Toric Varieties and Toric Vector Bundles

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Certificate

This is to certify that this dissertation entitled Toric Varieties and Toric Vector Bundles towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Komban Subhas Christopher at Indian Institute of Science Education and Research under the supervision of Dr. Vivek Mohan Mallick, Associate Professor, Department of Mathematics , during the academic year 2025-2026.

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Declaration

I hereby declare that the matter embodied in the report entitled Toric Varieties and Toric Vector Bundles are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Vivek Mohan Mallick and the same has not been submitted elsewhere for any other degree.

Christopher

15 May 2026
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Abstract

Toric varieties form a subfield of algebraic geometry That provides with a lot of examples to text theory in, since a lot of their geometric data can be expressed as combinatoric information that is easier to compute and study. In this thesis, we study the basics of toric varieties, look at some examples, and then see how this combinatoric data about toric varieties extend to vector bundles defined on them.

Contents

Chapter 1

Toric Varieties

Much of this chapter, much like the rest of this thesis, is based on material from [CLS11] and [Ful93]. The basics of algebraic geometry is assumed from [Har77].

1.1 Some Preliminaries

1.1.1 The Algebraic Torus

Here we introduce some basic facts of an algebraic torus T, which is considered to be an affine variety isomorphic to $(\mathbb{C}^*)^n$ for a natural number n, and inherits its group structure through this isomorphism. We introduce the notions of the lattices of characters and cocharacters as follows:

A *character* χ is a morphism (of varieties) from T to $\mathbb{C}^*$ which is also a group homomorphism. For the torus $T = (\mathbb{C}^*)^n$, if we have an element $m \in \mathbb{Z}^n$ with $m = (m_1, \ldots, m_n)$, we have the following character:

$$\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$$

$$(t_1, \ldots, t_n) \mapsto t_1^{m_1} \ldots t_n^{m_n}$$

We can show that all characters of T arise similarly from elements of $M = \mathbb{Z}^n$, and hence M is called the *character lattice* of T.

Remark 1.1.1.1. By the word *lattice*, we mean a free abelian group of finite rank, which means it is isomorphic to $\mathbb{Z}^n$, where n is the rank of the lattice.

Dual to this concept is the idea of cocharacter. A *cocharacter* (also called a *one parameter subgroup*) λ is a morphism from $\mathbb{C}^*$ to t which is also a group homomorphism. Like characters, we have a similar lattice $N \cong \mathbb{Z}^n$, called the *cocharacter lattice*. For the torus $T = (\mathbb{C}^*)^n$, the element $u \in N$ with $u = (u_1, \ldots, u_n)$ corresponds to the following cocharacter:

$$\lambda^u : \mathbb{C}^* \to (\mathbb{C}^*)^n$$

$$t \mapsto (t^{u_1}, \ldots, t^{u_n})$$

We can also show that every cocharacter comes from an element of N, like we had for characters. For an arbitrary torus T, its character lattice M and cocharacter lattice N are free abelian group whose ranks are equal to the dimension of the torus.

There also exists a natural bilinear pairing between N and M. Given that $T = (\mathbb{C}^*)^n$, one way to describe the bilinear product for $m \in M$ and $u \in N$, is to observe the following composition:

$$\chi^m \circ \lambda^u : \mathbb{C}^* \to \mathbb{C}^*$$

$$t \mapsto t^k$$

for some $k \in \mathbb{Z}$. We then write

$$\langle m, u \rangle = k$$

.

Another more explicit way to describe this bilinear product, for $m = (m_1 \ldots, m_n) \in M$ and $u = (u_1, \ldots, u_n) \in N$ (since $M \cong \mathbb{Z}^n$ and $N \cong \mathbb{Z}^n$), is as follows:

$$\langle m, u \rangle := \sum_{i=1}^{n} m_i u_i$$

We can see that this is equivalent to the first description.

We note that due to this bilinear pairing, $N \cong \hom_{\mathbb{Z}}(M, \mathbb{Z})$ and $M \cong \hom_{\mathbb{Z}}(N, \mathbb{Z})$. We also have the canonical isomorphism from $N \otimes_{\mathbb{Z}} \mathbb{C}^* \to T$ given by $u \otimes t \mapsto \lambda^u(t)$. Due to this, given a lattice of finite rank N, we write the associated torus as T_N.

We also have this important theorem about tori in other tori:

Proposition 1.1.1.2.

1. Given two tori T_1 and T_2 and a morphism of varieties $\phi : T_1 \to T_2$ which is also a group homomorphism, $\phi(T_1)$ is a torus in T_2.

2. If T_1 is a torus containing an closed irreducible subvariety H, H is also a torus.

Finally, we end with a discussion of what happens when a torus T acts linearly on a vector space W of finite dimension. Given such an action, for every $t \in T$, we have a linear map $X(t) : W \to W$, where $w \mapsto t \cdot w$. We find that these $X(t)$ diagonalise simultaneously at all $t \in T$. That is, if we consider the following subspace for each $m \in M$, where M is the character lattice of T:

$$W_m(t) = \{w \in W | t \cdot w = \chi^m(t)w\}$$

We see that $W_m(t) = W_m(s) = W_m$ for all $t, s \in T$. We also see that if $W_m \neq \{0\}$, $w \in W_m$ is a simultaneous eigenvector for all $X(t)$. Hence, we have the following decomposition:

Proposition 1.1.1.3. For a finite dimensional vector space W with a torus T acting linearly on it, we have that :

$$W = \bigoplus_{m \in M} W_m$$

where W_m and M are the same as what has been described in the paragraph above.

1.1.2 Convex Geometry

In this section we will discuss about convex polyhedral cones and polytopes, after defining the notion of affine semigroups. We will be working with two lattices M and N, which are dual to each other and are embedded in dual real vector spaces $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

Affine Semigroups

A semigroup S is a set with an associative binary operation, along with an identity for that operation. We define an affine semigroup as follows:

Definition 1.1.2.1. *A semigroup S with binary operation $+$ is called an affine semigroup if it satisfies the following conditions:*

- *The binary operation $+$ is commutative.*
- *S is generated by finite subset $\mathscr{A}$, i.e., $S = \mathbb{Z}_{\geq 0}\mathscr{A}$, and hence can be embedded into a lattice M.*

An affine semigrpup $S \subset M$ is called saturated if it for all $k \in \mathbb{N}$ and $m \in M$, $km \in S \Rightarrow m \in S$.

Convex Polyhedral Cones

Definition 1.1.2.2. *For a finite subset $A \subset N_{\mathbb{R}}$, we define the convex polyhedral cone generated by A as follows:*

$$\sigma = Cone(A) := \{\sum_{a_i \in A} \lambda_i a_i | \lambda_i \geq 0 \forall i\}$$

From now on, whenever we say the word 'cone', we mean a convex polyhedral cone. The dimension of a cone σ is the dimension of the vector space $\mathbb{R}\sigma$. We now define the dual of a cone $\sigma \subset N_{\mathbb{R}}$ as follows:

$$\sigma^{\vee} := \{m \in M_{\mathbb{R}} | \langle m, u \rangle \geq 0 \forall u \in \sigma\}$$

We define the dual for every convex set similarly. We see that for every point $m \in \sigma^{\vee}$, we have a hyperplane $H_m = \{u \in N_{\mathbb{R}} | \langle m, u \rangle = 0\} \subset N_{\mathbb{R}}$, and a corresponding supporting half space $H_m^+ = \{u \in N_{\mathbb{R}} | \langle m, u \rangle \geq 0\}$. We define *faces* of the cone σ as the intersection $\tau := H_m \cap \sigma$ for some $m \in \sigma^{\vee}$. We denote τ as a face of σ by:

$$\tau \preceq \sigma$$

Note that σ is a face of itself. A face τ is called a proper face of σ if $\tau \neq \sigma$. We note the following facts about faces, which follow after assuming that $(\sigma^{\vee})^{\vee} = \sigma$:

Proposition 1.1.2.3. Let $\sigma = \text{Cone}(\{u_1, \dots, u_s\}) \subset N_{\mathbb{R}}$. Then:

1. A face $\tau = H_m \cap \sigma$ for $m \in \sigma^{\vee}$ of σ is also a cone, generated by the subset of generators of σ contained in τ.

2. Intersection of faces τ and τ' of σ is also a face of σ, as well as a face of τ and τ'.

3. Facets of σ are faces of codimension 1, and all proper faces of σ are contained in a facet.

4. A proper face of σ is the intersection of all the facets containing it.

5. If $\dim(\sigma) = \dim(N_{\mathbb{R}})$, then the topological boundary of σ is formed by the union of its facets.

The relative interior of a cone σ, denoted by relint(σ), and refers to the topological interior of σ in the vector space $\mathbb{R}\sigma$. We state the following facts about the dual of a cone σ, which utilises the notion of the relative interior in their proofs:

Proposition 1.1.2.4. Let σ be a cone in $N_{\mathbb{R}}$. Then the following are true:

1. The dual of the cone σ is also a convex polyhedral cone.

2. Let τ be a face of the cone σ. Let $\tau^{\perp} = \{m \in M_{\mathbb{R}} | \langle m, u\rangle = 0 \forall u \in \tau\}$. Then, $\tau^* = \tau^{\perp} \cap \sigma^{\vee}$ is a face of $\sigma^{\vee}$, and more over, this relation forms a order reversing bijection between faces of σ and $\sigma^{\vee}$.

