Logarithmic Connections and their Residues

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

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May, 2026

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Certificate

This is to certify that this dissertation entitled Logarithmic Connections and their Residues towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Sanchit Nayak at the Indian Institute of Science Education and Research Pune under the supervision of Prof. Mainak Poddar, Professor, Department of Mathematics, during the academic year 2025-2026.

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This thesis is dedicated to my parents

Declaration

I hereby declare that the matter embodied in the report entitled Logarithmic Connections and their Residues are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Prof. Mainak Poddar and the same has not been submitted elsewhere for any other degree.

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Acknowledgments

My thanks are due, first and foremost, to Prof. Mainak Poddar for introducing me to the beautiful subject of complex geometry and for guiding me through this thesis. Working on it has been a wonderful experience and has motivated me to continue pursuing my interest in geometry. I am also grateful to Prof. Vivek Mohan Mallick for his enthusiasm for my thesis and his support throughout the year.

My thanks are also due to the members of Mainak's research group – Eshwar, Abir, Sandip, Prashant and others. The weekly seminars and many engaging discussions have helped cultivate my interest and appreciation for geometry as a whole.

Of course, I must mention my family – my parents, Bharati and Suresh Nayak, my grandmother Sumitra Nayak, and my cousin brother Kishan Prabhakar – without whose constant support, academic or otherwise, I would not be where I am today.

Last, but definitely not least, I thank my dear friends – Mukta, Vishnu, Anay, Christopher, Madhukar, Rajat, Clifford, my frisbee and football teammates, and many others whom I have not been able to name – for being a part of my life, and for their companionship throughout my journey at IISER Pune.

Abstract

A central theme in geometry is the construction of characteristic classes of vector bundles from geometric data. This has its roots in Chern-Weil theory, where smooth connections and their curvature are used to construct representatives of these classes. An analogous theory exists in the complex analytic category, whose study was first initiated by Michael Atiyah in [1].

In his 1982 paper [16], Makoto Ohtsuki showed that characteristic classes can also be described using residues. Specifically, given a holomorphic vector bundle E over a compact complex manifold M and a logarithmic connection D on E with pole along a normally crossing divisor Z, he showed the existence of a formula relating the Chern classes of E with the residues of D.

In this thesis, we present a mild generalisation of the notion of residue to holomorphic principal bundles over complex manifolds. We also show that Ohtsuki's formula generalises, under reasonable assumptions on the structure group, to principal bundles over compact complex manifolds.

Contents

Introduction

As suggested by the title, the subject of this thesis is the study of the residues of logarithmic connections on vector and principal bundles over complex manifolds. We aim to provide a fairly comprehensive overview of the subject, including a detailed overview of its background and motivation. The content is therefore largely expository in nature, with new results appearing only in the final sections.

As prerequisites, we shall assume the reader is familiar with basic smooth manifold theory (at the level of [19]), introductory Riemannian geometry (first three chapters of [20]), and basic complex analysis (at the level of [3]). We also assume some familiarity with algebraic topology, particularly cohomology theory, as well as basic module theory.

Outline of the Thesis

The first chapter of the thesis is a brief and rapid overview of basic complex manifold theory. The emphasis is on breadth rather than depth, so that we may introduce all the essential concepts and results that we will require later. For a more complete and satisfactory exposition, the reader may consult the excellent sources [9], [12] and [13].

The second chapter is an overview of Chern-Weil theory, mainly following [20] and [6]. Chern-Weil theory serves as the first instance where we see geometric data (specifically, smooth connections and their curvature) being used to construct characteristic classes. We discuss the theory for both vector bundles and principal bundles.

Chapter 3 is based on M. F. Atiyah's 1957 paper [1], where he introduces the theory of complex analytic connections on principal and vector bundles. We focus primarily on the first five sections of the paper. Additionally, we illustrate a method for moving between complex analytic connections on a vector bundle and connections on its associated principal GL-bundle.

Logarithmic connections and their residues appear in Chapter 4. We first discuss the technical machinery required to define such connections, following [9] and [4]. We then develop logarithmic analogues of the concepts encountered in Chapter 3, following [7] and [10].

Towards the end of the chapter, we define the residue of a logarithmic connection on a holomorphic vector bundle over a complex manifold. We also show that this concept generalises to any holomorphic principal bundle over the manifold, with an arbitrary complex Lie group as structure group. This has been discussed by S. Gurjar and A. Paul in [10], where they give a global formulation; we give an alternate local formulation, more in the spirit of [16], using coordinate transformations of the bundle.

The final chapter of the thesis is dedicated to Ohtsuki's formula. We first state the main theorem ([16], Thm. 3) in the case of vector bundles. We then present a simple generalisation of the formula to principal bundles, under reasonable assumptions on the structure group G. This result is likely well known to experts in the field; however, as there is no explicit exposition available in the literature, we include a complete proof with all necessary details. As we shall see, the proof is essentially a modification of Ohtsuki's original proof in the vector bundle case.

Chapter 1

Preliminaries: Complex Manifolds

1.1 Complex Manifold Basics

A ***complex n-manifold*** is a topological $2n$-manifold endowed with a maximal holomorphic atlas (that is, the transition maps between any two charts are holomorphic maps between open subsets of $\mathbb{C}^n$). In particular, it admits the structure of a smooth $2n$-manifold. The local behaviour of a complex manifold is governed by complex analysis, making it somewhat "rigid" compared to smooth manifolds.

We will usually use the letters M, N and/or X to denote a complex manifold. Throughout, we shall use the terms "holomorphic" and "complex analytic" interchangeably.

1.1.1 Examples and Basic Properties

Standard examples of complex manifolds include $\mathbb{C}^n$ and its open subsets, $\mathbb{CP}^n$ (the n-dimensional complex projective space), complex tori (complex n-space modulo a lattice), and complex Lie groups (complex manifolds endowed with group structures that are compatible with their complex structures), among many others. Complex 1-manifolds are also often called ***Riemann surfaces***.

Holomorphic maps between complex manifolds are defined in a similar way as smooth maps: $f : M \to N$ is holomorphic if for every $p \in M$, there exist charts (U, ϕ) on M and (V, ψ) on N, containing p and $f(p)$ respectively, such that $\psi \circ f \circ \phi^{-1}$ is holomorphic. As one would expect, holomorphic maps are well-behaved with respect to composition and restriction to open subsets. An invertible holomorphic map is called a ***biholomorphism***.

Finally, we state a few essential results concerning complex manifolds that are consequences of standard several-variable complex analysis:

Theorem 1.1.1 (Identity Theorem; [13], Cor.1.29)**.** *Let M and N be complex manifolds with M connected, and $f, g : M \to N$ holomorphic maps that agree on a nonempty subset of M. Then $f \equiv g$ on M.*

Proposition 1.1.2 (Maximum Principle; [13], Prop.1.32)**.** *Suppose $f : U \to \mathbb{C}$ is a holomorphic map on a connected open set $U \subseteq \mathbb{C}^n$. If $|f(z)|$ attains a maximum value at some point in U, then f is constant.*

Theorem 1.1.3 ([13], Cor.1.33)**.** *Let M be a connected compact complex manifold. Then every global holomorphic function $f : M \to \mathbb{C}$ is constant.*

1.1.2 Complexified Tangent and Cotangent Bundles

Let M be a complex n-manifold. A ***smooth complex vector bundle*** over M is defined analogously to the real case, but with the fibres now being complex vector spaces ([13], p.24). Any smooth real vector bundle $E \to M$ of real rank r can be complexified (by complexifying each fibre) to obtain a smooth complex vector bundle $E_{\mathbb{C}} \to M$ of complex rank r. In particular, one can form the ***complexified tangent bundle*** $T_{\mathbb{C}}M$ and the ***complexified cotangent bundle*** $T^*_{\mathbb{C}}M$ of M; these are both of complex rank $2n$.

Consider the special case of $\mathbb{C}^n$, where one has the ***standard holomorphic coordinates*** $z^j = x^j + iy^j$. Define the smooth complex vector fields

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right), \qquad \frac{\partial}{\partial \overline{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right).$$

One can show that these form a smooth global frame for $T_{\mathbb{C}}\mathbb{C}^n$; the associated dual coframe is given by the smooth covector fields

$$dz^j = dx^j + idy^j, \qquad d\overline{z}^j = dx^j - idy^j.$$

This utility of this frame comes from its ability to characterise holomorphicity: let $U \subseteq \mathbb{C}^n$ be open, and $f : U \to \mathbb{C}$ be smooth. Then f is holomorphic if and only if $\partial f/\partial \overline{z}^j = 0$ for all $1 \leq j \leq n$.

Going back to a general complex manifold M, let $\phi : U \to \mathbb{C}^n$. Then one can transfer all the objects and concepts defined above to U using the holomorphic coordinate maps $(z^1 \circ \phi, \dots, z^n \circ \phi)$, which we (by abuse of notation) again call $(z^1, \dots, z^n)$.

An interesting consequence of the above computations is that **all complex manifolds admit canonical orientations**:

Proposition 1.1.4 ([13], Prop.1.49)**.** *Every complex manifold M has a canonical orientation, uniquely determined by the following two properties:*

(i) *The canonical orientation of $\mathbb{C}^n$ is determined by the $2n$-form $dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n$.*

(ii) *Every local biholomorphism from M to $\mathbb{C}^n$ is orientation-preserving.*

1.1.3 Complex and Almost Complex Structures

Let M be a complex n-manifold. A ***complex structure*** on a smooth real vector bundle $E \to M$ is a smooth bundle endomorphism $J : E \to E$ satisfying $J^2 = -\mathrm{Id}_E$.

The real tangent bundle $T\mathbb{C}^n$ admits a standard complex structure, which we call $J_{\mathbb{C}^n}$. It is determined by the equations

$$J_{\mathbb{C}^n} \frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j}, \qquad J_{\mathbb{C}^n} \frac{\partial}{\partial y^j} = -\frac{\partial}{\partial x^j}.$$

This, in turn, enables us to define a unique complex structure on TM that is "compatible" with $J_{\mathbb{C}^n}$ (see [13], Prop.1.55 for the precise meaning of "compatibility"). The complexified tangent bundle $T_{\mathbb{C}}M$ then splits as

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$$

where $T^{1,0}M$ and $T^{0,1}M$ are the $+i$ and $-i$ eigen-subbundles of the complexification of J_M, respectively. They are called the ***holomorphic*** and ***antiholomorphic tangent***

bundles of M, respectively. On a local holomorphic coordinate chart $(U, z^1, \ldots, z^n)$, the complex vector fields $\{\partial/\partial z^1, \ldots, \partial/\partial z^n\}$ form a smooth local frame for $T^{1,0}U$, and $\{\partial/\partial \overline{z}^1, \ldots, \partial/\partial \overline{z}^n\}$ form a smooth local frame for $T^{0,1}U$.

Similarly, one may show that the complexified cotangent bundle of M splits as

$$T^*_{\mathbb{C}}M = (T^{1,0}M)^* \oplus (T^{0,1}M)^*.$$

The holomorphic cotangent bundle $(T^{1,0}M)^*$ is locally spanned by $\{dz^1, \ldots, dz^n\}$, and the antiholomorphic cotangent bundle $(T^{0,1}M)^*$ by $\{d\overline{z}^1, \ldots, d\overline{z}^n\}$.

Now, let M be a smooth $2n$-manifold. An ***almost complex structure*** on M is a smooth bundle endomorphism $J : TM \to TM$ satisfying $J^2 = -\mathrm{Id}_{TM}$; in this case, M is called an ***almost complex manifold***. The complex structure on a complex manifold is, in particular, an almost complex structure on the underlying smooth manifold; additionally, it satisfies a condition called ***integrability***: for any two vector fields $V, W \in \Gamma(M, T^{1,0}M)$, $[V, W] \in \Gamma(M, T^{1,0}M)$.

When does an almost complex structure on a smooth $2n$-manifold arise from a complex structure? A famous result of Newlander-Nirenberg tells us that an integrable almost complex structure on a smooth $2n$-manifold is, in fact, induced by a holomorphic structure on it. (See [13], p.40).

1.1.4 Complex Submanifolds

In smooth manifold theory, the primary tools for constructing new manifolds out of existing ones are the inverse function theorem and its variations (namely, the implicit function theorem, the constant rank theorem, and the immersion and submersion theorems). All of them generalize, in a straightforward manner, to the holomorphic category (see [13], Chapter 2).

Let M be a complex manifold. An (***embedded***) ***complex submanifold of*** M is a subset $S \subseteq M$ that is a topological manifold in the subspace topology, and is endowed with holomorphic structure that makes the inclusion $S \hookrightarrow M$ a holomorphic emebedding. The holomorphic rank theorem gives us an equivalent way to define a complex k-submanifold of M:

Proposition 1.1.5 ([13], Cor.2.13). *Let M be a complex manifold, and $S \subseteq M$. The following are equivalent:*

(a) *S is a complex k-submanifold of M;*

(b) *each $p \in S$ has a neighbourhood U in M such that $S \cap U$ is the zero set of a holomorphic submersion $F : U \to \mathbb{C}^{n-k}$ (called a **local defining function for** S).*

Examples of complex submanifolds include open submanifolds of complex manifolds, images of holomorphic embeddings, regular level sets of holomorphic maps, and complex Lie subgroups of complex Lie groups.

1.1.5 Holomorphic Vector Bundles

A holomorphic vector bundle over a complex manifold M is a smooth complex vector bundle $\pi : E \to M$, where E is a complex manifold, π a holomorphic map, and the local trivializations can be chosen to be biholomorphisms ([13], p.71). A familiar example is the holomorphic tangent space $T^{1,0}M$ of a complex manifold M.

Holomorphic bundles can equivalently be described via their ***transition functions*** with respect to an open cover, as follows:

(1) Start with a holomorphic vector bundle $\pi : E \to M$ of rank r. Choose an open cover $\mathscr{U} = \{U_\alpha\}$ admitting holomorphic local trivializations $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$. On nonempty overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$, we may write

$$\begin{aligned} \phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \times \mathbb{C}^r &\longrightarrow U_{\alpha\beta} \times \mathbb{C}^r \\ (x, v) &\longmapsto (x, g_{\alpha\beta}(x)v\,) \end{aligned}$$

where $g_{\alpha\beta} : U_{\alpha\beta} \to GL_r(\mathbb{C})$ is a holomorphic map, called the ***transition function from ϕ_β to ϕ_α***.

These transitions satisfy an additional property called the ***cocycle condition***: on nonempty triple overlaps $U_{\alpha\beta\gamma}$, one has

$$g_{\alpha\gamma} = g_{\alpha\beta} \cdot g_{\beta\gamma}.$$

(2) Conversely, given an open cover $\mathscr{U} = \{U_\alpha\}$ and a collection of holomorphic maps $g_{\alpha\beta} : U_{\alpha\beta} \to GL_r(\mathbb{C})$ satisfying the above cocycle condition, one can recover the bundle E up to a unique isomorphism ([13], Prop.3.5).

The above characterization is extremely useful, and will be utilised, with more or less the same notations, throughout the remaining chapters.

One can construct new holomorphic bundles from existing ones by using standard algebraic operations on their fibres, namely, direct sums, tensor products, quotients, dual, Hom, End, and others ([13], p.79 - 81).

Holomorphic *line* bundles, in particular, exhibit special behaviour. The set of isomorphism classes of holomorphic line bundles over a complex manifold M can be given the structure of an abelian group, called the ***Picard group*** of M (denoted $\mathrm{Pic}(M)$). The group operation is (induced by) the tensor product, and the inverse of a class $[L]$ is represented by its dual, $[L^*]$ ([13], Thm.3.29).

Finally, we introduce the notion of a ***Hermitian metric*** h on a smooth complex vector bundle $E \to M$ ([13], p.76). This is a choice h_x of Hermitian inner product on each fiber E_x that varies smoothly with x. An easy partition-of-unity argument shows that every smooth complex vector bundle admits a Hermitian metric.

1.2 Dolbeault Cohomology

The Dolbeault cohomology groups of a complex manifold M are important biholomorphism invariants of M built out of differential forms. They are sensitive to the holomorphic structure on M.

1.2.1 The Dolbeault Complex

Denote the kth-exterior power of $T^*_{\mathbb{C}}M$ by $\Lambda^k_{\mathbb{C}}M$. The space of smooth sections of this bundle over M is nothing but the space of ***smooth complex-valued k-forms on*** M; we denote it by $\mathscr{E}^k(M)$.

Recall that the complex structure on M gives us a canonical decompoisiton

$$T^*_{\mathbb{C}}M = (T^{1,0}M)^* \oplus (T^{0,1}M)^*$$

of the complexified cotangent bundle. This, in turn, yields a decomposition of the bundle $\Lambda^k_{\mathbb{C}}M$:

$$\Lambda^k_{\mathbb{C}}M = \bigoplus_{p+q=k} \Lambda^{p,q}M,$$

where $\Lambda^{p,q}M := \Lambda^p(T^{1,0}M)^* \otimes \Lambda^q(T^{0,1}M)^*$. The space of smooth global sections of $\Lambda^{p,q}M$ is called the space of smooth (p,q)-forms on M, and is denoted $\mathscr{E}^{p,q}(M)$. (Some authors, e.g. [9], use the notation $\Omega^{p,q}(M)$).

For each (p,q), there is therefore a projection operator $\pi^{p,q} : \mathscr{E}^k(M) \to \mathscr{E}^{p,q}(M)$. With some work, one may show that the (complexified) exterior derivative $d : \mathcal{E}^{p,q}(M) \longrightarrow \mathcal{E}^{p+1,q}(M) \oplus \mathcal{E}^{p,q+1}(M)$ also splits as $d = \partial + \overline{\partial}$, where

$$\partial := \pi^{p+1,q} \circ d, \qquad \overline{\partial} := \pi^{p,q+1} \circ d.$$

As it turns out,

$$0 \longrightarrow \mathscr{E}^{p,0}(M) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathscr{E}^{p,n}(M) \longrightarrow 0$$

is a cochain complex – called the ***pth Dolbeault complex*** of M. The qth cohomology of the pth Dolbeault complex is called the (p,q)***-th Dolbeault cohomology group of*** M, and is denoted $H^{p,q}(M)$. ([13], p.108).

One can show that for any $p \leq n$,

$$H^{p,0}(M) = \Omega^p(M),$$

where $\Omega^p(M)$ is the space of ***holomorphic*** p***-forms on*** M. This is the space of holomorphic global sections of the holomorphic vector bundle $\Lambda^{p,0}M$. In particular, for the case $p = 0$, we have

$$H^{p,0}(M) = \mathscr{O}(M),$$

the space of holomorphic functions on M. In this manner, the $\overline{\partial}$ operator characterises holomorphicity; it is therefore also referred to as the ***Cauchy-Riemann operator***.

Finally, the $\overline{\partial}$-Poincaré lemma ([12] Prop.1.3.8; [13] Thm.4.13) tells us that locally, every $\overline{\partial}$-closed (p,q)-form is $\overline{\partial}$-exact. Of course, this needn't hold globally; the degree to which it fails is precisely what Dolbeault cohomology captures.

1.2.2 Dolbeault Cohomology of Holomorphic Vector Bundles

Let $E \to M$ be a holomorphic vector bundle. Then one form the smooth bundle $\Lambda^k_{\mathbb{C}} M \otimes E$, whose global sections are called the space of E***-valued*** k***-forms on*** M, denoted $\mathscr{E}^k(M,E)$. Similarly, one may define the bundle $\Lambda^{p,q}_{\mathbb{C}} M \otimes E$, and the space $\mathscr{E}^{p,q}(M,E)$ of E***-valued*** k***-forms on*** M. ([13], p.114).

One can then show that the Cauchy-Riemann operator extends to E-valued forms on M, giving us the Dolbeault complexes of the form

$$0 \longrightarrow \mathscr{E}^{p,0}(M,E) \xrightarrow{\overline{\partial}_E} \cdots \xrightarrow{\overline{\partial}_E} \mathscr{E}^{p,n}(M,E) \longrightarrow 0$$

whose qth cohomology is the (p,q)***-th Dolbeault cohomology group of*** M ***with coefficients in*** E (denoted $H^{p,q}(M,E)$). (See [13], Prop.4.16 for the details). As before, for any $p \leq n$, we have

$$H^{p,0}(M,E) = \Omega^p(M,E),$$

where $\Omega^p(M,E)$ is the space of ***holomorphic*** E***-valued*** p***-forms on*** M. This is the space of holomorphic global sections of the holomorphic vector bundle $\Lambda^{p,0}M \otimes E$. We can also show the existence of a $\overline{\partial}_E$-Poincaré lemma for E-valued forms.

1.3 Sheaves and Čech Cohomology

The absence of a holomorphic analogue of a "bump function" has profound consequences for the theory of complex manifolds − one may not be able to extend locally defined objects to global ones, and the space of global functions/sections needn't carry sufficient information about the manifold!

Sheaves give us a way to systematically track local data on a manifold, and their cohomology groups define powerful invariants of the manifold.

1.3.1 Sheaves

A ***sheaf*** (***of abelian groups***) $\mathscr{F}$ on a topological space X assigns an abelian group to each open subset of X in a "consistent" manner (so that a certain gluing condition is obeyed). The local behaviour of $\mathscr{F}$ at a point $p \in X$ is captured by its stalk $\mathscr{F}_p$ at p. (See [11], II.1 for the precise definitions).

Example 1.3.1. Some examples of sheaves include (following the notation of [13]):

(i) $\underline{A}$, the constant sheaf on X with coefficients in an abelian group A.

(ii) $\mathscr{E}$, the sheaf of smooth functions on a smooth manifold M;

$\mathscr{E}^k$, the sheaf of smooth complex-valued k-forms on a smooth manifold M.

(iii) $\mathscr{E}^{p,q}$, the sheaf of smooth (p, q)-forms on a complex manifold M;

$\mathscr{O}$, the sheaf of holomorphic functions on a complex manifold M;

Ω^p, the sheaf of holomorphic p-forms on a complex manifold M.

(iv) If $E \to M$ is a holomorphic vector bundle, we have

$\mathscr{O}(E)$, the sheaf of holomorphic sections of E;

$\Omega^p(E)$, the sheaf of holomorphic p-forms valued in E;

$\mathscr{E}^k(E)$, the sheaf of smooth complex-valued k-forms valued in E;

$\mathscr{E}^{p,q}(E)$, the sheaf of smooth (p, q)-forms valued in E.

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Just as we defined sheaves of abelian groups, we may define the notion of a ***sheaf of (commutative) rings*** on X, where we now assign a commutative ring to each open subset of X, and the sheaf restrictions are ring homomorphisms. As a result, the stalk at any point also admits the structure of a commutative ring. Examples include the sheaves $\mathscr{O}$ and $\mathscr{E}$ above.

Once we fix a sheaf of rings $\mathscr{A}$ on X, we may talk about ***sheaves of $\mathscr{A}$-modules*** on X. A sheaf $\mathscr{F}$ of $\mathscr{A}$-modules assigns to each open set U a module $\mathscr{F}(U)$ over the ring $\mathscr{A}(U)$, in a manner that is consistent with the sheaf restrictions of $\mathscr{A}$. As a result, the stalk $\mathscr{F}_x$ at any point $x \in X$ is an $\mathscr{A}_x$-module. Examples include the sheaves $\mathscr{O}(E)$, Ω^p and $\Omega^p(E)$ (which are sheaves of $\mathscr{O}$-modules), and $\mathscr{E}$, $\mathscr{E}^k$, $\mathscr{E}^{p,q}$, $\mathscr{E}^k(E)$ and $\mathscr{E}^{p,q}(E)$ (which are sheaves of $\mathscr{E}$-modules).

A sequence of sheaves (of abelian groups)

$$\cdots \longrightarrow \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \longrightarrow \cdots$$

on X is ***exact*** if $\ker \varphi^i = \mathrm{im}\, \varphi^{i-1}$ for each i. Equivalently, the induced sequence of stalks should be exact at each $p \in X$.

Example 1.3.2. Exact sequences of sheaves abound; the examples of interest to us, that we will often encounter later, are:

(i) The ***de Rham sequence*** on a smooth manifold M:

$$0 \longrightarrow \underline{\mathbb{C}} \hookrightarrow \mathscr{E}^0 \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{E}^n \longrightarrow 0.$$

(ii) The p-***th Dolbeault sequence*** on a complex manifold M,

$$0 \longrightarrow \Omega^p \hookrightarrow \mathscr{E}^{p,0} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{E}^{p,n} \longrightarrow 0$$

and the p-***th Dolbeault sequence with coefficients in*** E, E being a holomorphic vector bundle over M:

$$0 \longrightarrow \Omega^p(E) \hookrightarrow \mathscr{E}^{p,0}(E) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{E}^{p,n}(E) \longrightarrow 0.$$

(iii) The (holomorphic) ***exponential sequence*** on a complex manifold M,

$$0 \longrightarrow \underline{\mathbb{Z}} \hookrightarrow \mathscr{O} \xrightarrow{\exp} \mathscr{O}^* \longrightarrow 0$$

and its smooth counterpart

$$0 \longrightarrow \underline{\mathbb{Z}} \hookrightarrow \mathscr{E} \xrightarrow{\exp} \mathscr{E}^* \longrightarrow 0.$$

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1.3.2 Čech Cohomology

Čech cohomology, loosely speaking, measures the degree of obstruction to patching up local sections of sheaves and obtaining global sections. We present a brief outline of the construction, and refer the reader to [13], chapter 6 for the precise definitions.

Start with an open cover $\mathscr{U} = \{U_\alpha\}$ of the space X, and a sheaf $\mathscr{F}$ on X. To each $k \geq 0$, we associate the ***kth-Čech cochain group***

$$C^k(\mathscr{U}; \mathscr{F}) := \prod_{(\alpha_0, \ldots, \alpha_k)} \mathscr{F}(U_{\alpha_0 \cdots \alpha_k}).$$

There is a way to define a coboundary operator δ at each level that turns

$$0 \longrightarrow C^0(\mathscr{U}; \mathscr{F}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^k(\mathscr{U}; \mathscr{F}) \xrightarrow{\delta} C^{k+1}(\mathscr{U}; \mathscr{F}) \xrightarrow{\delta} \cdots$$

into a cochain complex; its kth cohomology (denoted $\check{H}^k(X; \mathscr{F})$) is called the ***$k$th Čech cohomology group of $\mathscr{F}$, with respect to the cover $\mathscr{U}$***. The ***kth Čech cohomology group of $\mathscr{F}$*** is then obtained by taking the direct limit over all possible open covers of X, partially ordered by refinement:

$$\check{H}^k(X; \mathscr{F}) := \varinjlim_{\mathscr{U}} \check{H}^k(\mathscr{U}; \mathscr{F}).$$

The main tools to compute the Čech cohomology groups of a sheaf are ***acyclic resolutions***. A sheaf $\mathcal{A}$ is said to be ***acyclic*** if its higher Čech cohomologies (≥ 1) all vanish. An acyclic resolution of a sheaf $\mathscr{F}$ is nothing but an exact sheaf sequence

$$0 \longrightarrow \mathscr{F} \xrightarrow{\imath} \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \longrightarrow \cdots$$

where the sheaves $\mathcal{A}^i$ are all acyclic. The De Rham and Dolbeault sequences from the previous example are acyclic resolutions of the sheaves $\underline{\mathbb{C}}$ and Ω^p respectively; the sheaves $\mathscr{E}^k$ and $\mathscr{E}^{p,q}$ are acyclic since they are *fine* sheaves ([13], p.161-164).

We may now state the fundamental result that helps us compute Čech cohomology (see [13], Thm.6.11):

Theorem 1.3.1 (**De Rham-Weil**). *Let $\mathscr{F}$ be a sheaf on a paracompact Hausdorff space X, and*

$$0 \longrightarrow \mathscr{F} \xrightarrow{\imath} \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \longrightarrow \cdots$$

an acyclic resolution of $\mathscr{F}$. Then the global-section sequence $\mathcal{A}^*(X)$, given by

$$0 \longrightarrow \mathcal{A}^0(X) \xrightarrow{d} \mathcal{A}^1(X) \xrightarrow{d} \mathcal{A}^2(X) \longrightarrow \cdots$$

is a cochain complex, whose cohomology computes the cohomology of $\mathscr{F}$: for each q, there is a natural isomorphism

$$\check{H}^q(X, \mathscr{F}) \cong H^q(\mathcal{A}^*(X)).$$

1.3.3 Applications

Applying the De Rham-Weil theorem to the de Rham and Dolbeault sequences give us the following milestone theorems:

Theorem 1.3.2 ([13] Thm.6.19). *Suppose M is a complex manifold, and Ω^p its sheaf of holomorphic p-forms. For each $q \geq 0$, we have a **Dolbeault isomorphism***

$$\check{H}^q(M; \Omega^p) \cong H^{p,q}(M). \tag{1.1}$$

More generally, if $E \to M$ is a holomorphic vector bundle, and $\Omega^p(E)$ the sheaf of E-valued p-forms, then

$$\check{H}^q(M, \Omega^p(E)) \cong H^{p,q}(M, E). \tag{1.2}$$

Theorem 1.3.3 ([13] Thm.6.20)**.** *Suppose M is a smooth manifold. For each $k \geq 0$, we have a* ***de Rham isomorphism***

$$\check{H}^k(M, \underline{\mathbb{C}}) \cong H^k_{\mathrm{dR}}(M, \mathbb{C}).$$

This tells us that the de Rham cohomology (with complex coefficients) of a smooth manifold is independent of the smooth structure on the manifold. When paired with the fact that $\check{H}^k(M; \underline{\mathbb{C}}) \cong H^k_{sing}(M; \mathbb{C})$ (see [13], Thm.6.24), this gives a slick, sheaf-theoretic proof of de Rham's classic result.

1.3.4 Chern Classes of Line Bundles

Sheaf cohomology also serves an important tool in classifying line bundles over complex manifolds. The first result in this direction is as follows:

Theorem 1.3.4 ([13] Prop.6.28)**.** *Let M be a complex manifold. We have a natural isomorphism*

$$\check{H}^1(M, \mathscr{O}^*) \cong \mathrm{Pic}(M).$$

An analogous statement holds in the case of smooth line bundles over M, where we replace the sheaf $\mathscr{O}^*$ by $\mathscr{E}^*$.

Now, consider the exponential sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \hookrightarrow \mathscr{O} \xrightarrow{\exp} \mathscr{O}^* \longrightarrow 0.$$

This yields, via the long exact sequence construction, a group homomorphism $\delta^* : \check{H}^1(M, \mathscr{O}^*) \longrightarrow \check{H}^2(M, \underline{\mathbb{Z}})$. Given a line bundle $L \to M$, we may then define its ***sheaf-theoretic first Chern class*** $c_1(L)$ to be the image $-\delta^*([L])$ of its equivalence class $[L] \in \mathrm{Pic}(M) \cong \check{H}^1(M, \mathscr{O}^*)$. We shall study Chern classes in considerable depth in the coming chapters.

