} if it is infinite.
2. **Type II:** If it is semifinite and contains no non-zero abelian projection. Further,
   - It is Type II_1 if it is finite.
   - It is Type II_{∞} if it is not finite.
3. **Type III:** If it contains no non-zero finite projection.

It is easy to see that every subprojection of an abelian projection is finite. Let E be an abelian projection, then from (3.30), for $F \leq E$, $F = EZ(F)$. We want to show that F is finite. Let $G \leq F$ and $G \sim F$, then $G \leq E$, implies $G = EZ(G)$. But since $F \sim G$, $Z(G) = Z(F) \implies F = G$. Therefore, F is finite.
Now, let us use this fact in the classification above. If a von Neumann algebra contains an abelian projection (also contains a finite projection), then it is of Type I. If it does not contain any abelian projections but does contain some finite projections, then it is of Type II. Furthermore, if it contains neither abelian nor finite projections, then it is of Type III. Now, let us use this to prove the following theorem.

Theorem 3.32. ***(Decomposition theorem):***
Let $\mathcal{A}$ be the von Neumann algebra. Then there exists a unique decomposition of $\mathcal{A}$ into the direct sum

$$\mathcal{A} = \mathcal{A}_I \bigoplus \mathcal{A}_{II} \bigoplus \mathcal{A}_{III} \tag{3.6.27}$$

of type I, type II, and type III von Neumann algebras.

Proof. Let $\{E_n\}_{n\in I}$ be a maximal family of centrally orthogonal abelian projections. Then, by (3.31), $E := \bigvee_{n\in I} E_n$ is abelian. Let Z_I be the central support of E, i.e $Z_I = Z(E)$. Then $\mathcal{A}_I := \mathcal{A}Z_I$ is a von Neumann algebra of type I.

Indeed, if $Z \leq Z_I$, that is, if Z is a nonzero central projection in $\mathcal{A}_I$, then ZE is a nonzero abelian projection dominated by Z (if it were zero, then $Z_I - Z < Z_I$ would contradict the fact that Z_I is the central support of E).

By construction, $(1 - Z_I)\mathcal{A}$ is a von Neumann algebra with no abelian projections. Let $\{F_i\}$ be a maximal family of centrally orthogonal finite projections in $(1 - Z_I)\mathcal{A}$, and let $F = \sum F_i$, which is finite by (3.31). Now, let Z_{II} be the central support of F in $(1-Z_I)A$, i.e $Z_{II} = Z(F)$. Then $A_{II} = Z_{II}(1 - Z_I)A$ is a von Neumann algebra, and, as in the previous paragraph, one shows that it is of type II.

Finally, letting $Z_{III} = 1 - Z_I - Z_{II}$, we find that Z_{III} is central, and $A_{III} = Z_{III}A$ is a type III von Neumann algebra. Therefore,

$$\mathcal{A} = \mathcal{A}_I \oplus \mathcal{A}_{II} \oplus \mathcal{A}_{III} \tag{3.6.28}$$

The uniqueness follows from the uniqueness of $Z(E)$ and $Z(F)$ for any equivalence maximal family of orthogonal projections. □

For more details, see [22]. The theorem above shows that a general von Neumann algebra does not need to be of a single type; rather, it can be decomposed into components of different types. If algebra is a factor, then the following corollary follows.

Corollary 3.6.3. A factor $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is of exactly one of the following types: I_n, I_∞, II_1, II_∞, or III. It is of *type I* if it contains a minimal projection. Furthermore, if the identity projection I can be written as a sum of n mutually orthogonal minimal projections, then $\mathcal{A}$ is of type I_n. If $n = \infty$, then it is of type I_∞.

If $\mathcal{A}$ has no minimal projections but contains a non-zero finite projection, then it is of *type II.* Within type II, if the identity projection I is itself finite, then $\mathcal{A}$ is of type II_1; if I is infinite, then it is of type II_∞.

Finally, if $\mathcal{A}$ has no non-zero finite projections at all, then it is of *type III.*

Proof. If $\mathcal{A}$ is a factor, then its center is trivial. That is, the center contains only the identity projection I or the zero projection. From the decomposition theorem, we have

$$Z_I + Z_{II} + Z_{III} = I,$$

Which implies that only one of them can be nonzero. Hence, a factor von Neumann algebra can be of only one type.

Now, suppose Z_I is the only nonzero central projection. Then the algebra is of type I. Moreover, in every factor, every abelian projection is minimal. Since $Z_I = I$ dominates the collection of all mutually orthogonal abelian projections E, it follows that $E = Z_I = I$. If not, then the orthogonal complement $I - E$ would be a nonzero central projection. But because the algebra is type I, this complement must contain an abelian subprojection, which contradicts the maximality of E.

Therefore, in a type I factor, the identity projection can be written as a sum of minimal projections—specifically, $n \leq \dim \mathcal{H}$ minimal projections[9]. If n is finite, it will be type I_n, otherwise it will be type I_∞.
Now, suppose Z_{II} is the only non-zero central projection. Since Z_{II} does not support any abelian projections, it follows that the algebra lacks minimal projections and is therefore of type II. Furthermore, if the identity projection I is finite, then the algebra is of type II_1; otherwise, it is of type II_∞.

Finally, if Z_{III} is non-zero, then the algebra contains neither abelian nor minimal finite projections. Therefore, it is of type III. □

Thus, having classified von Neumann algebras via the theory of projections, we now return to our original question: Can a renormalized trace be defined that renders all or some projections finite? Moreover, is such a construction possible for every von Neumann algebra?

We have already introduced the definition of a general trace in (3.17). We have also seen that it allows us to define the expectation value of any operator in the algebra. However, the definition is quite general and imposes no restrictions on how the trace assigns values to positive operators.

Physically, we do not want our trace to assign zero to any nonzero positive operator or nonzero density matrix. This motivates the requirement that the trace be **faithful**. Furthermore, we want the trace to be compatible with the **spectral decomposition** of self-adjoint operators, since every observable can be written as the supremum over linear combinations of its spectral projections. That is, we expect

$$\tau\left(\sup_i \rho_i\right) = \sup_i \tau(\rho_i)$$

[9]If $\mathcal{H}$ is a separable Hilbert space with orthonormal basis $\{|e_i\rangle\}$, then $E_i = |e_i\rangle\langle e_i|$ is a minimal projection and $E = \sum_i^{dim\mathcal{H}} |e_i\rangle\langle e_i| = I$

whenever $\{\rho_i\}$ is an increasing net of positive operators (e.g., spectral projections). This is the property of **normality**.

Finally, our entire motivation for considering such a general notion of trace is to be able to define **density matrices**. Therefore, we want our trace to assign **finite values** to all **finite projections**. This is the condition of **semifiniteness**. These requirements motivate us to define the normalized trace as follows.

Definition 3.33. (Renormalized trace):
A trace $\tau : \mathcal{A}_+ \to [0, \infty]$ is a normalized trace if it follows:

1. **Faithful**: For any $\rho \in \mathcal{A}_+$, $\tau(\rho) = 0 \implies \rho = 0$.
2. **Semi-finitness**: For every $P \in \mathcal{A}_+$ there exists some non-zero $Q \in \mathcal{A}_+$, with $Q \leq P$ and $\tau(Q) < \infty$.
3. **Normal**: If $\{\rho_i\}$ is a family of positive operator in $\mathcal{A}_+$ with $\rho = \sup_i \rho_i$, then $\tau\left(\sup_i \rho_i\right) = \sup_i \tau(\rho_i)$.

This is the minimal requirement we expect from any well-defined trace. As emphasized earlier, projections can be thought of as the building blocks of positive operators. From the above definition, it follows,

1. Every non-zero projection will have a non-zero trace.
2. Equivalent projections have equal trace.
   Let $E, F \in \mathcal{P}_\mathcal{A}$ and $E \sim F$. The there will some partial isometry V such that, $E = V^+V$ and $F = VV^\dagger$. From (3.6.4), it follows that,
   $$\tau(E) = \tau(V^\dagger V) = \tau(VV^\dagger) = \tau(F)$$
3. If $E, F \in \mathcal{P}_\mathcal{A}$ and $E < F$, then $\tau(E) < \tau(F)$.
   Using linearity and the fact that $F = E + F - E$,
   $$\tau(F) = \tau(E) + \tau(F - E) \implies \tau(F) > \tau(E) \tag{3.6.29}$$
4. An infinite projection has infinite trace. For every infinite projection, there exists an equivalent proper subprojection. Let $F \in \mathcal{P}_\mathcal{A}$ be an infinite projection. Then, there exists a projection E such that $E < F$ and $E \sim F$. Suppose $\tau(E)$ is finite. Since

$F > E$, we have

$$\tau(F) = \tau(E) + \tau(F - E).$$

Since $E \sim F$, $\tau(F) = \tau(E)$, implies $\tau(F - E) = 0$. From property (1), this would mean $E = F$, which is a contradiction since E is a proper subprojection of F. Therefore, $\tau(E) = \infty$.

5. If $E, F \in \mathcal{P}_{\mathcal{A}}$ are finite projection with $\tau(E) = \tau(F)$. Then $E \sim F$.
Let us assume $F \preceq E$. Then there exists $F' \sim F$, such that $F' \leq E$. Therefore,

$$\tau(E) = \tau(F') + \tau(E - F').$$

This implies $\tau(E - F') = 0$. From property (1), it follows that $E = F' \sim F$.

Similarly, if we assume $F \preceq E$, the result will be the same. Hence, $E \sim F$.

It can be shown that any two infinite projections are equivalent. Furthermore, it follows from properties $(2) - (5)$ above that the trace preserves the algebraic comparison; that is, it respects both the von Neumann equivalence and the partial order on projections. That is, for any $E, F \in \mathcal{P}_{\mathcal{A}}$,

$$\tau(E) = \tau(F) \iff E \sim F$$

$$\tau(E) \leq \tau(F) \iff E \preceq F$$

Now, we would like to ask: which von Neumann algebras can admit the renormalized trace defined above?
Let us start with type III von Neumann algebras. From the algebraic classification of von Neumann algebras, we know that in a type III von Neumann algebra, there are no non-zero finite projections. This implies that for any projection E in a type III algebra, $\tau(E) \in \{0, \infty\}$. Therefore, there is no trace on a type III algebra that satisfies semi-finiteness, since semi-finiteness requires the existence of projections with finite, non-zero trace. As a result, there is no renormalized trace for type III von Neumann algebras.

Now, let the algebra be a type I_n factor. Since in a factor algebra all projections are comparable, and type I algebras have minimal projections, any two minimal projections must be equivalent. Then, from property (2), any minimal projection must have the same trace. Let λ be the trace for any nonzero minimal projection. Furthermore, we know that any projection can be written as a linear combination of at most n minimal mutually orthogonal projections. For any projection E in a type I_n algebra, we have

$$\tau(E) = C_n\lambda \quad \text{with } C_n \in \{0, 1, \ldots, n\},$$

and $\tau(I) = n\lambda$, which follows from the fact that the identity dominates every projection. We emphasize that n can be infinite, and then the algebra will be type I_∞. This clearly satisfies all the properties of the renormalized trace.

Now, let the algebra be of type II. In a type II von Neumann algebra, there are no minimal projections. However, there is a maximal projection, namely the identity I, which dominates every other projection. Let the trace of the identity be $\lambda \in \mathbb{R}$, which is finite in the case of type II_1 algebras and infinite in the case of type II_∞ algebras. Then for any projection E in a type II algebra, we have $\tau(E) \in [0, \lambda]$. Now, let us argue that for each $\alpha \in [0, \lambda]$, there exists a projection in the algebra such that the trace takes the value α. This implies that the trace takes values continuously.

In type II for any projection E with $\tau(E) \leq \lambda$, and for any $\alpha \in [0, \tau(E)]$, define the set:

$$\mathcal{S}_\alpha := \{F \leq E \,|\, F \text{ is a projection and } \tau(F) \leq \alpha\}.$$

This set is non-empty and directed. By Zorn's Lemma, there exists a maximal element $F_\alpha \in \mathcal{S}_\alpha$. Suppose $\tau(F_\alpha) < \alpha$. Since $E - F_\alpha \neq 0$ and $\mathcal{A}$ contains no minimal projections, we can find a projection $G \leq E - F_\alpha$ with $\tau(G) > 0$ and such that $\tau(F_\alpha) + \tau(G) < \alpha$, contradicting the maximality of F_α. Therefore, $\tau(F_\alpha) = \alpha$.

Furthermore, there is a remarkable theorem by von Neumann which states that the trace defined above is unique up to an overall scaling factor (for details, see [22, 23]). By an overall scaling, we mean that any other trace $\tau'(\cdot)$ must be of the form $\tau'(\cdot) = \alpha\tau(\cdot)$, where α is a positive constant.

We conclude this section by emphasizing that, through appropriate scaling of finite projections, we can interpret them as density matrices. Therefore, only type I and type II von Neumann algebras admit density matrices. Furthermore, in type I algebras, if we choose the normalization $\lambda = 1$, then each minimal projection corresponds to a pure state. Hence, only type I algebras admit pure states in this sense. This observation connects the discussion of the previous section with the current one, thereby completing our exposition.

3.7 Tomita Takesaki modular theory

Tomita–Takesaki modular theory is a powerful tool for investigating von Neumann algebras. It plays a crucial role in the mathematical development of von Neumann algebras and their applications to physical systems, such as systems described by statistical mechanics and quantum field theory. There are many outstanding works by Connes, Bisognano, Wichmann, Borchers, and others, where modular theory has played a pivotal role. It is modular theory that makes it possible to connect abstract von Neumann algebras with local algebras in quantum field theory in a useful and meaningful way. An excellent review on the use of modular theory in quantum field theory is provided by Borchers [42]. Modular theory also plays an important role in understanding black hole thermodynamics, as we will see later in this thesis.

Let us now begin by briefly introducing the key concepts of modular theory.

3.7.1 Brief Review of Modular Theory

Let $\mathcal{A}$ be a von Neumann algebra acting on Hilbert space $\mathcal{H}$. Let $\mathcal{A}'$ be the commutant of the algebra $\mathcal{A}$.

Definition 3.34. (Cyclic and Separating vector):
A vector $|\psi\rangle \in \mathcal{H}$ is said to be cyclic for algebra $\mathcal{A}$ if

$$\overline{\mathcal{A}|\psi\rangle} = \mathcal{H}$$

And it is called separating with respect to $\mathcal{A}$, if

$$\forall a \in \mathcal{A}, \quad a\,|\psi\rangle = 0 \implies x = 0. \tag{3.7.1}$$

The above definition tells us that if a vector is **cyclic**, then the action of the algebra $\mathcal{A}$ on this vector generates a dense subspace of the Hilbert space. In other words, a cyclic vector encodes enough information to approximate any state in the Hilbert space using vectors of the form $a|\psi\rangle$, with $a \in \mathcal{A}$. Thus, given an algebra $\mathcal{A}$, we can, in principle, generate the Hilbert space from a single vector (”cyclic vector”), also known as GNS reconstruction.

Furthermore, the **separating** property of a vector ensures that the vector can distinguish (separate) between distinct operators in $\mathcal{A}$. That is, if $a, b \in \mathcal{A}$ with $a \neq b$, then a separating vector $|\psi\rangle$ satisfies $a|\psi\rangle \neq b|\psi\rangle$. Equivalently, $a|\psi\rangle = 0$ implies $a = 0$.

Furthermore, cyclic and separate vectors also encode information about the commutant. We can prove the following proposition.

Proposition 3.35. *Let $\mathcal{A}$ be any von Neumann algebra and $\mathcal{A}'$ be its commutant. Then*

$$|\psi\rangle \in \mathcal{H} \;\; \textit{is cyclic for } \mathcal{A} \iff |\psi\rangle \;\; \textit{is Separating for } \mathcal{A}'$$

Proof. We will first prove $\implies$ direction.
Let $|\psi\rangle \in \mathcal{H}$ is cyclic for $\mathcal{A}$ and $b \in \mathcal{A}'$ is some operator, such that $b|\psi\rangle = 0$, then

$$ba|\psi\rangle = [b, a]|\psi\rangle = 0 \quad \forall a \in \mathcal{A} \tag{3.7.2}$$

Since $\mathcal{AH}$ is dense in $\mathcal{H}$, for every $|\phi\rangle \in \mathcal{H}$ there exists a sequence of operators $\{a_n\} \in \mathcal{A}$ such that $\lim_{n\to\infty} a_n|\psi\rangle = |\phi\rangle$. We know from the above equation that $ba_n|\psi\rangle = 0$ for any n. Therefore, from the continuity of bounded operators,

$$b|\phi\rangle = \lim_{n\to\infty} ba_n|\psi\rangle = 0 \tag{3.7.3}$$

Implies, $b = 0$. Hence $|\psi\rangle$ is separating for $\mathcal{A}'$.
Now let us prove the $\impliedby$ direction. We assume that $\mathcal{A}$ is *not cyclic*, and we aim to show that $\mathcal{A}'$ is *not separating*. Let $|\psi\rangle \in \mathcal{H}$ be a vector that is not cyclic for $\mathcal{A}$, so the cyclic subspace

$$\mathcal{H}^1 := \overline{\mathcal{A}|\psi\rangle} \subsetneq \mathcal{H}.$$

Define P to be the orthogonal projection onto the orthogonal complement $(\mathcal{H}^1)^\perp$. Since $|\psi\rangle \in \mathcal{H}^1$ (as $I \in \mathcal{A}$), it follows that

$$P|\psi\rangle = 0.$$

Moreover, P is a bounded operator and belongs to $\mathcal{A}'$; we have established this in (3.4). Therefore, $|\psi\rangle$ is *not separating* for $\mathcal{A}'$. $\square$

Since $\mathcal{A}$ can be any von Neumann algebra, and we know $\mathcal{A}'$ is always a von Neumann algebra. Therefore,

$$|\psi\rangle \in \mathcal{H} \;\; \text{is cyclic for } \mathcal{A}' \iff |\psi\rangle \;\; \text{is Separating for } \mathcal{A}''$$

Furthermore, it follows from the above proposition that if $|\psi\rangle$ is cyclic and separating for $\mathcal{A}$, then it is cyclic and separating for $\mathcal{A}'$, and the converse is also true.

Given a cyclic and separating state, Tomita and Takesaki showed that one can define an important operator that relates any operator in the algebra to its adjoint. This operator plays a crucial role in understanding the structure of von Neumann algebras. In particular, it is essential for analyzing Type III algebras, which are of physical importance to us, especially in the context of quantum field theory.

Definition 3.36. (Tomita operator):
Let $|\psi\rangle \in \mathcal{H}$ be a cyclic and separating vector, then the Tomita operator for the pair $(\mathcal{A}, \psi)$ is an antilinear operator,

$$S_\psi : \mathcal{A}\,|\psi\rangle \subset \mathcal{H} \to \mathcal{A}\,|\psi\rangle \subset \mathcal{H}$$
$$S_\psi a\,|\psi\rangle = a^\dagger\,|\psi\rangle\,, \forall a \in \mathcal{A}$$

We want to emphasize that the cyclic and separating vector plays a crucial role in the definition of the Tomita operator. Since $|\psi\rangle$ is separating, for any non-zero $a \in \mathcal{A}$, we have $a\,|\psi\rangle \neq 0$. This is important for S_ψ to be well-defined. Otherwise, if for some a, $a\,|\psi\rangle = 0$, then S_ψ would map it to $a^\dagger\,|\psi\rangle \neq 0$, violating well-definedness.

Additionally, due to the cyclic property of $|\psi\rangle$, the domain of S_ψ is dense in $\mathcal{H}$. Together, cyclicity and separability imply that S_ψ is closable. From now on, we will assume that S_ψ has been closed and denote the closed extension by the same symbol.

Furthermore, the Tomita operator is unbounded. If it were bounded, then by the bounded operator extension theorem, it could be extended to a bounded operator on the full Hilbert space. This would imply the existence of a bounded operator that relates every bounded operator to its adjoint, which is not true in general.

For example, in quantum field theory, we know that any state at very high energy resembles the vacuum. Therefore, we can use high-energy modes to construct a sequence of operators that approximate the annihilation operator. However, the adjoints of these operators will approximate the creation operator. So, S_ψ maps states very close to the vacuum to states that are not close to the vacuum, and hence it must be unbounded.
It is straightforward to see from the above definition that,

- $S_\psi^2 = I$
- $S_\psi\,|\psi\rangle = |\psi\rangle$

Therefore, S_ψ is an invertible operator. We also want to emphasize that it is a very special operator in the sense that it is an unbounded operator whose square is bounded.
It is natural to ask whether we can define a similar operator for the commutant. Let S'_ψ be the Tomita operator for the pair $(\mathcal{A}', \psi)$. We can easily show that $S'_\psi = S^\dagger_\psi$.
For any $a \in \mathcal{A}$ and $b' \in \mathcal{A}'$,

$$\langle S'_\psi b'\psi|a|\psi\rangle = \langle b'^\dagger\psi|a|\psi\rangle = \langle\psi|\, b'a\,|\psi\rangle = \langle a^\dagger\psi|b'|\psi\rangle = \langle S_\psi a\psi|\, b'\,|\psi\rangle = \langle S^\dagger_\psi b'\psi|a|\psi\rangle \quad (3.7.4)$$

In the last equality, we have used the antilinearity of the S_ψ [10]. Since the above equation is true for any $a \in \mathcal{A}$, $b' \in \mathcal{A}'$ and cyclic separating state $|\psi\rangle$. The Tomita operator S'_ψ of $(\mathcal{A}', |\psi\rangle)$ is $S^\dagger_\psi$.

Theorem 3.37. *Let T be a closed, densely defined operator on a Hilbert space $\mathcal{H}$. Then there exists a positive self-adjoint operator A, with $D_A = D_T$ and an isometric operator: $V : ran(A) \to \overline{ran(T)}$ where $ran(A) = ran(T)$ such that $T = VA$ and $A^2 = T^\dagger T$. Moreover, if ker(A)=ker(T) , such decomposition is unique.*

Proof. Since T is closed and densely defined, the adjoint $T^\dagger$ exists and is also densely defined. Then the operator $T^\dagger T$ is a positive, self-adjoint operator. By the spectral theorem for densely defined operators, we can define,

$$A := (T^\dagger T)^{1/2}.$$

Then A is positive, self-adjoint, and $A^2 = T^\dagger T$, with $D(A) = D(T)$.

For any $|\psi\rangle \in D(T)$, we compute:

$$\|T|\psi\rangle\|^2 = \langle T\psi|T|\psi\rangle = \langle T^\dagger T\psi|\psi\rangle = \langle A^2\psi, \psi\rangle = \|A\psi\|^2.$$

Therefore, the mapping

$$V : A\psi \mapsto T\psi$$

defines an isometry on $\mathrm{ran}(A)$, since it preserves the norm. Thus, V extends to a partial isometry from $\mathrm{ran}(A)$ to $\overline{\mathrm{ran}(T)}$, and we can write

$$T = VA.$$

[10] An antilinear operator K has the following property, $\langle\psi|\, K\,|\phi\rangle = \overline{\langle K^\dagger\psi|\phi\rangle} = \langle\phi|\, K^\dagger\,|\psi\rangle$.

Uniqueness: Suppose $T = V_1 A = V_2 A$ are two such decompositions with the same A, and assume $\ker(A) = \ker(T)$. Then $V_1 A|\psi\rangle = V_2 A|\psi\rangle$ for all $|\psi\rangle \in D(A)$, and hence $V_1 = V_2$ on $\mathrm{ran}(A)$. Therefore, V is uniquely determined under this condition. □

The more elaborate proof of the above theorem can be found in any standard book. Since S_ψ satisfies all the properties of the above theorem and it is antilinear and invertible, we can uniquely decompose S_ψ into

$$S_\psi = J_\psi \Delta_\psi^{1/2} \tag{3.7.5}$$

where J_ψ is antiunitary operator called modular conjugation and $\Delta_\psi^{1/2}$ is a self adjoint positive operator satisfying,

$$\Delta_\psi = S_\psi^\dagger S_\psi \tag{3.7.6}$$

Δ_ψ is known as a modular operator. Similarly, we can obtain the polar decomposition, $S'_\psi = J'_\psi \Delta_\psi'^{1/2}$. The modular operator and modular conjugation have many nice properties. We will list some of them here.

1. $\Delta_\psi \, |\psi\rangle = |\psi\rangle$.
   It follows from the fact that I belongs to both $\mathcal{A}$ and $\mathcal{A}'$. Therefore,

$$S_\psi|\psi\rangle = |\psi\rangle, \quad S_\psi^\dagger|\psi\rangle = |\psi\rangle \implies \Delta_\psi \, |\psi\rangle = S_\psi^\dagger S_\psi \, |\psi\rangle = |\psi\rangle$$

2. $J_\psi \Delta_\psi^{1/2} J_\psi = \Delta_\psi^{-1/2}$.
   Since $S_\psi^2 = I$, it follows from polar decomposition,

$$J_\psi \Delta_\psi^{1/2} J_\psi \Delta_\psi^{1/2} = I$$

   Implies,

$$J_\psi \Delta_\psi^{1/2} J_\psi = \Delta_\psi^{-1/2}$$

3. $J_\psi \Delta_\psi^{it} J_\psi = \Delta_\psi^{it}$, where $t \in \mathbb{R}$
   It follows from property (2) and antilinearity of J_ψ.

4. $J_\psi^2 = I$
   Since $J_\psi \Delta_\psi^{1/2} J_\psi = \Delta_\psi^{-1/2}$, and J_ψ has inverse,

$$J_\psi^2 (J_\psi^{-1} \Delta_\psi^{1/2} J_\psi) = \Delta_\psi^{-1/2} = I.\Delta_\psi^{-1/2}$$

Since $J_\psi^{-1}\Delta_\psi^{1/2}J_\psi$ is positive and from the uniqueness of polar decomposition of $\Delta_\psi^{-1/2}$, we must have $J_\psi^2 = I$. This also implies that $J_\psi^\dagger = J_\psi$.

5. $J'_\psi = J_\psi$ and $\Delta'_\psi = \Delta_\psi^{-1}$.
   Since,
   $$S'_\psi \equiv J'_\psi \Delta'^{1/2}_\psi = S_\psi^\dagger = \Delta_\psi^{1/2}J_\psi = J_\psi J_\psi \Delta_\psi^{1/2}J_\psi = J_\psi \Delta_\psi^{-1/2}$$
   where we have used the property (2) and (4). From the uniqueness of polar decomposition, $J'_\psi = J_\psi$ and $\Delta'_\psi = \Delta_\psi^{-1}$.

6. $\forall a, b \in \mathcal{A}$, $\langle\psi|\, a\Delta_\psi b\, |\psi\rangle$=$\langle\psi|\, ba\, |\psi\rangle$
   $$\langle\psi|\, a\Delta_\psi b\, |\psi\rangle = \langle\psi|\, aS_\psi^\dagger S_\psi b\, |\psi\rangle = \langle\psi|\, aS_\psi^\dagger b^\dagger\, |\psi\rangle = \overline{\langle S_\psi a^\dagger \psi | b^\dagger | \psi\rangle} = \langle\psi|ba|\psi\rangle$$

7. Let $a(t) \equiv \Delta_\psi^{it} a \Delta_\psi^{-it}$, where $a \in \mathcal{A}$ is a modular evolution of an operator. Then, all operators that evolve under a modular operator satisfy the **KMS condition**. That is, $\forall a, b \in \mathcal{A}$
   $$\langle\psi|a(t+i)b(t)|\psi\rangle = \langle\psi|b(t)a(t)|\psi\rangle$$
   We can easily prove the above relation using property (1) and 6),
   $$\begin{aligned}\langle\psi|a(t+i)b(t)|\psi\rangle &= \langle\psi|\Delta_\psi^{it-1}a\Delta_\psi^{-it+1}\Delta_\psi^{it}b\Delta_\psi^{-it}|\psi\rangle = \langle\psi|\Delta_\psi b|\psi\rangle = \langle\psi|ba|\psi\rangle \\ &= \langle\psi|\Delta_\psi^{it}b\Delta_\psi^{-it}\Delta_\psi^{it}a\Delta_\psi^{-it}|\psi\rangle = \langle\psi|b(t)a(t)|\psi\rangle\end{aligned}$$
   This tells us that the state is thermal with unit temperature with respect to the modular Hamiltonian $K_\psi = -\ln\Delta_\psi$.

8. $\forall a, b, c \in \mathcal{A}$, $[S_\psi a S_\psi, b]c|\psi\rangle = 0$
   $$S_\psi a S_\psi bc|\psi\rangle = S_\psi a c^\dagger b^\dagger|\psi\rangle = bca^\dagger\psi\rangle = bS_\psi ac^\dagger|\psi\rangle = bS_\psi a S_\psi c|\psi\rangle$$

9. The modular operator generates the automorphisms of the algebra, and modular conjugation acts as a reflection onto the commutant. That is, for all $a \in \mathcal{A}$ and $t \in \mathbb{R}$,
   $$\Delta_\psi^{it}a\Delta_\psi^{-it} \in \mathcal{A}, \quad J_\psi a J_\psi \in \mathcal{A}'.$$
   This famous theorem by Takesaki is not proven here, but interested readers can refer to Takesaki's book [43] or notes by Sorce [23, 44].

One might initially think that in property (8), we have shown that $S_\phi a S_\phi$ belongs to the commutant algebra. However, what we have actually shown is that $S_\phi a S_\phi$ commutes with all elements of the algebra when viewed as an operator on the subspace $\mathcal{A}\mathcal{H}$. Such operators are said to be *affiliated* with the commutant algebra: they commute with all elements of the algebra but do not necessarily belong to the algebra.

For an operator to belong to the algebra, it must be bounded and admit a bounded extension to the entire Hilbert space. In the case of $S_\phi a S_\phi$, we have not yet demonstrated that it is bounded or that it extends to a bounded operator on the full Hilbert space; establishing this would require additional analysis.

However, Takesaki's theorem that proves property (9) also establishes that $S_\phi a S_\phi$ does indeed belong to the commutant algebra. Since such operators frequently appear and behave almost like elements of the algebra, we now provide the definition of an operator affiliated with a von Neumann algebra, to avoid further confusion.

Definition 3.38. (Operator Affiliated to the Algebra):
Let $\mathcal{A}$ be a von Neumann algebra with commutant $\mathcal{A}'$. A closed operator T (possibly unbounded) is said to be affliated to the algebra $\mathcal{A}$, if it comutes with all the element of $\mathcal{A}'$ on every vector where both $a'T$ and Ta' is defined, for $a' \in \mathcal{A}'$.

We will conclude the introduction by quoting the very useful theorem by Takesaki [43].

Theorem 3.39. *If $\mathcal{A}$ is a von Neumann algebra and $|\Psi\rangle$ is a cyclic and separating vector for $\mathcal{A}$, then there exists a unique one-parameter automorphism group $\sigma_t : a \in \mathcal{A} \to \sigma_t(a) \in \mathcal{A}$, where $t \in \mathbb{R}$, such that:*

1. *The vector $|\Psi\rangle$ is invariant under the automorphism group. That is,*

$$\sigma_t(|\Psi\rangle\langle\Psi|) = |\Psi\rangle\langle\Psi|.$$

2. *The state $|\Psi\rangle$ satisfies the KMS condition with respect to σ_t. That is,*

$$\langle\Psi|\sigma_{t+i}(a)\,\sigma_t(b)|\Psi\rangle = \langle\Psi|\sigma_t(b)\,\sigma_t(a)|\Psi\rangle,$$

for all $a, b \in \mathcal{A}$ and $t \in \mathbb{R}$.

We will not prove this theorem here; the proof can be found in Takesaki's book [43]. However, it is important to observe that the modular operator generates an automorphism of the algebra, leaves the cyclic and separating state invariant, and satisfies the KMS condition—see properties (1), (7), and (9). Therefore, the modular automorphism is the unique automorphism associated with the pair $(\mathcal{A}, |\Psi\rangle)$ that satisfies the conditions of the theorem. Moreover, we can use this theorem to determine the modular operator associated with $(\mathcal{A}, |\Psi\rangle)$.

3.7.2 The Relative Modular Operator

The modular theory can be extended to what we can call the relative modular theory. This extension involves defining Tomita and modular operators relative to two states [45]. Let us begin with the definition.

Definition 3.40. (Relative Tomita Operator):
Let $|\psi\rangle$ be cyclic and separating for an algebra $\mathcal{A}$ and $|\phi\rangle$ be another state. We can define an antilinear relative Tomita operator $S_{\phi|\psi}$ for the algebra $\mathcal{A}$,

$$S_{\phi|\psi} : \mathcal{A}|\psi\rangle \subset \mathcal{H} \to \mathcal{A}|\phi\rangle \subset \mathcal{H}$$

$$S_{\phi|\psi} a\,|\psi\rangle = a^\dagger\,|\phi\rangle$$

Since $|\psi\rangle$ is cyclic and separating for the $\mathcal{A}$, $S_{\phi|\psi}$ is densely defined. It also follows that $S_{\phi|\psi}$ is closable. We will assume that $S_{\phi|\psi}$ has been closed and denote it by the same symbol. We emphasize that $|\phi\rangle$ can be any state. But, if $|\phi\rangle$ is cyclic and separating, then we can define,

$$S_{\psi|\phi} a\,|\phi\rangle = a^\dagger\,|\psi\rangle \tag{3.7.7}$$

In this case, $S_{\psi|\phi}S_{\phi|\psi} = I$, and therefore $S_{\phi|\psi}$ is invertible. However, let us not assume this for now; we will return to it after introducing a few more definitions that do not require this assumption.

Following the steps in (3.7.4) we can show that the relative Tomita operator $S'_{\phi|\psi}$ of $\mathcal{A}'$ follows $S'_{\phi|\psi} = S^\dagger_{\phi|\psi}$. Furthermore, we can apply the polar decomposition theorem on $S_{\phi|\psi}$. We can write,

$$S_{\phi|\psi} = J_{\phi|\psi}\Delta^{1/2}_{\phi|\psi} \tag{3.7.8}$$

where,

$$\Delta_{\phi|\psi} = S^\dagger_{\phi|\psi}S_{\phi|\psi} \tag{3.7.9}$$

is a self-adjoint positive operator called a relative modular operator. Notice that if $|\phi\rangle = |\psi\rangle$, then

$$S_{\psi|\psi} = S_\psi \qquad J_{\psi|\psi} = J_\psi \qquad \Delta_{\psi|\psi} = \Delta_\psi$$

If $|\phi\rangle$ is not separating, then $S_{\phi|\psi}$ will have a non-trivial kernel, and the polar decomposition in (3.7.8) will not be unique. However, we can make the polar decomposition unique by defining $J_{\phi|\psi}$ in such a way that it annihilates this kernel. Furthermore, if $|\phi\rangle$ is not cyclic, then the image of $S_{\phi|\psi}$ is not dense in the Hilbert space. This implies that $J_{\phi|\psi}$ will be an antilinear map only on $\mathrm{ran}(S_{\phi|\psi})$. However, if $|\phi\rangle$ is cyclic, then $J_{\phi|\psi}$ is antiunitary. Following the steps in (3.7.4), we can show that,

$$S'_{\phi|\psi} = S^\dagger_{\phi|\psi} \tag{3.7.10}$$

where $S'_{\phi|\psi}$ is the Tomita operator of the commutant. The relative modular operator and conjugation have many interesting properties. We will list here some of them,

1. If both $|\psi\rangle$ and $|\phi\rangle$ are cyclic and separating, then $J_{\psi|\phi}\Delta^{1/2}_{\psi|\phi}J_{\phi|\psi} = \Delta^{-1/2}_{\phi|\psi}$.
   Since $S_{\psi|\phi}S_{\phi|\psi} = I$, it follows from polar decomposition,

   $$J_{\psi|\phi}\Delta^{1/2}_{\psi|\phi}J_{\phi|\psi}\Delta^{1/2}_{\phi|\psi} = I$$

   Implies,

   $$J_{\psi|\phi}\Delta^{1/2}_{\psi|\phi}J_{\phi|\psi} = \Delta^{-1/2}_{\phi|\psi}$$

2. If both $|\psi\rangle$ and $|\phi\rangle$ are cyclic and separating, then $J_{\psi|\phi}J_{\phi|\psi} = I$.
   This follows from the property (1) and the uniqueness of polar decomposition.

3. $J'_{\phi|\psi} = J^\dagger_{\phi|\psi}$ and $(\Delta'_{\phi|\psi})^{1/2} = \Delta^{-1/2}_{\psi|\phi}$.

   Since,

   $$S'_{\phi|\psi} \equiv J'_{\phi|\psi}(\Delta'_{\phi|\psi})^{1/2} = S^\dagger_{\phi|\psi} = \Delta^{1/2}_{\phi|\psi}J^\dagger_{\phi|\psi} = J^\dagger_{\phi|\psi}J^\dagger_{\psi|\phi}\Delta^{1/2}_{\phi|\psi}J^\dagger_{\phi|\psi} = J^\dagger_{\phi|\psi}\Delta^{-1/2}_{\psi|\phi}$$

   where we have used the property (1) and (2). From the uniqueness of polar decomposition, $J'_{\phi|\psi} = J^\dagger_{\phi|\psi}$ and $(\Delta'_{\phi|\psi})^{1/2} = \Delta^{-1/2}_{\psi|\phi}$.

4. Let a' be any unitary operator in $\mathcal{A}'$, then $\Delta_{a'\phi|\psi} = \Delta_{\phi|\psi}$.
   Let us apply the definition of a relative Tomita operator to state $a'|\phi\rangle$, $S_{a'\phi|\psi}b\,|\psi\rangle =$

$a^\dagger b' |\phi\rangle = b' a^\dagger |\phi\rangle$ for any $b \in \mathcal{A}$. Implies,$S_{a'\phi|\psi} = a' S_{\phi|\psi}$. Therefor,

$$\Delta_{a'\phi|\psi} = S^\dagger_{a'\phi|\psi} S_{a'\phi|\psi} = S^\dagger_{\phi|\psi} a'^\dagger a' S_{\phi|\psi} = S^\dagger_{\phi|\psi} S_{\phi|\psi} = \Delta_{\phi|\psi}$$

5. $\forall a, b \in \mathcal{A}$, $\langle\psi| a\Delta_{\phi|\psi} b |\psi\rangle$=$\langle\phi| ba |\phi\rangle$.
Since,

$$\langle\psi| a\Delta_{\phi|\psi} b |\psi\rangle = \langle\psi| aS^\dagger_{\phi|\psi} S_{\phi|\psi} b |\psi\rangle = \langle\psi| aS^\dagger_{\phi|\psi} b^\dagger |\phi\rangle = \overline{\langle S_{\phi|\psi} a^\dagger \psi | b^\dagger |\phi\rangle} = \langle\phi|ba|\phi\rangle$$

6. There are some fascinating relations between the modular operator and the relative modular operator due to Connes, known as the **Connes Cocycle Theorem**. The key statements are as follows:

   (a) $u_{\phi|\psi}(s) \equiv \Delta^{is}_{\phi|\psi} \Delta^{-is}_{\psi} = \Delta^{is}_{\phi} \Delta^{-is}_{\psi|\phi} \in \mathcal{A}$ and $s \in \mathbb{R}$.

   (b) $u'_{\phi|\psi}(s) \equiv \Delta^{-is}_{\phi|\psi} \Delta^{is}_{\phi} = \Delta^{-is}_{\psi} \Delta^{is}_{\psi|\phi} \in \mathcal{A}'$ and $s \in \mathbb{R}$.

   We will not prove this theorem here, but a very nice proof can be found in [46]. Though it is true for all von Neumann algebras, it is easy to prove it for the type I case, as we will see in the next section.

Let us make a few remarks about the beautiful Connes Cocycle theorem. The first one is, $u_{\phi|\psi}(s)$ and $u'_{\phi|\psi}(s)$ are unitary operators. The second one is that the relative modular operators relate the modular flow of two states, that is

$$\Delta^{is}_{\phi|\psi} \Delta^{-is}_{\psi} = \Delta^{is}_{\phi} \Delta^{-is}_{\psi|\phi} \implies \Delta^{is}_{\phi|\psi} \Delta^{-is}_{\psi} \Delta^{is}_{\psi|\phi} = \Delta^{is}_{\phi} \tag{3.7.11}$$

Another interesting property that follows from the fact that $u'_{\phi|\psi} \in \mathcal{A}'$ is that for $a \in \mathcal{A}$,

$$u'_{\phi|\psi}(s) a u'_{\phi|\psi}(-s) = a \implies \Delta^{is}_{\psi|\phi} a \Delta^{-is}_{\psi|\phi} = \Delta^{is}_{\psi} a \Delta^{-is}_{\psi} \tag{3.7.12}$$

$\Delta^{is}_{\psi|\phi}$ generate same modular flow that Δ^{is}_{ψ} generates. Hence, the relative modular operators have very nice applications.
Another important application of the relative modular operator is that it allows us to define *relative entropy*. Moreover, we want to emphasize that the modular operator is defined for all von Neumann algebras, and therefore, the relative entropy defined using it will also be well-defined for all types of von Neumann algebras. It was first introduced by Araki and is hence known as *Araki relative entropy*.

Definition 3.41. (Araki's relative entropy):
Let $|\psi\rangle$ be a cyclic and separating state and $|\phi\rangle$ be another state, then Araki's relative entropy is defined as

$$S^{\mathcal{A}}(\phi||\psi) = -\langle\phi|\log\Delta^{\mathcal{A}}_{\psi|\phi}|\phi\rangle \tag{3.7.13}$$

where $\Delta^{\mathcal{A}}_{\phi|\psi}$ is a relative modular operator associated with von Neumann algebra $\mathcal{A}$.

The above definition of Araki relative entropy is a generalization of relative entropy from quantum mechanics to the infinite-dimensional case. It has many nice properties.

1. It is defined in all types of von Neumann algebras.

   This follows from the fact that the relative modular operator is well-defined in all von Neumann algebras.

2. The Araki relative entropy vanishes when both states are the same.

   If $|\phi\rangle$ is equals to $|\psi\rangle$, then

$$S^{\mathcal{A}}(\psi||\psi) = -\langle\psi|\log\Delta^{\mathcal{A}}_{\psi|\psi}|\psi\rangle = \langle\psi|\log\Delta^{\mathcal{A}}_{\psi}|\psi\rangle = 0$$

3. The Araki relative entropy is non-negative. That is

$$S^{\mathcal{A}}(\phi||\psi) \geq 0 \tag{3.7.14}$$

   The above statement follows from the fact that $\log\Delta^{\mathcal{A}}_{\psi|\phi} \leq I - \Delta^{\mathcal{A}}_{\psi|\phi}$. Therefore, $S^{\mathcal{A}}(\phi||\psi) \geq \langle\phi|I - \Delta^{\mathcal{A}}_{\psi|\phi}|\phi\rangle = 0$[11]. Hence, Araki's relative entropy is always non-negative.

4. Araki's relative entropy is Monotonic under the inclusion of the algebra. That is, Let $\mathcal{A}$ be the von Neumann algebra and $\mathcal{B}$ be the von Neumann subalgebra of $\mathcal{A}$, then

$$S^{\mathcal{B}}(\phi||\psi) \geq S^{\mathcal{A}}(\phi||\psi) \tag{3.7.15}$$

   The proof can be found in [24].

[11] We assume that $\langle\psi|\psi\rangle = \langle\phi|\phi\rangle = 1$

It is called a *relative* entropy because it quantifies the distinguishability between two quantum states, as should be evident from properties (2) and (3). We emphasize again that the Araki relative entropy reduces to the familiar finite-dimensional expression,

$$\mathrm{Tr}_{\mathcal{H}}(\rho \log \rho - \rho \log \sigma),$$

when ρ and σ are density matrices. These properties have profound implications in physics. For instance, there exists a beautiful derivation of the Bekenstein bound using relative entropy by Casini [47]. Moreover, the Quantum Null Energy Condition (QNEC) can be rigorously proven using relative entropy, as demonstrated in [48]. The relative entropy also plays a crucial role in various proofs of the Generalized Second Law (GSL) [5, 3, 4], as we will see later in this thesis. Many other important applications exist across quantum field theory and quantum gravity.

3.8 Finite-Dimensional Quantum Systems And Some Lessons

In this section, we explicitly construct the modular operator and modular conjugation in a finite-dimensional quantum system. The example is taken from Witten's lecture notes [24]. We include it here because it is simple and serves the purpose of building intuition for understanding these objects. By exploring modular operators in finite-dimensional settings, we aim to draw some valuable lessons that will help us later in more general contexts.

3.8.1 The Modular Operators in Finite-Dimensional Interacting System

In finite dimensions, each system is described by a matrix algebra (3.1), and the interesting case is that of a bipartite quantum system (3.2) defined on a Hilbert space given by the tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.

Let $\mathcal{A}$ be the algebra of linear operators acting on $\mathcal{H}_1$, and let $\mathcal{A}'$ be the algebra of linear operators acting on $\mathcal{H}_2$, as described in (3.2). To apply modular theory, we require a cyclic vector $|\Psi\rangle$ for $\mathcal{A}$. From (3.35), we know that such a vector will be separating for $\mathcal{A}'$, and vice versa. Let us understand when a vector in $\mathcal{H}$ will be cyclic.

From Schmidt decomposition theorem, we know that any $|\Psi\rangle \in \mathcal{H}$ admits an expansion,

$$|\Psi\rangle = \sum_{k=1}^{n} c_k \, |\psi_k\rangle \otimes \left|\psi_k'\right\rangle , \tag{3.8.1}$$

where $n = \min[\dim(\mathcal{H}_1), \dim(\mathcal{H}_2)]$, $|\psi_k\rangle$ are orthogonal unit vectors in $\mathcal{H}_1$ and ψ_k' are orthogonal unit vectors in $\mathcal{H}_2$. Furthermore, the action of operator $a \otimes I \in \mathcal{A}$ on the above state is defined as,

$$(a \otimes 1) \, |\Psi\rangle = \sum_{k} c_k \alpha \, |\psi_k\rangle \otimes \left|\psi_k'\right\rangle , \tag{3.8.2}$$

Now, we want to determine when the state is cyclic and separating for $\mathcal{A}$, or equivalently, separating for both $\mathcal{A}$ and its commutant $\mathcal{A}'$. Suppose that $a \otimes I$ annihilates the state $|\Psi\rangle$. This is possible if and only if a annihilates all of the vectors $|\psi_k\rangle$.

Now, assume that all coefficients c_k are nonzero. Then, the set $\{|\psi_k\rangle\}$ forms a complete basis for $\mathcal{H}_1$. This implies that $a = 0$. Thus, $|\Psi\rangle$ is separating for the algebra $\mathcal{A}$ if and only if the $|\psi_k\rangle$ form a basis of $\mathcal{H}_1$. Similarly, it is separating for $\mathcal{A}'$ if and only if the $|\psi_k'\rangle$ form a basis for $\mathcal{H}_2$.