3. $\dim(\tau) + \dim(\tau^*) = \dim(N_{\mathbb{R}}) = \dim(M_{\mathbb{R}})$

4. If $\dim(\sigma) = \dim(N_{\mathbb{R}})$, then σ is the intersection of the supporting halfspaces corresponding to facets of σ. i.e.

$$\sigma = \{u \in N_{\mathbb{R}} | \langle m_{\tau}, u\rangle \geq 0 \text{ for } \tau \text{ corresponding to facets and } \tau = \sigma \cap H^{+}_{m_{\tau}}\}$$

We call a cone in $N_{\mathbb{R}}$ a rational polyhedral cone, when it is generated by points in the lattice N. Note that the dual of a rational polyhedral cone σ in $N_{\mathbb{R}}$ corresponding to the lattice N is also a rational polyhedral cone in $M_{\mathbb{R}}$. For such a σ, we see that $\sigma^{\vee} \cap M$ is a semigroup with a commutative group operation. The following proposition, called Gordan's Lemma, shows that it forms an affine semigroup.

Proposition 1.1.2.5 (Gordan's Lemma)**.** For a rational polyhedral cone $\sigma \in N_{\mathbb{R}}$, $S_{\sigma} = \sigma^{\vee} \cap M$ is a finitely generated semigroup, and hence is an affine semigroup.

Sketch. We do this by taking the union of the set of $\mathbb{R}$-generators of $\sigma^{\vee}$ and the set $S_{\sigma} \cap [0, 1]^n$, which is finite as it is the intersection of a discrete set and a compact set. This union gives us a set of $\mathbb{Z}$-generators of S_{σ}. $\square$

Now we will state a result where we relate the affine semigroups of a rational polyhedral cone σ and a face τ of σ (which is also a rational cone):

Proposition 1.1.2.6. Let $\tau = \sigma \cap H_m$ be a face of a cone σ, for $m \in \sigma^{\vee}$. Then:

1. $\tau^{\vee} = \sigma^{\vee} + \mathbb{R}\{-m\}$

2. If σ is a rational polyhedral cone, then there exists $m \in S_\sigma$ such that $\tau = \sigma \cap H_m$. And for this m, $S_\tau = S_\sigma + \mathbb{Z}\{m\}$.

Most of the cones we will deal with will be strongly convex, which we define as follows:

Definition 1.1.2.7. *A strongly convex polyhedral cone $\sigma \subset N_\mathbb{R}$ is a cone that satisfies any of the following equivalent criteria:*

1. *σ contains no non zero subspace of $N_\mathbb{R}$.*

2. *$\{0\}$ is a face of σ, i.e. there exists $m \in \sigma^\vee$ such that $\{0\} = \sigma \cap H_m$.*

3. *$\sigma \cap (-\sigma) = \{0\}$*

4. *$\sigma^\vee$ spans $M_\mathbb{R}$.*

We discuss one more fact, called the separation lemma, which deals with two cones σ and σ', whose intersection is another cone τ, which is a face of each. The following result holds:

Proposition 1.1.2.8. Let σ, σ' and τ be as described above. Then:

1. There exists $m \in \sigma^\vee \cap (-\sigma')^\vee$, such that $\tau = \sigma \cap H_m = \sigma' \cap H_m$

2. If σ and σ' were rational polyhedral cones, the above result implies that for the associated semigroups, $S_\tau = S_\sigma + S_{\sigma'}$.

Something special about strongly convex rational polyhedral cones is the following proposition that relates them to saturated affine semigroups.

Proposition 1.1.2.9. For a strongly convex rational polyhedral cone $\sigma \in N_\mathbb{R}$, S_σ is a saturated affine semigroup. And the converse is also true, i.e., every saturated affine semigroup $S \subset M$ is of the form S_σ for some strongly convex rational polyhedral cone $\sigma \subset N_\mathbb{R}$.

Polytopes

In this section, we will again consider a finite set A, but consider the polytope generated by A, i.e., its convex hull.

Definition 1.1.2.10. *Let $A = \{m_1, \dots, m_s\} \subset M_{\mathbb{R}}$ be a finite subset. The polytope generated by A is defined as follows:*

$$P = Conv(A) = \{\sum_{i=1}^{s} \lambda_i m_i | \sum_{i=1}^{s} \lambda_i = 1, \lambda_i \geq 0 \forall i\}$$

The dimension of a polytope is the dimension of the vector space $\mathbb{R}P$. For $u \in N_{\mathbb{R}}$, and $b \in \mathbb{R}$, we consider the affine hyperplane $H_{u,b} = \{m \in M_{\mathbb{R}} | \langle m, u \rangle = b\}$, and the corresponding halfspace $H^+_{u,b} = \{m \in M_{\mathbb{R}} | \langle m, u \rangle \geq b\}$. If there exists $u \in N_{\mathbb{R}}, b \in \mathbb{R}$, such that $Q = P \cap H_{u,b}$ and $P \subset H^+_{u,b}$, then Q is called a face of P, and we denote this by $Q \preceq P$. P is a face of itself, and every face of P is a polytope of the form $Q = \text{Conv}(A \cap H_{u,b})$. The notions of a proper face and a facet are similar to those discussed for cones. We also have edges and vertices, which are faces of dimensions 1 and 0 respectively. We now state a few facts related to polytopes:

Proposition 1.1.2.11. Let $P = \text{Conv}(A)$ be a polytope in $M_{\mathbb{R}}$. Then:

1. P is the convex hull of its vertices.
2. The vertices of P are contained in A.
3. If $Q \preceq P$ is a proper face, it is the intersection of the facets containing it.

There is a way to write P as an intersection of finitely many supporting halfspaces corresponding to facets. When P is full dimensional, i.e., $\dim(P) = \dim(M_{\mathbb{R}})$, this representation becomes unique, because every facet corresponds to a unique u $N_{\mathbb{R}}$ up to a multiple of u. This representation is given as follows:

$$P = \bigcap_{F \text{ is a facet of P}} H^+_{u_F, -a_F} = \{m \in M_{\mathbb{R}} | \langle m, u_F \rangle \geq a_F \forall F \text{ is a facet of } P\}$$

Where the facets of P are of the form $F = P \cap H_{u_F, -a_F}$, unique up to a positive multiple of $(u_F, -a_F)$.

Notice that any polytope hence becomes an intersection of finitely many halfspaces. However, every set that is an intersection of finitely many half spaces is a polytope only if it is bounded. In general, such sets are called *polyhedra*.

If a polytope $P \subset M_{\mathbb{R}}$ has dimension d, then:

- It is called a *simplex* if it has $d + 1$ vertices.

- It is called *simplicial* if all of its facets are simplices, i.e they have $d-1$ vertices.
- It is called *simple* if all of its vertices are the intersection of exactly d facets.

We also note that multiples $rP = \text{Conv}(rA)$ of polytopes as well as Minkowski sums of two polytopes $P_1 + P_2 = \{a \in M_{\mathbb{R}} | a = m_1 + m_2 \text{ for some } m_1 \in P_1, m_2 \in P_2\}$ are polytopes as well.

Polytopes of special interest to us are lattice polytopes. These are polytopes P whose vertices lie inside the lattice M. Their faces are also special because $Q \preceq P$ can be written in the form $Q = P \cap H^+_{u_Q, -a_Q}$, such that $u_Q \in N$ and $-a_Q \in \mathbb{Z}$.

These lattice polytopes are of further interest based on if they contain 'enough' points of the lattice M. Thus we make the following definitions:

Definition 1.1.2.12. *A lattice polytope $P \subset M_{\mathbb{R}}$ is said to be:*

1. ***Normal*** *if for any two positive integers k, l, $(kP \cap M) + (lP \cap M) = (k+l)P \cap M$.*
2. ***Very ample*** *if for every vertex m of P, the affine semigroup $S_{P,m} = \mathbb{Z}_{\geq 0}(P \cap M - \{m\})$ is saturated.*

Now, the following facts hold:

Proposition 1.1.2.13.

1. Every normal polytope is very ample.
2. Let P be a full dimensional lattice polytope in $M_{\mathbb{R}}$ of dimension $d \geq 2$. Then kP is normal, and hence, very ample for $k \geq d-1$. In particular, lattice polygons are normal and very ample in $M_{\mathbb{R}}$ of dimension 2.

1.2 Affine Toric Varieties

In this section, we will look at the definition and different equivalent ways to describe affine toric varieties.

1.2.1 Definition and some Examples

Definition 1.2.1.1 (Affine Toric Variety). *An irreducible affine variety U is called an affine toric variety, if it contains a Zariski open subset T isomorphic to a torus $T_N \cong (\mathbb{C}^*)^n$, such that the action of T on itself extend to an algebraic action of T on U.*

We now describe a few examples:

Example 1.2.1.2. Consider the irreducible variety $V = Z(\langle x^2 - y^3 \rangle) \subset \mathbb{C}^{\nvDash}$. We see that the torus $\mathbb{C}^* \cong U = V \backslash \{0\}$, via the isomorphism $t \mapsto (t^3, t^2)$, is an open subset of V, and the torus action extends to all of V. Note that the coordinate ring $\mathbb{C}[x, y]/\langle x^2 - y^3 \rangle$ is not integrally closed. ♢♦♢

Example 1.2.1.3. We can also consider the irreducible variety $V = Z(\langle xy - zw \rangle) \subset \mathbb{C}^4$. Here, we have the torus $(\mathbb{C}^*)^3 \cong U = V \backslash Z(\langle xyzw \rangle)$ as an open set in V, via the isomorphism $(x, y, z) \mapsto (x, y, z, \frac{xy}{z})$. The torus action extends to the whole of V by component wise multiplication. ♢♦♢

1.2.2 Lattice points of a Torus, and Toric Ideals

Given a torus T_N with character lattice M, we can construct an affine toric variety by taking a finite subset $\mathscr{A} = \{m_1, \dots, m_s\} \subset M$ and constructing the following map:

$$\Phi_{\mathscr{A}} : T_N \to \mathbb{C}^s$$

$$t \mapsto (\chi^{m_1}(t), \dots, \chi^{m_s}(t))$$

We then consider the Zariski closure $Y_{\mathscr{A}}$of $\Phi_{\mathscr{A}}(T_N)$ in $\mathbb{C}^s$ and by the following theorem, show that $Y_{\mathscr{A}}$ is an affine toric variety:

Proposition 1.2.2.1. $Y_{\mathscr{A}}$ as described earlier, is an affine toric variety, with torus $\Phi_{\mathscr{A}}(T_N)$, whose character lattice is $\mathbb{Z}\mathscr{A}$.