Remark 1.3.5. One can analogously consider the smooth exponential sequence on M and obtain a homomorphism $\delta^* : \check{H}^1(M, \mathscr{E}^*) \longrightarrow \check{H}^2(M, \underline{\mathbb{Z}})$. In this case, however, δ^* becomes an isomorphism, since $\mathscr{E}$ and $\mathscr{E}^*$ are fine sheaves! Thus, smooth complex line bundles over M are classified, up to smooth isomorphism, by their (smooth) sheaf-theoretic first Chern classes.

1.4 Smooth Connections

Among the most fundamental tools in Riemannian geometry are connections, which help us compare tangent spaces at different points of a manifold. Connections lead to the notions of curvature, holonomy, parallel transport and geodesics on a Riemannian manifold − all concepts of fundamental importance. They can also be used to construct certain invariants of bundles, known as characteristic classes (see the next chapter).

In this section, we provide a quick overview of smooth connections on vector bundles over complex manifolds. We show that a Hermitian holomorphic vector bundle admits a unique connection compatible with its metric and holomorphic structure, much like the Levi-Civita connection in real differential geometry. We also introduce the first Chern class of a smooth vector bundle, using the curvature of a connection on the bundle.

1.4.1 Smooth Connections on Vector Bundles

Let E be a smooth complex vector bundle of rank r over a smooth manifold M. A ***smooth connection*** on E is a method of differentiating the sections of E along vector fields on M. Precisely, it is an operator

$$\begin{aligned} \nabla : \Gamma(T_{\mathbb{C}}M) \times \Gamma(E) &\longrightarrow \Gamma(E) \\ (X, s) &\longmapsto \nabla_X s \end{aligned}$$

that is $\mathscr{E}$-linear in X, and satisfies the ***Leibniz rule*** in s:

$$\nabla(f \cdot s) = X(f) \cdot s + f \cdot \nabla(s) \tag{1.3}$$

for any smooth function f on M.

Using the fact that a smooth connection (in the above sense) is a *point operator* (see [20], section 7), and the canonical isomorphism $\mathrm{Hom}(TM, E) \cong T^*M \otimes E$, one can equivalently define a smooth connection as a $\mathbb{C}$-linear sheaf homomorphism

$$\nabla : \mathscr{E}(E) \longrightarrow \mathscr{E}^1 \otimes \mathscr{E}(E) \tag{1.4}$$

that satisfies the ***Leibniz rule***

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s) \tag{1.5}$$

for any local function f and local section s of E.

Proposition 1.4.1 ([12], Cor.4.2.4)**.** *The set of all smooth connections on E is in a natural way an affine space over the (infinite-dimensional) complex vector space $\mathscr{E}^1(M, \mathrm{End}(E))$.*

There is a nice way to describe connections locally, using frames. Let $e = (e_1, \ldots, e_r)$ be a local frame for E on an open set U. Then, one may write

$$\nabla e_j = \sum_{i=1}^{n} \theta_{ij} \otimes e_i \tag{1.6}$$

where θ_{ij}s are smooth 1-forms on U. This gives us a 1-form $\theta_e \in \mathscr{E}^1(U, \mathrm{End}(E)) \cong \mathscr{E}^1(U, \mathfrak{gl}_r(\mathbb{C}))$, called the ***connection matrix*** of ∇ with respect to (U, e). One may go on to show that locally on U, the operator ∇ may be expressed as $d + \theta_e$, d being the usual exterior derivative on $\mathscr{E}(E)$.

Suppose now that E is endowed a Hermitian metric $\langle -, - \rangle$. A connection ∇ on E is ***compatible with the metric*** (or a ***metric connection***) if for all $X \in \Gamma(T_{\mathbb{C}}M)$ and $s, t \in \Gamma(E)$,

$$X\langle s, t \rangle = \langle \nabla_X s, t \rangle + \langle s, \nabla_{\overline{X}} t \rangle. \tag{1.7}$$

One can show ([13], Prop.7.1) that the connection matrix of a metric connection with respect to an *orthonormal* local frame is skew-Hermitian.

Finally, we show how one can induce connections on new vector bundles that one gets by performing algebraic operations on existing vector bundles (with existing connections on them). Let E_1 and E_2 be two vector bundles, with connections ∇^1 and ∇^2, respectively. Let s_1, s_2 be local sections of E_1 and E_2, respectively. Then

(i) On $E_1 \oplus E_2$, we have the direct sum connection

$$\nabla(s_1 \oplus s_2) := \nabla^1(s_1) \oplus \nabla^2(s_2).$$

(ii) On $E_1 \otimes E_2$, we have the tensor product connection

$$\nabla(s_1 \otimes s_2) := \nabla^1(s_1) \otimes s_2 + s_1 \otimes \nabla^2(s_2).$$

(iii) The dual connection on E_1^* acts as

$$(\nabla_X^1 \varphi_1)(s_1) = X(\varphi_1(s_1)) - \varphi_X^1(\nabla_1(s_1))$$

where X is a smooth local vector field and φ_1 is a local section of E_1^*.

(iv) On $\mathrm{Hom}(E_1, E_2)$, we have the Hom connection

$$\nabla(f)(s_1) := \nabla^2(f(s_1)) - f(\nabla^1(s_1))$$

where f is a local section of $\mathrm{Hom}(E_1, E_2)$.

See Ex.4.2.6 of [12] and Problems 7-4, 7-5 and 7-6 of [13] for more details.

Remark 1.4.2. Once we fix a connection ∇ on the tangent bundle TM, we can develop a notion of ***parallel transport*** on M. If TM also admits a Riemannian metric, and ∇ is the associated Levi-Civita connection, we may go on further and introduce the theory of geodesics, lengths, volumes, measurements and holonomy – all classical objects of study in Riemannian geometry! We shall not delve further into this theory, however, and instead refer the reader to [20] for more details.

1.4.2 Curvature

Let $E \to M$ be a smooth complex vector bundle, and ∇ a connection on E. The *curvature* of E is the map

$$\Theta : \Gamma(T_{\mathbb{C}}M) \times \Gamma(T_{\mathbb{C}}M) \times \Gamma(E) \longrightarrow \Gamma(E)$$

defined by

$$\Theta(X,Y)s := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s. \tag{1.8}$$

Using the canonical isomorphism $\mathrm{End}(E) \cong E^* \otimes E$ and the fact that Θ is $\mathscr{E}(M)$-linear in all three arguments X, Y and s ([13], p.199), one can show that Θ determines a smooth $\mathrm{End}(E)$-valued 2-form on M: $\Theta \in \mathscr{E}^2(M, \mathrm{End}(E))$. A connection ∇ is said to be ***flat*** if its curvature is identically zero.

We can also describe the curvature of a connection locally, as follows. Let (U, e) be as before. Then, one may write

$$\Theta(s_j) = \sum_{i=1}^{n} \Theta_{ij} \otimes s_i \tag{1.9}$$

where Θ_{ij} are now smooth 2-forms on U. They determine the 2-form $\Theta_e \in \mathscr{E}^2(U, \mathrm{End}(E)) \cong \mathscr{E}^2(U, \mathfrak{gl}_r(\mathbb{C}))$, called the ***curvature matrix*** of ∇ with respect to (U, e). In fact, Θ_e is simply obtained by restricting the global form Θ to U, and using the isomorphism $\mathrm{End}(E)|_U \cong U \times \mathfrak{gl}_r(\mathbb{C})$ induced by the choice of frame e.

The connection and curvature matrices with respect to a local frame (U, e) are related by the ***second structural equation*** ([20], p.85)

$$\Theta = d\theta + \theta \wedge \theta. \tag{1.10}$$

The above equality is one of endomorphism valued 2-forms. If E is endowed with a Hermitian metric and ∇ is a metric connection on E, then the curvature matrix of Θ with respect to any local orthonormal frame is also skew-Hermitian.

Finally, we show how the curvatures of the induced connections from the previous subsection are related to the existing curvatures. Let Θ^1 and Θ^2 be the curvatures of the connections ∇^1 and ∇^2, respectively.

(i) The curvature of the direct sum connection is

$$\Theta := \Theta^1 \oplus \Theta^2.$$

(ii) The curvature of the tensor product connection is

$$\Theta := \Theta^1 \otimes \mathbb{1}_{E_1} + \mathbb{1}_{E_2} \otimes \Theta^2.$$

(iii) The curvature of the dual connection on E_1^* is the negative of its adjoint:

$$\Theta = -(\Theta^1)^\dagger.$$

1.4.3 The First Chern Class

For any connection ∇ on a smooth complex vector bundle $E \to M$, the curvature $\Theta \in \mathscr{E}^2(M, \mathrm{End}(E))$ is a global endomorphism-valued 2-form. Because the trace of an endomorphism is independent of the choice of basis, we can form a global *scalar* 2-form

$$c_1(\nabla) = \frac{-1}{2\pi i}\mathrm{tr}(\Theta) \qquad \in \mathscr{E}^2(M)$$

called the ***first Chern form of*** ∇. This is a closed 2-form on M; furthermore, its de Rham cohomology class is independent of the choice of connection ∇ on E! (See [13], Thm.7.12).

Now, suppose E has a Hermitian metric. If ∇ is a metric connection on E, one can show ([13] Prop.7.13) that $c_1(\nabla)$ is in fact a *real* 2-form! The cohomology class determined by $c_1(\nabla)$ is therefore in $H^2_{\mathrm{dR}}(M, \mathbb{R})$. We define the ***first real Chern class of*** $\boldsymbol{E}$ to be this cohomology class $c_1^{\mathbb{R}}(E) \in H^2_{dR}(M, \mathbb{R})$.

In the case of line bundles, one may ask: how is the first real Chern class related to the sheaf-theoretic first Chern class? This is answered by the following theorem:

Theorem 1.4.3 ([13], Thm.7.14)**.** *Let $L \to M$ be a smooth complex line bundle. Under the composition*

$$\check{H}^2(M, \underline{\mathbb{Z}}) \longrightarrow \check{H}^2(M, \underline{\mathbb{R}}) \xrightarrow{\simeq} H^2_{\mathrm{dR}}(M, \mathbb{R}),$$

where the first map is induced by the sheaf inclusion $\underline{\mathbb{Z}} \hookrightarrow \underline{\mathbb{R}}$ and the second map is the de Rham isomorphism, the sheaf theortic first Chern class $c_1(L)$ maps to the first real Chern class $c_1^{\mathbb{R}}(L)$.

Note that $\check{H}^2(M, \underline{\mathbb{Z}}) \to \check{H}^2(M, \underline{\mathbb{R}})$ need not be injective; hence the sheaf-theoretic first Chern class generally carries more information than the first real Chern class.

Remark 1.4.4. The terminology "first Chern class" suggests that one can define higher Chern classes of bundles. These are discussed in the next chapter.

1.4.4 The Chern Connection

Recall from Riemannian geometry that the tangent bundle of a Riemannian manifold admits a unique connection, called the *Levi-Civita connection*, that is both torsion-free and compatible with the Riemannian metric ([20], Thm.6.6). One may then ask − given a Hermitian vector bundle E over a complex manifold M, does there exist a unique connection on E compatible with the Hermitian structure? As it turns out, this is true when E is holomorphic, and provided we impose the following additional condition:

Definition 1.4.5. *A smooth connection ∇ on a holomorphic vector bundle $E \to M$ is said to be* ***compatible with the holomorphic structure*** *if any of the following equivalent conditions hold:*

(a) $\nabla^{(0,1)} = \overline{\partial}_E$ *as operators* $\mathscr{E}^0(E) \to \mathscr{E}^{0,1}(E)$*;*

(a) *The connection matrix of ∇ with respect to any holomorphic local frame of E is an* $\mathrm{End}(E)$*-valued form of type* $(1, 0)$.

On the Hermitian holomorphic bundle $(E, \langle -, - \rangle)$, there is a unique smooth connection Γ, called the ***Chern connection***, that is compatible with both the metric and the holomorphic structure of E ([13], Thm. 7.17). The connection matrix θ of this connection with respect to a holomorphic local frame (U, e) is given by

$$\theta = H^{-1}\partial H$$

where H is the matrix whose entries are the smooth functions $H_{ij} = \langle e_i, e_j \rangle$. Furthermore, one may show that the curvature of the Chern connection is locally given by $\Theta = \overline{\partial}_E\theta$; thus it is a $\overline{\partial}_E$-closed endomorphism-valued form of type $(1, 1)$. As a result, the first Chern class of E lives in the Dolbeault cohomology group $H^{1,1}(M)$. ([13], Prop. 7.18).

Finally, let us specialise to the case of holomorphic line bundles. Let $L \to M$ be a holomorphic line bundle, and (U, e) a holomorphic framed open set for L. The metric on $E|_U$ is completely determined by the smooth positive function $h := \langle e, e \rangle = |e|^2$, and the connection matrix with respect to (U, e) simply becomes

$$\theta = h^{-1}\partial h = \partial(\log h).$$

The corresponding curvature matrix then becomes

$$\Theta = \overline{\partial}\partial(\log h)$$

and the Chern form admits the local expression

$$c_1(\Gamma) = \frac{-1}{2\pi i}\overline{\partial}\partial(\log h).$$

1.5 Currents

In this section, we develop the theory of currents on smooth and complex manifolds. Our exposition mainly follows section I.2 of [5], and Chapter 3 of [9].

1.5.1 Currents on Smooth Manifolds

Start with a smooth orientable n-manifold M. For this subsection only, fix the notation $\mathscr{E}^p_{\mathbb{R}}(M)$ for the space of *real-valued* p-forms on M.

One can endow the space $\mathscr{E}^p_{\mathbb{R}}(M)$ with a topology, called the C^∞ ***topology***, that turns it into a Frechét space. Within this, we can identify the subspace $\mathscr{D}^p(M)$ of ***compactly supported*** p***-forms on*** M.

Definition 1.5.1 ([5], Def.I.2.3)**.** *The space of* ***currents of dimension*** p (***or degree*** $n-p$) *is the space*

$$\mathscr{D}'^{\,n-p}(M) = \mathscr{D}'_p(M) := \textit{topological dual } (\mathscr{D}^p(M))'.$$

The action of a current $T \in \mathscr{D}'_p(M)$ on a "test" form $u \in \mathscr{D}^p(M)$ will be denoted $\langle T, u\rangle$.

Example 1.5.1 (Current of Integration; [5], Ex.I.2.4)**.** Let $Z \subseteq M$ be a closed, oriented submanifold of codimension p. The *current of integration* over Z is the current in $\mathscr{D}'^{\,p}(M)$ given by

$$\int_Z u$$

for any test form $u \in \mathscr{D}^{n-p}(M)$. ♢♦♢

Example 1.5.2 ([5], Ex.I.2.5)**.** Let $f \in \mathscr{E}^q_{\mathbb{R}}(M)$ be a smooth q-form. To this, we can associate a current T_f of dimension $n-q$, as follows:

$$\langle T_f, u\rangle := \int_M f \wedge u$$

for any $u \in \mathscr{D}^{n-q}(M)$. ♢♦♢

Using duality, one can define an ***exterior derivative*** operator on currents as follows: given $T \in \mathscr{D}'^{\,q}(M)$, define $dT \in \mathscr{D}'^{\,q+1}(M)$ by

$$\langle dT, u\rangle = (-1)^{q+1}\langle T, du\rangle$$

for any $u \in \mathscr{D}^{n-q-1}(M)$. This gives us a cochain complex of $\mathbb{R}$-vector spaces

$$0 \longrightarrow \mathscr{D}'^{\,0}(M) \xrightarrow{d} \mathscr{D}'^{\,1}(M) \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{D}'^{\,n}(M) \longrightarrow 0.$$

We may therefore define the q***th cohomology of currents on*** M, denoted $H^q(\mathscr{D}^*(M))$, to be the qth cohomology of the above cochain complex. ([9], p.382).

Example 1.5.3. Let $Z \subseteq M$ be a closed, oriented submanifold. One can show using Stokes' theorem that

$$d\left(\int_Z -\right) = (-1)^{n-p+1}\int_{\partial Z} -$$

where ∂Z denotes the boundary of Z. In particular, **the current of integration over a closed submanifold without boundary is closed**. ♢♦♢

One may also define ***wedge products*** of currents with differential forms, as follows. Let $T \in \mathscr{D}'^{\,q}(M)$ and $g \in \mathscr{E}^r_{\mathbb{R}}(M)$. Define their wedge product $T \wedge g \in \mathscr{D}'^{\,q+r}(M)$ by

$$\langle T \wedge g, u\rangle := \langle T, g \wedge u\rangle$$

for $u \in \mathscr{D}^{n-q-r}(M)$ ([5] I.2.B.2). The wedge product and exterior derivative are compatible operations, in the sense that they satisfy the following ***Leibniz rule***:

$$d(T \wedge g) = dT \wedge g + (-1)^{\deg T}(T \wedge dg).$$

Finally, we discuss the relation between the cohomology of currents on M and the de Rham cohomology of M. First, note that we have a natural map of cochain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{D}^0(M) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \mathscr{D}^n(M) & \longrightarrow & 0 \\
 & & \downarrow T & & & & \downarrow T & & \\
0 & \longrightarrow & \mathscr{D}'^{\,0}(M) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \mathscr{D}'^{\,n}(M) & \longrightarrow & 0.
\end{array}$$

where $T : \mathscr{D}^q(M) \to \mathscr{D}'^{\,q}(M)$ is the assignment $f \mapsto T_f$. This, by [9], p.382, induces an isomorphism of cohomology rings

$$\begin{aligned} H^*_{\mathrm{dR}}(M) &\xrightarrow{\simeq} H^*(\mathscr{D}^*(M)) \\ [f] &\longmapsto [T_f] = \left[\int_M f \wedge -\right]. \end{aligned}$$

Let $Z \subseteq M$ be a closed submanifold (without boundary) of codimension p. Then, by the above, the current of integration $\int_Z - \in \mathscr{D}'^{\,p}(M)$ gives rise to a unique cohomology class $c_p(Z) \in H^p_{\mathrm{dR}}(M)$, which we call the ***Poincaré dual*** of $\int_Z -$.

Remark 1.5.2. An alternate proof of the above statement can be given by Poincaré duality: see [2], p.50-51.

1.5.2 Currents on Complex Manifolds

Now, let M be a complex n-manifold. The same constructions as before go through, but now with the sheaves $\mathscr{E}^p(M)$ of *complex*-valued forms instead of real-valued ones.

Definition 1.5.3 ([5], Def.I.3.20)**.** *The space of* ***currents of bidimension*** (p, q) (***or bidegree*** $(n-p, n-q)$) *is the space*

$$\mathscr{D}'^{\,n-p,\,n-q}(M) = \mathscr{D}'_{p,q}(M) := \text{topological dual } (\mathscr{D}^{p,q}(M))'.$$

where $\mathscr{D}^{p,q}(M)$ is again endowed with the C^∞ topology.

In fact, one has a decomposition

$$\mathscr{D}'_k(M, \mathbb{C}) = \bigoplus_{p+q=k} \mathscr{D}'_{p,q}(M).$$

Using duality, one can define a ***Dolbeault operator*** on currents: for any $T \in \mathscr{D}'^{\,p,q}(M)$, define $\overline{\partial} T \in \mathscr{D}'^{\,p,\,q+1}(M)$ by

$$\langle dT, u \rangle = (-1)^{q+1} \langle T, du \rangle$$

for any $u \in \mathscr{D}^{n-p,\, n-q-1}(M)$. Once again, we get a cochain complex of $\mathbb{C}$-vector spaces

$$0 \longrightarrow \mathscr{D}'^{\,p,0}(M) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathscr{D}'^{\,p,n}(M) \longrightarrow 0.$$

We accordingly define the (p,q)***th Dolbeault cohomology of currents on*** M, denoted $H^{p,q}(\mathscr{D}^{p,*}(M))$, to be the qth cohomology of the above cochain complex.

One may also define ***wedge products*** as before: let $T \in \mathscr{D}'^{\,p,q}(M)$ and $g \in \mathscr{E}^{r,s}(M)$. Define their wedge product $T \wedge g \in \mathscr{D}'^{\,p+r,\,q+s}(M)$ by

$$\langle T \wedge g, u \rangle := \langle T, g \wedge u \rangle$$

for $u \in \mathscr{D}^{n-p-r,\, n-q-s}(M)$. Again, we have a ***Leibniz rule***

$$\overline{\partial}(T \wedge g) = \overline{\partial}T \wedge g + (-1)^{\deg T}(T \wedge \overline{\partial}g).$$

Similar arguments to before also show us that we have (for each p) an isomorphism of cohomology rings

$$\begin{aligned} H^{p,*}_{\overline{\partial}}(M) &\xrightarrow{\simeq} H^{p,*}(\mathscr{D}^{p,*}(M)) \\ [f] &\longmapsto [T_f] = \left[\int_M f \wedge -\right]. \end{aligned}$$

Example 1.5.4. Let $Z \subseteq M$ be a closed, oriented submanifold of codimension p. Its current of integration defines a $\overline{\partial}$-closed current of bidegree (p,p), given by

$$\int_Z u$$

where $u \in \mathscr{D}^{n-p,\, n-p}(M)$ is a *real* test form on M. By the above isomorphism, $[Z]$ defines a unique $\overline{\partial}$-cohomology class $c_p(Z) \in H^{p,p}(M)$, which we again call its ***Poincaré dual***. ◊♦◊

1.6 Divisors and Line Bundles

The final section of this chapter introduces the theory of divisors on complex manifolds. Since compact complex manifolds do not admit any global holomorphic functions, their structure is better reflected by meromorphic functions defined on them. These meromorphic functions may blow up along certain codimension one subsets, called analytic hypersurfaces. We therefore study such hypersurfaces, and more generally, linear combinations of such hypersurfaces − called divisors.

1.6.1 Analytic Hypersurfaces

First, let us introduce the theory of meromorphic functions on complex manifolds. A ***meromorphic function*** f on a complex manifold M is a holomorphic map $f : M\backslash S \to \mathbb{C}$ such that

(i) $S \subseteq M$ is nowhere-dense;

(ii) there exists an open cover $\{U_\alpha\}$ and a collection of holomorphic local functions $g_\alpha, h_\alpha \in \mathscr{O}(U_\alpha)$ such that $f|_{U_\alpha\backslash S} \cdot h|_{U_\alpha\backslash S} = g|_{U_\alpha\backslash S}$.

The sheaf of meromorphic functions on M is denoted $\mathscr{M}$.

Let us now quickly recall some facts about analytic hypersurfaces. An ***analytic hypersurface*** V on a complex manifold M is a subset of M that is given locally by the vanishing of a single holomorphic function. That is, each point $p \in V$ admits a neighbourhood U such that $V \cap U = \{z \in U \,|\, f(z) = 0\}$ for some holomorphic function $f \in \mathscr{O}(U)$. The function f is called a ***local defining function*** for V; it is unique up to multiplication by a function nonzero at p. ([9], p.130).

V is said to be ***irreducible*** if it cannot be written as a union of two strictly smaller nonempty analytic hypersurfaces. One can show ([9], p.12) that locally, every analytic hypersurface is a finite union of irreducible analytic hypersurfaces.

Typically, analytic hypersurfaces are not complex submanifolds, since they admit singular points. The smooth locus of V is denoted V^*; it is a basic fact ([9], p.21) that V^* is a complex submanifold of codimension one, and that V is irreducible if and only if V^* is connected.

One may define intergation on analytic hypersurfaces, too. Given a smooth $(2n-2)$-form u, we define

$$\int_V u := \int_{V*} u.$$

As before, this defines a current on M of bidegree $(1,1)$; by Poincaré duality, it corresponds to a unique cohomology class $c_1(Z) \in H^{1,1}(M)$, which we (again) call the ***Poincaré dual*** of Z.

Example 1.6.1. An embedded complex submanifold of codimension one is a smooth analytic hypersurface. ◊♦◊

1.6.2 Divisors

A ***divisor*** D on M is a *locally finite* $\mathbb{Z}$-linear combination of irreducible analytic hypersurfaces on M:

$$D = \sum_{\nu} a_\nu \cdot V_\nu$$

where ν runs over all possible irreducible analytic hypersurfaces. The set of all divisors on M forms an abelian group, and is denoted $\mathrm{Div}(M)$. It is sometimes called the set of ***Weil Divisors*** on M.

Divisors can also be described sheaf-theoretically, as follows. Let

- $\mathscr{M}^*$ denote the sheaf of meromorphic functions that do not identically vanish on M;
- $\mathscr{O}^*$ denote the sheaf of nowhere-vanishing holomorphic functions on M.

Both are multiplicative sheaves of abelian groups on M. One may then show ([9], p.131) that there is a canonical isomorphism $\check{H}^0(M, \mathscr{M}^*/\mathscr{O}^*) = \mathrm{Div}(M)$!

A global section of $\mathscr{M}^*/\mathscr{O}^*$ is called a ***Cartier divisor*** on M; the previous isomorphism shows that Cartier divisors and Weil divisors are the same on complex manifolds.

1.6.3 The Line Bundle Associated to a Divisor

On a compact complex manifold, one may not be able to find a global defining function for an arbitrary hypersurface, since the only global holomorphic functions are constants. One way to remedy this is by "adding an extra dimension" and considering line bundles over the manifold. As it turns out, this is the right approach; the global sections of these line bundles then help us capture all possible hypersurfaces on the manifold.

Consider the exact sheaf sequence

$$0 \longrightarrow \mathscr{O}^* \xrightarrow{\imath} \mathscr{M}^* \xrightarrow{\jmath} \mathscr{M}^*/\mathscr{O}^* \longrightarrow 0$$

whose associated long exact cohomology sequence contains the part

$$\check{H}^0(M, \mathscr{M}^*) \xrightarrow{\jmath^*} \check{H}^0(M, \mathscr{M}^*/\mathscr{O}^*) \xrightarrow{\delta} \check{H}^1(M, \mathscr{O}^*).$$

The second group in this sequence is simply $\mathrm{Div}(M)$, and the third group $\mathrm{Pic}(M)$. Hence, associated to any Cartier divisor D on M is a holomorphic line bundle $[D]$, unique up to isomorphism − called the ***line bundle associated to*** D.

The importance of the line bundle $[D]$ stems from the fact that it admits a meromorphic global section (i.e., a global section of the sheaf $\mathscr{M} \otimes_{\mathscr{O}} \mathscr{O}(D)$) that recovers back the divisor D. (See [9], p.136).

1.6.4 The First Chern Class of the Associated Line Bundle

We conclude the chapter by comparing all the definitions of the first Chern class encountered so far, in the case of the line bundle associated to a divisor.

Recall that each irreducible analytic hypersurface Z defines a unique cohomology class $c_1(Z) \in H^{1,1}(M)$, called its Poincaré dual. This extends to a group homomorphism $\mathrm{Div}(M) \longrightarrow H^{1,1}(M)$, and therefore allows us to define the Poincaré dual of any divisor on M. The final result is then as follows:

Theorem 1.6.1 ([9], p.141)**.** *Let $D \in Div(M)$. Then*

$$c_1(D) = c_1([D]) \quad \in H^{1,1}(M)$$

where the left-hand side is the Poincaré dual of D, and the right-hand side the first (real) Chern class of the line bundle $[D]$ associated to D.

Thus, for a divisor D, there are three equivalent definitions of the first Chern class of its associated line bundle $[D]$:

1. the Poincaré dual of D;
2. the de Rham class defined by the first Chern form $\frac{-1}{2\pi i}\Theta$, Θ being the curvature of some connection on $[D]$;
3. the sheaf theoretic first Chern class of $[D]$.

Chapter 2

Chern-Weil Theory

Chern-Weil theory deals with the construction of **characteristic classes**, which are certain isomorphism invariants of a vector bundle on a smooth manifold. They are constructed using the (locally defined) curvature forms associated with some connection on the bundle, but turn out to be independent of the choice of the connection. We have already seen an instance of this in the previous chapter, when we defined the first Chern class of a smooth complex line bundle over a complex manifold.

Traditionally, characteristic classes are constructed using purely topological methods – without any reference to a smooth structure! Chern-Weil theory, therefore, reveals a deep connection between algebraic topology and differential geometry. The idea of constructing characteristic classes from geometric data forms the overarching theme of this thesis and will be a recurring concept in the chapters to come.

Along the way, we will also introduce principal bundles, which are, in some sense, a generalisation of vector bundles. We discuss the theory of smooth connections on principal bundles and show that an analogous version of Chern-Weil theory also holds in this setting.

Our main sources for this chapter are [20] and [6]. For the topological treatment (via classifying spaces), we refer the reader to [15].

2.1 Chern-Weil Theory: Vector Bundles

For the rest of this chapter, M will denote a smooth n-manifold, and $E \to M$ a smooth vector bundle of rank r. For this chapter only, we shall denote by $\mathscr{E}^k$ the sheaf of smooth real-valued k-forms on M, rather than complex-valued forms (in any case, the theory does not change by much).

2.1.1 Connection and Curvature Matrices

Let ∇ be a (smooth) connection on E. With respect to the framed open cover $\mathscr{U} = \{(U_\alpha, e_\alpha)\}$ (1.1.5), ∇ is represented by the connection matrices $\{\theta_\alpha \in \mathscr{E}^1(U, \mathfrak{gl}_r(\mathbb{R}))\}$:

$$\nabla e_{\alpha j} = \sum_{i=1}^{n} (\theta_\alpha)_{ij} \otimes e_{\alpha i}. \tag{2.1}$$

How do the matrices corresponding to overlapping frames compare? If $g_{\alpha\beta} : U_{\alpha\beta} \to GL_r(\mathbb{R})$ denotes the smooth transition between the overlapping frames e_α and e_β, with $e_\alpha = e_\beta \cdot g_{\alpha\beta}$, then one has ([20] Thm.22.1)

$$\theta_\beta = g_{\alpha\beta}^{-1} \theta_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}. \tag{2.2}$$

One may also show that the curvature matrices transform as

$$\Theta_\beta = g_{\alpha\beta}^{-1} \Theta_\alpha g_{\alpha\beta}\,. \tag{2.3}$$

That is, they are related by conjugation.

2.1.2 Invariant Polynomials

To build a global invariant out of the local curvature matrices, we introduce the theory of invariant polynomials on the space $\mathfrak{gl}_r(\mathbb{R})$ of $r \times r$ matrices.

A ***polynomial of degree*** k on $\mathfrak{gl}_r(\mathbb{R})$ is an element of $\mathrm{Sym}^k(\mathfrak{gl}_r(\mathbb{R}))^\vee$ (more simply, it is a polynomial $F(X)$ of degree k in the entries X_{ij} of an $r \times r$ indeterminate matrix X). A ***polynomial*** on $\mathfrak{gl}_r(\mathbb{R})$ is simply an $\mathbb{R}$-linear combination of such homogeneous polynomials.

$F(X)$ is said to be $\mathrm{ad}(GL_r(\mathbb{R}))$***-invariant*** (or simply ***invariant***) if it is under the adjoint representation of $GL_r(\mathbb{R})$ (which is basically matrix conjugation):

$$P(A^{-1}XA) = P(X)$$

for all $A \in GL_r(\mathbb{R})$. The collection of all such invariant polynomials is an $\mathbb{R}$-algebra, denoted $\mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$ ([20], 23.1).

The two most useful types of invariant polynomials are as follows:

(i) The ***Chern polynomials*** $c_k(X)$ $(1 \leq k \leq r)$. These are the coefficients of the characteristic polynomial of $-X$, i.e., of $\det(I + tX)$.