This is possible precisely when $\mathcal{H}_1$ and $\mathcal{H}_2$ have equal dimension. The converse is also true: when $\mathcal{H}_1$ and $\mathcal{H}_2$ both have dimension n, a generic vector will admit a Schmidt decomposition (3.8.1), and if all the coefficients c_k are nonzero, then the vector will be cyclic and separating for the algebras of both subsystems.

For simplicity, we will write $|\psi_k\rangle = |k\rangle$, $|\psi_k'\rangle = |k\rangle'$ and $|j\rangle \otimes |k\rangle'$ as $|j, k\rangle$. Thus

$$|\Psi\rangle = \sum_{k=1}^{n} c_k |k\rangle |k\rangle' = \sum_{k=1}^{n} c_k |k, k\rangle. \tag{3.8.3}$$

One interesting point we wish to emphasize is that precisely when the Hilbert spaces of both subsystems have the same dimension, the above state defines an isomorphism from the algebra $\mathcal{A}$ to the Hilbert space $\mathcal{H}$, as also explained by Witten in [24].
Now, we would like to find the modular data associated with the finite-dimensional system. So, we start with the definition of Tomita's operator $S_\Psi : \mathcal{H} \to \mathcal{H}$, defined by the following action of the operator in $\mathcal{A}$,

$$S_\Psi((a \otimes I)) \, |\Psi\rangle = (a^\dagger \otimes I) \, |\Psi\rangle \tag{3.8.4}$$

Now, pick some i and j in the set $\{1, 2, \cdots, n\}$, and choose some elementary matrix $a \in \mathcal{A}$ that acts on $\mathcal{H}_1$ by

$$a|i\rangle = |j\rangle, \quad \alpha|k\rangle = 0 \text{ if } k \neq i \tag{3.8.5}$$

and its adjoint acts by,

$$a^\dagger|j\rangle = |i\rangle, \quad a^\dagger|k\rangle = 0 \quad \text{if } \ k \neq j \tag{3.8.6}$$

Now, we compute the action of a on $|\Psi\rangle$,

$$(a \otimes I)\Psi = c_i|j, i\rangle, \quad (a^\dagger \otimes I)|\Psi\rangle = c_j|i, j\rangle \tag{3.8.7}$$

It follows from the definition of S_Ψ that,

$$S_\Psi(c_i|j, i\rangle) = c_j|i, j\rangle \tag{3.8.8}$$

From the antilinearity of S_Ψ,

$$S_\Psi|j, i\rangle = \frac{c_j}{\bar{c}_i}|i, j\rangle \tag{3.8.9}$$

This defines the action of S_Ψ on the arbitrary state and therefore completely specifies the Tomita operator. Infact, we can write,

$$S_\Psi = \sum_{i,j=1}^{n} \frac{c_j}{\bar{c}_i}|i, j\rangle\langle i, j| \tag{3.8.10}$$

Similarly the adjoint $S_\Psi^\dagger$ acts by

$$S_\Psi^\dagger|i, j\rangle = \frac{c_j}{\bar{c}_i}|j, i\rangle \tag{3.8.11}$$

Since modular operator $\Delta_\Psi = S_\Psi^\dagger S_\Psi$,

$$\Delta_\Psi|j, i\rangle = \frac{|c_j|^2}{|c_i|^2}|j, i\rangle \tag{3.8.12}$$

To obtain the above equation, we used the antilinearity of $S_\Psi^\dagger$. Therefore, we can write,

$$\Delta_\Psi = \sum_{i,j=1}^{n} \frac{|c_j|^2}{|c_i|^2}|j, i\rangle\langle i, j| \tag{3.8.13}$$

Now, we can find the modular conjugate J_Ψ, using the polar decomposition $S_\Psi = J_\Psi \Delta_\Psi^{1/2}$. Since

$$\Delta_\Psi^{1/2} \; |j, i\rangle = \sqrt{\frac{|c_j|^2}{|c_i|^2}} \;\; |j, i\rangle, \tag{3.8.14}$$

We will get,

$$J_\Psi \; |j, i\rangle = \sqrt{\frac{c_j c_i}{\bar{c}_j \bar{c}_i}} \;\; |i, j\rangle \tag{3.8.15}$$

or equivalently,

$$J_\Psi = \sum_{i,j=1}^{n} \sqrt{\frac{c_j c_i}{\bar{c}_j \bar{c}_i}} \; |j, i\rangle\langle j, i| \tag{3.8.16}$$

We have now constructed the modular operator and modular conjugation. While this construction is straightforward in finite dimensions, we wish to emphasize that the Tomita operator and the modular operator are bounded operators in this setting.

Now, we would like to construct the relative Tomita operator $S_{\Psi|\Phi}$ and the modular operator $\Delta_{\Psi|\Phi}$ associated with it. But for that we need another state. Let $|\Phi\rangle$ be another state in $\mathcal{H}$. In some orthonormal bases ϕ_i of $\mathcal{H}_1$, which we want to denote as $|\alpha\rangle$ and ϕ'_α of $\mathcal{H}_2$, which we will denote with $|\alpha'\rangle$, where $\alpha = \{1, \ldots, n\}$.

$$|\Phi\rangle = \sum_{\alpha=1}^{n} d_\alpha |\alpha\rangle \otimes |\alpha\rangle' = \sum_{\alpha=1}^{n} d_\alpha |\alpha, \alpha\rangle, \tag{3.8.17}$$

where d_α is some coefficient. Further for notational convenience, we write $|\alpha\rangle \otimes |i\rangle' = |\alpha, i\rangle$, $|i\rangle \otimes |\alpha\rangle' = |i, \alpha\rangle$, etc. Again, the state $|\Phi\rangle$ is cyclic and separating for both algebras if and only if the d_α are all nonzero. But we will not assume this, as already emphasized in the introduction of relative modular theory. The relative Tomita operator $S_{\Psi|\Phi}$ define through the equation,

$$S_{\Phi|\Psi}((a \otimes I) \, |\Psi\rangle = (a^\dagger \otimes 1) \, |\Phi\rangle \quad \forall a \in \mathcal{A} \tag{3.8.18}$$

Now pick some i and α from $\{1, 2, \cdots, n\}$ and matrix operator $a \in \mathcal{A}$ such that,

$$a|i\rangle = |\alpha\rangle, \quad a|j\rangle = 0 \;\; \text{for } j \neq i \tag{3.8.19}$$

Then the adjoint acts by

$$a^\dagger|\alpha\rangle = |i\rangle, \quad a^\dagger|\beta\rangle = 0 \;\; \text{for } \beta \neq \alpha. \tag{3.8.20}$$

then its action $|\Psi\rangle$ and $|\Phi\rangle$ is given by,

$$(a \otimes 1)|\Psi\rangle = c_i|\alpha, i\rangle, \quad (a^\dagger \otimes 1)|\Phi\rangle = d_\alpha|i, \alpha\rangle \tag{3.8.21}$$

therefore,

$$S_{\Phi|\Psi}|\alpha, i\rangle = \frac{d_\alpha}{\bar{c}_i}|i, \alpha\rangle \tag{3.8.22}$$

or equivalently,

$$S_{\Phi|\Psi} = \sum_{i,j=1}^{n} \frac{d_\alpha}{\bar{c}_i}|i, \alpha\rangle\langle i, \alpha| \tag{3.8.23}$$

The adjoint is given by,

$$S^\dagger_{\Phi|\Psi}|i, \alpha\rangle = \frac{d_\alpha}{\bar{c}_i}|\alpha, i\rangle \tag{3.8.24}$$

Since $\Delta_{\Psi|\Phi} = S^\dagger_{\Psi|\Phi} S_{\Psi|\Phi}$, we get

$$\Delta_{\Phi|\Psi}|\alpha, i\rangle = \frac{|d_\alpha|^2}{|c_i|^2}|\alpha, i\rangle \tag{3.8.25}$$

or equivalently,

$$\Delta_{\Phi|\Psi} = \sum_{i,j=1}^{n} \frac{|d_\alpha|^2}{|c_i|^2}|\alpha, i\rangle\langle i, \alpha| \tag{3.8.26}$$

Now we will write some of the above formulas in terms of density matrices. For our convenience we will assume that $|\Psi\rangle$ and $|\Phi\rangle$ are normalized. That is,

$$\sum_i |c_i|^2 = \sum_\alpha |d_\alpha|^2 = 1 \tag{3.8.27}$$

To the state $|\Psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$, We can associate a density matrix,

$$\rho_{12} = |\Psi\rangle\langle\Psi| = \sum_{i=1}^{n} |c_n|^2 |i, i\rangle\langle i, i| \tag{3.8.28}$$

It is basically a projection operator onto the subspace generated by $|\Psi\rangle$. It is a density matrix because it is positive and has unit trace with respect Hilbert space trace.

$$\mathrm{Tr}_{12}\, \rho_{12} = 1 \tag{3.8.29}$$

where Tr_{12} represents the trace over $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. We can define reduced density matrix on $\mathcal{H}_1$ and $\mathcal{H}_2$ by taking a partial trace over $\mathcal{H}_1$ or $\mathcal{H}_2$, respectively.

$$\rho_1 = \mathrm{Tr}_2\, \rho_{12}, \qquad \rho_2 = \mathrm{Tr}_1\, \rho_{12} \tag{3.8.30}$$

Where Tr_1 and Tr_2 are trace on $\mathcal{H}_1$ and $\mathcal{H}_2$. The explicit form is,

$$\rho_1 = \sum_i |c_i|^2 |i\rangle\langle i|, \quad \rho_2 = \sum_i |c_i|^2 |i\rangle'\langle i|' \tag{3.8.31}$$

It can easily be seen from the above equation that ρ_1 and ρ_2 are positive matrices acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. It follows from (3.8.29) that they have unit trace, ie $\mathrm{Tr}_1\,\rho_1 = \mathrm{Tr}_1\mathrm{Tr}_2\,\rho_{12} = \mathrm{Tr}_{12}\,\rho_{12} = 1$. It is also evident from (3.8.31) that, ρ_1 and ρ_2 are invertible if and only if the c_i are all nonzero, that is if and only if $|\Psi\rangle$ is cyclic separating for both algebras. Furthermore, we use (3.8.31) to write modular operator Δ_Ψ in (3.8.13) and density matrix ρ_{12} in (3.8.28), as

$$\Delta_\Psi = \rho_1 \otimes \rho_2^{-1} \qquad \rho_{12} = \sqrt{\rho_1} \otimes \sqrt{\rho_2} \tag{3.8.32}$$

Similarly, we can define a density matrix $\sigma_{12} = |\Phi\rangle\langle\Phi|$ associated to the state $|\Phi\rangle$ and reduced density matrices $\sigma_1 = \mathrm{Tr}_2\,\sigma_{12}$, $\sigma_2 = \mathrm{Tr}_1\,\sigma_{12}$. The reduced density matrices of $|\Phi\rangle$ are

$$\sigma_1 = \sum_\alpha |d_\alpha|^2 |\alpha\rangle\langle\alpha|, \quad \sigma_2 = \sum_\alpha |d_\alpha|^2 |\alpha\rangle'\langle\alpha|'. \tag{3.8.33}$$

Using (3.8.33) and (3.8.31), we can write relative modular operator $\Delta_{\Psi|\Phi}$ in terms of the reduced density matrices,

$$\Delta_{\Phi|\Psi} = \sigma_1 \otimes \rho_2^{-1} \qquad \Delta_{\Psi|\Phi} = \rho_1 \otimes \sigma_2^{-1} \tag{3.8.34}$$

Here, we would like to emphasize that both the modular operator and the relative modular operator factorize in terms of the reduced density matrices of the system and its commutant. This is a very general feature of the modular operator: whenever a density matrix associated with a cyclic and separating state exists, the modular operator factorizes.

What is special in the finite-dimensional case—or, equivalently, in the case of a type I von Neumann algebra—is that the reduced density matrix belongs to the algebra. In contrast, for type II algebras, while certain density matrices may exist, they do not belong to the algebra. At most, they can be affiliated with the algebra.
Now we would like to use the above expression to compute Araki's relative entropy. Since

$$\begin{aligned} S(\Phi||\Psi) = -\langle\Phi|\log\Delta_{\Psi|\Phi}|\Phi\rangle &= \langle\Phi| -\log\Delta_{\Psi|\Phi} + \log\Delta_\Phi |\Phi\rangle & (3.8.35)\\ &= \langle\Phi|\log\sigma_1 - \log\rho_1|\Phi\rangle & (3.8.36)\\ &= \mathrm{Tr}_1(\sigma_1\log\sigma_1 - \sigma_1\log\rho_1) & (3.8.37) \end{aligned}$$

So, the Araki relative entropy is the usual relative entropy in quantum mechanics. We also want to emphasize that $S_{ent}(\sigma_1) = -\mathrm{Tr}_1(\sigma_1 \log \sigma_1)$ is the von Neumann entropy or Entanglement entropy associated with the density matrix σ_2. Therefore, relative entropy also knows about the entanglement encoded in the state. We can also check that,

$$\begin{aligned} u_{\phi|\psi}(s) &\equiv \Delta^{is}_{\phi|\psi}\Delta^{-is}_{\psi} = \Delta^{is}_{\phi}\Delta^{-is}_{\psi|\phi} = \sigma_1^{is}\rho_1^{is} \in \mathcal{A} \\ u'_{\phi|\psi}(s) &\equiv \Delta^{-is}_{\phi|\psi}\Delta^{is}_{\phi} = \Delta^{-is}_{\psi}\Delta^{is}_{\psi|\phi} = \rho_2^{-is}\sigma_2^{-is} \in \mathcal{A}' \\ \Delta^{is}_{\psi|\phi} a \Delta^{-is}_{\psi|\phi} &= \Delta^{is}_{\psi} a \Delta^{-is}_{\psi} = \rho_1^{is} a \rho_1^{-is} \end{aligned} \tag{3.8.38}$$

Hence, all the statements of Connes Cocyle are satisfied.

3.8.2 Modular operator for the Rindler Algebra

Let $x^\mu = (t, x, \vec{y_\perp})$ be the Cartesian coordinate in $d+1$ dimensional Minkowski spacetime. The right rindler wedge W_R is defined as,

$$W_R = \{(t, x, \vec{y_\perp}) \in \mathbb{R}^{(1,d)} | \quad x > |t|\} \tag{3.8.39}$$

Now, we can construct the von Neumann algebra of the scalar quantum field $\phi(x)$ in W_R as (3.5.0.1),

$$\mathcal{A}(W_R) = \{^* \text{ algebra generated by } \phi(f) \text{ such that support } f \subset W_R \}'' \tag{3.8.40}$$

where $''$ is double commutant to ensure the weak closure of the algebra (3.2.1). It can easily be shown that,

$$\mathcal{A}(W_R)' = \mathcal{A}(W_L) \tag{3.8.41}$$

where W_L is causal complement of W_R. We also know that if $U(\Lambda)$ represents the Lorentz transformation on spacetime, then

$$U(\Lambda)\phi(f)U^\dagger(\Lambda) = \phi(f_{\Lambda^{-1}}) \tag{3.8.42}$$

where $f_{\Lambda^{-1}} = f((\Lambda^{-1})^{\mu}_{\nu}x^{\nu})$. We also know that the boost in t and x by amount τ is given by

$$t(\tau) = t\cosh\tau + x\sinh\tau \tag{3.8.43}$$

$$x(\tau) = t\sinh\tau + x\cosh\tau \tag{3.8.44}$$

$$\vec{y}_{\perp}(\tau) = \vec{y}_{\perp} \tag{3.8.45}$$

It can further be checked the if $x \geq |t| \implies x(\tau) \geq |t(\tau)|$ for all $\tau \in \mathbb{R}$. Implies if $x^{\mu} \in W_R$ then $x^{\mu}(\tau) \in W_R$. Now, if K_B denotes the generator of the boost transformation, then,

$$\phi(f) \in \mathcal{A}_{W_R} \implies e^{iK_B\tau}\phi(f)e^{-iK_B\tau} \in \mathcal{A}(W_R) \quad \forall\tau \in \mathbb{R} \tag{3.8.46}$$

Therefore, boost generates one parameter group of automorphisms of $\mathcal{A}(W_R)$. Also notice that,

$$t(\tau + i\alpha) = t(\tau)\cos\alpha + i\sin\alpha\ x(\tau) \tag{3.8.47}$$

$$x(\tau + i\alpha) = x(\tau)\cos\alpha + i\sin\alpha\ t(\tau) \tag{3.8.48}$$

which implies,

$$t(\tau + 2\pi i) = t(\tau) \qquad\qquad x(\tau + 2\pi i) = x(\tau) \tag{3.8.49}$$

furthermore, if $x^{\mu} \in W_R$ then $Im(x^{\mu}(\tau+i\alpha))$ lies in future lightcone for $0 < \alpha < \pi$. Now it follows from the spectrum condition of local quantum field theory that for the Minkowski vacuum $|\Omega\rangle$, $e^{iK_Bs}\phi(g)|\Omega\rangle$ is analytic for $0 \leq Im(s) \leq \pi$ and $\langle\Omega|\phi(f)e^{-iK_Bs}$ is analytic in $-\pi \leq Im(s) \leq 0$. Hence, $\langle\Omega|\phi(f)e^{iK_Bs}\phi(g)|\Omega\rangle$ is analytic in $s \in \mathbb{R} + i[0, 2\pi]$[49]. Now, using the Schwartz reflection principle,

$$\lim_{\epsilon\to 0_+}\langle\Omega|\phi(f)e^{iK_B(\tau+2\pi i-i\epsilon)}\phi(g)|\Omega\rangle = \lim_{\epsilon\to 0_+}\langle\Omega|\phi(g)e^{-iK_B(\tau-i\epsilon)}\phi(f)|\Omega\rangle \tag{3.8.50}$$

for all $\tau \in \mathbb{R}$. Hence, automorphisms satisfy the KMS condition [12] with $\beta = 2\pi$. Therefore, the Minkowski vacuum is thermal with respect to the boost generator. This is precisely the statement of the Unruh effect. We can define a new generator $K_W = 2\pi K_B$. Notice that it satisfy *a)* $e^{-K_W}|\Omega\rangle = |\Omega\rangle$, b) e^{-K_W}generates autmorphism of $\mathcal{A}(W_R)$ and c)$|\Omega\rangle$ is thermal thermal with respect to K_W with $\beta = 1$. Now from (3.39) it follows that $\Delta_{\Omega} = e^{-K_W}$ is the modular operator of $(\mathcal{A}(W_R), \Omega)$. Now, let us compute the modular conjugation J_{Ω}.

[12] Similarly, it can be shown for a polynomial of $\phi(f)$ and therefore for the general algebra elements.

We know that,

$$J_\Omega \phi(f)|\Omega\rangle = \Delta_\Omega^{1/2}\phi(f)^\dagger|\Omega\rangle \tag{3.8.51}$$

From (3.8.43) we know that under $\Lambda_B(-i/2)(t, x, \vec{y}_\perp) = (-t, -x, \vec{y}_\perp) = \Theta R_x(\pi)(t, x, \vec{y}_\perp)$, where Θ is CPT operator and $R_x(\pi)$ is rotation about the x axis. Therefore, noting the fact that $\Delta_\Omega^{1/2}$ acts as $\Lambda_B(-i/2)$ on the points on which f has support, $J_\Omega = \Theta U(R_x(\pi))$ is antiuntary operator satisfying (3.8.51), where $U(R_x(\pi)$ is a unitary representation of $R_x(\pi)$.

So, we have learned that the modular Hamiltonian for the Rindler algebra in the Minkowski vacuum state is the boost operator, i.e. $\Delta_\Omega = e^{-K_B}$. Now, we would like to ask whether we can factorize the modular operator as some operator in the right wedge (algebra) times some operator in the left wedge (commutant), as in (3.8.32). Although we can write the boost operator in terms of the stress tensor $T_{\mu\nu}$ on some Cauchy surface.

$$K_B = \int_\Sigma d\Sigma^\mu T_{\mu\nu}\xi^\nu \tag{3.8.52}$$

We can also write K_B, on $t = 0$ surface and split it as,

$$K_B = \int_{t=0} (xT_{tt})dxdy^{d-2} = K_B^R - K_B^L \tag{3.8.53}$$

where

$$K_B^R = \int_{t=0,x\geq 0} (xT_{tt})dxdy^{d-2} \qquad K_B^L = -\int_{t=0,x\leq 0} (xT_{tt})dxdy^{d-2} \tag{3.8.54}$$

But the action of K_B^R and K_B^L is not well-defined on any state, as the norm of the state $K_B^{R/L}|\Psi\rangle$ diverges for any $|\Psi\rangle$. It can be shown that this divergence is universal and arises from the region near $x = 0$. Nevertheless, the matrix elements of $K_B^{R/L}$ are well-defined (or well-defined as quadratic forms) in any finite-energy state. This is because, if $|\phi\rangle$ and $|\psi\rangle$ are finite-energy states, then $\langle\phi|K_B^{R/L}|\psi\rangle$ effectively projects $K_B^{R/L}|\psi\rangle$ onto the finite-energy state $|\phi\rangle$. As a result, it eliminates the high-energy modes and avoids UV divergences.

Hence, the modular operator for the Rindler wedge does not factorize and cannot be written purely in terms of operators defined individually on the right and left wedges. Algebraically, this implies that the modular automorphism is generated by the boost operator acting as an outer automorphism. It is well known that for type I and type II algebras, all modular automorphisms are inner; see Theorem 3.14 in [50]. Hence, the

Rindler algebra must be of type III factor. Furthermore, we know that the boost operator has a continuous spectrum[13] and the spectrum is $\mathbb{R}$. Therefore, the spectrum of the modular operator $\sigma(\Delta_\Omega) \in [0, \infty]$. Additionally, it is known that every state resembles the vacuum at short wavelengths. Thus, for any state $|\psi\rangle$, we expect $\sigma(\Delta_\psi) \in [0, \infty]$. By Connes classification of type III factors, it then follows that the Rindler algebra is of type III_1. For completeness, we would like to briefly talk about Connes classification of type III factor [51]. Connes classification is based on spectral property of modular operator. In particular, Conne studied,

$$S(\mathcal{A}) = \bigcap_{|\psi\rangle} \sigma(\Delta_\psi) \tag{3.8.55}$$

where $\mathcal{A}$ is von Neumann algebra. Connes showed that $S(\mathcal{A})$, the *modular spectrum* of a von Neumann algebra $\mathcal{A}$, is a multiplicative subgroup of $\mathbb{R}_+$. Since it is also closed, the only possibilities are:

- $S(\mathcal{A}) = \{0, 1\}$,
- $S(\mathcal{A}) = \{0\} \cup \{\lambda^n \mid n \in \mathbb{Z}\}$ for some $0 < \lambda < 1$,
- $S(\mathcal{A}) = \mathbb{R}_+$.

This leads to a finer classification of type III factors. A type III factor $\mathcal{A}$ is called:

1. **Type III_0** if $S(\mathcal{A}) = \{0, 1\}$.
2. **Type III_λ** if $S(\mathcal{A}) = \{0\} \cup \{\lambda^n \mid n \in \mathbb{Z}\}$ for some $0 < \lambda < 1$.
3. **Type III_1** if $S(\mathcal{A}) = \mathbb{R}_+$.

From the above classification, it must be clear that the Rindler algebra should be type III_1.

3.9 Crossed Product Algebra

As we have seen, in type III algebras there is no semifinite trace (i.e., a trace that assigns a finite value to at least some nonzero projection), whereas type II and type I algebras admit a unique semifinite trace, up to scaling. This makes type II and type I algebras

[13] In Rindler coordinates, the boost acts like a translation operator.

more tractable and interesting, particularly in the context of assigning finite entanglement measures.

One might ask whether it is possible to study type III algebras using type II algebras. In other words, can we map a type III algebra to a type II algebra? If so, this would provide greater control and may potentially lead to a deeper understanding of type III algebras. As it turns out, certain crossed product constructions achieve precisely this. Therefore, the crossed product is an essential tool for probing type III algebras. Moreover, we will see that crossed product algebras naturally arise in gravitational theories, making them even more compelling to study.

Let us now begin by defining crossed product algebras.
Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and let T be a self-adjoint operator on $\mathcal{H}$ that generates a one-parameter group of automorphisms on $\mathcal{A}$. That is,

$$\forall a \in \mathcal{A} \quad e^{iTs} a e^{-iTs} \in \mathcal{A}, \quad \text{where} \quad s \in \mathbb{R} \tag{3.9.1}$$

where e^{iTs} can be viewed as an additive, possibly non-faithful, unitary representation of $\mathbb{R}$[14]. Furthermore, an automorphism is called *inner* if $e^{iTs} \in \mathcal{A}$; otherwise, it is called *outer*. Inner automorphisms are considered trivial in the sense that they are implemented by elements within the algebra itself. In contrast, outer automorphisms are nontrivial, as they define an action by an external operator on the algebra that preserves its structure. Now, let us define the crossed product algebra.

Definition 3.42. (Crossed product algebra):
The crossed product algebra $\mathcal{A} \rtimes_T \mathbb{R}$, of a von Neumann algebra $\mathcal{A}$ by $\mathbb{R}$, is defined on the Hilbert space $\mathcal{H}^{ext} = \mathcal{H} \otimes L^2(\mathbb{R})$ as follows:

$$\mathcal{A} \rtimes_T \mathbb{R} = \left\{ e^{iT\hat{p}} a e^{-iT\hat{p}},\, e^{i\hat{q}s} \mid a \in \mathcal{A},\, s \in \mathbb{R} \right\}'' \tag{3.9.2}$$

Here, $\hat{q}$ and $\hat{p}$ are the canonical conjugate operators satisfying $[\hat{q}, \hat{p}] = i$, and they act on $L^2(\mathbb{R})$.
Remark: The double commutant ensures that the resulting algebra is a von Neumann algebra, i.e., it is weakly closed.

[14] We are defined here for $\mathbb{R}$, but one can consider other groups.

From the above definition, it must be clear that the crossed product algebra is generated by the operator of the type $e^{iT\hat{p}}ae^{-iT\hat{p}}$ and $e^{i\hat{q}s}$[15]. A priori, it is not clear that it's an algebra, but nevertheless, it can be checked easily. It can easily be shown using the Baker–Campbell–Hausdorff (BCH) formula that for any $a, b \in \mathcal{A}$ and $s_1, s_2 \in \mathbb{R}$,

$$(e^{iT\hat{p}}ae^{-iT\hat{p}}e^{i\hat{q}s_1})(e^{iT\hat{p}}be^{-iT\hat{p}}e^{i\hat{q}s_2}) = e^{iT\hat{p}}\{a(e^{-iTs_1}be^{iTs_1})\}e^{-iT\hat{p}}e^{i\hat{q}(s_1+s_2)} \tag{3.9.3}$$

Since T generates the automorphism of algebra, $a(e^{-iTs_1}be^{iTs_1}) \in \mathcal{A}$. Therefore, $\mathcal{A} \rtimes_T \mathbb{R}$ is indeed an algebra, and closure ensures that it is a von Neumann algebra.

By adjoining with respect to $e^{-iT\hat{p}}$ and using the BCH formula, we can easily show that,

$$\mathcal{A} \rtimes_T \mathbb{R} \cong \left\{a \otimes I,\, e^{-iTs} \otimes e^{i\hat{q}s} \mid a \in \mathcal{A},\, s \in \mathbb{R}\right\}'' \tag{3.9.4}$$

Notice that $T - \hat{q}$ belong to $\mathcal{A} \rtimes_T \mathbb{R}$. Furthermore, the above representation is particularly interesting in the sense that in this representation, it is explicit that the automorphism is now inner. That is,

$$e^{iTs} \otimes e^{i\hat{q}s}(a \otimes I)e^{-iTs} \otimes e^{-i\hat{q}s} = e^{iTs}ae^{-iTs} \otimes I \tag{3.9.5}$$

One way to think about what we have just done by constructing the crossed product is that we started with an outer automorphism, which can be viewed as the action of some symmetry on the algebra that leaves the algebra invariant. Then, we added an additional degree of freedom in a specific way, namely $\hat{q}$, which transforms the outer automorphism into an inner one. This procedure is analogous to *gauging* the automorphism, and it closely resembles what is done in gauge theory. Another way of saying this is the famous commutation theorem [52], which says that crossed product algebra is the subalgebra of the algebra $\mathcal{A} \otimes B(L^2(\mathbb{R}))$that is invariant under $T + \hat{q}$. That is,

$$\mathcal{A} \rtimes_T \mathbb{R} = \{\mathbf{a} \in \mathcal{A} \otimes B(L^2(\mathbb{R})) : e^{i(T+\hat{q})t}\mathbf{a}e^{-i(T+\hat{q})t} = \mathbf{a} \;\; \forall t \in \mathbb{R}\} \tag{3.9.6}$$

This is like gauging the symmetry whose generator is $T + \hat{q}$.
It can easily be shown that the commutant of (3.9.2) is,

$$(\mathcal{A} \rtimes_T \mathbb{R})' = \{a' \otimes I,\, e^{iTs} \otimes e^{i\hat{q}s} \mid a \in \mathcal{A}',\, s \in \mathbb{R}\}'' \tag{3.9.7}$$

[15] Here, we have made a slight abuse of notation: by $e^{iT\hat{p}}ae^{-iT\hat{p}}$, we actually mean $e^{i(T\otimes\hat{p})}(a \otimes I_{L^2})e^{-i(T\otimes\hat{p})}$, and by $e^{i\hat{q}s}$, we mean $I_{\mathcal{H}} \otimes e^{i\hat{q}s}$.

This follows from the fact that $e^{iT\hat{p}}ae^{-iT\hat{p}} \in \mathcal{A}$ and commute with $a' \otimes I$ for all $a' \in \mathcal{A}'$. Also,

$$[e^{iT\hat{p}}ae^{-iT\hat{p}}, T+\hat{q}] = e^{iT\hat{p}}[a,T]e^{-iT\hat{p}} + e^{iT\hat{p}}[T,a]e^{-iT\hat{p}} = 0 \tag{3.9.8}$$

where we have used that $[e^{iT\hat{p}}, \hat{q}] = Te^{iT\hat{p}}$. Notice that $T+\hat{q}$ belongs to the commutant $(\mathcal{A} \rtimes_T \mathbb{R})'$. We also emphasize that equation (3.9.7) describes a crossed product algebra as well. It should be clear from equations (3.9.2) and (3.9.7) that if $\mathcal{A}$ is a factor, then $\mathcal{A} \rtimes_T \mathbb{R}$ is also a factor.

Till now, we have considered the general automorphisms on the algebra, but we can very well choose it to be a modular automorphism (an automorphism by the modular operator of some state). Then, there is a beautiful theorem by Takesaki, see corollary 9.7 of [53].

Theorem 3.43. ***(Duality theorem of type III$_1$ and type II$_\infty$)**:*
Let $\mathcal{A}$ be a type III$_1$ factor von Neumann algebra, then the crossed product of $\mathcal{A}$ by any of its modular Hamiltonians $K = -\log\Delta$, where Δ is a modular operator of some state, is a type II$_\infty$ factor von Neumann algebra.

$$\mathcal{A} \rtimes_K \mathbb{R} \quad \textit{is a type } II_\infty \textit{ von Neumann algebra.}$$

The proof of the theorem can be found in [53]. The type III$_1$ refers to one of the subclasses in Connes' further classification of type III von Neumann algebras [51]. In particular, it is a type III algebra for which every modular operator has spectrum supported on $\mathbb{R}^+ \cup \{0\}$. The beauty of the theorem is that it allows us to define both the trace and the density matrix. As we have seen earlier, type III von Neumann algebras do not admit a (renormalized) trace, whereas type II algebras possess a unique trace up to rescaling. The crossed product construction enables us to relate type III algebras to type II, thereby allowing us to define a renormalized trace on the extended algebra, which in turn makes it possible to define a density matrix. This should be understood as an algebraic method of renormalizing the infinite quantities that arise in type III algebras or possibly in quantum field theory. The trace on the crossed product has already been studied by Takesaki [53], and we will write it explicitly following Witten [1]. As we also emphasized, the crossed product construction is very similar to the construction of algebras in gauge theory. Later in the thesis, we will see that it arises naturally in the construction of gravitational algebras.

3.9.1 Modular theory in Classical Quantum States

One interesting question that one might want to ask is whether we can write the modular operator of the crossed product algebra in $\mathcal{A} \rtimes_K \mathbb{R}$ in terms of the modular data of $\mathcal{A}$. It is shown in [1] that the modular operator can be explicitly obtained for some class of classical-quantum states. So, let us first define the classical quantum state.

Definition 3.44. Classical-quantum states:
Consider a bipartite system with Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. The state is called a classical-quantum state if the density matrix ρ associated with the state in $\mathcal{H}$ is of the form:

$$\rho = \sum_i p_i |i\rangle_1 \langle i|_1 \otimes \rho_{i,2}$$

where $\sum_i P_i = 1$,$\{|i\rangle_1\}$ some basis in $\mathcal{H}_1$ and ρ_2 is a density in $\mathcal{H}_2$.

One of the reasons it is called a classical quantum state is that it is non-entangled. In particular, we want to take the state of type,

$$|\hat{\Phi}\rangle = |\Phi\rangle \otimes |f\rangle \in \mathcal{H}^{ext} \quad \text{where } |\Phi\rangle \in \mathcal{H} \;\&\; |f\rangle \in L^2(\mathbb{R}) \tag{3.9.9}$$

It is clearly, classical quantum state. We can choose the position representation of $|f\rangle$, and write,

$$|\hat{\Phi}\rangle = |\Phi\rangle \otimes |f\rangle = \int dq f(q) |\Phi\rangle \otimes |q\rangle \equiv \int dq f(q) |\Phi, q\rangle = |\Phi, f\rangle \tag{3.9.10}$$

Now, let $|\Psi\rangle \in \mathcal{H}$ be cyclic and separating in $\mathcal{A}$ for which we have constructed a crossed product. We would like to make a convention that the operator $\hat{a}_s = \Delta_\Psi^{-i\hat{p}} a \Delta_\Psi^{i\hat{p}} e^{i\hat{q}s}$, where Δ_Ψ is a modular operator of $(\mathcal{A}, |\Psi\rangle)$. Then the Tomita operator of the state $|\hat{\Phi}\rangle$ for modular crossed product algebra $\mathcal{A} \rtimes_{K_\Psi} \mathbb{R}$, where $K_\Psi = -\log \Delta_\Psi$, is

$$S_{\hat{\Phi}} \hat{a}_s |\hat{\Phi}\rangle = \hat{a}_s^\dagger |\hat{\Phi}\rangle \tag{3.9.11}$$

then,

$$\begin{aligned}
\hat{a}_s^\dagger|\hat{\Phi}\rangle &= e^{-i\hat{q}s}\Delta_\Psi^{-i\hat{p}}a^\dagger\Delta_\Psi^{i\hat{p}}\;|\Phi,f\rangle \\
&= e^{-is\hat{q}}\Delta_\Psi^{-i\hat{p}}a^\dagger\Delta_\Psi^{i\hat{p}}S_{\Phi|\Psi}S_\Psi\;|\Psi,f\rangle \\
&= S_{\Phi|\Psi}S_\Psi e^{-is\hat{q}}\Delta_\Psi^{-i\hat{p}}a^\dagger\Delta_\Psi^{i\hat{p}}\;|\Psi,f\rangle \\
&= S_{\Phi|\Psi}S_\Psi e^{is\hat{q}}\Delta_\Psi^{-i\hat{p}}S_\Psi a\;|\Psi,f^*\rangle \\
&= S_{\Phi|\Psi}e^{is\hat{q}}\Delta_\Psi^{i\hat{p}}a\;\int dq f^*(q)|\Psi,q\rangle \\
&= S_{\Phi|\Psi}e^{is\hat{q}}\Delta_\Psi^{i\hat{p}}a\;\int dq f^*(q)\int dp e^{ipq}|\Psi,p\rangle \\
&= S_{\Phi|\Psi}\;\int dq f^*(q)\int dp e^{ipq}e^{-iK_\Psi p}ae^{is\hat{q}}|\Psi,p\rangle \\
&= S_{\Phi|\Psi}\;\int dp\tilde{f}^*(p)e^{i(\hat{q}-K_\Psi)p}a|\Psi,s\rangle \\
&= S_{\Phi|\Psi}f^*(\hat{q}-K_\Psi)a|\Psi,s\rangle
\end{aligned} \tag{3.9.12}$$

In the third step, we use the fact that $S_{\Phi|\Psi}S_\Psi$ is affiliated to $\mathcal{A}'$. That is,

$$S_{\Phi|\Psi}S_\Psi ba|\Psi\rangle = S_{\Phi|\Psi}a^\dagger b^\dagger|\Psi\rangle = ba|\Phi\rangle = bS_{\Phi|\Psi}S_\Psi a|\Phi\rangle \implies [S_{\Phi|\Psi}S_\Psi, b]a|\Psi\rangle = 0$$

for all $a, b \in \mathcal{A}$[16]. In the fourth step, we have used the antilinearity of the Tomita operator and its action on the algebra elements. In the six-step, we went to the Fourier basis so that we can act $\hat{p}$. Now, let us compute,

$$\begin{aligned}
\hat{a}_s|\hat{\Phi}\rangle &= \Delta_\Psi^{-i\hat{p}}a\Delta_\Psi^{i\hat{p}}e^{i\hat{q}s}\;|\Phi,f\rangle \\
&= \Delta_\Psi^{-i\hat{p}}a\Delta_\Psi^{i\hat{p}}e^{i\hat{q}s}S_{\Phi|\Psi}S_\Psi\;|\Psi,f\rangle \\
&= S_{\Phi|\Psi}S_\Psi\Delta_\Psi^{-i\hat{p}}a\Delta_\Psi^{i\hat{p}}e^{i\hat{q}s}\;|\Psi,f\rangle \\
&= S_{\Phi|\Psi}S_\Psi\Delta_\Psi^{-i\hat{p}}a\Delta_\Psi^{-i\hat{p}}e^{i\hat{q}s}\;\int f(q)|\Psi,q\rangle \\
&= S_{\Phi|\Psi}S_\Psi\;\int e^{iqs}\Delta_\Psi^{-i\hat{p}}af(q)|\Psi,q\rangle \\
&= S_{\Phi|\Psi}S_\Psi\Delta_\Psi^{-i\hat{p}}f(\hat{q})a\;|\Psi,s\rangle
\end{aligned}$$

since $|\Psi\rangle$ is cyclic and separating state, it follows from (3.9.11),that

$$S_{\hat{\Phi}}S_{\Phi|\Psi}S_\Psi\Delta_\Psi^{-i\hat{p}}f(\hat{q}) = S_{\Phi|\Psi}f^*(\hat{q}-K_\Psi) \tag{3.9.13}$$

[16] Similarly, it can be shown that $S_{\Phi|\Psi}S_\Psi$

Therefore, we get,

$$S_{\hat{\Phi}} = S_{\Phi|\Psi} f^*(\hat{q} - K_\Psi) \frac{1}{f(\hat{q})} \Delta_\Psi^{i\hat{p}} S_\Psi S_{\Psi|\Phi} \tag{3.9.14}$$

where, we used the fact that $S_\Psi^{-1} = S_\Psi$ and $S_{\Phi|\Psi}^{-1} = S_{\Psi|\Phi}$. We can further write the above equation as

$$\begin{aligned}
S_{\hat{\Phi}} &= J_{\Phi|\Psi} \Delta_{\Phi|\Psi}^{1/2} f^*(\hat{q} - K_\Psi) \frac{1}{f(\hat{q})} \Delta_\Psi^{i\hat{p}} J_\Psi \Delta_\Psi^{1/2} J_{\Psi|\Phi} \Delta_{\Psi|\Phi}^{1/2} \\
&= J_{\Phi|\Psi} \Delta_{\Phi|\Psi}^{1/2} f^*(\hat{q} - K_\Psi) \frac{1}{f(\hat{q})} \Delta_\Psi^{i\hat{p}} \Delta_\Psi^{-1/2} J_\Psi J_{\Psi|\Phi} \Delta_{\Psi|\Phi}^{1/2} \\
&= J_{\Phi|\Psi} \Delta_{\Phi|\Psi}^{1/2} f^*(\hat{q} - K_\Psi) e^{\hat{q}/2} \Delta_\Psi^{i\hat{p}} e^{-\hat{q}/2} \frac{1}{f(q \hat{+} K_\Psi)} J_\Psi J_{\Psi|\Phi} \Delta_{\Psi|\Phi}^{1/2} \\
&= \left(J_{\Phi|\Psi} \Delta_\Psi^{i\hat{p}}\right) \left(\Delta_\Psi^{-i\hat{p}} \Delta_{\Phi|\Psi}^{1/2} f^*(\hat{q} - K_\Psi) e^{\hat{q}/2} \Delta_\Psi^{i\hat{p}}\right) \left(e^{-\hat{q}/2} \frac{1}{f(q \hat{+} K_\Psi)} J_\Psi J_{\Psi|\Phi} \Delta_{\Psi|\Phi}^{1/2}\right)
\end{aligned} \tag{3.9.15}$$

Here in the second step, we used $J_\psi \Delta_\psi^{1/2} = \Delta_\Psi^{-1/2} J_\Psi$. In the third step, we used the relation $e^{\hat{q}/2} \Delta^{i\hat{p}} e^{-\hat{q}/2} = \Delta^{i\hat{p}} \Delta^{-1/2}$ and $[q, \Delta_\Psi^{i\hat{p}}] = K_\Psi \Delta_\Psi^{i\hat{p}}$. We can write the second term in the bracket as,

$$\left(\Delta_\Psi^{-i\hat{p}} \Delta_{\Phi|\Psi}^{1/2} f^*(\hat{q} - K_\Psi) e^{\hat{q}/2} \Delta_\Psi^{i\hat{p}}\right) = \left(\Delta_\Psi^{-i\hat{p}} \Delta_{\Phi|\Psi}^{1/2} \Delta_\Psi^{-1/2} f^*(\hat{q} - K_\Psi) e^{\frac{q \hat{-} K_\Psi}{2}} \Delta_\Psi^{i\hat{p}}\right) \tag{3.9.16}$$

Since the bounded function $\hat{q} - K_\Psi$ belongs to the crossed product algebra, see equation (3.9.4), $f^*(\hat{q} - K_\Psi) e^{\frac{q \hat{-} K_\Psi}{2}}$ must be affiliated to the algebra. Moreover, it can be shown that $\Delta_{\Phi|\Psi}^{1/2} \Delta_\Psi^{-1/2}$ is affiliated to $\mathcal{A}$. Therefore, the whole term in the above equation is affiliated with crossed product algebra. Similarly, it can be shown that the third term in the equation (3.9.15) is affiliated with the commutant of the crossed product algebra [17]. Now, we can find a modular operator using the relation, $\Delta_{\hat{\Phi}} = S_{\hat{\Phi}}^\dagger S_{\hat{\Phi}}$. It can easily be shown that a modular operator takes the following form,

$$\Delta_{\hat{\Phi}} = \rho_{\hat{\Phi}} \otimes (\rho'_{\hat{\Phi}})^{-1} \tag{3.9.17}$$

where,

$$\rho_{\hat{\Phi}} = \Delta_\Psi^{-i\hat{p}} f(\hat{q} - K_\Psi) \Delta_{\Phi|\Psi} \Delta_\Psi^{-1/2} e^{\frac{\hat{q} - K_\Psi}{2}} f^*(\hat{q} - K_\Psi) \Delta_\Psi^{i\hat{p}} \tag{3.9.18}$$

$$\rho'_{\hat{\Phi}} = \Delta_{\Phi|\Psi}^{-1/2} J_{\Phi|\Psi} J_\Psi e^{\hat{q}/2} |f(\hat{q} + K_\Psi)|^2 J_\Psi J_{\Phi|\Psi} \Delta_{\Phi|\Psi}^{-1/2} \tag{3.9.19}$$

In order to obtained above equations, we used that $\Delta_{\Psi|\Phi}^{-1} = \Delta_{\Phi|\Psi}$ and $J_{\Psi|\Phi}^{-1} = J_{\Phi|\Psi}$. It must be clear, following the same argument presented in the last paragraph, that $\rho_{\hat{\Phi}}$ is affiliated to the crossed product algebra and $\rho'_{\hat{\Phi}}$ is affiliated to its commutant. Notice that

[17]It follows from the fact that $\hat{q}+K_\Psi$ and $J_\Psi J_{\Phi|\Psi}$ is affiliate to the commutant of crossed product algebra

the modular operator factorizes into operators affiliated to the type II crossed product algebra and its commutant. We would like to remind that the modular operator gets factorized even in type I, but there the operator corresponding to $\rho_{\hat{\Phi}}$ and $\rho'_{\hat{\Phi}}$ was the density matrix in the algebra and its commutant (3.8.32). This tells us that maybe we should interpret the $\rho_{\hat{\Phi}}$ and $\rho'_{\hat{\Phi}}$ as density matrix of type II crossed product algebra and its commutant respectively, keeping in mind that they at most can be affiliated to the algebra and the commutant. We also want to emphasize that the density matrix is ambiguous up to overall scaling, because scaling of $\rho_{\hat{\Phi}}$ by e^c and $\rho'_{\hat{\Phi}}$ by e^{-c}, where c is some constant, gives same modular operator. These are the expressions obtained by authors in [41, 54]. Now, let's say we want to find the modular of the state $|\hat{\Psi}\rangle = |\Psi\rangle \otimes |f\rangle$, then we just need to put $|\Phi\rangle = |\Psi\rangle$ in the equation (3.9.18) and (3.9.19), which will yield,

$$\rho_{\hat{\Psi}} = e^{\hat{q}}|f(\hat{q})|^2 \tag{3.9.20}$$

$$\rho'_{\hat{\Psi}} = e^{(\hat{q}+K_\Psi)}|f(\hat{q}+K_\Psi)|^2 \tag{3.9.21}$$

Therefore, the modular operator takes the following form,

$$\Delta_{\hat{\Psi}} = \Delta_\Psi \frac{|f(\hat{q}|^2}{|f(q+\hat{K}_\Psi)|^2} \tag{3.9.22}$$

Witten obtained this expression in [1]. We can also obtain the Tomita operator by putting $|\Phi\rangle = |\Psi\rangle$ in (3.9.15).

$$S_{\hat{\Psi}} = J_\Psi \Delta_\Psi^{i\hat{p}} \frac{f^*(\hat{q})}{f(\hat{q}+K_\Psi)} \Delta_\Psi^{1/2} \tag{3.9.23}$$

We can use the above equation along with (3.9.22) to get,

$$J_{\hat{\Psi}} = J_\Psi \Delta_\Psi^{i\hat{p}} \Big(\frac{f^*(\hat{q})f^*(\hat{q}+K_\Psi)}{f(\hat{q})f(\hat{q}+K_\Psi)}\Big)^{1/2} \tag{3.9.24}$$

Hence, we have obtained the modular data for the type II crossed product algebra in classical quantum states.