Sketch. We first see that $T = \Phi_{\mathscr{A}}(T_N)$ is a closed torus in $(\mathbb{C}^*)^s$ by 1.1.1.2. And since T is irreducible, its Zariski closure $Y_{\mathscr{A}}$ in $\mathbb{C}^s$ is also irreducible. To show that the action of T on itself extends to an algebraic action on $Y_{\mathscr{A}}$ we use the fact that $Y_{\mathscr{A}}$ is the Zariski closure and hence the smallest subvariety of $\mathbb{C}^s$ containing T as well as the following fact:

$$T = t \cdot T \subset t \cdot Y_{\mathscr{A}}$$

Finally, to show that the character lattice M' of T is $\mathbb{Z}\mathscr{A}$, we consider the following commutative diagram:

$$\begin{tikzcd}
T_n \arrow[r, "\Phi_{\mathscr{A}}"] \arrow[dr, two heads] & (\mathbb{C}^*)^s \\
& T \arrow[u, hook]
\end{tikzcd}$$

Where the map $\Phi_{\mathscr{A}}$ has image T in $(\mathbb{C}^*)^s$. We have a corresponding map $\hat{\Phi}_{\mathscr{A}}$ from character lattice $\mathbb{Z}^{\sim}$ of $(\mathbb{C}^*)^s$ to the character lattice M of T_N that factors through M' as follows:

$$\begin{tikzcd} M & \mathbb{Z}^s \arrow[l, "\hat{\Phi}_{\mathscr{A}}"'] \arrow[d, two heads] \\ & M' \arrow[ul, hook] \end{tikzcd}$$

This commutative diagram implies that M' is isomorphic to the image of the dual map $\hat{\Phi}_{\mathscr{A}}$. This dual map $\hat{\Phi}_{\mathscr{A}}$ between the character lattices is the map corresponding to the one that takes the character $\xi^n : (\mathbb{C}^*)^s \to \mathbb{C}^*$ to the character $\chi^m = \xi^n \circ \Phi_{\mathscr{A}} : T_N \to \mathbb{C}^*$. Because of the way we defined $\Phi_{\mathscr{A}}$, the map $\hat{\Phi}_{\mathscr{A}}$ is the one that takes the basis elements e_i (for $i = 1, \ldots, s$) of $\mathbb{Z}^s$ to elements m_i of M, which are precisely the elements of $\mathscr{A}$. Hence, $\hat{\Phi}_{\mathscr{A}}(\mathbb{Z}^s) = \mathbb{Z}\mathscr{A}$.

Therefore, we see that $M' \cong \mathbb{Z}\mathscr{A}$, and we are hence done. □

As we have seen in the proof of this theorem, once we have the map $\hat{\Phi}$ from $\mathbb{Z}^{\sim}$ to M. If we were to consider the kernel of this map, we get a sublattice $L \subset \mathbb{Z}^s$. We see that for $l = (l_1, \ldots, l_s) \in L$, $\sum_{i=1}^{s} l_i m_i = 0$. For each element $l \in L$, let $l_+ := \sum_{l_i > 0} l_i e_i$, and $l_- := \sum_{l_i < 0} -l_i e_i$, where $e_i, i = 1, \ldots, s$ are the basis elements of $\mathbb{Z}^s$ such that $l = \sum_{i=1}^{s} l_i e_i$. Now if we were to consider all polynomials of the form $x^{l+} - x^{l-}$ (Here, by x^l, where $l = (l_1, \ldots, l_s) = \sum_{i=1}^{s} l_i e_i \in \mathbb{Z}^s_{\geq 0}$, we mean the monomial $x_1^{l_1} \cdots x_s^{l_s}$ in $\mathbb{C}[x_1, \ldots x_s]$) we can see that all elements of $\Phi_{\mathscr{A}}(T_n)$ are roots of all such polynomials, since $\frac{x^{l_+}}{x^{l_-}} = 1$ are satisfied by all elements of the image of $\Phi_{\mathscr{A}}$. We can hence define the ideal generated by the sublattice L as follows:

$$I_L := \langle x^{l_+} - x^{l_-} | l \in L \rangle = \langle x^\alpha - x^\beta | \alpha, \beta \in \mathbb{Z}^s_{\geq 0}, \alpha - \beta \in L \rangle$$

We can show rather easily that the two ideals are in fact the same, and hence, we prove the following claim about the ideal of $Y_{\mathscr{A}}$:

Proposition 1.2.2.2. The ideal of $Y_{\mathscr{A}}$ is given by:

$$I(Y_{\mathscr{A}}) = I_L$$

Sketch. We can easily see that $I_L \subset I(Y_{\mathscr{A}})$. To show the converse, we apply a monomial ordering on $\mathbb{C}[x_1, \ldots, x_s]$, and find a contradiction when $I(Y_{\mathscr{A}}) \backslash I_L$ is non empty. We do this by considering a polynomial f with a minimal leading term, and then show that we can construct a polynomial whose leading term is less than the minimal leading term of f with respect to the monomial ordering, as per [CLS11]. $\square$

We now generalize to obtain all ideals that give us an affine toric variety. We begin with the following definitions:

Definition 1.2.2.3. *Consider a sublattice $L \subset \mathbb{Z}^s$, and consider the coordinate ring $\mathbb{C}[x_1, \dots, x_s]$*

1. *An ideal is called a lattice ideal if it is of the form $I_L = \langle x^\alpha - x^\beta | \alpha, \beta \in \mathbb{Z}^s_{\geq 0}, \alpha - \beta \in L\rangle$ for some sublattice $L \subset \mathbb{Z}^s$.*

2. *An ideal is called a toric ideal if it is a lattice ideal that is prime.*

We can see that toric ideals always give rise to affine toric varieties, because of the discussion we had earlier. We now discuss a characterization of toric ideals.

Proposition 1.2.2.4. An ideal $I \in \mathbb{C}[x_1, \dots, x_s]$ is toric if and only if it is prime and generated by binomials of the form $x^\alpha - x^\beta, \alpha, \beta \in \mathbb{Z}^s_{\geq 0}$.

Sketch. We already see that a toric ideal is prime and generated by binomials of the above form, based on our previous discussion. To show the opposite implication, we begin with showing that for a prime ideal I of the above form, the associated variety V contains the point $(1, \dots, 1)$, and hence $(\mathbb{C}^*)^s \cap V$ is non empty, and also a subgroup of $(\mathbb{C}^*)^s$. Then we use the fact that V is irreducible, to show that $(\mathbb{C}^*)^s \cap V$ is a closed subvariety of $(\mathbb{C}^*)^s$, and hence, by 1.1.1.2, $T = (\mathbb{C}^*)^s \cap V$ is a torus, and an open subset of V. Finally, to show that $I = I(Y_{\mathscr{A}})$ for an a finite subset $\mathscr{A} \subset M$ for the character lattice M of T, we use the inclusion map of $T \hookrightarrow (\mathbb{C}^*)^s$ to obtain the inverse image of the characters χ^{e_i} corresponding to the coordinate wise projection map on $(\mathbb{C}^*)^s$. This gives us the finite set $\mathscr{A} \subset M$, after which showing that $V \cong Y_{\mathscr{A}}$ is easy, and this implies that $I = I(Y_{\mathscr{A}})$ by the Nullstellensatz. Hence, I is a toric ideal. $\square$

Example 1.2.2.5. We can show that the ideal $I \subset \mathbb{C}[x_1, \dots, x_d]$ of the form $\langle x_i x_{j+1} - x_{i+1} x_j | 1 \leq i < j \leq d$ of the form is toric since the variety $Z(I)$ is irreducible, by applying 1.2.2.1 to the map $\Phi : \mathbb{C}^2 \to \mathbb{C}^d$ restricted to $(\mathbb{C}^*)^2$, where $(x, y) \mapsto (x^d, x^{d-1}y, \dots, xy^{d-1}, y^d)$. We see that the ideal I is the ideal of the image of Φ. Hence I is a toric ideal, and $Z(I)$ is an affine toric variety. ◊♦◊

1.2.3 Semigroup algebras and Equivalence of Definitions

Now, we describe a fourth way to define an affine toric variety, that is, as the maximal spectrum of a algebra generated by an affine semigroup, which we define as follows:

Definition 1.2.3.1. *For an affine semigroup $S \subset M$, where M is a lattice, we define the algebra generated by the semigroup as the algebra generated by the set $\{\chi^m | m \in S\}$, with the multiplication between two elements of the set obtained by extending $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$. In other words, it is the following algebra:*

$$\mathbb{C}[S] := \{\sum_{m_i \in S} c_i \chi^{m_i} | c_i = 0 \textit{ for all but finitely many } i\}$$

with multiplication as defined earlier.

Note that this means that $\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_s}]$, if $S = \mathbb{Z}_{\geq 0}\mathscr{A}$, and $\mathscr{A} = \{m_1, \dots, m_s\}$.

Example 1.2.3.2. Let $S = \mathbb{Z}^s_{\geq 0}$. Then, one can see that $\mathbb{C}[S] = \mathbb{C}[\chi^{e_1}, \dots, \chi^{e_s}]$. This ring is isomorphic to $\mathbb{C}[x_1, \dots, x_s]$, and hence, $\mathrm{Spec}(\mathbb{C}[S]) \cong \mathbb{C}^s$. ◊♦◊

Example 1.2.3.3. For $S = M = \mathbb{Z}^s$, we see that $S = \mathbb{Z}_{\geq 0}\{\pm e_1, \dots, \pm e_s\}$. Hence, the semigroup algebra $\mathbb{C}[M] = \mathbb{C}[\chi^{\pm e_1}, \dots, \chi^{\pm e_s}]$ is isomorphic to the Laurent polynomial ring $\mathbb{C}[x_1^{\pm 1}, \dots, x_s^{\pm 1}]$, and hence we have that $\mathrm{Spec}(\mathbb{C}[M])$ is isomorphic to the torus $T_N = (\mathbb{C}^*)^s$. ◊♦◊

Now we can define an affine variety as the maximal spectrum of $\mathbb{C}[S]$. We will show that this variety is also an affine toric variety, with a torus having character lattice M.