(ii) The ***trace polynomials*** $\mathfrak{t}_k(X) := \mathrm{tr}(X^k)$.

One can in fact show that both sets of polynomials $\{c_1(X), \ldots, c_r(X)\}$ and $\{\mathfrak{t}_1(X), \ldots, \mathfrak{t}_r(X)\}$ generate $\mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$ as an $\mathbb{R}$-algebra.

Finally, we introduce the concept of polarisation. Associated to every homogoeneous degree k polynomial $F(X)$ on $\mathfrak{gl}_r(\mathbb{R})$ is a polynomial $F(X_1, \ldots, X_k)$, called its ***completely polarised form***, that is uniquely determined by the following properties:

(i) $F(X_1, \ldots, X_k)$ is multilinear (i.e., is linear separately in each X_k);

(ii) $F(X_1, \ldots, X_k)$ is symmetric;

(iii) $F(X, \ldots, X) = F(X)$ (normalization condition).

If $F(X)$ were an invariant polynomial, then one can show that its completely polarized form $F(X_1, \ldots, X_k)$ is also invariant.

2.1.3 The Chern-Weil Homomorphism

With the machinery of invariant polynomials at our hand, we may state the main result of this section. Let E, M, ∇ and Θ be as before. We saw in the start of the section that the curvature matrices $\{\Theta_\alpha\}$ with respect to the open cover $\mathscr{U}$ are related by conjugation: $\Theta_\beta = g_{\alpha\beta}^{-1}\Theta_\alpha g_{\alpha\beta}$, with $g_{\alpha\beta}$ taking values in $GL_r(\mathbb{R})$. Therefore, any homogeneous degree k invariant polynomial $F(X) \in \mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$ can be evaluated on Θ_α to give us a well-defined *global* $2k$-form $P(\Theta)$.

Theorem 2.1.1 ([20], Thm.23.3)**.** *Let the notations and assumptions be as above. Then*

(i) *the global $2k$-form $F(\Theta)$ is closed;*

(i) *the cohomology class $[F(\Theta)] \in H^{2k}_{\mathrm{dR}}(M)$ is independent of the connection.*

We thus obtain an $\mathbb{R}$-algebra homomorphism

$$\begin{aligned} c_E : \mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R})) &\longrightarrow H^*_{\mathrm{dR}}(M) \\ F(X) &\longmapsto [F(\Theta)] \end{aligned}$$

called the ***Chern-Weil homomorphism***. The image of the Chern-Weil homomorphism is called the ***characteristic ring*** of E.

Definition 2.1.2. *A **characteristic class** is a natural transformation between the functors*

$$c : \mathrm{Vect}_k(-) \longrightarrow H^*_{\mathrm{dR}}(-)$$

on the category of smooth manifolds.

One can show that each invariant polynomial $F(X) \in \mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$ defines a characteristic class ([20], p.221).

Remark 2.1.3. (i) The same definitions and constructions hold in the case where E is a smooth *complex* vector bundle over M. We only need replace $GL_r(\mathbb{R})$ and $\mathfrak{gl}_r(\mathbb{R})$ by $GL_r(\mathbb{C})$ and $\mathfrak{gl}_r(\mathbb{C})$, respectively; the Chern-Weil homomorphism takes on the form

$$c_E : \mathrm{Inv}(\mathfrak{gl}(r, \mathbb{C})) \longrightarrow H^*_{\mathrm{dR}}(M, \mathbb{C}).$$

(ii) Suppose E were now a holomorphic vector bundle over a complex manifold M. Then we saw that the curvature of the Chern conection on E (with respect to some Hermitian metric) is of type $(1, 1)$. The Chern-Weil homomorphism therefore assumes the form

$$c_E : \mathrm{Inv}(\mathfrak{gl}(r, \mathbb{C})) \longrightarrow \bigoplus_{k=0}^{\infty} H^{k,k}(M).$$

2.1.4 The Pontrjagin, Euler and Chern Classes

Specific choices of invariant polynomials lead us to the Pontrjagin and Euler classes of a real vector bundle, and Chern classes of a complex vector bundle ([20], 24-25).

(1) The k***th Pontrjagin class*** of a real vector bundle $E \to M$ is defined to be

$$p_k(E) := \left[c_{2k}\left(\frac{-1}{2\pi i}\Theta\right)\right] \in H^{4k}_{\mathrm{dR}}(M).$$

The ***total Pontrjagin class*** of E is

$$p(E) := \det\left(I + \frac{-1}{2\pi i}\Theta\right) = 1 + p_1(E) + \cdots p_{\lfloor r/2 \rfloor}(E).$$

(2) The ***Euler class*** of an oriented real vector bundle $E \to M$ of even rank $2m$ is defined to be

$$e(E) := \left[\mathrm{Pf}\left(\frac{1}{2\pi}\Omega\right)\right] \in H^{2m}_{\mathrm{dR}}(M)$$

where $\mathrm{Pf}(X)$ is the *Pfaffian* of the matrix X.

(3) The k***th Chern class*** of a complex vector bundle $E \to M$ is defined to be

$$c_k(E) := \left[f_k\left(\frac{i}{2\pi}\Omega\right)\right] \in H^{2k}_{\mathrm{dR}}(M, \mathbb{C}).$$

The ***total Chern class*** of E is

$$c(E) := \det\left(I + \frac{-1}{2\pi i}\Theta\right) = 1 + c_1(E) + \cdots c_{\lfloor r/2 \rfloor}(E).$$

The total Pontrjagin and Chern classes satisfy an additional property called the ***Whitney Product Formula***: if E and E' are two vector bundles over M, then

$$p(E \oplus E') = p(E) \wedge p(E');$$
$$c(E \oplus E') = c(E) \wedge c(E').$$

2.2 Principal Bundles

In this section, we introduce (smooth) principal bundles over a manifold, with the structure group being a real Lie group. These arise naturally even when one is studying vector bundles; in fact, we show that there is a one-to-one correspondence between vector bundles and principal GL-bundles over a manifold. Principal bundles are therefore, in a sense, a generalization of the concept of vector bundles.

2.2.1 Principal Bundles

Let G be a (real) Lie group and $L(G)$ its Lie algebra. A (smooth) ***principal*** G***-bundle*** is a smooth fiber bundle $\pi : P \to M$ with fiber G such that

(i) P admits a smooth right-action of G, and M admits the trivial action of G;

(ii) $\pi : P \to M$ admits (right) G-equivariant local trivializations of the form $\varphi : \pi^{-1}(U) \longrightarrow U \times G$, where $U \subseteq M$ is open.

G is called the ***structure group*** of P.

As in the case of vector bundles (1.1.5), principal bundles can be described via their transition functions with respect to an open cover. Start with a principal G-bundle $\pi : P \to M$. Choose an open cover $\mathscr{U} = \{U_\alpha\}$ admitting smooth, G-equivariant local trivializations $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$. On nonempty overlaps $U_{\alpha\beta}$,

$$\begin{aligned} \varphi_\alpha \circ \varphi_\beta^{-1} : U_{\alpha\beta} \times G &\longrightarrow U_{\alpha\beta} \times G \\ (x, a) &\longmapsto (x,\ g_{\alpha\beta}(x) \cdot a\ ) \end{aligned}$$

where $g_{\alpha\beta} : U_{\alpha\beta} \to G$ is a smooth map, called the ***transition function from*** φ_β ***to*** φ_α. (The action is via left-multiplication). Again, these transitions satisfy the ***cocycle condition*** $g_{\alpha\gamma} = g_{\alpha\beta} \cdot g_{\beta\gamma}$ on nonempty triple overlaps $U_{\alpha\beta\gamma}$.

Conversely, given an open cover $\mathscr{U} = \{U_\alpha\}$ and a collection of smooth maps $g_{\alpha\beta} : U_{\alpha\beta} \to G$ satisfying the above cocycle condition, one can recover P.

While each fiber of a principal bundle is diffeomorphic to G, there is no canonical group structure on a particular fiber. For this reason, there is no consistent choice of 'identity' across fibers. In fact, one has the following:

Theorem 2.2.1 ([14], Prop.2.1, 2.2)**.**

(i) *Every morphism of principal bundles is an isomorphism.*

(ii) *A principal bundle admits a global section if and only if it is trivial.*

2.2.2 Frame Bundles

Let $E \to M$ be a vector bundle of rank r. Naturally associated to E is a principal $GL_r(\mathbb{R})$-bundle, called its ***frame bundle*** (denoted $\mathrm{Fr}(E)$). The fiber $\mathrm{Fr}(E)_x$ over a point $x \in M$ consists of all ordered bases of E_x. Note that sections of $\mathrm{Fr}(E)$ correspond to frames of E.

$\mathrm{Fr}(E)$ has a nice local description in terms of a trivializing open cover $\mathscr{U}$ for E. A local trivialization $\phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{C}^r$ induces an (equivariant) local trivialization of Fr(E), given by

$$\begin{aligned} \varphi_\alpha : \mathrm{Fr(E)}|_{U_\alpha} &\longrightarrow U_\alpha \times GL_r(\mathbb{R}) \\ [v_1, \ldots, v_r] \in \mathrm{Fr}(E_x) &\longmapsto (x, [(\phi_\alpha)_x v_1, \ldots, (\phi_\alpha)_x v_r]). \end{aligned}$$

Suppose $g_{\alpha\beta} : U_{\alpha\beta} \to GL_r(\mathbb{R})$ is the transition from ϕ_β to ϕ_α. A simple calculation then shows that the same map serves as a transition from φ_β to φ_α, but with $GL_r(\mathbb{R})$ now acting on itself instead of $\mathbb{C}^r$. ([20], 27.2).

Finally, we remark that a vector bundle $E \to M$ is trivial if and only if its frame bundle $\mathrm{Fr}(E) \to M$ is trivial.

2.2.3 The Vector Bundle Associated to a Representation

We now exhibit a method to construct vector bundles out of principal bundles, using representations of the structure group. Let $P \to M$ be a principal G-bundle, and $\rho : G \to GL(V)$ a finite-dimensional representation of G. There exists a vector bundle $E := P \times_\rho V$, called the ***associated bundle*** of P, whose total space is the quotient of $P \times V$ by the equivalence relation

$$(p, v) \sim (p \cdot g,\ \rho(g^{-1})v).$$

The fiber E_x over each point is isomorphic to V. One can show that a trivialization $\mathscr{U} = (U_\alpha, \varphi_\alpha)$ for P with transitions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ naturally gives rise to a trivialization $\mathscr{U} = (U_\alpha, \psi_\alpha)$ for $E = P \times_\rho V$, with transitions $\rho \circ g_{\alpha\beta} : U_{\alpha\beta} \to V$. ([20], Problem 31.1).

Here are some important examples of associated bundles:

(i) The trivial representation $\mathbb{1} : G \to GL_r(\mathbb{R})$ gives rise to the trivial bundle of rank r: $E = M \times \mathbb{R}^r$ ([20], Prop.31.1).

(ii) The bundle associated to the adjoint representation of G is called the ***adjoint bundle*** of P, and is denoted $\mathrm{ad}(P)$.

The adjoint bundle in particular is a key player in the study of connections on principal bundles, and will keep reappearing in the chapters to come.

An important fact about the associated bundle is that the space $\mathscr{E}^k(M, E)$ of E-valued k-forms is naturally isomorphic to a subcollection of k-forms on P, called the space of ***tensorial forms of type*** ρ (denoted $\mathscr{E}^k_\rho(P, V)$). (See [20], Thm.31.9).

Finally, we remark that there is a one-to-one correspondence between vector bundles of rank r and principal $GL_r(\mathbb{R})$-bundles on a manifold, given by

$$\left\{ \begin{array}{c} \text{Isomorphism Classes of} \\ \text{Vector Bundles of Rank } r \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Isomorphism Classes of} \\ \text{Principal } GL_r(\mathbb{R})\text{-bundles} \end{array} \right\}$$

$$E \longmapsto \mathrm{Fr}(E)$$

$$P \times_{\mathrm{Id}_{GL_r(\mathbb{R})}} \mathbb{R}^r \longleftarrow P$$

where $\mathrm{Id}_{GL_r(\mathbb{R})}$ is the identity map of $GL_r(\mathbb{R})$.

2.3 Smooth Connections: Principal Bundles

The lack of a linear structure on principal bundles leads us to an alternate approach to defining connections. As it turns out, the tangent bundle of any principal bundle naturally admits a "vertical" subbundle; a connection, essentially, is a choice of a complementary "horizontal" subbundle to the vertical subbundle. The idea is motivated by the notion of parallel transport, which tells us how to move frames in a parallel, or horizontal manner, along curves on a manifold.

We also show that these connections can be described in terms of smooth 1-forms taking values in the Lie algebra of the structure group. Using this approach, one can illustrate a method for going back and forth between connections on vector bundles and their associated principal bundles.

2.3.1 The Ehresmann Connection

Let $\pi : P \to M$ be a principal G-bundle. Since π is a surjective submersion, the induced map of tangent bundles $\pi_* : TP \to TM$ is also a surjective. As it turns out, the kernel $\mathcal{V} = \ker(\pi_*)$ forms a subbundle of TP, called the ***vertical subbundle*** of TP. We therefore have a **canonical exact sequence** of vector bundles over P, given by

$$0 \longrightarrow \mathcal{V} \longrightarrow TP \longrightarrow \pi^*T \longrightarrow 0. \tag{2.4}$$

The vertical subbundle $\mathcal{V}$, in fact, naturally has the structure of the trivial bundle $P \times L(G)$ on with fiber $L(G)$. This is because a choice of basis $A_1, \dots, A_l$ of $L(G)$ gives rise to a global frame $\underline{A_1}, \dots, \underline{A_l}$ for $\mathcal{V}$, where $\underline{A_j}$ is the ***fundamental vector field*** of A_j (see [20], p.251 for the definition). Furthermore, $\mathcal{V}$ admits a right action of G, which happens to be given by

$$(r_g)_*\underline{A} = \underline{\mathrm{ad}(g^{-1})A}.$$

(See [20], Prop.27.13).

We are now ready to define connections on principal bundles. A ***principal Ehresmann connection*** on P is given by any one of the following equivalent notions:

(1) a **smooth, G-equivariant splitting** of the canonical exact sequence (2.4)

$$0 \longrightarrow \mathcal{V} \longrightarrow TP \longrightarrow \pi^*T \longrightarrow 0;$$

(2) a **smooth, G-invariant horizontal subbundle** $\mathcal{H}$ of TP that is complementary to $\mathcal{V}$;

(3) a **smooth, G-equivariant $L(G)$-valued 1-form that reproduces the Lie algebra generators of the fundamental vector fields on P.** That is, a smooth form $\theta \in \mathscr{E}^1(P, L(G))$ satisfying

  (i) for any $A \in L(G)$ and $p \in P$, $\theta_p(\underline{A}_p) = A$;

  (ii) (G-equivariance) for any $g \in G$, $r_g^*\theta = (\operatorname{ad} g^{-1})\omega$.

The equivalence of (1) and (2) is straightforward from the definition; for the proof of (2) $\iff$ (3), see [20], sections 28.1, 28.3.

2.3.2 Connections on the Frame Bundle

Let $E \to M$ be a vector bundle with connection ∇. We show that ∇ naturally induces, via parallel transport, a unique principal Ehresmann connection ω on its frame bundle $\mathrm{Fr}(E)$. This gives us a beautiful geometric interpretation of the notion of a principal Ehresmann connection on the frame bundle (and is in fact what motivates the definition in the general case, of an arbitrary structure group). The contents of this subsection are lifted almost verbatim from Section 29 of [20].

Start with a point x on M, a frame e_x of E_x, and a curve c on M with initial point x. There is a unique way to parallel transport e_x along c (induced by ∇); this gives us a unique lift of c to a curve $\tilde{c}$ on $\mathrm{Fr}(E)$, whose initial point is e_x, called the ***horizontal lift*** of c.

Consider now the collection of all possible curves on M with initial point x. The space of initial velocities of all the (unique) horizontal lifts (with initial frame e_x) is a vector subspace of $T_{e_x}\mathrm{Fr}(E)$ that is complementary to $\mathcal{V}_{e_x}$. Varying e_x and x, we get a smooth, right-invariant horizontal distribution $\mathcal{H}$ on TP – an Ehresmann connection on $\mathrm{Fr}(E)$! (See [20], Thm.29.9).

A local section of E is now parallel if and only if it is a horizontal section of $\mathrm{Fr}(E)$. Moreover, if (U, e) is a framed open set for E, one can show that the Ehresmann connection θ (now thought of as a $L(G)$-valued 1-form on $\mathrm{Fr}(E)$) pulls back, via e, to the connection matrix ω_e of ∇ with respect to (U, e)! That is,

$$\theta_e = e^*\theta.$$

2.3.3 Covariant Derivatives on the Associated Bundle

In the other direction, let P be a principal G-bundle, and $\rho : G \to GL(V)$ a finite-dimensional representation. Then every smooth connection θ on P gives rise to a covariant derivative D on the associated bundle $E = P \times_\rho V$ ([20], Prop.31.16). That is, we obtain $\mathbb{R}$-linear maps

$$D : \mathscr{E}^k(M, E) \longrightarrow \mathscr{E}^{k+1}(M, E)$$

that form an antiderivation of degree +1: for any $\omega, \tau \in \mathscr{E}^*(M, E)$,

$$D(\omega \wedge \tau) = D\omega \wedge \tau + (-1)^{\deg \omega} \omega \wedge D\tau.$$

In particular, restricting to the case $k = 0$ gives us a connection on E.

2.3.4 A Local Description of Principal Connections

While one can describe a principal Ehresmann connection as a Lie algebra-valued global 1-form on P, it is often convenient to describe it as a collection of *local* Lie algebra-valued 1-forms defined on the base manifold M. This gives us a nice way to move between connections on a vector bundle and its corresponding GL-bundle, as we shall see.

Connections on the Trivial Bundle

We shall first describe connections on the trivial bundle $P = M \times G$. The canonical exact sequence (2.4) in this case looks like

$$0 \longrightarrow \mathcal{V} \longrightarrow \pi^*T \oplus \mathcal{V} \longrightarrow \pi^*T \longrightarrow 0.$$

P therefore admits a canonical connection, called the ***Maurer-Cartan connection***. It can be described as

(i) the splitting $\pi^*T \to \pi^*T \oplus \mathcal{V}$, sending $t \mapsto t \oplus 0$;

(ii) the horizontal subbundle $\mathcal{H}_{MC} = TM \times G$, where G here represents the zero section of TG;

(ii) the smooth 1-form $\omega_{MC} \in \Omega^1(M, L(G))$ that acts as $\omega_{MC}(t) = t \oplus 0$.

The reason behind the terminology is as follows: let $\pi_2 : M \times G \to G$ denote the natural projection onto the second factor. Then, the Maurer-Cartan connection is the pullback $\omega_{MC} = \pi_2^*\omega_0$ of the ***Maurer-Cartan form*** ω_0 on G!

Proposition 2.3.1 ([6], Prop.6.8)**.** *Let $P = M \times G$ be the trivial bundle. There is a one-to-one correspondence*

$$\begin{aligned} \{\text{Connections on } P\} &\longleftrightarrow \Omega^1(M, L(G)) \\ \omega_{MC} + \theta &\longleftrightarrow \theta. \end{aligned}$$

Hence the set of smooth connections on P is an affine space over $\Omega^1(M, L(G))$.

Notice how we are now dealing with differential forms defined on the base M, rather than the bundle P.

Connections on an Arbitrary Bundle

We now investigate the local picture of connections on an *arbitrary* principal bundle P. As before, we fix a trivialization $\mathscr{U} = \{(U_\alpha, \varphi_\alpha)\}$ of P, with transitions $g_{\alpha\beta}$.

Does P always admit a smooth connection? The answer is yes, as we see from the following partition-of-unity argument. For each α, choose the Maurer-Cartan connection ω_{MC}^α on $U_\alpha \times G$. Pull them back via the trivializing maps to get local connections $\eta_\alpha \in \mathscr{E}^1(P|_{U_\alpha}, L(G))$. We may now take a partition of unity $\{\rho_\alpha\}$ subordinate to $\mathscr{U}$, and obtain the locally finite sum $\eta := \sum_\alpha \rho_\alpha \eta_\alpha$; this is a connection on P, as one can check ([20], Problems 28.1 - 28.3).

In fact, the set of smooth connections on P is completely characterized by the following theorem ([6], Cor.6.9):

Theorem 2.3.2. *There is a one-to-one correspondence between the set of complex analytic connections on P and collections of 1-forms $\{\theta_\alpha \in \mathscr{E}^1(U_\alpha, L(G))\}$ satisfying*

$$\theta_\beta = \mathrm{ad}(g_{\alpha\beta}^{-1})\theta_\alpha + g_{\alpha\beta}^*\omega_0 \tag{2.5}$$

on nonempty overlaps $U_{\alpha\beta}$.

The 1-forms θ_α are called the ***local connection forms*** of the connection with respect to the cover $\mathscr{U}$.

Corollary 2.3.2.1 ([6], Cor.6.14)**.** *The set of connections on P is an affine space over the real vector space $\mathscr{E}^1(M, \mathrm{ad}(P))$.*

Relating Connections on Vector Bundles and Principal GL-Bundles

Let $E \to X$ be a vector bundle of rank r, and P the associated principal $GL_r(\mathbb{R})$-bundle. We shall now illustrate how a connection on P naturally induces one on E, and vice versa.

(1) Start with a connection ∇ on E. Then, by the procedure in 2.3.2 (using horizontal lifts), ∇ induces an Ehresmann connection θ on $P = \mathrm{Fr}(E)$.

(2) Conversely, a connection on P induces a covariant derivative D on $E = P \times_{\mathrm{Id}_{GL_r(\mathbb{R})}} \mathbb{R}^r$, by the arguments in 2.3.3.

One can show that the above operations are inverses of each others. □

There is an alternate approach to proving this using the local connection forms (resp., local connection matrices) associated to a connection on P (resp., on E). We shall quickly state the result here, without supplying too many details; a complete proof in the complex analytic setting will be provided in 3.4.3.

Start with an open covering $\mathscr{U} = \{U_\alpha\}$ of X that admit local trivializations $\phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^r$ and transitions $g_{\alpha\beta} : U_{\alpha\beta} \to GL_r(\mathbb{R})$. Let e_α denote the holomorphic local frame of E_α that is taken to the standard frame of $U_\alpha \times \mathbb{C}^r$ by ϕ_α. The frame bundle P then admits trivializations $\varphi_\alpha : P|_{U_\alpha} \to U_\alpha \times GL_r(\mathbb{R})$, whose transitions are again $g_{\alpha\beta}$, but with $GL_r(\mathbb{R})$ now acting on itself via matrix multiplication.

(1) Start with a connection ∇ on E. The connection matrices of ∇ with respect to $\mathscr{U}$ give us a collection on 1-forms $\{\theta_\alpha \in \mathscr{E}^1(U_\alpha, L(G))\}$, satisfying the transformation relation

$$\theta_\beta = g_{\alpha\beta}^{-1}\theta_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}$$

on overlaps. By (2.3.2) this gives us an Ehresmann connection θ on $P = \mathrm{Fr}(E)$.

(2) Conversely, a connection on P induces one on E by simply reversing the above steps. $\square$

Example 2.3.1. Apply the above method to the trivial bundle $E = M \times \mathbb{R}^r$. We see that the Maurer-Cartan connection $\omega_{MC} = \pi_2^*\omega_0$ on $P = M \times GL_r(\mathbb{R})$,

$$\begin{aligned} \omega_{MC} :\ \pi^*T &\longrightarrow \pi^*T \oplus L(G) \\ t &\longmapsto t \oplus 0 \end{aligned}$$

corresponds to the exterior derivative

$$\begin{aligned} d :\ (\mathscr{E}(M))^r &\longrightarrow (\mathscr{E}(M))^r \otimes \mathscr{E}^1(M) \\ (s_1, \ldots, s_r) &\longmapsto (ds_1, \ldots, ds_r). \end{aligned}$$

The connection matrix/form associated to them with respect to the (flat) global frame $(e_1, \ldots, e_r)$ is just the zero matrix.

◊♦◊

2.4 Chern-Weil Theory: Principal Bundles

The premise of Chern-Weil theory for principal bundles is the same as that of vector bundles − to construct characteristic classes using curvature. As before, this is achieved by evaluating invariant polynomials on the curvature form of some connection; the cohomology class of the resulting scalar form turns out to be independent of the choice of connection.

2.4.1 Curvature

The *curvature* of an Ehresmann connection θ on a principal bundle P is defined as

$$\Theta := d\theta + \frac{1}{2}[\theta, \theta]$$

where $[-,-] : \Omega^1(M, \mathfrak{g}) \times \Omega^1(M, \mathfrak{g}) \to \Omega^2(M, \mathfrak{g})$ is an extension of the Lie bracket to differential forms valued in a Lie algebra, called the ***Frölicher-Nijenhuis bracket***. The definition is motivated from the second structural equation $\Theta = d\theta + \theta \wedge \theta$ (1.10) for a connection ∇ on a vector bundle $E \to M$; indeed, for any 1-form τ valued in the Lie algebra $\mathfrak{gl}_r(\mathbb{R})$, one has $\tau \wedge \tau = \frac{1}{2}[\tau, \tau]$.

Let $E \to M$ be a vector bundle with connection ∇, and θ the corresponding Ehresmann connection on $P = \mathrm{Fr}(E)$. One can show that this correspondence takes the curvature of ∇ to the curvature of θ. In particular, if (U, e) is a framed open set for E, then the curvature form $\Theta \in \mathscr{E}^2(P, L(G))$ pulls back, via e, to the curvature matrix Θ_e of ∇ over U:

$$\Theta_e = e^*\Theta.$$

It so happens that Θ is also a tensorial 2-form of type ad ([20], Ex.31.10); one can therefore view the curvature as a 2-form defined on the base, that is,

$$\Theta \in \mathscr{E}^2(M, \mathrm{ad}(P)) = \mathscr{E}^2_{\mathrm{ad}}(P, L(G)).$$

2.4.2 Characteristic Classes of Principal Bundles

Let G, $L(G)$, $\pi : P \to M$, θ and Θ be as before. Invariant polynomials on $L(G)$ are defined in a similar manner as before (see 2.1.2): one simply replaces $GL_r(\mathbb{R})$ by G, and $\mathfrak{gl}_r(\mathbb{R})$ by $L(G)$. The space of $\mathrm{ad}(G)$-invariant polynomials on $L(G)$ will be denoted $\mathrm{Inv}(L(G))$. An invariant polynomial $F \in \mathrm{Inv}(L(G))$, when evaluated on the curvature Θ of some connection on P, gives us a usual/scalar differential form $F(\Theta)$ on P.

Theorem 2.4.1 ([20], Thm. 23.3)**.** *Fix $F \in \mathrm{Inv}(L(G))$. Then*

(i) *There exists a $2k$-form Λ on M such that $F(\Theta) = \pi^*\Lambda$.*

(ii) *Λ is closed.*

(iii) *The cohomology class $[\Lambda] \in H^{2k}_{\mathrm{dR}}(M)$ is independent of the connection.*

The Chern-Weil homomorphism thus takes on the avatar

$$\begin{aligned} c_P : \mathrm{Inv}(L(G)) &\longrightarrow H^*_{\mathrm{dR}}(M) \\ F &\longmapsto [\Lambda]. \end{aligned}$$

The cohomology class $[\Lambda]$ is called the ***characteristic class*** of P associated to F.

Remark 2.4.2. The $2k$-form Λ can also be directly obtained by evaluating F on Θ, where $\Theta \in \mathscr{E}^2(M, \mathrm{ad}(P))$ is now though of as a 2-form on M valued in the adjoint bundle $\mathrm{ad}(P)$.

Remark 2.4.3. If $E \to M$ is a vector bundle and P the associated frame bundle, one can show that a fixed polynomial $f \in \mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$ determines the same characteristic class for both E and P. This is done by choosing connections on E and P that correspond to each other via the equivalence described in the previous section.

Chapter 3

Complex Analytic Connections

In this chapter, we introduce the theory of complex analytic connections on holomorphic vector and principal bundles. The treatment is based on M.F. Atiyah's seminal 1957 paper [1].

Let us first recall the situation in the smooth setup. Connections always exist in the category; one can always construct integrable/flat connections locally and patch them up using partitions of unity. The resulting global connection, however, need not be integrable; this obstruction to integrability is precisely its curvature. What Chern-Weil theory then shows is that the curvature generates the characteristic cohomology ring of the bundle.

The picture changes quite drastically in the complex analytic setting, however. Complex analytic connections need not always exist − one no longer has the privilege of using partitions of unity to patch local connections. The obstruction to the existence of a complex analytic connection on a bundle naturally defines a cohomology class, called the *Atiyah class* of the bundle. Moreover, under suitable restrictions, this class generates the characteristic ring (with complex coefficients) of the bundle. A virtue of this approach is that it is canonical, and unlike the smooth case, does not require an arbitrary choice of connection.

3.1 Vector Bundles and Locally Free Sheaves

The first section of this chapter elaborates on the correspondence between vector bundles and locally free sheaves on a manifold. Translating algebraic questions about vector bundles into the language of (sheaves of) modules allows us to apply the powerful machinery of homological algebra to tackle them. In particular, extensions of vector bundles can be characterised using the cohomology groups of their corresponding sheaves. This perspective will be used when defining complex analytic connections as splittings of a specific canonical short exact sequence of bundles.

For the rest of the chapter, we shall denote by

- X, a complex n-manifold;
- $E \to X$, a holomorphic vector bundle;
- $\mathscr{O}$, the sheaf of holomorphic functions on X.

Henceforth, all manifolds and vector bundles will be holomorphic, unless explicitly stated otherwise.

3.1.1 Locally Free Sheaves

Associated to any holomorphic vector bundle E on X is its ***sheaf of holomorphic sections***, denoted $\mathscr{O}(E)$, or $\mathcal{E}$. This is a sheaf of $\mathscr{O}$-modules on X (see 1.3).

A vector bundle also has the property of being *locally trivial*: each point of X admits a neighbourhood U such that

$$E|_U \cong U \times \mathbb{C}^r$$

as holomorphic vector bundles. This endows its corresponding sheaf with the following property:

Definition 3.1.1 (Free and Locally Free Sheaves)**.** *Let $\mathcal{F}$ be a sheaf of $\mathscr{O}$-modules. $\mathcal{F}$ is said to be*

(i) ***free of rank r*** *if it is isomorphic (as an $\mathscr{O}$-module) to $\mathscr{O}^r$;*

(i) ***locally free of rank r*** *if each point of X admits a neighbourhood U such that $\mathcal{F}|_U$ is locally free of rank r.*

We then have the following result (Thm. 5.14. of [13]):

Theorem 3.1.2. *Every locally free sheaf $\mathcal{F}$ of rank r on X determines a vector bundle E, unique up to isomorphism, such that $\mathcal{F} \cong \mathcal{E}$ as $\mathscr{O}$-modules.*
We thus have a bijective correspondence

$$\left\{ \begin{array}{c} \text{Isomorphism Classes of} \\ \text{Vector Bundles of Rank } r \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Isomorphism Classes of} \\ \text{Locally Free Sheaves of Rank } r \end{array} \right\}$$
$$E \longmapsto \mathcal{E} = \mathscr{O}(E)$$

Let E, F be vector bundles on X. As discussed in 1.1.5, we may define the bundles E^*, $E \oplus F$, $E \otimes F$ and $\mathrm{Hom}(E, F)$, whose fibres at x are the vector spaces E_x^*, $E_x \oplus F_x$, $E_x \otimes F_x$ and $\mathrm{Hom}(E_x, F_x)$. We also have a canonical isomorphism $\mathrm{Hom}(E, F) \cong E^* \otimes F$.