3.9.2 Trace in Crossed Product

As we have seen in the last subsection, the modular operator of the type II crossed product algebra $\mathcal{A} \rtimes_{K_\Psi} \mathbb{R}$ in classical quantum state factories. We have also argued that $\rho_{\hat{\Phi}}$ and $\rho_{\hat{\Psi}}$ can be thought of as density matrix affiliated to the algebra $\mathcal{A} \rtimes_{K_\Psi} \mathbb{R}$. The identification of $\rho_{\hat{\Psi}}$ as a density matrix immediately implies the existence of the trace. We can define trace following [1], as

$$\mathrm{tr}[\hat{a}] = \langle\hat{\Psi}|\hat{a}\rho_{\hat{\Psi}}^{-1}|\hat{\Psi}\rangle \qquad \forall\hat{a} \in \mathcal{A} \rtimes_{K_\Psi} \mathbb{R} \tag{3.9.25}$$

It is evident from the above definition that the above tr satisfies linearity and positivity. The cyclicity follows from the property (6) of the modular operator. For any $\hat{a}, \hat{b} \in \mathcal{A} \rtimes_{K_\Psi} \mathbb{R}$,

$$\begin{aligned}\mathrm{tr}(\hat{a}\hat{b}) &= \langle\hat{\Psi}|\hat{a}\hat{b}\rho_{\hat{\Psi}}^{-1}|\hat{\Psi}\rangle = \langle\hat{\Psi}|\hat{b}\rho_{\hat{\Psi}}^{-1}\Delta_{\hat{\Psi}}\hat{a}|\hat{\Psi}\rangle = \langle\hat{\Psi}|\hat{b}\rho_{\hat{\Psi}}^{-1}\rho_{\hat{\Psi}} \otimes (\rho'_{\hat{\Psi}})^{-1}\hat{a}|\hat{\Psi}\rangle = \langle\hat{\Psi}|\hat{b}\hat{a}\rho_{\hat{\Psi}}\Delta_{\hat{\Psi}}|\hat{\Psi}\rangle \\ &= \mathrm{tr}(\hat{b}\hat{a})\end{aligned} \tag{3.9.26}$$

In the above equation, we used the fact that $\rho'_{\hat{\Psi}}$ is affiliated with commutant and $\Delta_{\hat{\Psi}}|\hat{\Psi}\rangle = |\hat{\Psi}\rangle$. Therefore, (3.9.25) satisfies all the properties of trace defined in (3.17). We can also put an explicit form of $\rho_{\hat{\Psi}}$ from (3.9.20) and $|\hat{\Psi}\rangle = \int_{-\infty}^{\infty} dq f(q)|\Psi\rangle \otimes |q\rangle$, we get

$$\mathrm{tr}[\hat{a}] = \int_{-\infty}^{\infty} dq e^{q}\langle\Psi|\hat{a}(q)|\Psi\rangle \tag{3.9.27}$$

where $\hat{a}(q)$ is operator valued function of q. Notice that the above trace is not well defined for all elements of $\mathcal{A} \rtimes_{K_\Psi} \mathbb{R}$, for example, if $\hat{a} = \Delta_{\Psi}^{-i\hat{p}} a \Delta_{\Psi}^{i\hat{p}} \in \mathcal{A} \rtimes_{K_\Psi} \mathbb{R}$, then trace is not finite. One might wonder whether it even assigns a finite value to any operator. If we assume, $\langle\hat{\Psi}|\hat{\Psi}\rangle = 1$, then $\mathrm{tr}[\rho_{\hat{\Psi}}] = 1$. Hence $\rho_{\hat{\Psi}}$ is traceclass with respect to (3.9.25).
Further, we know that the algebra is generated by the operator of type $\Delta_{\Psi}^{-i\hat{p}} a \Delta_{\Psi}^{i\hat{p}} e^{i\hat{q}s}$, where $s \in \mathbb{R}$. Therefore, we can take an arbitrary linear combination of such operators, such as

$$\hat{a} = \int_{-\infty}^{\infty} ds \Delta_{\Psi}^{-i\hat{p}} a(s) \Delta_{\Psi}^{i\hat{p}} e^{i\hat{q}s} \tag{3.9.28}$$

where $a(s)$ for each s pick an operator for $\mathcal{A}$. Let us further assume that $a(s)$ goes to zero at $|s| \to \infty$ and holomorphic for $0 \leq Im(s) \leq 1$. Then,

$$\mathrm{tr}[\hat{a}] = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} ds e^{q+iqs}\langle\Psi|\hat{a}(q)|\Psi\rangle = 2\pi\langle\Psi|\hat{a}(i)|\Psi\rangle < \infty \tag{3.9.29}$$

We obtained the second equality by shifting the contour of s from $\mathbb{R}$ to $\mathbb{R}+i\mathbb{R}$. So, indeed, there are some trace-class operators with respect to (3.9.25) that assign a finite value to them. Since the density matrix is ambiguous up to overall scaling, by (3.9.25), the trace will also be ambiguous up to overall scaling. If crossed product algebra is a factor, then we know that there is a unique trace up to overall scaling. Hence, for the crossed product factor, any trace will be related to the trace in (3.9.25) by overall scaling. Once we have the trace and density matrix, we can define the **Algebra entropy** for the state $\rho_{\hat{\Phi}}$ by [18],

$$S(\hat{\Phi}) = -\mathrm{tr}[\rho_{\hat{\Phi}} \log \rho_{\hat{\Phi}}] \tag{3.9.30}$$

Since the trace is ambiguous up to scaling, the entropy will be ambiguous up to an additive constant. But the entropy difference will be unambiguous, like in thermodynamics. It can easily be shown that the above expression can also be written as,

$$S(\hat{\Phi}) = -\langle\hat{\Phi}| \log \rho_{\hat{\Phi}} |\hat{\Phi}\rangle \tag{3.9.31}$$

To obtain the above equation, we used the proper (5) of the relative modular operator. To further simplify the above expression, let us first simplify the $\rho_{\hat{\Phi}}$ given in (3.9.18),

$$\begin{aligned}
\rho_{\hat{\Phi}} &= \Delta_\Psi^{-i\hat{p}} f(\hat{q}-K_\Psi)\Delta_{\Phi|\Psi}\Delta_\Psi^{-1/2} e^{\frac{\hat{q}-K_\Psi}{2}} f^*(\hat{q}-K_\Psi)\Delta_\Psi^{i\hat{p}} \\
&= f(\hat{q})e^{\hat{q}/2}\Delta_\Psi^{-i\hat{p}}\Delta_\Psi^{-1/2}\Delta_{\Phi|\Psi}\Delta_\Psi^{-1/2}\Delta_\Psi^{i\hat{p}} e^{\hat{q}/2} f^*(\hat{q}) \\
&= f(\hat{q})e^{\hat{q}/2}\Delta_\Psi^{-i\hat{p}}\Delta_{\Psi|\Phi}^{-1/2}\Delta_\Phi\Delta_{\Psi|\Phi}^{-1/2}\Delta_\Psi^{i\hat{p}} e^{\hat{q}/2} f^*(\hat{q}) \\
&= \Delta_{\Psi|\Phi}^{-i\hat{p}} f(\hat{q}-K_{\Psi|\Phi})e^{\hat{q}}\Delta_\Phi f^*(\hat{q}-K_{\Psi|\Phi})\Delta_{\Psi|\Phi}^{i\hat{p}}
\end{aligned} \tag{3.9.32}$$

In the second line, we used the fact that $\hat{p}$ acts on $\hat{q}$ as a shift operator. In the third line, we used the Conne cocycle theorem that $\Delta_\Psi^{-1/2}\Delta_{\Phi|\Psi}^{1/2} = \Delta_{\Psi|\Phi}^{-1/2}\Delta_\Phi^{1/2}$. In the last line, we shift property of $\hat{p}$ and $K_{\Psi|\Phi} = -\log\Delta_{\Psi|\Phi}$. Now using the fact the for any unitary operator U and some operator A, $\log\big(UAU^\dagger\big) = U(\log A)U^\dagger$, we get,

$$\begin{aligned}
\log\rho_{\hat{\Phi}} &= \Delta_{\Psi|\Phi}^{-i\hat{p}}\Big\{q + \log\Big(f(\hat{q}-K_{\Psi|\Phi})\Delta_\Phi f^*(\hat{q}-K_{\Psi|\Phi})\Big)\Big\}\Delta_{\Psi|\Phi}^{i\hat{p}} \\
&= \hat{q} + K_{\Psi|\Phi} + \log\Big(f(\hat{q})\Delta_{\Psi|\Phi}^{-i\hat{p}}\Delta_\Phi\Delta_{\Psi|\Phi}^{i\hat{p}} f^*(\hat{q})\Big)
\end{aligned} \tag{3.9.33}$$

Putting the above equation in (3.9.31), we get

$$S(\hat{\Phi}) = -\langle\hat{\Phi}|\hat{q}|\hat{\Phi}\rangle - \langle\hat{\Phi}|K_{\Psi|\Phi}|\hat{\Phi}\rangle - \langle\hat{\Phi}|\log\Big(f(\hat{q})\Delta_{\Psi|\Phi}^{-i\hat{p}}\Delta_\Phi\Delta_{\Psi|\Phi}^{i\hat{p}} f^*(\hat{q})\Big)|\hat{\Phi}\rangle \tag{3.9.34}$$

This expression is also obtained in [41, 55, 1]. The crossed product construction and the algebra entropy defined above will play a vital role in understanding entropy in semiclassical gravity.

[18] It can easily be shown using $\langle\Psi|\Delta_{\Phi|\Psi}|\Psi\rangle = \langle\Phi|\Phi\rangle$, that $\mathrm{tr}[\rho_{\hat{\Phi}}] = \langle\hat{\Phi}|\hat{\Phi}\rangle$. Hence, for $\langle\hat{\Phi}|\hat{\Phi}\rangle$=1, it is a state.

Chapter 4

Black Holes in GR, Local Algebra of Observables and GSL

The material presented in this chapter is based on the work of the author in [4]

In this chapter, we show a local generalized second law (the generalized entropy is nondecreasing) in crossed product constructions for maximally extended static and Kerr black holes using modular theory. The new ingredient is the use of results from a recent paper discussing the entropy of the algebra of operators in subregions of arbitrary spacetimes [41]. These results rely on an assumption which we show is true in our setting. In the last part of this chapter, we look at modular Hamiltonians of deformed half-spaces in a class of static spacetimes, including the Schwarzschild spacetime. These are computed using path integrals, and we primarily compute them to investigate whether these non-local modular Hamiltonians can be made local by subtracting off pieces from the algebra and its commutant, as has been surmised in the literature. Along the way, the averaged null energy condition (ANEC) also follows in this class of spacetimes.

4.1 Introduction

Bekenstein proposed the Generalized Second Law (GSL) [56], [57] for black hole spacetimes with quantum matter in the expectation that the second law of thermodynamics would be valid near black holes. This is the statement that the generalized entropy is nondecreasing, $\frac{dS_{gen}}{dv} \geq 0$, where v is the null coordinate on the horizon. Here,

$$S_{gen} = < \frac{A}{4\hbar G} > + S_{QFT}, \tag{4.1.1}$$

where A is the black hole horizon area at an arbitrary cut of the event horizon and S_{QFT} is the entanglement entropy of the quantum fields in the black hole exterior. We put the $<>$ on the area term to emphasize that the graviton correction is included. As is also well known, both terms in (4.1.1) are individually ultraviolet (UV) divergent (the first term due to loop effects which renormalize G and the second term, entanglement entropy, which is UV divergent), but there is a lot of evidence that the sum is UV finite [18]—[58]. The GSL for Einstein gravity was proved by Wall [5] under the assumption that there exists some renormalization scheme for the boost energy and the entropy in quantum field theory (QFT). Explicitly constructing such a renormalization scheme and proving GSL where each step is manifestly finite is always good. It can also provide insight on why the S_{gen} is UV finite and independent of the UV cutoff, and the role of gravity in making (4.1.1) well defined.

The QFT algebra in the exterior of the black hole in strict $G_N \to 0$ is of type III_1 [1]. Recently, it was shown that the perturbative correction in G_N changes the algebra from type III_1 to type II_∞ crossed product algebra [1, 2, 3]. Furthermore, the entropy of the algebra of observables (3.9.34) in the AdS-Schwarzschild exterior, defined in the crossed product construction, is equal to the generalized entropy at the bifurcation surface up to a state-independent constant [3]. This construction was extended to asymptotically flat black holes in [2], where including the ADM Hamiltonian and a timeshift degree of freedom yields a crossed product with the modular automorphism group of the original algebra. This is further generalized for the case of Kerr in [60]. We want to emphasize that these results are obtained only at the bifurcation surface.

Our primary goal in this chapter is to extend the above results to arbitrary cuts on the horizon and to show that a *local* version of the generalized second law (GSL) is indeed true

[1]The boundary version of this statement in the AdS/CFT correspondence was found by Leutheusser and Liu [14], [13] (see also [59]). They studied the holographic boundary operator algebra of the CFT dual to gravity in the asymptotically anti-de Sitter (AdS) black hole spacetime. They found an emergent type III_1 von Neumann algebra for single trace operators in the large N limit of the CFT boundary.

in crossed product constructions for maximally extended static black holes and Kerr geometries. We have also discussed the asymptotically AdS black holes in Section V. Hence, the crossed product gives the renormalization scheme needed for the proof of GSL. In order to prove this, we utilised the construction of Jensen, Sorce, and Speranza (JSS) [41], who studied operator algebras associated with domains of dependence of arbitrary partial Cauchy slices in Einstein gravity coupled to matter. These algebras are of Type III, but can be promoted to Type II via a crossed product with the modular automorphism group. Their construction relies on a conjecture that a certain local gravitational Hamiltonian, generating flow along a specially chosen vector field on the Cauchy slice, serves as the modular Hamiltonian for some state. JSS support this by arguing that non-local modular Hamiltonians can be rendered local through appropriate subtractions, and the converse of Connes' cocycle theorem then ensures such a local integral defines the modular Hamiltonian of a state. Under this conjecture, one can associate an entropy to the algebra of a subregion, which matches the generalized entropy up to a constant. In the setting of our application of the JSS results, we have shown that the conjecture is indeed true. By considering a slight generalization of the JSS construction to include an observer even for wedge-shaped regions with an asymptotic boundary, we have shown that we obtain a local GSL. We also give evidence for this conjecture for more general modular Hamiltonians in section (4.7) - specifically, seeing how a non-local modular Hamiltonian can be made local by subtracting off appropriate terms.

The chapter is organized as follows. In Section 2, we briefly review the crossed product in black hole spacetime. In Section 3, we define a half-sided modular inclusion, which is used crucially to obtain modular Hamiltonians at the arbitrary cut on the horizon in black hole spacetimes. In Section 4, we discuss modular Hamiltonians in black hole spacetimes, both static and Kerr black holes. In Section 5, we review the salient results of JSS on algebra entropy of subregions of spacetime, which are domains of dependence of partial Cauchy slices. In Section 6, we derive a local GSL using these results. In Appendix A (7.2), we discuss the modular Hamiltonians of wedges in Minkowski spacetime as a warm-up example for the application of half-sided modular inclusions. In section (4.7), we compute (one-sided) modular Hamiltonians for various subregions of a general class of spacetimes and discuss whether they satisfy the conjecture of JSS. In the Discussion section, we provide a summary and discussion of the results presented in this chapter.

4.2 Crossed Product in Black Hole Spacetime

The series of recent papers by CLPW, Witten, and CPW [2, 1, 3] has helped to understand the generalized entropy introduced by Bekenstein [56] better. They have addressed the question of why the generalized entropy is well-defined, whereas the gravity contribution and the quantum field contribution in the generalized entropy are not well-defined separately. CPW showed for an eternal black hole that is either asymptotically flat and asymptotically AdS, that the exterior algebra of quantum field (including gravitons) is a type II crossed product algebra. Further, they showed that the type II algebra entropy defined by (3.9.34) in semi-classical state is the generalized entropy at the bifurcation surface of the black hole. They have also discussed the monotonicity of the generalized entropy of asymptotically AdS black holes by using techniques from von Neumann algebras.

The general construction of crossed products is reviewed in the previous chapter (3.9)[2]. In this subsection, we will briefly see how the algebra in the exterior of a black hole in $G_N \to 0$ is a crossed product algebra as shown by CPW.

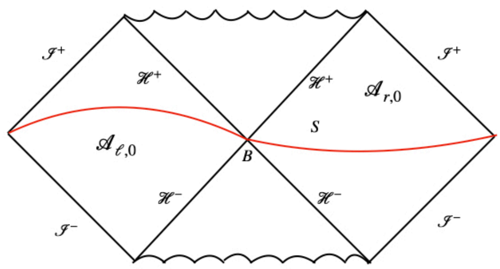


FIGURE 4.1: Maximally extended stationary black hole with Cauchy surface S. The bulk algebra of the left and the right exterior region is $\mathcal{A}_{\ell,0}$ and $\mathcal{A}_{r,0}$ respectively.

[2]Gravitational crossed product constructions have been explored in [61]. For crossed product constructions without gravity, and for a connection between the crossed product and extended phase space, see [62], [63]. Crossed product constructions for quantum field theories on subregions are discussed in [64]. Approximations to the crossed product for Type I algebras are explored in [65]. The use of the crossed product as a covariant regulator is discussed in [55].

Let M be the asymptotically flat, maximally extended Schwarzschild black hole in Einstein's theory or the maximally extended AdS-Schwarzschild black hole. We consider quantum fields in this spacetime, including gravitons. The left and right exterior regions of M will be denoted by ℓ and r respectively, while L and R will be used to denote left and right spatial infinity. Let $\mathcal{H}_0$ be the Hilbert space of this theory that we get by quantizing the fields and the local algebra of observables of the left and right exterior region be $\mathcal{A}_{\ell,0}$ and $\mathcal{A}_{r,0}$ respectively as shown in Figure 4.1. It is well known that algebras $\mathcal{A}_{\ell,0}$ and $\mathcal{A}_{r,0}$ are Type III$_1$ factors (their centers are trivial) [66, 24, 67][3]. Moreover, $\mathcal{A}_{\ell,0}$ and $\mathcal{A}_{r,0}$ are each other's commutants (i.e all the operators of $\mathcal{A}_{\ell,0}$ commute with all the operators in $\mathcal{A}_{r,0}$).

The spacetime is stationary and equipped with a time translation Killing field V. V is future directed in the right exterior region and past-directed in the left exterior region. Due to background diffeomorphism invariance, one can define a conserved quantity $\hat{h}$ associated with the time translation vector field V. Let S be the bulk Cauchy surface going from the spatial infinity of the right exterior region to the spatial infinity of the left exterior region, through the bifurcation surface as shown in Figure 4.1. Then $\hat{h}$ can be defined as

$$\hat{h} = \int_S d\Sigma^\mu V^\nu T_{\mu\nu}. \tag{4.2.1}$$

Here, $T_{\mu\nu}$ is the stress-energy tensor of the bulk fields[4]. In Tomita-Takesaki theory of the quantum fields in the black hole exterior, $\beta\hat{h}$ is the modular Hamiltonian associated with the Hartle-Hawking state $|\Psi_{HH} >$ of the black hole and β is the Hawking temperature[5][68, 69]. It is well known that $\hat{h}$ in Einstein's gravity is the difference between the right ADM Hamiltonian H_R and the left ADM Hamiltonian H_L i.e. $\hat{h} = H_R - H_L$.

CLPW and CPW now extend the Type III$_1$ algebra $\mathcal{A}_{\ell,0}$ and $\mathcal{A}_{r,0}$ by including one more operator h_L with $\mathcal{A}_{\ell,0}$ and h_R similarly for the right algebra [6]. This extended (crossed product) algebra acts on an extended Hilbert space $\mathcal{H} = \mathcal{H}_0 \otimes L^2(\mathbb{R})$ where the extra degree of freedom that has been introduced is the time-shift (the sum of the times in the left and the right exteriors). The extended crossed product right algebra is denoted $\mathcal{A}_r = \mathcal{A}_{r,0} \rtimes \mathbb{R}_h$ and similarly for the left algebra. Here,

$$h_L = H_L - M_0 \qquad h_R = H_R - M_0 \tag{4.2.2}$$

[3]The algebra of operators in quantum field theory in a causal wedge is always a von Neumann algebra of Type III [14]. When the center is trivial it is a Type III$_1$ algebra.

[4]$T_{\mu\nu}$ includes the contribution from the pseudo-stress tensor of gravitons.

[5]The modular operator is defined as $\Delta = \exp\{(-H_{mod})\}$ and for $|\Psi_{HH}\rangle$, $H_{mod} = \beta\hat{h}$.

[6]As discussed in (3.9), the crossed product algebra can be understood as an extension of the original algebra by adjoining the operator $T + \hat{q}$, where T generates an automorphism of the algebra.

M_0 is the ADM mass of some reference black hole. CPW work in a micro-canonical ensemble i.e. an energy eigenstate centered around some energy M_0 (mass of the reference black hole)[7]. The algebra of observables for the right exterior region is studied in a semi-classical limit i.e. $G \to 0$. In this limit, the ADM masses H_R and H_L diverge because the black hole mass M_0 (Schwarzschild radius divided by $2G$) diverges. Also note that the modular Hamiltonian $\hat{h}$ depends only on the fluctuation of the ADM Hamiltonian, i.e., on h_L and h_R. So, CPW works with the non-divergent subtracted Hamiltonians h_L and h_R.

We can also write $\hat{h} = h_r - h_\ell$ where

$$h_r = \int_{S_1} d\Sigma^\mu V^\nu T_{\mu\nu} \tag{4.2.3}$$

$$h_\ell = -\int_{S_2} d\Sigma^\mu V^\nu T_{\mu\nu} \tag{4.2.4}$$

where S_1 and S_2 are the right exterior and the left exterior part of the Cauchy surface S

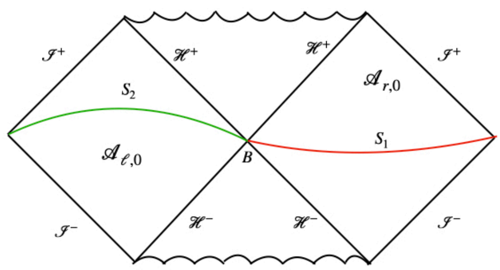


FIGURE 4.2: This figure depicts the split of Cauchy surface S into union of the red Cauchy surface S_1 in the right exterior and the green Cauchy surface S_2 in the left exterior.

as shown in Figure 4.2. As pointed out by CPLW, h_r and h_ℓ have divergent fluctuations. Thus, such a splitting is not true, strictly speaking [1], but in the extended algebra of

[7] CPW explicitly do the microcanonical ensemble construction in the boundary CFT using a thermofield double state.

Witten, the modular operator has a factorization into a product of operators in the left and right algebra.

$$\hat{h} = h_r - h_\ell = h_R - h_L \tag{4.2.5}$$

Further,

$$h_R = \hat{h} + h_L \equiv \frac{h_\psi}{\beta} + x \tag{4.2.6}$$

Let $h_\Psi = H_{\text{mod}}$ denote the modular Hamiltonian for the Hartle–Hawking state $|\Psi\rangle = |\Psi_{\text{HH}}\rangle$. The algebra $\mathcal{A}_r = \mathcal{A}_{r,0} \rtimes \mathbb{R}_h$ is the crossed product of the algebra $\mathcal{A}_{r,0}$ by the modular group associated with the cyclic and separating vector $|\Psi\rangle$. As discussed in the previous chapter, the crossed product of a type III_1 von Neumann algebra by its modular group yields a type II_∞ von Neumann algebra. Moreover, a type II_∞ crossed product algebra admits a notion of trace, and thus we can define the notion of density matrix and entropy, referred to entropy of algebra (see (3.9)). One might think that the operator measuring fluctuations in the ADM mass is added in the algebra by hand without any reason. However, we emphasize that this is not the case. Gravity is a diffeomorphism-invariant theory, and there are gauge constraints associated with diffeomorphisms. Importantly, some of these gauge constraints must be imposed even in the $G_N \to 0$ limit—for example, the gravitational Gauss law. The crossed product algebra constructed by CPW is the algebra of observables that commute with these constraints, and therefore represents the correct algebra of observables in a gravitational setting.

Following the discussion in (3.9), we can write operator in $\mathcal{A}_r$ denoted by $\hat{a}$ has the form $\hat{a} = ae^{(ish_\psi/\beta)} \otimes e^{(isx)}$, where $a \in \mathcal{A}_{r,0}$. The states on which this operator acts has the form $\left|\hat{\Psi}\right\rangle = |\Psi\rangle \otimes g(x) \in \mathcal{H}$ where $|\Psi\rangle \in \mathcal{H}_0$ and $g(x) \in L^2(\mathbb{R})$. The most generic operator in $\mathcal{A}_r$ can be written as

$$\hat{a} = \int_{-\infty}^{\infty} a(s) e^{is(x+\hat{h})} ds \tag{4.2.7}$$

where $a(s) \in \mathcal{A}_{r,0}$. Similarly, we can write the most general state as

$$\left|\hat{\Psi}\right\rangle = \int dx f(x) \left|\Psi\right\rangle \left|x\right\rangle \tag{4.2.8}$$

We can also write algebra entropy using (3.9.34). As emphasized earlier, $S(\hat{\Phi})$ defined in (3.9.34) should not be thought of as the entanglement entropy of $\mathcal{A}_r$ but is a renormalized entropy. Also, because of the ambiguity in the definition of trace (trace is defined up to a scaling), it is only the entropy differences that are unambiguous, not the entropy itself. CPW now works with a semi-classical state, i.e, the state with fluctuation in timeshift p, $\Delta p \sim O(\varepsilon)$, where ε is some parameter much smaller than unity. Then, with $x = h_L$,

$\Delta x \sim O(\frac{1}{\varepsilon})$. CPW consider the AdS-Schwarzschild black hole and write down the semi-classical state in the *boundary* CFT. However, it could have equally been defined in the bulk and we will assume that the formulae below correspond to the equivalent bulk statement. The general form of such a state is,

$$\left|\hat{\Phi}\right\rangle = \int_{\infty}^{\infty} \varepsilon^{\frac{1}{2}} g(\varepsilon x) |\Phi\rangle |x\rangle \quad \text{where} \quad |\Phi\rangle \in \mathcal{H}, \;\; g(x) \in L^2(\mathbb{R}) \tag{4.2.9}$$

It is shown by CPW that the density matrix $\rho_{\hat{\Phi}}$ for the state $\left|\hat{\Phi}\right\rangle$ is approximately [8] given by

$$\rho_{\hat{\Phi}} \approx \varepsilon \bar{g}(\varepsilon h_R) e^{-\beta x} \Delta_{\Phi|\Psi} g(\varepsilon h_R) \tag{4.2.10}$$

where $\Delta_{\Phi|\Psi} = e^{-h_{\Psi|\Phi}}$ is a relative modular operator and $h_{\Psi|\Phi}$ is the relative modular Hamiltonian [9]. The relative modular Hamiltonian $h_{\Phi|\Psi}$ is defined such that $h_{\Psi|\Psi} = h_\Psi$. As we have already mentioned, the Type II algebra modular operator factorizes, i.e $\Delta_{\hat{\Phi}|\hat{\Psi}} = \rho_{\hat{\Psi}} \rho'^{-1}_{\hat{\Phi}}$ (where prime denotes the element of the commutant of the algebra $\mathcal{A}_r$)[3]. Putting (4.2.10) in (3.9.31) or (3.9.34) yields

$$S(\hat{\Phi})_{\mathcal{A}_r} = \left\langle \hat{\Phi} \middle| \beta h_R \middle| \hat{\Phi} \right\rangle - \left\langle \hat{\Phi} \middle| h_{\Psi|\Phi} \middle| \hat{\Phi} \right\rangle - \left\langle \hat{\Phi} \middle| \log\left(\varepsilon |g(\varepsilon h_R)|^2\right) \middle| \hat{\Phi} \right\rangle + O(\varepsilon) \tag{4.2.11}$$

By definition, the second term in the above equation is the relative entropy,

$$S_{rel}(\Phi||\Psi) = -\left\langle \hat{\Phi} \middle| h_{\Psi|\Phi} \middle| \hat{\Phi} \right\rangle \tag{4.2.12}$$

The expression in (4.2.11) is shown to be equal to S_{gen} at the bifurcation surface up to an additive constant; for details, see [2, 3]. This completes the discussion of the algebra of observables in black hole spacetime. Now, we introduce concepts of the half-sided modular inclusion, which will play a key role in proving GSL.

4.3 The half-sided modular inclusion (HSMI)

In this section, we want to introduce the definition and properties of the half-sided modular inclusion (HSMI). Let A be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$, with the cyclic and separating vector $\Omega \in \mathcal{H}$. The modular operator for A is Δ_A.
1) $\mathcal{H}_{smi}(A)^-$ is a von Neumann sub-algebra B of A with the properties: a) Ω is cyclic and separating for B.

[8] The expression is only valid up to corrections suppressed by $O(\varepsilon)$ terms.
[9] Ψ is the Hartle Hawking state and Φ is any arbitrary state of quantum fields in the black hole spacetime.

b) $\Delta_A^{it} B \Delta_A^{-it} \subset B$ for $t \leq 0$.

In this case B is called the positive half-sided modular inclusion of A.

2) $\mathcal{H}_{smi}(A)^+$ is a von Neumann sub-algebra B of A with the properties: a) Ω is cyclic and separating for B.

b) $\Delta_A^{it} B \Delta_A^{-it} \subset B$ for $t \geq 0$.

In this case B is called the negative half-sided modular inclusion of A.

Let Δ_B be the modular operator of B. There is a theorem ensuring the existence of the one-parameter continuous unitary $U(t)$ such that $U(1)$ maps A and B [42].

Theorem 1: If A and B are von Neumann algebras such that B is the half-sided modular inclusion of A, then there exists a one parameter continuous unitary $U(t)$ with $t \in \mathbb{R}$ with the following properties:

When inclusion is negative

a) $U(t)$ has a positive generator, i.e we can write

$$U(t) = \exp[iHt], \quad \text{with} \qquad H \geq 0 \tag{4.3.1}$$

b) $U(t)\Omega = \Omega \ \forall t \in \mathcal{R}$

c) $U(t)AU(-t) \subset A$ and $t \leq 0$

d) $B = U(-1)AU(1)$

e) $\Delta_A^{-it} \Delta_B^{it} = U(1 - e^{-2\pi t})$.

h) $\Delta_A^{it} U(s) \Delta_A^{-it} = U(e^{(2\pi t)} s)$

When inclusion is positive

a) $U(t)$ has a positive generator, i.e, we can write

$$U(t) = \exp[iHt], \quad \text{with} \qquad H \geq 0 \tag{4.3.2}$$

b) $U(t)\Omega = \Omega \ \forall t \in \mathcal{R}$

c) $U(t)AU(-t) \subset A$ and $t \geq 0$

d)$B = U(1)AU(-1)$

e) $\Delta_A^{-it} \Delta_B^{it} = U(e^{2\pi t} - 1)$.

h) $\Delta_A^{it} U(s) \Delta_A^{-it} = U(e^{(-2\pi t)} s)$

The conditions a,b and c in (4.3.1) and (4.3.2) define what is known as half-sided modular translation in the literature. As a warm up example, we have shown in the appendix 7.2 how one can obtain the modular operator for arbitrary wedges in Minkowski spacetime using modular inclusions. In particular, we obtain the modular operator of the wedge which is a translated version of the original wedge at the origin by constant amount. One can also obtain the modular operator of a null translated wedge where the null translation depends on the transverse coordinate. The latter case is a simpler case of a null translated wedge in a black hole spacetime which will be used while proving the GSL.

4.4 Half-sided modular inclusions in black hole spacetimes

In this section, we want to obtain the relation between the modular Hamiltonian of two wedges in both static and Kerr black holes. For static black holes, the wedges are A and B, as shown in Figure 4.3. We will restrict ourselves to black holes with a bifurcate Killing horizon and a smooth bifurcation surface. Both asymptotically flat and asymptotically AdS black holes are considered. We will also obtain such a relation for the two wedges on $\mathcal{H}^+$ for the Kerr black hole.

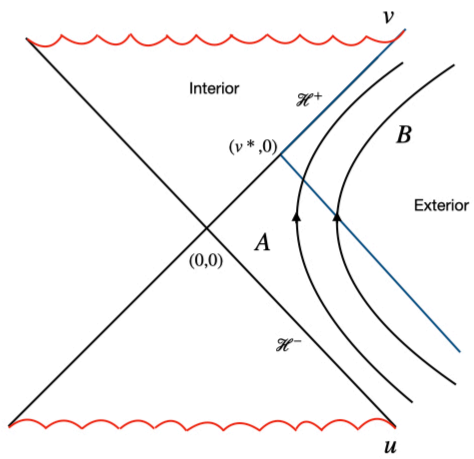


FIGURE 4.3: The figure depicts a black hole spacetime with Kruskal-like coordinates.

Static Black holes:
Let $\mathcal{M}_\mathcal{A}$ and $\mathcal{M}_\mathcal{B}$ be the von Neumann algebras associated with the wedge A and B, respectively, as shown in Figure 4.3. Here, $v^*(y) > 0$ can be a function of the transverse coordinates y. Sewell's work in [69] established that the modular Hamiltonian of the right exterior in the static black hole spacetime generates time translation with respect to the

asymptotically timelike Killing vector in the right exterior (for e.g, the Schwarzschild time in a Schwarzschild black hole). The time translation Killing field behaves like a boost vector field on the event horizon of the black hole. Let ξ^μ be the Killing field associated with the time translation in this spacetime and let $T_{\mu\nu}$ be the stress tensor of all quantum fields present in this spacetime. Then, let us define

$$K_A = \frac{2\pi}{\kappa} \int_\Sigma T_{\mu\nu} \xi^\mu d\Sigma^\nu \tag{4.4.1}$$

where Σ represents a Cauchy surface in spacetime and κ, the surface gravity, is specific to the black hole. In this spacetime, there exists a Hartle-Hawking state Ω_{HH} that is a unique stationary state with respect to Killing time and is regular at the horizon [70]. This state is also KMS when restricted to the wedge A. The KMS condition for operators a and b in $\mathcal{M}_\mathcal{A}$ is

$$< \alpha_t(a)b >=< b\alpha_{t+i\beta}(a) > \tag{4.4.2}$$

Here, α_t is the automorphism of the algebra generated by the isometry of translations of the time (generated by the asymptotically timelike Killing vector). The modular operator for $(\mathcal{M}_\mathcal{A}, \Omega_{HH})$ for the right exterior is $\Delta_A = \exp[-K_A]$. Modular flow $\Delta_A^{it} = \exp[-iK_A t]$, corresponds to boost-like flow near the horizon and timelike flow inside A. $\alpha_t(a) = \exp[-iK_A t] a \exp[+iK_A t]$, and we know from modular theory that this is an automorphism of the algebra. Now, since $\mathcal{M}_\mathcal{B} \subset \mathcal{M}_\mathcal{A}$, this implies Ω_{HH} is separating for $\mathcal{M}_\mathcal{B}$. Furthermore, the fact that Ω_{HH} is cyclic with respect to $\mathcal{M}_\mathcal{A}$ and the spacetime possesses a global timelike Killing field implies Ω_{HH} is cyclic for $\mathcal{M}_\mathcal{B}$ [71, 72]. In this section, we will choose for the Cauchy slice Σ of the black hole right exterior, $\mathcal{H}^+ \cup \mathcal{I}^+$, the union of the future event horizon and future null infinity (both on the left and right for the maximally extended black hole).

As seen in Figure 4.3, Δ_A^{it} has a local geometrical action on the operators in $\mathcal{M}_\mathcal{B}$, causing them to move along integral curves of the boost Killing field. Since the flow is null on the horizon and timelike inside, the forward boost cannot take the local operator in $\mathcal{M}_\mathcal{B}$ outside it. This implies: $\Delta_A^{it} \mathcal{M}_\mathcal{B} \Delta_A^{-it} \subset \mathcal{M}_\mathcal{B}$ for $t \leq 0$. Therefore, according to the definition of positive half-sided modular inclusion, $\mathcal{M}_\mathcal{B}$ is the positive half-sided modular inclusion (HSMI) of $(\mathcal{M}_\mathcal{A}, \Omega)$. Once the inclusion holds, Theorem 1 guarantees the existence of a unitary $U(t)$ such that

$$\Delta_A^{-it} \Delta_B^{it} = U(e^{2\pi t} - 1) \tag{4.4.3}$$

where $U(t) = \exp[i\mathcal{E}_{v*} t]$ where $\mathcal{E}_{v*}$ is a positive operator. From property (d) of a positive half-sided modular inclusion, it follows that $\mathcal{E}_{v*}$ is the generator of a null translation and can be written as

$$\mathcal{E}_{v*} = \int_{-\infty}^{\infty} \int d^2\Omega \; dv \; v^*(y) T_{vv} \tag{4.4.4}$$

Now following the same steps as in the appendix (7.2) for the Minkowski wedges, we can easily show that

$$K_B = K_A - 2\pi \mathcal{E}_{v*} \tag{4.4.5}$$

and the modular flow of the wedge B

$$\Delta_B^{it} = e^{-iK_B t}. \tag{4.4.6}$$

We know there is no global translation symmetry in this spacetime, but null translation is a symmetry on the horizon. As a result, we anticipate that the modular Hamiltonian of the wedge B will be expressed in local form, at least on the horizon (by local form, we mean as a local integral over the three dimensional Cauchy slice, which in this case is a portion of the horizon). If one chooses any other partial Cauchy slice in the wedge B other than the horizon ($\mathcal{H}^+ \cup \mathcal{I}^+$ for asymptotically flat black holes), the modular Hamiltonian will be non-local. It is straightforward to verify that the one-sided modular Hamiltonian when computed on $\mathcal{H}^+_{v*} \cup \mathcal{I}^+$ in asymptotically flat spacetime and $\mathcal{H}^+_{v*}$ in AdS, is identical to the form of a conjectured modular Hamiltonian in [41], where $\mathcal{H}^+_{v*}$ is the portion of the future horizon for $v > v*$. This is because

$$K_B^{\mathcal{H}^+_{v*} \cup \mathcal{I}^+} = 2\pi \int_{v*}^{\infty} dv \int d^2\Omega \; T_{vv}(v - v^*) + K(\mathcal{I}^+). \tag{4.4.7}$$

Here, $K(\mathcal{I}^+)$ is the contribution to the modular operator from the partial Cauchy slice $\mathcal{I}^+$, which is common to K_B and K_A. Note that the first term on the right in (4.4.7) is exactly of the form of a local modular operator proposed in [41] in a perturbative gravity expansion where T_{ab} also contains the stress-energy of gravitons. In [41], it is conjectured that for the domain of dependence of a partial Cauchy slice Σ_A in any spacetime, an expression such as

$$\int_{\Sigma_A} T_{ab} V^a d\Sigma^b \tag{4.4.8}$$

is proportional to the modular Hamiltonian for some state, if the vector field V in the integral obeys certain properties: it acts like a boost close to the entangling surface separating the partial Cauchy slice from its complement, and has a certain prescribed form on the domain of dependence of the complement of this Cauchy slice. Further, the vector V is such that $\nabla_a V_b|_S = \kappa n_{ab}$ where S is the entangling surface and n_{ab} is the binormal to the surface satisfying $n_{ab}n^{ab} = -2$. κ is a constant. It can be checked that the vector field $V = \kappa(v - v^*)\frac{\partial}{\partial v}$ satisfies all the requirements of [41] on the horizon — further, it can be suitably extended both to the rest of the wedge (by choosing for example $\kappa((v - v^*)\frac{\partial}{\partial v} - u\frac{\partial}{\partial u}))$ and to the domain of dependence of the complement to the Cauchy slice. κ is exactly the surface gravity on the horizon. The condition $\nabla_a V_b|_S = \kappa n_{ab}$ is satisfied if we take v^* to be a constant. When $v^* = f(y)$, then we need to modify the condition in [41] to be

$$n^{ab}\nabla_a V_b|_S = -2\kappa. \tag{4.4.9}$$

As it happens, all the derivations in [41] go through with this modification - further, it is possible to change coordinates and satisfy the condition $\nabla_a V_b|_S = \kappa n_{ab}$. Now, if we have $v^*(y)$ being a non-trivial function of the transverse coordinates y, then $\nabla_a V_b$ will not get any contributions from the terms proportional to the Christoffel symbols since those terms are proportional to the components of V which vanishes at the entangling surface S. The terms of the form $\partial_a V_b$ will contain derivatives with respect to the transverse coordinates. However, those terms get projected out when multiplied by the binormal n^{ab} in (4.4.9). So, V satisfies (4.4.9).

Here, we know exactly that K_B is the modular Hamiltonian of wedge B for the *same* state Ω_{HH} as K_A and is obtained via the half-sided modular inclusion. Thus, this provides an example where the conjecture of JSS in [41] is exactly true — further, in their conjecture, the state for which this expression is the modular Hamiltonian is not known in general, whereas in this example, we know this.

Kerr black hole:

The Kerr spacetime is stationary, and has a Killing horizon. The Killing field associated with the horizon is not timelike everywhere in the exterior of the black hole. As a result, there is no global KMS state on this spacetime. There is no Hartle-Hawking state for which we can repeat the procedure which we did for the Schwarzschild black hole. However, one can define a stationary state in the interior until the Cauchy horizon, and the exterior of the black hole [73]. The Kerr spacetime is not globally hyperbolic when extended beyond the inner Cauchy horizon. Nevertheless, there is a well defined initial value problem for the exterior region R and the region between the Cauchy horizon and the exterior horizon F. This has been discussed in great detail by Kudler-Flam, Leutheusser and Satishchandran (KLS) in the paper [60]. Figure 4.4 depicts one of the Cauchy surfaces for these regions.
In this section, we will be primarily interested in the right exterior of the black hole. Let us consider linear fields in this spacetime which we generically denote $\Phi(x)$. First of all, it can be shown that any local field smeared with respect to some smooth function with

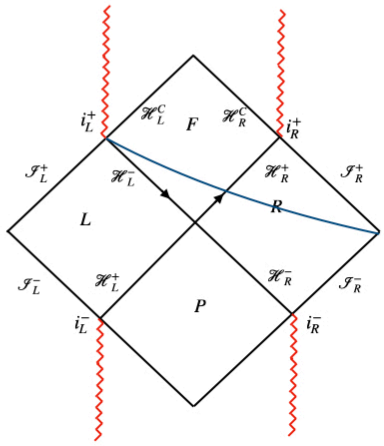


FIGURE 4.4: The figure depicts the Kerr black hole, where the red curvy line represents the singularity and the blue line represents the Cauchy surface for $R \cup F$.

compact support can be written in terms of smeared local fields on some Cauchy slice [74]. We are interested in a Cauchy slice for the right exterior. It was shown by Wall [5] that one can take the Cauchy slice to be the union of $\mathcal{H}_{\mathcal{R}}^+$ and $\mathcal{I}^+$. We can construct the algebra of observables $\mathcal{A}_{\mathcal{H}_{\mathcal{R}}^+}$ on $\mathcal{H}_{\mathcal{R}}^+$ and $\mathcal{A}_{\mathcal{I}_{\mathcal{R}}^+}$ on $\mathcal{I}_R^+$ and then the algebra of observables in the right exterior $\mathcal{A}_R \simeq \mathcal{A}_{\mathcal{H}_{\mathcal{R}}^+} \otimes \mathcal{A}_{\mathcal{I}_{\mathcal{R}}^+}$. This decomposition of bulk algebra in terms of boundary algebra has also been done by KLS in [60] for the Cauchy slice which is a union of $\mathcal{H}^-$ and $\mathcal{I}^-$. KLS consider a Gaussian state in the black hole right exterior [10]. They assume that the state is Hadamard, stationary with respect to the horizon Killing field and has zero energy with respect to it. This state can be written as a state on the von Neumann algebra $\omega_0 : \mathcal{A}_R \to C$. This state is denoted $\omega_0 = \omega^{\mathcal{H}^-} \otimes \omega^{\mathcal{I}^-}$, where $\omega^{\mathcal{H}^-}$ and $\omega^{\mathcal{I}^-}$ are invariant with respect to the Killing field, and are Gaussian at $\mathcal{H}^-$ and $\mathcal{I}^-$ respectively. Then, they take the unique Gaussian state invariant under affine time translations on $\mathcal{H}^-$ and $\mathcal{I}^-$. They show this obeys the KMS condition (4.4.2) on the horizon. This enables

[10]These are states for which one-point functions vanish and n point functions can be written as products of 2 point functions.

them to determine the form of the modular Hamiltonian on $\mathcal{H}^-$ and $\mathcal{I}^-$.

$$K_- = \frac{2\pi}{\kappa}\int_{\mathcal{H}^- \cup\, \mathcal{I_R}^-} T_{\mu\nu}\xi^\mu d\Sigma^\nu \tag{4.4.10}$$

where $\xi^\mu = t^\mu + \Omega_H \psi^\mu$. t^μ is the time translation Killing field and ψ^μ is the azimuthal Killing field in the Kerr spacetime. We can do the same at $\mathcal{H}^+$ and $\mathcal{I}^+$. Thus, we assume the existence of a quasifree Hadamard, stationary state with zero Killing energy and consider the unique state invariant under affine time translations on $\mathcal{H}_L^-$, $\mathcal{H}_R^+$ and $\mathcal{I}^+$. Such a state is automatically KMS on $\mathcal{H}_R^+$. The two-point function in this state is given on $\mathcal{H}_R^+$ with coordinates (V, x^A) by

$$\omega_0(\Pi(x_1)\Pi(x_2)) = -\frac{2}{\pi}\frac{\delta_{S^2}(x_1^A, x_2^A)}{(V_1 - V_2) - i0^+}. \tag{4.4.11}$$

Here, $\Pi(x) = \partial_V \Phi$ are operators supported on $\mathcal{H}_R^+$. Considering the geometric flow of the Killing time translation $(V, x^A) \to (e^{\kappa t}V, x^A)$ on $\mathcal{H}^+$ — this generates an automorphism of the algebra, α_t. With respect to this flow, it can be checked that the state ω_0 is KMS [60]. A similar observation can be made at $\mathcal{I}_\mathcal{R}^+$. Added to the assumption that this state has zero Killing energy on the horizon, this implies that this flow is modular flow and the modular Hamiltonian is given on $\mathcal{H}_L^- \cup \mathcal{H}_R^+ \cup\, \mathcal{I}_\mathcal{R}^+$ by

$$K = \frac{2\pi}{\kappa}\int_{\mathcal{H}_L^- \cup \mathcal{H}_R^+ \cup\, \mathcal{I}_\mathcal{R}^+} T_{\mu\nu}\xi^\mu d\Sigma^\nu \tag{4.4.12}$$

We note that the modular Hamiltonian on some other Cauchy slice may not have this nice, local form and away from this slice, modular flow may not be a geometric flow. We can formally split the above equation as an integral on $\mathcal{H}_R^+ \cup \mathcal{I}^+$ and $\mathcal{H}_L^-$ and call it K_R and $-K_L$,

$$K = K_R - K_L \tag{4.4.13}$$

This split is just formal because the entanglement across the bifurcation surface is infinite. This give rise to a type III von Neumann algebra $\mathcal{U}(\mathcal{H}_R^+, \omega_0)$. Further, using the fact that for linear fields, one can always write any observable in the bulk region in terms of an observable on the Cauchy slice pertaining to that bulk, the algebra of the right exterior can be written as [11],

$$\mathcal{U}(R, \omega_0) \simeq \mathcal{U}(\mathcal{H}_R^+, \omega_0) \otimes \mathcal{U}(\mathcal{I}_R^+, \omega_0) \tag{4.4.14}$$

Further, the algebra on the $\mathcal{H}_L^-$, $\mathcal{U}(\mathcal{H}_L^-, \omega_0)$ is the commutant of $\mathcal{U}(\mathcal{H}_R^+, \omega_0)$. More details can be found in the paper of KLS [60].