Proposition 1.2.3.4. For an affine semigroup $S = \mathbb{Z}_{\geq 0}\mathscr{A}$, for $\mathscr{A} = \{m_1, \dots, m_S\}$, inside a lattice M;

1. $\mathbb{C}[S]$ is an integral domain and finitely generated.
2. $\mathrm{Spec}(\mathbb{C}[S])$ is an affine toric variety, whose torus has character lattice $\mathbb{Z}\mathscr{A}$, and is isomorphic to the affine toric variety $Y_{\mathscr{A}}$.

Sketch. To prove 1., we already see that $\mathbb{C}[S]$ is finitely generated by the discussion above, and since $\mathbb{C}[S] \subset \mathbb{C}[M]$, we see $\mathbb{C}[S]$ is an integral domain by 1.2.3.3. For 2., we construct the map $\hat{\Phi}_{\mathscr{A}} : \mathbb{C}[x_1, \dots, x_s] \to \mathbb{C}[M]$, where $x_i \mapsto \chi^{m_i}$. We show that the image of this map is $\mathbb{C}[S]$, and this map corresponds to the map $\Phi_{\mathscr{A}} : T_N \to \mathbb{C}^s$, as described in 1.2.2.1. We use this to show that the affine toric variety $Y_{\mathscr{A}} \cong \mathrm{Spec}(\mathbb{C}[S])$, and we are hence done. $\square$

We now discuss the action of a torus T_N on its coordinate ring $\mathbb{C}[M]$, so that we can see what happens when a subspace of $\mathbb{C}[M]$ is invariant under the action of the torus and we then use this to explain how any affine toric variety can be expressed as the maximal spectrum of the algebra generated by an affine semigroup. We define the action of T_N on $\mathbb{C}[M]$ as follows:

$$\cdot : T_N \times \mathbb{C}[M] \to \mathbb{C}[M]$$

$$t \cdot f(p) \mapsto f(t^{-1}p)$$

Now we note the following result about subspaces of $\mathbb{C}[M]$ invariant under this torus action.

Proposition 1.2.3.5. If A is a subspace of $\mathbb{C}[M]$ invariant under the action of torus described above, then:

$$A = \bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m$$

Sketch. We see that the inclusion $\bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m \subset A$ is obvious. To show the inclusion the other way, we use the fact that A and any subset B of $\mathbb{C}[M]$ spanned by finite characters is invariant under the action of the torus, and hence, using 1.1.1.3, we decompose them as a direct sum of eigenspaces. But for B, these eigenspaces are those of spanned by individual characters, and we use this to show that each element of A is spanned by characters already in A. $\square$

We now use this to prove the equivalence of all the definitions given earlier.

Theorem 1.2.3.6. Given an affine irreducible variety U, the following are equivalent:

1. U is an affine toric variety with torus T_N, whose character lattice is M.
2. U is isomorphic to the affine toric variety $Y_{\mathscr{A}}$ for a finite set $\mathscr{A}$ in the character lattice of a torus T.
3. U is obtained from a toric ideal.

4. U is isomorphic to the maximal spectrum of the algebra $\mathbb{C}[S]$ of an affine semigroup $S \subset M$

Sketch. The equivalence between 2., 3., and 4. is obvious from 1.2.3.4, 1.2.2.4, and 1.2.2.2. And we already know that 2. implies 1. using 1.2.2.1. Now, if we show that 1. implies 4., we are done. To show this, we use the inclusion of the torus T_N as a Zariski open subset of the affine toric variety U to describe the coordinate ring $\mathbb{C}[U]$ as a subset of the coordinate ring $\mathbb{C}[M]$. Then, since the torus action extends to all of U, $\mathbb{C}[U]$ is a torus invariant subspace of $\mathbb{C}[M]$, and hence, using 1.2.3.5, we can write it as a semigroup algebra $\mathbb{C}[S]$, for the semigroup $S \subset M$, which is finitely generated since $\mathbb{C}[U]$ is finitely generated. Hence, we are done. □

1.2.4 Relating Cones and Affine Toric Varieties, and some other Properties

We discussed the affine semigroup that arises from the dual of a rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$, namely $S_\sigma = \sigma^\vee \cap M$. Hence we have the following fact:

Proposition 1.2.4.1. Let σ be a rational polyhedral cone in $N_{\mathbb{R}}$. Then, $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ is an affine toric variety, whose torus has character lattice $\mathbb{Z}S_\sigma$. Moreover, the torus of U_σ is T_N if and only if σ is strongly convex.

Sketch. The only part left to prove is that $\sigma^\vee$ being full dimensional corresponds to the torus T_N being the torus of U_σ. This is true because the character lattice $\mathbb{Z}S_\sigma$ is isomorphic to M if and only if $\sigma^\vee$ is full dimensional, and hence, corresponds to the torus T_N. □

Furthermore, we can relate the semigroup algebras corresponding to a rational cone σ and its face $\tau = H_m \cap \sigma$ (where $m \in S_\sigma$) as follows:

Proposition 1.2.4.2. For the above situation, the semigroup algebra $\mathbb{C}[S_\tau]$ is the localization $\mathbb{C}[S_\sigma]_{\chi^m}$.

Sketch. This is proved by applying the result mentioned in 1.1.2.6 to the semigroup algebra generated by the cones τ and σ. □

We now study points in affine toric varieties in terms of the semigroups they come from. We get the following important result:

Proposition 1.2.4.3. Given an affine toric variety $U = \mathrm{Spec}(\mathbb{C}[S])$ for an affine semigroup S, we see that points in U correspond to semigroup morphisms $\gamma : S \to \mathbb{C}$, where $\mathbb{C}$ is a semigroup under multiplication.

Sketch. To prove this, given a point $p \in U$, we get a semigroup morphism $\gamma : S \to \mathbb{C}$ by taking $m \mapsto \chi^m(p)$ for those m that generate S. And given a semigroup morphism $\gamma : S \to \mathbb{C}$, we get a surjective map $\hat{\gamma} : \mathbb{C}[S] \to \mathbb{C}$, whose kernel corresponds to a maximal ideal of $\mathbb{C}[S]$, and hence a point $p \in U$. □

Once we have this, we describe the torus action on an affine toric variety in terms of these semigroup morphisms:

Proposition 1.2.4.4. For an affine toric variety $U = \mathrm{Spec}(\mathbb{C}[S])$, with a point $\gamma : S \to \mathbb{C}$, the action of of the torus $T_N \subset U$ is described by:

$$t \cdot \gamma : S \to \mathbb{C}$$

$$m \mapsto \chi^m(t)\gamma(m)$$

Sketch. We get this action by observing that the action of T_N on U corresponds to a $\mathbb{C}$-algebra homomorphism from $\mathbb{C}[U] \to \mathbb{C}[T_N \times V] \cong \mathbb{C}[M] \otimes \mathbb{C}[S]$, which corresponds to the map $\chi^m \mapsto \chi^m \otimes \chi^m$. From this, we get the aforementioned action on semigroup morphisms. □

We then describe a distinguished point for $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ for the semigroup $S_\sigma = \sigma^\vee \cap M$ obtained from the rational polyhedral cone $\sigma \subset N_\mathbb{R}$ which corresponds to the following map:

$$x_\sigma : S \to \mathbb{C}$$

$$x_\sigma(m) = \begin{cases} 1 \text{ if } m \in \sigma^\perp \\ 0 \text{ otherwise} \end{cases}$$

A related result about the affine toric variety corresponding to a pointed semigroup (an affine semigroup S where $S \cap (-S) = \{0\}$), will prove useful:

Proposition 1.2.4.5. Let S be an affine semigroup and let $U = \mathrm{Spec}(\mathbb{C}[S])$ be the associated affine toric variety. For such a U, the point $\gamma : S \to \mathbb{C}$:

$$\gamma(m) = \begin{cases} 0 \text{ if } m \neq 0 \\ 1 \text{ if } m = 0 \end{cases}$$

is the only fixed point of U if and only if S is a pointed semigroup.

Sketch. The proof of this hinges on the fact that the given map is a semigroup morphism if and only if S is a pointed semigroup. Then, we notice that for a map on the semigroup S to be fixed by the torus action, we always have some $m \neq 0 \in S$ such that $\chi^m(t) \neq 1$ for some $t \in T_N$, and so $\gamma(m) = 0$ for $m \neq 0$. Hence, the map $\gamma : S \to \mathbb{C}$ takes the aforementioned form if it is fixed by the torus action. □

As a result of this proposition, the distinguished point of an affine toric variety U_σ is its only fixed point if and only if the cone σ has the same dimension as $N_{\mathbb{R}}$.

We finally describe how normal affine toric varieties (i.e., its coordinate ring is integrally closed) correspond to saturated affine semigroups, and therefore to strongly convex rational polyhedral cones $\sigma \in N_{\mathbb{R}}$.

Proposition 1.2.4.6. An affine toric variety $U = \mathrm{Spec}(\mathbb{C}[S])$ is normal if and only if the affine semigroup S is saturated.