One may now ask: what do their corresponding sheaves of sections look like? Indeed, we have canonical isomorphisms

$$\begin{aligned} \mathscr{O}(E^*) &\cong \mathscr{O}(E)^* \\ \mathscr{O}(E \oplus F) &\cong \mathscr{O}(E) \oplus \mathscr{O}(F) \\ \mathscr{O}(E \otimes F) &\cong \mathscr{O}(E) \otimes_{\mathscr{O}} \mathscr{O}(F) \\ \mathscr{O}(\mathrm{Hom}(E, F)) &\cong \mathcal{H}om(\mathscr{O}(E), \mathscr{O}(F)) \end{aligned}$$

where $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ denotes the ***sheaf of germs of homomorphisms of $\mathcal{F}$ into $\mathcal{G}$*** ([11], Chapter II, Ex.1.15). In particular, we note that locally free $\mathscr{O}$-modules are closed under the operations of taking duals, direct sums, tensor products, and Homs.

At the level of stalks, we have $\mathscr{O}_x$-isomorphisms

$$\begin{aligned} (\mathcal{E}^*)_x &\cong (\mathcal{E}_x)^* \\ (\mathcal{E} \oplus \mathcal{F})_x &\cong \mathcal{E}_x \oplus \mathcal{F}_x \\ (\mathcal{E} \otimes \mathcal{F})_x &\cong \mathcal{E}_x \otimes_{\mathscr{O}_x} \mathcal{F}_x \\ (\mathcal{H}om(\mathcal{E}, \mathcal{F}))_x &\cong \mathrm{Hom}(\mathcal{E}_x, \mathcal{F}_x) \end{aligned}$$

for any point x of X. While the first three isomorphisms are obvious, the fourth isomorphism makes use of the fact that $\mathscr{O}$ is ***coherent*** (see [17], Prop.5, p.209). We shall say a word or two about coherence in the coming paragraphs.

We now come to the issue of morphisms. The collection of vector bundle morphisms $E \to F$ can be identified with $\Gamma(X; \mathrm{Hom}(E, F))$, the space of global sections of the bundle $\mathrm{Hom}(E, F)$. Let $h : E \to F$ be one such morphism; it induces an $\mathscr{O}$-morphism $h : \mathcal{E} \to \mathcal{F}$.

An arbitrary bundle map need not have locally constant rank, as we see in the following example:

Example 3.1.1. Let $X = \mathbb{C}$, and consider the trivial line bundle $X \times \mathbb{C}$. Then the endomorphism

$$\begin{aligned} h : X \times \mathbb{C} &\longrightarrow X \times \mathbb{C} \\ (z, w) &\longmapsto (z, zw) \end{aligned}$$

has rank 1 on $\mathbb{C}^*$, and rank 0 at the point $\{0\}$. One can then easily see that $\mathrm{Ker}(h)$ and $\mathrm{Im}(h)$ are not bundles on X, and h is not an inclusion of bundles. However, one may verify that the induced sheaf map $h : \mathscr{O} \to \mathscr{O}$ is injective! ♢♦♢

As a result, kernels, images, and cokernels of bundle maps need not be vector bundles themselves. The same is true of locally free sheaves of $\mathscr{O}$-modules. Moreover, an inclusion at the level of sheaves of sections need not correspond to an inclusion of bundles.

How does one remedy this situation? One approach is to only allow morphisms of constant, or locally constant rank; however, this turns out to be very restrictive. The second, more algebraic approach is to enlarge our category from locally free $\mathscr{O}$-modules to that of ***coherent*** $\mathscr{O}$***-modules***.

As it turns out, coherent $\mathscr{O}$-modules are closed under all the algebraic operations listed previously (namely, duals, direct sums, tensor products, and Homs). Moreover, kernels, cokernels, and images of homomorphisms between coherent $\mathscr{O}$-modules are also coherent. Two important results about coherent modules relevant to us are as follows:

Theorem 3.1.3 (Oka)**.** *$\mathscr{O}_X$ is a coherent sheaf of rings (i.e., is coherent as a module over itself) for any complex analytic variety X.*

Proposition 3.1.4. *Locally free $\mathscr{O}$-modules are coherent.*

For the proofs of the above statements, and a general, in-depth discussion on coherent sheaves, we refer the reader to [17].

3.1.2 Extensions

We now study exact sequences and extensions of vector bundles. A sequence of vector bundles and bundle maps

$$0 \longrightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \longrightarrow 0 \tag{3.1}$$

is said to be ***exact*** if for each $x \in X$, the sequence of fibres

$$0 \longrightarrow E'_x \xrightarrow{f_x} E_x \xrightarrow{g_x} E''_x \longrightarrow 0$$

is exact. In this case, E' is called *subbundle* of E, and E'' a *factor bundle* of E. We also have the relation $E'' \cong E/E'$.

Definition 3.1.5 (Splittings)**.** *The sequence (3.1) is said to* ***split*** *if any one of the following equivalent conditions hold:*

(i) *there exists a bundle map $r : E \to E'$ such that $rf = \mathbb{1}_{E'}$;*

(ii) *there exists a bundle map $s : E'' \to E$ such that $gs = \mathbb{1}_{E''}$.*

In this case, $E \cong E' \oplus E''$.

Choose an open set U on which E' admits a frame $(s_1, \ldots, s_p)$ that extends to a frame $(s_1, \ldots, s_q)$ of E. Then $(\overline{s}_{p+1}, \ldots, \overline{s}_q)$ is a frame for E'', and we obtain a splitting $E''_U \longrightarrow E_U$ by lifting each $\overline{s}_j$ to s_j. Thus, splittings can always be constructed locally. Furthermore, if

$$\rho, \sigma : E'_U \oplus E''_U \longrightarrow E_U$$

are two local isomorphisms generated by two such possibly distinct splittings, their difference

$$\sigma^{-1}\rho : E'_U \oplus E''_U \longrightarrow E'_U \oplus E''_U$$

is of the form $I+\phi$, where I is the identity, and $\phi : E''_U \longrightarrow E_U$. This is an important observation that will help us study the local structure of extensions (see next).

In the smooth setup, one can always utilise partitions of unity to patch up such local splittings and generate a global one. This is no longer possible in the holomorphic world, and hence, holomorphic global splittings do not always exist. In what follows, we shall develop a method to measure this obstruction to splitting using cohomological techniques. The first step is to translate questions about exact sequences into the language of $\mathscr{O}$-modules:

Proposition 3.1.6 ([1], Prop.1)**.**

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

is an exact sequence of bundles if and only if

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

is an exact sequence of sheaves. Furthermore, one of them splits if and only if the other does.

Definition 3.1.7 ([1], p.184)**.**

(a) *An **extension** of E'' by E' is an exact sequence of bundles*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0.$$

(b) *Two extensions of E'' by E' are **equivalent** if we have an isomorphism of exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & E' \arrow[r] \arrow[d, "I'"] & E_1 \arrow[r] \arrow[d, "\simeq"] & E'' \arrow[r] \arrow[d, "I''"] & 0 \\ 0 \arrow[r] & E' \arrow[r] & E_2 \arrow[r] & E'' \arrow[r] & 0 \end{tikzcd}$$

where I' and I'' are the identity morphisms of E' and E'', respectively.

If the exact sequence in (i) splits, we call the extension ***trivial***. Alternately, a trivial extension is one that is equivalent to the extension

$$0 \longrightarrow E' \longrightarrow E' \oplus E'' \longrightarrow E'' \longrightarrow 0.$$

Similar definitions hold for the corresponding sheaves, and by (3.1.6), the extensions of E'' by E' correspond one-to-one with the extensions of $\mathcal{E}''$ by $\mathcal{E}'$, equivalence being preserved in this correspondence. We now make use of the observation about differences of splittings (see the discussion after defining splittings), and argue as in p.184 of [1] to obtain the following result:

Proposition 3.1.8 ([1], Prop.2)**.** *The equivalence classes of extensions of E'' by E' are in one-to-one correspondence with the elements of $\check{H}^1(X, \mathcal{H}om(\mathcal{E}'', \mathcal{E}'))$, the trivial extension corresponding to the zero element.*

Remark 3.1.9. (a) A similar result holds in the smooth case, upon replacing all holomorphic sheaves with their smooth counterparts. Note, however, that $\mathcal{H}om(\mathcal{E}'', \mathcal{E}')$ becomes a fine sheaf, and so $\check{H}^1(X, \mathcal{H}om(\mathcal{E}'', \mathcal{E}')) = 0$! Thus, every smooth extension is trivial. We also saw this directly via a partition-of-unity argument.

(b) The result in (a) is also true if X is a ***Stein manifold*** ([13], p.65); this follows from Cartan's Theorem B and the fact that $\mathcal{H}om(\mathcal{E}'', \mathcal{E}')$ is coherent.

(c) If X is a compact complex manifold, $\check{H}^1(X, \mathcal{H}om(\mathcal{E}'', \mathcal{E}'))$ is generally known to be nontrivial.

We shall now identify the cohomology class corresponding to a chosen extension $\mathcal{E}$ of $\mathcal{E}''$ by $\mathcal{E}'$. As one would in module theory, we apply the covariant functor $\mathcal{H}om(\mathcal{E}'', -)$ to the exact sequence

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0.$$

Since $\mathcal{E}''$ is locally free (and therefore projective), $\mathcal{H}om(\mathcal{E}'', -)$ is an exact functor. We get an exact sheaf sequence

$$0 \longrightarrow \mathcal{H}om(\mathcal{E}'', \mathcal{E}') \longrightarrow \mathcal{H}om(\mathcal{E}'', \mathcal{E}) \longrightarrow \mathcal{H}om(\mathcal{E}'', \mathcal{E}'') \longrightarrow 0.$$

The associated long exact sequence of cohomology groups contains the segment

$$\check{H}^0(X,\, \mathcal{H}om(\mathcal{E}'', \mathcal{E})) \longrightarrow \check{H}^0(X,\, \mathcal{H}om(\mathcal{E}'', \mathcal{E}'')) \xrightarrow{\delta} \check{H}^1(X,\, \mathcal{H}om(\mathcal{E}'', \mathcal{E}')).$$

The identity element $I'' \in \mathcal{H}om(\mathcal{E}'', \mathcal{E}'')$ maps to the class $\delta(I'')$, which represents the extension $\mathcal{E}$.

A more explicit, geometric construction is as follows. Choose an open cover $\mathscr{U} = \{U_\alpha\}$ of X on which $\mathcal{E}'_\alpha, \mathcal{E}_\alpha$ and $\mathcal{E}''_\alpha$ are all free. (These are the restrictions of $\mathcal{E}', \mathcal{E}$ and $\mathcal{E}''$ to U_α, respectively). Choose local splittings $h_\alpha : \mathcal{E}''_\alpha \to \mathcal{E}_\alpha$. Then (h_α) defines a Čech 0-cochain of $\mathcal{H}om(\mathcal{E}'', \mathcal{E}')$ that lifts the 1-cocycle I'' of $\mathcal{H}om(\mathcal{E}'', \mathcal{E}'')$. By the definition of the connecting homomorphism δ,

$$(h_\beta - h_\alpha) \in \mathscr{Z}^1(\mathscr{U}, \mathcal{H}om(\mathcal{E}'', \mathcal{E}'))$$

is a representative 1-cocycle for $\delta(I'')$.

On the other hand, these splittings define local isomorphisms

$$\begin{aligned} u_\alpha : \mathcal{E}'_\alpha \oplus \mathcal{E}''_\alpha &\longrightarrow \mathcal{E}_\alpha \\ s' \oplus s'' &\longmapsto s' + h_\alpha(s'') \end{aligned}$$

with transitions

$$\begin{aligned} g_{\alpha\beta} := u_\alpha^{-1} u_\beta \ : \ \mathcal{E}'_{\alpha\beta} \oplus \mathcal{E}''_{\alpha\beta} &\longrightarrow \mathcal{E}'_{\alpha\beta} \oplus \mathcal{E}''_{\alpha\beta} \\ s' \oplus s'' &\longmapsto (s' + h_\beta(s'') - h_\alpha(s'')) \oplus s''. \end{aligned}$$

That is, $g_{\alpha\beta} = I + (h_\beta - h_\alpha)$. Following the arguments in [1], p.184, the extension $\mathcal{E}$ is determined by the class $[(h_\beta - h_\alpha)] = \delta(I'')$.

Finally, we address the converse: given a class in $\check{H}^1(X,\, \mathcal{H}om(\mathcal{E}'', \mathcal{E}'))$, how does one determine the corresponding extension? Again, choose an open covering $\mathscr{U} = \{U_\alpha\}$ so that the given class is represented by a 1-cocycle $(\phi_{\alpha\beta})$ w.r.t. $\mathscr{U}$. This determines a collection of coordinate transformations

$$\psi_{\alpha\beta} := I + \phi_{\alpha\beta} \ : \ \mathcal{E}'_{\alpha\beta} \oplus \mathcal{E}''_{\alpha\beta} \longrightarrow \mathcal{E}'_{\alpha\beta} \oplus \mathcal{E}''_{\alpha\beta}.$$

The collection $\mathcal{E}'_{\alpha\beta} \oplus \mathcal{E}''_{\alpha\beta}$ can then be glued together via the maps $\psi_{\alpha\beta}$ (in the manner described in [1], p.184) to obtain an extension $\mathcal{E}$.

3.2 The Atiyah Class

As before, let X be a complex manifold. Let G be a connected complex Lie group, whose Lie algebra we denote $L(G)$. Fix a holomorphic principal G-bundle $\pi : P \to X$. In this section, we will show that associated with P is a canonical exact sequence of vector bundles over X, called the ***Atiyah sequence of*** P. Complex analytic connections on P will then be defined as splittings of the Atiyah sequence of P. Of course, these need not always exist, and the obstruction is precisely the cohomology class determined by the Atiyah sequence of P; it is called the ***Atiyah class*** of P.

We will then determine explicit representatives for the Atiyah class with respect to an open cover of X trivializing P. In the case that the Atiyah class equals zero, we shall see how to describe a complex analytic connection using a collection of ***local connection forms*** associated with the trivializing open cover.

3.2.1 The Atiyah Sequence

Let G, $L(G)$, and P be as before. Denote by T the holomorphic tangent bundle of X, and B the holomorphic tangent bundle of P. Each element $g \in G$ determines a right-multiplication map r_g, which is a fiber-preserving biholomorphism of G. The corresponding right differentials $D'r_g$ then determine a free, fiber-preserving action $B \times G \to B$.

Consider the space B/G.

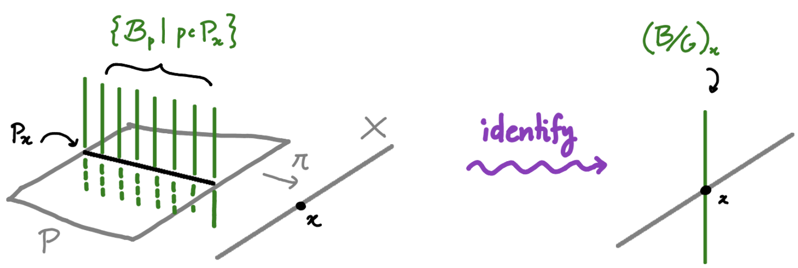

Figure 3.1: Visual representation of B/G. Choose a point $x \in X$, and look at the fiber P_x. Over each $p \in P_x$, we have the tangent space B_p; identify all of these together using the action of G via right-differentials.

Following the arguments in [1], p.186, B/G has the structue of a holomorphic vector bundle over X. A point in the fiber $(B/G)_x$ corresponds precisely to a right-invariant holomorphic vector field on P_x: indeed, the set of vectors identified with $v \in B_p$ is $\{(D'r_g)_p(v) \mid g \in G\}$.

Definition 3.2.1. *The holomorphic vector bundle B/G is called the* ***Atiyah bundle*** *of P, and is denoted* $\mathrm{At}(P)$*.*

Now, consider the *vertical subbundle* $\mathcal{V} = \ker(D'\pi)$ of B. The right action of G on B descends to $\mathcal{V}$, as we see by performing an explicit computation:

Fix $p \in P$. Let $A \in L(G)$ be an element of the Lie algebra, and consider the fundamental vector $\underline{A}_p \in \mathcal{V}_p$. By G-equivariance of the right-differentials,

$$(D'r_g)_p(\underline{A}_p) = \underline{\mathrm{ad}(g^{-1})A}_{p \cdot g}$$

([20] Prop.27.13, adapted to the holomorphic setting) which is again in $\mathcal{V}_{p\cdot g}$. We may thus form the space $\mathcal{V}/G$; similar arguments to before show that this too admits the structure of a holomorphic vector bundle on X. Furthermore, one may observe from the above calculation that this is simply the *adjoint bundle* $P \times_{\mathrm{ad}} G = \mathrm{ad}(P)$ of P!

Now, recall (2.4, adapted to the holomorphic setting) that we have a canonical exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathcal{V} \longrightarrow B \longrightarrow \pi^*T \longrightarrow 0$$

over P. Note that G acts on B and $\mathcal{V}$ via right-differentials, and on T trivially. (In the smooth setup, connections were defined as G-equivariant splittings of this sequence). Going modulo G and arguing as in [1], p.187 gives us the following result:

Theorem 3.2.2 ([1], Thm.1)**.** *There exists a canonical exact sequence $\mathscr{A}(P)$ of vector bundles over X*

$$0 \longrightarrow \mathrm{ad}(P) \longrightarrow \mathrm{At}(P) \longrightarrow T \longrightarrow 0$$

called the ***Atiyah sequence*** *of P.*

Definition 3.2.3. *A **complex analytic connection** on the principal bundle P is a holomorphic splitting of the Atiyah sequence $\mathscr{A}(P)$.*

Remark 3.2.4. (i) One can easily show that $\pi^*\mathrm{At}(P) \cong B$, and $\pi^*\mathrm{ad}(P) \cong \mathcal{V}$. Thus, the Atiyah sequence $\mathscr{A}(P)$ is obtained from (3.2.1) by going modulo G, and the sequence (3.2.1) is obtained as the pullback of $\mathscr{A}(P)$ by π.

(ii) How are (holomorphic) splittings of (3.2.1) and $\mathscr{A}(P)$ related? We can answer this in one direction: clearly, a G-equivariant splitting of (3.2.1) gives a splitting of $\mathscr{A}(P)$.

3.2.2 The Atiyah Class

When does the principal bundle P admit a complex analytic connection? The Atiyah sequence $\mathscr{A}(P)$ is a canonical extension of T by $\mathrm{ad}(P)$, which by (3.1.8), defines a class

$$a(P) \in \check{H}^1(X, \mathcal{H}om(\mathcal{T}, \mathrm{ad}(\mathcal{P})))$$

where $\mathcal{T}$ and $\mathrm{ad}(\mathcal{P})$ are the sheaves corresponding to T and $\mathrm{ad}(P)$, respectively. Moreover, $\mathscr{A}(P)$ splits if and only if $a(P) = 0$. Let Ω^p denote the sheaf of holomorphic p-forms on X. Identifying the sheaves $\mathcal{H}om(\mathcal{T}, \mathrm{ad}(\mathcal{P}))) \cong \Omega^1 \otimes \mathrm{ad}(\mathcal{P})$ via the canonical isomorphism, we have:

Theorem 3.2.5 ([1], Thm.2)**.** *A principal bundle P over X defines an element*

$$a(P) \in \check{H}^1(X, \Omega^1 \otimes \mathrm{ad}(\mathcal{P})),$$

*called the **Atiyah class** of P. P admits a complex analytic connection if and only if $a(P) = 0$.*

Remark 3.2.6. Combining this with (3.1.9), we can say:

(a) Connections always exist in the smooth setup.

(b) If X is Stein, then every principal bundle over X admits a complex analytic connection.

Again, (a) can be proved directly via a partition-of-unity argument.

Functorial Properties

We now list some functorial properties of $\mathscr{A}(P)$ and $a(P)$, following p.188 of [1].

(1) Let (P, X, G) be as before. Suppose we have a Lie group homomorphism $h : G \to G'$. Then h naturally induces a principal G'-bundle P' over X, whose total space $(P \times G)/\sim$ is the quotient by the relation

$$(p, g) \sim (p \cdot g,\ h(g^{-1}) \cdot g).$$

This induces a natural bundle map $h : P \to P'$, $p \mapsto [(p, e)]$, which then induces a map of (holomorphic) tangent bundles $D'h : B \to B'$. One may then check ([1], p.188) that we obtain a homomorphism of Atiyah sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{ad}(P) \arrow[r] \arrow[d, "\overline{h}"] & \mathrm{At}(P) \arrow[r] \arrow[d, "\overline{h}"] & T \arrow[r] \arrow[d, "I"] & 0 \\ 0 \arrow[r] & \mathrm{ad}(P') \arrow[r] & \mathrm{At}(P') \arrow[r] & T \arrow[r] & 0 \end{tikzcd}$$

where $\overline{h} : \mathrm{At}(P) \to \mathrm{At}(P')$ is induced by h and I is the identity map.

(2) Suppose we have a holomorphism $f : X' \to X$. We may pull P back via f to obtain a principal G-bundle $P' := f^*P$ over X', and there is a commutative diagram

$$\begin{tikzcd} P' = f^*P \arrow[r, "\tilde{f}"] \arrow[d, "\pi'"] & P \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X. \end{tikzcd}$$

Again, $\tilde{f}$ induces a map of tangent bundles $D'\tilde{f} : B \to B'$. We then obtain a homomorphism of Atiyah sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{ad}(P') \arrow[r] \arrow[d, "\overline{f}"] & \mathrm{At}(P') \arrow[r] \arrow[d, "\overline{f}"] & T' \arrow[r] \arrow[d, "D'f"] & 0 \\ 0 \arrow[r] & \mathrm{ad}(P) \arrow[r] & \mathrm{At}(P) \arrow[r] & T \arrow[r] & 0 \end{tikzcd}$$

where $\overline{f} : \mathrm{At}(P') \to \mathrm{At}(P)$ is induced by $D'\tilde{f}$. Note, however, that this is a morphism between exact sequences of bundles over different base spaces; it would be more precise to pull back $\mathscr{A}(P)$ via f, and then express the morphism as one of exact sequences over the common base X'.

The above functorial properties of the Atiyah sequence give us the following functorial properties of the Atiyah class ([1], p.189):

(1) With notations and assumptions as in (1) above, let

$$h^{\#} : \check{H}^1(X,\, \Omega^1 \otimes \mathrm{ad}(\mathcal{P})) \longrightarrow \check{H}^1(X,\, \Omega^1 \otimes \mathrm{ad}(\mathcal{P}'))$$

be the map induced by the sheaf map $h : \mathrm{ad}(\mathcal{P}) \to \mathrm{ad}(\mathcal{P}')$. Then $a(P') = h^{\#}a(P)$.

(2) Notations and assumptions as in (2) above, let

$$f^{\#} : \check{H}^1(X,\, \Omega^1_X \otimes \mathrm{ad}(\mathcal{P})) \longrightarrow \check{H}^1(X',\, \Omega^1_{X'} \otimes \mathrm{ad}(\mathcal{P}'))$$

be the map induced by the sheaf map $f : \Omega^1_X \otimes \mathrm{ad}(\mathcal{P}) \to \Omega^1_{X'} \otimes \mathrm{ad}(\mathcal{P}')$. Then $a(P') = f^{\#}a(P)$.

The proofs of these statements follow immediately once we construct explicit representatives for the Atiyah class, which is done next.

Explicit Representatives of the Atiyah Class

We calculate explicit representatives of the Atiyah class $a(P)$ in terms of coordinate transformations of P, as follows. Let $\mathscr{U} = \{U_\alpha\}$ be an open covering of X, with local trivializations $\varphi_\alpha : P_\alpha \to U_\alpha \times G$ and transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$; that is,

$$\begin{aligned} \varphi_\alpha\varphi_\beta^{-1} : U_{\alpha\beta} \times G &\longrightarrow U_{\alpha\beta} \times G \\ (x, a) &\longmapsto (x,\; g_{\alpha\beta}(x) \cdot a). \end{aligned}$$

(As before, the subscript α denotes the restriction of a bundle to U_α). Each trivialization induces an isomorphism $\hat{\varphi}_\alpha : \mathrm{At}(P)_\alpha \to T_\alpha \oplus \mathrm{ad}(P)_\alpha$. Now, we have an obvious local splitting of $\mathscr{A}(P_\alpha)$: simply take the map

$$\begin{aligned} \eta_\alpha : T_\alpha &\longrightarrow \mathrm{At}(P)_\alpha \\ t &\longmapsto \hat{\varphi}_\alpha^{-1}(t \oplus 0). \end{aligned}$$

Set $\eta_{\alpha\beta} = \eta_\beta - \eta_\alpha$. Then, by the discussion in (3.1.2),

$$(\eta_{\alpha\beta}) \in \mathscr{Z}^1(\mathscr{U}, \Omega^1 \otimes \mathrm{ad}(\mathcal{P})) \tag{3.2}$$

is a representative cocycle for $a(P)$.

Remark 3.2.7. Beware of the notational differences with [1]. Atiyah uses the indexing letters i, j, and denotes his trivializations as $u_i : U_i \times G \to P_i$ (which we would have called φ_i^{-1}).

3.2.3 Local Connection Forms

Suppose now that $a(P) = 0$, so that P admits a complex analytic connection. We shall show how to describe a splitting of $\mathscr{A}(P)$ as a collection of ***local connection forms*** with respect to a trivializing open cover of X. The approach is similar in spirit to (2.3.4); the difference being that we use the Atiyah sequence instead of (3.2.1).

Connections on the Trivial Bundle

We shall first describe connections on the trivial bundle $P = X \times G$. Recall that in this case, $\mathrm{At}(P) = T \oplus \mathrm{ad}(P)$; furthermore, $\mathrm{ad}(P)$ itself is the trivial bundle L with fibre $L(G)$, a canonical isomorphism being

$$\begin{aligned} L &\longrightarrow \mathrm{ad}(P) \\ (x, A) &\longmapsto [(x, e), A]. \end{aligned}$$

The Atiyah sequence therefore looks like

$$0 \longrightarrow \mathrm{ad}(P) \longrightarrow T \oplus \mathrm{ad}(P) \longrightarrow T \longrightarrow 0.$$

P therefore admits a canonical connection, which we again call the (holomorphic) ***Maurer-Cartan connection***. It can be described as

(i) the splitting $T \to T \oplus \mathrm{ad}(P)$, sending $t \mapsto t \oplus 0$;

(ii) the holomorphic 1-form $\omega_{MC} \in \Omega^1(X, \mathrm{At}(\mathcal{P}))$ that acts as $\omega_{MC}(t) = t \oplus 0$.

The splittings η_α constructed previously were simply the pullbacks of the Maurer-Cartan connections on $U_\alpha \times G$ by the trivializing maps φ_α. By the general theory of (3.1.8), and the structure of the Atiyah sequence for $P = X \times G$, we have the following:

Proposition 3.2.8. *$P = X \times G$ the trivial bundle. There is a one-to-one correspondence*

$$\begin{aligned} \{\text{Connections on } P\} &\longleftrightarrow \Omega^1(X, \mathrm{ad}(P)) \\ \omega_{MC} + \theta \ &\longleftrightarrow \theta. \end{aligned}$$

Hence the set of complex analytic connections on P is an affine space over the $\mathbb{C}$-vector space $\Omega^1(X, \mathrm{ad}(P)) \cong \Omega^1(X, L)$.

Connections on an Arbitrary Bundle with Vanishing Atiyah Class

We now investigate the local picture of connections on an *arbitrary* principal bundle whose Atiyah class vanishes. We start with a lemma:

Lemma 3.2.9. *Let $P = X \times G$ be the trivial bundle, and let $\psi : P \to P$ be an endomorphism of the form $\psi(x, a) = (x,\ g(x) \cdot a)$, where $g : X \to G$. (For instance, the transition between overlapping trivial open sets of an arbitrary principal bundle).*

(i) *The induced isomorphism of Atiyah bundles is*

$$\begin{aligned} \hat{\psi} :\ \mathrm{At}(P) &\xrightarrow{\simeq} \mathrm{At}(P) \\ [\, t \oplus A\,] &\longmapsto [\, t \oplus \mathrm{ad}(g)(g^*\omega_0(t) + A)\,]. \end{aligned}$$

*where ω_0 is the holomorphic **Maurer-Cartan form** on G.*

(ii) *The above map restricted to $\mathrm{ad}(P)$ yields an isomorphism*

$$\begin{aligned} \hat{\psi} :\ \mathrm{ad}(P) &\xrightarrow{\simeq} \mathrm{ad}(P) \\ [A] &\longmapsto [\mathrm{ad}(g)(A)]. \end{aligned}$$

Proof. Let us first compute the differential $D'\psi$ at the point (x, e): this is

$$\begin{aligned}(D'\psi)_{(x,e)}: \ T_x \oplus L(G) &\longrightarrow T_x \oplus T_{g(x)}G \\ t \oplus 0 &\longmapsto t \oplus (D'g)(t) \\ 0 \oplus A &\longmapsto 0 \oplus (D'l_{g(x)})(A).\end{aligned}$$

Now, go modulo G; this identifies $T_{g(x)}G$ with $L(G)$ vis $D'r_{g(x)^{-1}}$. The induced isomorphism of Atiyah bundles therefore reads

$$\begin{aligned}\hat{\psi}_{(x,e)}: \ \mathrm{At}(P)_x &\longrightarrow \mathrm{At}(P)_x \\ t \oplus A &\longmapsto t \oplus (\, D'r_{g(x)^{-1}} \circ D'g(t) + D'r_{g(x)^{-1}} \circ D'l_{g(x)}(A)\,).\end{aligned}$$

This is simplified by the following observations. First, $D'r_{g(x)^{-1}} \circ D'l_{g(x)}$ is simply $\mathrm{ad}(g(x))$; second, $D'r_{g(x)^{-1}} \circ D'g = \mathrm{ad}(g) \cdot g^*\omega_0$! This proves (i), and (ii) immediately follows. □

Now, let P be a principal bundle on X, such that $a(P) = 0$. As before, let $\mathscr{U} = \{U_\alpha\}$ be the open cover of X with local trivializations $\varphi_\alpha : P_\alpha \to U_\alpha \times G$ for P and transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$. The set of complex analytic connections on P is completely characterized by the following result:

Theorem 3.2.10. *There is a one-to-one correspondence between the set of complex analytic connections on P and collections of 1-forms $\{\theta_\alpha \in \Omega^1(U_\alpha, L)\}$ satisfying*

$$\theta_\beta = \mathrm{ad}(g_{\alpha\beta}^{-1})\theta_\alpha + g_{\alpha\beta}^*\omega_0 \tag{3.3}$$

on nonempty overlaps $U_{\alpha\beta}$.