[11] We assume that we can specify the initial data independently on $\mathcal{H}_R^+$ and $\mathcal{I}_R^+$.

In the null coordinates on $\mathcal{H}^+$, the modular Hamiltonian (4.4.12) generates dilatation on the horizon, see Figure 4.4. Consider another wedge R' that has one of its future null boundaries overlapping with the part of $\mathcal{H}^+_R$, as shown in Figure 4.5.

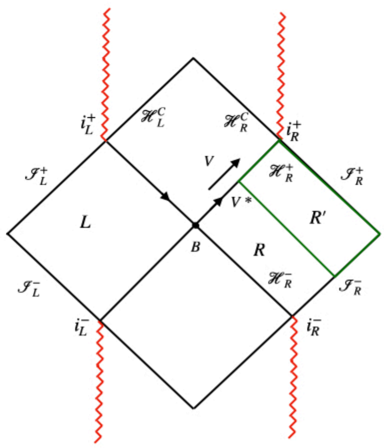


FIGURE 4.5: In this diagram, the green wedge represents bulk wedge R', with the vertex at $V = V*$.

Let V be an affine time on $\mathcal{H}^+$ which goes from 0 to ∞ as we move from the bifurcation surface B to i_R^+, as shown in the Figure 4.5. Let the vertex of the new wedge be at $V*$ and $\mathcal{H}^+(V*) = \mathcal{H}^+ \cap (V \geq V*)$. Following the above discussion, the algebra of the wedge can be represented by the boundary algebra $\mathcal{U}(R', \omega_0) \simeq \mathcal{V}(\mathcal{H}^+(V*), \omega_0) \otimes \mathcal{U}(\mathcal{I}_R^+, \omega_0)$. The state ω_0 here is a restriction of ω_0 to R'. The state Ω is clearly separating and cyclic [71] for $\mathcal{V}(\mathcal{H}^+(V*), \omega_0)$ by construction. Since modular flow produces dilatation on the horizon, $\mathcal{U}(R', \omega_0)$ is the HSMI of the algebra $\mathcal{U}(R, \omega_0)$. Following the same steps as for the Schwarzschild and Rindler spacetime, the modular operator of R' is,

$$K(V*) = K - 2\pi \mathcal{E}_{V*} \tag{4.4.15}$$

where $K(V*)$ and K the modular Hamiltonian for the wedge R' and R and $\mathcal{E}_{V*}$ is the generator of the null translation connecting them. This modular operator (4.4.15) is the modular operator for the state ω_0 and it has a local geometric action on the boundary algebra, since the null translation is a symmetry on the horizon. It will not have a local geometrical action in the exterior R' away form the horizon.

4.5 Review: Type II construction for gravitational subregion

In this section, we would like to summarize the construction of Jensen, Sorce and Speranza (JSS) in [41] which we will use to prove the GSL in the next section. This construction is for Einstein gravity coupled to quantum fields. The construction generalizes recent work which studies QFT in a static black hole background in [1, 2, 3], where the entropy of an algebra is discussed. This algebra is that of fields in the black hole exterior. In [1, 2, 3], this algebra is enlarged using the crossed product construction in von Neumann algebra. The operator that is added in the enlarged algebra is the ADM mass which now acts on an enlarged Hilbert space which includes square integrable functions of a new degree of freedom, the timeshift. This crossed product (by the modular automorphism group) changes the algebra of fields in the right exterior from a Type III$_1$ von Neumann algebra to a Type II algebra. Consequently, one has a renormalized trace on the algebra which can be used to obtain a von Neumann entropy in any state for the algebra. [1, 2, 3] showed that this entropy is the generalized entropy of the black hole at the bifurcation surface modulo a constant. In this construction, the isometry group of the static black hole is implemented as a set of constraints - the time translation generator then equals a boundary term — the difference of the left and right ADM Hamiltonians.

In [41], JSS have generalized the crossed product construction of [1, 2, 3] for arbitrary subregions to obtain the entropy of the algebra of domains of dependence of partial Cauchy slices. This is for subregions in theories of Einstein gravity coupled to matter in the Newton's constant $G_N \to 0$ limit. The construction depends on a specific vector field and relies on the existence of a conjectured state whose modular flow is local and geometrical on some Cauchy slice.

JSS construction:

Let A be a subregion and A' be the causal complement as shown in the figure 4.6. Since the observables must be diffeomorphism invariant, they must commute with the constraint associated with these diffeomorphisms. In particular, JSS consider a class of subregion-preserving diffeomorphisms that act both on A and A', with the following properties:

1) They generate boosts around the entangling surface.

2) The vector field ξ^a, which generates this diffeomorphism should be future directed in A

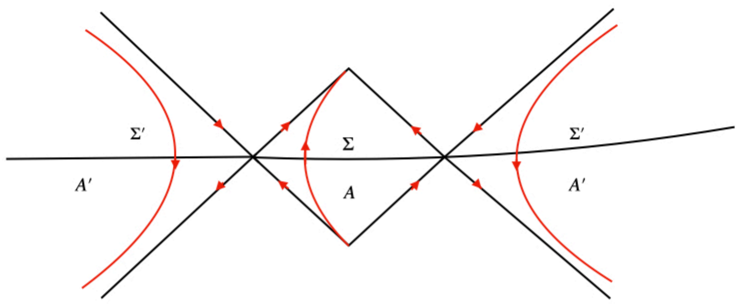


FIGURE 4.6: In this diagram, A is the subregion of interest and A' is its complement. Σ and Σ' are partial Cauchy slices associated with A and A'. Red lines in the diagram represent the vector field ξ^a and its direction.

and past directed in A', and should be tangent to the null boundaries of the subregions.
3) ξ^a must vanish at the entangling surface $\partial\Sigma$ and have constant surface gravity κ on $\partial\Sigma$, given by

$$\nabla_a \xi_b \overset{\partial\Sigma}{=} \kappa n_{ab} \tag{4.5.1}$$

where n_{ab} is the binormal to $\partial\Sigma$.
The gravitational algebra is obtained by imposing the diffeomorphism as a constraint on the algebra of observables order by order in the full nonlinear theory of gravity. However, as shown in [2], directly imposing constraints on $\mathcal{A}_{QFT}$ and $\mathcal{A}'_{QFT}$ trivializes the algebra. Instead, one must introduce an observer in the subregion and extend the algebra by adding Hamiltonian $H_{obs} = \hat{q}$ of the observer in the algebra of observables. Also, one must extend the Hilbert space by tensoring the Hilbert space of the QFT with the observer Hilbert space $\mathcal{H}_{obs} = L^2(\mathcal{R})$. When the subregion does not contain any asymptotic boundary [12], we need an observer to define the location of subregion. We must also add an observer in A', but since A' contains an asymptotic boundary, the role of the observer is played by the

[12] In all the cases considered in this chapter, the subregion will contain an asymptotic boundary.

ADM Hamiltonian H_{ADM} [13]. Now, the full algebra is $(\mathcal{A}_{QFT} V \mathcal{A}'_{QFT}) \otimes \mathcal{A}_{obs} \otimes \mathcal{A}_{ADM}$, and it acts on the Hilbert space $\mathcal{H} = \mathcal{H}_{QFT} \otimes \mathcal{H}_{obs} \otimes \mathcal{H}_{ADM}$. It can be shown that the gravitational constraint associated with ξ^a is given by

$$C[\xi] = H^g_\xi + H_{obs} + H_{ADM}. \tag{4.5.2}$$

Here, H^g_ξ is the operator generating the flow ξ^a on the quantum field algebra $\mathcal{A}_{QFT}$ and $\mathcal{A}'_{QFT}$ instantaneously on the Cauchy slice $\Sigma_c = \Sigma \cup \Sigma'$ as shown in the figure 4.6. One can write it as a local integral of the matter and graviton stress tensors (i.e., as an integral on the Cauchy surface), which at leading order has the form (linear order constraint having already been implemented):

$$H^g_\xi = \int_{\Sigma_c} T_{\mu\nu} \xi^\nu d\Sigma^\mu \tag{4.5.3}$$

Further, JSS have shown that implementing the constraint at the level of the subregion algebra gives the von Neumann algebra $\mathcal{A}^C$, which is the crossed product of $\mathcal{A}_{QFT}$ by the flow generated by H^g_ξ.

$$\mathcal{A}^C = \{e^{iH^g_\xi \hat{p}} a e^{-iH^g_\xi \hat{p}}, e^{i\hat{q}t} | a \in A_{QFT}, t \in \mathcal{R}\}'' \tag{4.5.4}$$

where $\hat{p}$ is the canonical conjugate to the observer Hamiltonian and S'' denotes the smallest von Neumann algebra containing the set S. **Further, JSS assume that H^g_ξ is the modular Hamiltonian for some state on the algebra** $\mathcal{A}_{QFT}$. With this assumption, the algebra in (4.5.4) becomes type II, for more details, see [41]. The assumption that H^g_ξ is a modular Hamiltonian is a key assumption for obtaining the type II algebra — there are cases for which this assumption is **exactly true,** and these situations are the focus of our attention in the previous and next sections. Once the type II algebra is obtained, it is straightforward to define a renormalized trace and to associate entropy with the algebra as done in [1, 2, 3],

$$S(\rho_{\hat{\phi}}) = -S_{rel}(\Phi || \Psi) - \beta \Big\langle H_{obs} \Big\rangle_f + S^f_{obs} + \log \beta \tag{4.5.5}$$

where β is the inverse temperature associated with the KMS state Ψ for which H^g_ξ is the modular operator and $\hat{\Phi} = \Phi \otimes f$ is the state in the crossed product construction of the type II algebra, see [41]. Here, we have assumed that the state $\hat{\Phi}$ is semiclassical. This is defined as a eigenstate of the conjugate momentum operator $\hat{p}$ peaked around zero momentum or, equivalently, a state with a slowly varying position wavefunction $f(\hat{q})$. This assumption is

[13] This needs to be generalized to include an observer even for subregions with an asymptotic boundary, as we discuss in the next section.

crucial because, in crossed product algebras, operators are dressed with respect to $e^{iH^g_\xi \hat{p}}$. When acting on states of the form $|\Phi\rangle \otimes |f\rangle$, this dressing induces entanglement between the observer and quantum field degrees of freedom whenever the modular energy is non-zero. For the state with non-zero modular energy, the semiclassical assumption ensures this entanglement remains small, and simplifies the entropy of the algebra $S(\rho_\Phi)$, enabling the derivation of equation (4.5.5). Further it can be shown that Araki's Type III relative entropy of the state Φ with respect to state Ψ is [66]

$$S_{rel}(\Phi||\Psi) = \beta\Big\langle H^\Sigma_\xi \Big\rangle_\phi - S^{QFT}_\phi - \beta\Big\langle H^\Sigma_\xi \Big\rangle_\psi + S^{QFT}_\Psi \tag{4.5.6}$$

where $H^\Sigma_\xi = \int_\Sigma T_{\mu\nu}\xi^\nu d\Sigma^\mu$ is the one-sided modular Hamiltonian. Finally, JSS show that the algebra entropy is the generalized entropy modulo a state independent constant. JSS assume that the gravitational constraint $C[\xi] = 0$ holds locally on the partial Cauchy slice for the subregion. This allows them to obtain an integrated first law of local subregions,

$$H^\Sigma_\xi + H_{obs} = -\frac{1}{16\pi G_N}\int_{\partial\Sigma} n_{ab}\nabla^a\xi^b = -\frac{\kappa}{2\pi}\frac{A_{\partial\Sigma}}{4G_N} \tag{4.5.7}$$

where $A_{\partial\Sigma}$ is the area of the entangling surface [14]. Further, in quantum theory, the constraint should be implemented as an operator equation, which will give

$$\Big\langle H^\Sigma_\xi \Big\rangle_\Phi + \Big\langle H_{obs} \Big\rangle_f = -\frac{\kappa}{2\pi}\Big\langle \frac{A_{\partial\Sigma}}{4G_N}\Big\rangle_{\hat{\Phi}}. \tag{4.5.8}$$

Using (4.5.7) and (4.5.8), it can be shown that

$$S(\rho_{\hat{\phi}}) = \Big\langle \frac{A_{\partial\Sigma}}{4G_N}\Big\rangle_{\hat{\Phi}} + S^{QFT}_\Phi + S^{obs}_f + c \tag{4.5.9}$$

where c is a state-independent constant. The above equation is the relation between the entropy of the algebra and generalized entropy for the subregion.

$$c = \log\beta - \beta\Big\langle H^\Sigma_\xi \Big\rangle_\psi + S^{QFT}_\Psi \tag{4.5.10}$$

[14]This is exactly the place where $\nabla_a\xi_b \overset{\partial\Sigma}{=} \kappa n_{ab}$ is used, but notice that we will get same answer even if we assume the weaker condition $n^{ab}\nabla_a\xi_b \overset{\partial\Sigma}{=} -2\kappa$. By changing coordinates, we can also get exactly the JSS condition $\nabla_a\xi_b \overset{\partial\Sigma}{=} \kappa n_{ab}$

4.6 The Generalized Second Law (GSL)

In this section, we will show that a local generalized second law holds true in the crossed product constructions for static and Kerr black holes with bifurcate Killing horizons and in Rindler spacetime. While we will do the computation for asymptotically flat black holes, it can be done for asymptotically AdS black holes as well. The idea is to use the construction of JSS [41], but with a modification. As mentioned in the previous section, their construction relies on an assumption that their proposed local Hamiltonian is the modular Hamiltonian of some state. In our computation of modular Hamiltonians, we have shown that the modular Hamiltonian obtained using HSMI of the algebra is for the Hartle Hawking state of static black holes and Minkowski vacuum for Rindler (and for the particular Gaussian state we assumed in the Kerr case) - furthermore, it precisely matches with the form of the JSS local Hamiltonian (4.4.8), when the Cauchy slice contains the future horizon. This is true for both wedges A and B in Figure 4.3. For wedge B, the modular Hamiltonian is (4.4.7) which is of the form (4.4.8) with vector field $V = \kappa(v - v^*(y))\frac{\partial}{\partial v}$ on the horizon. This satisfies nearly all of the conditions of JSS (when suitably extended to the rest of the wedge as described in previous sections and in the complement Cauchy slice) and the modified condition (4.4.9) which, as we have already mentioned in previous sections, is all we need for the JSS construction to go through. Thus, we have the modular Hamiltonians of both wedges A and B corresponding to the *same* state. They both have the JSS form. So the assumption of JSS for a local modular Hamiltonian is explicitly realized in this case. Our aim is to use the JSS construction along with the positivity of relative entropy to show $S_{gen}(\infty) \geq S_{gen}(v*)$ for any $v^*(y) \geq 0$. Then, using monotonicity of relative entropy, we can establish that $S_{gen}(v**) \geq S_{gen}(v*)$ for $v^{**}(y) \geq v^*(y)$. This is a local GSL.

Before we get into the computation, we employ the JSS construction with a generalization which we describe in the next sub-section.

4.6.1 Adding an observer to the calculation:

We now wish to discuss a slight generalization of the JSS construction where we add an observer to the calculation. We will use the JSS construction to get relative entropy in two different wedges. For wedges with asymptotic regions, in the JSS construction, the ADM Hamiltonian associated with the chosen vector field plays the role of an observer in section 4. However, since this is a boundary term on a codimension 2 surface at infinity, it only depends on the asymptotic form of the vector field which is the same for the two wedges

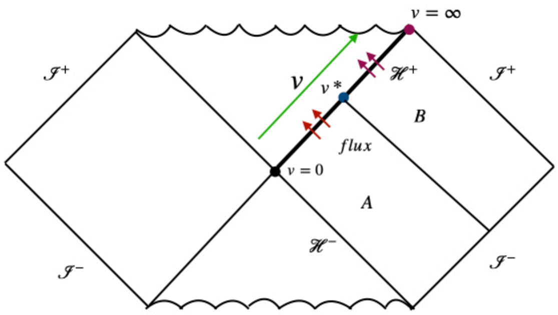


FIGURE 4.7: The figure represents accretion of quantum matter into the black hole.

we are considering (or indeed for any wedge-shaped regions). To make the construction and dressing of operators specific to the wedges we are considering, we add an observer with Hamiltonian $H_{obs} = q \geq 0$ *in addition to* H_{ADM}. The commutant has an observer with Hamiltonian $H'_{obs} = q' \geq 0$. This has also been discussed in [60]. To begin with, we then have the Hilbert space $\mathcal{H} = \mathcal{H}_{QFT} \otimes \mathcal{H}_{obs} \otimes \mathcal{H}'_{obs} \otimes \mathcal{H}_{ADM} \otimes \mathcal{H}_{ADM'}$. Then, the constraint is

$$C[\xi] = H^g_\xi + q - q' - H_{ADM} + H_{ADM'}. \tag{4.6.1}$$

Properly implementing the constraint at the level of the Hilbert space [2] leads to the Hilbert space $\mathcal{H} = \mathcal{H}_{QFT} \otimes \mathcal{H}_{obs} \otimes \mathcal{H}_{ADM} \otimes \mathcal{H}_{ADM'}$. We now dress operators with respect to the **observer** to make the dressing specific to the wedge. This produces the crossed product algebra $\mathcal{A}^C$, which is the crossed product of $\mathcal{A}_{QFT}$ by the flow generated by H^g_ξ [15].

$$\mathcal{A}^C = \{e^{iH^g_\xi \hat{p}} a e^{-iH^g_\xi \hat{p}}, e^{i\hat{q}t}, H_{ADM} | a \in A_{QFT}, t \in \mathcal{R}\}'' \tag{4.6.2}$$

where $\hat{p}$ is the canonical conjugate to the observer Hamiltonian and S'' denotes the smallest von Neumann algebra containing the set S. The inclusion of both the observer and the

[15] Here we have only added an observer in the right exterior algebra and dressed the operator with respect to it.

ADM Hamiltonian still leads to a Type II_∞ von Neumann algebra as discussed in [60]. Considering the states $\hat{\Phi} = \Phi \otimes g \otimes f$ and $\hat{\Omega} = \Omega \otimes g \otimes f$ (the second factor relates to the observer degree of freedom, and the third factor to the H_{ADM} degree of freedom), we can now discuss relative entropies of the two wedges. We note that implementing the constraint on each partial Cauchy slice now yields

$$\Big\langle K_A(v*)\Big\rangle_{\Phi} + \beta\Big\langle q\Big\rangle_g - \beta\Big\langle H_{ADM}\Big\rangle_f = -\Big\langle \frac{A}{4G_N}\Big\rangle_{\hat{\Phi}} \tag{4.6.3}$$

First, we will discuss the static black hole case. We have a static black hole with Killing horizon which is perturbed away from stationarity by quantum fields (including gravitons) in the spacetime. We will assume that the black hole settles down to a stationary state at late times. This is plausible because all the flux of matter would either have crossed the horizon or would have escaped to future null infinity. In the case of AdS, all matter will eventually cross the horizon. So, at late times, the state must be indistinguishable from the vacuum Ω_{HH}. Let us now apply the modified JSS construction to the wedge B for the black hole. This will yield a type II von Neumann algebra of fields in the wedge B. Once we have obtained the type II algebra, the modular Hamiltonian can be factorized and we can write a one-sided modular Hamiltonian. Further, using the definition of Araki's relative entropy, we can write $S(\hat{\Phi}||\hat{\Omega})$ in terms of the one-sided modular operator in the type II algebra, with the states $\hat{\Phi} = \Phi \otimes g \otimes f$ and $\hat{\Omega} = \Omega \otimes g \otimes f$ in the type II algebra corresponding to some quantum field state Φ and cyclic, separating state Ω as defined in the previous section. Using (5.13) and (5.14) in the paper [41] and the fact that we are working with a semiclassical state [16], it can be shown that $S(\hat{\Phi}||\hat{\Omega})$= $S(\Phi||\Omega)$. This explicitly shows the well-known fact that Araki's relative entropy is well-defined and finite even in a type III algebra. It allows us to write the relative entropy in terms of the one-sided modular Hamiltonian in the type III algebra (which is well-defined as a Hermitian form [55]). The partial Cauchy slice on which we want to write the modular Hamiltonian is $\mathcal{H}^+(v*) \cup \mathcal{I}^+$, where $\mathcal{H}^+(v*)$ represents $v \geq v*$ part of the horizon. The modular Hamiltonian on this partial Cauchy slice can be written as $K_A(v*) = K_{\mathcal{H}^+}(v*) + K_{\mathcal{I}^+}$. Here, $K_{\mathcal{I}^+}$ is the modular Hamiltonian at $\mathcal{I}^+$ and

$$K_{\mathcal{H}^+}(v_*) = 2\pi \int_{v_*}^{\infty} dv \int_{\mathcal{H}^+} d^{D-2}x\sqrt{h}(v - v_*)T_{vv} \tag{4.6.4}$$

which can easily be obtained using the equation (4.4.5) and the fact that $\mathcal{E}_{v*}$ is a generator of null translation and is local on the horizon. Note that $K_{\mathcal{I}^+}$ is independent of $v*$. This

[16] One needs to use the fact that in a semiclassical state, $g(q)$ is slowly varying or equally its Fourier transform is peaked around zero momentum.

implies that $K_{\mathcal{I}^+}$ will not contribute to the difference in the modular Hamiltonian between two cuts. The state of the system restricted to the wedge B can be obtained by specifying the density matrix at $\mathcal{H}^+(v*)$ and $\mathcal{I}^+$,i.e $\Psi(\mathcal{H}^+(v*) \cup \mathcal{I}^+) = \rho_{\mathcal{H}^+(v*)} \otimes \sigma_{\mathcal{I}^+}$ [5]. Let Φ be the state of the quantum field, which is indistinguishable from Ω_{HH} at late times. Now as shown by JSS in [41],

$$S_{rel}(\Phi||\Omega_{HH}) = \Big\langle K_A(v*)\Big\rangle_\Phi - S_\Phi^{QFT} - \Big\langle K_A(v*)\Big\rangle_{\Omega_{HH}} + S_{\Omega_{HH}}^{QFT} \tag{4.6.5}$$

where $S_{rel}(\Phi||\Psi)$ is Araki's relative entropy of the state Φ and Ω_{HH}, S_Φ^{QFT} and $S_{\Omega_{HH}}^{QFT}$ are the entropy of the QFT in the state Φ and Ω_{HH} respectively. While each term may not be finite, but since relative entropy in type II and type III algebras are equal, all divergent terms must come in pairs in such a way that the final answer is finite. Further, the one-sided modular Hamiltonian is well-defined as a sesquilinear form on a dense set of states [55].

Imposing constraints as an operator equation and for some state Φ,

$$\Big\langle K_A(v*)\Big\rangle_\Phi + \beta\Big\langle q\Big\rangle_g - \beta\Big\langle H_{ADM}\Big\rangle_f = -\Big\langle \frac{A}{4G_N}\Big\rangle_{\hat{\Phi}} \tag{4.6.6}$$

where β is inverse temperature associated with the KMS state Ω_{HH} and $\hat{\Phi} = \Phi \otimes g \otimes f$ is the state in the crossed product construction of the type II algebra, see [41] . Similarly, $\hat{\Omega} = \Omega \otimes g \otimes f$. The f and g in the type II state are square integrable wavefunctions and $\beta\Big\langle H_{ADM}\Big\rangle_f$ is the expectation value of the ADM Hamiltonian in f while $\beta\Big\langle q\Big\rangle_g$ is the expectation value of the observer Hamiltonian in state g. Notice that f and g are independent of the state Φ and therefore if we consider the difference of the area operator in two different quantum field states, than it will be independent of both f and g. Finally, we see that the relative entropy of the states $\hat{\Phi}$ and $\hat{\Omega}$ is the same as the type III entropy, and the terms dependent on the observer and ADM degrees of freedom *cancel out in a single wedge*[17]. We will be doing this computation for two wedges that we define below, and inclusion of the observer degree of freedom explicitly defines the wedges. Using (4.6.6) and (4.6.5), we get

$$S_{rel}(\Phi||\Omega_{HH}) = \Big\langle \frac{A}{4G_N}\Big\rangle_{\Omega_{HH}} - \Big\langle \frac{A}{4G_N}\Big\rangle_\Phi + S_{\Omega_{HH}}^{QFT} - S_\Phi^{QFT} \tag{4.6.7}$$

[17]We could think of the addition of the observer as a regularization tool since the observer degrees of freedom do not play a role in the relative entropy. The crossed product with respect to the observer as a regularization tool has also been discussed for QFTs in [75]

Since Ω_{HH} is a stationary state, the area at any cut is the same as at $v* \to \infty$. Using this fact, we can write the equation as

$$S_{rel}(\Phi||\Omega_{HH})(v*) = \Big\langle \frac{A}{4G_N}\Big\rangle(\infty) - \Big\langle \frac{A}{4G_N}\Big\rangle_{\Phi}(v*) + S^{QFT}_{\Omega_{HH}} - S^{QFT}_{\Phi} \tag{4.6.8}$$

We put $v*$ in the equation above to emphasize that it is for the wedge at $v*$. Further, the above equation can also be written as:

$$S_{rel}(\Phi||\Omega_{HH})(v*) = S_{gen}(\infty) - S_{gen}(v*). \tag{4.6.9}$$

The positivity of relative entropy implies $S_{gen}(\infty) \geq S_{gen}(v*)$. Since the equation (4.6.9) holds for any $v* \geq 0$, we can write the same equation for some wedge which is at $v{*}{*}(y) \geq v*(y)$, i.e

$$S_{rel}(\Phi||\Omega_{HH})(v{*}{*}) = S_{gen}(\infty) - S_{gen}(v**) \tag{4.6.10}$$

Since the wedge at $v{*}{*}$ is contained in the wedge at $v*$, the monotonicity of Araki's Type III relative entropy for QFTs implies

$$S_{rel}(\Phi||\Omega_{HH})(v*) - S_{rel}(\Phi||\Omega_{HH})(v**) \geq 0. \tag{4.6.11}$$

Using (4.6.10) and (4.6.9), we get

$$S_{gen}(v**) \geq S_{gen}(v*) \tag{4.6.12}$$

for all $v{*}{*} \geq v* \geq 0$. This is the local version of GSL that we wish to obtain. Since the relative entropy has been used, at every step, we are dealing with finite quantities. The computation will continue to hold in AdS [18], with the only difference being that the partial Cauchy slice will be $\mathcal{H}^+(v*)$. The computation is also identical for the Rindler wedge; the only difference is that the cyclic separating state at late times will be the Minkowski vacuum, and the modular Hamiltonian in equation (4.6.5) is with respect to this vacuum. Furthermore, this technique can be simply extended to any spacetime having a Killing horizon like Kerr, modulo our assumption about the existence of a special Gaussian state on the horizon. Notice that this computation is fundamentally dependent on modular inclusion and the fact that the null translation is a symmetry on the horizon — this mainly results in the local modular Hamiltonian for all wedges of type B (wedges whose future null boundary coincides with part of the future horizon). As we have seen, the JSS conjecture is true for any wedge with a boundary which overlaps with a part of the Killing

[18]A subtlety in the AdS case is discussed in the next paragraph.

horizon. The generalized entropy at each cut is equal to the entropy of the type II algebra of the wedge associated with that cut in the sense of JSS.

In an asymptotically AdS spacetime, a question that can be asked is the boundary dual of this construction. Modular Hamiltonians on time bands in the boundary have been discussed in [76] (see also eq.(98) in [77]). However, we have been informed by Prof. E. Witten that the modular Hamiltonian of a proper subregion in the boundary does not have a splitting into left and right parts. Thus, in the asymptotically AdS case, we cannot merely use H_{ADM} to implement a crossed product construction. It is clear that one has to add an observer in addition to H_{ADM} and implement the crossed product with respect to the observer. If we add the observer, then the question is what is the boundary dual of the observer. This is a question which we hope to address in the future. The meaning of the observer has also been extensively discussed in [41].

Finally, we end this section with a note comparing this derivation with the proof of the GSL by Wall [5] who also used the monotonicity of the relative entropy. The type II crossed product construction provides a natural renormalization scheme which was an assumption in Wall's proof. Further, the relative entropy used in this section is Araki's type III relative entropy and all computations are done in modular theory.

4.7 Modular Hamiltonian of deformed half-spaces in general spacetimes

In this section, we depart somewhat from the techniques of the previous sections and consider (one-sided) modular Hamiltonians (for the vacuum state) computed using path integrals rather than Tomita-Takesaki theory. The issue then is that the one-sided modular Hamiltonian may be formally infinite, however we will assume that this can be renormalized, since we will finally be interested in two-sided modular Hamiltonians [19]. The purpose is to compute some examples of general modular Hamiltonians to confirm/check the expectation of JSS [41] that a non-local modular Hamiltonian H of some state $|\Psi>$ may be made local by subtracting off an element of the algebra a and an element of its commutant, b' such that $H' = H - a - b'$ is local. By local, we mean a local integral over some Cauchy slice. Then the converse of the cocycle derivative theorem implies that H' is the modular Hamiltonian of some other state $|\Psi_{ab'}>$.

[19] The one-sided modular Hamiltonian is well defined as a Hermitian form on a dense set of states.

The expectation that the modular Hamiltonian can be made local is crucial to the conjecture of JSS that H^g_ξ is the modular Hamiltonian for some state on the algebra $\mathcal{A}_{QFT}$. In order to check this, we need modular Hamiltonians at our disposal, which are hard to calculate in situations without a lot of symmetry. In cases where, for example, there is Killing symmetry, the modular Hamiltonian for the vacuum can be computed in most situations and is a local expression on a Cauchy slice. However, other than these examples, the modular Hamiltonian will in general, be non-local and will not generate a geometric flow on the spacetime.

The simplest example is to consider the half-space in Minkowski spacetime, whose domain of dependence is the Rindler wedge. We know the modular Hamiltonian — it is the generator of a boost and therefore can be written as a local integral on the $t = 0$ surface. Consider an arbitrary deformation of the $t = 0$ surface by a small amount which, in particular, perturbs the entangling surface itself. As is well-known, the modular Hamiltonian on this surface will be non-local, but we can do a perturbative expansion about the modular Hamiltonian associated with the half-space at $t = 0$ and obtain a relation between them. This was obtained by Faulkner, Leigh, Parrikar and Wang (FLPW) in [77] and later by Balakrishnan and Parrikar (BP)[78] for Minkowski spacetime and in the paper [79] by Rosso for $AdS_2 \times S^{d-2}$. This technique gives, at least perturbatively, the modular Hamiltonian for more general wedges in the spacetime.

We would like to see if the perturbative correction to the half-space modular Hamiltonian can be made local as surmised by JSS by subtracting off a piece a from the modular Hamiltonian (and a piece b' from the commutant when one considers the two-sided modular Hamiltonian). But first, we want to apply the technique of FLPW to a more general class of spacetimes and find the relation between the half-space modular operator and the deformed one. We will work with the Wick-rotated Euclidean metric. The class of metrics we are interested in is the class of Wick-rotated metrics with the form

$$ds^2 = g_{\mu\nu} dX^\mu dX^\nu = \exp[\Omega(\rho)]\Big(d\rho^2 + f(\rho) d\tau^2\Big) + h_{ab}(\rho^2, \vec{x}) dx^a dx^b \tag{4.7.1}$$

where $\Omega(\rho)$ is a smooth function for all ρ and $f(\rho)$ is a positive function, which goes like $\kappa^2\rho^2$ for small ρ where κ is some constant. We will also assume h_{ab} is a Riemannian metric which is smooth at $\rho = 0$. τ is Euclidean time with periodicity $\beta = \frac{2\pi}{\kappa}$. Note, the metric components are independent of τ. The periodicity ensures that there is no conical singularity. Since the metric is independent of τ, the metric after the Wick rotation corresponds to a static Lorentzian metric. Furthermore, the metric is degenerate at $\rho = 0$ indicating the existence of a horizon associated with these coordinates. After the Wick rotation, one can analytically continue the coordinates to obtain its maximal extension. The metric ansatz accommodates many interesting cases. The metric is conformally Rindler if $f(\rho) = \rho^2$ and transverse metric is flat, Rindler if we also have $\Omega(\rho) = 0$. It is $AdS_2 \times M_{transverse}$ when $f(\rho) = \sinh^2 \rho$ and $\Omega(\rho) = 0$. The Schwarzschild metric can also be put in this form.

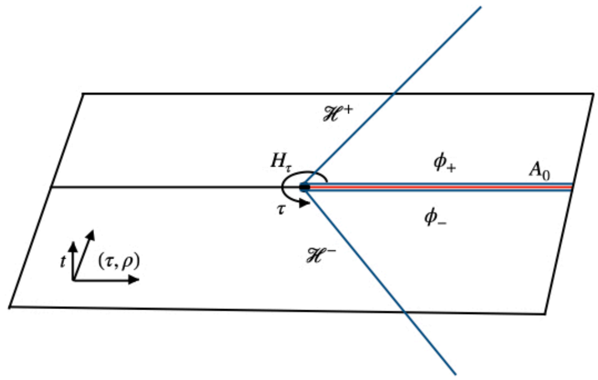


FIGURE 4.8: The red line represents the surface A_0 on which density matrix is computed in (τ, ρ) plane. H_τ is the generator that maps ϕ_+ to ϕ_- on (τ, ρ). t in the figure is Lorentzian time and $\mathcal{H}^\pm$ are the Cauchy (and Killing) horizons of the Cauchy slice.

We are interested in computing the modular Hamiltonian for some Cauchy slice which is not a half space in these class of spacetimes. Consider a QFT in the above spacetime. It is well-known how to compute the density matrix of the vacuum state using the Euclidean path integral on any generic spacetime with the time translation symmetry. The spacetime should have a well-defined Wick rotation, and the metric should be smooth everywhere after the rotation. The density matrix for the vacuum state on an arbitrary surface is obtained in [80] and is non-local, as expected. The nonlocality arises from the fact that the generator that maps the configuration of fields above and below the surface of interest on which the density matrix has to be computed is not a symmetry. In our case, we want to compute the density matrix for a surface which is a small deformation of the $\tau = 0$

surface. But, first let us compute the density matrix for the $\tau = 0$ surface as shown in Figure 4.8. Since τ translation is an isometry of the spacetime, the result will be local. In general, we will get

$$\rho_{A_0,g} = \exp[-\beta H_\tau] \tag{4.7.2}$$

$\rho_{A_0,g}$ represents the density matrix [20]. H_τ is a generator of τ translation. H_τ can be written in terms of the QFT stress tensor and vector field ∂_τ [80]. Now the one-sided modular Hamiltonian of the vacuum can be obtained using $K_{A_0,g} = -\log \rho_{A_0,g} = \beta H_\tau$. This is the modular Hamiltonian associated with the domain of dependence $D(A_0)$. The modular Hamiltonian derived by Sewell for the right exterior of the Schwarzschild black hole in [69] is of this form. In fact, the class of (Lorentzian) metrics we consider in this section are exactly of the form assumed by Sewell. So we can simply use Sewell's computation for getting the modular Hamiltonian. Now, we are interested in the modular Hamiltonian of the deformed region $D(A)$ associated with the Cauchy surface A, which is obtained via a small diffeomorphism of A_0. Let $X^\mu = X'^\mu - \zeta(\rho, \vec{x})$ be the diffeomorphism which maps A_0 to A, where ζ is generator of this infinitesimal diffeomorphism. Note, we also assume it to be independent of τ. We will assume that ζ is non-vanishing and smooth at $\rho = 0$. The unitary operator that implements this diffeomorphism on the Hilbert space is [77, 79],

$$U = \exp\left[\int_{\tau=0} d\Sigma^\mu T_{\mu\nu}\zeta^\nu\right] \tag{4.7.3}$$

where $T_{\mu\nu}$ is the stress tensor. Note that there is no i in the above equation, since we are working with the Euclidean theory. By applying a general identity for computing the derivative of the log of an operator, FLPW [77] have shown that

$$K_{A,g} = K_{A_0,g} + [K_{A_0,g}, \delta U] + \delta K_{A_0,g} + O(\zeta^2) \tag{4.7.4}$$

where $K_{A,g}$ is the modular Hamiltonian of the deformed surface, δU is a linear order term in ζ when U is expanded in ζ and $\delta K_{A_0,g}$ is,

$$\delta K_{A_0,g} = \int_{-\infty+i\alpha}^{\infty+i\alpha} \frac{dz}{4\sinh^2 z/2} \int_{\partial M_E} dS^\mu \rho_{A_0,g}^{-\frac{iz}{2\pi}} T_{\mu\nu}\zeta^\nu \rho_{A_0,g}^{\frac{iz}{2\pi}} \tag{4.7.5}$$

where $\alpha \in (0, 2\pi)$ is a free parameter, $z = s + i\alpha$ where $s \in \mathbb{R}$ and ∂M_E is the boundary of the Euclidean manifold M_E we are working on. Here we have put $\beta = 2\pi$, so that we do not have to track it at each step, but one can introduce it and it will just change

[20] g in $\rho_{A_0,g}$ is just to emphasize that it is defined in the metric g.

the answer by scaling of β[21]. The above integral does not get a contribution from the conformal boundary[22]. The only contributions we get are from the branch cut and C , i.e $\partial M_E = C \cup R_+ \cup R_-$ as shown in the Figure 4.9, for more details, see [77, 79]. Further,

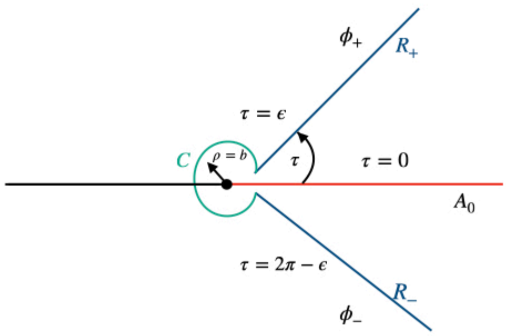


FIGURE 4.9: The red line represents branch cut in the (ρ, τ) plane, the blue line $R_\pm$ will give the contribution from the branch cut and C is the $\rho = b$ surface which will give the contribution from the entangling surface as $b \to 0$.

we can split the contribution as coming from C and $R_+ \cup R_-$,

$$\delta K_{A_0,g} = \delta K_{A_0,g,C} + \delta K_{A_0,g,R_+\cup R_-} \tag{4.7.6}$$

Contribution from C:

We are interested in computing the contribution of C as $b \to 0$. Since ρ is very small on the contour C, we can work we the metric

$$ds^2 = \exp[\Omega(\rho)]\Big(d\rho^2 + \rho^2 d\tau^2\Big) + h_{ab}(\rho^2, \vec{x})dx^a dx^b \tag{4.7.7}$$

[21]To do the computation with β, let $\rho_{A_0,g}^{\frac{iz}{2\pi}} \to \rho_{A_0,g}^{\frac{iz}{\beta}}$ in the equation (4.7.5).

[22]We can take ζ be non-vanishing only for very small ρ and at $\tau = 0$, the contribution coming from the spatial boundary dies off due to appropriate fall-off of the stress tensor.

It will be more convenient to work in the following variable,

$$x_{\pm} = \rho \exp[\mp i\tau] \tag{4.7.8}$$

It is straightforward to show

$$ds^2 = \exp[\Omega(\rho)]\Big(dx_+ dx_-\Big) + h_{ab}(\rho^2, \vec{x}) dx^a dx^b, \tag{4.7.9}$$

the translation in τ is scaling in $x_{\pm}$, i.e $\tau \to \tau + iz$ becomes $x_{\pm} \to e^{\pm z} x_{\pm}$. For computing $\delta K_{A_0,g,C}$, we need the inward unit normal n^{μ} to C, which can easily be obtained,

$$n^{\mu} = -\exp[\Omega(b)/2](e^{-i\tau}\delta^{\mu}_{+} + e^{i\tau}\delta^{\mu}_{-}) \tag{4.7.10}$$

We know that $\rho_{A_0}^{-\frac{iz}{2\pi}}$ generates the diffeomorphism $x_{\pm} \to \bar{x}_{\pm} = e^{\pm z} x_{\pm}$. We can write

$$\rho_{A_0,g}^{-\frac{iz}{2\pi}} T_{\mu\nu}(x_{\pm}, \vec{x}) \rho_{A_0,g}^{\frac{iz}{2\pi}} = \frac{\partial \bar{x}^{\gamma}}{\partial x^{\mu}} \frac{\partial \bar{x}^{\beta}}{\partial x^{\nu}} T_{\gamma\beta}(\bar{x}_{\pm}, \vec{x}) \tag{4.7.11}$$

Since α in the limits of integration in (4.7.5) is a free parameter, we will work with the choice $\alpha = \tau$. Using (4.7.10) and (4.7.11), we can show $B = n^{\mu}\zeta^{\nu}\rho_{A_0,g}^{-\frac{iz}{2\pi}} T_{\mu\nu} \rho_{A_0,g}^{\frac{iz}{2\pi}}\Big|_C$ is

$$B = -e^{-\Omega(b)/2}\Big(\zeta^+ e^{s+i\tau}(T_{++}e^s + T_{+-}e^{-s}) + \zeta^- e^{-(s+i\tau)}(T_{--}e^{-s} + T_{-+}e^s) + \zeta^a(T_{+a}e^s + T_{-a}e^{-s})\Big). \tag{4.7.12}$$

Since

$$\delta K_{A_0,g,C} = \lim_{b\to 0} \int \sqrt{h} dx^{d-2} \int_{-\infty}^{\infty} \frac{ds}{4\sinh^2(\frac{s+i\tau}{2})} \int_0^{2\pi} b e^{\Omega(b)/2} d\tau B, \tag{4.7.13}$$

note that $e^{\Omega(b)/2}$ in the equation cancels the $e^{-\Omega(b)/2}$ in B. Therefore the contribution from C is independent of the conformal factor in the metric (4.7.1). We can split the above equation in the components of ζ, as

$$\delta K_{A_0,g,C} = \delta K_{A_0,g,+} + \delta K_{A_0,g,-} + \delta K_{A_0,g,a} \tag{4.7.14}$$

where

$$\delta K_{A_0,g,\pm} = -\lim_{b\to 0} \int \sqrt{h} dx^{d-2} \zeta^{\pm}(b, \vec{x}) \int_{-\infty}^{\infty} ds I_{\pm}(s) b \Big(T_{\pm\pm}e^{\pm s} + T_{+-}e^{\mp s}\Big) \tag{4.7.15}$$

where

$$I_{\pm}(s) = \int_0^{2\pi} d\tau \frac{e^{\pm(s+i\tau)}}{4\sinh^2(\frac{s+i\tau}{2})} \tag{4.7.16}$$

and

$$\delta K_{A_0,g,a} = -\lim_{b\to 0}\int \sqrt{h}dx^{d-2}\zeta^a(b,\vec{x})\int_{-\infty}^{\infty} dsI_a(s)b\Big(T_{+a}e^s + T_{-a}e^{-s}\Big) \tag{4.7.17}$$

where

$$I_a(s) = \int_0^{2\pi} d\tau \frac{1}{4\sinh^2(\frac{s+i\tau}{2})} \tag{4.7.18}$$

Now to compute I, we will use the well-known identity

$$\bar{I}_J = \oint d\omega \frac{\omega^J}{\omega - e^{-s}} = 2\pi i\Theta_J e^{-Js} \tag{4.7.19}$$

where J is integer and

$$\Theta_J = \begin{cases} \Theta(s) & J \geq 0 \\ -\Theta(-s) & J < 0 \end{cases}$$

where Θ is the step function. It is easy to show that $I_a = i\frac{\partial \bar{I}_0}{\partial s} = 2\pi\delta(s)$ and $I_\pm = i\frac{\partial \bar{I}_{\pm 1}}{\partial s}e^{\pm s}$, where

$$\frac{\partial \bar{I}_{\pm 1}}{\partial s} = 2\pi i(\delta(s)e^{\mp s} - \pm e^{\mp s}\Theta_{\pm 1}) \tag{4.7.20}$$

We eventually want to compute the integrals (4.7.14) and (4.7.17) as $b \to 0$, but they are non-vanishing only if $b \to 0$ is compensated by $s \to \pm\infty$. Since $I_a = 2\pi\delta(s)$, $\delta K_{A_0,g,a}$ vanishes as $b \to 0$. Similarly, $\delta(s)$ terms in $I_\pm$ do not contribute, leaving only Θ terms to contribute. Notice that the term with T_{+-} does not contribute. Therefore, we are left with

$$\delta K_{A_0,g,\pm} = -2\pi\int \sqrt{h}dx^{d-2}\zeta^\pm(\vec{x})\int_0^{\pm\infty} dx_{L\pm}T_{\pm\pm}(x_{L\pm},\vec{x}) \tag{4.7.21}$$

where $x_{L\pm} = be^{\pm s}$ are Lorentzian null coordinates. Therefore

$$\delta K_{A_0,g,C} = -2\pi\int \sqrt{h}dx^{d-2}\Big(\zeta^+(\vec{x})\int_0^{\infty} dx_{L+}T_{++}(x_{L\pm},\vec{x}) + \zeta^-(\vec{x})\int_0^{-\infty} dx_{L-}T_{--}(x_{L\pm},\vec{x})\Big) \tag{4.7.22}$$

Contribution from $R_+ \cup R_-$: Let the unit normal to the $R_\pm$ be denoted by $n_\pm$. This is given by

$$n^\mu_\pm = \mp\frac{\delta^\mu_\tau}{\sqrt{e^{\Omega(\rho)}f(\rho)}} \tag{4.7.23}$$

and the metric induced on this surface is

$$ds^2|_{R_\pm} = \exp[\Omega(\rho)]d\rho^2 + h_{ab}(\rho^2,\vec{x})dx^a dx^b \tag{4.7.24}$$

In computing the contribution from $R_\pm$, we will choose the free parameter $\alpha = \epsilon$ for R_+ and $\alpha = 2\pi - \epsilon$ for R_-. $\rho_{A_0,g}^{-iz}$ generates τ translation. Using (4.7.11), we can write

$$\delta K_{A_0,g,R_\pm} = \int \sqrt{h} dx^{d-2} \int_b^\infty e^{\Omega(\rho)/2} d\rho \zeta^\nu n_\pm^\mu \int_{-\infty}^\infty \frac{ds}{4\sinh^2\frac{(s\pm i\epsilon)}{2}} \rho_{A_0,g}^{-is} T_{\mu\nu}(0,\rho,\vec{x}) \rho_{A_0,g}^{is} \tag{4.7.25}$$

since the contours for $R_\pm$ are oriented oppositely and $n_-^\mu = -n_+^\mu$. One can close the contour and from residue theorem the contribution comes only from the double pole at $s = 0$. Further, one can show[23]

$$\delta K_{A_0,g,R_-\cup R_+} = [\delta U, K_{A_0,g}]. \tag{4.7.26}$$

Using (4.7.26), (4.7.22) and (4.7.6), we can obtain $\delta K_{A_0,g}$. Further, putting in (4.7.4), we get

$$K_{A,g} = K_{A_0,g} - 2\pi \int \sqrt{h} dx^{d-2} \Big(\zeta^+(\vec{x}) \int_0^\infty dx_{L+} T_{++} + \zeta^-(\vec{x}) \int_0^{-\infty} dx_{L-} T_{--} \Big) + O(\zeta^2) \tag{4.7.27}$$

We have obtained the modular Hamiltonian of the deformed surface at leading order in the deformation field using the FLPW technique. $K_{A_0,g}$ is a local expression

$$\int_{A_0} d\Sigma^\mu V^\nu T_{\mu\nu}$$

where V is the Killing vector field. It is conserved (from the fact that the energy momentum tensor is conserved and that V is Killing). Therefore, we can also evaluate it on a Cauchy slice that consists of the Killing horizon $\mathcal{H}^+$ and $\mathcal{I}^+$ (for asymptotically flat spacetimes). Then we can attempt to combine the second integral on the right in (4.7.27) which is also an integral on $\mathcal{H}^+$ with the contribution to $K_{A_0,g}$ from $\mathcal{H}^+$. Indeed, in a situation where $\zeta^-(\vec{x}) = 0$, we can do the computation to higher orders, and all of these will be integrals on $\mathcal{H}^+$. Presumably, they can be resummed [78] to get the one-sided modular Hamiltonian for a null translated wedge along $\mathcal{H}^+$ exactly as in the previous sections. But when $\zeta^-(\vec{x}) \neq 0$, we will not be able to combine the second integral in this manner. This is the signature, at leading order in the deformation field, that the modular Hamiltonian is really non-local. It can be made more explicit at higher orders in the deformation field, where the non-local products of the stress tensor will appear. This can be seen even in the Minkowski spacetime, for example, in the equation (3.21) in [78]. We note also that the integrals are over the horizons of the domain of dependence of the slice A_0 and differ

[23] For the residue at double pole, we have to compute the first derivative of the non-singular part at $s = 0$,and will result in the commutator.

from similar integrals on the horizon of the domain of dependence of A only at quadratic order in the deformation field.