Sketch. The proof that S is saturated if U is normal is given by the fact that we can write χ^m as the solution of a monic polynomial in $\mathbb{C}[S]$ if $km \in S$ for some natural number k. The proof of the converse is done by using the associated strongly convex rational polyhedral cone σ, and the rays that generate this cone, as explained in [CLS11]. □

Example 1.2.4.7. We see that if we refer to 1.2.1.2, that affine toric variety corresponds to the affine semigroup $S \subset \mathbb{Z}$ generated by the points 2 and 3. Note that this means that S is not a saturated affine semigroup, and we can see that $Z(\langle x^2 - y^3 \rangle)$ is not normal. ◊♦◊

1.3 Projective Toric Varieties

1.3.1 Definitions, Facts about the Affine Cone, Character Lattice, and Affine Pieces

We first start with our definition of what $\mathbb{P}^n$ is:

$$\mathbb{P}^n := (\mathbb{C}^{n+1} \backslash \{0\})/\mathbb{C}^*$$

Here, we identify the lines of $\mathbb{C}^{n+1}$ that pass through the origin as points in $\mathbb{P}^n$. We observe that $\mathbb{P}^n$ has the following torus inside it:

$$T_{\mathbb{P}^n} := \mathbb{P}^n \backslash V(x_0 x_1 \cdots x_n) = \{(a_o : a_1 : \cdots : a_n) | a_0 a_1 \cdots a_n \neq 0\}$$

We notice that the action of $T_{\mathbb{P}^n}$ on itself extends in an obvious manner to $\mathbb{P}^n$. We can describe the lattices of characters and cocharacters $\mathscr{M}_n$ and $\mathscr{N}_n$ of $T_{\mathbb{P}^n}$ by looking at the following exact sequence:

$$1 \longrightarrow \mathbb{C}^* \longrightarrow (\mathbb{C}^*)^n \longrightarrow T_{\mathbb{P}^n} \longrightarrow 1$$

We obtain the fact that $\mathscr{M}_n = \{m = (m_0, \ldots, m_n) | m \in \mathbb{Z}^{n+1} \& \sum_{i=0}^n m_i = 0\}$ and $\mathscr{N}_n = \mathbb{Z}^{n+1}/(\mathbb{Z}\{(1, \ldots, 1)\})$.

We now define a projective toric variety in a similar way to an affine toric variety, when we were given a torus T_N and a finite subset $\mathscr{A} = \{m_1, \ldots, m_s\}$ of its character lattice M. We then have the map $\Phi_{\mathscr{A}} : T_N \to (\mathbb{C}^*)^s$ as discussed earlier. We then have the projection map $\pi : (\mathbb{C}^*)^s \to T_{\mathbb{P}^{s-1}} \subset \mathbb{P}^{s-1}$.

Definition 1.3.1.1. *Given $T_N, M, \mathscr{A}, \Phi_{\mathscr{A}}, \pi$ as above, we define the projective toric variety $X_{\mathscr{A}}$ as the Zariski closure of $\pi \circ \Phi_{\mathscr{A}}(T_N)$ in $\mathbb{P}^{s-1}$.*

Given a projective variety $X \subset \mathbb{P}^n$, we know that there exists a homogeneous ideal of $I(X) \subset \mathbb{C}[x_0, \ldots, x_n]$ that corresponds to X by the projective Nullstellensatz. The affine variety $\hat{X}$ corresponding to $I(X)$ in $\mathbb{C}^{n+1}$ is called the affine cone of X. Now, we can discuss how the affine cone $\hat{X}_{\mathscr{A}}$ of $X_{\mathscr{A}}$ behaves with respect to the affine toric variety $Y_{\mathscr{A}}$ which we described in an earlier section. We use some facts about the kernel L of the map $\hat{\Phi}_{\mathscr{A}} : \mathbb{Z}^s \to M$ of character lattices which we used earlier to define the toric ideal $I_l = I(Y_{\mathscr{A}})$ of $Y_{\mathscr{A}}$.

Proposition 1.3.1.2. Consider a finite subset $\mathscr{A}$ of the character lattice M of a torus T_N, and the affine and projective toric varieties, $Y_{\mathscr{A}}$ and $X_{\mathscr{A}}$ respectively, obtained from $\mathscr{A}$. Then the following are equivalent:

1. The affine cone of $X_{\mathscr{A}}$ is $Y_{\mathscr{A}}$.

2. $I_L = I(X_{\mathscr{A}})$

3. I_L is a homogeneous ideal.

4. There exists $u \in N$ and $k > 0$ such that $\langle m_i, u\rangle = k$ for $i = 1, \dots, s$

Sketch. We can see that the definition of the affine cone of a projective variety proves that 1. and 2. are equivalent. We also see that 2. implies 3. is obvious. To show that 3. implies 4., we first see that the ideal I_L is generated by homogeneous polynomials of the form $x^\alpha - x^\beta$ we tensor the following exact sequence with $\mathbb{Q}$, and dualize it:

$$0 \to L \to \mathbb{Z}^s \to M$$

to get the exact sequence:

$$N_{\mathbb{Q}} \to \mathbb{Q}^s \to \hom_{\mathbb{Q}}(L \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Q}) \to 1$$

We now see that the image of $(1, \dots, 1)$ in $\hom_{\mathbb{Q}}(L \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Q})$ is 0, since the ideal I_L is generated by homogeneous polynomials of the form $x^\alpha - x^\beta$, and hence, $\langle l, (1, \dots, 1)\rangle = 0$ for all $l \in L$. Therefore, for the element $\hat{u} \in N_{\mathbb{Q}}$ which maps to $(1, \dots, 1) \in \mathbb{Q}^s$, we have that $\langle m_i, \hat{u}\rangle = 1$ for all $m_i \in \mathscr{A}$. To show that 4. implies 1., we only have to show that $\hat{X}_{\mathscr{A}} \cap (\mathbb{C}^*)^s \subset Y_{\mathscr{A}}$, since it is obvious that $Y_{\mathscr{A}} \subset \hat{X}_{\mathscr{A}}$. To show this, we use the fact that any $p \in \hat{X}_{\mathscr{A}} \cap (\mathbb{C}^*)^s$ is of the form $\mu(\chi^{m_0}(t), \dots, \chi^{m_s}(t))$. We use the element $u \in N$ to show that the image of the point $\lambda^u(\tau)t$ in the map $\Phi_{\mathscr{A}} : T_N \to \mathbb{C}^{n+1}$ can be manipulated to get p. And hence, $p \in Y_{\mathscr{A}}$. $\square$

Now, another way to think about this condition is that there exists an affine hyperplane in M that contains the points in $\mathscr{A}$ that does not pass through the origin. And hence, when we are given any finite subset $\mathscr{A}$ of the character lattice M of a torus T_N, we can consider the subset $\mathscr{A} \times \{1\} \subset M \oplus \mathbb{Z}$, which is the character lattice of the torus $T_N \times \mathbb{C}^*$. We therefore have the map $\Phi_{\mathscr{A}\times\{1\}} : T_N \times \mathbb{C}^* \to \mathbb{C}^s$, where, $(t, \mu) \mapsto (\mu\chi^{m_1}(t), \dots, \mu\chi^{m_s}(t) = \mu(\chi^{m_1}(t), \dots, \chi^{m_s}(t))$. Hence, we have that the projective toric variety $X_{\mathscr{A}}$ is the same as that of the projective toric variety $X_{\mathscr{A}\times\{1\}}$, and the affine cone of $X_{\mathscr{A}\times\{1\}}$ is the affine toric variety $Y_{\mathscr{A}\times\{1\}}$, since the set $\mathscr{A} \times \{1\}$ lies on an affine hyperplane in $M \oplus \mathbb{Z}$ that does not lie on the origin.

An example to consider here is as follows:

Example 1.3.1.3. We described the affine toric variety $Z(I)$ in 1.2.2.5. We see that we can define an analogous projective toric variety by using $\mathscr{A} = \{(d, 0), (d-1, 1), \dots, (1, d-1), (0, d)\}$, and using this description, we have a projective toric variety $X_{\mathscr{A}}$ that satisfies the conditions of 1.3.1.2.

Now, if we were to consider $\mathscr{B} = \{0, 1, \ldots, d\} \subset \mathbb{Z}$, we can construct the affine toric variety $Y_{\mathscr{B}}$ and the projective toric variety $X_{\mathscr{B}}$. In this case however, $I(Y_{\mathscr{B}}$ is not homogeneous and does not satisfy 1.3.1.2, even though $X_{\mathscr{A}} \cong X_{\mathscr{B}}$. ◊♦◊

We have not yet described how the character lattice of a projective toric variety $X_{\mathscr{A}}$ looks. We will answer this question with the following proposition:

Proposition 1.3.1.4. Consider a projective toric variety $X_{\mathscr{A}}$. Then the character lattice of the torus contained in $X_{\mathscr{A}}$ is given by:

$$\mathbb{Z}'\mathscr{A} := \{\sum_{i=1}^{s} a_i m_i | a_i \in \mathbb{Z}, \sum_{i=1}^{s} a_i = 0\}$$

Sketch. We prove this in a similar way to 1.2.2.1, by considering the maps between the different character lattices mentioned there, along with the projection map $\pi : (\mathbb{C}^*)^s \to T_{\mathbb{P}^{s-1}}$, and corresponding map of character lattices to obtain the following commutative diagram:

$$\begin{tikzcd}
M & \mathbb{Z}^s \arrow[l, "\hat{\Phi}_{\mathscr{A}}"'] & \mathscr{M}_{s-1} \arrow[l, hook'] \arrow[d, two heads] \\
 & & M' \arrow[ull, hook']
\end{tikzcd}$$

Where M' is the character lattice of the torus in the projective toric variety $X_{\mathscr{A}}$. Again, by using a similar argument to 1.2.2.1, we get the required isomorphism of lattices. □

Another special thing about projective space $\mathbb{P}^{s-1}$ is it has a decomposition into open sets $U_i := \mathbb{P}^{s-1} \backslash V(x_i)$ each of which are isomorphic to the affine space $\mathbb{C}^{s-1}$ as follows:

$$\Psi_i : \mathbb{P}^{s-1} \backslash V(x_i) \cong \mathbb{C}^{s-1}$$
$$(a_1 : \cdots : a_i : \cdots : a_s) \mapsto \left(\frac{a_1}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_s}{a_i}\right)$$

.