Proof. Let $\theta : T \to \mathrm{At}(P)$ be a connection of P. On each U_α, the trivialization φ_α induces an isomorphism of Atiyah sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{ad}(P)_\alpha \arrow[r] \arrow[d] & \mathrm{At}(P)_\alpha \arrow[r] \arrow[d, "\hat{\varphi}_\alpha"] & T_\alpha \arrow[r] \arrow[d, "\gamma_\alpha"] & 0 \\ 0 \arrow[r] & L_\alpha \arrow[r] & T_\alpha \oplus L_\alpha \arrow[r] & T_\alpha \arrow[r] & 0. \end{tikzcd}$$

θ then induces a splitting

$$\begin{aligned} \omega^\alpha_{MC} + \theta_\alpha :\ T_\alpha &\longrightarrow T_\alpha \oplus L_\alpha \\ t &\longmapsto t \oplus \theta_\alpha(t) \end{aligned}$$

where $\theta_\alpha \in \Omega^1(U_\alpha, L)$. (We have used (3.2.8)).

Now, suppose U_α and U_β overlap; we wish to compare the forms θ_α and θ_β on the overlap. First, note that the transition map

$$\begin{aligned} \varphi_\alpha \circ \varphi_\beta^{-1} : U_{\alpha\beta} \times G &\longrightarrow U_{\alpha\beta} \times G \\ (x, a) &\longmapsto (x,\ g_{\alpha\beta}(x) \cdot a). \end{aligned}$$

induces a commutative diagram of Atiyah sequences

$$\begin{tikzcd} 0 \arrow[r] & L_{\alpha\beta} \arrow[r] & T_{\alpha\beta} \oplus L_{\alpha\beta} \arrow[r] & T_{\alpha\beta} \arrow[r] & 0 \\ 0 \arrow[r] & \mathrm{ad}(P)_{\alpha\beta} \arrow[r] \arrow[u] \arrow[d] & \mathrm{At}(P)_{\alpha\beta} \arrow[r] \arrow[u, "\hat\varphi_\beta"] \arrow[d, "\hat\varphi_\alpha"] & T_{\alpha\beta} \arrow[r] \arrow[u, "\gamma_\beta"] \arrow[d, "\gamma_\alpha"] & 0 \\ 0 \arrow[r] & L_{\alpha\beta} \arrow[r] & T_{\alpha\beta} \oplus L_{\alpha\beta} \arrow[r] & T_{\alpha\beta} \arrow[r] & 0. \end{tikzcd}$$

Moreover, by the previous lemma,

(i) $\hat\varphi_\alpha \circ \hat\varphi_\beta^{-1}(t \oplus A) = t \oplus \mathrm{ad}(g_{\alpha\beta})(g^*_{\alpha\beta}\omega_0(t) + A)$;

(ii) $\gamma_\alpha \circ \gamma_\beta^{-1} = \mathbb{1}_{T_{\alpha\beta}}$.

We use this to compute, for $s \in T_{\alpha\beta}$,

$$\begin{aligned} \gamma_\alpha(s) \oplus \theta_\alpha(\gamma_\alpha(s)) &= \hat\varphi_\alpha(\beta(s)) \\ &= \hat\varphi_\alpha \circ \hat\varphi_\beta^{-1} \circ \hat\varphi_\alpha(\beta(s)) \\ &= \hat\varphi_\alpha \circ \hat\varphi_\beta^{-1}(\gamma_\beta(s) \oplus \theta_\beta(\gamma_\beta(s))) \\ &= t \oplus \mathrm{ad}(g_{\alpha\beta})(g^*_{\alpha\beta}\omega_0(\gamma_\beta(s)) + \theta_\beta(\gamma_\beta(s))). \end{aligned}$$

Replace s by $\gamma_\beta^{-1}(t)$; we get

$$t \oplus \theta_\alpha(t) \ = t \oplus \mathrm{ad}(g_{\alpha\beta})(g^*_{\alpha\beta}\omega_0(t) + \theta_\beta(t)).$$

Applying $\mathrm{ad}(g^{-1}_{\alpha\beta})$ on both sides gives us (3.3).

Conversely, suppose we are given 1-forms $\{\theta_\alpha \in \Omega^1(U_\alpha, L)\}$ satisfying

$$\theta_\beta = \mathrm{ad}(g_{\alpha\beta}^{-1})\theta_\alpha + g_{\alpha\beta}^*\omega_0$$

on nonempty overlaps $U_{\alpha\beta}$. Then, we may reverse-engineer the previous steps and show that the local splittings

$$\begin{aligned} T_\alpha &\longrightarrow \mathrm{At}(P)_\alpha \\ t &\longmapsto \hat{\varphi}_\alpha^{-1}(t \oplus \theta_\alpha(t)) \end{aligned}$$

agree on nonempty overlaps. These patch to yield a global splitting of $\mathscr{A}(P)$.

$\square$

Definition 3.2.11. *The 1-forms θ_α are called the **local connection forms** of the connection θ with respect to the cover $\mathscr{U}$.*

Remark 3.2.12. Part of the above compuations are performed by Atiyah in [1], p.190, albeit with some differences in notation.

3.3 The Characteristic Ring

Mirroring Chern-Weil theory (2), we show in this section that the Atiyah class of a principal bundle generates, under suitable restrictions, its characteristic ring with complex coefficients.

3.3.1 Curvature and the Atiyah Class

Let $P \to X$ be a holomorphic principal bundle, and $\mathrm{ad}(P) \to X$ its adjoint bundle. Recall (1.3.2) that we have a **Dolbeault isomorphism**

$$H^{p,q}(X, \mathrm{ad}(P)) \cong \check{H}^q(X, \Omega^p \otimes \mathrm{ad}(\mathcal{P})). \tag{3.4}$$

where the left-hand side is the (p,q)-th Dolbeault cohomology group of $\mathrm{ad}(P)$, and the right-hand side is the qth cohomology of the sheaf $\Omega^p \otimes \mathrm{ad}(\mathcal{P})$. Recall further from (3.2) that $a(P)$ is represented by

$$(\eta_{\alpha\beta}) \in \mathscr{Z}^1(\mathscr{U}, \Omega^1 \otimes \mathrm{ad}(\mathcal{P}))$$

where $\eta_{\alpha\beta}$ are as described before. Consider the underlying smooth sheaf $\mathscr{E}^1 \otimes_{\mathscr{E}} \mathscr{E}(\mathrm{ad}(P))$; since this is fine, all of its higher cohomologies vanish! In particular, the 1-cocycle

$$(\eta_{\alpha\beta}) \in \mathscr{Z}^1(\mathscr{U}, \mathscr{E}^1 \otimes \mathscr{E}(\mathrm{ad}(P)))$$

becomes a coboundary: $(\eta_{\alpha\beta}) = (f_\beta - f_\alpha)$ for some collection of *smooth* $\mathrm{ad}(P)$-valued 1-forms (f_α). Using these, one can construct (follow [1] p.190, or the method in (3.2.10)) a collection of 1-forms $\{\theta_\alpha \in \mathscr{E}^1(U_\alpha, L(G))\}$ satisfying

$$\theta_\beta = \mathrm{ad}(g_{\alpha\beta}^{-1})\theta_\alpha + g_{\alpha\beta}^*\omega_0.$$

These are the local connection forms of a *smooth* connection θ on P. Its curvature Θ is locally given by

$$\Theta_\alpha = d\theta_\alpha + \frac{1}{2}[\theta_\alpha, \theta_\alpha].$$

Suppose now that each θ_α is of type $(1,0)$, so that $\Theta_\alpha^{1,1} = \overline{\partial}\theta_\alpha$. Then, using the above relations, one may show that the $\overline{\partial}$-cohomology class $[\Theta^{1,1}]$ is independent of the choice of smooth connection on P, provided it be of type $(1,0)$! We thus obtain the following proposition (Prop.4 of [1]):

Proposition 3.3.1. *Let θ be a C^∞ connection of type $(1,0)$ on P, $\Theta^{1,1}$ the $(1,1)$-component of its curvature, and $[\Theta^{1,1}] \in H^{1,1}(X, \mathrm{ad}(P))$ its $\overline{\partial}$-cohomology class. Then $[\Theta^{1,1}]$ corresponds to $a(P)$ under the Dolbeault isomorphism (3.4):*

$$H^{p,q}(X, \mathrm{ad}(P)) \cong \check{H}^q(X,\, \mathrm{ad}(\mathcal{P}) \otimes \Omega^p)$$
$$[\Theta^{1,1}] \leftrightarrow a(P).$$

Proof. The same collection of $(1,0)$-forms θ_α represents

(i) $[\Theta^{1,1}]$ on the Dolbeault side, via $\Theta^{1,1}_\alpha = \overline{\partial}\theta_\alpha$;

(ii) $a(P)$ on the Čech side, via $\delta(\theta_\alpha) = (\theta_\beta - \theta_\alpha)$.

The problem now becomes one of unraveling the Dolbeault isomorphism (3.4), for which we follow the procedure outlined in Problem 6-3, p.189 of [13]. This gives us the desired result. □

Corollary 3.3.1.1. *If P admits a flat (integrable) C^∞ connection of type $(1,0)$, it admits a complex analytic connection.*

(The converse to the above need not hold since the $(2,0)$- and $(0,2)$- components of the curvature of a C^∞ connection needn't vanish, in general).

One may now ask: when does a principal G-bundle admit a C^∞ connection of type $(1,0)$? When is its curvature of type $(1,1)$? For what choice of structure group G can one always have such a connection?

For the case $G = GL_r(\mathbb{C})$, this is always possible: let E denote the vector bundle associated to P. Choose a Hermitian metric on E, and then take the Chern connection with respect to that metric. This, by (1.4.4), induces a C^∞ connection on P of type $(1,0)$. Moreover, its curvature is $\overline{\partial}$-closed and of type $(1,1)$.

More generally, assume G is either complex semisimple or $GL_r(\mathbb{C})$. Let G_1 be the maximal compact subgroup of G. We have the following relations between G and G_1 ([1], p.191):

(i) Every C^∞ principal bundle with group G is equivalent to one with group G_1;

(ii) $L(G)$ is the complexification of $L(G_1)$;

(iii) Every $\mathrm{ad}(G_1)$-invariant polynomial $F_1 : L(G_1) \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} L(G_1) \to \mathbb{R}$ induces an $\mathrm{ad}(G)$-invariant polynomial $F : L(G) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} L(G) \to \mathbb{C}$, and the relation $F_1 \leftrightarrow F$ sets up a one-to-one corresspondence between the invariant polynomials of G_1 and those invariant polynomials of G whose restriction to $L(G_1)$ is real.

We then have the following result, by Singer [18]:

Proposition 3.3.2 ([1], Prop.5)**.** *Let P be a complex analytic principal bundle with group G, and P_1 a C^∞ principal bundle with group G_1 which induces P. Then there is a unique C^∞ connection θ_1 in P_1 such that the induced C^∞ connection θ in P is of type $(1,0)$. Moreover, the curvature of θ is of type $(1,1)$.*

3.3.2 The Characteristic Ring

We now show that the Atiyah class of a principal bundle generates its characteristic ring (with complex coefficients), provided we assume the base X is compact Kähler. Let P, P_1, P, P_1, θ and θ_1 be as before. Let Θ, and Θ_1 denote the corresponding curvatures in P and P_1, and let F, F_1 be as above. Consider the cohomology classes

$$\begin{aligned} [\,F_1(\Theta_1)\,] &= [\,F_1(\Theta_1, \cdots, \Theta_1)\,] \qquad && \in\ H^*(X, \underline{\mathbb{R}}); \\ [\,F(\Theta)\,] &= [\,F(\Theta, \cdots, \Theta)\,] \qquad && \in\ H^*(X, \underline{\mathbb{C}}). \end{aligned}$$

Let $\imath : \underline{\mathbb{R}} \hookrightarrow \underline{\mathbb{C}}$ be the natural embedding. Then, by (iii) above, we have

$$[\,F(\Theta)\,] = \imath^*[\,F_1(\Theta_1)\,].$$

Let $\mathscr{R}(P)$ denote the characteristic cohomology ring of P with $\mathbb{R}$-coefficients. By Chern-Weil theory (2), this is the image of the Chern-Weil homomorphism

$$\begin{aligned} c_P : \mathrm{Inv}(L(G_1)) &\longrightarrow H^*(X, \underline{\mathbb{R}}) \\ F_1 &\longmapsto [\,F_1(\Theta_1)\,]. \end{aligned}$$

Define $\mathscr{C}(P)$ to be the characteristic cohomology ring of P with $\mathbb{R}$-coefficients; i.e., $\mathscr{C}(P) = \imath^*\mathscr{R}(P) \otimes_{\mathbb{R}} \mathbb{C}$. Then, by the previous discussion, $\mathscr{C}(P)$ is the image of the (complexified) Chern-Weil homomorphism

$$\begin{aligned} c_P : \mathrm{Inv}(L(G)) &\longrightarrow H^*(X, \underline{\mathbb{C}}) \\ F &\longmapsto [\, F(\Theta)\,]. \end{aligned}$$

Now, any invariant polynomial F of G defines a sheaf homomorphism $\mathcal{F} : \mathrm{ad}(\mathcal{P})\otimes\cdots\otimes\mathrm{ad}(\mathcal{P}) \to \mathscr{O}$. This allows us to define, via cup-product multiplication, the class

$$\mathcal{F}(a(P)) \in \check{H}^k(X, \Omega^k),$$

k being the degree of F. We then have the following result:

Theorem 3.3.3 ([1], Thm.3)**.** *Let X be a compact Kähler manifold, and P a holomorphic principal bundle over X with group G (semisimple or GL_r). Let*

$$a(P) \in \check{H}^1(X,\, \Omega^1 \otimes \mathrm{ad}(\mathcal{P}))$$

be its Atiyah class. Then, the characteristic ring $\mathscr{C}(P)$ of P with complex coefficients is generated by the image of the Chern-Weil homomorphism

$$\begin{aligned} c_P : \mathrm{Inv}_{\mathrm{ad}(G)}(L(G)) &\longrightarrow H^*(X, \underline{\mathbb{C}}) \\ F &\longmapsto \mathcal{F}(a(P)). \end{aligned}$$

Proof. Use (3.3.1) and (3.3.2), and the fact that the Dolbeault homomorphism (3.4) preserves multiplicative structures up to sign. We restrict X to be compact Kähler so that we can use *Hodge decomposition* ([13], Thm.9.44) and identify $\check{H}^k(X, \Omega^k)$ with a subspace of $H^{2k}(X, \underline{\mathbb{C}})$. (For the definition of a Kähler manifold, see Ch.8 of [13]. Compact Kähler manifolds are, for a variety of reasons, among the most important and well-studied classes of complex manifolds.) $\square$

Combining (3.2.5) and (3.3.3), we also obtain:

Theorem 3.3.4 ([1], Thm.4)**.** *Let X be a compact Kähler manifold, and P a holomorphic principal bundle over X with group G (semisimple or GL_r). If P admits* *a complex analytic connection, then the characteristic ring $\mathscr{C}(P)$ of P with complex coefficients is zero.*

The converse to the above theorem is false in general; a counterexample is given in Prop.22 of [1].

3.4 The $\mathscr{D}$-Functor

Recall that we have one-to-one correspondences

$$\left\{\begin{array}{c}\text{Isomorphism Classes of}\\ \text{Vector Bundles of Rank } r\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Isomorphism Classes of}\\ \text{Principal } GL_r(\mathbb{C})\text{-bundles}\end{array}\right\}$$
$$E \longmapsto \mathrm{Fr}(E)$$
$$\left\{\begin{array}{c}\text{Isomorphism Classes of}\\ \text{Vector Bundles of Rank } r\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Isomorphism Classes of}\\ \text{Locally Free Sheaves of Rank } r\end{array}\right\}$$
$$E \longmapsto \mathcal{E} = \mathscr{O}(E).$$

Therefore, there are two natural generalisations of a vector bundle: (i) to a bundle with an alternate structure group, and (ii) to an arbitrary coherent sheaf. So far, we have focused on (i); we now shift our attention on (ii) and illustrate how to generalise the notions developed in the previous sections in this direction, via the $\mathscr{D}$-functor.

A consequence of the $\mathscr{D}$-functor approach is that one can think of a complex analytic connection on a vector bundle as a differential operator acting on its space of sections. This is somewhat more in line with the definition in [12], or in the smooth case (see 1.4.1). Towards the end, we shall also describe a method to go back and forth between complex analytic connections on a principal GL-bundle P and its associated vector bundle E.

3.4.1 The $\mathscr{D}$- and $\mathscr{B}$- Functors

Let X and $\mathscr{O}$ be as before, and let $\mathcal{S}$ be a coherent sheaf of $\mathscr{O}$-modules on X. We define a new sheaf $\mathscr{D}(\mathcal{S})$ as follows:

(i) As a sheaf of $\underline{\mathbb{C}}$-modules, define

$$\mathscr{D}(\mathcal{S}) := \mathcal{S} \oplus (\mathcal{S} \otimes \Omega^1).$$

(ii) Over an open set U, define a $\mathscr{O}(U)$-action on $\mathscr{D}(\mathcal{S})(U)$ as follows:

$$f \cdot \alpha \;:=\; fs \oplus (f\beta + s \otimes df),$$

where $\alpha = s \oplus \beta \in \mathscr{D}(\mathcal{S})(U)$.

That is, we modify the usual $\mathscr{O}$-action so that it satisfies the **Leibniz rule**. One can check that $\mathscr{D}(\mathcal{S})$ with the above $\mathscr{O}$-action becomes a sheaf of $\mathscr{O}$-modules.

Furthermore, we have an exact sequence $\mathscr{B}(\mathcal{S})$ of $\mathscr{O}$-modules

$$0 \longrightarrow \mathcal{S} \otimes \Omega^1 \xrightarrow{\ i\ } \mathscr{D}(\mathcal{S}) \xrightarrow{\ p\ } \mathcal{S} \longrightarrow 0 \tag{3.5}$$

where $i : \beta \longmapsto 0 \oplus \beta$ and $p : s \oplus \beta \longmapsto s$ are the natural inclusion and projection, respectively. (These are both $\mathscr{O}$-homomorphisms, as one can easily check). Since $\mathcal{S}$ and $\mathcal{S} \otimes \Omega^1$ are both coherent, by [17] (Thm.1, p.208), $\mathscr{D}(\mathcal{S})$ is coherent.

Remark 3.4.1. (i) The natural $\underline{\mathbb{C}}$-homomorphisms $j : \mathcal{S} \to \mathscr{D}(\mathcal{S})$ and $q : \mathscr{D}(\mathcal{S}) \to \mathcal{S} \otimes \Omega^1$ are no longer $\mathscr{O}$-homomorphisms.

(ii) As a consequence of (i), $\mathscr{B}(\mathcal{S})$ does not split as a sequence of $\mathscr{O}$-modules in general. It does split as a sequence of $\underline{\mathbb{C}}$-modules, however.

(iii) A local section of $\mathscr{D}(\mathcal{S})$ can still be uniquely represented as $s \oplus \beta$, where $s \in \mathcal{S}$ and $\beta \in \mathcal{S} \otimes_{\mathscr{O}} \Omega^1$.

Proposition 3.4.2. *$\mathscr{D}$ defines a covariant endofunctor on the category of coherent $\mathscr{O}$-modules; it sends*

- *a coherent module $\mathcal{S}$ to $\mathscr{D}(\mathcal{S})$;*
- *an $\mathscr{O}$-morphism $\phi : \mathcal{S} \to \mathcal{S}'$ to*

$$\begin{aligned} \mathscr{D}(\phi) : \mathscr{D}(\mathcal{S}) &\longrightarrow \mathscr{D}(\mathcal{S}') \\ s \oplus \beta &\longmapsto \phi(s) \oplus (\phi \otimes \mathbb{1})(\beta) \end{aligned}$$

where $\mathbb{1}$ denotes the identity map on Ω^1.

Moreover, $\mathscr{D}$ is exact, and therefore additive.

Proposition 3.4.3. *$\mathscr{B}$ defines a covariant functor that sends*

- *a coherent module $\mathcal{S}$ to the exact sequence $\mathscr{B}(\mathcal{S})$;*
- *an $\mathscr{O}$-morphism $\phi : \mathcal{S} \to \mathcal{S}'$ to*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S}\otimes\Omega^1 & \xrightarrow{i} & \mathscr{D}(\mathcal{S}) & \xrightarrow{p} & \mathcal{S} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \phi\otimes\mathbb{1}} & & \Big\downarrow{\scriptstyle \mathscr{D}(\phi)} & & \Big\downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & \mathcal{S}'\otimes\Omega^1 & \xrightarrow{i'} & \mathscr{D}(\mathcal{S}') & \xrightarrow{p'} & \mathcal{S}' & \longrightarrow & 0. \end{array}$$

Moreover, $\mathscr{B}$ is exact, and therefore additive.

For proofs of the above statements, see Propositions 6 and 7 of [1].

Proposition 3.4.4 ([1], Prop.8)**.** *If $\mathcal{S}$ is locally free, then so is $\mathscr{D}(\mathcal{S})$.*

Proof. Since $\mathcal{S}$ is locally free, so is $\mathcal{S}\otimes\Omega^1$. Exactness of $\mathscr{B}(\mathcal{S})$ tells us that $\mathscr{D}(\mathcal{S})$ is an extension of $\mathcal{S}$ by $\mathcal{S}\otimes\Omega^1$, and hence, by (3.1.6), $\mathscr{D}(\mathcal{S})$ is locally free. □

3.4.2 The Class $b(E)$

Let $E \to X$ be a vector bundle of rank r. Since the sheaf $\mathcal{E}$ is locally free, by (3.4.4), so is $\mathscr{D}(\mathcal{E})$. It therefore corresponds to a vector bundle $\mathscr{D}(E)$ on X. We also have a natural exact sequence of bundles $\mathscr{B}(E)$, given by

$$0 \longrightarrow E\otimes T^* \longrightarrow \mathscr{D}(E) \longrightarrow E \longrightarrow 0. \tag{3.6}$$

The extension $\mathscr{B}(E)$ defines a cohomology class

$$\begin{aligned} b(E) &\in \check{H}^1(X,\, \mathcal{H}om(\mathcal{E},\mathcal{E}\otimes\Omega^1)) \\ &\in \check{H}^1(X,\, \mathcal{E}^*\otimes\mathcal{E}\otimes\Omega^1) \\ &\in \check{H}^1(X,\, \mathcal{E}nd(\mathcal{E})\otimes\Omega^1). \end{aligned}$$

Let us calculate explicit representatives of the class $b(E)$ in terms of coordinate transformations of E. Let $\mathscr{U} = \{U_\alpha\}$ be an open covering of X, with local trivializations $\phi_\alpha : E_\alpha \to U_\alpha \times \mathbb{C}^r$ and transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to GL_r(\mathbb{C})$. Each trivialization induces a sheaf isomorphism $\hat{\phi}_\alpha : \mathcal{E}_\alpha \to \mathscr{O}_\alpha^r$. The right-hand side, moreover, has a natural exterior derivative d_α acting on its space of sections, which we can transfer to $\mathcal{E}_\alpha$ to obtain

$$\begin{aligned} D_\alpha : \mathcal{E}_\alpha &\longrightarrow \mathcal{E}_\alpha\otimes\Omega^1_\alpha \\ s &\longmapsto \hat{\phi}_\alpha^{-1} d_\alpha \hat{\phi}_\alpha s. \end{aligned}$$

This is a $\underline{\mathbb{C}}$-linear sheaf map that obeys Leinbiz rule. Using this, we can now define a local splitting of $\mathscr{B}(E_\alpha)$ by

$$\begin{aligned} \mu_\alpha : \mathcal{E}_\alpha &\longrightarrow \mathscr{D}(E)_\alpha \\ s &\longmapsto s \oplus D_\alpha(s). \end{aligned}$$

Set $\mu_{\alpha\beta} = \mu_\beta - \mu_\alpha$. Then

$$(\mu_{\alpha\beta}) \in \mathscr{Z}^1(\mathscr{U}, \Omega^1 \otimes \mathcal{E}nd(\mathcal{E})) \tag{3.7}$$

is a representative cocycle for $b(E)$.

We will now show that the $b(E) = -a(P)$, where P is the principal $GL_r(\mathbb{C})$-bundle associated with E. First, we have a proposition:

Proposition 3.4.5 ([1], Prop.9)**.** *Let E be a vector bundle over X, and P the corresponding principal bundle. There is a canonical isomorphism* $\mathrm{End}(E) \cong \mathrm{ad}(P)$*.*

Thus, $a(P)$ and $b(E)$ can be regarded as elements of the same cohomology group. We may now state the main result of this section:

Theorem 3.4.6 ([1], Thm.5)**.** *Let E be a vector bundle over X, and P the corresponding principal bundle. Let $a(P)$ and $b(E)$ be the obstruction classes defined by the extensions $\mathscr{A}(P)$ and $\mathscr{B}(E)$, respectively. Then $a(P) = -b(E)$.*

Proof. See [1], Thm.5. The proof is by comparing the local expressions for $a(P)$ and $b(E)$ with respect to the aforementioned open cover $\mathscr{U}$, by using (3.4.5). □

Corollary 3.4.6.1. *With notations and assumptions as above, the Atiyah class of E is represented by the cocycle*

$$(\phi_\beta^{-1} \circ (g_{\alpha\beta}^{-1} dg_{\alpha\beta}) \circ \phi_\alpha) \in \mathscr{Z}(\mathscr{U}, \Omega^1 \otimes \mathcal{E}nd(\mathcal{E})).$$

This is the definition of the Atiyah class given in some books (for instance, [12]). The local expression of $b(E)$ described above is quite useful, and can be applied to prove the following properties:

Proposition 3.4.7 ([1] Prop.11)**.** *Let E, E' be vector bundles on X. Then*

(i) $b(E \oplus E') = b(E) + b(E')$ *as elements of* $\check{H}^1(X, \mathcal{E}nd(\mathcal{E} \oplus \mathcal{E}') \otimes \Omega^1)$*;*

(ii) $b(E \otimes E') = b(E) \otimes I' + I \otimes b(E')$ *as elements of* $\check{H}^1(X, \mathcal{E}nd(\mathcal{E} \otimes \mathcal{E}') \otimes \Omega^1)$*, where I, I' are the identity maps on E and E', respectively.*

3.4.3 Complex Analytic Connections on Vector Bundles

Let E be a vector bundle over X, and P the associated principal bundle. The previous theorem (3.4.6) tells us that P admits a complex analytic connection if and only if the sequence $\mathscr{B}(E)$ splits holomorphically.

Let us analyze the splittings of $\mathscr{B}(E)$ more carefully. These correspond to $\mathscr{O}$-linear maps of the form

$$\mu : \mathcal{E} \longrightarrow \mathscr{D}(\mathcal{E})$$

satisfying $p \circ \mu = \mathbb{1}_{\mathcal{E}}$ (p is the map in (3.5)). Given a local section s of $\mathcal{E}$, μ can be expressed as

$$\mu(s) = s \oplus D(s)$$

where the map $D : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1$ is $\underline{\mathbb{C}}$-linear and satisfies the Leibniz rule! Furthermore, the difference between two splittings $\mu = \mathbb{1}_{\mathcal{E}} \oplus D$ and $\mu' = \mathbb{1}_{\mathcal{E}'} \oplus D'$ is the element $D - D' \in \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1$.

We may thus recast the definition of a complex analytic connection on E as follows:

Definition 3.4.8. *A complex analytic connection on E is a $\underline{\mathbb{C}}$-linear sheaf map*

$$D : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1$$

that satisfies the Leibniz rule: given local sections s of E and f of $\mathscr{O}$,

$$D(fs) = df \otimes s + fD(s).$$

This definition of a connection (as a differential operator acting on sections) is similar in spirit to what we have seen in the smooth case (1.4.1). In fact, some authors (such as [12]) directly use this definition. As before, we have

Corollary 3.4.8.1. *The set of complex analytic connections on E is an affine space over the $\mathbb{C}$-vector space $\Gamma(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1)$.*

Example 3.4.1. The usual **exterior derivative** on the trivial bundle $E = X \times \mathbb{C}^r$ is an example of a complex analytic connection. At the level of sheaves, this corresponds to the map

$$\begin{aligned} d = \partial :\ \mathscr{O}^r &\longrightarrow \mathscr{O}^r \otimes \Omega^1 \\ (s_1, \ldots, s_r) &\longmapsto (ds_1, \ldots, ds_r) = (\partial s_1, \ldots, \partial s_r). \end{aligned}$$

($d = \partial$ since the sections are holomorphic). Furthermore, by the previous corollary, every connection on $X \times \mathbb{C}^r$ is of the form $\partial + \theta$, where θ is a holomorphic section of $\mathcal{E}nd(\mathcal{E}) \otimes \Omega^1$.

We made use of this observation when constructing explicit representatives for the class $b(E)$ in the previous subsection. ◊♦◊

Now, let $\mathscr{U}$ be an open cover of X trivializing E. As in the smooth case, a complex analytic connection D can be described by a collection of ***connection matrices***, as follows: on the framed open set (U_α, e_α), define $\theta_\alpha \in \Gamma(U_\alpha,\, \Omega^1 \otimes \mathfrak{gl}_r(\mathbb{C}))$ by

$$De_{\alpha j} = \sum_{i=1}^{r} (\theta_\alpha)_{ij} \otimes e_{\alpha i}.$$

On (U_α, e_α), one may write $D = \partial + \theta_\alpha$.

Furthermore, if (U_α, e_α) and (U_β, e_β) overlap with transition $g_{\alpha\beta}$ (i.e., $e_\beta = e_\alpha g_{\alpha\beta}$), we have the transformation rule

$$\theta_\beta = g_{\alpha\beta}^{-1} \theta_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} \partial g_{\alpha\beta}. \tag{3.8}$$

Conversely, given such a collection θ_α on the cover $\mathscr{U}$ satisfying the above transformation rule, one can recover D. The proofs of all the above statements are exactly as in the smooth case (1).

Lemma 3.4.9. *D naturally gives rise to a C^∞ connection $\nabla : \mathscr{E}^0(E) \to \mathscr{E}^1(E)$, of type $(1,0)$, such that*

(i) $\nabla^{0,1} = \overline{\partial}_E$ *;*

(ii) $\nabla^{1,0} = D$ *when restricted to* $\mathscr{O}(E)$*;*

(iii) *the connection matrices of* ∇ *and* D *with respect to any holomorphic framed open set* (U, e) *coincide.*

Proof. Choose a cover $\mathscr{U}$ like before. Each e_α is both a holomoprhic frame for E, and a smooth frame for the underlying smooth bundle of E. The holomorphic connection matrices θ_α can therefore be viewed as smooth matrices, which patch (with the help of the transformation rule) to give a smooth connection ∇ of E. One can easily check that this is independent of the cover $\mathscr{U}$, and the above properties follow in a straightforward manner. □

Relating Connections on Vector Bundles and Principal GL-Bundles

Let $E \to X$ be a vector bundle of rank r, and P the associated principal $GL_r(\mathbb{C})$-bundle. We saw previously that $a(P) = -b(E)$, which means that P admits a complex analytic connection if and only if E does so. We shall now illustrate how a connection on P naturally induces one on E, and vice versa. A key ingredient in the correspondence will be the natural isomorphism $\mathrm{End}(E) \cong \mathrm{ad}(P)$.