Since the integrands in the terms in $K_{A,g} - K_{A_0,g}$ depend on the quantum fields due

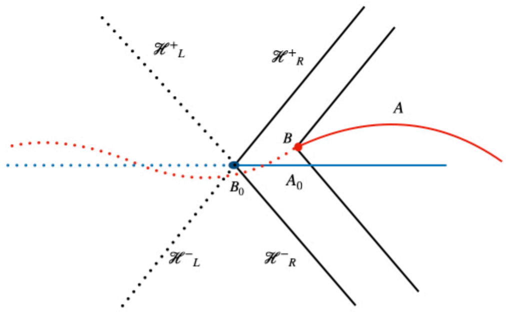


FIGURE 4.10: In this diagram A_0 is the undeformed partial Cauchy slice of the half space and A is the deformed partial Cauchy slice. B_0 represents bifurcation surface associated with wedge with A_0 being Cauchy slice where B represents bifurcation surface associated with the wedge with A being Cauchy slice.

to the energy-momentum tensor, we can think of these terms as affiliated to the right exterior algebra (at linear order in the deformation field) and for the two-sided modular Hamiltonian, we have an identical term in the left half space affiliated to the commutant. Here by affiliated, we mean affiliated to the algebra or its commutant as a Hermitian form on some Hilbert space $\mathcal{H}$ [55] [24]. The above computation is a perturbative computation in the deformation field, and it only makes sense in the regime of perturbation theory. It restricts the state in which one can compute the expectation value of the deformed modular Hamiltonian. For example, if we work with states such that $|\int_{\mathcal{H}_\pm} < T_{\pm\pm} > \xi^\pm| < 1$, this presumably can always be met by appropriately choosing the magnitude of the deformation field. It tells us that maybe we can think of the deformation field as a smoothing function. Notice that the difference of the two-sided modular Hamiltonians

[24] See also Corollary 2.12 in [23] for operators affiliated to a von Neumann algebra.

under the deformation depends on precisely $\int_{\mathcal{H}_\pm} T_{\pm\pm}\xi^{\pm}$ types of quantities coming from the wedge and its commutant. So, it might be possible that there exists some a in the algebra and b' in its commutant such that the difference of the modular Hamiltonians is $a + b'$ in the states where $|\int_{\mathcal{H}_\pm} < T_{\pm\pm} > \xi^{\pm}| < 1$. We also see that these terms have support near the original entangling surface since the deformation actually changes the entangling surface — see Figure (4.10). The distinction between the horizon of the original wedge and the deformed wedge only comes at quadratic order in the deformation. This result is suggestive of the statement of JSS in [41], that a non-local two-sided modular Hamiltonian can be written as the sum of a local modular Hamiltonian of some other state and some operator from the algebra plus one from the commutant which makes the full operator non-local.

The construction of FLPW [77] can be done for higher order terms in the deformation. It is expected that this can be resummed for null deformations given only by $\zeta^+(\vec{x})$ and one will get [25],

$$K_{A,g} = 2\pi \int \sqrt{h} dx^{d-2} \int_{\zeta(\vec{x})}^{\infty} (x_{L+} - \zeta^+(\vec{x})) T_{++} dx_{L+}. \tag{4.7.28}$$

We have already seen this in previous sections for the Schwarzschild black hole, using HSMI. Translation symmetry in τ corresponds to a boost in null coordinates. Null translation at the horizon is a symmetry for the Schwarzschild metric. Therefore $K_{A_0} - K_A = \mathcal{E}_\zeta$ where $\mathcal{E}_\zeta$ is the generator of the null translation on the horizon. Now, we can easily obtain the equation (4.7.28) by extracting out the one-sided modular Hamiltonian for the new wedge. Further, one can write the full modular Hamiltonian and use the fact from modular theory that if $\mathcal{M}_{D(A)} \subset \mathcal{M}_{D(A_0)}$ then $K_{A_0} - K_A > 0$ for any state. For the class of metrics we consider in this section as well, such a result follows on the Cauchy (Killing) horizon by resumming terms when we only have $\zeta^+(\vec{x})$ deformations or by using HSMI. Further, for this class of metrics, Sewell's result [69] for the modular Hamiltonian of the wedge corresponding to the half-space can be used and it is local. This along with the fact that $K_{A_0} - K_A > 0$ is true for any $\zeta^+ > 0$ implies,

$$\int_{-\infty}^{\infty} dx_{L+} T_{++}(\vec{x}, x_{L+}) \geq 0. \tag{4.7.29}$$

This is just the Averaged Null energy condition (ANEC) integrated along a null generator of the Cauchy (Killing) horizon. A similar relation can be obtained on $\mathcal{H}^-$. Therefore, in particular, HSMI along with the monotonicity of K implies ANEC along null generators

[25] It seems that if one works with Gaussian null coordinates, a computation similar to [78] will go through.

of the future horizon of the Schwarzschild spacetime and for the class of metrics we are working with in this section. This has already been noted by FLPW [77].

4.8 Discussion

Crossed product constructions have proved to be very useful in renormalizing quantities such as one-sided modular Hamiltonians and associating an entropy with the algebra of field operators in subregions. However, so far, it has only been possible to obtain a weak form of a GSL in black hole spacetimes in crossed product constructions [3]. This involves considering asymptotically AdS black holes and proving that for a very large time gap between early and late times, the generalized entropy at late times is greater than at early times. In this chapter, we primarily show that a local version of the GSL, namely $\frac{dS_{gen}}{dv} \geq 0$, follows from crossed product constructions. The new ingredient is the application of recent results on the entropy of the algebra of operators on subregions of general spacetimes by JSS [41]. We discuss a slight generalization of the JSS construction in the case of asymptotic wedges, where we explicitly introduce an observer and implement the crossed product with respect to the observer. This also allows us to obtain a GSL for asymptotically AdS and flat black holes, for which, as discussed in section V, we have added an observer to make the construction specific to a wedge. Such extra degrees of freedom do not change our results, since these degrees of freedom cancel out in a single wedge when considering the relative entropy in the wedge.

We first use half-sided modular inclusions to obtain expressions for modular Hamiltonians for algebras of null-shifted wedges along the future horizons in maximally extended static black hole spacetimes. We also outline a similar computation for the horizon of the Kerr black hole. Then we apply the result of JSS to these wedges. The results of JSS rely on the conjecture that the Hamiltonian generating the flow of a specific vector field on the Cauchy slice is a modular Hamiltonian of some state. This conjecture is true in the setting to which we apply these results. *It also allows us to interpret the generalized entropy at each cut on the horizon as the entropy of the algebra of the wedge associated with that cut* in the sense of JSS. Further, we are able to compare relative entropy for two different subregions (specifically, wedges along the horizon) using the JSS results since the modular Hamiltonian used for the crossed product construction in both the wedges is *for the same state*. How to compare, for example, algebra entropy of two different subregions of a spacetime in general in the JSS construction is an interesting open problem, with many potential applications.

4.8.1 Modular Hamiltonians of deformed half-spaces:

In the section (4.7), we also compute modular Hamiltonians in a class of static spacetimes (including the Schwarzschild spacetime), which are modular Hamiltonians for the domains of dependence of deformed Cauchy slices of half-space using path integrals. The purpose is to check whether such a modular Hamiltonian, which is expected to be non-local, can be made local by adding two operators, one from the algebra, and one from the commutant (for the two-sided modular Hamiltonian) as surmised by JSS. We compute these modular Hamiltonians using the path integral method. The results produced for the two-sided Hamiltonian, apart from a local integral on a Cauchy surface, are operators of the form $\int_{\mathcal{H}_\pm} T_{\pm\pm}\xi^\pm$. Since perturbation theory requires $|\int_{\mathcal{H}_\pm} < T_{\pm\pm} > \xi^\pm| < 1$, so it might be possible that there exists some operator a in the algebra and an operator b' in its commutant such that the difference of the deformed and undeformed modular Hamiltonians is $a + b'$. Along the way, we see that the averaged null energy condition (ANEC) also holds for null generators of the Cauchy horizon in the class of static spacetimes we have considered, which includes the Schwarzschild spacetime.

Chapter 5

Black Holes in Higher Curvature theory and Local algebra of Observer

The material presented in this chapter is based on the work of the author in [6]

In this chapter, we generalize the result of Chandrasekaran, Penington, and Witten (CPW) to higher curvature theory. In particular, we have shown that the generalized entropy at the bifurcation surface of any static black hole with a causal horizon in an arbitrary diffeomorphism-invariant theory of gravity is equal to the entropy of the algebra of observables of the exterior. We have also presented a version of the generalized second law for an arbitrary diffeomorphism-invariant theory of gravity that follows.

5.1 Introduction

In an arbitrary diffeomorphism invariant theory of gravity, the entropy of a black hole is not $\frac{A}{4G_N}$, where A is the area of the horizon at some horizon cut. Instead, it will have some higher curvature corrections. Therefore, the generalized entropy is not given by the formula (4.1.1). But, we can analogously define the generalized entropy for a black hole with quantum fields to be the sum of its horizon entropy S in that theory and the entanglement entropy of quantum fields in the black hole exterior,

$$S_{gen} = S + S_{QFT}. \tag{5.1.1}$$

A candidate for the horizon entropy for a stationary black hole is the Wald entropy [81], [82]. The Wald entropy is ambiguous for a non-stationary black hole — these ambiguities were first discussed in [83]. A linearized GSL (ignoring gravitons) was proved for Lovelock gravity in [84].

As we have seen in the last chapter, the generalized entropy at the arbitrary cut on the horizon in Einstein gravity is equal to the entropy of a von Neumann algebra of observables in the black hole exterior up to an additive constant. We have also seen that it helps us to prove the local GSL in crossed product construction, where each step is manifestly finite.

In this chapter, we study the generalization of these results to an arbitrary diffeomorphism-invariant theory of gravity. Our aim is to prove that the relation between generalized entropy and the entropy of the algebra of observables is true even in higher curvature theory, at least at the bifurcation surface. We first write the black hole entropy in such a theory at an arbitrary horizon cut, which is the Wald entropy [81], [82] with an extra term representing an ambiguity in the Wald entropy for a non-stationary black hole [83]. We work in semiclassical gravity. We consider a static (therefore stationary) black hole that is slightly perturbed due to infalling quantum matter and gravitons. In the limit when the cut $v \to \infty$, the perturbed black hole approaches a stationary black hole (v is the affine parameter along the null generator of the horizon). We compute the entropy at $v \to \infty$ minus the entropy at the bifurcation surface up to quadratic order in the perturbation, and we take into account the contribution due to gravitons to the stress-energy tensor. It is possible to fix the ambiguity in the Wald entropy in such a way that this difference of entropies takes a simplified form proportional to the vv component of the stress-energy tensor. Generalizing the computations of CPW [3], we find the difference of *generalized* entropies at $v \to \infty$ and at the bifurcation surface to be proportional to the relative entropy, which is non-negative. By computing the entropy of the extended von

Neumann algebra of the black hole exterior [1], we show that the entropy of the algebra is the generalized entropy at the bifurcation surface just as in [2] for Einstein gravity. All the above constructions can be done for asymptotically flat static black holes [2]. Finally, we discuss the monotonicity result of CPW [3], who show that the monotonicity of relative entropy under trace-preserving inclusions can be used to argue that the generalized entropy at late times is more than that at early times. For this, we need asymptotically AdS black holes with a holographic dual, but modulo this change, the monotonicity result of CPW goes through for the generalized entropy of a black hole in an arbitrary diffeomorphism-invariant theory of gravity.

In section II, we discuss the difference of entropies at $v \to \infty$ and at the bifurcation surface for a slightly perturbed black hole. We use boost arguments, which we summarize in section II.1, to simplify this difference of entropies. By expanding the Raychaudhuri equation order by order in the perturbation parameter, we compute the change in entropies to quadratic order in section II.4, both without graviton contributions to the stress-energy tensor, and with the graviton contribution included. In section III, we discuss the entropy of the algebra of operators in the black hole exterior. We first generalized salient results from earlier papers of Witten [1], Chandrasekaran, Longo, Penington, and Witten [2], and CPW [3], who discussed how the entropy of the algebra was related to the generalized entropy in Einstein gravity, to higher curvature gravity. We then generalize these results to an arbitrary diffeomorphism-invariant theory of gravity. In Section IV, we conclude with a discussion.

5.2 Entropy Change in Higher Curvature Theory

In what follows and the rest of the chapter, we work in units where $G = 1$. Consider an entropy function for a black hole horizon in an arbitrary diffeomorphism invariant theory of gravity with matter,

$$S = \frac{1}{4} \int \rho \sqrt{h} d^{D-2}x \tag{5.2.1}$$

where $\rho = 1 + \rho_w + \Omega$, where h is the induced metric on the $D-2$ dimensional transverse cut on the horizon and $1 + \rho_w$ is the Wald local entropy density [81], [82]. As is well-known, the Wald entropy is unambiguously defined for a stationary black hole, but suffers from ambiguities when evaluated on a non-stationary black hole. These ambiguities were pointed out by Jacobson, Kang and Myers (JKM) [83] and by Iyer and Wald [82]. Ω is a correction to Wald entropy density representing this JKM ambiguity, such that it

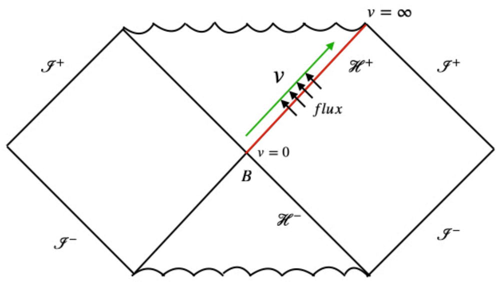


FIGURE 5.1: Accretion of matter across the horizon in an asymptotically flat black hole.

vanishes for a stationary solution. We are interested in a black hole spacetime with a regular bifurcation surface $\mathcal{B}$, which is slightly perturbed from stationarity by throwing some quantum matter. Let v be an affine parameter along the null generator of the future horizon, such that $v = 0$ is the bifurcation surface as shown in Figure 5.1. Then, the entropy at an arbitrary horizon cut (given by $v = constant$) is

$$S[v] = \frac{1}{4} \int_v \rho \sqrt{h} d^{D-2}x \tag{5.2.2}$$

where the subscript v in the integral indicates that the integral is over the transverse space at fixed v on the horizon. We can compute the change in the entropy along the horizon,

$$\frac{dS}{dv} = \frac{1}{4} \int_v \sqrt{h} d^{D-2}x \Big(\frac{d\rho}{dv} + \theta \rho \Big) \tag{5.2.3}$$

where expansion $\theta \equiv \frac{1}{\sqrt{h}} \frac{d\sqrt{h}}{dv}$. To compute change in the entropy from $v = 0$ to $v \to \infty$, we can integrate both the sides with respect to v. This yields

$$\Delta S = \frac{1}{4} \int_0^\infty dv \int_v \sqrt{h} d^{D-2}x \Big(\frac{d\rho}{dv} + \theta \rho \Big). \tag{5.2.4}$$

Here, $\Delta S = S(\infty) - S(0)$. Using integration by parts,

$$\Delta S = \frac{1}{4}\int_v \left\{ v\sqrt{h}\left(\frac{d\rho}{dv}+\theta\rho\right)\right\}\Big|_{v=0}^{v\to\infty} d^{D-2}x \; -\frac{1}{4}\int_0^\infty dv\int \sqrt{h}d^{D-2}xv\Big\{\frac{d^2\rho}{dv^2}+\frac{d\theta}{dv}\rho+2\frac{d\rho}{dv}\theta+\theta^2\rho\Big\}. \tag{5.2.5}$$

We will assume that $\sqrt{h}\Big(\frac{d\rho}{dv}+\theta\rho\Big)$ goes to zero faster than $\frac{1}{v}$, therefore the first term in the above equation is identically zero and we are left with

$$\Delta S = -\frac{1}{4}\int_0^\infty dv\int \sqrt{h}d^{D-2}xv\Big\{\frac{d^2\rho}{dv^2}+\frac{d\theta}{dv}\rho+2\frac{d\rho}{dv}\theta+\theta^2\rho\Big\}. \tag{5.2.6}$$

To compute ΔS order by order, we will now consider the metric perturbation sourced by a stress-energy tensor of order ϵ, i.e, $< T_{vv} > \sim O(\epsilon)$. We will also assume that the perturbation is about a stationary black hole background and at late times, the black hole will again settle down into a stationary state. The perturbation expansion we are interested in is

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \epsilon^{\frac{1}{2}} g^{(\frac{1}{2})}_{\mu\nu} + \epsilon g^{(1)}_{\mu\nu} + O(\epsilon^{\frac{3}{2}}), \tag{5.2.7}$$

where the zeroth order term corresponds to the stationary black hole solution with regular bifurcation surface, the $\sqrt{\epsilon}$ term is due to quantized graviton fluctuations, and the ϵ term is due to the gravitational field of matter or gravitons. We can think of ϵ as $\hbar$. We want to emphasize that Ω vanishes at order $\sqrt{\epsilon}$ at the bifurcation surface [85], a fact which will be useful later in the calculations.

5.2.1 Boost Argument

We now use boost arguments first used in [86] and later in [87], [88]. The metric near any null hypersurface and therefore near the event horizon can be given in Gaussian null coordinates as

$$ds^2 = 2dvdu - u^2X(u,v,x^k)dv^2 + 2u\omega_i(u,v,x^k)dvdx^i + h_{ij}(u,v,x^k)dx^idx^j \tag{5.2.8}$$

where v is an affine parameter along the null generator of the horizon, x^i corresponds to coordinates on the $D-2$ transverse surface (cut) and u is chosen in a way that $\partial_v.\partial_u = 1$ and $\partial_u.\partial_i = 0$. In this coordinate system, $u = 0$ is the future horizon and $u = 0, v = 0$ corresponds to the bifurcation surface $\mathcal{B}$. These coordinates may not cover the entire spacetime, but the near-horizon region of any dynamical black hole spacetime can always be written in this form. Now, the black hole spacetime we consider in this chapter is a static (therefore stationary) black hole spacetime which is a solution in a diffeomorphism invariant theory of gravity. Then, the black hole event horizon is a Killing horizon [81]. First consider the case where this horizon is a Killing horizon with respect to the boost field $\xi = v\partial_v - u\partial_u$ (see [87]). This is true for any stationary black hole spacetime which, near the horizon, looks like a Rindler spacetime, hence the terminology 'boost field'. The near-horizon metric of this stationary black hole will then be boost invariant, i.e the Lie derivative of the metric $\mathcal{L}_\xi g_{\mu\nu} = 0$. Then, the near-horizon metric (5.2.8) is of the form

$$ds^2_{st} = 2dvdu - u^2 X(uv, x^k)dv^2 + h_{ij}(uv, x^k)dx^i dx^j \tag{5.2.9}$$

Here, $\omega_i = 0$ since the spacetime is static. This is the most general form of a static spacetime with a Killing horizon near the horizon. It can easily be seen from (5.2.8) that along the horizon, any non-zero tensor A, which is constructed out of metric components can always be written as $A = \partial_v^n \partial_u^m B$, where m, n are integers and B is constructed out of metric components X, ω_i, h_{ij} and their derivatives with respect to ∇_i . Then, we can associate a boost weight with these tensors as boost weight $= \#v$ index $- \#u$ index. Furthermore, we can write the schematic form for the vv component of any 2-tensor A_{vv} constructed from metric components as

$$A_{vv} = \tilde{X}\partial_v^2 Y + C\partial_v A\partial_v B. \tag{5.2.10}$$

Here, $\tilde{X}, Y, C, A, B$ have boost weight zero and are constructed out of metric components. Now for the stationary black hole spacetime, the above equation reduces down to

$$A_{vv}|_{st} = u^2\tilde{X}\partial_{uv}^2 Y + u^2 C\partial_{uv} A\partial_{uv} B. \tag{5.2.11}$$

This is because the stationary black hole has a Killing symmetry which reduces on the horizon to a scaling symmetry under $u \to pu$ and $v \to v/p$. Thus, the metric components in the stationary case only depend on uv at the horizon. This implies that the vv component of any 2-tensor A_{vv} constructed from metric components in a stationary black hole spacetime vanishes at the future horizon i.e at $u = 0$.

$$A_{vv}|_{st}^{u=0} = 0. \tag{5.2.12}$$

Now, the vv component of the equation of motion for any higher curvature theory takes the following form,

$$R_{vv} + H_{vv} = 8\pi T_{vv} \tag{5.2.13}$$

where H_{vv} corresponds to a higher curvature contribution to the equation of motion. Using (5.2.11) for the stationary black hole $R_{vv} = 0$ and $H_{vv} = 0$, this implies that

$$T_{vv}|^{u=0} = 0 \tag{5.2.14}$$

for any *classical* matter stress-energy tensor. Furthermore, whenever a v derivative acts on the stationary background metric component, it gives a factor of u as well, since the metric component depends on v only through uv. Hence such a term will vanish at the future horizon $u = 0$. Therefore, from (5.2.10), the vv component of any 2- tensor A_{vv} *linear* in the metric perturbation at the future horizon $u = 0$ can always be written in the following form,

$$A_{vv}|^{u=0} = \partial_v^2 \zeta, \tag{5.2.15}$$

where ζ has boost weight zero and is constructed from the background metric and the linear perturbation over stationarity.

5.2.2 Semi-classical gravity equations

Following Chandrasekaran, Penington and Witten [2] and Wall [5], we will work in semi-classical gravity where the expectation value of the matter stress energy tensor is a source term in the gravity equations. Now let us look at the semiclassical equations of motion. The vv component is

$$R_{vv} + H_{vv} = 8\pi < T_{vv} > \tag{5.2.16}$$

Writing this order by order in ϵ, we get:

$$\epsilon^0 : \qquad R_{vv} + H_{vv} = 0 \tag{5.2.17}$$

$$\epsilon^{\frac{1}{2}} : \qquad R_{vv}^{(\frac{1}{2})} + H_{vv}^{(\frac{1}{2})} = 0 \tag{5.2.18}$$

$$\epsilon^1 : \qquad R_{vv}^{(1)} + H_{vv}^{(1)} = 8\pi < T_{vv}^{Q} > . \tag{5.2.19}$$

where $T_{vv}^{Q} = T_{vv}^{M} + t_{vv}$, $< T_{vv}^{M} > \sim O(\epsilon)$ is a matter stress energy tensor and $< t_{vv} > \sim O(\epsilon)$, the pseudo-stress-energy tensor of the graviton. Further, $R_{vv}^{(\frac{1}{2})}$, $H_{vv}^{(\frac{1}{2})}$ are linear in $g_{\mu\nu}^{(\frac{1}{2})}$ perturbation, and $R_{vv}^{(1)}$, $H_{vv}^{(1)}$ are linear in $g_{\mu\nu}^{(1)}$ perturbation.

5.2.3 Raychaudhuri equation order by order

As is well-known, the Raychaudhuri equation (5.2.18) plays a key role in the proof of the second law and we will use it later in our computation. The Raychaudhuri equation is given by,

$$\frac{d\theta}{dv} = -\Big\{\frac{\theta^2}{D-2} + \sigma^{\alpha\beta}\sigma_{\alpha\beta} + R_{vv}\Big\} \tag{5.2.20}$$

Now, if we write it order by order in ϵ, we get,

$$\epsilon^0 : \qquad \frac{d\theta^{(0)}}{dv} = 0 \tag{5.2.21}$$

$$\epsilon^{\frac{1}{2}} : \qquad \frac{d\theta^{(\frac{1}{2})}}{dv} = -R_{vv}^{(\frac{1}{2})} \tag{5.2.22}$$

$$\epsilon^1 : \quad \frac{d\theta^{(1)}}{dv} = -\Big\{\frac{\theta^{(\frac{1}{2})}\theta^{(\frac{1}{2})}}{D-2} + \sigma^{\alpha\beta}_{(\frac{1}{2})}\sigma^{(\frac{1}{2})}_{\alpha\beta} + R_{vv}^{(1)} + R_{vv}^{(\frac{1}{2},\frac{1}{2})}\Big\} \tag{5.2.23}$$

Further using (5.2.19), (5.2.23) can be written as

$$\frac{d\theta^{(1)}}{dv} = -\Big\{\frac{\theta^{(\frac{1}{2})}\theta^{(\frac{1}{2})}}{D-2} + \sigma^{\alpha\beta}_{(\frac{1}{2})}\sigma^{(\frac{1}{2})}_{\alpha\beta} + 8\pi < T^Q_{vv} > -H_{vv}^{(1)} + R_{vv}^{(\frac{1}{2},\frac{1}{2})}\Big\}. \tag{5.2.24}$$

Furthermore, if we compute $R_{vv}^{(\frac{1}{2},\frac{1}{2})}$ in TT (transverse traceless) gauge at the horizon [1], we will get,

$$R_{vv}^{(\frac{1}{2},\frac{1}{2})} \underset{TT}{=} -\frac{1}{4}\frac{dg^{ij}_{(\frac{1}{2})}}{dv}\frac{dg^{(\frac{1}{2})}_{ij}}{dv} + \frac{1}{2}\frac{d}{dv}\Big(g^{ij}_{(\frac{1}{2})}\frac{dg^{(\frac{1}{2})}_{ij}}{dv}\Big). \tag{5.2.25}$$

Now using the fact that $\frac{1}{2}\frac{dg^{(\frac{1}{2})}_{ij}}{dv} = \sigma^{(\frac{1}{2})}_{ij} + \frac{1}{D-2}g^{(0)}_{ij}\theta^{(\frac{1}{2})}$ [5] (i,j are transverse coordinate indices), we can write equation (5.2.25) as

$$R_{vv}^{(\frac{1}{2},\frac{1}{2})} \underset{TT}{=} -\Big(\frac{\theta^{(\frac{1}{2})}\theta^{(\frac{1}{2})}}{D-2} + \sigma^{\alpha\beta}_{(\frac{1}{2})}\sigma^{(\frac{1}{2})}_{\alpha\beta}\Big) + \frac{1}{4}\frac{d^2}{dv^2}\Big(g^{ij}_{(\frac{1}{2})}g^{(\frac{1}{2})}_{ij}\Big) \tag{5.2.26}$$

which yields

$$\frac{d\theta^{(1)}}{dv} = -\Big\{8\pi < T^Q_{vv} > -H_{vv}^{(1)} + \frac{1}{4}\frac{d^2}{dv^2}\Big(g^{ij}_{(\frac{1}{2})}g^{(\frac{1}{2})}_{ij}\Big)\Big\}. \tag{5.2.27}$$

(5.2.21) follows from the fact that the background solution is stationary. The other equations are obtained by expanding the Raychaudhuri equation order by order.

[1]The perturbative expansion of the Ricci tensor to quadratic order can be found in [89].

5.2.4 Entropy change due to accretion of quantum matter across the horizon

In this subsection, we will compute the order-by-order change in the entropy due to the accretion of quantum matter and gravitons across the horizon. We will assume that the background black hole solution is stationary, as well as the final state of the black hole at late times. Also, we assume the perturbation will fall off fast enough, so that all boundary terms at late times vanish [2]. To compute the entropy change order by order, first we will write the perturbative expansion of entropy density as

$$\rho = \rho^{(0)} + \epsilon^{\frac{1}{2}}\rho^{(\frac{1}{2})} + \epsilon\rho^{(1)} + O(\epsilon^{\frac{3}{2}}) \tag{5.2.28}$$

Now, we have all the tools to compute the change in entropy. First we will do the change in entropy computation in the absence of the graviton. Then we will do the computation in which we will include gravitons.

5.2.4.1 Entropy change without graviton contribution

When there is no graviton, all the terms with $g_{\mu\nu}^{(\frac{1}{2})}$ perturbation will go away in all of the above equations. Also, the stress-energy tensor will have only the matter contribution i.e, $< T_{vv}^{Q} >=< T_{vv}^{M} >$, which we will take to be $O(\epsilon)$. The background solution is a stationary black hole, with a Killing horizon and regular bifurcation sphere. For the stationary black hole, the expansion coefficient of the horizon is zero and the entropy density will be independent of the chosen horizon cut. Therefore

$$\epsilon^0 : \qquad \Delta S^{(0)} = 0 \tag{5.2.29}$$

Now, we will compute the change in the entropy due to accretion of matter by the stationary black hole, which takes it away from stationarity. As we have already mentioned, the black hole will settle down into a new stationary state at late times. Now using (5.2.6) and (5.2.27) with the fact that there is no $\rho^{(\frac{1}{2})}$ and $\Theta^{\frac{1}{2}}$ we will get

$$\epsilon : \qquad \Delta\delta S^{(1)} = -\frac{1}{4}\int_0^\infty dv \int d^{D-2}x\sqrt{h}v\Big\{\frac{d^2\rho^{(1)}}{dv^2} + \frac{d\theta^{(1)}}{dv}\rho^{(0)}\Big\}. \tag{5.2.30}$$

[2] This implies that at late times all the perturbations would have either crossed the horizon or gone to asymptotic infinity. For AdS black hole spacetime with reflecting boundary conditions, all perturbations will cross the horizon. Also, we dynamically impose the gauge (5.2.8) at all times.

Here, δ is the perturbation away from stationarity. Since $\rho^{(0)} = 1 + \rho_w^{(0)}$,

$$\frac{d\theta^{(1)}}{dv}\rho^{(0)} = \frac{d\theta^{(1)}}{dv} + \frac{d\theta^{(1)}}{dv}\rho_w^{(0)} = -8\pi < T_{vv}^Q > + H_{vv}^{(1)} - R_{vv}^{(1)}\rho_w^{(0)}. \tag{5.2.31}$$

The equation (5.2.31) is obtained using (5.2.23),(5.2.27) and the fact that there is no $g^{(\frac{1}{2})}$ perturbation, which will make $\frac{d\theta^{(1)}}{dv} = -R_{vv}^{(1)}$. Further, we used the equation of motion to rewrite $R_{vv}^{(1)}$ in terms of the stress energy tensor. Putting the equation (5.2.31) in the equation (5.2.30), we get

$$\epsilon : \quad \Delta\delta S^{(1)} = -\frac{1}{4}\int_0^\infty dv \int d^{D-2}x\sqrt{h}v\Big\{\Big(\frac{d^2\rho^{(1)}}{dv^2} - R_{vv}^{(1)}\rho_w^{(0)} + H_{vv}^{(1)}\Big) - 8\pi < T_{vv}^Q > \Big\}. \tag{5.2.32}$$

We note that $\Big(\frac{d^2\rho^{(1)}}{dv^2} - R_{vv}^{(1)}\rho_w^{(0)} + H_{vv}^{(1)}\Big)$ is constructed out of background metric components and the perturbation and is linear in the perturbation [3]. Therefore, using boost arguments we can write $\Big(\frac{d^2\rho^{(1)}}{dv^2} - R_{vv}^{(1)}\rho_w^{(0)} + H_{vv}^{(1)}\Big) = \partial_v^2\zeta_{(1)}$, which will yield

$$\Delta\delta S^{(1)} = -\frac{1}{4}\int_0^\infty dv \int d^{D-2}x\sqrt{h}v\Big\{\partial_v^2\zeta_{(1)} - 8\pi < T_{vv}^Q > \Big\}. \tag{5.2.33}$$

We can simplify the above term using integration by parts,

$$\begin{aligned}\Delta\delta S^{(1)} = 2\pi\int_0^\infty dv \int d^{D-2}x\sqrt{h}v < T_{vv}^Q > &+ \frac{1}{4}\int d^{D-2}x\sqrt{h}\int_0^\infty dv\partial_v\zeta_{(1)} \\ &- \frac{1}{4}\int d^{D-2}x\sqrt{h}\Big(v\partial_v\zeta_{(1)}\Big)\Big|_{v=0}^{v\to\infty}.\end{aligned} \tag{5.2.34}$$

Now we assume fall-off conditions at late times i.e, all perturbations and their derivatives should fall fast enough such that this boundary term goes to zero at late times. The contribution from the last term in (5.2.34) also vanishes at $v = 0$. Let us recall that we are in the gauge (5.2.8) in which the horizon is always at $u = 0$. Then we will get

$$\Delta\delta S^{(1)} = 2\pi\int_0^\infty dv \int d^{D-2}x\sqrt{h}v < T_{vv}^Q > -\frac{1}{4}\int d^{D-2}x\sqrt{h}\zeta_{(1)}\Big|_{v=0}. \tag{5.2.35}$$

We can get rid of the second term by assuming that we can fix the JKM ambiguity in such a way that

$$\frac{d^2\rho^{(1)}}{dv^2} - R_{vv}^{(1)}\rho_w^{(0)} + H_{vv}^{(1)} = \partial_v^2\zeta_{(1)} = 0 \tag{5.2.36}$$

everywhere on the horizon. This will get rid of the term which is giving rise to the second term in (5.2.35). This is because then $\zeta_{(1)}(v) = av + b$, where a and b are only functions

[3] We can replace the ordinary derivatives with respect to v in the first term with covariant derivatives in the gauge we are in.

of transverse coordinate. $\zeta_{(1)}$ is constructed out of the background metric and the linear perturbation in the gauge (5.2.8). We have to further impose the fall-off conditions on the perturbation i.e., the perturbation and its derivatives with respect to v must go to zero at late times. Thus $\zeta_{(1)}(v) = 0$. There is no contradiction with the fact that Ω vanishes at the bifurcation surface in linear order. It is shown in [90] that the second term in (5.2.35) is zero for $F(R)$ gravity and arbitrary order Lovelock gravity. It is also argued there that this will be true for an arbitrary diffeomorphic theory at linear order. Therefore, assuming this,

$$\Delta\delta S^{(1)} = 2\pi \int_0^\infty dv \int d^{D-2}x\sqrt{h}v < T^Q_{vv} > \tag{5.2.37}$$

The above derivation is of course true even when the accreting matter is classical. For classical matter, imposing the null energy condition i.e $T^Q_{vv} \geq 0$ will give the second law.

5.2.4.2 Entropy change with the graviton contribution included

In this section, we include the graviton contribution, and therefore we will work with the full perturbation expansion. We will again do an order by order expansion. Using (5.2.6) and the fact that background solution is stationary,

$$\epsilon^0 : \qquad \Delta S^{(0)} = 0 \tag{5.2.38}$$

Now, let us compute change in entropy at $\epsilon^{\frac{1}{2}}$ order. Writing (5.2.6) at $\epsilon^{\frac{1}{2}}$ will give

$$\epsilon^{\frac{1}{2}} : \qquad \Delta\delta S^{(\frac{1}{2})} = -\frac{1}{4}\int_0^\infty dv \int d^{D-2}x\sqrt{h}v\Big\{\frac{d^2\rho^{(\frac{1}{2})}}{dv^2} + \frac{d\theta^{(\frac{1}{2})}}{dv}\rho^{(0)}\Big\}. \tag{5.2.39}$$

Here, δ corresponds to entropy change due to a perturbation that takes the solution away from stationarity. Using (5.2.22), we can write (5.2.39) as

$$\Delta\delta S^{(\frac{1}{2})} = -\frac{1}{4}\int_0^\infty dv \int d^{D-2}x\sqrt{h}v\Big\{\frac{d^2\rho^{(\frac{1}{2})}}{dv^2} - R^{(\frac{1}{2})}_{vv}\rho^{(0)}\Big\} \tag{5.2.40}$$

Furthermore using the boost argument in (5.2.15), we can write $R^{(\frac{1}{2})}_{vv} = \partial_v^2\zeta_{(\frac{1}{2})}$, where $\zeta_{(\frac{1}{2})}$ is constructed out of the background metric and linear perturbation in $g^{(\frac{1}{2})}_{\mu\nu}$ where we work in the gauge (5.2.8). Further, $\rho^{(0)}$ is independent of v. Therefore, the equation (5.2.40) becomes

$$\Delta\delta S^{(\frac{1}{2})} = -\frac{1}{4}\int_0^\infty dv \int d^{D-2}x\sqrt{h}v\frac{d^2}{dv^2}\Big\{\rho^{(\frac{1}{2})} - \rho^{(0)}\zeta_{(\frac{1}{2})}\Big\}. \tag{5.2.41}$$

Using integration by parts,

$$\Delta\delta S^{(\frac{1}{2})} = -\frac{1}{4}\int d^{D-2}x\sqrt{h}v\frac{d}{dv}\Big\{\rho^{(\frac{1}{2})}-\rho^{(0)}\zeta_{(\frac{1}{2})}\Big\}\Big|_{v=0}^{v\to\infty}+\frac{1}{4}\int_0^\infty dv\int d^{D-2}x\sqrt{h}\frac{d}{dv}\Big\{\rho^{(\frac{1}{2})}-\rho^{(0)}\zeta_{(\frac{1}{2})}\Big\}. \tag{5.2.42}$$

Using the fall-off condition on the perturbation at late times, the first term vanishes. Therefore, we get

$$\Delta\delta S^{(\frac{1}{2})} = \frac{1}{4}\int_0^\infty dv\int d^{D-2}x\sqrt{h}\frac{d}{dv}\Big\{\rho^{(\frac{1}{2})}-\rho^{(0)}\zeta_{(\frac{1}{2})}\Big\}. \tag{5.2.43}$$

Integrating and using fall-off condition gives

$$\Delta\delta S^{(\frac{1}{2})} = -\frac{1}{4}\int d^{D-2}x\sqrt{h}\Big\{\rho^{(\frac{1}{2})}-\rho^{(0)}\zeta_{(\frac{1}{2})}\Big\}\Big|_{v=0}. \tag{5.2.44}$$

Using the fact that $\theta = \frac{d}{dv}\log(\sqrt{h})$ and (5.2.22), we can write $\delta\log(\sqrt{h}) = -\zeta_{(\frac{1}{2})}$. Therefore, we can write (5.2.44) as

$$\Delta\delta S^{(\frac{1}{2})} = -\frac{1}{4}\int_{\mathcal{B}} d^{D-2}x\sqrt{h}\Big\{\rho^{(\frac{1}{2})}+\delta\log(\sqrt{h})\rho^{(0)}\Big\}. \tag{5.2.45}$$

Using (5.2.1), it is straightforward to verify that

$$\Delta\delta S^{(\frac{1}{2})} = -\delta S^{(\frac{1}{2})}\Big|_{\mathcal{B}} \tag{5.2.46}$$

$\mathcal{B}$ is the bifurcation surface. Now, we use Theorem 6.1 in Iyer and Wald (IW) [82], i.e $(\delta S = \delta\mathcal{E} - \Omega_H\delta\mathcal{J})\Big|_{\mathcal{B}}$, where $\mathcal{E}$ is the canonical energy and $\mathcal{J}$ is the canonical angular momentum of the black hole in the covariant phase space formalism [4]. This was proved by IW at the bifurcation surface for any non-stationary perturbation satisfying the linearized equation of motion[5]. Now in our case, there is no stress-energy tensor at $\epsilon^{(\frac{1}{2})}$ order. This implies

$$\Delta\delta S^{(\frac{1}{2})} = -\delta S^{(\frac{1}{2})}\Big|_{\mathcal{B}} = 0 \tag{5.2.47}$$

Hence, $\delta S^{(\frac{1}{2})}(0) = \delta S^{(\frac{1}{2})}(\infty) = 0$. Therefore, from (5.2.47), if $\delta S^{(\frac{1}{2})}$ is non-zero at any cut, then for some range of v entropy will definitely decrease. This violates the second law. The only way for the second law to be true is to assume that $\delta S^{(\frac{1}{2})}$ will vanish at arbitrary cut. It was shown explicitly by the authors in [85, 90] that

$$\mathcal{R}_{vv} \equiv \frac{d^2\rho}{dv^2} - R_{vv}\rho_w^{(0)} + H_{vv} \tag{5.2.48}$$

[4] Since we have a static black hole, angular velocity at the horizon is zero.

[5] IW's first law at the bifurcation surface $\mathcal{B}$ is unaffected by the JKM ambiguity.

vanishes for $F(R)$ theory and Lovelock theory of arbitrary order, at the linear order in perturbation theory about the stationary black hole (perturbation can be non-stationary). Using (5.2.18), it can can be checked that the term in curly brackets in (5.2.39) is the same as $\mathcal{R}_{vv}^{(\frac{1}{2})}$. The authors in [85, 90] also argued that this relation may be true for an arbitrary theory of gravity with an appropriate definition of local entropy density. Vanishing of $\delta S^{(\frac{1}{2})}$ in general will yield,

$$\frac{d^2\rho^{(\frac{1}{2})}}{dv^2} = -\rho^{(0)}\partial_v\theta_{(\frac{1}{2})} \tag{5.2.49}$$

which after integration and using the boundary condition that the perturbation vanishes at late times will give $\rho^{(\frac{1}{2})} = \rho^{(0)}\zeta_{(\frac{1}{2})}$.