We see that each affine piece U_i contains the torus $T_{\mathbb{P}^{s-1}}$, which we see maps to the torus $(\mathbb{C}^*)^{s-1}$. We can then show that the affine pieces of a projective toric variety $X_{\mathscr{A}}$, which are given by $X_{\mathscr{A}} \cap U_i$, correspond to the following affine toric varieties:

Proposition 1.3.1.5. Let $X_{\mathscr{A}}$ be the projective toric variety for $\mathscr{A} = \{m_1, \ldots, m_s\} \subset M$, where M is the character lattice for the torus T_N. Then if U_i are the affine pieces of $\mathbb{P}^{s-1}$, $X_{\mathscr{A}} \cap U_i$ is the affine toric variety:

$$Y_{\mathscr{A}_i} = \operatorname{Spec}(\mathbb{C}[S_i])$$

where $\mathscr{A}_i = \mathscr{A} - m_i = \{m_j - m_i | j \neq i\}$ and $S_i = \mathbb{Z}_{\geq 0}\mathscr{A}_i$ Moreover the inclusions $X_{P\cap M} \cap U_j \supset X_{P\cap M} \cap U_j \cap U_i \subset X_{P\cap M} \cap U_i$ is given by the following localizations:

$$\operatorname{Spec}(\mathbb{C}[S_i]) \supset \operatorname{Spec}(\mathbb{C}[S_i]_{\chi^{m_i - m_j}}) = \operatorname{Spec}(\mathbb{C}[S_j]_{\chi^{m_j - m_i}}) \subset \operatorname{Spec}(\mathbb{C}[S_j])$$

Sketch. This description of $X_{\mathscr{A}} \cap U_i$ comes from the morphism $\Phi_{\mathscr{A}_i} : T_N \to \mathbb{C}^{s-1}$, as described in 1.2.2.1, which we see is the same as the map $\Psi_i \circ \pi \circ \Phi_{\mathscr{A}}$, which gives us the affine piece $X_{\mathscr{A}} \cap U_i$. Showing that the intersection of affine pieces take the form mentioned in the theorem is just noticing that $\chi^{m_i - m_j} \in \mathbb{C}[S_i]$ is always non zero when we restrict it to $X_{\mathscr{A}} \cap U_i \cap U_j$, and then using that. $\square$

Finally, we describe the affine pieces of a projective toric variety $X_{\mathscr{A}}$ as those corresponding to vertices of the polytope generated by $\mathscr{A}$.

Proposition 1.3.1.6. Given $\mathscr{A} = \{m_a, \ldots, m_s\} \subset M$ for the character lattice M of the torus T_N, and let $P = \operatorname{Conv}(\mathscr{A}) \subset M_{\mathbb{R}}$, i.e., P is the convex hull of $\mathscr{A}$ and hence is a polytope generated by $\mathscr{A}$. Let $J = \{i \in \{1, \ldots, s\} | m_i \text{ is a vertex of } P\}$. Then $X_{\mathscr{A}} = \bigcup_{j \in J} X_{\mathscr{A}} \cap U_j$.

Sketch. The proof of this theorem is to show that when m_i is not a vertex, m_i can be written as the convex linear combination of vertices of P. Clearing the denominators in this linear combination so that it is a linear combination with integral coefficients, and using this, we can show that for some vertex $m_j, j \in J$, $m_i - m_j \in S_i$. Hence, $\chi^{m_i - m_j} \in \mathbb{C}[S_i]$, due to which $\mathbb{C}[S_i]_{\chi^{m_i - m_j}} = \mathbb{C}[S_i]$. We therefore have that $X_{\mathscr{A}} \cap U_i = X_{\mathscr{A}} \cap U_i \cap U_j \subset X_{\mathscr{A}} \cap U_j$, by 1.3.1.5, and we are hence done. $\square$

1.3.2 Relationship with Polytopes, and the Notion of a Normal Fan

We start by considering a polytope $P \subset M_{\mathbb{R}}$, which is very ample relative to the lattice M, which is the character lattice for the torus T_N. Then we get the finite subset $P \cap M = \{m_1, \ldots, m_s\}$, and we can look at the projective toric variety $X_{P\cap M}$.

Proposition 1.3.2.1. Consider the projective toric variety $X_{P\cap M}$, for a full dimensional very ample lattice polytope P. Then the following facts are true:

1. The affine pieces of $X_{P\cap M}$ are affine toric varieties of the form $X_{P\cap M} \cap U_i = \mathrm{Spec}(\mathbb{C}[\sigma_i^\vee \cap M])$, where σ_i is the strongly convex rational polyhedral cone dual to the cone $C_i = \mathrm{Cone}\left((P \cap M) - \{m_i\}\right)$, where m_i is a vertex of P.

2. The torus of $X_{P\cap M}$ has character lattice M.

Sketch. To prove 1., we show that the cone $\mathrm{Cone}\left((P \cap M) - \{m_i\}\right)$ and its dual σ_i is strongly convex, by considering the hyperplane $H_{u,0}$, if the vertex m_i is the intersection $P \cap H_{u,b}$. Then, using the fact that $S_i = \mathbb{Z}_{\geq 0}(P \cap M - \{m_i\})$ is saturated since P is very ample, we show that $\sigma_i^\vee \cap M = S_i$, and we get the required description of the affine pieces of $X_{\mathscr{A}}$ using 1.3.1.5 and 1.3.1.6. 2. follows by using 1.2.4.1. □

Example 1.3.2.2. We can consider the lattice polytope $P = \mathrm{Conv}\left(\{(0,0),(1,0),(0,1)\}\right) \subset \mathbb{Z}^2$. We can see that since P is a polygon, our given construction works, and we find out that the projective toric variety $X_{P\cap M}$ is isomorphic to the projective space $\mathbb{P}^2$. ◊♦◊

Now, we study how these cones σ_i patch together on $N_\mathbb{R}$. For a full dimensional lattice polytope $P = \mathrm{Conv}(A) \subset M_\mathbb{R}$, we know there exists a facet presentation of P of the form $\{m \in M_\mathbb{R} | \langle m, u_F\rangle \geq a_F \forall u_F, a_F\}$, where u_F are vectors in N, and $-a_F \in \mathbb{Z}$. Now, for a vertex m_i of P, we can show that faces $Q \preceq P$ containing m_i bijectively correspond to faces $Q_{m_i} \preceq \sigma_i^\vee$ by the following relation:

$$Q \mapsto \mathrm{Cone}((Q \cap M) - \{m_i\}) = Q_{m_i}$$

Now, in particular, using the facets that contain m_i, we have that every facet of P that contains m_i corresponds to a facet of $\sigma_i^\vee$, and we hence have a facet presentation of $\sigma_i^\vee$:

$$\sigma_i^\vee = \{m \in M_\mathbb{R} | \langle m, u_F\rangle \geq 0 \forall \text{ facets } F \ni m_i\}$$

What is special about this is that this gives us an alternative presentation for σ_i by duality:

$$\sigma_i = \mathrm{Cone}(\{u_F | F \text{ is a facet of } P \text{ containing } m_i\})$$

We can similarly define the corresponding cone $\sigma_Q \subset N_{\mathbb{R}}$ for each face $Q \preceq P$ as follows:

$$\sigma_Q = \text{Cone}(\{u_F | F \text{ is a facet of } P \text{ containing } Q\})$$

Now, let us denote the collection of these cones as Σ_P. We define a collection of cones in $N_{\mathbb{R}}$ with special properties as follows:

Definition 1.3.2.3. *Let Σ be a set of rational polyhedral cones in $N_{\mathbb{R}}$. It is called a **normal fan** if it satisfies the following properties:*

1. *Each cone $\sigma \in \Sigma$ is a strongly convex cone.*
2. *For a cone $\sigma \in \Sigma$, every face τ of σ is also a cone in Σ.*
3. *The intersection of two cones $\sigma, \sigma' \in \Sigma$ is a face of each.*

We call the union of cones $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$ in a normal fan Σ as the support of the fan Σ. The collection of all r-dimensional cones in Σ is denoted by $\Sigma(r)$. We now state the following theorem:

Theorem 1.3.2.4. Let P be a full dimensional lattice polytope in $M_{\mathbb{R}}$. Then the following are true:

1. The collection of cones Σ_P is a normal fan of cones in $N_{\mathbb{R}}$.
2. The support of Σ_P is the whole space $N_{\mathbb{R}}$.

Example 1.3.2.5. For the polytope P, the fan Σ has the following cones:

$$\Sigma = \{\{(0,0)\}.\rho_1, \rho_2, \rho_3, \sigma_1, \sigma_2, \sigma_3\}$$

Where $\rho_i = \text{Cone}(\{v_i\})$ and $\sigma_i = \text{Cone}(\{v_a, v_b\})$ such that $a \neq i$ and $b \neq i$. Here, $v_1 = e_1, v_2 = e_2, v_3 = -e_1 - e_2$ for the standard basis $\{e_1 = (1,0), e_2 = (0,1)\}$ of $\mathbb{R}^2$. This is visualised in figure 1.1

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This notion of a normal fan is the foundation for how we define abstract toric varieties, and this will be explored deeply in the next few sections. Now, we will discuss how this fan is important in the context of 1.3.2.1, namely how the intersection of affine pieces corresponds to the toric variety of a cone in the fan Σ_P, since we already saw how the affine piece $X_{P \cap M} \cap U_i$ is the affine toric variety obtained from the cone σ_i,which is in the fan Σ_P.