Before we begin, recall the definition of the holomorphic Maurer-Cartan form ω_0 on $GL_r(\mathbb{C})$. This is the $\mathfrak{gl}_r(\mathbb{C})$-valued 1-form

$$(\omega_0)_a = a^{-1}da = a^{-1}\partial a.$$

Start with the usual open covering $\mathscr{U} = \{U_\alpha\}$ of X with local trivializations $\phi_\alpha : E_\alpha \to U_\alpha \times \mathbb{C}^r$ and transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to GL_r(\mathbb{C})$. Let e_α denote the holomorphic local frame of E_α that is taken to the standard frame of $U_\alpha \times \mathbb{C}^r$ by ϕ_α. The frame bundle P then admits trivializations $\varphi_\alpha : P_\alpha \to U_\alpha \times GL_r(\mathbb{C})$, whose transitions are again $g_{\alpha\beta}$, but with $GL_r(\mathbb{C})$ now acting on itself via matrix multiplication. Furthermore, ϕ_α and φ_α induce isomorphisms

$$\mathrm{End}(E)_\alpha \cong U_\alpha \times \mathfrak{gl}_r(\mathbb{C}) \cong \mathrm{ad}(P)_\alpha$$

the composite being the canonical isomorphism of (3.4.5).

We now illustrate the method to move back and forth between complex analytic connections on P and on E.

(1) Start with a complex analytic connection

$$D : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1.$$

The connection matrices of D with respect to $\{(U_\alpha, e_\alpha)\}$ give us a collection of 1-forms

$$\theta_\alpha \in \Gamma(U_\alpha,\ \Omega^1 \otimes \mathfrak{gl}_r(\mathbb{C})).$$

On overlaps, these satisfy the transformation rule

$$\theta_\beta = g_{\alpha\beta}^{-1}\theta_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}.$$

We now make two observations. First, since the representation $\mathrm{ad} : GL_r(\mathbb{C}) \to \mathfrak{gl}_r(\mathbb{C})$ is nothing but matrix conjugation, we can write $g_{\alpha\beta}^{-1}\theta_\alpha g_{\alpha\beta} = \mathrm{ad}(g_{\alpha\beta}^{-1})\theta_\alpha$. Second, the term $g_{\alpha\beta}^{-1} dg_{\alpha\beta}$ is simply the pullback $g_{\alpha\beta}^{*}\omega_0$! The above transformation rule may thus be written as

$$\theta_\beta = \mathrm{ad}(g_{\alpha\beta}^{-1})\theta_\alpha + g_{\alpha\beta}^{*}\omega_0.$$

By (3.2.10), we obtain a complex analytic connection on P. One can check (by refining if necessary) that our answer is independent of the cover $\mathscr{U}$.

(ii) Conversely, a complex analytic connection θ on the frame bundle gives us a natural connection D on E, by simply reversing the above steps.

In summary, the local data describing both kinds of connections, including the transition formulae (3.3) and (3.8), can be identified via the canonical isomorphism (3.4.5).

Example 3.4.2. Apply the above method to the trivial bundle $E = X \times \mathbb{C}^r$. We see that the *Maurer-Cartan connection* ω_{MC} on $P = X \times GL_r(\mathbb{C})$,

$$\begin{aligned} \omega_{MC} :\ T &\longrightarrow T \oplus \mathrm{ad}(P) \\ t &\longmapsto t \oplus 0 \end{aligned}$$

corresponds to the exterior derivative

$$\begin{aligned} d = \partial : \ \mathscr{O}^r &\longrightarrow \mathscr{O}^r \otimes \Omega^1 \\ (s_1, \ldots, s_r) &\longmapsto (ds_1, \ldots, ds_r) = (\partial s_1, \ldots, \partial s_r). \end{aligned}$$

The connection matrix/form associated to them with respect to the (flat) global frame $(e_1, \ldots, e_r)$ is just the zero matrix.

◊♦◊

3.5 Chern Classes

In the previous sections, we showed how the Atiyah class $a(P)$ of a principal bundle P generated, under suitable restrictions, the characteristic ring of P with complex coefficients. In the final section of this chapter, we give an alternate treatment in the case of the structure group $GL_r(\mathbb{C})$. Again, we shall assume throughout that X is compact Kähler.

Line Bundles

We start with the simplest case: a line bundle $E \to X$. Note that $\mathrm{End}(E)$ is trivial here, since the identity map on E constitutes a global frame of $\mathrm{End}(E)$.

Proposition 3.5.1. *Let E be a line bundle over X. The natural inclusion*

$$\check{H}^1(X, \Omega^1) \xrightarrow{\simeq} H^{1,1}(X) \hookrightarrow \check{H}^2(X, \underline{\mathbb{C}})$$

takes $b(E)$ to $-2\pi i\, c_1(E)$, where $c_1(E)$ is the first Chern class of E.

Proof. See [1], Prop.12. Note that the class $b(E)$ is represented by the 1-cocycle $(-\, dg_{\alpha\beta} \cdot g_{\alpha\beta}^{-1})$, where $g_{\alpha\beta}$ are the transitions of E with respect to some trivialization of E. $\square$

Split Vector Bundles

A vector bundle $E \to X$ of rank r is said to ***split*** if it is a direct sum of line bundles: $E = E_1 \oplus \cdots \oplus E_r$. If the transitions of the component E_j are denoted $g_{\alpha\beta}^{E_j}$, then the transitions of E are diagonal matrices of the form $g_{\alpha\beta} = \mathrm{diag}(g_{\alpha\beta}^{E_1}, \ldots, g_{\alpha\beta}^{E_r})$. We show that the Chern classes of a split vector bundle can be computed from the Chern classes of its individual components, as follows.

Consider the total Chern class $c(E)$ of the split vector bundle E. By the Whitney product formula, this splits as

$$c(E) = c(E_1) \wedge \cdots \wedge c(E_r).$$

Expanding both sides out and comparing degrees, we have

$$c_k(E) = \sigma_k(c_1(E_1), \dots, c_1(E_r))$$

where σ_k is the kth elementary symmetric polynomial in r variables.

Let $\Sigma_k : \mathfrak{gl}_r(\mathbb{C}) \to \mathbb{C}$ be the polynomial of degree k that assigns to a matrix $A \in \mathfrak{gl}_r(\mathbb{C})$ the number $\sigma_k(A_1, \dots, A_r)$, where $A_1, \dots, A_r$ are the ***characteristic roots*** of A. This is a an invariant polynomial on $\mathfrak{gl}_r(\mathbb{C})$ (since the characteristic polynomial is itself an invariant polynomial), and is homogeneous of degree k. We may therefore consider its completely polarized form $S_k(A, \dots, A) = \sigma_k(A)$.

Form the cohomology class $S_k[\, b(E)\,] \in \check{H}^2(X, \underline{\mathbb{C}})$, as in (3.3.3). (We use the fact that X is compact Kähler here). Since the transitions of E are block diagonal,

$$dg_{\alpha\beta} \cdot g_{\alpha\beta}^{-1} = \operatorname{diag}(dg_{\alpha\beta}^{E_1} \cdot (g_{\alpha\beta}^{E_1})^{-1}, \dots, dg_{\alpha\beta}^{E_1} \cdot (g_{\alpha\beta}^{E_1})^{-1})$$

is again of diagonal form. As a result, the characteristic roots of $b(E)$ are simply $b(E_1), \dots, b(E_r)$, and we may write

$$S_k[b(E)] = \sigma_k(b(E_1), \dots, b(E_r)).$$

Combining the above facts with (3.5.1), we obtain

Proposition 3.5.2 ([1], Prop.13)**.** *Let E be a split vector bundle. Then the kth Chern class of E is given in terms of $b(E)$ by the formula*

$$S_k[b(E)] = (-2\pi i)^k c_k(E).$$

Proof. Let $E = E_1 \oplus \cdots \oplus E_r$. Then

$$\begin{aligned} S_k[b(E)] &= \sigma_k(b(E_1), \dots, b(E_r)) \\ &= \sigma_k(-2\pi i c_1(E_1), \dots, -2\pi i c_r(E_r)) \\ &= (-2\pi i)^k \sigma_k(c_1(E_1), \dots, c_r(E_r)) \\ &= (-2\pi i)^k c_k(E). \end{aligned}$$

□

Arbitrary Vector Bundles

Finally, we show how one can reduce the case of an arbitrary vector bundle to a split vector bundle. The main ingredient is the **Splitting Principle**:

Theorem 3.5.3 (Splitting Principle)**.** *Let $E \to X$ be a vector bundle of rank r (not necessarily split). There exists a space $Y = \mathrm{Fl}(E)$, called the **flag bundle** associated to E, and a map $p : Y \to X$, such that*

(i) *the induced map of cohomologies*

$$\check{H}^*(X, \underline{\mathbb{C}}) \xrightarrow{p^*} \check{H}^*(Y, \underline{\mathbb{C}})$$

is injective;

(ii) *the pullback bundle $p^*E \to X$ is split.*

Proof. See [2], section 21. □

In our case, one may in fact choose Y to be compact Kähler and $p : Y \to X$ to be a holomorphism. Now, by the functorial properties of Chern classes and the Atiyah class, we have

$$b(p^*E) = p^*b(E) \qquad \& \qquad c_k(p^*E) = p^*c_k(E).$$

Combining this with (3.5.2), we finally obtain:

Theorem 3.5.4 ([1], Thm.6)**.** *Let E be a holomorphic vector bundle over a compact Kähler manifold X. Then the kth Chern class of E is given by*

$$c_k(E) = \frac{1}{(-2\pi i)^k} S_k[b(E)].$$

Chapter 4

Logarithmic Connections

We continue the theme of constructing characteristic classes using connections, and introduce the theory of logarithmic connections and their residues. Residues are an old and well-studied phenomenon in complex analysis, and the residue theorem for meromorphic 1-forms on a compact Riemann surface will serve as an archetype for the main theorem of the next chapter.

We first introduce the complex of logarithmic differential forms on a manifold, with poles along normally crossing divisors. We then give two definitions of a logarithmic connection on a holomorphic vector bundle: one as a holomorphic splitting of the log Atiyah sequence, and the other as a $\mathbb{C}$-linear sheaf map taking sections of the bundle to logarithmic 1-forms valued in the bundle. The existence of such a connection is captured by a logarithmic analogue of the Atiyah class.

Once we have such a connection at our disposal, we can define its residue along each component of the given divisor. The chapter then concludes with a generalisation of this concept from vector bundles to principal bundles with arbitrary structure group.

4.1 Residue Theorems in Complex Analysis

To motivate the theory that follows, let us revisit some single-variable complex analysis (see [3] for more details). Let f be a meromorphic function on $\mathbb{C}$ with a pole of order m at some point $a \in \mathbb{C}$. f admits a Laurent series expansion about a that looks something like this:

$$f = \sum_{n=-m}^{\infty} c_n (z-a)^n.$$

The ***residue*** of f at a, denoted $\mathrm{Res}_a(f)$, is then defined to be the coefficient c_{-1}. It represents the first-order approximation of the function blowing up to infinity.

An interesting theorem involving residues is the **Residue Theorem** ([3], Thm.V.1.1); in its simplest avatar, it states

$$\mathrm{Ind}_a(\gamma) \cdot \mathrm{Res}_a(f) = \frac{1}{2\pi i} \oint_{\gamma} f(z) dz$$

where γ is a smooth closed curve that encircles a, and $\mathrm{Ind}_a(\gamma)$ is its ***winding number*** about a. As we know, $\mathrm{Ind}_a(\gamma)$ is a topological entity $-$ it is the homotopy class represented by γ in the fundamental group of $\mathbb{C}\backslash\{a\}$. The residue theorem is therefore a basic example of residues and integrals of meromorphic functions capturing topological information.

4.1.1 The Residue Theorem on a Compact Riemann Surface

The archetype for Ohtsuki's formula is the Residue Theorem for meromorphic 1-forms on a compact Riemann surface, which we introduce in this subsection. Throughout, M will denote a Riemann surface. The material is taken from [8].

Let $Y \subseteq M$ be an open set, $a \in Y$, and ω a holomorphic 1-form defined on $Y \setminus \{a\}$. Let (U, z) be a coordinate chart on M centered at a. We may write $\omega = f dz$, where f is holomorphic on $Y \setminus \{a\}$. Denote the Laurent series expansion of f about a with respect to z by $f = \sum_{n=-m}^{\infty} c_n (z-a)^n$.

Definition 4.1.1. *ω is said to have a **pole** of order/multiplicity n at a if f has a pole of order n at a. The coeffiecient c_{-1} is called the **residue** of ω at a, and is denoted $\mathrm{Res}_a(\omega)$.*

One can show that the definition is independent of the choice of coordinate chart about a ([8], p.63-64). Clearly, it matches with the notion of residue introduced in one-variable complex analysis.

Definition 4.1.2. *A meromorphic 1-form on $Y \subseteq M$ is a holomorphic* 1*-form on $Y \setminus \{a_1, \ldots, a_m\}$ that admits poles of finite order at each point a_k.*

Theorem 4.1.3 (The Residue Theorem)**.** *Let M be a compact Riemann surface, and ω a meromorphic* 1*-form on X with poles at finitely many points $a_1, \ldots a_m$. Then*

$$\sum_{k=1}^{m} \mathrm{Res}_{a_k}(\omega) = 0.$$

Proof. See [8], Thm. 10.21. □

Corollary 4.1.3.1. *Any non-constant meromorphic function f on a compact Riemann surface M has, counting multiplicities, as many zeroes as poles.*

Proof. [8], Cor. 10.22. □

We now recast the Residue Theorem in the language of connections, so that it becomes a special case of Ohtsuki's more general result. One may interpret the meromorphic 1-form ω as the connection form of a "meromorphic connection" on the trivial line bundle $M \times \mathbb{C} \to M$. Since this has trivial Chern classes, the Residue theorem establishes an equality

$$\sum_{k=1}^{m} \mathrm{Res}_{a_k}(\omega) = 0 = c_1(M \times \mathbb{C}). \tag{4.1}$$

4.1.2 Ohtsuki's Formula: A Prelude

Let us try to generalize the previous situation to higher dimensions. Let E be a holomorphic vector bundle over a compact complex manifold M. If D is a "meromorphic connection" on E, is there a formula that relates the residues of D and the Chern classes of E?

Ohtsuki, in [16], shows that if D is "logarithmic", and the pole Z of D satisfies certain nice conditions, there is such a formula. The first version of the theorem is as follows:

Theorem 4.1.4 ([16], p.13)**.** *If the pole Z of a logarithmic connection D on $E \to M$ is normally crossing, and the intersection of any finite number of irreducible components of Z is connected, the following relation holds in the cohomology group $\check{H}^k(M, \Omega^k)$:*

$$c_k(E) = (-1)^k \sum_{j_1,\dots,j_k} c_k(\mathrm{Res}_{Z_{j_1}} D, \dots, \mathrm{Res}_{Z_{j_k}} D) \prod_{t=1}^{k} c_1([Z_{j_t}]) \tag{4.2}$$

where $c_k(E)$ is the kth Chern class of E, $Res_{Z_j} D$ is the residue of D along the irreducible component Z_j of Z, $c_k(A_1, \dots, A_k)$ is the completely polarized form of the kth Chern polynomial $c_k(A)$, $c_1([W])$ is the first Chern class of the line bundle associated with the divisor W, and the summation is taken over all ordered k-tuples of irreducible components of Z.

We shall precisely state the main theorem in (5).

4.2 Logarithmic Differential Forms

In this section, we set up the technical machinery needed to define logarithmic connections. We begin by introducing the concept of a normally crossing divisor, and focus on meromorphic differential forms with poles along such divisors. Of these, the ones admitting the "mildest" possible singularities along the divisor constitute logarithmic forms.

Logarithmic forms were first introduced by Deligne in [4]; our exposition is based on his article, along with p.449-443 of [9]. We shall denote by

- M, a complex n-manifold;
- $\mathscr{O}_M$, the sheaf of holomorphic functions on M;
- $\mathcal{M}_M$, the sheaf of meromorphic functions on M;
- Ω^p_M, the sheaf of holomorphic p-forms on M.

4.2.1 Normally Crossing Divisors

Recall the notion of a divisor on a complex manifold (1.6).

Let $Z = \sum_\nu a_\nu Z_\nu$ be a divisor on M. Assume Z is *reduced* (that is, each a_ν is 0 or 1), and write $Z = \sum_\nu Z_\nu$.

Definition 4.2.1 (Normally Crossing Divisor). *Z is **normally crossing** if*

- *each Z_ν is smooth;*
- *Z_νs intersect transversely.*

By local finiteness of the summation, only finitely many Z_νs intersect in a neighbourhood of any point of Z. Moreover, each irreducible component Z_ν is a connected, codimension-1 complex submanifold of M, by [9], p.21.

The local picture of a normally crossing divisor is as follows:
Let $p \in Z$, and suppose k many of the components intersect at p ($1 \leq k \leq n$). Transversality implies that we may choose holomorphic coordinates $(U, z^1, \dots, z^n)$ about p such that

(i) $U = \{|z^i| < 1\}$ and $p = (0, \ldots 0)$ (i.e., U is a polydisc centered at p);

(ii) $Z \cap U = \{z^1 \cdots z^k = 0\}$ is a union of coordinate hyperplanes.

Remark 4.2.2. Some authors prefer the term "simple normal crossing" for such divisors, and allow "normal crossing" divisors to have singular components. We shall, however, stick to the terminology of [4] and [9].

4.2.2 The Log Complex

Throughout, Z will denote a normally crossing divisor on M.

Definition 4.2.3. *We denote by $\Omega^p_M(*Z)$ the sheaf of meromorphic p-forms, holomorphic on $M\backslash Z$, admitting poles (of any finite order) exclusively along Z.*

By definition, this is an $\mathscr{O}_M$-submodule of the sheaf $\mathcal{M}_M \otimes \Omega^p_M$ of meromorphic p-forms on M. In fact, more is true: let $j : M\backslash Z \hookrightarrow M$ denote the inclusion map, and d the exterior derivative on M. Then

(i) $\Omega^p_M(*Z)$ is a subsheaf of the direct image sheaf $j_*\Omega^p_{M\backslash Z}$.

(ii) $\left(j_*\Omega^*_{M\backslash Z}, d\right)$ is a complex of sheaves, and $(\Omega^*_M(*Z), d)$ forms a subcomplex.

See [9] p.445 for the definition of a complex of sheaves. The proofs of the above statements are straightforward, and therefore omitted. We must note, however, that the above inclusion is strict in general, since there may exist meromorphic forms with essential singularities along the divisor.

Definition 4.2.4 ([4], Def.3.1)**.** *We define the **logarithmic de Rham complex** of M along Z, denoted $\Omega^*(\log Z)$, to be the smallest sheaf subcomplex of $\Omega^*_M(*Z)$ such that*

(i) $\Omega^*(\log Z)$ *is stable under $\wedge$, the wedge product;*

(ii) $\Omega^*(\log Z)$ *contains and Ω^*_M, the complex of holomorphic forms, as a subcomplex;*

(iii) *for every local section $f \in \Omega^0(*Z)$, df/f is a local section of $\Omega^1(\log Z)$.*

A local section of $\Omega^p(\log Z)$ is called a *logarithmic p-form*, and is said to present a *logarithmic pole* along Z. The local picture of a logarithmic form is quite simple, as the next proposition shows:

Proposition 4.2.5. (i) *Let $\alpha \in \Omega^p(*Z)$. The following are equivalent:*

(a) $\alpha \in \Omega^p(\log Z)$ *(i.e., α presents a logarithmic pole along Z);*

(b) *α and $d\alpha$ present, at worst, simple poles along Z;*

(c) *for any local defining function f of Z, both $f\alpha$ and $fd\alpha$ are holomorphic.*

(ii) $\Omega^1(\log Z)$ *is a locally free. More specifically, let $(U, z^1, \ldots, z^n)$ be a coordinate chart on M such that $Z \cap U = \{z^1 \cdots z^k = 0\}$. Then $\Omega^1(\log Z)|_{U \cap Z}$ is a free $\mathscr{O}$-module of rank n, with basis $\{dz^1/z^1, \ldots, dz^k/z^k, dz^{k+1}, \ldots, dz^n\}$.*

(iii) $\Omega^p(\log Z) = \bigwedge^p \Omega^1(\log Z)$.

Proof. [4], Lemma 3.2.1. □

In particular, $\Omega^p(\log Z)$ is locally free of rank $\binom{n}{p}$, and therefore corresponds to a holomorphic vector bundle on E. We call the bundle $\Omega^1(\log Z)$ the ***logarithmic cotangent bundle*** of M, with pole along Z.

A meromorphic section α of $\Omega^p(\log Z)$ on $(U, z_1, \ldots, z_n)$ can be expressed as

$$\alpha \;=\; \sum_{\substack{J \subseteq (1,\ldots,k),\; K \subseteq (k+1,\ldots,n) \\ |J \cup K| = p}} \alpha_{JK} \cdot \frac{dz^J}{z^J} \wedge dz^K$$

where α_{JK} are holomorphic functions on $U \cap Z$.

Finally, we note that the proposition helps explain the terminology "logarithmic form": on a simply-connected open set, each dz^k/z^k is nothing but $d(\log z^k)$.

Remark 4.2.6. In dimension one, we have already seen that logarithmic forms can capture sufficient information about the complement of a divisor. After all, the logarithmic 1-form $dz/(z-a)$ generates the cohomology ring $H^*_{\mathrm{dR}}(\mathbb{C}\backslash\{a\})$ for any point $a \in \mathbb{C}$.

It was Deligne's insight that in higher dimensions too, the cohomology of the complement of a normally crossing divisor can be captured by the logarithmic forms along it. He showed the existence of an isomorphism ([4] II.6, [9] p.453)

$$\mathbb{H}^*(M, \log Z) \cong H^*_{\mathrm{dR}}(M \backslash Z; \mathbb{C})$$

where the left-hand side is the *hypercohomology* of the sheaf complex $(\Omega^*(\log Z), d)$ (see [9] p.446 for the definition).

4.3 Logarithmic Connections

We now introduce the theory of logarithmic connections on principal and vector bundles over a complex manifold, with poles along a normally crossing divisor. As in the complex analytic case, such a connection need not always exist; the corresponding obstruction class is called the logarithmic Atiyah class of the bundle.

The outline of this section closely follows Chapter 3; most constructions and proofs carry over, mutatis mutandis, into the logarithmic setup. For more details, we refer the reader to [10] and [7].

4.3.1 Logarithmic Tangent Sheaves

Let X be a complex n-manifold, and Z a normally crossing divisor.

Definition 4.3.1. *The* ***logarithmic tangent sheaf of*** (X, Z)*, denoted* $\mathcal{T}(-\log Z)$*, is the sheaf associated to the submodule of derivations on* X *preserving the ideal sheaf of* Z*.*

In other words, let $\mathscr{F}$ denote the presheaf

$$\mathscr{F}(U) := \{t \in \mathcal{T} \mid t(\mathscr{I}_Z) \subseteq \mathscr{I}_Z\}$$

where $\mathscr{I}_Z$ is the ideal sheaf of Z. Then $\mathcal{T}(-\log Z)$ is the sheaf associated to $\mathscr{F}$. In particular, it is an $\mathscr{O}$-submodule of the tangent sheaf $\mathcal{T}$.

What does a section of $\mathcal{T}(-\log Z)$ locally look like? As before, let $(U, z^1, \ldots, z^n)$ be a coordinate chart on M such that $Z \cap U = \{z^1 \cdots z^k = 0\}$. One may then show ([10], p.7) that $\mathcal{T}(-\log Z)|_U$ is a free $\mathscr{O}|_U$-submodule of $\mathcal{T}|_U$ of rank n, with basis

$$\left\{z^1 \frac{\partial}{\partial z^1}, \ldots, z^k \frac{\partial}{\partial z^k}, \frac{\partial}{\partial z^{k+1}}, \ldots, \frac{\partial}{\partial z^n}\right\}$$

.

$\mathcal{T}(-\log Z)(U)$ therefore comprises of those holomorphic vector fields that admit logarithmic zeroes along the divisor Z, i.e., vanish up to order 1 on Z.

The above picture also gives us an isomorphism

$$\mathcal{T}(-\log Z) \cong (\Omega^1(\log Z))^\vee \cong \mathcal{H}om(\Omega^1(\log Z), \mathscr{O}).$$

This is because we can identify the dual basis of

$$\left\{\frac{dz^1}{z^1}, \ldots, \frac{dz^k}{z^k}, dz^{k+1}, \ldots, dz^n\right\}$$

with the above basis of $\mathcal{T}(-\log Z)|_U$ via the natural inclusion of sheaves $(\Omega^1(\log Z))^\vee \hookrightarrow (\Omega^1)^\vee = \mathcal{T}$.

Finally, we note that since $\mathcal{T}(-\log Z)$ is locally free of rank n, it corresponds to a rank n bundle over X. This is called the ***logarithmic tangent bundle of*** (X, Z), and is denoted $T(-\log Z)$.

Remark 4.3.2. The sheaf inclusion $\mathcal{T}(-\log Z) \hookrightarrow \mathcal{T}$ does not correspond to an inclusion $T(-\log Z) \hookrightarrow T$ at the level of bundles; look, for instance, at the fibres over Z. This brings us back to the point raised in (3.1.1).

4.3.2 The Logarithmic Atiyah Class

Recall that for a complex Lie group G and a holomorphic principal G-bundle $P \to X$, we have a canonical exact sheaf sequence

$$0 \longrightarrow \mathrm{ad}(\mathcal{P}) \longrightarrow \mathrm{At}(\mathcal{P}) \xrightarrow{D'\pi} \mathcal{T} \longrightarrow 0$$

coming from the Atiyah sequence (3.2.2) of P. Consider the submodule $\mathrm{At}(\mathcal{P})(-\log Z)$ of $\mathrm{At}(\mathcal{P})$ whose sections over an open set U are given by

$$\mathrm{At}(\mathcal{P})(-\log Z)(U) = (D'\pi)^{-1}(\mathcal{T}(-\log Z))(U).$$

Let $\varphi : P|_U \to U \times G$ be a trivialization, and $\hat{\varphi} : \mathrm{At}(\mathcal{P})|_U \to \mathcal{T}|_U \oplus \mathrm{ad}(\mathcal{P})|_U$ the induced isomorphism. Restricting this to the above submodule tells us that $\hat{\varphi} : \mathrm{At}(\mathcal{P})(-\log Z)|_U \longrightarrow \mathcal{T}(-\log Z)|_U \oplus \mathrm{ad}(\mathcal{P})|_U$ is an isomorphism $\mathscr{O}$-modules! We may therefore conclude the following:

(1) $\mathrm{At}(\mathcal{P})(-\log Z)$ is a locally free sheaf of $\mathscr{O}$-modules on X. We call it the ***logarithmic Atiyah sheaf of*** P (associated to Z).

(2) There is a canonical exact sheaf sequence

$$0 \longrightarrow \mathrm{ad}(\mathcal{P}) \longrightarrow \mathrm{At}(\mathcal{P})(-\log Z) \xrightarrow{D'\pi} \mathcal{T}(-\log Z) \longrightarrow 0 \qquad (4.3)$$

called the ***logarithmic Atiyah sheaf sequence of*** P (associated to Z).

We also have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{ad}(\mathcal{P}) \arrow[r] \arrow[d, "="] & \mathrm{At}(\mathcal{P})(-\log Z) \arrow[r, "D'\pi"] \arrow[d, hook] & T \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathrm{ad}(\mathcal{P}) \arrow[r] & \mathrm{At}(\mathcal{P}) \arrow[r, "D'\pi"] & T \arrow[r] & 0. \end{tikzcd}$$

Definition 4.3.3. *A* ***logarithmic connection of*** P ***with pole along*** Z *is an* $\mathscr{O}$*-linear splitting of the logarithmic Atiyah sheaf sequence of* P*.*

Logarithmic connections need not always exist; the obstruction is given by

Definition 4.3.4. *The* ***logarithmic Atiyah class of*** P (*associated to the divisor* Z) *is the cohomology class*

$$a_{\log Z}(P) \in \check{H}^1(X, \Omega^1(\log Z) \otimes \mathrm{ad}(\mathcal{P})) \tag{4.4}$$

determined by the extension $At(\mathcal{P})(-\log Z)$*.*

Finally, we construct explicit representatives of the log Atiyah class of P with respect to coordinate transformations of P. As in (3.2), let $\mathscr{U} = \{U_\alpha\}$ be an open cover of P admitting trivializations $\varphi : P_\alpha \to U_\alpha \times G$. We have isomorphisms $\hat{\varphi}_\alpha : \mathrm{At}(\mathcal{P})(-\log Z)_\alpha \longrightarrow \mathcal{T}(-\log Z)_\alpha \oplus \mathrm{ad}(\mathcal{P})_\alpha$.
Now, we have obvious choices for local splittings; take

$$\begin{aligned} \eta_\alpha : \mathcal{T}(-\log Z)_\alpha &\longrightarrow \mathrm{At}(P)(-\log Z)_\alpha \\ t &\longmapsto \hat{\varphi}_\alpha^{-1}(t \oplus 0). \end{aligned}$$

Set $\eta_{\alpha\beta} = \eta_\beta - \eta_\alpha$. Then

$$(\eta_{\alpha\beta}) \in \mathscr{Z}^1(\mathscr{U}, \Omega^1(\log Z) \otimes \mathrm{ad}(\mathcal{P})) \tag{4.5}$$

is a representative cocycle for $a_{\log Z}(P)$.

4.3.3 Logarithmic Connections on Principal Bundles

Following (3.2), we give a local description of a logarithmic connection on a principal bundle with respect to its coordinate transformations.

The Trivial Bundle

Consider the trivial bundle $P = X \times G$, for which $\mathrm{At}(\mathcal{P})(-\mathrm{log}Z) = \mathcal{T}(-\mathrm{log}Z) \oplus \mathrm{ad}(\mathcal{P})$. Its Atiyah sheaf sequence looks like

$$0 \longrightarrow \mathrm{ad}(\mathcal{P}) \longrightarrow \mathcal{T}(-\mathrm{log}Z) \oplus \mathrm{ad}(\mathcal{P}) \longrightarrow \mathcal{T}(-\mathrm{log}Z) \longrightarrow 0.$$

P therefore admits a canonical log connection, which we call the ***logarithmic Maurer-Cartan connection***. It can be described as

(i) the splitting $\mathcal{T}(-\mathrm{log}Z) \to \mathcal{T}(-\mathrm{log}Z) \oplus \mathrm{ad}(\mathcal{P})$, sending $t \mapsto t \oplus 0$;

(ii) the holomorphic 1-form $(\omega_{MC})_{\log Z} \in \Omega^1(X, \mathrm{At}(\mathcal{P})(-\log Z))$ that acts as $(\omega_{MC})_{\log Z}(t) = t \oplus 0$.