Now, let us compute the ϵ order change in entropy, writing (5.2.6) to the ϵ order. We get

$$\Delta\delta S^{(1)} = -\frac{1}{4}\int_0^\infty dv\int v\Big\{\frac{d^2\rho^{(1)}}{dv^2}+\rho^{(0)}\frac{d\theta^{(1)}}{dv}+\frac{d\theta^{(\frac{1}{2})}}{dv}\rho^{(\frac{1}{2})}+2\theta^{(\frac{1}{2})}\frac{d\rho^{(\frac{1}{2})}}{dv}+\rho^{(0)}\theta^{(\frac{1}{2})}\theta^{(\frac{1}{2})}\Big\}\sqrt{h}d^{D-2}x. \tag{5.2.50}$$

Using (5.2.19), (5.2.23) and (5.2.48), the first two terms in the above expression can be written as

$$\frac{d^2\rho^{(1)}}{dv^2}+\rho^{(0)}\frac{d\theta^{(1)}}{dv} = -8\pi < T^Q_{vv} > +\mathcal{R}^{(1)}_{vv} - \frac{1}{4}\frac{d^2}{dv^2}\Big(g^{ij}_{(\frac{1}{2})}g^{(\frac{1}{2})}_{ij}\rho^{(0)}\Big). \tag{5.2.51}$$

Using (5.2.48), (5.2.15) and the fact that $H^{(1)}_{vv}$ and $R^{(1)}_{vv}$ are constructed out of background metric components and the perturbation and are linear in $g^{(1)}_{\mu\nu}$ perturbation in the gauge (5.2.8), $\mathcal{R}^{(1)}_{vv}$ can be written as $\mathcal{R}^{(1)}_{vv} = \partial_v^2\zeta_{(1)}$. This yields

$$\frac{d^2\rho^{(1)}}{dv^2}+\rho^{(0)}\frac{d\theta^{(1)}}{dv} = -8\pi < T^Q_{vv} > +\partial_v^2\zeta'_{(1)}. \tag{5.2.52}$$

where, $\zeta'_{(1)} = \zeta_{(1)} - \frac{1}{4}\Big(g^{ij}_{(\frac{1}{2})}g^{(\frac{1}{2})}_{ij}\rho^{(0)}\Big)$.Putting the above equation in (5.2.50) we get,

$$-\frac{1}{4}\int_0^\infty dv\int v\sqrt{h}d^{D-2}x\Big(\frac{d^2\rho^{(1)}}{dv^2}+\rho^{(0)}\frac{d\theta^{(1)}}{dv}\Big) = 2\pi\int_0^\infty dv\int v\sqrt{h}d^{D-2}x < T^Q_{vv} > \\ -\frac{1}{4}\int\sqrt{h}\zeta'_{(1)}d^{D-2}x\Big|_{v=0}. \tag{5.2.53}$$

We get the above equation using integration by parts in the $\partial_v^2\zeta'_{(1)}$ integral and the fact that the term at $v\to\infty$ will vanish due to the fall-off condition. Let us consider the rest of the terms in (5.2.50), we will call it $A(\frac{1}{2},\frac{1}{2})$,

$$A(\frac{1}{2},\frac{1}{2}) = -\frac{1}{4}\int_0^\infty dv\int v\Big\{\frac{d\theta^{(\frac{1}{2})}}{dv}\rho^{(\frac{1}{2})}+2\theta^{(\frac{1}{2})}\frac{d\rho^{(\frac{1}{2})}}{dv}+\rho^{(0)}\theta^{(\frac{1}{2})}\theta^{(\frac{1}{2})}\Big\}\sqrt{h}d^{D-2}x. \tag{5.2.54}$$

After integrating (5.2.49) once, we get $\frac{d\rho^{(\frac{1}{2})}}{dv} = -\rho^{(0)}\theta_{(\frac{1}{2})}$. Putting this in (5.2.54) will yield

$$A(\frac{1}{2},\frac{1}{2}) = -\frac{1}{4}\int_0^\infty dv \int v\Big\{\frac{d\theta^{(\frac{1}{2})}}{dv}\rho^{(\frac{1}{2})} + \theta^{(\frac{1}{2})}\frac{d\rho^{(\frac{1}{2})}}{dv}\Big\}\sqrt{h}d^{D-2}x \tag{5.2.55}$$

which can be further simplified using integration by parts and using fall-off conditions as $v \to \infty$,

$$A(\frac{1}{2},\frac{1}{2}) = \frac{1}{4}\int_0^\infty dv \int \theta^{(\frac{1}{2})}\rho^{(\frac{1}{2})}\sqrt{h}d^{D-2}x. \tag{5.2.56}$$

Using $\frac{d\rho^{(\frac{1}{2})}}{dv} = -\rho^{(0)}\theta_{(\frac{1}{2})}$ in (5.2.56), we get

$$A(\frac{1}{2},\frac{1}{2}) = -\frac{1}{8}\int_0^\infty dv \int \frac{d}{dv}\Big(\frac{(\rho^{(\frac{1}{2})})^2}{\rho^{(0)}}\Big)\sqrt{h}d^{D-2}x. \tag{5.2.57}$$

After integrating and using fall-off at late times, we will get

$$A(\frac{1}{2},\frac{1}{2}) = \frac{1}{8}\int \Big(\frac{(\rho^{(\frac{1}{2})})^2}{\rho^{(0)}}\Big)\sqrt{h}\Big|_{v=0}d^{D-2}x \tag{5.2.58}$$

This quantity is thus manifestly positive. From (5.2.53) and (5.2.58), we get

$$\Delta\delta S^{(1)} = 2\pi\int_0^\infty dv \int v\sqrt{h}d^{D-2}x < T^Q_{vv} > -\frac{1}{4}\int \sqrt{h}\zeta'_{(1)}d^{D-2}x\Big|_{v=0} + A(\frac{1}{2},\frac{1}{2}). \tag{5.2.59}$$

We know that the Wald entropy has JKM ambiguities when the metric is not stationary. That was a motivation for putting Ω as the correction to the Wald entropy in the definition of local entropy density. We now fix Ω such that the last two terms in (5.2.59) vanish. These terms are anyway zero for a stationary black hole, as can be seen from the expression for $A(\frac{1}{2},\frac{1}{2})$ in (5.2.58) and $\mathcal{R}^{(1)}_{vv}$ is zero for a stationary black hole from results in the subsection on the boost argument [6]. So Ω will be non-zero only when the metric is not stationary. For the cases when it is possible to set these two terms to zero by a choice of Ω, we will get

$$\Delta\delta S^{(1)} = 2\pi\int_0^\infty dv \int v\sqrt{h}d^{D-2}x < T^Q_{vv} > \tag{5.2.60}$$

Some of the ambiguities in Wald's entropy were fixed for some class of theories at linear order in perturbation theory[86]. Moreover, this entropy was shown to be equal to holographic entanglement entropy computed by Dong [91].It is also pointed out in [86] that considering the second law at linear order does not fix the ambiguities at higher order. Therefore these results are not in contradiction with our computation. One can also get rid

[6] For a stationary black hole, v derivatives of ρ are zero and R_{vv} and H_{vv} are zero for a stationary metric as discussed using boost arguments.

of the second and third term in (5.2.59), by restricting the perturbation to a special class of perturbations which vanishes at $v = 0$. One physical case in which such perturbation can be realised is when matter falls after $v = 0$.

5.3 The Entropy of Algebra And Generalized Entropy in Higher Curvature Theory

In this section, we utilize the algebraic approach to quantum field theory, specifically the constructions of Chandrasekaran, Penington, Longo, and Witten (CPLW) in [2], and Chandrasekaran, Penington, and Witten (CPW) in [3], to study black holes in higher curvature theories. We have employed their construction to prove a version of the GSL (Generalized Second Law) for an arbitrary diffeomorphism invariant theory of gravity, with certain appropriate assumptions. The setup that we are interested in involves both the asymptotically flat and asymptotically AdS stationary black hole solutions in an arbitrary diffeomorphism-invariant theory. Throughout this section, we follow the notation of CPW.

5.3.1 Generalization To Higher Curvature Gravity

We now generalize the construction of the subsection (4.2) to an arbitrary diffeomorphism-invariant theory of gravity. We note that some of the constructions in the previous subsection such as the semi-classical state were done by CPW for the AdS-Schwarzschild black hole in the *boundary* CFT. But we can analogously define such a semi-classical state in the bulk using the same construction. In fact, only in the sections on monotonicity of generalized entropy in CPW, are results in the boundary theory crucially used. Therefore, except while discussing monotonicity of the generalized entropy, we can confine our analysis to the bulk, and we can even work with an asymptotically flat black hole, as discussed by CPLW. So, in our case, M is the $(3 + 1)$ dimensional, asymptotically flat, maximally extended static (therefore stationary) black hole solution in an arbitrary diffeomorphism invariant theory of gravity. Therefore, its horizon is a Killing horizon. The equation (4.2.1) will define a conserved quantity even in the arbitrary theory of gravity, since it is the consequence of invariance of the action under background diffeomorphisms. Let us define the 1-form $J = T_{\mu\nu}V^{\nu}dx^{\mu}$ where V is the timelike Killing vector of M. Then, divergence of J_{μ} is zero, i.e., J^{μ} is covariantly conserved. This implies $d * J = 0$, where $*$ is the Hodge dual. Since the spacetime does not have non-trivial topology, this implies $*J = dQ$ for some 2 form Q. Also, notice that the integral of $*J$ over the 3 dimensional

Cauchy surface is $\hat{h}$. Since $*J = dQ$, this reduces to an integral over the codimension 2 surface which is the boundary of the Cauchy slice. Therefore we can write $\hat{h} = H_R - H_L$ where H_R and H_L are codimension 2 integrals at right and left spatial infinity respectively. We note that the canonical energy in the covariant phase space formalism is given by (4.2.1) apart from a surface term ambiguity (see appendix of IW [82]).

$$\mathcal{E} = \int_\Sigma J + \text{Surface term} = \int_\Sigma d\Sigma^\mu T_{\mu\nu}\xi^\nu + \text{Surface term} \tag{5.3.1}$$

where $\mathcal{E}$ is canonical energy in covariant phase space formalism. So we can think of H_R and H_L as being the right and left canonical energy, respectively, apart from ambiguities in the canonical energy [7]. We will call them right and left Hamiltonian. It can be shown that these statement will go through even in semi classical regime as discussed in appendix (7.2). Now, $\hat{h}$ is the modular Hamiltonian corresponding to the Hartle-Hawking state as before. This follows from the analysis of Sewell [69] for any metric of the following form:

$$ds^2 = A(t^2 - w^2, y)(-dt^2 + dw^2) + B(t^2 - w^2, y)d\sigma^2(y). \tag{5.3.2}$$

The Schwarzschild spacetime in Kruskal coordinates is of this form. We will assume that our static black hole solution also has this form (i.e., we assume the existence of Kruskal-like coordinates).

Now we can proceed by defining h_L and h_R as defined in (4.2.2). Following the argument in the previous section that including gravity changes the algebra to Type II, we can split $\hat{h}$ as done in (4.2.3) and (4.2.4). Further, we can straightforwardly obtain the equation in (4.2.5) and (4.2.6). The only difference is now h_R in (4.2.6) is the renormalized Hamiltonian in the higher curvature theory which generates the time translation on the boundary of the right exterior region. Afterwards, the construction of the crossed -product algebra (extended algebra) and other constructions like defining the trace and entropy will analogously go through as done in the previous section. We will work with a semi-classical state as defined in (4.2.9). Therefore we can define the entropy of the algebra in the right exterior region by the same formula (4.2.11) i.e.

$$S(\hat{\Phi})_{\mathcal{A}_r} = \left\langle \hat{\Phi} \middle| \beta h_R \middle| \hat{\Phi} \right\rangle - \left\langle \hat{\Phi} \middle| h_{\Psi|\Phi} \middle| \hat{\Phi} \right\rangle - \left\langle \hat{\Phi} \middle| \log\left(\varepsilon |g(\varepsilon h_R)|^2\right) \middle| \hat{\Phi} \right\rangle + O(\varepsilon) \tag{5.3.3}$$

[7] As discussed in the Appendix of IW [82], the ambiguity in the canonical energy is the sum of two surface terms, one of which vanishes for common matter theories in a background spacetime. There is an ambiguity due to the second surface term which is a function of the background metric, Killing vector, matter fields and their derivatives. In what follows, we ignore this ambiguity.

where $S_{rel}(\Phi||\Psi) = -\left\langle \hat{\Phi} \middle| h_{\Psi|\Phi} \middle| \hat{\Phi} \right\rangle$ is relative entropy as defined earlier. So the form of the equation (4.2.11) remains intact —the only change is that h_R is the renormalized canonical energy in higher curvature theory and $h_{\Phi|\Psi}$ is the relative modular Hamiltonian in that particular theory.

In our case of interest, the black hole settles down to a stationary state at very late times. This is plausible since at late times all the flux of matter would either have crossed the horizon or would have escaped through future null infinity. So, at late times we will not be able to distinguish between $|\Psi\rangle$ and $|\Phi\rangle$. We get

$$S_{bulk}(\infty)_\Phi = S_{bulk}(\infty)_\Psi = S_{bulk}(b)_\Psi \tag{5.3.4}$$

where S_{bulk} denotes the entanglement entropy of quantum fields in the exterior region of the black hole [8]. Now let us analyze h_r, using (4.2.3) and the fact that the deformation of Cauchy surfaces S does not affect the conserved quantity $\hat{h}$. We deform S_1 such that $S_1' = \mathcal{H}^+ \cup \mathcal{I}^+$, where $\mathcal{H}^+$ is the future horizon and $\mathcal{I}^+$ is future null infinity [9]. Therefore

$$\beta h_r^{\mathcal{H}^+} = \beta(h_r - h_r^{\mathcal{I}^+}) = \int_0^\infty dv \int_{\mathcal{H}^+} d^{D-2}x\sqrt{h} v T_{vv} \tag{5.3.5}$$

where $\beta h_r^{\mathcal{I}^+}$ is the time translation generator at future null infinity and $\beta h_r^{\mathcal{H}^+}$ is the boost generator on the horizon. The second equality in the above equation can be obtained using (4.2.3) and the fact that $h_r^{\mathcal{I}^+}$ is just the integral of the stress tensor supported at future null infinity. Let us define a one-sided modular operator (boost operator) at arbitrary cut $v = v_*$ (which is the $D-2$ dimensional transverse surface) at the horizon. It is well known that the density matrix $(\rho_r)_{HH}$ of the Hartle Hawking state in the region r is thermal with respect to [68, 5]

$$K_r(v_*) = \int_{v_*}^\infty dv \int_{\mathcal{H}^+} d^{D-2}x\sqrt{h}(v - v_*)T_{vv} + K_r^{\mathcal{I}^+} \tag{5.3.6}$$

where, $K_r^{\mathcal{I}^+} = \beta h_r^{\mathcal{I}^+}$ is the modular energy at $\mathcal{I}^+$, which accounts for energy which goes to $\mathcal{I}^+$ without crossing the horizon. Also notice that $K_r^{\mathcal{I}^+}$ is independent of v_*. When $v_* = 0$, then the first term in the above equation will become $\beta h_r^{\mathcal{H}^+}$ as defined in (5.3.5), and therefore

$$K_r(b) = \beta h_r = \beta h_r^{\mathcal{H}^+} + \beta h_r^{\mathcal{I}^+} \tag{5.3.7}$$

[8] Earlier, we denoted this quantity by S_{QFT}.

[9] In the case of an asymptotically AdS black hole, the deformed Cauchy surface is just $\mathcal{H}^+$

where b is the bifurcation surface $v_* = 0$. We can also define

$$K_r(\infty) = \lim_{v_* \to \infty} K_r(v_*) \tag{5.3.8}$$

Using the result from the previous section and equation (5.3.8), we get

$$\langle \Phi | \, \beta h_r^{\mathcal{H}^+} \, | \Phi \rangle = \Delta \delta S \tag{5.3.9}$$

If a density matrix were to exist for the algebra $\mathcal{A}_{r,0}$, then using the definition of modular Hamiltonian for state $|\Psi\rangle$ and the fact that $\Delta_\Psi = \rho_\Psi \rho_\Psi'^{-1}$, we will be able to write

$$\log \rho_\Psi = -K_r(b) + C \tag{5.3.10}$$

where C is some constant. The density matrix of the Hartle Hawking state in region r can be written as $\Psi(\mathcal{H}^+ \cup \mathcal{I}^+) = (\rho_r)_{HH} \otimes \sigma$, which corresponds to the ground state [10] ρ_{HH} at $\mathcal{H}^+$, product taken with some arbitrary density matrix defining a faithful state at $\mathcal{I}^+$ [5]. Therefore,

$$\langle \Psi | \log \rho_\Psi | \Psi \rangle = - \langle \Psi | \, K_r^{\mathcal{I}^+} \, | \Psi \rangle + C \tag{5.3.11}$$

Here we use the fact that $\langle \Psi | \, h_r^{\mathcal{H}^+} \, | \Psi \rangle = 0$ since it is the Hartle Hawking state. Further, since $S_{bulk}(b)_\Psi = - \langle \Psi | \log \rho_\Psi | \Psi \rangle$, we get $S_{bulk}(b)_\Psi = \langle \Psi | \, K_r^{\mathcal{I}^+} \, | \Psi \rangle - C$.

As mentioned before, it is not strictly true of the algebra $\mathcal{A}_{r,0}$ that the modular operator factorizes, but by extending the algebra to $\mathcal{A}_r$, it is true that the modular operator factorizes as $\hat{\Delta}_\Psi = \rho_{\hat{\Psi}} \rho_{\hat{\Psi}}'^{-1}$ (in the notation of[1]). We will ignore this detail just for illustrative purposes following [2] .

$$S_{gen}(\infty) - S_{gen}(b) = S(\infty) - S(b) + S_{bulk}(b)_\Psi - S_{bulk}(b)_\Phi. \tag{5.3.12}$$

It can be written in terms of the one-sided modular operator,

$$S_{gen}(\infty) - S_{gen}(b) = - \langle \Phi | \left(K_r(\infty) - K_r(b) \right) | \Phi \rangle + S_{bulk}(b)_\Psi - S_{bulk}(b)_\Phi. \tag{5.3.13}$$

Putting $S_{bulk}(b)_\Psi$ and $\langle \Phi | \, K_r(b) \, | \Phi \rangle$ using the equation (5.3.10) , we get

$$S_{gen}(\infty) - S_{gen}(b) = - \langle \Phi | \log \rho_\Psi | \Phi \rangle + C - \langle \Phi | \, K_r(\infty) \, | \Phi \rangle + \langle \Psi | \, K_r^{\mathcal{I}^+} \, | \Psi \rangle - C - S_{bulk}(b)_\Phi \tag{5.3.14}$$

[10] Hartle Hawking state is a ground state with respect to the time v.

Now, we use the fact that at late times, every state is indistinguishable from $|\Psi\rangle$ and $K_r(\infty) = K_r^{\mathcal{I}^+}$. Further, $K_r^{\mathcal{I}^+}$ is independent of the cut. The expectation value $K_r^{\mathcal{I}^+}$ in state $|\Phi\rangle$ will be equal to its expectation value in state $|\Psi\rangle$. Therefore, we get

$$S_{gen}(\infty) - S_{gen}(b) = -\langle\Phi|\log\rho_\Psi|\Phi\rangle - S_{bulk}(b)_\Phi \tag{5.3.15}$$

therefore we got,

$$S_{gen}(\infty) - S_{gen}(b) = S_{rel}(\Phi||\Psi) \tag{5.3.16}$$

As we see, the difference of generalized entropies in (5.3.16) is manifestly finite and non-negative. For Einstein gravity, the above expression has been already obtained by Wall in [5]. The result (5.3.16) is in an arbitrary theory of gravity — the difference between generalized entropy at late times and generalized entropy at the bifurcation surface is relative entropy of the state of the black hole with respect to the Hartle Hawking state. We now need to show, as in [2], that the generalized entropy at the bifurcation surface is the entropy of the algebra. We thus need to show

$$S_{gen}(\infty) = \left\langle\hat{\Phi}\right|\beta h_R\left|\hat{\Phi}\right\rangle - \left\langle\hat{\Phi}\right|\log\left(\varepsilon|g(\varepsilon h_R)|^2\right)\left|\hat{\Phi}\right\rangle + \text{Const} \tag{5.3.17}$$

Since both terms in the above equation are only functions of h_R, and since we have interpreted h_R as the renormalized canonical energy, the above terms are some distributions of energy in the semi-classical state $\left|\hat{\Phi}\right\rangle$. Also, these terms are independent of the state $|\Phi\rangle$. To see that, choose $a(s)$ such that

$$a(s) = \int e^{-ih'_R s} f(h'_R)dh'_R \tag{5.3.18}$$

where $f(h_R)$ is some chosen function. Putting the equation (5.3.18) in the equation (4.2.7) and using the fact that $h_R = \hat{h_\Psi} + x$, will yield $\hat{a} = f(h_R)$. Now let us compute the expectation value $\left\langle\hat{\Phi}\right|\beta\hat{a}\left|\hat{\Phi}\right\rangle$ for (5.3.18) with the semi-classical state $\left|\hat{\Phi}\right\rangle$ defined in (4.2.9). Using the results (3.25) and (3.26) in CPW [2],

$$\left\langle\hat{\Phi}\right|\hat{a}\left|\hat{\Phi}\right\rangle = \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}ds|\varepsilon g(\varepsilon x)|^2 e^{isx}\langle\Psi|\Delta_{\Phi|\Psi}a(s)|\Psi\rangle \tag{5.3.19}$$

Now put (5.3.18) in (5.3.19). Using the fact that $h_R = \hat{h_\Psi} + x$ and $\hat{h_\Psi}|\Psi\rangle = 0$, we get

$$\left\langle\hat{\Phi}\right|f(h_R)\left|\hat{\Phi}\right\rangle = \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}ds\int_{-\infty}^{\infty}dy|\varepsilon g(\epsilon x)|^2 e^{is(x-y)}f(y)\langle\Psi|\Delta_{\Phi|\Psi}|\Psi\rangle \tag{5.3.20}$$

By definition, $\langle\Psi|\Delta_{\Phi|\Psi}|\Psi\rangle = 1$. Therefore the above equation is independent of $|\Phi\rangle$, it

will only depend on $f(h_R)$ and $g(\varepsilon x)$. Therefore, both the terms on the right-hand side of (5.3.17) will give the same result either when we compute them in the state $|\Phi\rangle$ or in the Hartle Hawking state $|\Psi\rangle$ at late times. Since both terms can be determined from the late-time behavior of the black holes, the relation (5.3.17) is plausible. This is because at late times, all the fields have either fallen across the horizon or to infinity.
Using equation (4.2.2), we can write $\left\langle\hat{\Psi}\middle|\beta h_R\middle|\hat{\Psi}\right\rangle = \beta(\Delta\mathcal{E})$, where $\Delta\mathcal{E}$ is the energy difference between the black hole we are studying and the reference black hole. Since both the black holes are taken in an equilibrium state, we can apply the first law of black hole mechanics for two equilibrium configurations in phase space which yields

$$\left\langle\hat{\Psi}\middle|\beta h_R\middle|\hat{\Psi}\right\rangle = \Delta S \tag{5.3.21}$$

where ΔS is the difference of entropy of the equilibrium black hole state we get at late times to the reference black hole. Therefore, the first term in (5.3.17) describes the change in $S_{gen}(\infty)$ due to a change in black hole entropy. At very late times, all the matter would have either crossed the horizon or would have escaped to null infinity. The second term should be thought of as the contribution of entropy of fluctuations in black hole entropy [2]. Finally, we add and subtract the entanglement entropy of the quantum fields in the Hartle-Hawking state and lump one of the pieces in the constant in (5.3.17) using (5.3.11). This is because at late times, all the fields have either fallen across the horizon or to infinity.

Combining everything, we get

$$S(\hat{\Phi})_{\mathcal{A}_r} = S_{gen}(b) + \text{Const.} \tag{5.3.22}$$

The equation (5.3.22) tells us that $S_{gen}(b)$ for black holes at the bifurcation surface in the arbitrary theory of gravity can be thought of as the entropy of the algebra $\mathcal{A}_r$ modulo a constant. Notice, we have shown that the generalized entropy at the bifurcation surface is equal to the entropy of the algebra up to a constant, but we are not making any statement about entropy at an arbitrary cut of the horizon. In algebraic QFT, relative entropy is positive. This implies $S_{gen}(\infty) - S_{gen}(b) \geq 0$. Can we go beyond this result and prove that the generalized entropy is monotonic? The entropy of the algebra is monotonic under trace-preserving inclusions [92]. To obtain a GSL (monotonicity of the generalized entropy), CPW consider an AdS Schwarzschild black hole with a holographic dual CFT. Then, they have the following clever argument: In the dual CFT, they consider operator algebras at two different times (early and late times), $\mathcal{A}_{R,0}$ and $\mathcal{B}_{R,0}$ respectively, separated by a timescale much larger than the thermal time scale β. The correlation functions of operators at these different times factorize into a product of correlators of early and late times. Thus, the algebra generated by both early and late time operators is $\mathcal{C}_{R,0} = \mathcal{A}_{R,0} \otimes \mathcal{B}_{R,0}$. The Hilbert space factorizes similarly. These algebras are extended similar to what was done before, to obtain a Type II algebra which has an associated entropy. Now, consider three different situations: first, the quantum fields at both early and late times are in an arbitrary state, a second situation where the fields at early times have fallen into the horizon, so that the state of these fields is the vacuum times any state of the late time fields, and finally, a situation where both sets of fields have fallen into the horizon and the state of the fields in the exterior is the vacuum. CPW argue that the generalized entropies at these three horizon cuts is the generalized entropy for the extended algebra $\mathcal{C}_R$ for these three different states at a hypothetical *bifurcation surface* in the limit of large time gap. From the property of the monotonicity of the entropy of the algebra under trace preserving inclusions, it follows that the generalized entropy is increasing in going from the first to the third situation above. This argument is then a version of the GSL. We can use these results to prove this version of the GSL in an arbitrary diffeomorphism invariant theory of gravity if we start with an asymptotically AdS black hole which has a holographic dual. We can repeat all the steps in this section for such a black hole. The only thing we need is for the first law as in the paper of Iyer and Wald [82] to be true in this situation. Although the statement of the First law is only for asymptotically flat black holes, the same will be true for an asymptotically AdS black hole provided the integrals involved in the presymplectic form and the canonical energy are finite after assuming appropriate fall-offs for the fields. In this case, the computation of CPW generalizes to these black holes in a higher curvature theory of gravity, and a version of the increase of generalized entropy (comparing the entropy at early and late times) is true. The ambiguity in the Wald entropy, which we fixed in the bifurcation surface in a previous section, was in terms of a quadratic function of the half-order perturbation. This is not affected by the different states of the quantum fields in the argument of CPW, so their argument goes over to our case. Can we show a stronger monotonicity result $\frac{dS_{gen}}{dv} \geq 0$? This is what Wall [5] has done for Einstein gravity, using an expression for the entropy at any cut of the horizon. Due to JKM ambiguities in the Wald entropy, this expression will probably need one to specify the particular theory of gravity.

5.4 Discussion

In the context of Einstein gravity, it was shown by CPW [3] that for the system of quantum fields in a perturbed Schwarzschild black hole spacetime in the $G \to 0$ limit with infalling quantum matter across the horizon, the generalized entropy at the bifurcation surface was equal to the entropy of the von Neumann algebra of operators in the black hole exterior. This was achieved by enlarging the operator algebra by including the ADM Hamiltonian and by enlarging the Hilbert space. This had the effect of changing the algebra of operators to a Type II_∞ von Neumann algebra, to which we can associate a notion of trace and entropy. Furthermore, CPW showed that the difference of the generalized entropy of an arbitrary cut of the horizon, in the limit when the cut $v \to \infty$ and the generalized entropy at the bifurcation surface was equal to the relative entropy, and therefore nonnegative. For this, CPW worked in semiclassical gravity and up to quadratic order in perturbations. They then obtained a monotonicity result (GSL) for the generalized entropy from the monotonicity of relative entropy under trace-preserving inclusions. In this chapter, we consider quantum fields in a slightly perturbed static black hole with a causal horizon in an arbitrary diffeomorphism-invariant theory of gravity in the $G \to 0$ limit. In this setup, generalized entropy is the sum of Wald entropy (including JKM correction) and entanglement entropy of quantum fields in the black hole exterior. We consider the difference in Wald entropy at infinity and at the bifurcation surface up to quadratic order in the perturbations and obtain (5.2.59). Wald entropy has ambiguities on non-stationary geometries. We fix the ambiguity in order to get (5.2.60), which matches the result for Einstein gravity in the paper of CPW, and we obtain a simplified result for the difference of entropies, which enables us to employ the CPW construction. We then consider the difference in *generalized* entropies and show that this difference equals the relative entropy of the state of the quantum fields and the Hartle-Hawking state — it is thus non-negative. We next consider the von Neumann algebra of the quantum fields in the black hole exterior, extended to include the Hamiltonian and an enlarged Hilbert space as in CPW. Evaluated on the semiclassical states defined by CPW, we show that the entropy of the von Neumann algebra equals the generalized entropy of the bifurcation surface. Finally, we see that the derivation of the increase of generalized entropy by CPW in Einstein gravity goes through for black holes in an arbitrary gravity theory, provided the black hole is asymptotically AdS, which has a holographic dual.

It would be interesting to extend the above results to arbitrary cuts on the horizon and to establish the validity of the generalized second law (GSL). However, this extension is subtle. The subtlety arises from the fact that the causal structure in higher curvature theories differs significantly from that of general relativity. In particular, defining the local algebra of observables for a given region requires us to understand the causal structure (see the discussion in the section (3.5.0.1)). This issue will be the focus of the next chapter.

Chapter 6

Causal Structure of Higher Curvature theory

The material presented in this chapter is based on the work of the author in [19]

In this chapter, we analyze the causal structure of Generalized Quadratic Gravity (GQG) and Einsteinian Cubic Gravity (ECG). It is well known that gravitons in higher-curvature theories can exhibit superluminal propagation, rendering the conventional definition of causal structures based on null curves inadequate. Instead, the causal structure must be defined using the fastest propagating modes, which travel along characteristic surfaces. The superluminal propagation in higher-curvature theories has significant implications for black holes. Specifically, if the Killing horizon of a black hole is not a characteristic surface corresponding to the fastest propagating mode, the horizon can no longer function as a causal barrier. Here, we present a detailed characteristic analysis of GQG and ECG, and discuss their implications for holography and the algebra of observables.

6.1 Introduction

General Relativity (GR) is one of the most elegant and experimentally validated theories, providing a robust framework for describing gravitational phenomena in the weak-field, low-energy regime [93]. However, its validity in strong-field regimes remains an open question, as we currently lack sufficient experimental or observational evidence to confirm its applicability under such extreme conditions. This uncertainty makes it imperative to explore extensions of GR, such as higher curvature theories, which naturally arise in many approaches to quantum gravity. These theories are not only motivated by attempts to quantize gravity but also serve as fertile ground for exploring deviations from GR in regimes where it has yet to be tested.

In higher curvature theories, the equations of motion (EoM) incorporate higher derivative terms, leading to a more intricate spectrum of propagating modes and the causal structure. These modes often have distinct propagation speeds depending on fields and polarization, with some modes propagating superluminally[94, 95]. In the presence of such superluminal propagation, the usual GR notion of causal structure, defined with respect to null curves, becomes inadequate. Instead, the causal structure must be redefined in terms of the fastest propagating modes. This causal structure, defined with respect to the fastest propagating, is what we should use to define the local algebra of observables.

For many partial differential equations (PDEs), the fastest propagating modes travel along the characteristic surfaces, which are determined by the principal symbol of the PDE—corresponding to the highest derivative terms in the EoM. As a result, the causal structure of higher curvature theories is encoded in the principal symbol of their governing equations. In GR, the fastest propagating modes travel along null curves, and therefore it makes sense to define the causal structure with respect to null curves, which is not true in generic higher curvature theory. This shift in how causality is understood underscores the importance of analyzing the principal symbol in such theories. This fact has implications for the behavior of black holes in higher curvature theories. Let M be the manifold representing the entire spacetime. If we extend the definition of a black hole region from GR as $M \setminus J^-(\mathcal{I}^+)$, where $J^-(\mathcal{I}^+)$ is the causal past of future null infinity $\mathcal{I}^+$, this definition inherently relies on the causal structure defined with respect to null curves associated with the spacetime metric. However, in higher curvature theories that admit superluminal propagating modes, the causal structure is determined by the fastest propagating modes rather than the null structure of the spacetime metric. In these theories, if the horizon is not a characteristic surface corresponding to the fastest mode,

the black hole boundary can no longer act as a causal barrier. This motivates us to study the causal structure of the higher curvature theories.
Generic higher curvature theory has lots of pathologies, like ill-posed initial value problems, instabilities, and perturbative ghosts. But there are higher curvature theories with second-order EoM and, which therefore, are free from such instabilities and ghosts, like Lovelock theories. Moreover, it is shown by Izumi in [96] that in the Gauss-Bonnet theory, the Killing horizon is a characteristic surface for all polarization modes of the graviton and therefore, no modes can leak from the horizon. Reall, Tanahashi, and Way in [94] have generalized this result to the full Lovelock class of theories.
Generally, the theories with the EoM higher than the second order have linearized ghosts and are also considered to be ill-posed. Nevertheless, there are theories with ghosts that have well-posed initial value problems and stable dynamical properties [97, 98, 99]. This makes the higher derivative theories more interesting and worth exploring. One of the main goals of studying these theories is to come up with some physically motivated criteria to define good gravitational theories. For example, the causality criteria of Camanho, Edelstein, Maldacena, and Zhiboedov (CEMZ) require the theory not to have a Shapiro time *advance* [100]. Edelstein, Ghosh, Laddha, and Sarkar in [101, 102] have shown that Generalized Quadratic Gravity (GQG) in a shock wave background for some class of couplings has Shapiro time delay and therefore satisfies CEMZ causality criteria.
In this chapter, we have analyzed the causal structure in GQG, whose Lagrangian has arbitrary linear combinations of squares of the Riemann tensor, Ricci tensor and Ricci scalar. We have also analyzed the causal structure of Einsteinian Cubic Gravity (ECG), a special cubic curvature theory that has only a massless graviton in the spectrum when linearized about the maximally symmetric background. To study the causal structure, we have used the method of characteristics. In Section (6.2), we consider a theory with fourth-order EoM, described by a diffeomorphism invariant Lagrangian. Following Reall in [103], we define the principal symbol and study its symmetries, as well as the implications of the action principle and diffeomorphism invariance on the principal symbol. In Section (6.3), we analyze the characteristics of the Riemann-squared theory in D-dimensions and their implications for Killing horizons. In Section (6.4), we examine GQG in D-dimensions and study its characteristics when the theory has genuinely a fourth-order EoM. The characteristics analysis of linearized perturbations in some higher derivative theories in specific backgrounds is also studied in [104]. In Section (6.5), we present a characteristic analysis of Einsteinian Cubic Gravity. First, we analyze ECG on an arbitrary background but are only able to analyze the null characteristics. To analyze the non-null case, we perform a characteristics analysis on Type N spacetimes in the Weyl classification. In

Section (6.6), we discuss the results and outline future directions.

6.2 Principal Symbol and its Symmetries

Let us consider theories described by the action below with metric $g_{\mu\nu}$ and matter Φ_I.

$$S[g_{\mu\nu}, \Phi_I] = \frac{1}{16\pi G}\int d^D x\sqrt{-g}L(g, \Phi_I). \tag{6.2.1}$$

The gravitational EoM obtained from the above action is

$$E_g^{\mu\nu} = -\frac{-16\pi G}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}} = 0. \tag{6.2.2}$$

Similarly, the matter EoM is

$$E_m^I = -\frac{-16\pi G}{\sqrt{g}}\frac{\delta S}{\delta \Phi_I} = 0 \tag{6.2.3}$$

where g denotes metric and m denotes matter. Let us assume that the EoM is fourth order, then the variation of EoM takes the following form,

$$\delta E_g^{\mu\nu} = M_{gg}^{\mu\nu\alpha\beta,\gamma\rho\delta\zeta}\ \partial_\gamma\partial_\rho\partial_\delta\partial_\zeta\delta g_{\alpha\beta} + M_{gm}^{\mu\nu J,\gamma\rho\delta\zeta}\ \partial_\gamma\partial_\rho\partial_\delta\partial_\zeta\delta\Phi_J + ... \tag{6.2.4}$$

$$\delta E^I = M_{mm}^{IJ,\gamma\rho\delta\zeta}\ \partial_\gamma\partial_\rho\partial_\delta\partial_\zeta\delta\Phi_J + M_{mg}^{I\alpha\beta,\gamma\rho\delta\zeta}\ \partial_\gamma\partial_\rho\partial_\delta\partial_\zeta\delta g_{\alpha\beta} + ... \tag{6.2.5}$$

where $M_{gg}^{\mu\nu\alpha\beta,\gamma\rho\delta\zeta}$, $M_{gm}^{\mu\nu J,\gamma\rho\delta\zeta}$, $M_{mm}^{IJ,\gamma\rho\delta\zeta}$ and $M_{mg}^{I\alpha\beta,\gamma\rho\delta\zeta}$ denote the coefficients of the highest derivatives of the metric and matter fields in the equations of motion (EoM), which we refer to as the principal tensors. The ellipses in the above equation denote the terms with less than four derivatives. We define the matrix of the principal symbol for the EoM by contracting the $\gamma\rho\delta\zeta$ indices of M with an arbitrary covector K_μ as

$$\boldsymbol{\mathcal{P}(K)} = \begin{pmatrix} M_{gg}^{\mu\nu\alpha\beta,\gamma\rho\delta\zeta}\ K_\gamma K_\rho K_\delta K_\zeta & M_{gm}^{\mu\nu J,\gamma\rho\delta\zeta}\ K_\gamma K_\rho K_\delta K_\zeta \\ M_{mg}^{I\alpha\beta,\gamma\rho\delta\zeta}\ K_\gamma K_\rho K_\delta K_\zeta & M_{mm}^{IJ,\gamma\rho\delta\zeta}\ K_\gamma K_\rho K_\delta K_\zeta \end{pmatrix} \tag{6.2.6}$$

which we can write as,

$$\boldsymbol{\mathcal{P}(K)} = \begin{pmatrix} P_{gg}^{\mu\nu\alpha\beta}(K) & P_{gm}^{\mu\nu J}(K) \\ P_{mg}^{I\alpha\beta}(K) & P_{mm}^{IJ}(K) \end{pmatrix} \tag{6.2.7}$$

where $P_{AB}^{****}(K) = M_{AB}^{****,\gamma\rho\delta\zeta} K_\gamma K_\rho K_\delta K_\zeta$ with $A, B \in \{g, m\}$, is the principal symbol of the field B coming from E_A^{**}[1]. The principal symbol matrix $\boldsymbol{\mathcal{P}(K)}$ is an endomorphism on the vector space V_{pol} of "polarization" vectors of the form $\mathbf{T} = (T_{\mu\nu}, T_I)$ where $T_{\mu\nu}$ is symmetric. As we will see, $\boldsymbol{\mathcal{P}(K)}$ is always degenerate due to diffeorphism invariance, but then we can define a gauge equivalence class of polarizations, which we can call "physical polarization." The covector K_μ is called a characteristic covector if there exists a non-zero $\mathbf{T}$ in the gauge equivalence class of polarizations such that it satisfies the characteristic equation $\boldsymbol{\mathcal{P}(K)}\mathbf{T} = 0$. Furthermore, a hypersurface with a normal covector that is a characteristic covector everywhere on the hypersurface is called a characteristic hypersurface. On these surfaces, the coefficient of the highest derivative in (6.2.4) and (6.2.5) is non-invertible. Therefore, one cannot express the highest derivative in the EoM in terms of lower derivatives. Thus, characteristic surfaces represent the boundary of the Cauchy development. There can be multiple characteristics, and the modes propagating along the outermost characteristic surfaces are the fastest-moving modes. Consequently, information about the causal structure of a theory is encoded in the principal symbol of the EoM of the theory.

6.2.1 Symmetries of Principal symbol

In this section, we will follow [103] to find the principal symbol's symmetries. It is clear from the (6.2.2)–(6.2.5)that,

$$M_{gg}^{\mu\nu\alpha\beta,\gamma\rho\delta\zeta} = M_{gg}^{(\mu\nu)\alpha\beta,\gamma\rho\delta\zeta} = M_{gg}^{\mu\nu(\alpha\beta),\gamma\rho\delta\zeta} = M_{gg}^{\mu\nu\alpha\beta,(\gamma\rho\delta\zeta)} \tag{6.2.8}$$

$$M_{gm}^{\mu\nu J,\gamma\rho\delta\zeta} = M_{gm}^{(\mu\nu)J,\gamma\rho\delta\zeta} = M_{gm}^{\mu\nu J,(\gamma\rho\delta\zeta)} \tag{6.2.9}$$

$$M_{mm}^{IJ,\gamma\rho\delta\zeta} = M_{mm}^{IJ,(\gamma\rho\delta\zeta)} \tag{6.2.10}$$

$$M_{mg}^{I\alpha\beta,\gamma\rho\delta\zeta} = M_{mg}^{I(\alpha\beta),\gamma\rho\delta\zeta} = M_{mg}^{I\alpha\beta,(\gamma\rho\delta\zeta)} \tag{6.2.11}$$

6.2.1.1 Implication of action principle on Principal symbol

Let $g_{\mu\nu}(x)$ and $\Phi_I(x)$ be any background field configuration (need not be a solution to the EoM) in the configuration space. Let $g_{\mu\nu}(x, \lambda_1, \lambda_2)$ and $\Phi_1(x, \lambda_1, \lambda_2)$ be the compactly supported two-parameter family of field configurations in the configuration space such that

[1] $*$ here is a proxy for the index

$g_{\mu\nu}(x,0,0) = g_{\mu\nu}(x)$ and $\Phi_I(x,0,0) = \Phi_I(x)$. We will denote derivatives with respect to λ_1 and λ_2 as δ_1 and δ_2 respectively. It can easily be shown that

$$\begin{aligned}\delta_2\delta_1 S = -\frac{1}{16\pi G}\int d^Dx\sqrt{-g}\Big[&(E_g^{\mu\nu}\delta_2\delta_1 g_{\mu\nu} + E_m^I\delta_2\delta_1\Phi_I)\\ &+ (M_{gg}^{\mu\nu\alpha\beta,\gamma\rho\delta\zeta}\,\partial_\gamma\partial_\rho\partial_\delta\partial_\zeta\delta_2 g_{\alpha\beta} + M_{gm}^{\mu\nu J,\gamma\rho\delta\zeta}\,\partial_\gamma\partial_\rho\partial_\delta\partial_\zeta\delta_2\Phi_J + ...)\delta_1 g_{\mu\nu}\\ &+ (M_{mm}^{IJ,\gamma\rho\delta\zeta}\,\partial_\gamma\partial_\rho\partial_\delta\partial_\zeta\delta_2\Phi_J + M_{mg}^{I\alpha\beta,\gamma\rho\delta\zeta}\,\partial_\gamma\partial_\rho\partial_\delta\partial_\zeta\delta_2 g_{\alpha\beta} + ...)\delta_1\Phi_I\Big]\end{aligned} \tag{6.2.12}$$

where ellipses involve terms with less than four derivatives in total acting on the variation of fields. In order to get the above equation, we have used (6.2.4) and (6.2.5), and the fact that the variation is compactly supported, and therefore, we can neglect the boundary terms. Since the above equation is covariant and we are not interested in terms with fewer than four derivatives, we can replace partial derivatives with covariant derivatives in the equation above. Further, by integrating by parts twice

$$\begin{aligned}\delta_2\delta_1 S = -\frac{1}{16\pi G}\int d^Dx\sqrt{-g}\Big[&(E_g^{\mu\nu}\delta_2\delta_1 g_{\mu\nu} + E_m^I\delta_2\delta_1\Phi_I)\\ &+ (M_{gg}^{\mu\nu\alpha\beta,\gamma\rho\delta\zeta}\,\nabla_\gamma\nabla_\rho\delta_1 g_{\mu\nu}\nabla_\delta\nabla_\zeta\delta_2 g_{\alpha\beta} + M_{gm}^{\mu\nu J,\gamma\rho\delta\zeta}\,\nabla_\gamma\nabla_\rho\delta_1 g_{\mu\nu}\nabla_\delta\nabla_\zeta\delta_2\Phi_J + ...)\\ &+ (M_{mm}^{IJ,\gamma\rho\delta\zeta}\,\nabla_\gamma\nabla_\rho\delta_1\Phi_I\nabla_\delta\nabla_\zeta\delta_2\Phi_J + M_{mg}^{I\alpha\beta,\gamma\rho\delta\zeta}\,\nabla_\gamma\nabla_\rho\delta_1\Phi_I\nabla_\delta\nabla_\zeta\delta_2 g_{\alpha\beta} + ...)\Big]\end{aligned} \tag{6.2.13}$$

Now antisymmetrizing the variation in λ_1 and λ_2 and then computing the equation at the background configuration gives,

$$\begin{aligned}0 = -\frac{1}{16\pi G}\int d^Dx\sqrt{-g}\Big[&(M_{gg}^{\mu\nu\alpha\beta,\gamma\rho\delta\zeta} - M_{gg}^{\alpha\beta\mu\nu,\gamma\rho\delta\zeta})\nabla_\gamma\nabla_\rho\delta_1 g_{\mu\nu}\nabla_\delta\nabla_\zeta\delta_2 g_{\alpha\beta}\\ &+ (M_{gm}^{\mu\nu J,\gamma\rho\delta\zeta} - M_{mg}^{J\mu\nu,\gamma\rho\delta\zeta})\nabla_\gamma\nabla_\rho\delta_1 g_{\mu\nu}\nabla_\delta\nabla_\zeta\delta_2\Phi_J\\ &+ (M_{mm}^{IJ,\gamma\rho\delta\zeta} - M_{mm}^{JI,\gamma\rho\delta\zeta})\nabla_\gamma\nabla_\rho\delta_1\Phi_I\nabla_\delta\nabla_\zeta\delta_2\Phi_J\\ &+ (M_{mg}^{I\alpha\beta,\gamma\rho\delta\zeta} - M_{gm}^{\alpha\beta I,\gamma\rho\delta\zeta})\nabla_\gamma\nabla_\rho\delta_1\Phi_I\nabla_\delta\nabla_\zeta\delta_2 g_{\alpha\beta} + ...\Big]\end{aligned} \tag{6.2.14}$$

The above expression has to hold for an arbitrary compactly supported variation, which implies

$$M_{gg}^{\mu\nu\alpha\beta,\gamma\rho\delta\zeta} = M_{gg}^{\alpha\beta\mu\nu,\gamma\rho\delta\zeta} \qquad M_{gm}^{\mu\nu J,\gamma\rho\delta\zeta} = M_{mg}^{J\mu\nu,\gamma\rho\delta\zeta} \qquad M_{mm}^{IJ,\gamma\rho\delta\zeta} = M_{mm}^{JI,\gamma\rho\delta\zeta} \tag{6.2.15}$$

which for covector K_μ gives,

$$P_{gg}^{\mu\nu\alpha\beta}(K) = P_{gg}^{\alpha\beta\mu\nu}(K) \qquad P_{gm}^{\mu\nu J}(K) = P_{mg}^{J\mu\nu}(K) \qquad P_{mm}^{IJ}(K) = P_{mm}^{JI}(K) \tag{6.2.16}$$

Thus, the principal symbol is symmetric. We emphasize that one can always derive the above equation from the action principle for any action whose EoM is of even order. For the odd-order EoM, the action principle will not give a symmetric principal symbol.

6.2.1.2 Implication of Diffeomorphism invariance on the principal symbol

As we know, under diffeomorphisms with compact support, the action is invariant. In a diffeomorphism invariant theory, the diffeomorphism invariance implies the Bianchi identity,

$$\nabla_\mu E_g^{\mu\nu} - L_I^\nu E_m^I = 0 \tag{6.2.17}$$

where L_I^ν is the coefficient of the infinitesimal change in Φ_I under an infinitesimal diffeomorphism [2]. This must hold for an arbitrary configuration of fields. Using (6.2.4) and (6.2.5) we can write (6.2.17) as

$$M_{gg}^{\mu\nu\alpha\beta,\gamma\rho\delta\zeta}\,\partial_\mu\partial_\gamma\partial_\rho\partial_\delta\partial_\zeta\delta g_{\alpha\beta} + M_{gm}^{\mu\nu J,\gamma\rho\delta\zeta}\,\partial_\mu\partial_\gamma\partial_\rho\partial_\delta\partial_\zeta\delta\Phi_I + ... = 0 \tag{6.2.18}$$

where the ellipsis denotes terms with fewer than four derivatives. Since the above equation is true for an arbitrary configuration, the coefficient of the highest derivative must vanish.

$$M_{gg}^{\nu(\mu|\alpha\beta|,\gamma\rho\delta\zeta)} = 0 \qquad M_{gm}^{\nu(\mu|J|,\gamma\rho\delta\zeta)} = 0 \tag{6.2.19}$$

where $(\mu|\alpha\beta|,\gamma\rho\delta\zeta)$ means that upon fixing $\alpha\beta$, it is symmetric in $\mu\gamma\rho\delta\zeta$. For an arbitrary covector K_μ, the above equation implies that

$$K_\mu P_{gg}^{\mu\nu\alpha\beta}(K) = 0 \qquad K_\mu P_{gm}^{\mu\nu J}(K) = 0 \tag{6.2.20}$$

This relation will hold for any higher curvature theory because it is just an outcome of the diffeomorphism invariance of the action.