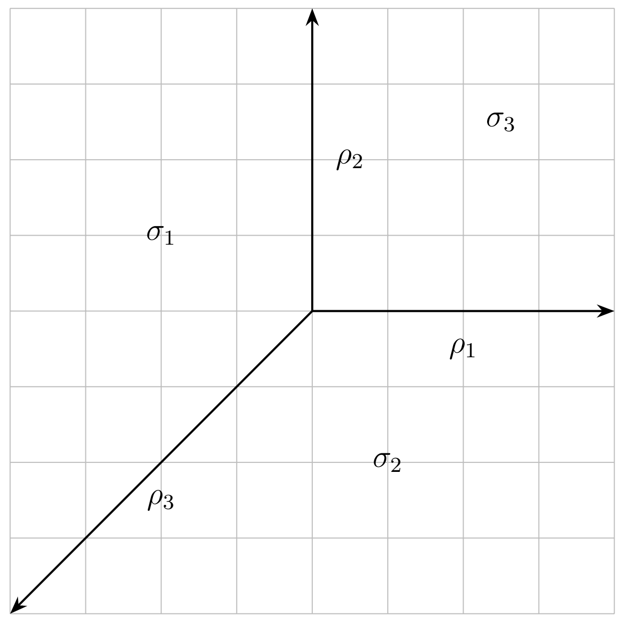

Figure 1.1: The fan Σ_P embedded in $N_{\mathbb{R}}$.

Proposition 1.3.2.6. Let P be a full dimensional and very ample lattice polytope in $M_{\mathbb{R}}$, and let $X_{P\cap M}$ be the associated projective toric variety. Let m_j and m_i be two vertices in P, corresponding to the affine pieces $X_{P\cap M} \cap U_i = U_{\sigma_i}$ and $X_{P\cap M} \cap U_j = U_{\sigma_j}$. Let $Q \preceq P$ be the smallest face of P containing both m_i and m_j. Then the intersection of the affine pieces U_{σ_i} and U_{σ_j} is given by the following toric variety:

$$X_{P\cap M} \cap U_j \cap U_i = \mathrm{Spec}(\mathbb{C}[\sigma_Q^\vee \cap M]) = U_{\sigma_Q}$$

Moreover, the inclusion $X_{P\cap M} \cap U_j \supset X_{P\cap M} \cap U_j \cap U_i \subset X_{P\cap M} \cap U_i$ is given by the localisation:

$$U_{\sigma_i} \supset (U_{\sigma_i})_{\chi^{m_j - m_i}} = U_{\sigma_Q} = (U_{\sigma_j})_{\chi^{m_i - m_j}} \subset U_{\sigma_j}$$

Sketch. We can show this by using 1.3.1.5, and applying 1.1.2.8 with σ_i, σ_j and $\sigma_Q = \sigma_i \cap \sigma_j$. $\square$

We now define the projective toric variety of a full dimensional lattice polytope P as follows:

Definition 1.3.2.7. *The toric variety defined by the full dimensional lattice polytope $P \subset M_{\mathbb{R}}$ is the projective toric variety $X_{kP\cap M}$, where $k \in \mathbb{N}$ is chosen such that the polytope kP is very ample.*

1.4 (Normal) Toric Varieties

1.4.1 Abstract Varieties, and The Toric Variety of a Normal Fan

We first begin by defining an abstract variety as a glueing of finitely many affine varieties. We then talk about how for a normal fan Σ on $N_{\mathbb{R}}$, the affine toric varieties defined by each of the cones $\sigma \in \Sigma$ glue together to give us an abstract toric variety that is normal.

We first start with how to define an abstract variety, referring to [CLS11]. For this, we begin with a collection of finitely many affine varieties $\{V_a\}_{a\in A}$, for a finite indexing set A. We define an equivalence relation on the disjoint union of $\{V_a\}_{a\in A}$, after we define a group of gluing functions.

For every pair of affine varieties V_a, V_b, we have Zariski open sets $V_{b,a} \subset V_a, V_{a,b} \subset V_b$ and corresponding morphisms $g_{b,a} : V_{b,a} \to V_{a,b}$ and $g_{a,b} : V_{a,b} \to V_{b,a}$, which are inverses of each other and hence form isomorphisms between $V_{b,a}$ and $V_{a,b}$. The other property these functions have is that for any three $a, b, c \in A$, $g_{b,a}(V_{b,a} \cap V_{c,a}) = V_{a,b} \cap V_{c,b}$ and $g_{c,a} = g_{c,b} \circ g_{b,a}$.

Now if we define a relation $\sim$ on $\bigsqcup_{a\in A} V_a$, where $x \sim y$ when $y = g_{b,a}(x)$ for some $b, a \in A$. Then the conditions we discussed earlier will ensure that $\sim$ is an equivalence relation. We now define the *abstract variety* X as the set $\bigsqcup_{a\in A} V_a/\sim$ with the quotient topology. We notice that the set $U_a = \{[a] \in X | a \in V_a\}$ is an open set of X, and is isomorphic to V_a. Thus, locally, X looks like an affine variety.

We call an irreducible abstract variety X normal if all the corresponding local rings $\mathcal{O}_{X,p}$ are integrally closed. This is equivalent to the condition that for a cover of affine open sets of X, each affine open set is a normal affine variety.

We can now define an abstract variety X_Σ if we have a normal fan Σ as defined in 1.3.2.3. In this case, we consider the collection of affine varieties U_σ obtained from every cone $\sigma \in \Sigma$. We obtain the gluing morphisms from 1.2.4.2 and the following relationship between $U_\sigma, U_{\sigma'}, U_\tau$, for three cones $\sigma, \sigma', \tau \in \Sigma$, such that $\tau = \sigma \cap \sigma'$:

$$U_\sigma \supset (U_\sigma)_{\chi^m} \cong U_\tau \cong (U_{\sigma'})_{\chi^{-m}} \subset U_{\sigma'}$$

Where m is such that $\tau = \sigma \cap H_m = \sigma' \cap H_m$, which we obtain from 1.1.2.8.

We now define a general toric variety as follows, and then see that X_Σ is a toric variety for a normal fan Σ.

Definition 1.4.1.1. *A toric variety is an irreducible variety X that contains a torus $T_N \cong (\mathbb{C}^*)^n$ as a Zariski open subset, such that the action of T_N on itself extends to an algebraic action of T_N on X.*

Now, X_Σ is a toric variety since for each affine toric variety U_σ for a cone $\sigma \in \Sigma$, the corresponding torus is T_N, where N is the lattice corresponding to $N_\mathbb{R}$, where the cones in Σ are contained. And when gluing these varieties together, T_N is contained in the intersection of all these affine varieties, and are hence the same torus in X_Σ. The action of T_N extends on each U_σ, and then one just has to show that they patch together with the gluing morphisms to obtain an algebraic action of T_N on X_Σ.

X_Σ is also a normal variety since each cone σ is a strongly convex rational polyhedral cone, and hence, U_σ is a normal affine variety. Thus X_Σ is normal based on our discussion earlier.

We can see that for every full dimensional polytope $P \subset M_\mathbb{R}$, the projective toric variety X_P is isomorphic to the toric variety X_{Σ_P}. Similarly, if we take the collection of all faces of a strongly convex rational polyhedral cone $\sigma \subset N_\mathbb{R}$, it forms a fan and we get an associated toric variety which is isomorphic to U_σ. Henceforth, all toric varieties we study will derive from normal fans Σ of cones in $N_\mathbb{R}$, and hence will be normal toric varieties.

1.4.2 Limits of Cocharacters, and the Orbit-Cone Correspondence

In this section, we first want to recover the cones and hence the fan that gives us a toric variety X_Σ. We do this by taking a look at the limits of cocharacters $v \in N$ as $z \to 0$. This works because for $v \in N$, the corresponding cocharacter $\lim_{z\to 0} \lambda^v(z)$ goes to the distinguished point $x_\tau \in U_\tau$ for the cone τ in fan Σ such that $v \in \mathrm{relint}(\tau)$. This is covered in the following proposition:

Proposition 1.4.2.1. Let Σ be a normal fan in N, and X_Σ be the corresponding toric variety. Let $v \in N$ be a point in the cocharacter lattice. Then:

1. $v \in \mathrm{relint}(\tau)$ if and only if $\lim_{z\to 0} \lambda^v(z) = x_\tau$.

2. If $v \notin |\Sigma|$, then $lim_{z\to 0}\lambda^v(z)$ does not exist in X_Σ

Sketch. In this proof, if $v \in \sigma$, we use the following identification: $T_N \cong \hom_{\mathbb{Z}}(M, \mathbb{C}^*)$, and the corresponding identification between points of U_σ and semigroup morphisms from S_σ to $\mathbb{C}$. We then check the case for the limiting morphism $\gamma : S_\sigma \to \mathbb{C}$, which we see corresponds to the distinguished point x_τ. The second point is true because the limit point of $\lambda^v(z)$ is not defined in U_σ, as $z \to 0$, if $v \notin \sigma$. □

These distinguished points of each cone τ help us identify the T_N orbits of X_Σ. We define the orbit $O(\sigma)$ of the distinguished point x_σ for a cone $\sigma \in \Sigma$ as follows:

$$O(\sigma) = T_N \cdot x_\sigma$$

Example 1.4.2.2. For the projective space $\mathbb{P}^2$, we have the fan described in 1.3.2.5. The map of the cocharacter λ^u to $\mathbb{P}^2$ for $u = (a, b) \in N = \mathbb{Z}^2$ is given by $\lambda^u(t) = (1 : t^a : t^b)$. And hence, for each cone $\sigma \in \Sigma$, the limit of the cocharacter for $u \in \text{relint}(\sigma)$, and hence the distinguished point x_σ, is given as follows:

1. $\lim_{z\to 0} \lambda^v(z) = (1 : 1 : 1)$ for $v \in \text{relint}(\{0\})$.
2. $\lim_{z\to 0} \lambda^v(z) = (1 : 0 : 1)$ for $v \in \text{relint}(\rho_1)$.
3. $\lim_{z\to 0} \lambda^v(z) = (1 : 1 : 0)$ for $v \in \text{relint}(\rho_2)$.
4. $\lim_{z\to 0} \lambda^v(z) = (0 : 1 : 1)$ for $v \in \text{relint}(\rho_1)$.
5. $\lim_{z\to 0} \lambda^v(z) = (0 : 1 : 0)$ for $v \in \text{relint}(\sigma_1)$.
6. $\lim_{z\to 0} \lambda^v(z) = (0 : 0 : 1)$ for $v \in \text{relint}(\sigma_2)$.
7. $\lim_{z\to 0} \lambda^v(z) = (1 : 0 : 0)$ for $v \in \text{relint}(\sigma_3)$.