By the general theory of 3.1.8, and the structure of the log Atiyah sequence for $P = X \times G$, we have:

Proposition 4.3.5. *Let $P = X \times G$ be the trivial bundle. There is a one-to-one correspondence*

$$\{\text{Logarithmic Connections on } P\} \longleftrightarrow \Gamma(X,\, \Omega^1(\log Z) \otimes \mathrm{ad}(P))$$
$$(\omega_{MC})_{\log Z} + \theta \;\longleftrightarrow \theta.$$

Hence the set of logarithmic connections on P is an affine space over the $\mathbb{C}$-vector space $\Gamma(X,\, \Omega^1(\log Z) \otimes \mathrm{ad}(P)) \cong \Gamma(X,\, \Omega^1(\log Z) \otimes \mathcal{L})$. (Here, $\mathcal{L}$ is the sheaf associated to the trivial bundle $X \times L(G)$).

An Arbitrary Bundle with Vanishing Log Atiyah Class

Let P be a principal bundle with vanishing log Atiyah class. Let $(U_\alpha, \varphi_\alpha)$ be as before, and let the transitions be $g_{\alpha\beta}$. Just as in 3.2.10, we have the following local description of logarithmic connections on P:

Theorem 4.3.6. *There is a one-to-one correspondence between the set of logarithmic connections on P and collections of logarithmic 1-forms $\{\theta_\alpha \in \Gamma(X, \Omega^1(\log Z) \otimes \mathcal{L})\}_\alpha$ satisfying*

$$\theta_\beta = \mathrm{ad}(g_{\alpha\beta}^{-1})\theta_\alpha + g_{\alpha\beta}^{*}\omega_0 \tag{4.6}$$

on nonempty overlaps $U_{\alpha\beta}$. (ω_0 denotes the Maurer-Cartan form on G).

The 1-forms $\{\theta_\alpha\}$ are called the ***local connection forms*** of the logarithmic connection with respect to the cover $\mathscr{U}$.

4.3.4 Logarithmic Connections on Vector Bundles

As in 3.4, one may develop logarithmic analogues of the $\mathscr{D}$- and $\mathscr{B}$-functors and define the class $b_{\log Z}(E)$ of a vector bundle E. The same proof as in the complex analytic case shows that $b_{\log Z}(E) = -a_{\log Z}(P)$, where P is the GL-bundle associated to E. This allows us to think of logarithmic connections on vector bundles as differential operators acting on sections:

Definition 4.3.7. *A meromorphic connection D on E with a simple logarithmic pole along Z (in short,* ***logarithmic connection****) is a $\mathbb{C}$-linear sheaf map*

$$D : \mathscr{O}(E) \longrightarrow \Omega^1_M(\log Z) \otimes_{\mathscr{O}} \mathscr{O}(E)$$

that satisfies the Leibniz rule:

$$D(f \cdot s) = df \otimes s + f \cdot D(s)$$

for any local sections f of $\mathscr{O}$ and s of $\mathscr{O}(E)$.

Example 4.3.1. The exterior derivative on $E = X \times \mathbb{C}^r$:

$$\begin{aligned} d = \partial :\ \mathscr{O}^r &\longrightarrow \mathscr{O} \otimes \Omega^1(\log Z) \\ (s_1, \ldots, s_r) &\longmapsto (ds_1, \ldots, ds_r) = (\partial s_1, \ldots, \partial s_r) \end{aligned}$$

is a canonical choice of logarithmic connection on E. ◊♦◊

Connection Matrices

Let $\mathscr{U}$ be an open cover of X trivializing E. As in the complex analytic case, a logarithmic connection D can be described by a collection of *connection matrices*, as follows: on the framed open set (U_α, e_α), define $\theta_\alpha \in \Gamma(U_\alpha, \Omega^1(\log Z) \otimes \mathfrak{gl}_r(\mathbb{C}))$ by

$$De_{\alpha j} = \sum_{i=1}^{r} (\theta_\alpha)_{ij} \otimes e_{\alpha i}.$$

On (U_α, e_α), one may write $D = \partial + \theta_\alpha$.

Furthermore, if (U_α, e_α) and (U_β, e_β) overlap with transition $g_{\alpha\beta}$, we have the transformation rule

$$\theta_\beta = g_{\alpha\beta}^{-1} \theta_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}. \tag{4.7}$$

Conversely, given such a collection θ_α on the cover $\mathscr{U}$ satisfying the above transformation rule, one can recover D.

Relating Logarithmic Connections on Vector Bundles and Principal GL-Bundles

Let $E \to X$ be a rank r vector bundle, and $P \to X$ the associated $GL_r(\mathbb{C})$-bundle. Following the exact procedure outlined in the complex analytic case (see (3)), one can go back and forth between log connections on E and P as follows:

(1) Start with a logarithmic connection

$$D : \mathscr{O}(E) \longrightarrow \mathscr{O}(E) \otimes \Omega^1(\log Z).$$

The connection matrices of D with respect to a trivializing cover $\{(U_\alpha, e_\alpha)\}$ give us a collection of logarithmic 1-forms

$$\theta_\alpha \in \Gamma(U_\alpha,\ \Omega^1(\log Z) \otimes \mathfrak{gl}_r(\mathbb{C})).$$

On overlaps, these satisfy the transformation rule

$$\begin{aligned} \theta_\beta &= g_{\alpha\beta}^{-1} \theta_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta} \\ &= \operatorname{ad}(g_{\alpha\beta}^{-1}) \theta_\alpha + g_{\alpha\beta}^{*} \omega_0. \end{aligned}$$

This determines a logarithmic connection on P. One can check (by refining if necessary) that our answer is independent of the cover chosen.

(ii) Conversely, a log connection on the frame bundle P gives us a choice of connection D on E, by simply reversing the above steps.

When applied to the trivial bundle $E = X \times \mathbb{C}^r$, we find that the logarithmic Maurer-Cartan connection $(\omega_{MC})_{\log Z}$ corresponds to the exterior derivative d.

4.4 Residues

In this section, we introduce the notion of residues of logarithmic connections on vector bundles, following [16]. We also provide a natural generalization of residues to logarithmic connections on principal bundles.

Henceforth, we will make a few minor changes in our notation, in order to match that of [16]. For the rest of the section, we shall fix the following:

- M is a complex n-manifold;
- $\pi : E \longrightarrow M$ is a holomorphic vector bundle of rank q;
- $\mathscr{O}$ is the sheaf of holomorphic functions on M;
- $\mathscr{O}(E)$ is the sheaf of holomorphic sections of E;
- Z is an analytic hypersurface satisfying the following conditions:
  (i) Z is normally crossing;
  (ii) $Z = \bigcup_{j \in N} Z_j$ is the irreducible decomposition of Z (where N is some indexing set).
- $\Omega^p_M(\log Z)$ is the sheaf of meromorphic p-forms on M, with simple logarithmic poles along Z.

We shall now use the indices λ and μ instead of α and β when dealing with an open cover $\mathscr{U}$.

4.4.1 Residues: Vector Bundles

Let $\pi : E \longrightarrow M$ be a holomorphic vector bundle of rank q, as above. Assume that the logarithmic Atiyah class of E (associated to Z) vanishes, so that E admits a logarithmic connection

$$D : \mathscr{O}(E) \longrightarrow \Omega^1_M(\log Z) \otimes_{\mathscr{O}} \mathscr{O}(E).$$

Choose a sufficiently fine covering $\mathscr{U} = \{U_\lambda\}$ of M by *coordinate* open sets, such that the following conditions hold:

(i) $E|_{U_\lambda}$ is trivial;

(ii) for each Z_j, a defining function $f_{\lambda j}$ is chosen in U_λ and it is a coordinate function of the local coordinate system in U_λ.

Note that (ii) is possible because of our assumptions on Z. If $Z_j \cap U_\lambda = \emptyset$, we set $f_{\lambda j} \equiv 1$.

Let $e_\lambda = (e_{\lambda 1}, \ldots, e_{\lambda q})$ be a holomorphic frame of E on U_λ, thought of as a column vector. As before, the connection matrix D_λ of D with respect to the frame e_λ is defined by

$$D(e_{\lambda j}) = \sum_i (D_\lambda)_{ij} \otimes e_{\lambda i}.$$

D_λ is a matrix, whose entries are sections of $\Omega^1_M(\log Z)$. Let $e_{\lambda\mu} : U_{\lambda\mu} \to GL_q(\mathbb{C})$ be the transition between the overlapping frames e_λ and e_μ, satisfying the relation $e_\lambda = e_{\lambda\mu} e_\mu$.

Lemma 4.4.1. *The connection matrices on $U_{\lambda\mu}$ are related by the familiar formula*

$$D_\lambda e_{\lambda\mu} = de_{\lambda\mu} + e_{\lambda\mu} D_\mu. \tag{4.8}$$

Proof. Exactly the same as [20], Thm.22.1. Note that there is a slight notational difference, since e_λ denotes a column vector here, rather than a row vector. □

Now, since Z is normally crossing, we can uniquely express a logarithmic 1-form $\omega \in \Omega^1_M(\log Z)$ as

$$\omega = a_{\lambda j} \frac{df_{\lambda j}}{f_{\lambda j}} + b_{\lambda j}$$

where $a_{\lambda j} \in \mathscr{O}(U)$ is a holomorphic function and $b_{\lambda j} \in \Omega^1_M(\log Z)$ is logarithmic with a pole along $\bigcup_{l \neq j} Z_l$. Hence, for each λ and j, one may *uniquely* write

$$D_\lambda = A_{\lambda j} \frac{df_{\lambda j}}{f_{\lambda j}} + B_{\lambda j} \tag{4.9}$$

where $A_{\lambda j}$ is a $q \times q$ matrix of holomorphic functions on U_λ, and $B_{\lambda j}$ is a $q \times q$ matrix of logarithmic 1-forms on U_λ with poles along $\bigcup_{l \neq j} Z_l$.

Definition 4.4.2. *The **residue of** D_λ **along** Z_j is defined by*

$$\operatorname{Res}_{Z_j} D_\lambda = A_{\lambda j}|_{Z_j}. \tag{4.10}$$

This is a $q \times q$ matrix of holomorphic functions on $Z_j \cap U_\lambda$. We need to check that this is independent of the choice $f_{\lambda j}$ of a local defining function for Z_j. Suppose $g_{\lambda j}$ were another such choice. By the irreducibility of Z_j, it follows that $g_{\lambda j} = h \cdot f_{\lambda j}$, where $h \in \mathscr{O}(U_\lambda)$ is nowhere vanishing. Therefore, by (4.9),

$$\begin{aligned} D_\lambda &= A_{\lambda j}\frac{df_{\lambda j}}{f_{\lambda j}} + B_{\lambda j} \\ &= A_{\lambda j}\frac{dg_{\lambda j}}{g_{\lambda j}} - A_{\lambda j}\frac{dh}{h} + B_{\lambda j} \\ &= A_{\lambda j}\frac{dg_{\lambda j}}{g_{\lambda j}} + \tilde{B}_{\lambda j} \end{aligned}$$

where $\tilde{B}_{\lambda j} = B_{\lambda j} - A_{\lambda j} \cdot dh/h$ has poles along $\bigcup_{l \neq j} Z_l$. Well-definedness follows from the uniqueness of this expression.

$\operatorname{Res}_{Z_j} D_\lambda$ does depend on the frame e_λ, however (since D_λ is defined with respect to this frame). When two frames overlap, the corresponding residues are related by conjugating by the corresponding transition function:

Lemma 4.4.3. *On $U_{\lambda\mu} \cap Z_j$, we have*

$$\operatorname{Res}_{Z_j} D_\lambda \cdot e_{\lambda\mu}|_{Z_j} = e_{\lambda\mu}|_{Z_j} \cdot \operatorname{Res}_{Z_j} D_\mu. \tag{4.11}$$

Proof. By (4.8) and (4.9),

$$\begin{aligned} A_{\lambda j} e_{\lambda\mu}\frac{df_{\lambda j}}{f_{\lambda j}} + B_{\lambda j} e_{\lambda\mu} &= D_\lambda e_{\lambda\mu} \\ &= de_{\lambda\mu} + e_{\lambda\mu} D_\mu \\ &= de_{\lambda\mu} + e_{\lambda\mu}\left(A_{\mu j}\frac{df_{\mu j}}{f_{\mu j}} + B_{\mu j}\right) \\ &= (e_{\lambda\mu} A_{\mu j})\frac{df_{\mu j}}{f_{\mu j}} + de_{\lambda\mu} + e_{\lambda\mu} B_{\mu j}. \end{aligned}$$

Comparing the first and last expressions, using uniqueness, and restricting to Z_j gives us the desired result. $\square$

Since the collection $\{\text{Res}_{Z_j} D_\lambda\}_\lambda$ obeys the change-of-frame formula stated above, and the open sets $\{U_\lambda \cap Z_j\}_\lambda$ cover Z_j, we may patch the local residues and define:

Definition 4.4.4 (Residue)**.** *The **residue** of the logarithmic connection D along Z_j is the element*

$$\text{Res}_j D \in \check{H}^0(Z_j, \mathscr{O}(\text{End}(E|_{Z_j})))$$

defined by the above collection.

The definition is independent of the trivializing open cover $\{(U_\lambda, e_\lambda)\}_\lambda$ chosen, as one can check using (4.4.3).

Example 4.4.1. As a sanity check, let us see if the above definition of residue aligns with the usual single-variable version for meromorphic 1-forms on a Riemann surface (as defined in (4.1)).

Let M and $E = M \times \mathbb{C}$ be as in (4.1). Let $Z = \{p_1, \ldots, p_m\}$ be a divisor on M (vacuously normally crossing), and let $\omega \in \Omega^1(\log Z)$ be a logarithmic 1-form. Consider the logarithmic connection $D := \omega \otimes e$ on M. We wish to show that $\text{Res}_{p_j}(\omega) = \text{Res}_{p_j}(D)$.

Let (U, z) be a local coordinate centered at the point p_j (so that z becomes a local defining function for the hypersurface p_j), and assume that no other p_k appears in U. Then, one can uniquely write

$$\omega|_U = a\frac{dz}{z} + b,$$

where $a \in \mathscr{O}(U)$ and $b \in \Omega^1(U)$ are holomorphic. Thus

(i) $\text{Res}_{p_j}(\omega)$, in the sense of (4.1.1), is the -1th coefficient of the Laurent series expansion of ω w.r.t. z. This, clearly, is $a(p_j)$.

(ii) $\text{Res}_{p_j}(D)$, in the sense of (4.4.4), is the endomorphism of $E|_{p_j}$ determined by a. Identifying $E|_{p_j}$ with $\mathbb{C}$ via the isomorphism that takes $e(p_j)$ to 1, $\text{Res}_{p_j}(D)$ simply becomes multiplication by $a(p_j)$.

◊♦◊

4.4.2 Residues: Principal G-Bundles

We now provide a simple generalization of the concept of residues to (logarithmic connections on) principal bundles. As in the previous section, our description of the residue is local, with respect to a trivialization of the bundle; we later show that the definition is independent of the trivialization chosen. For a more global formulation (without reference to any trivializing open cover), the reader may refer to [10].

Let our notations be as before. Fix a complex analytic structure group G and a principal G-bundle $P \to M$. Assume that the logarithmic Atiyah class of P vanishes, so that it admits a logarithmic connection

$$D : \mathcal{T}(-\log Z) \longrightarrow \mathrm{At}(\mathcal{P})(\log Z).$$

Let $\mathscr{U} = \{(U_\lambda, \varphi_\lambda)\}_\lambda$ be a trivializing open cover for P with transitions $g_{\lambda\mu}$. With respect to $\mathscr{U}$, D can be uniquely represented by its local connection forms

$$D_\lambda \in \Gamma(U_\lambda,\, \Omega^1(\log Z) \otimes \mathrm{ad}(\mathcal{P})) \cong \Gamma(U_\lambda,\, \Omega^1(\log Z) \otimes \mathcal{L})$$

that satisfy the transformation relations

$$D_\mu = \mathrm{ad}(g_{\lambda\mu}^{-1}) D_\lambda + g_{\lambda\mu}^* \omega_0. \tag{4.12}$$

Since Z is normally crossing, we can uniquely express, for each λ and j,

$$D_\lambda = A_{\lambda j} \frac{df_{\lambda j}}{f_{\lambda j}} + B_{\lambda j}$$

where $A_{\lambda j} \in \Gamma(U_\lambda,\, \Omega^1 \otimes \mathrm{ad}(P))$ is a *holomorphic* $\mathrm{ad}(P)$-valued 1-form on U_λ, and $B_{\lambda j}$ is an $\mathrm{ad}(P)$-valued 1-form on U_λ with poles along $\bigcup_{l \neq j} Z_l$.

Definition 4.4.5. *The* ***residue of*** D_λ ***along*** Z_j *is defined by*

$$\mathrm{Res}_{Z_j} D_\lambda := A_{\lambda j}|_{Z_j}. \tag{4.13}$$

This is now a holomorphic section of the bundle $U_\lambda \times L(G) \cong U_\lambda \times \mathrm{ad}(P)$, restricted to the set $Z_j \cap U_\lambda$. The same reasoning as before shows that this definition is independent of the choice $f_{\lambda j}$ of a local defining function for Z_j.

$\mathrm{Res}_{Z_j} D_\lambda$ does depend on the choice of trivialization φ_λ, however. On overlaps, we have the following transformation rule:

Lemma 4.4.6. *On $U_{\lambda\mu} \cap Z_j$, we have*

$$\mathrm{ad}(g_{\lambda\mu}) \cdot \mathrm{Res}_{Z_j} D_\mu = \mathrm{Res}_{Z_j} D_\lambda. \tag{4.14}$$

Proof. By the previous equalities,

$$\begin{aligned} \mathrm{ad}(g_{\lambda\mu}) \cdot \left(A_{\mu j} \frac{df_{\mu j}}{f_{\mu j}} + B_{\mu j} \right) &= \mathrm{ad}(g_{\lambda\mu}) \cdot D_\mu \\ &= D_\lambda + \mathrm{ad}(g_{\lambda\mu}) \cdot (g^*_{\lambda\mu}\omega_0) \\ &= A_{\lambda j} \frac{df_{\lambda j}}{f_{\lambda j}} + B_{\lambda j} + \mathrm{ad}(g_{\lambda\mu}) \cdot (g^*_{\lambda\mu}\omega_0). \end{aligned}$$

Comparing the first and last expressions, using uniqueness, and restricting to Z_j gives us the desired result. □

Since the collection $\{\mathrm{Res}_{Z_j} D_\lambda\}_\lambda$ obeys the transition formula stated above, and the transitions of the adjoint bundle $\mathrm{ad}(P)$ are precisely $\mathrm{ad}(g_{\lambda\mu})$, we may patch the local residues and define:

Definition 4.4.7 (Residue)**.** *The **residue** of the logarithmic connection D along Z_j is the element*

$$\mathrm{Res}_j D \in \check{H}^0(Z_j, \mathrm{ad}(P)|_{Z_j})$$

defined by the above collection.

The definition is independent of the trivializing open cover $\mathscr{U}$ chosen, as one can check using (4.4.6).

Example 4.4.2. Let $E \to M$ be a holomorphic vector bundle of rank q, and P the associated $GL_q(\mathbb{C})$-bundle. By (4.3.4), a logarithmic connection D^E on E corresponds to a unique logarithmic connection D^P on P, and vice-versa. Compute both their residues along the divisor Z_j; one can easily check that under the canonical isomorphism $\mathrm{End}(E) \cong \mathrm{ad}(P)$, $\mathrm{Res}_j D^E$ equals $\mathrm{Res}_j D^P$. ◊♦◊

Chapter 5

Ohtsuki's Formula

The final chapter of the thesis is about Ohtsuki's formula. In the first section, we will state Ohtsuki's main theorem in the case of vector bundles ([16], Thm.3). In the second section, we will present a simple generalisation of this formula to principal bundles, allowing for a slightly wider range of structure group (other than $GL_q(\mathbb{C})$). This will make use of the generalized concept of residue introduced in Section 4.4. The proof of this statement, which we illustrate with all relevant details, is essentially a modification of Ohtsuki's original proof in [16].

5.1 Ohtsuki's Formula: Vector Bundles

We present Ohtsuki's formula for vector bundles, as stated by him in [16]. Our notations and assumptions will be exactly as in the last section of the previous chapter (4.4).

Let $J = (j_1, \dots, j_k) \in N^k$. Suppose $j_1^*, \dots, j_p^*$ are the distinct elements of J. Set $J^* := \{j_1^*, \dots, j_p^*\}$, and denote the multiplicity of j_m^* in J by a_m. We have the relation $\sum_{m=1}^{p} a_m = k$.

Now, define $Z_{J^*} := \bigcap_{m=1}^{p} Z_{j_m^*}$. By our assumptions on Z, Z_{J^*} is either empty or a submanifold of codimension p. It is compact, since M is compact; it need not be connected, however. We write its decomposition into connected components as $Z_{J^*} = \bigcup_i Z_{J^*}^{(i)}$. By local connectedness, the components $Z_{J^*}^{(i)}$ are both open and closed in Z_{J^*}; compactness of Z_{J^*} then forces them to be finite in number.

Now, recall the definition of the ***Chern Polynomials*** $c_k(A)$ from (2.1.2). We denote by $c_k(A_1, \dots, A_k)$ the completely polarized form of $c_k(A)$. This, as we know, is $\mathrm{ad}(GL_r(\mathbb{C}))$-invariant:

$$c_k(BA_1B^{-1}, \dots, BA_kB^{-1}) = c_k(A_1, \dots, A_k)$$

for any invertible B. We now have a proposition:

Proposition 5.1.1. *For any $J = (j_1, \dots, j_k) \in N^k$,*

$$c_k(\mathrm{Res}_{j_1} D, \dots, \mathrm{Res}_{j_k} D) \;\in\; \mathscr{O}(Z_{J^*})$$

is constant on each component $Z_{J^*}^{(i)}$.

Proof. First, let us check that this is a well-defined holomorphic function on Z_{J^*}. By (4.4.3), for each $1 \le r \le k$, the matrices $\mathrm{Res}_{j_r} D_\lambda$ and $\mathrm{Res}_{j_r} D_\mu$ are conjugate on $U_{\lambda\mu}$ via the invertible matrix $e_{\lambda\mu}$. Invariance of the Chern polynomials then implies that $c_k(\mathrm{Res}_{j_1} D, \dots, \mathrm{Res}_{j_k} D)$ is well-defined on all of Z_{J^*}.

Now, each component $Z_{J^*}^{(i)}$ is a compact connected manifold. A corollary of the maximum principle (1.1.3) therefore shows that $c_k(\mathrm{Res}_{j_1} D, \dots, \mathrm{Res}_{j_k} D)$ is constant on each $Z_{J^*}^{(i)}$. □

For each i, we denote this constant value by $c_k(\mathrm{Res}_J D)^{(i)}$. Note that for any permutation $\widetilde{J}$ of J, $c_k(\mathrm{Res}_J D)^{(i)} = c_k(\mathrm{Res}_{\widetilde{J}} D)^{(i)}$ by the symmetry property of the completely polarised form of the kth Chern polynomial.

Now we recall the theory of currents (see 1.5). Let W be a complex submanifold of M of codimension p (endowed with the natural orientation). W defines a current

$$\int_W -$$

of type (p,p), called the ***current of integration along*** W. Since W has empty boundary, by Stokes' theorem, its current of integration is $\overline{\partial}$-closed; it can therefore be identified with a unique cohomology class in $H^{p,p}(M) \cong \check{H}^p(M,\Omega^p)$, called its Poincaré dual. We denote this by $c_p(W)$.

We may now finally state the main theorem (Thm. 3 of [16]), in the vector bundle case:

Theorem 5.1.2 (Ohtsuki's Formula)**.** *Let E be a holomorphic vector bundle over a compact complex manifold M and let D be a logarithmic connection of E with the pole Z. Suppose Z is normally crossing. Then the following relation holds in the cohomology group $\check{H}^k(M,\Omega^k)$:*

$$c_k(E) \;=\; (-1)^k \sum_{J\in N^k} \sum_i \left(c_k(\mathrm{Res}_J D)^{(i)} \cdot c_p(Z_{J^*}^{(i)}) \right) \prod_{m=1}^{p} c_1([Z_{j_m^*}])^{a_m - 1} \tag{5.1}$$

where $c_k(E)$ is the kth Chern class of E.

Proof. See [16], pages 17-21. We shall provide a proof of the more general statement, for principal bundles, which is but a slight modification of Ohtsuki's original. □

Remark 5.1.3. If Z_{J^*} is connected for each $J \in N^k$, the statement reduces to (4.1.4). This is because of the relation $c_p(Z_{J^*}) = c_1(Z_{j_1^*}) \wedge \cdots \wedge c_1(Z_{j_p^*})$ (see the remark on p.17 of [16]).

5.2 Ohtsuki's Formula: Principal Bundles

Using the generalised definition of residue from 4.4, we will state and prove a generalised version of Ohtsuki's formula for principal bundles. For reasons apparent later, we will assume that the structure group G is either complex semisimple or $GL_r(\mathbb{C})$.

5.2.1 Statement of the Main Theorem

Fix a principal G-bundle $P \to M$, with G as above. Assume that the log Atiyah class of P vanishes, and choose a logarithmic connection D on P. Let all other notations and assumptions be as in the previous section and 4.4.

Choose a homogeneous invariant polynomial $F_k(A) \in \mathrm{Inv}(L(G))$ of degree k, and denote its completely polarised form (which is again invariant) by $F_k(A, \ldots, A)$.

Proposition 5.2.1. *For any $J = (j_1, \ldots, j_k) \in N^k$,*

$$F_k(\mathrm{Res}_{j_1} D, \ldots, \mathrm{Res}_{j_k} D) \;\in\; \mathscr{O}(Z_{J^*})$$

is constant on each component $Z_{J^*}^{(i)}$.

Proof. Same as (5.1.1); use the transition formula (4.4.6) instead of (4.4.3). The constant value of $F_k(\mathrm{Res}_{j_1} D, \ldots, \mathrm{Res}_{j_k} D)$ on $Z_{J^*}^{(i)}$ will hereon be denoted $F_k(\mathrm{Res}_J D)^{(i)}$. □

Theorem 5.2.2 (Ohtsuki's Formula: Principal Bundles)**.** *Let M be a compact complex manifold, and P a complex analytic principal bundle over M with structure group G (semisimple or $GL_q(\mathbb{C})$). Let D be a logarithmic connection of P with the pole Z. Suppose Z is normally crossing. Then, for any homogeneous invariant polynomial $F_k \in \mathrm{Inv}(L(G))$ of degree k, the following relation holds in the cohomology group $\check{H}^k(M, \Omega^k)$:*

$$\left(\frac{-1}{2\pi i}\right)^k F_k(P) \;=\; (-1)^k \sum_{J\in N^k} \sum_i \left(F_k(\mathrm{Res}_J D)^{(i)} \cdot c_p(Z_{J^*}^{(i)})\right) \prod_{m=1}^{p} c_1([Z_{j_m^*}])^{a_m - 1} \quad (5.2)$$

where $F_k(P)$ is the characteristic class of P determined by F_k.

5.3 Proof of the Main Theorem

5.3.1 A Smooth $(1,0)$-connection on P

- Choose a C^∞ connection Γ on P, of type $(1,0)$. This is always possible by Singer's result (see [18], or (3.3.2)), since we assumed G to be semisimple or $GL_q(\mathbb{C})$ for some q.

- Let $\mathscr{U} = \{(U_\lambda, \varphi_\lambda)\}_\lambda$ be a trivializing open cover for P, as in 4.4. The corresponding local connection form on U_λ is denoted

$$\Gamma_\lambda \in \mathscr{E}^{1,0}(U_\lambda, \mathrm{ad}(P)).$$

- Let K be the curvature of Γ. By (2.4.1) and Singer's result, K is a global, $\overline{\partial}$-closed $\mathrm{ad}(P)$-valued $(1,1)$-form on M that is locally given by

$$K_\lambda = d\Gamma_\lambda - \Gamma_\lambda \wedge \Gamma_\lambda = \overline{\partial}\Gamma_\lambda.$$

- By Chern-Weil theory (see 2),

$$\left(\frac{-1}{2\pi i}\right)^k F_k(P) = \left[F_k\left(\frac{-1}{2\pi i}K\right)\right] \in H^{k,k}(M).$$

5.3.2 A Metric on $[Z_j]$

Recall that the local data $(f_{\lambda j})$ defines the divisor Z_j. The associated line bundle $[Z_j]$ is then constructed in such a way that it has transitions $f_{\lambda\mu j} := f_{\lambda j}/f_{\mu j}$. (CAUTION: this notation, while in line with [9], is the opposite of that used in [16]).
Choose a frame $(s_{\lambda j})$ on U_λ for $[Z_j]$. On overlaps, the frames obey the transformation rule (see [13], Lemma 3.25)

$$s_{\mu j} = f_{\lambda\mu j} s_{\lambda j}\ .$$

Let h be a hermitian metric on $[Z_j]$. h is completely determined on U_λ by the positive function $h_{\lambda j} = h(s_{\lambda j}, s_{\lambda j})$. On overlaps, these functions satisfy

$$h_{\mu j} = |f_{\lambda\mu j}|^2 h_{\lambda j}.$$

Conversely, any collection of smooth positive functions $(h_{\lambda j})_\lambda$ satisfying the above relation on overlaps defines a Hermitian metric on $[Z_j]$.

Having fixed one such choice, define

(i) $\omega_{\lambda j} := \partial h_{\lambda j} \cdot h_{\lambda j}^{-1} = \partial(\log h_{\lambda j})$;

(ii) $\Theta_{\lambda j} := \overline{\partial}\omega_{\lambda j} = \overline{\partial}\partial(\log h_{\lambda j})$.

$\omega_{\lambda j}$ is the matrix of the Chern connection (with respect to h) on $(U_{\lambda j}, s_{\lambda j})$, and $\Theta_{\lambda j}$ the corresponding curvature matrix ([13], p.209).

Furthermore, if we call the global curvature form Θ_j, then

(i) Θ_j is a global $\overline{\partial}$-closed (1,1)-form ([13] Prop.7.18)

(ii) $\frac{-1}{2\pi i}\Theta_j$ is the first real Chern form of the Chern connection;

(iii) $\frac{-1}{2\pi i}\Theta_j$ is $\overline{\partial}$-cohomologous to the current of integration of Z_j ([9], p.141).