[2] If Φ_I is a n-tensor field then $L_{\nu I} = \frac{\delta(\mathcal{L}_\xi \Phi_I)}{\delta \xi^\nu}$, where $\mathcal{L}_\xi$ is the Lie derivative with respect to vector field ξ^ν .

6.3 Riemann squared theory

For illustration of the method, in this section, we are interested in studying the causal structure of a theory with the following action:

$$S = \int \sqrt{-g}\, d^D x (R + \lambda R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}) \tag{6.3.1}$$

where λ is the coupling constant associated with a higher curvature term, D is the spacetime dimension, and we will assume that $D \geq 4$. In generic higher curvature theory, the analysis of causal structure based on null curves (with respect to spacetime metric) does not make sense. Therefore, we must do a characteristic analysis of the differential equation obtained from the above Lagrangian. The equation of motion (EoM) for the above Lagrangian is,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda \Big(2R_{\mu}^{\ \alpha\beta\gamma} R_{\nu\alpha\beta\gamma} + 2\nabla^\alpha \nabla^\beta R_{\mu\alpha\nu\beta} - \frac{1}{2} g_{\mu\nu} R_{\gamma\delta\alpha\beta} R^{\gamma\delta\alpha\beta} \Big) = 0 \tag{6.3.2}$$

It is clear from the above equation that it is a fourth-order quasi-linear PDE. Therefore, we can write the above equation as,

$$M_{\mu\nu}^{\ \ \ \alpha\beta,\gamma\rho\delta\zeta}(g)\, \partial_\gamma \partial_\rho \partial_\delta \partial_\zeta g_{\alpha\beta} + O(\partial^3 g) = 0 \tag{6.3.3}$$

where $M_{\mu\nu}^{\ \ \ \alpha\beta,\gamma\rho\delta\zeta}$ is the coefficient of the highest derivative term in the EoM. It is clear from (6.3.2) that the highest derivative term will come from $\nabla^\alpha \nabla^\beta R_{\mu\alpha\nu\beta}$, and it will only depend on the metric. Let Σ be a co-dimension 1 surface with the normal K^μ. We can define the principal symbol for the above PDE as

$$P_{\mu\nu}^{\ \ \ \alpha\beta}(x, K) = M_{\mu\nu}^{\ \ \ \alpha\beta,\gamma\rho\delta\zeta}(g)\, K_\gamma K_\rho K_\delta K_\zeta \tag{6.3.4}$$

We know from the method of characteristics that the fastest propagating modes are tangent to the characteristic surface. The characteristic equation for (6.3.3) is given by

$$det(P(x, k)) = 0 \tag{6.3.5}$$

The above equation tells us that the kernel of the principal symbol will give us the modes moving along the characteristic surface. Let $T_{\alpha\beta}$ be a symmetric tensor corresponding to a polarization mode of the graviton. The possible $T_{\alpha\beta}$ satisfying

$$\lambda Q_{\mu\nu}(K, T) = P_{\mu\nu}^{\ \ \ \alpha\beta}(x, K) T_{\alpha\beta} = 0 \tag{6.3.6}$$

will give the kernel of the principal symbol, and its dimension will give the number of modes propagating along characteristics. It can easily be shown that the $Q_{\mu\nu}(K,T)$ for Riemann Squared theory is,

$$Q_{\mu\nu}(K,T) = -2K^2K^2T_{\mu\nu} + 2K^2K_\mu K^\alpha T_{\alpha\nu} + 2K^2K_\nu K^\alpha T_{\alpha\mu} - 2K_\mu K_\nu K^\alpha K^\beta T_{\alpha\beta} \quad (6.3.7)$$

It is straightforward to check that $K^\mu Q_{\mu\nu}(K,T) = 0$ and for $T'_{\mu\nu} = T_{\mu\nu} + K_{(\mu}X_{\nu)}$ where X_ν is any covector field, $Q_{\mu\nu}(K,T') = Q_{\mu\nu}(K,T)$. Therefore $K_{(\mu}X_{\nu)}$ is pure gauge. Now, we will split the characteristic analysis into two parts, as in [94].

Null Case : $K^2 = 0$

In this case, the equation (6.3.6) reduces to

$$K_\mu K_\nu K^\alpha K^\beta T_{\alpha\beta} = 0 \quad (6.3.8)$$

which implies $K^\alpha K^\beta T_{\alpha\beta} = 0$. Therefore, the characteristic equation only fixes one component of the $T_{\alpha\beta}$, but D of the components of $T_{\alpha\beta}$ are just pure gauge, and we can fix $D-1$ by suitable gauge choice. To see this more explicitly, we can choose a null basis $\{k^\mu, l^\mu, m_i^\mu\}$ such that,

$$k.k = l.l = k.m_i = l.m_i = 0 \;\&\; k.l = -1,\; m_i.m_j = \delta_{ij} \quad (6.3.9)$$

We will denote contraction with respect to $\{k^\mu, l^\mu, m_i^\mu\}$ as $\{0, 1, i\}$. Let us choose one of the null basis vectors $k = K$. The equation in 6.3.8 implies that $T_{00} = 0$; further, in this basis, $T_{1\mu}$ are pure gauge modes due to diffeomorphism invariance [3] . As a result, the total degeneracy of the principal symbol for the null characteristics is $\frac{D(D+1)}{2} - D - 1 = \frac{(D-2)(D+1)}{2}$, where $\frac{(D-2)(D+1)}{2}$ are the degrees of freedom (DoF) associated with physical propagating modes in the space of symmetric two tensors (total minus pure gauge). Notice that the degeneracy is equal to the number of DoF for massive gravitons. But it is important to remember that we are not in a 2 derivative theory, and therefore derivatives of the metric may not be canonically conjugate to the metric; some of them will be independent degrees of freedom. Further, we are in the eikonal limit, where all the information about the spectrum is encoded in the allowed polarization [4]. So we cannot associate $\frac{(D-2)(D+1)}{2}$

[3]In the null basis $T_{\mu\nu} = T_{00}l_\mu l_\nu + T_{11}K_\mu K_\nu + 2T_{1i}K_{(\mu}m^i_{\nu)} + 2T_{0i}l_{(\mu}m^i_{\nu)} + 2T_{01}K_{(\mu}l_{\nu)} + T_{ij}m^i_{(\mu}m^j_{\nu)}$, notice that $T_{1\mu} = \{T_{11}, T_{10}, T_{1i}\}$ are the coefficient of the term of type $K_{(\mu}X_{\nu)}$, where $X_\nu = \{K_\nu, l_\nu, m^i_\nu\}$. Therefore they are pure gauge modes.

[4]In the eikonal limit, although the mass term is irrelevant, the polarization retains information distinguishing between massive and massless fields.

DoF to massive gravitons directly; one must reduce the theory to second-order theory, and then these $\frac{(D-2)(D+1)}{2}$ propagating components will split into different particles that can occur in the spectrum of the theory. The crucial thing to notice is that this is true for any null surface in this theory without assuming any condition on curvature components. So, unlike the Gauss-Bonnet theory [96] , all null surfaces are characteristic. This also tells us the Killing horizon in this theory is a characteristic surface.

Non Null Case: $K^2 \neq 0$

In the non-null case, it is always possible to write $T_{\mu\nu} = \hat{T}_{\mu\nu} + K_{(\mu}X_{\nu)}$, for some X and where $\hat{T}_{\mu\nu}$ is transverse, i.e $K^{\mu}\hat{T}_{\mu\nu} - \frac{1}{2}\hat{T}K_b = 0$[5]. Using the fact that $K_{(\mu}X_{\nu)}$ is pure gauge and $\hat{T}_{\alpha\beta}$ is transverse, we can simplify the expression in (6.3.6) for the non-null case:

$$Q_{\mu\nu}(K,T) = -2K^2K^2\hat{T}_{\mu\nu} + K^2K_{\mu}K_{\nu}\hat{T} = 0. \tag{6.3.10}$$

As we want $Q_{\mu\nu}(K,T) = 0$, its trace will also vanish, implies

$$Q(K,T) = -K^2K^2\hat{T} = 0. \tag{6.3.11}$$

Since $K^2 \neq 0$, this implies $\hat{T} = 0$, and putting it back to the equation (6.3.10), we get

$$Q_{\mu\nu}(K,T) = -2K^2K^2\hat{T}_{\mu\nu} = 0. \tag{6.3.12}$$

The only solution to the above equation is $T_{\mu\nu} = 0$. When $K^2 \neq 0$, the symbol is not degenerate; therefore, Riemann squared theory has no non-null characteristics. The above analysis suggests that in the Riemann-squared theory, all the characteristics are null. Furthermore, there are $\frac{(D+1)(D-2)}{2}$ allowed polarization modes that move along the characteristic surface.

6.4 Generalized Quadratic Gravity (GQG

In this section, we are interested in studying the causal structure of theory with the following action:

$$S = \frac{1}{16\pi G}\int \sqrt{g}\, d^Dx(R + \lambda_3 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \lambda_2 R_{\mu\nu}R^{\mu\nu} + \lambda_1 R^2) \tag{6.4.1}$$

[5] $\hat{T}$ is the trace of $\hat{T}_{\alpha\beta}$

where λ_i are the coupling constants and $D \geq 4$. The EoM for the above Lagrangian is,

$$\begin{aligned} E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + (\lambda_2 + 4\lambda_3)\Box R_{\mu\nu} + \frac{1}{2}(\lambda_2 + 4\lambda_1)g_{\mu\nu}\Box R \\ -(2\lambda_1 + \lambda_2 + 2\lambda_3)\nabla_\mu\nabla_\nu R + 2\lambda_3 R_{\alpha\beta\gamma\mu}R_\nu^{\alpha\beta\gamma} + 2(\lambda_2 + 2\lambda_3)R_{\alpha\mu\gamma\nu}R^{\alpha\gamma} \\ - 4\lambda_3 R_{\mu\alpha}R_\nu^\alpha + 2\lambda_1 RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(\lambda_3 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \lambda_2 R_{\mu\nu}R^{\mu\nu} + \lambda_1 R^2) = 0 \end{aligned} \quad (6.4.2)$$

It is evident from the above equation that the EoM is a fourth-order quasi-linear PDE of the form (6.3.3). The terms that contribute to the fourth-order derivative are shown in red color in the equation (6.4.2). Since we are interested in causal structure determined by the principal symbol, these are the only terms of relevance. For any covector K_μ and symmetric 2-tensor $T_{\alpha\beta}$, it is straight forward to show that

$$\begin{aligned} Q_{\mu\nu}(K,T) = \lambda_1(-2K_\mu K_\nu K^\alpha K^\beta T_{\alpha\beta} + 2g_{\mu\nu}K^2K^\alpha K^\beta T_{\alpha\beta} + 2K_\mu K_\nu K^2 T - 2g_{\mu\nu}K^4 T) \\ +\lambda_2(-\frac{1}{2}K^4T_{\alpha\beta} + \frac{1}{2}K_\nu K^2 T_{\mu\beta}K^\beta + \frac{1}{2}K_\mu K^2 T_{\nu\beta}K^\beta - K_\mu K_\nu K^\alpha K^\beta T_{\alpha\beta} + \frac{1}{2}g_{\mu\nu}K^2K^\alpha K^\beta T_{\alpha\beta} \\ +\frac{1}{2}K^2K_\mu K_\nu T - \frac{1}{2}g_{\mu\nu}K^4T) + \lambda_3(-2K^2K^2T_{\mu\nu} + 2K^2K_\mu K^\alpha T_{\alpha\nu} + 2K^2K_\nu K^\alpha T_{\alpha\mu} - 2K_\mu K_\nu K^\alpha K^\beta T_{\alpha\beta}) \end{aligned} \quad (6.4.3)$$

where $K^4 = K^2K^2$. We can write the above equation in terms of $\Pi_{\mu\nu}(K) = g_{\mu\nu}K^2 - K_\mu K_\nu$, this is K^2 times a projector that projects the vector onto transverse space to K.

$$\begin{aligned} Q_{\mu\nu}(K,T) = -2\lambda_1(\Pi_{\mu\nu}(K)\Pi^{\alpha\beta}(K))T_{\alpha\beta} \\ -\frac{1}{2}\lambda_2(\Pi_\mu{}^\alpha(K)\Pi_\nu{}^\beta(K) + \Pi_{\mu\nu}(K)\Pi^{\alpha\beta}(K))T_{\alpha\beta} - 2\lambda_3(\Pi_\mu{}^\alpha(K)\Pi_\nu{}^\beta(K))T_{\alpha\beta}. \end{aligned} \quad (6.4.4)$$

Another way of writing this equation is

$$Q_{\mu\nu}(K,T) = \Big(-(2\lambda_1 + \frac{1}{2}\lambda_2)\Pi_{\mu\nu}(K)\Pi^{\alpha\beta}(K) - (\frac{1}{2}\lambda_2 + 2\lambda_3)\Pi_\mu{}^\alpha(K)\Pi_\nu{}^\beta(K)\Big)T_{\alpha\beta}. \quad (6.4.5)$$

Notice that the term inside the bracket is the principal symbol. It is evident from the above equation that that $K^\mu Q_{\mu\nu}(K,T) = 0$ and for $T'_{\mu\nu} = T_{\mu\nu} + K_{(\mu}X_{\nu)}$ where X_ν is any covector, $Q_{\mu\nu}(K,T') = Q_{\mu\nu}(K,T)$. Therefore $T_{\mu\nu} = K_{(\mu}X_{\nu)}$ is pure gauge. Further, the principal symbol vanishes for any K_μ, when $\lambda_3 = -\frac{1}{4}\lambda_2 = \lambda_1$. This choice of couplings corresponds to the Gauss-Bonnet term in the action whose EoM is second order. Here, we restrict ourselves to theories where the equation in (6.4.5) is genuinely fourth order. It requires either $\lambda_3 + \frac{1}{4}\lambda_2 \neq 0$ or $\frac{1}{4}\lambda_2 + \lambda_1 \neq 0$. We will assume that $\lambda_3 + \frac{1}{4}\lambda_2 \neq 0$, as this condition will also appear in the case of the characteristics analysis for non-null case. Let us analyze the characteristic equations for GQG,

Null Case: $K^2 = 0$

In this case, the equation (6.4.5) reduces to,

$$Q_{\mu\nu}(K,T) = -(2\lambda_1 + \lambda_2 + 2\lambda_3)K_\mu K_\nu K^\alpha K^\beta T_{\alpha\beta} = 0 \tag{6.4.6}$$

As we already mentioned, we don't want $2\lambda_1 + \lambda_2 + 2\lambda_3 = 0$. Otherwise, the principal symbol will be completely degenerate for $K^2 = 0$. Using the null basis defined in Section (6.3) and the equation (6.4.5), we get $T_{00} = 0$. Using the fact that in the null basis, $T_{1\mu}$ are pure gauge modes, the total degeneracy of the principal symbol for null characteristics is $\frac{(D-2)(D+1)}{2}$. Following the logic presented in the QG, these are the number of polarizations allowed in this theory. Similar to the Riemann squared theory, this is true without putting any conditions on the curvature components.

Non-null Case: $K^2 \neq 0$

In non-null cases, we will follow the same steps as in the last section. Without loss of generality, we can take $T_{\mu\nu} = \hat{T}_{\mu\nu}$ as the symmetric tensor that is transverse. The equation for characteristics will become $Q_{\mu\nu}(K,\hat{T}) = 0$, where we can use $K^\mu \hat{T}_{\mu\nu} = \frac{1}{2}K_\nu \hat{T}$. It can easily be seen that

$$Q(K,\hat{T}) = (\lambda_3 - \lambda_1 + D(\lambda_1 + \frac{1}{4}\lambda_2))K^4\hat{T} = 0. \tag{6.4.7}$$

Assuming that $\lambda_3 - \lambda_1 + D(\lambda_1 + \frac{1}{4}\lambda_2) \neq 0$, this implies $\hat{T} = 0$. Putting this back into $Q_{\mu\nu}(K,\hat{T}) = 0$, we will get

$$Q_{\mu\nu}(K,\hat{T}) = -\frac{1}{2}(\lambda_2 + 4\lambda_3)K^4\hat{T}_{\mu\nu} = 0 \tag{6.4.8}$$

Since $\lambda_2 + 4\lambda_3 \neq 0$, the above equation implies that $\hat{T}_{\mu\nu} = 0$. In this, all the degeneracy of the principal symbol is now lifted, and therefore, there are no non-null characteristics for GQG in the considered coupling space. The allowed couplings in D spacetime dimensions are

$$(\frac{1}{2}\lambda_2 + 2\lambda_3) \neq 0 \tag{6.4.9}$$

$$(2\lambda_1 + \lambda_2 + 2\lambda_3) \neq 0 \tag{6.4.10}$$

$$(\lambda_3 - \lambda_1 + D(\lambda_1 + \frac{1}{4}\lambda_2)) \neq 0 \tag{6.4.11}$$

In the figure (6.1) we have shown the forbidden couplings in dimension $D = 5$.

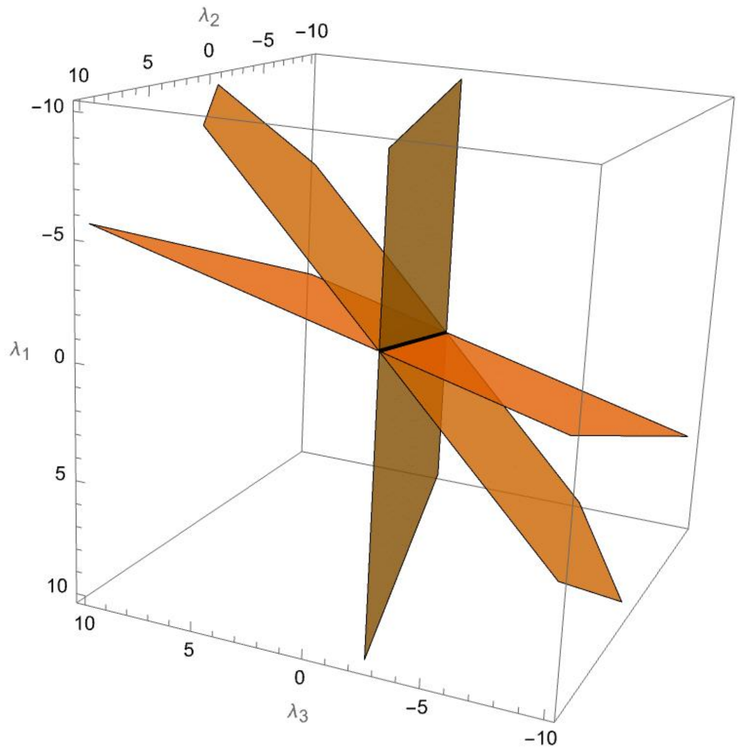


FIGURE 6.1: This figure depicts the space of couplings, with the colored regions indicating forbidden couplings. Here, $\{\lambda_1, \lambda_2, \lambda_3\}$ represent the couplings in the action of GQG, with the dimension D set to 5. From the above analysis, the equation for the forbidden is $(\frac{1}{2}\lambda_2 + 2\lambda_3)(2\lambda_1 + \lambda_2 + 2\lambda_3)(\lambda_3 - \lambda_1 + 5(\lambda_1 + \frac{1}{4}\lambda_2)) = 0$.

6.5 Einsteinian cubic gravity (ECG)

Recently, there has been a lot of interest in Einsteinian cubic gravity proposed by Bueno and Cano in the paper [105]. This theory is a special higher curvature theory of gravity (cubic in curvature) defined in a way that, when linearized about a maximally symmetric background, it gives an Einsteinian spectrum. It exists for spacetime dimension $D \geq 4$, and there is some evidence that it possesses Schwarzschild-like black hole solutions [106]. ECG does not have an Einsteinian spectrum when linearized about generic backgrounds[107, 108]. The action of ECG is

$$S = \frac{1}{16\pi G}\int d^Dx\sqrt{-g}(R - \Lambda_0 + \alpha\mathcal{L}_4 + \beta\mathcal{L}_6 + \lambda\mathcal{P}) \tag{6.5.1}$$

where

$$\mathcal{P} = 12R_{\mu\ \nu}^{\ \rho\ \sigma}R_{\rho\ \sigma}^{\ \gamma\ \delta}R_{\gamma\ \delta}^{\ \mu\ \nu} + R_{\mu\nu}^{\rho\sigma}R_{\rho\sigma}^{\gamma\delta}R_{\gamma\delta}^{\mu\nu} - 12R_{\mu\nu\rho\sigma}R^{\sigma\rho}R^{\nu\sigma} + 8R_{\mu}^{\nu}R_{\nu}^{\rho}R_{\rho}^{\mu} \tag{6.5.2}$$

$$\mathcal{L}_4 = \frac{1}{4}\delta^{\mu_1\mu_2\mu_3\mu_4}_{\nu_1\nu_2\nu_3\nu_4}R^{\nu_1\nu_3}_{\mu_1\mu_3}R^{\nu_2\nu_4}_{\mu_2\mu_4} \tag{6.5.3}$$

$$\mathcal{L}_6 = \frac{1}{8}\delta^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6}R^{\nu_1\nu_4}_{\mu_1\mu_4}R^{\nu_2\nu_5}_{\mu_2\mu_5}R^{\nu_3\nu_6}_{\mu_3\mu_6} \tag{6.5.4}$$

Λ_0 is the cosmological constant and $\delta^{\mu_1\mu_2\ldots\mu_i}_{\nu_1\nu_2\ldots\nu_i}$ is the generalized antisymmetric Kronecker delta. Notice that $\mathcal{L}_4$ and $\mathcal{L}_6$ are quadratic and cubic Lovelock terms. As we have summarized, ECG has a lot of nice properties. It is worthwhile to study the causality structure of ECG. Let us start by writing the EoM for ECG.

$$\mathcal{E}_{\mu}^{\ \alpha\beta\gamma}R_{\nu\alpha\beta\gamma} - \frac{1}{2}g_{\mu\nu}\mathcal{L} + 2\nabla^{\alpha}\nabla^{\beta}\mathcal{E}_{\mu\alpha\nu\beta} = 0 \tag{6.5.5}$$

where $\mathcal{L} = (R - \Lambda_0 + \alpha\mathcal{L}_4 + \beta\mathcal{L}_6 + \lambda\mathcal{P})$ and $\mathcal{E}^{\mu\alpha\nu\beta} = \frac{\partial\mathcal{L}}{\partial R_{\mu\alpha\nu\beta}}$. We can write

$$\mathcal{E}^{\mu\alpha\nu\beta} = \alpha\mathcal{E}_4^{\mu\alpha\nu\beta} + \beta\mathcal{E}_6^{\mu\alpha\nu\beta} + \lambda\mathcal{E}_{\mathcal{P}}^{\mu\alpha\nu\beta} \tag{6.5.6}$$

where $\mathcal{E}_4^{\mu\alpha\nu\beta}$ and $\mathcal{E}_6^{\mu\alpha\nu\beta}$ are terms coming from Lovelock terms in Lagrangian and $\mathcal{E}_{\mathcal{P}}^{\mu\alpha\nu\beta}$ comes from $\mathcal{P}$. Notice that the highest derivative term in (6.5.5) will come from $\nabla^{\alpha}\nabla^{\beta}\mathcal{E}_{\mu\alpha\nu\beta}$. Further, it is well-known that

$$\nabla_{\alpha}\nabla_{\beta}\mathcal{E}_4^{\mu\alpha\nu\beta} = \nabla_{\alpha}\nabla_{\beta}\mathcal{E}_6^{\mu\alpha\nu\beta} = 0 \tag{6.5.7}$$

since $\mathcal{E}_4$ and $\mathcal{E}_6$ are coming from Lovelock terms. Now the highest derivative contribution will come from $\nabla_{\alpha}\nabla_{\beta}\mathcal{E}_{\mathcal{P}}^{\mu\alpha\nu\beta}$ and it will be fourth order. It can easily be shown that

$$\begin{aligned}\lambda\mathcal{E}^{\mathcal{P}}_{\alpha\beta\mu\nu} = 6\lambda\Big(&R_{\alpha\nu}R_{\beta\mu} - R_{\alpha\mu}R_{\beta\nu} + g_{\beta\nu}R_{\alpha}^{\ \sigma}R_{\mu\sigma} - g_{\alpha\nu}R_{\beta}^{\ \sigma}R_{\mu\sigma} - g_{\beta\mu}R_{\alpha}^{\ \sigma}R_{\nu\sigma} + g_{\alpha\mu}R_{\beta}^{\ \sigma}R_{\nu\sigma}\\ &- g_{\beta\nu}R^{\sigma\gamma}R_{\alpha\sigma\mu\gamma} + g_{\beta\mu}R^{\sigma\gamma}R_{\alpha\sigma\nu\gamma} + g_{\alpha\nu}R^{\sigma\gamma}R_{\beta\sigma\mu\gamma}\\ &- 3R_{\alpha\ \nu}^{\ \sigma\ \gamma}R_{\beta\sigma\mu\gamma} - g_{\alpha\mu}R^{\sigma\gamma}R_{\beta\sigma\nu\gamma} + 3R_{\alpha\ \nu}^{\ \sigma\ \gamma}R_{\beta\sigma\nu\gamma} + \frac{1}{2}R_{\alpha\beta}^{\ \ \sigma\gamma}R_{\mu\nu\sigma\gamma}\Big).\end{aligned} \tag{6.5.8}$$

Using the above equation and with a bit of algebra, we can compute the principal symbol acting on the symmetric two-tensor $T_{\mu\nu}$ for the EoM in (6.5.5),

$$
\begin{aligned}
Q_{\alpha\beta}(T,K) = P_{\alpha\beta}{}^{\mu\nu}(x,K)T_{\mu\nu} &= 12K^2K^\mu K^\nu R_{\mu\nu}T_{\alpha\beta} + 6K^2K^\mu K^\nu R_{\beta\mu}T_{\alpha\nu} - 6K^4 R^\sigma_\beta T_{\alpha\sigma} \\
&- 18K_\beta K^\sigma K^\mu K^\nu R_{\sigma\nu}T_{\alpha\mu} + 6K_\beta K^2 K^\mu R^\nu_\mu T_{\alpha\nu} + 6K^2K^\mu K^\nu R_{\beta\mu\nu\sigma}T^\sigma_\alpha + 6K^2K^\mu K^\nu R_{\alpha\mu}T_{\beta\nu} \\
&- 6K^4R^\sigma_\alpha T_{\beta\sigma} - 18K_\alpha K^\sigma K^\mu K^\nu R_{\sigma\mu}T_{\beta\nu} + 6K_\alpha K^2K^\mu R^\nu_\mu T_{\beta\nu} + 6K^2K^\mu K^\nu R_{\alpha\mu\nu\sigma}T^\sigma_\beta \\
&- 18K^\sigma K^\mu K^\nu K^\rho R_{\alpha\nu\beta\rho}T_{\sigma\mu} - 18K_\beta K^\sigma K^\mu K^\nu R_{\alpha\mu\nu\rho}T^\rho_\sigma - 18K_\alpha K^\sigma K^\mu K^\nu R_{\beta\mu\nu\rho}T^\rho_\sigma \\
&- 6K^2K^\mu K^\nu R_{\alpha\beta}T_{\mu\nu} + 6K_\beta K^2K^\mu R^\nu_\alpha T_{\mu\nu} + 6K_\alpha K^2K^\mu R^\nu_\beta T_{\mu\nu} - 6K_\alpha K_\beta K^2R^{\mu\nu}T_{\mu\nu} \\
&+ 12K^2K^\mu K^\nu R_{\alpha\nu\beta\sigma}T^\sigma_\mu + 12K^2K^\mu K^\nu R_{\alpha\sigma\beta\nu}T^\sigma_\mu + 18g_{\alpha\beta}K^\sigma K^\mu K^\nu K^\rho R_{\sigma\mu}T_{\nu\rho} \\
&- 12g_{\alpha\beta}K^2K^\mu K^\nu R^\rho_\mu T_{\nu\rho} + 6g_{\alpha\beta}K^4R^{\mu\nu}T_{\mu\nu} + 6K^4R_{\alpha\beta}T - 6K_\beta K^2K^\mu R_{\alpha\mu}T \\
&- 6K_\alpha K^2K^\mu R_{\beta\mu}T + 18K_\alpha K_\beta K^\mu K^\nu R_{\mu\nu}T - 12K^4R_{\alpha\nu\beta\mu}T^{\mu\nu} + 12K_\beta K^2K^\nu R_{\alpha\mu\nu\rho}T^{\mu\rho} \\
&+ 12K_\alpha K^2K^\mu R_{\beta\nu\mu\rho}T^{\nu\rho} - 18K_\alpha K_\beta K^\mu K^\nu R_{\mu\sigma\nu\rho}T^{\sigma\rho} - 12g_{\alpha\beta}K^2K^\mu K^\nu R_{\mu\nu}T \\
&+ 6K^2K^\mu K^\nu R_{\alpha\mu\beta\nu}T + 6g_{\alpha\beta}K^2K^\mu K^\nu R_{\mu\rho\nu\sigma}T^{\rho\sigma} \qquad (6.5.9)
\end{aligned}
$$

where K_μ is an arbitrary covector . It can easily be checked that the above principal symbol satisfies the Bianchi identity $K^\alpha Q_{\alpha\beta}(T,K) = 0$ and is invariant under pure gauge transformations. Furthermore, it can be checked that for maximally symmetric space-times, the principal symbol vanishes as expected. This is related to the fact that when ECG is linearized about a maximally symmetric background, its EoM is almost that of the Einstein gravity[105].

As it must be clear from the expression of the principal symbol, the ECG will not have an Einsteinian spectrum about a generic background. We also want to emphasize that unlike GQG, the principal symbol here depends on the curvature tensors, and therefore, in principle, it can lead to very different causal structures depending on the background. Hence, doing characteristic analysis is extremely important. Let us start with null characteristics.

Null Case:$K^2 = 0$

Again, we will use the same null basis that we have defined in the section (6.3). By using this null basis and putting $K^2 = 0$, the principal symbol simplifies.

$$
\begin{aligned}
Q_{\alpha\beta}(T,K) = &-18T_{\alpha0}R_{00}K_\beta - 18T_{\beta0}R_{00}K_\alpha - 18T_{00}R_{\alpha0\beta0} - 18T^\rho_0R_{\alpha00\rho}K_\beta - 18K_\alpha T^\rho_0 R_{\beta00\rho} \\
&+ 18g_{\alpha\beta}T_{00}R_{00} + 18K_\alpha K_\beta TR_{00} - 18K_\alpha K_\beta R_{0\mu0\nu}T^{\mu\nu} = 0 \qquad (6.5.10)
\end{aligned}
$$

Let us write the above equation in component form,

$$Q_{00}(T,K)=0,\quad Q_{01}(T,K)=0,\quad Q_{0i}(T,K)=0 \tag{6.5.11}$$

$$Q_{11}(T,K)=18T_{ii}R_{00}-18R_{0i0j}T_{ij}=0 \tag{6.5.12}$$

$$Q_{1i}(T,K)=-18T_{i0}R_{00}-18T_{00}R_{10i0}-18T_0^{\rho}R_{i00\rho}=0 \tag{6.5.13}$$

$$Q_{ij}(T,K)=18\delta_{ij}T_{00}R_{00}-18T_{00}R_{i0j0}=0 \tag{6.5.14}$$

The equation (6.5.11) is due to the Bianchi identity. Further notice that $T_{1\mu}$ does not appear anywhere, which is consistent with the fact that it is pure gauge. Assuming $T_{00}=T_{0i}=0$, the equations (6.5.13) and (6.5.14) are trivially satisfied. Therefore, we are able to satisfy the null characteristic equation for any $T_{\mu\nu}$ satisfying $T_{00}=T_{0i}=0$ and the equation (6.5.12). This implies that the degeneracy of the principal symbol is $\frac{D(D-3)}{2}$. This is the same as the DoF of a massless graviton in D dimensional spacetime. Therefore, any null surface is a characteristic surface in ECG. We want to emphasize that our analysis is for any spacetime for which the fourth derivative terms do not vanish for all $T_{\mu\nu}$. Now, let us analyze the case of a Killing horizon. As we know, at the Killing horizon, $R_{i0j0}=R_{0ijk}=0$. Using this fact, we can easily show that all the equations for the null characteristics are satisfied for all T_{ab} in the equivalence class of symmetric tensors up to gauge. This implies that the principal symbol is degenerate for all T_{ab}, and therefore, the dynamics is governed by a lower-order differential equation. We emphasize that this holds true for any Killing horizon. For any such solution which is also asymptotic to a maximally symmetric spacetime, the order of the differential equation interpolates from 2 in the asymptotic region, to 4 in the bulk, and less than 4 on the horizon.

Furthermore, it is well known in the case of ECG that for spacetimes of the form $M_{D'}\times M_{D-D'}$, where $D'<D$, and $M_{D'}$ and $M_{D-D'}$ are maximally symmetric spaces, the equations of motion for gravitational perturbations reduce to a linearized Einstein equation with an effective Newton's constant G_{eff} [109]. This indicates that, in all such spacetimes, null surfaces are characteristic surfaces.

This observation has significant implications for black holes in this theory. It is well known that there exists a large class of black holes where the near-horizon geometry has this product form. Consequently, for all such black holes, the horizon is a characteristic surface. This means that none of the propagating modes can classically escape or leak through spacelike paths from inside the horizon.

Non-null Case: $K^2\neq 0$

In this case, we can use the fact that $T_{\alpha\beta}$ can be split into pure gauge and transverse parts, as in Section (6.3). Using the transverse property of $T_{\alpha\beta}$ we can show

$$\begin{aligned} Q_{\alpha\beta}(T,K) = 12K^2 R_{\mu\nu}K^\mu K^\nu T_{\alpha\beta} - 6K^4 R^\mu_\beta T_{\alpha\mu} + 6R^\mu_\nu T_{\alpha\mu}K^\nu K_\beta K^2 + 6K^2 R_{\beta\mu\nu\rho}T^\rho_\alpha K^\mu K^\nu \\ - 6K^4 R^\mu_\alpha T_{\mu\beta} + 6K^2 R_{\alpha\mu\nu\rho}T^\rho_\beta K^\mu K^\nu - 6K_\alpha K_\beta K^2 R^{\mu\nu}T_{\mu\nu} + 6g_{\alpha\beta}K^4 R^{\mu\nu}T_{\mu\nu} \\ 3K^4 R_{\alpha\beta}T - 12K^4 R_{\alpha\mu\beta\nu}T^{\mu\nu} + 12K_\beta K^2 R_{\alpha\mu\nu\rho}T^{\mu\rho}K^\nu + 12K_\alpha K^2 R_{\beta\mu\nu\rho}T^{\mu\rho}K^\nu \\ - 18K_\alpha K_\beta R_{\mu\rho\nu\sigma}T^{\rho\sigma}K^\mu K^\nu - 9g_{\alpha\beta}K^2 R_{\mu\nu}K^\mu K^\nu T + 9K^2 R_{\alpha\mu\beta\nu}K^\mu K^\nu T \\ + 6g_{\alpha\beta}K^2 R_{\mu\rho\nu\sigma}T^{\rho\sigma}K^\mu K^\nu + 6K^2 K_\alpha K^\nu R_{\nu\sigma}T^\sigma_\beta \end{aligned} \tag{6.5.15}$$

Unlike GQG, the principal symbol of ECG depends on the Riemann curvature tensor, making it challenging to analyze non-null cases on arbitrary backgrounds. Therefore, we focus on analyzing it in the Ricci-flat Type N spacetime in the algebraic classification of spacetimes using the Weyl tensor.

6.5.1 ECG in Type N Spacetimes

As is well known from the algebraic classification of the spacetimes, the type N spacetime is the simplest spacetime with a nontrivial Riemann tensor [110, 111]. Let us introduce a null basis $\{l^\mu, n^\mu, m_i^\mu\}$ such that,

$$n.n = l.l = n.m_i = l.m_i = 0 \ \& \ n.l = 1, \ m_i.m_j = \delta_{ij} \tag{6.5.16}$$

The spacetime is type N iff in some null basis; the Riemann tensor takes the following form,

$$R_{\alpha\beta\mu\nu} = 4\Omega_{ij} l_{[\alpha} m^i_{\beta]} l_{[\mu} m^j_{\nu]} \tag{6.5.17}$$

where Ω_{ij} is a $(D-2)$x$(D-2)$ matrix. If Ω is traceless, then the spacetime will be Ricci flat, which is the case of interest to us. Now, we will write the equation (6.5.9) in a Ricci flat type N spacetime. Further, using the Ricci flatness and the fact that the contraction of Weyl curvature with itself vanishes in type N spacetime, it can easily be shown that it is a solution to ECG. It is more convenient to work in the basis $\{K^\alpha, l^\alpha, m_i^\alpha\}$. It can be shown that the principal symbol in this case is

$$\begin{aligned}
Q_{\alpha\beta}(T,K) = P_{\alpha\beta}{}^{\mu\nu}(x,K)T_{\mu\nu} \\
= 2\Omega_{ij}\Big\{ -3K^2(K.l)^2 m^i_\beta T^j_\alpha - 3K^2(K.l)^2 m^i_\alpha T^j_\beta - 9(K.l)^2 m^i_\alpha m^j_\beta T_{00} \\
+12K^2(K.l)m^j_\beta m^i_\alpha T_{10} - 6m^i_\alpha m^j_\beta K^4 T_{11} - 6K^2(K.l)l_\beta m^j_\alpha T^j_0 + 9K_\beta m^i_\alpha (K.l)^2 T^j_0 \\
+6K^4 m^i_\alpha l_\beta T^j_1 - 6K_\beta m^i_\alpha K^2(K.l)T^j_1 - 6K^2 T^j_0 m^i_\beta (K.l) + 9K_\alpha m^i_\beta (K.l)^2 T^j_0 \\
+6K^4 l_\alpha m^i_\beta T^j_1 - 6K^2(K.l)K_\alpha m^i_\beta T^j_1 - 6K^4 l_\alpha l_\beta T^{ij} + 6K^2(K.l)K_\beta l_\alpha T^{ij} \\
+6K_\alpha l_\beta K^2(K.l)T^{ij} - 9K_\alpha K_\beta (K.l)^2 T^{ij} + 3g_{\alpha\beta}K^2(K.l)^2 T^{ij} + 3K^2(K.l)^2 m^i_\alpha m^j_\beta T\Big\}
\end{aligned} \tag{6.5.18}$$

where $\{0,1,i\}$ indexes in the above equation are contractions with respect to $\{K^\alpha, l^\alpha, m^\alpha_i\}$ respectively. We can always write K_α as a linear superposition of n_α and l_α. It can easily be checked that the above equation satisfies the Bianchi identity and is invariant under pure gauge transformation, i.e. $T_{\alpha\beta} \to T_{\alpha\beta} + K_{(a}X_{b)}$, for arbitrary X_α. As shown earlier in this section, we can analyze the null case $K^2 = 0$ of ECG in an arbitrary background. In this part, we will focus on the non-null case, using a Ricci flat type N spacetime.
Let $K^2 \neq 0$, which means that K_a is a non-trivial superposition of l_α and n_α and therefore $K.l \neq 0$. Since $K^2 \neq 0$, we can decompose $T_{\alpha\beta}$ into a transverse part and pure gauge. The invariance of the equation (6.5.18) under pure gauge transformations allows us to choose $T_{\alpha\beta}$ transverse, i.e $K^\alpha T_{\alpha\beta} = \frac{1}{2}K_\beta T$. Further, for solving the equation $Q_{\alpha\beta}(T,K) = 0$ for $T_{\alpha\beta}$, we will assume $T_{11} = T_1{}^i = 0$. With this assumption and the transverse property of $T_{\alpha\beta}$, the characteristics equation reduces to

$$\begin{aligned}
Q_{\alpha\beta}(T,K) = P_{\alpha\beta}{}^{\mu\nu}(x,K)T_{\mu\nu} \\
= 2\Omega_{ij}\Big\{ -3K^2(K.l)^2 m^i_\beta T^j_\alpha - 3K^2(K.l)^2 m^i_\alpha T^j_\beta + 6K^2(K.l)^2 m^j_\beta m^i_\alpha T \\
-6K^4 l_\alpha l_\beta T^{ij} + 6K^2(K.l)K_\beta l_\alpha T^{ij} + 6K_\alpha l_\beta K^2(K.l)T^{ij} - \frac{9}{2}K^2(K.l)^2 m^i_\alpha m^j_\beta T \\
-9K_\alpha K_\beta (K.l)^2 T^{ij} + 3g_{\alpha\beta}K^2(K.l)^2 T^{ij} + 3K^2(K.l)^2 m^i_\alpha m^j_\beta T\Big\} = 0
\end{aligned} \tag{6.5.19}$$

Now, we will solve the characteristics equation component by component. It can easily be checked that $Q_{0\beta}(K,T) = 0$ and

$$Q_{1\beta}(K,T) = 18\Omega_{ij}T^{ij}(K.l)^2\Big(K^2 l_\beta - K_\beta K.l\Big) \tag{6.5.20}$$

Since we want to solve for $Q_{1\beta}(K,T) = 0$, this implies either $\left(K^2 l_\beta - K_\beta K.l\right) = 0$ or $\Omega_{ij}T^{ij} = 0$. If $\left(K^2 l_\beta - K_\beta K.l\right) = 0$, then contraction with respect to l^β both sides and using $l.l = 0$ implies $K.l = 0$, which contradicts the fact that K_α is not parallel to l_α. Hence $Q_{1\beta}(K,T) = 0$ implies $\Omega_{ij}T^{ij} = 0$. If we consider T_{ij} as a $(D-2)$ x $(D-2)$ matrix, then this condition is the same as $\mathrm{tr}(\Omega T) = 0$, where tr is a trace in the transverse directions. Now, we are left to solve for the characteristics equations in $(D-2)$ transverse directions. Let p and q be the directions in the null basis which are orthogonal to l^α and n^α. Then,

$$Q_{pq}(K,T) = -6K^2(K.l)^2\left(T_p{}^j\Omega_{jq} + T_q{}^j\Omega_{jp} - \frac{3}{2}T\Omega_{pq}\right) \tag{6.5.21}$$

where T is the trace T^α_α. Since $T_{\alpha\beta}$ is transverse, it can easily be shown that $T^\alpha_\alpha = 2T_{01}(K.l)^{-1}$ and $T_{ii} = 0$[6]. In order to obtain the above equation, we used the fact that $\Omega_{ij}T^{ij} = 0$. The characteristics equation $Q_{pq}(K,T) = 0$ implies

$$T_{pj}\Omega_{jq} + T_{qj}\Omega_{jp} = \frac{3}{2}T_{01}(K.l)^{-1}\Omega_{pq} \tag{6.5.22}$$

Since all the indices take values in the transverse directions, we can write the equation (6.5.22) in $(D-2)$ x $(D-2)$ matrix notation as

$$T\Omega + \Omega T = \frac{3}{2}T_{01}(K.l)^{-1}\Omega. \tag{6.5.23}$$

Ω is symmetric and therefore invertible. Multiplying both sides by Ω inverse, we get,

$$\Omega^{-1}T\Omega + T = \frac{3}{2}T_{01}(K.l)^{-1}\mathbb{1}_{D-2} \tag{6.5.24}$$

Taking trace on both sides and using the fact that $T_{ii} = 0$, we get $T_{01} = 0$. This reduces the equation (6.5.23) to

$$T\Omega + \Omega T = 0 \tag{6.5.25}$$

The above certainly has a nontrivial solution space. For example, in D = 4, one can use a similarity transformation to set $\Omega = C_1\sigma_3$, where C_1 is some constant and σ_3 is the third Pauli matrix. Then it is clear that $T = C_2\sigma_1$, where σ_1 is the first Pauli matrix, is the solution to the equation (6.5.25). Further, one can use this solution to construct solutions in higher dimensions. Therefore, we have shown that ECG has non-null characteristics and it can have superluminal propagation.

[6]Using the fact that K is a linear combination of n and l, we can write $K^\alpha = n^\alpha K.l + l^\alpha K.n$. Since, $T_{11} = 0$, it can easily be shown that $n^\alpha T_{\alpha 1} = T_{01}(K.l)^{-1}$. Further, $T = 2n^\alpha T_{\alpha 1} + T_{ii} = 2T_{01}(K.l)^{-1} + T_{ii}$. Now, using the transverse condition, $T_{01} = TK.l$, implying $T_{ii} = 0$.

6.6 Discussion

In this chapter, we have analyzed the causal structure of Generalized Quadratic Gravity (GQG) and Einsteinian Cubic Gravity (ECG). Firstly, we analyze the Riemann-squared theory and show that the theory only possesses null characteristics, independent of the background metric. This background-independent analysis is possible because the principal symbol of GQG does not depend on curvature. We further extend this result to genuinely fourth-order GQG. By "genuinely fourth order," we mean that the theory has a non-trivial fourth-order principal symbol in both the null and non-null cases. This condition excludes the Gauss-Bonnet theory. Furthermore, we demonstrate that such GQG has $\frac{(D+1)(D-2)}{2}$ polarization modes. These modes correspond to massive spin-2, massless spin-2, and scalar fields in the spectrum.
GQG is a fourth-order theory, and therefore, the metric and its derivatives may not correspond to canonically conjugate variables. Some of the metric derivatives can represent independent degrees of freedom. The best way to analyze the spectrum is to reduce the equations of motion to second order; we will address this separately elsewhere. However, since we are analyzing the principal symbol of a fourth-order partial differential equation (PDE), this is equivalent to analyzing the theory in the eikonal limit. The massive and massless polarizations get mixed, and we can only count the number of helicities allowed in the theory. Further, our results imply for black holes with Killing horizons, the Killing horizon in GQG is a characteristic surface with no polarization modes of the graviton traversing it along a spacelike path.
We have also shown that for ECG, all null surfaces are characteristic surfaces. This result is demonstrated on an arbitrary background. It does not appear possible to analyze the non-null characteristics on an arbitrary background. To address the non-null case, we consider Type N spacetimes in the Weyl classification. We show that in ECG, there exist non-null characteristic surfaces. Further using the fact that, $R_{00} = R_{i0j0} = 0$ on the Killing horizon, we showed that the null characteristic equations are satisfied for any $T_{\mu\nu}$ in the equivalence class $T_{\mu\nu} \sim T_{\mu\nu} + X_{(\mu}K_{\nu)}$, where X_μ is an arbitrary covector. This tells us that on the horizon the EoM for dynamical degrees of freedom is lower order. For ECG, it is known that the linearized EOM for gravitational perturbations has the Einstein gravity form on spacetimes which are a product $M_{D'} \times M_{D-D'}$, where $D' < D$, $M_{D'}$ and $M_{D-D'}$ are maximally symmetric spaces. We argued that the killing horizon is a characteristic surface for black holes, and therefore, no modes can escape the black hole in ECG.