5.3.3 First Step of the Proof

Recall the statement of the main theorem (5.2.2): we want to establish an equality of cohomology classes

$$\left(\frac{-1}{2\pi i}\right)^k F_k(P) \;=\; (-1)^k \sum_{J\in N^k}\sum_i \left(c_k(\mathrm{Res}_J D)^{(i)} \cdot c_p(Z_{J^*}^{(i)})\right) \prod_{m=1}^{p} c_1([Z_{j_m^*}])^{a_m-1} \quad .$$

The left-hand side $\left(\frac{-1}{2\pi i}\right)^k F_k(P)$ is represented by the $\overline{\partial}$-closed (k,k)-form

$$F_k\left(\frac{-1}{2\pi i}K\right) = \left(\frac{-1}{2\pi i}\right)^k F_k(K,\ldots,K)$$

The right-hand side is represented by

$$(-1)^k \sum_{J\in N^k}\left\{\sum_i F_k(\mathrm{Res}_J D)^{(i)} \cdot \left(\prod_{m=1}^{p}\left(\frac{-1}{2\pi i}\Theta_{j_m^*}\right)^{a_m-1}\right) Z_{J^*}^{(i)}\right\}.$$

We need to show that these two are $\overline{\partial}$-cohomologous. (We suppress the notation of wedge products for brevity).

The Smooth 1-Form L

For each open set U_λ, define $L_\lambda := D_\lambda - \Gamma_\lambda$. This is a C^∞ $\mathrm{ad}(P)$-valued $(1,0)$-form on U_{lambda}, having a pole along Z. We may therefore express L_λ uniquely as

$$L_\lambda = A_{\lambda j}\frac{df_{\lambda j}}{f_{\lambda j}} + B_{\lambda j}$$

where $A_{\lambda j}$ is a holomorphic section of $\mathrm{ad}(P)$ on U_λ, and $B_{\lambda j}$ a C^∞ $\mathrm{ad}(P)$-valued 1-form on U_λ with a simple pole along $\bigcup_{l\neq j} Z_l$. As before, we define

$$\mathrm{Res}_{Z_j} L_\lambda := A_{\lambda j}. \tag{5.3}$$

Lemma 5.3.1. *On the overlap $U_{\lambda\mu}$, we have the relations*

(i) $L_\lambda = \mathrm{ad}(g_{\lambda\mu})L_\mu$.

(ii) $\mathrm{Res}_{Z_j} L_\lambda \;=\; \mathrm{ad}(g_{\lambda\mu})|_{Z_j} \cdot \mathrm{Res}_{Z_j} L_\mu$.

Proof. (i) Using the transformation rules (4.12) for D and (2.3.2) for Γ,

$$\begin{aligned}
L_\mu &= D_\mu - \Gamma_\mu \\
&= \mathrm{ad}(g_{\lambda\mu}^{-1})D_\lambda + g_{\lambda\mu}^*\omega_0 - \mathrm{ad}(g_{\lambda\mu}^{-1})\Gamma_\lambda - g_{\lambda\mu}^*\omega_0 \\
&= \mathrm{ad}(g_{\lambda\mu}^{-1})(D_\lambda - \Gamma_\lambda) \\
&= \mathrm{ad}(g_{\lambda\mu}^{-1})L_\lambda.
\end{aligned}$$

Apply $\mathrm{ad}(g_{\lambda\mu})$ on both sides.

(ii) Copy the proof of (4.4.6).

□

We may therefore patch the local forms L_λ to obtain a smooth, global $\mathrm{ad}(P)$-valued $(1,0)$-form, which we call L. Similarly, the local sections $\mathrm{Res}_{Z_j} L_\lambda$ patch to gives us a holomorphic global section $\mathrm{Res}_{Z_j} L$ of $\mathrm{ad}(P)|_{Z_j}$.

Lemma 5.3.2. *L and $\mathrm{Res}_{Z_j} L$ satisfy the following relations:*

(i) $\overline{\partial} L = -K$ *on* $M\backslash Z$.

(ii) $\mathrm{Res}_{Z_j} L = Res_{Z_j} D$.

Proof. (i) D is holomorphic on $M\backslash Z$; hence $\overline{\partial} L = \overline{\partial} D - \overline{\partial}\Gamma = -K$.

(ii) Clear from the definition.

□

First Steps

Fix a C^∞ Riemannian metric on M. For a subset $X \subseteq M$, we denote the ε-neighbourhood of X by X^ε.
Let $\varphi \in \mathscr{E}^{n-k,n-k}(M)$ be a C^∞ test form. We evaluate the LHS on φ:

$$\begin{aligned}
&\left(\frac{-1}{2\pi i}\right)^k F_k(K,\ldots,K)[\varphi] && \\
&= \left(\frac{-1}{2\pi i}\right)^k \int_M F_k(K,\ldots,K)\wedge\varphi && \textit{Poincaré Duality} \\
&= \left(\frac{-1}{2\pi i}\right)^k \lim_{\varepsilon\to 0}\int_{M\backslash Z^\varepsilon} F_k(K,\ldots,K)\wedge\varphi && \textit{Z has zero measure} \\
&= \left(\frac{-1}{2\pi i}\right)^k \lim_{\varepsilon\to 0}\int_{M\backslash Z^\varepsilon} F_k(-\overline{\partial}L,K,\ldots,K)\wedge\varphi && (5.3.2) \\
&= -\left(\frac{-1}{2\pi i}\right)^k \lim_{\varepsilon\to 0}\int_{M\backslash Z^\varepsilon} \overline{\partial}(F_k(L,K,\ldots,K))\wedge\varphi && \textit{expand; use } \overline{\partial}K = 0 \\
&= -\left(\frac{-1}{2\pi i}\right)^k \lim_{\varepsilon\to 0}\int_{M\backslash Z^\varepsilon} \overline{\partial}(F_k(L,K,\ldots,K)\wedge\varphi) && \textit{Leibniz rule} \\
&\quad -\left(\frac{-1}{2\pi i}\right)^k \lim_{\varepsilon\to 0}\int_{M\backslash Z^\varepsilon} F_k(L,K,\ldots,K)\wedge\overline{\partial}\varphi. &&
\end{aligned}$$

L is of type (1,0) on $M\backslash Z^\varepsilon$ since both D and Γ are so. Therefore, the differential form $\partial(F_k(L,K,\ldots,K)\wedge\varphi)$ is of type $(n+1,n-1)$, which means it is identically zero (the first component of its bidegree exceeds n). In particular, we have
$\partial(F_k(L,K,\ldots,K)\wedge\varphi) = d(F_k(L,K,\ldots,K)\wedge\varphi)$. The above expression thus equals

$$
\begin{aligned}
&- \left(\frac{-1}{2\pi i}\right)^k \lim_{\varepsilon\to 0} \int_{M\setminus Z^\varepsilon} d(F_k(L,K,\ldots,K)\wedge\varphi) \\
&- \left(\frac{-1}{2\pi i}\right)^k \lim_{\varepsilon\to 0} \int_{M\setminus Z^\varepsilon} F_k(L,K,\ldots,K)\wedge\overline{\partial}\varphi \\
= &- \left(\frac{-1}{2\pi i}\right)^k \lim_{\varepsilon\to 0} \int_{M\setminus Z^\varepsilon} d(F_k(L,K,\ldots,K)\wedge\varphi) \; - T[\overline{\partial}\varphi]
\end{aligned}
$$

where

$$
T[\psi] := \left(\frac{-1}{2\pi i}\right)^k \lim_{\varepsilon\to 0} \int_{M\setminus Z^\varepsilon} F_k(L,K,\ldots,K)\wedge\psi
$$

is a well-defined current of type $(k, k-1)$ on M, called a ***principal value current***. (Despite L having singularities along Z, the integral T is well-defined since the singularities are logarithmic, and therefore, controlled).

We now employ Stokes' Theorem:

$$
\begin{aligned}
&\left(\frac{-1}{2\pi i}\right)^k F_k(K,\ldots,K)[\varphi] \\
= &- \left(\frac{-1}{2\pi i}\right)^k \lim_{\varepsilon\to 0} \int_{M\setminus Z^\varepsilon} d(F_k(L,K,\ldots,K)\wedge\varphi) \; - T[\overline{\partial}\varphi] \\
= &\left(\frac{-1}{2\pi i}\right)^k \lim_{\varepsilon\to 0} \int_{\partial Z^\varepsilon} F_k(L,K,\ldots,K)\wedge\varphi \; - T[\overline{\partial}\varphi] \\
= &\left(\frac{-1}{2\pi i}\right)^k \lim_{\varepsilon\to 0} \int_{\partial Z^\varepsilon} F_k(L,K,\ldots,K)\wedge\varphi \; - \overline{\partial}T[\varphi]\;.
\end{aligned}
$$

The minus sign in line 2 disappears since the boundary orientation of $\partial(M\setminus Z^\varepsilon)$ is opposite to the boundary orientation of ∂Z^ε. In the last step, $T[\overline{\partial}\varphi] = \overline{\partial}T[\varphi]$ by definition of the $\overline{\partial}$ operator on currents.

We have thus shown that $\left(\frac{-1}{2\pi i}\right)^k F_k(K,\ldots,K)[\varphi]$ is $\overline{\partial}$-cohomologous to

$$
\left(\frac{-1}{2\pi i}\right)^k \lim_{\varepsilon\to 0} \int_{\partial Z^\varepsilon} F_k(L,K,\ldots,K)\wedge\varphi.
$$

We show how to modify this integral next.

5.3.4 An Iterated Integral

We now compute the integral

$$(\star) := \lim_{\varepsilon\to 0}\int_{\partial Z^\varepsilon} F_k(L, K, \dots, K)\wedge\varphi.$$

The key observation to make is that ∂Z^ε is a tubular neighbourhood of Z, which locally looks like a product $S^1\times Z$. This enables us to integrate iteratively.

Consider the neighborhood U_λ. We may assume, without loss of generality, that $U_\lambda \simeq \Delta^n$ is a polydisc, $U_\lambda\cap Z = Z_{j_1}\cup\cdots\cup Z_{j_r}$, and that $(f_{\lambda j_1},\dots,f_{\lambda j_r})$ are the first r coordinates of some coordinate system $(z_1,\dots,z_n)$ on U_λ.

As before, we can write

$$L_\lambda = \sum_{t=1}^{r} A_{\lambda t}\frac{dz_t}{z_t} + B_\lambda - \Gamma_\lambda$$

where $A_{\lambda t} = \mathrm{Res}_t D_\lambda$ and B_λ is holomorphic, with no poles. We compute

$$\begin{aligned}
(\star)|_{U_\lambda} &= \lim_{\varepsilon\to 0}\int_{\partial Z^\varepsilon\cap U_\lambda} F_k(L, K,\dots,K)\wedge\varphi\\
&= \lim_{\varepsilon\to 0}\int_{\partial Z^\varepsilon\cap U_\lambda} F_k(\sum_{t=1}^{r} A_{\lambda j}\frac{dz_t}{z_t} + B_\lambda - \Gamma_\lambda, K,\dots,K)\wedge\varphi\\
&= \sum_{t=1}^{r}\lim_{\varepsilon\to 0}\int_{\partial Z^\varepsilon\cap U_\lambda} F_k(A_{\lambda t}\frac{dz_t}{z_t}, K,\dots,K)\wedge\varphi\\
&+ \lim_{\varepsilon\to 0}\int_{\partial Z^\varepsilon\cap U_\lambda} F_k(B_\lambda - \Gamma_\lambda, K,\dots,K)\wedge\varphi\\
&= \sum_{t=1}^{r}\lim_{\varepsilon\to 0}\int_{\partial Z^\varepsilon\cap U_\lambda} F_k(A_{\lambda t}\frac{dz_t}{z_t}, K,\dots,K)\wedge\varphi
\end{aligned}$$

We used the mulitlinearity of F_k in the above. The term

$$\lim_{\varepsilon\to 0}\int_{\partial Z^\varepsilon\cap U_\lambda} F_k(B_\lambda - \Gamma_\lambda, K,\dots,K)\wedge\varphi = 0$$

since it is the integral of a $(n, n-1)$ form on the smooth $(2n-1)$-manifold ∂Z^ε, whose volume approaches zero as $\varepsilon\to 0$.

Let us now compute the integral

$$\lim_{\varepsilon \to 0} \int_{\partial Z^\varepsilon \cap U_\lambda} F_k(A_{\lambda 1} \frac{dz_1}{z_1}, K, \ldots, K) \wedge \varphi.$$

The singularities of the integrand lie solely along Z_1. Hence one may consider $\partial Z^\varepsilon \cap U_\lambda$ as a tubular ε-neighbourhood of Z_1, and integrate iteratively:

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_{|z_1|=\varepsilon} \int_{Z_1} F_k(A_{\lambda 1} \frac{dz_1}{z_1}, K, \ldots, K) \wedge \varphi \qquad \textit{multilinearity} \\
=&\lim_{\varepsilon \to 0} \int_{|z_1|=\varepsilon} \int_{Z_1} \frac{dz_1}{z_1} F_k(A_{\lambda 1}, K, \ldots, K) \wedge \varphi \\
=&\lim_{\varepsilon \to 0} \int_{|z_1|=\varepsilon} \frac{dz_1}{z_1} \int_{Z_1} F_k(A_{\lambda 1}, K, \ldots, K) \wedge \varphi \\
=&\lim_{\varepsilon \to 0} 2\pi i \int_{Z_1} F_k(\mathrm{Res}_1 D_\lambda, K, \ldots, K) \wedge \varphi \\
=&2\pi i \int_{Z_1} F_k(\mathrm{Res}_1 D_\lambda, K, \ldots, K) \wedge \varphi.
\end{aligned}$$

Doing so for each t, and summing up, we finally get

$$\begin{aligned}
(\star)|_{U_\lambda} &= \lim_{\varepsilon \to 0} \int_{\partial Z^\varepsilon} F_k(L, K, \ldots, K) \wedge \varphi \\
&= \sum_{t=1}^{r} 2\pi i \int_{Z_t} F_k(\mathrm{Res}_t D_\lambda, K, \ldots, K) \wedge \varphi.
\end{aligned}$$

We have thus shown that $\left(\frac{-1}{2\pi i}\right)^k F_k(K)$ is $\overline{\partial}$-cohomologous to the current

$$\frac{(-1)^k}{(2\pi i)^{k-1}} \sum_{j \in N} \int_{Z_j} F_k(\mathrm{Res}_j D_\lambda, K, \ldots, K) \wedge - \, .$$

To move towards the intersection Z_{J^*}, we would ideally like to induct and repeat the same process as before, with the appropriate restrictions of E, D and L:

$$\begin{aligned}\left(\frac{-1}{2\pi i}\right)^k F_k(K) &\sim \frac{(-1)^k}{(2\pi i)^{k-1}} \sum_{j\in N} \int_{Z_j} F_k(\mathrm{Res}_j D_\lambda, K, \ldots, K) \\ &\sim \frac{(-1)^k}{(2\pi i)^{k-2}} \sum_{(j_1,j_2)\in N^2} \int_{Z_{j_1^*}\cap Z_{j_2^*}} F_k(\mathrm{Res}_{j_1^*} D_\lambda, \mathrm{Res}_{j_2^*} D_\lambda, K, \ldots, K) \\ &\vdots \\ &\sim \frac{(-1)^k}{(2\pi i)^{k-p}} \sum_{J\in N^k} \int_{Z_{J^*}} F_k(\mathrm{Res}_{j_1^*} D_\lambda, \ldots, \mathrm{Res}_{j_p^*} D_\lambda, K, \ldots, K),\end{aligned}$$

where $\sim$ denotes the relation of being $\overline{\partial}$-cohomologous. These calculations do not go through, however, simply because $D|_{Z_{J^*}}$ can have a pole on all of Z_{J^*}. We would therefore like to subtract out terms of the form $df_{\lambda j}/f_{\lambda j}$ from D. This is elaborated on in the next subsection.

5.3.5 An Inductive Step

The connection $\widetilde{B}_J$

For this subsection, fix an element $J = (j_1, \ldots, j_k) \in N^k$. On U_λ, write

$$D_\lambda = \sum_{t\in J^*} A_{\lambda t} \frac{df_{\lambda t}}{f_{\lambda t}} + B_\lambda$$

where $A_{\lambda t} = \mathrm{Res}_t D_\lambda$ and B_λ is logarithmic with a pole along $\bigcup_{s\notin J^*}$.

Lemma 5.3.3. *The* 1*-forms B_λ transform according to the rule*

$$B_\mu = g_{\lambda\mu}^* \omega_0 + \mathrm{ad}(g_{\lambda\mu}^{-1}) B_\lambda + \sum_{t\in J^*} \mathrm{ad}(g_{\lambda\mu}^{-1}) \mathrm{Res}_t D_\lambda \frac{df_{\lambda\mu t}}{f_{\lambda\mu t}} \tag{5.4}$$

where ω_0 is the Maurer-Cartan form and $f_{\lambda\mu t} = f_{\lambda t}/f_{\mu t} \in \mathscr{O}^*(U_{\lambda\mu})$ is the transition function of the line bundle $[Z_t]$.

Proof. By (4.4.6),

$$
\begin{aligned}
\sum_{t\in J^*} A_{\mu t}\frac{df_{\mu t}}{f_{\mu t}} + B_\mu = D_\mu & \\
&= g_{\lambda\mu}^*\omega_0 + \mathrm{ad}(g_{\lambda\mu}^{-1})D_\lambda \\
&= g_{\lambda\mu}^*\omega_0 + \mathrm{ad}(g_{\lambda\mu}^{-1})\left(\sum_{t\in J^*} A_{\lambda t}\frac{df_{\lambda t}}{f_{\lambda t}} + B_\lambda\right) \\
&= g_{\lambda\mu}^*\omega_0 + \mathrm{ad}(g_{\lambda\mu}^{-1})\left(\sum_{t\in J^*} A_{\lambda t}\left(\frac{df_{\mu t}}{f_{\mu t}} + \frac{df_{\lambda\mu t}}{f_{\lambda\mu t}}\right) + B_\lambda\right)
\end{aligned}
$$

where we use the chain rule in the last step. Comparing the terms that do not involve $df_{\mu t}/f_{\mu t}$ on both sides gives us the desired result. □

Define $B_{\lambda J} := B_\lambda|_{Z_{J^*}}$, and set

$$\widetilde{B}_{\lambda J} := B_{\lambda J} - \sum_{j\in J^*}(\mathrm{Res}_j D_\lambda)|_{Z_{J^*}}\cdot \omega_{\lambda j}|_{Z_{J^*}}$$

where $\omega_{\lambda j}$ was the Chern connection on $[Z_j]$ with respect to the Hermitian metric $(h_{\lambda j})$. $\widetilde{B}_{\lambda J}$ is a smooth $\mathrm{ad}(P)$-valued 1-form on $Z_{J^*}\backslash(\bigcup_{s\notin J^*} Z_s)$, with logarithmic poles along $Z_{J^*}\cap(\bigcup_{s\notin J^*} Z_s)$.

Lemma 5.3.4. *$\widetilde{B}_J = \{\widetilde{B}_{\lambda J}\}$ defines a smooth connection of $P|_{Z_{J^*}}$ having a pole along $Z_{J^*}\cap(\bigcup_{s\notin J^*} Z_s)$.*

Proof. We need to check that the 1-forms $\widetilde{B}_{\lambda J}$ transform according to the rule (4.4.6). So far, we know that

(i) $B_{\lambda J}$s transform according to (5.4);

(ii) $\mathrm{Res}_j D_\lambda|_{Z_{J^*}}$s transform according to (4.4.6);

(iii) $\omega_{\mu j} = df_{\lambda\mu j}(f_{\lambda\mu j})^{-1} + f_{\lambda\mu j}\,\omega_{\lambda j}(f_{\lambda\mu j})^{-1} = df_{\lambda\mu j}/f_{\lambda\mu j} + \omega_{\lambda j}$.

Putting these together gives us (omitting the notation $|_{Z_{J^*}}$)

$$\begin{aligned}
\widetilde{B}_{\mu J} &= B_{\mu J} - \sum_{j\in J^*}(\mathrm{Res}_j D_\mu)\cdot\omega_{\mu j} \\
&= g_{\lambda\mu}^*\omega_0 + \mathrm{ad}(g_{\lambda\mu}^{-1})B_{\lambda J} + \sum_{t\in J^*}\mathrm{ad}(g_{\lambda\mu}^{-1})\mathrm{Res}_t D_\lambda\cdot\frac{df_{\lambda\mu t}}{f_{\lambda\mu t}} - \sum_{j\in J^*}(\mathrm{Res}_j D_\mu)\cdot\omega_{\mu j} \\
&= g_{\lambda\mu}^*\omega_0 + \mathrm{ad}(g_{\lambda\mu}^{-1})B_{\lambda J} + \sum_{t\in J^*}\mathrm{ad}(g_{\lambda\mu}^{-1})\mathrm{Res}_t D_\lambda\cdot\frac{df_{\lambda\mu t}}{f_{\lambda\mu t}} - \sum_{j\in J^*}\mathrm{ad}(g_{\lambda\mu}^{-1})(\mathrm{Res}_j D_\lambda)\cdot\omega_{\mu j} \\
&= g_{\lambda\mu}^*\omega_0 + \mathrm{ad}(g_{\lambda\mu}^{-1})B_{\lambda J} - \sum_{j\in J^*}\mathrm{ad}(g_{\lambda\mu}^{-1})(\mathrm{Res}_j D_\lambda)\cdot\omega_{\lambda j} \\
&= g_{\lambda\mu}^*\omega_0 + \mathrm{ad}(g_{\lambda\mu}^{-1})\widetilde{B}_{\lambda J}.
\end{aligned}$$

Thus, the $\widetilde{B}_{\lambda J}$s can be patched to give a smooth connection $\widetilde{B}_J$ on P, with a pole along $Z_{J^*}\cap(\bigcup_{s\notin J^*}Z_s)$. $\square$

Lemma 5.3.5. *Set* $L_J := \widetilde{B}_J - \Gamma|_{Z_{J^*}}$. *Then*

(0) L_J *is a* C^∞ $\mathrm{ad}(P)$*-valued 1-form on* Z_{J^*}*, having a pole along* $Z_{J^*}\cap(\bigcup_{s\in J^*}Z_s)$.

(i) $\overline{\partial}L_J = -\sum_{j\in J^*}(\mathrm{Res}_j D\cdot\Theta_j) - K$ *on* $Z_{J^*}\backslash(\bigcup_{s\in J^*}Z_s)$.

(ii) $L_{J\lambda} = \mathrm{ad}(g_{\lambda\mu})|_{Z_{J^*}}\cdot L_{J\mu}$ *on* $U_{\lambda\mu}\cap Z_{J^*}$.

(iii) $\mathrm{Res}_{Z_s}L_{J\lambda} = \mathrm{ad}(g_{\lambda\mu})|_{Z_s}\cdot\mathrm{Res}_{Z_s}L_{J\mu}$ *on* $U_{\lambda\mu}\cap Z_s$*, for* $s\notin J^*$.

(iv) $\mathrm{Res}_{Z_s}L_J = \mathrm{Res}_{Z_s}D$ *for* $s\notin J^*$.

Proof. The proof is an easy modification of the proofs of (5.3.1) and (5.3.2). $\square$

The Inductive Step

We may now proceed with the inductive step. Write $R_j := \mathrm{Res}_j D$. Let $l\le k$, $J=(j_1,\ldots,j_l)\in N^l$, and $J^*=\{j_1^*,\ldots,j_p^*\}$. Consider the integral

$$\begin{aligned}
I(\varphi) &:= \frac{(-1)^{l-p}}{(2\pi i)^{k-p}}\int_{Z_{J^*}}F_k(R_{j_1},\ldots,R_{j_l},K,\ldots,K)(\Theta_{j_1^*})^{a_1-1}\cdots(\Theta_{j_p^*})^{a_p-1}\,\varphi \\
&= \frac{(-1)^{l-p}}{(2\pi i)^{k-p}}\lim_{\varepsilon\to 0}\int_{Z_{J^*}\backslash X^\varepsilon}F_k(R_{j_1},\ldots,R_{j_l},K,\ldots,K)P_J(\Theta)\,\varphi
\end{aligned}$$

where $X := Z_{J^*} \cap (\bigcup_{s \notin J^*} Z_s)$, $P_J(\Theta) := (\Theta_{j_1^*})^{a_1-1} \cdots (\Theta_{j_p^*})^{a_p-1}$, and $\varphi \in \mathscr{E}^{n-p,n-p}(Z_{J^*})$ is a test form. We modify this integral in a series of steps quite similar to before.

By (5.3.5), $\overline{\partial} L_J = -\sum_{j \in J^*} R_j \Theta_j - K$. Substituting this in $I(\varphi)$,

$$\begin{aligned}
I(\varphi) =& \frac{(-1)^{l-p}}{(2\pi i)^{k-p}} \lim_{\varepsilon \to 0} \int_{Z_{J^*} \setminus X^\varepsilon} F_k(R_{j_1}, \dots, R_{j_l}, K, \dots, K) P_J(\Theta)\, \varphi \\
=& \frac{(-1)^{l-p+1}}{(2\pi i)^{k-p}} \lim_{\varepsilon \to 0} \int_{Z_{J^*} \setminus X^\varepsilon} F_k(R_{j_1}, \dots, R_{j_l}, \overline{\partial} L_J, K, \dots, K) P_J(\Theta)\, \varphi \\
&+ \frac{(-1)^{l-p+1}}{(2\pi i)^{k-p}} \lim_{\varepsilon \to 0} \sum_{j \in J^*} \int_{Z_{J^*} \setminus X^\varepsilon} F_k(R_{j_1}, \dots, R_{j_l}, R_j \Theta_j, K, \dots, K) P_J(\Theta)\, \varphi \\
=& \frac{(-1)^{l-p+1}}{(2\pi i)^{k-p}} \lim_{\varepsilon \to 0} \int_{Z_{J^*} \setminus X^\varepsilon} \overline{\partial}(F_k(R_{j_1}, \dots, R_{j_l}, L_J, K, \dots, K) P_J(\Theta))\, \varphi \\
&+ \frac{(-1)^{l-p+1}}{(2\pi i)^{k-p}} \lim_{\varepsilon \to 0} \int_{Z_{J^*} \setminus X^\varepsilon} F_k(R_{j_1}, \dots, R_{j_l}, L_J, K, \dots, K) P_J(\Theta)\, \overline{\partial}\varphi \\
&+ \frac{(-1)^{l-p+1}}{(2\pi i)^{k-p}} \lim_{\varepsilon \to 0} \sum_{j \in J^*} \int_{Z_{J^*}} F_k(R_{j_1}, \dots, R_{j_l}, R_j, K, \dots, K) \Theta_j P_J(\Theta)\, \varphi \\
=& \frac{(-1)^{l-p+1}}{(2\pi i)^{k-p}} (-2\pi i) \sum_{s \notin J^*} \int_{Z_s \cap Z_{J^*}} F_k(R_{j_1}, \dots, R_{j_l}, R_s, K, \dots, K) P_J(\Theta)\, \varphi \\
&+ T[\overline{\partial}\varphi] \\
&+ \frac{(-1)^{l-p+1}}{(2\pi i)^{k-p}} \lim_{\varepsilon \to 0} \sum_{j \in J^*} \int_{Z_{J^*} \setminus X^\varepsilon} F_k(R_{j_1}, \dots, R_{j_l}, R_j, K, \dots, K) \Theta_j P_J(\Theta)\, \varphi.
\end{aligned}$$

where

$$T[\psi] := \frac{(-1)^{l-p+1}}{(2\pi i)^{k-p}} \lim_{\varepsilon \to 0} \int_{Z_{J^*} \setminus X^\varepsilon} F_k(R_{j_1}, \dots, R_{j_l}, L_J, K, \dots, K) P_J(\Theta)\, \psi$$

is again a principal value current. The justifications for the above steps are analogous to those we have made in the previous subsections.

The inductive step thus tells us that as currents,

$$
\begin{aligned}
I := {} & \frac{(-1)^{l-p}}{(2\pi i)^{k-p}} \lim_{\varepsilon\to 0} \int_{Z_{J^*}\setminus X^\varepsilon} F_k(R_{j_1},\dots,R_{j_l},K,\dots,K)P_J(\Theta)\,\varphi \\
\sim {} & \frac{(-1)^{(l+1)-(p+1)}}{(2\pi i)^{k-(p+1)}} \sum_{J'\in N^{l+1}} \int_{Z_{J'^*}} F_k(R_{j_1},\dots,R_{j_l},R_s,K,\dots,K)P_{J'}(\Theta) \\
+ {} & \frac{(-1)^{(l+1)-p}}{(2\pi i)^{k-(p+1)}} \sum_{J''\in N^{l+1}} \int_{Z_{J''^*}} F_k(R_{j_1},\dots,R_{j_l},R_j,K,\dots,K)P_{J''}(\Theta)
\end{aligned}
$$

where $\sim$ denotes the relation of being $\overline{\partial}$-cohomologous. This is how one passes from l-fold intersections of hypersurfaces to $(l+1)$-fold intersections. Note that in the above summations,

- $J' = (j_1,\dots,j_l,s) = (J,s)$, with $s \notin J^*$
- $J'' = (J,j)$, with $j \in J^*$.

Together, they account for all the possible ways to intersect a single hypersurface of the decomposition with Z_{J^*}. The differences in the coefficients of the integrals involving J' and J'' come from the fact that J' contains p distinct elements of N, while J'' contains $p+1$ distinct elements of N.

5.3.6 Final Steps

We have, at last,

$$
\begin{aligned}
\frac{1}{(2\pi i)^k} F_k(K) &\sim \frac{1}{(2\pi i)^k} \int_M F_k(K,\dots,K) \\
&\sim \sum_{j\in N} \frac{(-1)}{(2\pi i)^{k-1}} \int_{Z_j} F_k(R_j,K,\dots,K) \\
&\sim \sum_{J\in N^2} \frac{(-1)^{2-p}}{(2\pi i)^{k-p}} \int_{Z_{J^*}} F_k(R_{j_1},R_{j_2},K,\dots,K)P_J(\Theta) \\
&\;\;\vdots \\
&\sim \sum_{J\in N^k} \frac{(-1)^{k-p}}{(2\pi i)^{k-p}} \int_{Z_{J^*}} F_k(R_{j_1},\dots,R_{j_k})P_J(\Theta).
\end{aligned}
$$

Now,

$$\begin{aligned}\left(\frac{-1}{2\pi i}\right)^{k-p} P_J(\Theta) &= \left(\frac{-1}{2\pi i}\right)^{k-p} (\Theta_{j_1^*})^{a_1-1}\cdots(\Theta_{j_p^*})^{a_p-1} \\ &= \left(\frac{-1}{2\pi i}\Theta_{j_1^*}\right)^{a_1-1}\cdots\left(\frac{-1}{2\pi i}\Theta_{j_p^*}\right)^{a_p-1}.\end{aligned}$$

Furthermore, $c_k(R_{j_1},\ldots,R_{j_k})$ is constant on each component $Z_{J^*}^{(i)}$ of Z_{J^*} (5.2.1). Combining these three facts together, we get

$$\left(\frac{-1}{2\pi i}\right)^k F_k(K) \sim (-1)^k \sum_{J\in N^k}\left\{\sum_i F_k(\mathrm{Res}_J D)^{(i)}\cdot\left(\prod_{m=1}^{p}\left(\frac{-1}{2\pi i}\Theta_{j_m^*}\right)^{a_m-1}\right) Z_{J^*}^{(i)}\right\}.$$

This completes the proof of the theorem. □

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