6.6.1 Implications on the Local Algebra of Observables

As discussed in Section (3.5.0.1), the causal diamond, defined with respect to null lines, is used to construct the commutant of the algebra. The bicommutant theorem (3.2.1) is then invoked to define the corresponding von Neumann algebras. However, in theories exhibiting superluminal propagation, the causal structure is not encoded by null curves, since superluminal modes allow for communication outside the null cone. In such cases, one must instead use characteristic surfaces (associated with the fastest propagating modes) to define both the causal structure and the corresponding local algebras. Our results indicate that in GQG, causality in the local algebra must still be implemented via null curves, as in general relativity. We also expect that the local GSL holds for black holes in GQG. In contrast, in ECG, the presence of spacelike characteristics leads to a causal structure and thus local algebras that deviate from those in GR.

6.6.2 Holographic Implications

One way to think about higher curvature theory is as the low-energy limit of a UV-complete theory of gravity. String theory is one such UV-complete quantum theory of gravity. In string theory, the semiclassical limit involves taking $G_N \to 0$ and the string length $\alpha' = l_s^2 \to 0$. In the case of AdS/CFT, this limit corresponds to $N \to \infty$ and the 't Hooft coupling $\lambda \to \infty$. The effective theory in the bulk is Einstein gravity plus matter, meaning all bulk fields see the same metric (the metric that describes the causal structure of the theory).

We can also define a *stringy regime* where $G_N \to 0$ and α' remains finite. In this regime, there are no quantum gravitational fluctuations, but spacetime is probed by strings rather than point particles. Perturbatively in α', the stringy regime corresponds to introducing higher derivative corrections, which implies that, in principle, different bulk fields may see different metrics. On the CFT side, this corresponds to $N \to \infty$ with λ finite.

Now, let us consider stringy black holes (i.e., black holes in the stringy regime) that are classically stable. For such solutions, one can ask if all fields see the same horizon, namely the event horizon of the black hole. If that is true, we should expect only those higher curvature corrections where the black hole horizon remains a characteristic surface for all the bulk fields. Our analysis indicates that GQG and ECG have this property. As shown by Liu and Gesteau in [112], the information about the causal structure of a region in the bulk is encoded in the associated time band algebra of operators in the boundary CFT. Gesteau and Liu propose a diagnostic of the presence of a horizon in the bulk, entirely using the boundary algebra. As already stated, this could in principle, lead to different horizons for different bulk fields. It would be interesting to investigate this diagnostic in the boundary algebras and see what it predicts for stringy horizons in the bulk at least for the different polarization modes of the graviton.

Chapter 7

Conclusions and future work

This section provides a synopsis of the results presented in this thesis, which are based on the following three papers by the author: [4, 6, 19]. We summarize the main findings and outline several open questions that we hope to explore in future work. This thesis is primarily an effort by the author to understand black holes and their behavior in the semiclassical limit. In particular, we employ techniques from algebraic quantum field theory and modular theory to investigate the notion of generalized entropy in this regime.

The first three chapters introduce the problem and review the tools and techniques used in the subsequent chapters to establish a relation between generalized entropy and algebraic entropy in the semiclassical limit using the crossed product construction. Our analysis encompasses black holes in both general relativity and higher curvature theories of gravity. To better understand the latter, we examine the causal structure of higher curvature theories and discuss its implications for black hole physics, holography, and the algebra of observables.

7.1 Summary of the Results

In chapter 4, we began with the elegant result of [2, 3], which shows that in general relativity, the generalized entropy at the bifurcation surface coincides up to a state-independent additive constant with the entropy of the algebra of observables of type II crossed product. The fact that the gravitational algebra in the exterior region is of type II, where entropy is well-defined, provides a natural explanation for the UV finiteness of the generalized entropy. However, since generalized entropy is well defined on arbitrary horizon cuts, it is essential to extend this construction beyond the bifurcation surface to arbitrary slices of the horizon.

Building on the conjecture in [41] regarding the existence of specific local modular Hamiltonians—which we explicitly verified in our setting and employing the half-sided modular inclusion (4.3), we demonstrated that the generalized entropy of a static black hole at an arbitrary horizon cut is equal to the algebraic entropy up to a state-independent additive constant. Utilizing the positivity and monotonicity of the Araki relative entropy, we provide an algebraic version of the local generalized second law (GSL)in the crossed product construction. We further extended this result to the Kerr spacetime, assuming the existence of a Hadamard stationary state. A key advantage of this algebraic approach is that each step in the argument remains manifestly finite, owing to the Type II nature of the crossed-product algebra. This finiteness offers a natural renormalization scheme, needed in Wall's proof of the GSL [5], which is realized naturally in crossed-product construction. In Section (4.7), we analyzed nonlocal modular flows in a class of spacetimes and investigated whether they can be rendered local by adding operators from the algebra and its commutant. Along the way, we see that the averaged null energy condition (ANEC) also holds for null generators of the Cauchy horizon in the class of static spacetimes we have considered, which includes the Schwarzschild spacetime.

In Chapter 5, we have extended the result to black holes in arbitrary diffeomorphic invariants (not necessarily the effective field theory). In particular, we have shown that indeed for any black hole with a regular bifurcation surface and causal horizon, the generalized entropy at the bifurcation surface is equal to the entropy of the algebra of observables up to a state-independent additive constant. The natural next step is to establish this correspondence between the generalized entropy and the algebraic entropy at an arbitrary cut on the horizon. However, this is a subtle issue because the causal structure of a generic higher curvature theory differs significantly from that of general relativity. In such theories, the definition of the local algebra must be adapted to the modified causal structure, which is determined by the fastest-propagating modes—that is, by the characteristic surfaces of the theory. This motivates a deeper investigation into the characteristics of higher curvature gravity theories.

In Chapter 6, we analyzed the causal structure of Generalized Quadratic Gravity (GQG) and Einsteinian Cubic Gravity (ECG). For genuinely fourth-order GQG, we found that the theory admits only null characteristic surfaces, and this result holds independently of the background metric. Furthermore, we demonstrated that GQG features $\frac{(D+1)(D-2)}{2}$ polarization modes corresponding to massless spin-2, massive spin-2, and scalar degrees of freedom in the spectrum. In the case of ECG, we showed that all null surfaces are characteristic surfaces, a result valid on arbitrary backgrounds. Additionally, by considering type N spacetimes in the Weyl classification, we found that ECG can also admit spacelike characteristic surfaces. We argued that in both GQG and ECG, the killing horizon of a black hole acts as a characteristic surface, implying that no modes can propagate out of the black hole. We also discussed its implications on the local algebra of observables and holography.

7.2 Future directions

Here we outline some future directions that we would like to explore in the future.

- **Quantum Focusing Conjecture (QFC) in Type II Algebras:** Recently, the authors in [48] established the quantum null energy condition (QNEC) and the Bekenstein bound using algebraic quantum field theory and modular theory. It would be intriguing to formulate the quantum focusing conjecture (QFC) within the framework of type II crossed-product algebras. This framework has a key technical advantage, as entanglement entropy in type II algebras is free from UV divergences, and therefore, modular theory provides a natural setting for proving QFC.

- **Exploring the Information Loss Paradox via von Neumann Algebras:** The information loss problem remains one of the key challenges in understanding the nature of quantum gravity. It has been suggested in [113] that the language of von Neumann algebras is particularly suitable for formulating the information paradox in the $G \to 0$ limit. The authors proposed a recovery protocol for retrieving black hole information, and it would be valuable to investigate this further and understand this recovery protocol.

- **Entanglement Entropy in String Perturbation Theory and Modular Theory:** In [114], the authors defined entanglement entropy in string perturbation theory using the orbifold method. They expressed entropy as a modular-invariant series, which was shown to be finite [115]. It would be intriguing to find generalizations of the modular theory in string theory and understand the entropy in an algebraic context, which could provide valuable insights into quantum gravity and semiclassical physics. It will also be interesting to study stringy effects using the algebraic techniques as in [112].

- **Higher Curvature Theory:** There are several interesting directions for further exploration. One is to investigate the well-posedness of GQG and ECG. From the perspective of classical theory, well-posedness is a fundamental criterion for a physically viable classical theory. Linearizing these theories around nontrivial backgrounds typically introduces a linearized ghost in the spectrum. If the theory admits a well-posed initial value formulation, it would be particularly interesting to examine the role of these linearized ghosts and their implications for the full non-linear theory and its quantization. Furthermore, Deser and Tekin in [116], have proven the positive mass theorem for full non-linear quadratic gravity (theory with R^2 and $R_{\mu\nu}R^{\mu\nu}$ in action). It would be worthwhile to investigate whether a similar theorem holds in the contexts of Generalized Quadratic Gravity (GQG) and Einsteinian Cubic Gravity (ECG). Such a study could shed light on the interplay between higher-curvature corrections and gravitational stability.

Appendix

A. Minkowski wedges

The objective of this section is to establish the relationship between the modular operator of the Rindler wedge **A**, whose null boundaries intersect at the origin of the Minkowski space, and another Rindler wedge **C**, which is contained within the wedge **A** and has no overlapping null boundaries with the wedge **A**, as shown in the Figure 1. A second wedge, **B**, is introduced for computational purposes and for a subsequent section. Its null boundary overlaps with the part of the future null boundary of that of the wedge **A**. Let

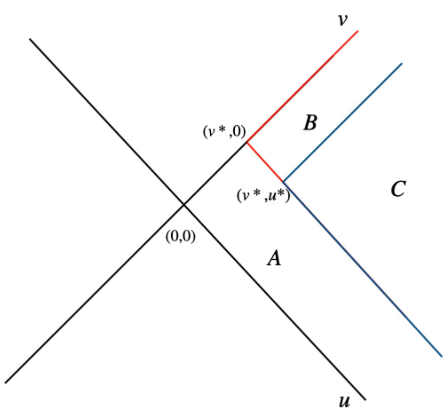


FIGURE 1: A,B and C are three Rindler wedges. A is the Rindler wedge at the centre, B is the wedge A shifted along the null coordinate v by $v*$ and C is the wedge B shifted along the null coordinate u by $u*$. The coordinates in the diagram are the null coordinates and transverse coordinates are suppressed.

$\mathcal{M}_{\mathcal{A}}$, $\mathcal{M}_{\mathcal{B}}$, and $\mathcal{M}_{\mathcal{C}}$ be the von Neumann algebras associated with the wedges **A**, **B**, and **C** correspondingly, and let these algebras act on the Hilbert space $\mathcal{H}$. In the Minkowski spacetime, the Reeh-Schlieder theorem provides us a cyclic and separating state Ω (the Minkowski vacuum) for the von Neumann algebra of any proper subregion in the spacetime. In modular theory, we may define (Δ_A, J_A), (Δ_B, J_B), and (Δ_C, J_C) as the modular operator and the modular conjugation associated with $(\mathcal{M}_{\mathcal{A}}, \Omega)$, $(\mathcal{M}_{\mathcal{B}}, \Omega)$, and $(\mathcal{M}_{\mathcal{C}}, \Omega)$, respectively.

We will obtain the relationship between the modular operator of $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{C}}$ in the following three steps:

i) We will prove that $\mathcal{M}_{\mathcal{B}}$ is the positive modular inclusion of $\mathcal{M}_{\mathcal{A}}$, and use its properties to derive the relationship between the modular Hamiltonians of $(\mathcal{M}_{\mathcal{A}}, \Omega)$ and $(\mathcal{M}_{\mathcal{B}}, \Omega)$. We will then demonstrate that Δ_B^{it} has a geometrical action on the wedge **B**.

ii) Following the same analysis as for the wedge A and B, we will obtain the relation between the modular Hamiltonian of $(\mathcal{M}_{\mathcal{B}}, \Omega)$ and $(\mathcal{M}_{\mathcal{C}}, \Omega)$, showing that $\mathcal{M}_{\mathcal{C}}$ is the negative half-sided modular inclusion of $\mathcal{M}_{\mathcal{B}}$.

iii) Using the previously obtained relation, we will obtain the relation between the modular Hamiltonian of $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{C}}$.

We have already defined modular inclusions in the main body of the thesis, following [42].

Claim: $\mathcal{M}_{\mathcal{B}}$ is the positive modular inclusion of $(\mathcal{M}_{\mathcal{A}}, \Omega)$.

Note that $\mathcal{M}_{\mathcal{B}} \subset \mathcal{M}_{\mathcal{A}}$. As previously stated, Ω is cyclic and separating for $\mathcal{M}_{\mathcal{B}}$. According to the Bisognano Wichmann theorem [68], Δ_A^{it} is the boost flow in the forward direction for the wedge A when $t \leq 0$. Thus, Δ_A^{it} has a geometrical action on the operators in $\mathcal{M}_{\mathcal{B}}$, i.e. it moves the operators along integral curves of the boost Killing field as shown in Figure 2. Because the boost is null on the Rindler horizon of wedge A and timelike inside, the forward boost cannot take the local operator in $\mathcal{M}_{\mathcal{B}}$ outside it. Therefore, $\Delta_A^{it}\mathcal{M}_{\mathcal{B}}\Delta_A^{-it} \subset \mathcal{M}_{\mathcal{B}}$ when $t \leq 0$. According to the definition of positive half-sided modular inclusion, $\mathcal{M}_{\mathcal{B}}$ is a positive half-sided modular inclusion of $(\mathcal{M}_{\mathcal{A}}, \Omega)$.

For a more detailed study, let the vertices of the wedge A and B be separated by $v*$ along the null direction v, as shown in Figure 2. Now, according to the results on modular inclusions, there exists a unitary $U(t)$ such that

$$\Delta_A^{-it}\Delta_B^{it} = U(e^{2\pi t} - 1) \tag{.0.1}$$

where $U(t) = \exp[i\mathcal{E}_{v*}t]$ and $\mathcal{E}_{v*}$ is a positive operator. $U(t)$ can be thought of as an

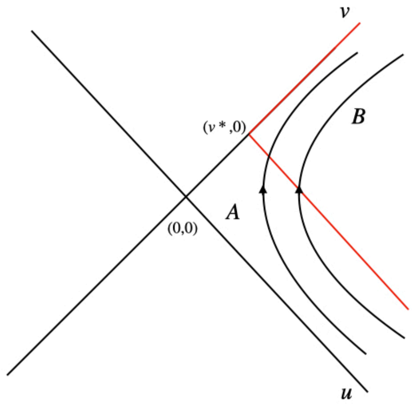


FIGURE 2: The figure represents boost integral curves, which also represent the modular flow in the wedge A.

operator that translates the wedge B in null direction v. Further, $\mathcal{E}_{v*}$ can be written as $v*$ times generator of null translation along the v. We can write the above equation as

$$\exp[-it\log[\Delta_A]]\exp[it\log[\Delta_B]] = \exp\left[i(e^{2\pi t}-1)\mathcal{E}_{v*}\right] \tag{.0.2}$$

Now, differentiating the above equation with respect to t and evaluating it at $t=0$ gives

$$\log[\Delta_A] - \log[\Delta_B] = -2\pi\mathcal{E}_{v*} \tag{.0.3}$$

Now, if we can define the modular Hamiltonian as $K = -\log[\Delta]$, then

$$K_B = K_A - 2\pi\mathcal{E}_{v*}. \tag{.0.4}$$

We want to emphasize that the above result is true even if $v*$ depends on the transverse coordinate. However, if $v*$ depends on the transverse coordinate, the modular flow generated by it will not have a local action on the wedge B but it will have local action along the null boundary (horizon) associated with the wedge. Nevertheless, for $v*$ independent of the transverse coordinate, the modular flow is local and that is what we will assume for rest of this section.

Claim: K_B is a boost generator associated with wedge B.

To show that K_B is a boost generator associated with wedge B, we use the fact that we can use Theorem 1. Differentiating the condition (h) in the theorem 1 first with respect to t and evaluating at $t = 0$ and then with respect to s and evaluating it at $s = 0$, we will get

$$[iK_A, i\mathcal{E}_{v*}] = 2\pi i\mathcal{E}_{v*}. \tag{.0.5}$$

From (.0.2), we can write

$$\exp[-itK_B] = \exp[-itK_A]\exp\left[i(e^{2\pi t} - 1)\mathcal{E}_{v*}\right] \tag{.0.6}$$

Now there is a well-known theorem which we will just use here.
Theorem 2: If $[X, Y] = sY$, where $s \in \mathbb{C}$ and $s \neq 2\pi in$ then

$$\exp[Y]\exp[-X]\exp[-Y] = \exp[-X]\exp[(\exp[s] - 1)Y] \tag{.0.7}$$

Now choose $X = iK_At$ and $Y = i\mathcal{E}_{v*}$. Then using (.0.5), one can identify $s = 2\pi t$. Since $t \in \mathbb{R}$, we can apply the theorem. This gives

$$\exp[-itK_A]\exp\left[i(e^{2\pi t} - 1)\mathcal{E}_{v*}\right] = \exp[i\mathcal{E}_{v*}]\exp[-iK_At]\exp[-i\mathcal{E}_{v*}] \tag{.0.8}$$

Now putting (.0.8) back in (.0.6), we get

$$\Delta_B^{it} = U(1)\Delta_A^{it}U(-1) \tag{.0.9}$$

So this is a null translated boost, which can still be thought of as a boost but this time associated with the wedge B. Furthermore, (.0.4) and (.0.9) establish its local and geometrical nature. There is another way to get (.0.9), because $\mathcal{M}_\mathcal{B} = U(1)\mathcal{M}_\mathcal{A}U(-1)$, and U is an Ω preserving unitary. It is straightforward to verify that the Tomita operator for B is $S_B = U(1)S_AU(-1)$.

$$U(1)S_AU(-1)\Big(U(1)aU(-1)\Big)\Omega = U(1)a^\dagger U(-1)\Omega \tag{.0.10}$$

For each $a \in \mathcal{M}_\mathcal{A}$, $U(1)aU(-1) \in \mathcal{M}_\mathcal{B}$. Now, using the definition of the modular operator $\Delta_B = S_B^\dagger S_B = U(1)\Delta_AU(-1)$. Further, using the spectral theorem for operators and the fact that $U(1)$ is unitary, we will get the equation in (.0.9).

We can now do the same with wedges B and C. As we already know, Ω is cyclic and separating for $\mathcal{M}_\mathcal{C}$. Δ_B^{it} is the boost associated with the wedge B and the past Rindler horizon of the wedge C overlaps with the portion of the past horizon of the wedge B, as shown in Figure 3. For $t \geq 0$, the boost Δ_B^{it} maps the wedge C into itself. Thus, $\mathcal{M}_\mathcal{C}$ is a

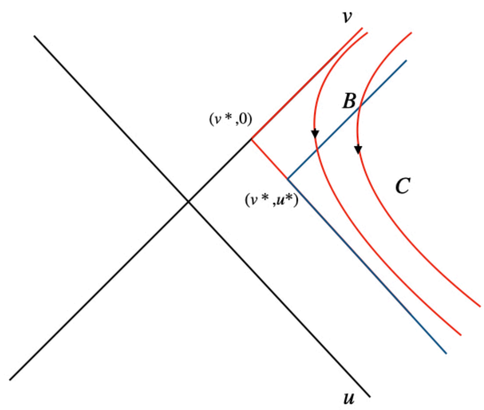


FIGURE 3: The figure represents boost integral curves associated with the wedge B which also represents the modular flow in the wedge B.

negative inclusion of $\mathcal{M}_{\mathcal{B}}$. Following the steps of previous analysis, using Theorem 1 we can write

$$\Delta_B^{-it}\Delta_C^{it} = V(1-e^{-2\pi t}) \tag{.0.11}$$

where $V(t) = \exp[i\mathcal{E}_{u*}t]$ and $\mathcal{E}_{u*}$ is a positive operator. $V(t)$ can be thought of as the operator that translates wedge B in the null direction u. Further $\mathcal{E}_{u*}$ can be written as $u*$ times generator of null translation along u. We obtain

$$K_C = K_B - 2\pi\mathcal{E}_{u*} \tag{.0.12}$$

and

$$\Delta_C^{it} = V(-1)\Delta_B^{it}V(1). \tag{.0.13}$$

The modular Hamiltonians of the algebra and the algebra related by the modular inclusion will differ by the generator of a one-parameter unitary group. If the algebras are wedge algebras with inclusion as a null translated wedge, then the generator that connects the two modular Hamiltonians is a null translation generator. Now we may express the modular Hamiltonian of the wedge C in terms of the modular Hamiltonian of A. Using (.0.12 and (.0.4), we obtain

$$K_C = K_A - 2\pi\mathcal{E}_{v*} - 2\pi\mathcal{E}_{u*}. \tag{.0.14}$$

Further using (.0.13) and (.0.9), we can write the modular flow

$$\Delta_C^{it} = V(-1)U(1)\Delta_A^{it}U(-1)V(1). \qquad (.0.15)$$

Since translations in Minkowski spacetime commute, we can define the unitary $W(s;a,b) = exp[2\pi i(a\mathcal{E}_{v*} + b\mathcal{E}_{u*})s]$ and the equation can be written as

$$\Delta_C^{it} = W(1;1,-1)\Delta_A^{it}W(-1;1,-1) \qquad (.0.16)$$

This is true for any $v*$ and $u*$; therefore, we have obtained a relation between the modular Hamiltonian of any wedge that can be reached via a series of null translations and the wedge at the origin. It is crucial to note that because null translation is a global isometry of spacetime, the resulting modular Hamiltonians for B and C are conserved and may be represented as a local integral on the Cauchy surface. This is the simplest example of a local modular Hamiltonian. The result (.0.4) is valid even if $v*$ depends on the transverse coordinate. The only difference is that the null translation that maps two wedges would not be a symmetry, and therefore the resultant modular Hamiltonian may not have local action everywhere inside the wedge B. But it will be local on the horizon, since the null translation depending on the transverse coordinate is still a symmetry on the horizon.

B. Quantum Canonical energy in covariant phase space formalism

As we are working with quantum fields, one might want to check whether IW relation [82] is true in the expectation value. The way to obtain the IW relation in expectation value is to write Schwinger Dyson's equation in some state for the quantum fields (including gravitons) about a static black hole spacetime and use diffeomorphism invariance of the Lagrangian. Using diffeomorphism invariance of the Lagrangian, we can write

$$\frac{\delta L}{\delta \phi}\mathcal{L}_\xi \phi = -dJ - \frac{\delta L}{\delta g}\mathcal{L}_\xi g \qquad (.0.17)$$

where J is same as in equation (49) in [82], ϕ here corresponds to all quantum fields and g is back ground metric. Since we want to compute the expectation value of the above equation in some state $|\Phi\rangle$ [117]. We will do that following the point split prescription as described in [117],

$$\lim_{y\to x} \langle\Phi| \frac{\delta L}{\delta \phi}(x)\mathcal{L}_\xi \phi(y) |\Phi\rangle = \lim_{y\to x} \langle\Phi| \Big(- dJ(x,y) - \frac{\delta L}{\delta g}(x)\mathcal{L}_\xi g(y)\Big) |\Phi\rangle \tag{.0.18}$$

From the Schwinger Dyson's equation, we know the left-hand side is zero up to a state-independent divergent term. Therefore if we consider the difference of the quantity $\Big(- dJ(x,y) - \frac{\delta L}{\delta g}(x)\mathcal{L}_\xi g(y)\Big)$ in any two state $|\Phi\rangle$ and $|\Psi\rangle$, then state independent divergent term will cancel out. Now we can take the coincident limit of $y \to x$.

$$\langle\Phi| \Big(- dJ(x) - \frac{\delta L}{\delta g}(x)\mathcal{L}_\xi g(x)\Big) |\Phi\rangle - \langle\Psi| \Big(- dJ(x) - \frac{\delta L}{\delta g}(x)\mathcal{L}_\xi g(x)\Big) |\Psi\rangle = 0 \tag{.0.19}$$

Now the above equation can be written as,

$$d\langle\Phi| (J + k.\epsilon) |\Phi\rangle - d\langle\Psi| (J + k.\epsilon) |\Psi\rangle = \nabla_\mu(\langle\Phi| T^{\mu\nu} |\Phi\rangle)\xi_\nu - \nabla_\mu(\langle\Psi| T^{\mu\nu} |\Psi\rangle)\xi_\nu \tag{.0.20}$$

above equation is obtained using the fact that $\frac{\delta L}{\delta g_{\mu\nu}} = T^{\mu\nu}\epsilon$, where ϵ is volume form and $k^\mu = T^{\mu\nu}\xi_\nu$. Notice that the left-hand side in the above equation is the total derivative, while the right-hand side is not. The only way this can happen is when $\nabla_\mu(\langle\Phi| T^{\mu\nu} |\Phi\rangle)\xi_\nu = \nabla_\mu(\langle\Psi| T^{\mu\nu} |\Psi\rangle)\xi_\nu$. Since this has to be true for any two states and any vector field ξ^μ, it can only be if $\nabla_\mu(\langle\Phi| T^{\mu\nu} |\Phi\rangle)$ vanishes for any state $|\Phi\rangle$ to a local term independent of state. We may modify our prescription to eliminate this extra state-independent term by performing background subtraction[117]. The same argument then leads to $d\langle\Phi| (J + k.\epsilon) |\Phi\rangle = 0$. Now you choose ξ^μ to be the killing field of the background spacetime (and we have killing fields since the background is static). Following IW [82], we will get

$$\langle\mathcal{E}\rangle_\Phi = \int_\sigma d\Sigma^{mu} \langle T_{\mu\nu}\rangle_\Phi \xi^\nu + \text{Surface term} \tag{.0.21}$$

where $\mathcal{E}$ is known as the canonical energy in the covariant phase space formalism, and it is independent of the choice of Cauchy slice.

Bibliography

[1] E. Witten, "Gravity and the crossed product," *JHEP*, vol. 10, p. 008, 2022.

[2] V. Chandrasekaran, R. Longo, G. Penington, and E. Witten, "An algebra of observables for de Sitter space," *JHEP*, vol. 02, p. 082, 2023.

[3] V. Chandrasekaran, G. Penington, and E. Witten, "Large N algebras and generalized entropy," *JHEP*, vol. 04, p. 009, 2023.

[4] M. Ali and V. Suneeta, "Local generalized second law in crossed product constructions," *Phys. Rev. D*, vol. 111, no. 2, p. 024015, 2025.

[5] A. C. Wall, "A proof of the generalized second law for rapidly changing fields and arbitrary horizon slices," *Phys. Rev. D*, vol. 85, p. 104049, 2012. [Erratum: Phys.Rev.D 87, 069904 (2013)].

[6] M. Ali and V. Suneeta, "Generalized entropy in higher curvature gravity and entropy of algebra of observables," *Phys. Rev. D*, vol. 108, no. 6, p. 066017, 2023.

[7] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras.* Theoretical and Mathematical Physics, Berlin, Heidelberg: Springer-Verlag, 2 ed., 1992.

[8] R. Haag and D. Kastler, "An Algebraic approach to quantum field theory," *J. Math. Phys.*, vol. 5, pp. 848–861, 1964.

[9] H. Araki, *Mathematical Theory of Quantum Fields*, vol. 101 of *International Series of Monographs on Physics.* Oxford: Oxford University Press, 1999.

[10] H. Araki, "Type of von neumann algebra associated with free field," *Progress of Theoretical Physics*, vol. 32, pp. 956–965, 12 1964.

[11] R. M. Wald, *Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics.* Chicago Lectures in Physics, Chicago, IL: University of Chicago Press, 1995.

[12] S. Hollands and R. M. Wald, "Quantum fields in curved spacetime," *Phys. Rept.*, vol. 574, pp. 1–35, 2015.

[13] S. Leutheusser and H. Liu, "Causal connectability between quantum systems and the black hole interior in holographic duality," *Phys. Rev. D*, vol. 108, no. 8, p. 086019, 2023.

[14] S. A. W. Leutheusser and H. Liu, "Emergent Times in Holographic Duality," *Phys. Rev. D*, vol. 108, no. 8, p. 086020, 2023.

[15] J. D. Bekenstein, "Black holes and entropy," *Phys. Rev. D*, vol. 7, pp. 2333–2346, 1973.

[16] J. D. Bekenstein, "Black holes and the second law," *Lett. Nuovo Cim.*, vol. 4, pp. 737–740, 1972.

[17] R. D. Sorkin, "On the entropy of the vacuum outside a horizon," in *Tenth International Conference on General Relativity and Gravitation, Contributed Papers, Vol. II* (B. Bertotti, F. de Felice, and A. Pascolini, eds.), (Roma), pp. 734–736, Consiglio Nazionale Delle Ricerche, 1983. Also available at `https://arxiv.org/abs/1402.3589`.

[18] L. Susskind and J. Uglum, "Black hole entropy in canonical quantum gravity and superstring theory," *Phys. Rev. D*, vol. 50, pp. 2700–2711, 1994.

[19] M. Ali and V. Suneeta, "Causal structure of higher curvature gravity," *arXiv preprint*, Feb. 2025. arXiv:2502.16527 [hep-th].

[20] B. C. Hall, *Quantum Theory for Mathematicians.* Springer, 2013.

[21] W. Rudin, *Functional Analysis.* McGraw-Hill, 2nd ed., 1991.

[22] S. Strătilă and L. Zsidó, *Lectures on von Neumann Algebras*, vol. 173 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, 2019.

[23] J. Sorce, "An intuitive construction of modular flow," *JHEP*, vol. 12, p. 079, 2023.

[24] E. Witten, "APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory," *Rev. Mod. Phys.*, vol. 90, no. 4, p. 045003, 2018.

[25] J. Sorce, "Notes on the type classification of von Neumann algebras," *Rev. Math. Phys.*, vol. 36, no. 02, p. 2430002, 2024.

[26] V. Moretti, *Fundamental Mathematical Structures of Quantum Theory: Spectral Theory, Foundational Issues, Symmetries, Algebraic Formulation.* Cham: Springer International Publishing, 2019.

[27] L. Schwartz, "Généralisation de la notion de fonction, de dérivation, de transformation de fourier et applications mathématiques et physiques," *Annales de l'Université de Grenoble*, vol. 21, pp. 57–74, 1945.

[28] L. Schwartz, "Théorie des noyaux," in *Proceedings of the International Congress of Mathematicians*, p. 220–230, 1950.

[29] V. S. Vladimirov, *Methods of the Theory of Generalized Functions.* Taylor & Francis, 2002.

[30] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Volume 1: Functional Analysis.* Academic Press, 1972.

[31] N. N. Bogolyubov, A. A. Logunov, and I. T. Todorov, *Introduction to Axiomatic Quantum Field Theory.* Benjamin-Cummings Publishing Company, 1975.

[32] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Volume 2: Fourier Analysis, Self-Adjointness.* Academic Press, 1975.

[33] H. Araki, "A lattice of von neumann algebras associated with the quantum theory of a free bose field," *Journal of Mathematical Physics*, vol. 4, no. 10, pp. 1343–1362, 1963.

[34] R. Haag, "Bemerkungen zum Nahwirkungsprinzip in der Quantenphysik," *Annalen der Physik*, vol. 11, pp. 29–34, 1963.

[35] J. J. Bisognano and E. H. Wichmann, "On the Duality Condition for Quantum Fields," *J. Math. Phys.*, vol. 17, pp. 303–321, 1976.

[36] B. Schroer, "A course on: 'an algebraic approach to nonperturbative quantum field theory'," *Lecture Notes in Physics*, vol. 1234, pp. 1–25, May 1998. Lecture notes, unpublished.

[37] H. Casini, M. Huerta, J. M. Magán, and D. Pontello, "Entanglement entropy and superselection sectors. Part I. Global symmetries," *JHEP*, vol. 02, p. 014, 2020.

[38] S. Doplicher, R. Haag, and J. E. Roberts, "Local observables and particle statistics i," *Communications in Mathematical Physics*, vol. 23, no. 3, pp. 199–230, 1971.

[39] S. Doplicher, R. Haag, and J. E. Roberts, "Local observables and particle statistics ii," *Communications in Mathematical Physics*, vol. 35, no. 1, pp. 49–85, 1974.

[40] S. Doplicher and J. E. Roberts, "Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics," *Communications in Mathematical Physics*, vol. 131, no. 1, pp. 51–107, 1990.

[41] K. Jensen, J. Sorce, and A. J. Speranza, "Generalized entropy for general subregions in quantum gravity," *JHEP*, vol. 12, p. 020, 2023.

[42] H.-J. Borchers, "On revolutionizing quantum field theory with tomita's modular theory," *Journal of Mathematical Physics*, vol. 41, no. 6, pp. 3604–3673, 2000.

[43] M. Takesaki, *Theory of Operator Algebras II*, vol. 125 of *Encyclopaedia of Mathematical Sciences*. Berlin: Springer, 2003.

[44] J. Sorce, "A short proof of tomita's theorem," *Journal of Functional Analysis*, vol. 286, no. 12, p. 110420, 2024.

[45] H. Araki, "Inequalities in von neumann algebras," *Publ. Res. Inst. Math. Sci. Kyoto*, vol. 11, no. 3, pp. 809–833, 1975.

[46] N. Lashkari, "Modular zero modes and sewing the states of QFT," *JHEP*, vol. 21, p. 189, 2020.

[47] H. Casini, "Relative entropy and the Bekenstein bound," *Class. Quant. Grav.*, vol. 25, p. 205021, 2008.

[48] J. Kudler-Flam, S. Leutheusser, A. A. Rahman, G. Satishchandran, and A. J. Speranza, "A covariant regulator for entanglement entropy: Proofs of the bekenstein bound and qnec," *Journal of High Energy Physics*, Dec. 2023. To appear.

[49] A. S. Wightman, "Quantum field theory in terms of vacuum expectation values," *Phys. Rev.*, vol. 101, pp. 860–866, Jan 1956.

[50] M. Takesaki, *Theory of Operator Algebras II*, vol. 125 of *Encyclopaedia of Mathematical Sciences*. Springer, 2003.

[51] A. Connes, "Une classification des facteurs de type iii," *Annales scientifiques de l'École Normale Supérieure*, vol. 6, no. 2, pp. 133–252, 1973.

[52] A. van Daele, *Continuous Crossed Products and Type III Von Neumann Algebras*, vol. 31 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1978.

[53] M. Takesaki, "Duality for crossed products and the structure of von neumann algebras of type iii," *Acta Mathematica*, vol. 131, pp. 249–310, 1973.

[54] T. Faulkner and A. J. Speranza, "Gravitational algebras and the generalized second law," *JHEP*, vol. 11, p. 099, 2024.

[55] J. Kudler-Flam, S. Leutheusser, A. A. Rahman, G. Satishchandran, and A. J. Speranza, "A covariant regulator for entanglement entropy: Proofs of the bekenstein bound and qnec," *Physical Review D*, vol. 108, no. 1, p. 014022, 2023.

[56] J. D. Bekenstein, "Black holes and entropy," *Physical Review D*, vol. 7, pp. 2333–2346, 1973.

[57] J. D. Bekenstein, "Black holes and the second law," *Lettere al Nuovo Cimento*, vol. 4, pp. 737–740, 1972.

[58] E. Gesteau, "Large n von neumann algebras and the renormalization of newton's constant," *Communications in Mathematical Physics*, vol. 406, no. 1, p. 40, 2025.

[59] K. Furuya, N. Lashkari, M. Moosa, and S. Ouseph, "Information loss, mixing, and emergent type iii$_1$ factors," *Journal of High Energy Physics*, vol. 2023, no. 8, p. 111, 2023.

[60] J. Kudler-Flam, S. Leutheusser, and G. Satishchandran, "Generalized black hole entropy is von neumann entropy," *Physical Review D*, vol. 111, no. 2, p. 025013, 2025.

[61] S. E. Aguilar-Gutierrez, E. Bahiru, and R. Espindola, "The centaur-algebra of observables," *JHEP*, vol. 03, p. 008, 2024.

[62] M. S. Klinger and R. G. Leigh, "Crossed products, conditional expectations, and constraint quantization," *Journal of High Energy Physics*, vol. 2023, no. 12, p. 67, 2023.

[63] M. S. Klinger and R. G. Leigh, "Crossed products, extended phase spaces and the resolution of entanglement singularities," *Nucl. Phys. B*, vol. 999, p. 116453, 2024.

[64] S. A. Ahmad and R. Jefferson, "Crossed product algebras and generalized entropy for subregions," *Physical Review D*, vol. 107, no. 12, p. 124042, 2023.

[65] R. M. Soni, "A type i approximation of the crossed product," *JHEP*, vol. 01, p. 123, 2024.

[66] H. Araki, "Relative entropy of states of von neumann algebras," *Publ. RIMS, Kyoto Univ.*, vol. 11, p. 809, 1976.

[67] E. Witten, "Why does quantum field theory in curved spacetime make sense? and what happens to the algebra of observables in the thermodynamic limit?," *arXiv*, 2021.

[68] J. Bisognano and E. Wichmann, "On the duality condition for a hermitian scalar field," *J. Math. Phys.*, vol. 16, pp. 985–1007, 1975.

[69] G. L. Sewell, "Quantum fields on manifolds: Pct and gravitationally induced thermal states," *Annals Phys.*, vol. 141, pp. 201–224, 1982.

[70] B. S. Kay and R. M. Wald, "Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on space-times with a bifurcate killing horizon," *Phys. Rept.*, vol. 207, pp. 49–136, 1991.

[71] A. Strohmaier, "The reeh-schlieder property for quantum fields on stationary space-times," *Commun. Math. Phys.*, vol. 215, pp. 105–118, 2000.

[72] A. Strohmaier, R. Verch, and M. Wollenberg, "Microlocal analysis of quantum fields on curved spacetimes: Analytic wavefront sets and reeh-schlieder theorems," *J. Math. Phys.*, vol. 43, pp. 5514–5530, 2002.

[73] W. G. Unruh, "Notes on black-hole evaporation," *Phys. Rev. D*, vol. 14, p. 870, 1976.

[74] R. M. Wald, *Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics.* Chicago Lectures in Physics, 1994. Available at: InspireHep.

[75] C. Gomez, "Entanglement, observers and cosmology: A view from von neumann algebras," *Journal of High Energy Physics*, vol. 45, pp. 122–145, 2023. Available at: arXiv:2302.14747.

[76] D. L. Jafferis, A. Lewkowycz, J. Maldacena, and S. J. Suh, "Relative entropy equals bulk relative entropy," *JHEP*, vol. 06, p. 004, 2016.

[77] T. Faulkner, R. G. Leigh, O. Parrikar, and H. Wang, "Modular hamiltonians for deformed half-spaces and the averaged null energy condition," *JHEP*, vol. 09, p. 038, 2016.

[78] S. Balakrishnan and O. Parrikar, "Modular hamiltonians for euclidean path integral states," *Journal of High Energy Physics*, 2020.

[79] F. Rosso, "Achronal averaged null energy condition for extremal horizons and (a)ds," *JHEP*, vol. 07, p. 023, 2020.

[80] G. Wong, I. Klich, L. A. P. Zayas, and D. Vaman, "Entanglement temperature and entanglement entropy of excited states," *JHEP*, vol. 12, p. 020, 2013.

[81] R. M. Wald, "Black hole entropy is noether charge," *Phys. Rev. D*, vol. 48, pp. 3427–3431, 1993.

[82] V. Iyer and R. M. Wald, "Some properties of noether charge and a proposal for dynamical black hole entropy," *Phys. Rev. D*, vol. 50, pp. 846–864, 1994.

[83] T. Jacobson, G. Kang, and R. C. Myers, "On black hole entropy," *Phys. Rev. D*, vol. 49, p. 6587, 1994.

[84] S. Sarkar and A. C. Wall, "Generalized second law at linear order for actions that are functions of lovelock densities," *Phys. Rev. D*, vol. 88, p. 044017, 2013.

[85] A. Chatterjee and S. Sarkar, "Physical process first law and increase of horizon entropy for black holes in einstein-gauss-bonnet gravity," *Phys. Rev. Lett.*, vol. 108, p. 091301, 2012.

[86] A. Wall, "A second law for higher curvature gravity," *Int. J. Mod. Phys. D*, vol. 24, p. 1544014, 2015.

[87] S. Bhattacharyya, P. Dhivakar, A. Dinda, N. Kundu, M. Patra, and S. Roy, "An entropy current and the second law in higher derivative theories of gravity," *JHEP*, vol. 09, p. 169, 2021.

[88] J. Bhattacharya, S. Bhattacharyya, A. Dinda, and N. Kundu, "An entropy current for dynamical black holes in four-derivative theories of gravity," *JHEP*, vol. 06, p. 017, 2020.

[89] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation.* W. H. Freeman and Company, 1973.

[90] A. Mishra, S. Chakraborty, A. Ghosh, and S. Sarkar, "On the physical process first law for dynamical black holes," *JHEP*, vol. 09, p. 034, 2018.

[91] X. Dong, "Holographic entanglement entropy for general higher derivative gravity," *JHEP*, vol. 01, p. 044, 2014.

[92] R. Longo and E. Witten, "A note on continuous entropy," *arXiv*, 2022.

[93] C. M. Will, "The confrontation between general relativity and experiment," *Living Rev. Rel.*, vol. 17, no. 4, 2014.

[94] H. Reall, N. Tanahashi, and B. Way, "Causality and hyperbolicity of lovelock theories," *Class. Quant. Grav.*, vol. 31, p. 205005, 2014.

[95] N. Tanahashi, H. S. Reall, and B. Way, "Causality, hyperbolicity and shock formation in lovelock theories," *Journal of High Energy Physics*, 2017. To appear.

[96] K. Izumi, "Causal structures in gauss-bonnet gravity," *Phys. Rev. D*, vol. 90, no. 4, p. 044037, 2014.

[97] S. M. C. Deffayet, A. Held and A. Vikman, "Global and local stability for ghosts coupled to positive energy degrees of freedom," *JCAP*, vol. 11, p. 031, 2023.

[98] A. Held and H. Lim, "Nonlinear evolution of quadratic gravity in 3+1 dimensions," *Phys. Rev. D*, vol. 108, no. 10, p. 104025, 2023.

[99] D. R. Noakes, "The initial value formulation of higher derivative gravity," *J. Math. Phys.*, vol. 24, pp. 1846–1850, 1983.

[100] J. M. X. O. Camanho, J. D. Edelstein and A. Zhiboedov, "Causality constraints on corrections to the graviton three-point coupling," *JHEP*, vol. 02, p. 020, 2016.

[101] A. L. J. D. Edelstein, R. Ghosh and S. Sarkar, "Causality constraints in quadratic gravity," *JHEP*, vol. 09, p. 150, 2021.

[102] J. D. Edelstein, R. Ghosh, A. Laddha, and S. Sarkar, "Restoring causality in higher curvature gravity," *Journal of High Energy Physics*, 2024.

[103] H. S. Reall, "Causality in gravitational theories with second order equations of motion," *Phys. Rev. D*, vol. 103, no. 8, p. 084027, 2021.

[104] L. D. K. Benakli, S. Chapman and Y. Oz, "Superluminal graviton propagation," *Phys. Rev. D*, vol. 94, no. 8, p. 084026, 2016.

[105] P. Bueno and P. A. Cano, "Einsteinian cubic gravity," *Phys. Rev. D*, vol. 94, no. 10, p. 104005, 2016.

[106] R. A. Hennigar and R. B. Mann, "Black holes in einsteinian cubic gravity," *Phys. Rev. D*, vol. 95, no. 6, p. 064055, 2017.

[107] A. D. Felice and S. Tsujikawa, "Excluding static and spherically symmetric black holes in einsteinian cubic gravity with unsuppressed higher-order curvature terms," *Phys. Lett. B*, vol. 843, p. 138047, 2023.

[108] P. A. C. P. Bueno and R. A. Hennigar, "On the stability of einsteinian cubic gravity black holes in eft," *Class. Quant. Grav.*, vol. 41, no. 13, p. 137001, 2024.

[109] P. A. C. M.-N. nirola, "Higher-curvature gravity, black holes and holography."

[110] V. Pravda, "On the algebraic classification of spacetimes," *J. Phys. Conf. Ser.*, vol. 33, pp. 463–468, 2006.

[111] A. P. M. Durkee, V. Pravda and H. S. Reall, "Generalization of the geroch-held-penrose formalism to higher dimensions," *Class. Quant. Grav.*, vol. 27, p. 215010, 2010.

[112] E. Gesteau and H. Liu, "Toward stringy horizons."

[113] J. van der Heijden and E. Verlinde, "An operator algebraic approach to black hole information," 2024. `https://arxiv.org/abs/2408.00071`.

[114] A. Dabholkar, "Quantum entanglement in string theory," 2022. `https://arxiv.org/pdf/2207.03624`.

[115] A. Dabholkar and U. Moitra, "Finite entanglement entropy in string theory," *Phys. Rev. D*, vol. 109, no. 12, 2024. `https://arxiv.org/abs/2306.00990`.

[116] S. Deser and B. Tekin, "Energy in generic higher curvature gravity theories," *Phys. Rev. D*, vol. 67, p. 084009, 2003.

[117] R. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics.* University of Chicago Press, 1994. Chicago Lectures in Physics.

[118] F. P. Schuller, "Lectures on quantum theory," 2019. Lecture notes, available at `https://www.youtube.com/playlist?list=PLPH7f_7ZlzxTi6kS4vCmv4ZKm9u8g5yic`.

[119] C. J. Fewster and K. Rejzner, "Algebraic quantum field theory – an introduction," in *Proceedings of the Fourteenth Marcel Grossmann Meeting on General Relativity* (M. Bianchi, R. T. Jantzen, and R. Ruffini, eds.), pp. 2113–2122, World Scientific, 2017.

[120] T. Jacobson, "Black hole entropy and induced gravity," *arXiv preprint gr-qc/9404039*, 1994.

[121] F. Larsen and F. Wilczek, "Renormalization of black hole entropy and of the gravitational coupling constant," *Nuclear Physics B*, vol. 458, pp. 249–266, 1996.

[122] V. P. Frolov, D. Fursaev, and A. Zelnikov, "Statistical origin of black hole entropy in induced gravity," *Nuclear Physics B*, vol. 486, p. 339, 1997.

[123] S. N. Solodukhin, "Entanglement entropy of black holes," *Living Reviews in Relativity*, vol. 14, p. 8, 2011.

[124] J. H. Cooperman and M. A. Luty, "Renormalization of entanglement entropy and the gravitational effective action," *Journal of High Energy Physics (JHEP)*, vol. 12, p. 045, 2014.

[125] R. Bousso, Z. Fisher, S. Leichenauer, and A. C. Wall, "Quantum focusing conjecture," *Physical Review D*, vol. 93, p. 064044, 2016.

[126] H. Araki, "Von neumann algebras of local observables for free scalar field," *J. Math. Phys.*, vol. 5, pp. 1–13, 1964.

[127] K. S. Stelle, "Classical gravity with higher derivatives," *Gen. Rel. Grav.*, vol. 9, pp. 353–371, 1978.

[128] K. S. Stelle, "Renormalization of higher derivative quantum gravity," *Phys. Rev. D*, vol. 16, pp. 953–969, 1977.