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Now, we can show that this is isomorphic to the torus obtained from the lattice construction $N(\sigma) = N/\mathbb{Z}(\sigma \cap N)$, whose dual lattice is $\sigma^\perp \cap M$. The torus obtained from this lattice $N(\sigma$ is denoted by $T_{N(\sigma)}$. We describe this isomorphism in the following proposition:

Proposition 1.4.2.3.

$$O(\sigma) = \{\gamma : S_\sigma \to \mathbb{C} | \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp\} \cong T_{N(\sigma)}$$

Sketch. We prove this after noticing that $x_\sigma \in O' = \{\gamma : S_\sigma \to \mathbb{C} | \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp\}$, and that $O' \cong \hom_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*)$, which we do by restricting $\gamma \in O'$ to $\sigma^\perp \cap M$. We can then see that $T_{N(\sigma)} \cong \hom_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*)$ implies our result. This last isomorphism can be obtained after working out that $N(\sigma)$ and $\sigma^\perp \cap M$ are dual lattices to each other, and hence by applying hom and tensor products with $\mathbb{C}^*$ as required, we get the isomorphism. □

With this above proposition, we can describe the torus orbits of $\mathbb{P}^2$ using 1.4.2.2. Now, we can state the main result of this section, which generalizes this description to any toric variety X_Σ for a fan $\Sigma \in N_{\mathbb{R}}$:

Theorem 1.4.2.4 (Orbit-Cone correspondence)**.** Let Σ be a normal fan in N, and let X_Σ be the associated toric variety. then the following facts are true:

1. There is bijection between the cones $\sigma \in \Sigma$ and the T_N orbits of X_Σ as follows:

$$\sigma \mapsto O(\sigma)$$

   Which means all torus orbits of X_Σ are of the form $O(\sigma)$ for some cone $\sigma \in \Sigma$. Moreover, if $\dim(N_{\mathbb{R}}) = n$, $\dim(O(\sigma)) = n - \dim(\sigma)$

2. The affine open variety $U_\sigma \subset X_\Sigma$ can be written as a union of torus orbits as follows:

$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau)$$

   .

3. For a cone $\tau \in \Sigma$, a cone σ has τ as a face if and only if the closure $\overline{O(\tau)}$ contains $O(\sigma)$. Moreover;

$$\overline{O(\tau)} = \bigcup_{\sigma \succeq \tau} O(\sigma)$$

Sketch. To prove 1., we see that any orbit O in X_Σ belongs to U_σ for some $\sigma \in \Sigma$, and if we pick the minimal such cone τ, we can prove that $O = O(\tau)$. The dimension of O comes from the dimension of $T_{N(\sigma)}$, as discussed in 1.4.2.3 To prove 2., we just notice that U_σ has to be a union of torus orbits, and then use the fact that $O(\tau) \subset U_\sigma$ only if $\tau \preceq \sigma$. To prove 3., we show the closure $\overline{O(\tau)}$ in the complex topology is given by the union, by using the fact that it is torus invariant, and using the limits of cocharacters $v \in \text{relint}(\sigma)$, such that $\sigma \succeq \tau$. Then we show that the closure in the complex topology as well as the Zariski topology coincide for $O(\tau)$. □

1.4.3 Complete Fans and Associated Toric Varieties

In this section, we try to relate when a toric variety X_Σ is compact in the classical topology obtained from $\mathbb{C}$ to the the support of the fan Σ. The main theorem of this section is as follows:

Theorem 1.4.3.1. The toric variety X_Σ is compact in the classical topology if and only if $|\Sigma| = N_{\mathbb{R}}$.

This is done by showing that a particular morphism obtained by a map between fans is proper, which can be specialized to a map from Σ to the $\{0\}$ fan. We state that result here, from which 1.4.3.1 is a simple corollary:

Proposition 1.4.3.2. For a lattice homomorphism $\phi : N' \to N$, which maps a fan Σ' in $N'_{\mathbb{R}}$ into a fan Σ in $N_{\mathbb{R}}$, the corresponding map of varieties $\phi' : X_{\Sigma'} \to X_\Sigma$ is proper if and only if $\phi(|\Sigma'|) = |\Sigma|$.

Sketch. To prove that $\phi(|\Sigma'|) = |\Sigma|$, we see that if there exists v' such that $\phi(v') = v \in |\Sigma|$, and $v' \notin |\Sigma'|$, then by 1.4.2.1, $\lambda^v(z)$ has a limit when $z \to 0$, but $\lambda^{v'}(z)$ does not, which contradicts properness. To prove the properness of the map when $\phi(|\Sigma'|) = |\Sigma|$, we use the valuative criterion of proper maps as described in [Har77]. □

Chapter 2

Toric Divisors and Vector Bundles

In this chapter, we will only be discussing separated irreducible normal varieties. Much of this chapter is based on similar sections in [CLS11], with some of the algebraic geometry being based on [Har77].

2.1 Divisors over a toric variety

2.1.1 Introduction to Weil and Cartier Divisors

For a variety X, a prime divisor D of X is an irreducible variety of codimension 1. Prime divisors are special because we get a ring $\mathcal{O}_{X,D} \subset \mathbb{C}(X)$ which is a discrete valuation ring on the field of fractions $\mathbb{C}(X)$ of X and hence gives a discrete valuation on $\nu_D : \mathbb{C}(X)^* \to \mathbb{Z}$. The DVR associated to the prime divisor D is of the following form:

$$\mathcal{O}_{X,D} = \{f \in \mathbb{C}(X) | f \text{ is defined on } U \subset X, \text{ where } U \text{ open and } U \cap D \neq \emptyset\}$$

We see that for an open subset $U \subset X$, $\mathcal{O}_{X,D} = \mathcal{O}_{U,U\cap D}$ which we get from the fact that $\mathbb{C}(U) = \mathbb{C}(X)$, since X is irreducible. We start with the following proposition:

Proposition 2.1.1.1. For $f \in \mathbb{C}(X)$, $\nu_D(f)$ is zero for all but finitely many prime divisors D of X.

We now define the group of Weil divisors over X:

Definition 2.1.1.2. *1. The free abelian group generated by prime divisors D of a variety X is called the group of* ***Weil Divisors*** *over X, and is denoted by $Div(X)$. Their elements are called Weil divisors, or just divisors, of X.*

2. *When a Weil divisor D, when represented as a sum of prime divisors, has only positive coefficients, then it is called an* ***effective divisor****, which is denoted by $D \geq 0$.*

3. *For every rational function $f \in \mathbb{C}(X)^*$, we have a Weil divisor $div(f) = \sum_{D_i \text{ is a prime divisor of } X} \nu_{D_i}(f) D_i$, which is well defined. Every divisor of this form are called* ***principal divisors*** *of X. The set of principal divisors is denoted by $Div_0(X)$*

4. *Two divisors D and E are said to be linearly equivalent if there exists a principal divisor $div(f)$ such that: $D = E + div(f)$.*

Since ν_D is a discrete valuation, $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$ and $\mathrm{div}(-f) = -\mathrm{div}(f)$ for $f, g \in \mathbb{C}(X)^*$. Hence $\mathrm{Div}_0(X)$ is a subgroup of $\mathrm{Div}(X)$.

For an open subset $U \subset X$, we define the restriction of a divisor $D = \sum_{D_i} \lambda_i D_i$ to U as:

$$D|_U = \sum_{D_i} \lambda_i (U \cap D_i)$$

where D_i are the prime divisors of X.

Another special subgroup of divisors are those called Cartier divisors. These are Weil divisors that are *locally principal.* i.e.:

Definition 2.1.1.3. *A Weil divisor D of X is called a Cartier divisor if there exists an open covering of X $\{U_i\}$, such that $D|_{U_i} = div(f_i)$ for some function $f_i \in \mathbb{C}(U_i)$. The subgroup of Cartier divisors is denoted by: $CDiv(X)$. The collection $\{(U_i, f_i)\}$ is called the local data of the Cartier divisor D.*

We can see that all principal divisors by definition are Cartier divisors. We can use the relation of linear equivalence of divisors to define the following two quotient groups:

Definition 2.1.1.4. *1. $Div(X)/Div_0(X)$ is called the class group of X, denoted by $Cl(X)$.*

2. *$CDiv(X)/Div_0(X)$ is called the Picard group of X, denoted by $Pic(X)$.*

Now we state the following result about the divisors of rational functions, which motivate our definition of the sheaf of $\mathcal{O}_X$-modules for a divisor D.

Proposition 2.1.1.5. For a function $f \in \mathbb{C}(X)$, the divisor $\operatorname{div}(f) \geq 0$ if and only if $f \in \mathcal{O}_X(X)$. Moreover, $\operatorname{div}(f) > 0$ if and only if $f \in \mathcal{O}^*_X(X)$.

We will now define a sheaf of $\mathcal{O}_X$ modules on X for each divisor $D \in \operatorname{Div}(X)$, by defining the module $\mathcal{O}_X(D)(U)$ for each open set $U \subset X$ as follows:

$$\mathcal{O}_X(D)(U) = \{f \in \mathbb{C}(U)) | \operatorname{Div}(f) + D \geq 0\}$$

This gives a sheaf of $\mathcal{O}_X$ modules $\mathcal{O}_X(D)$ over our variety X.

2.1.2 Torus Invariant Weil and Cartier Divisors

In this section, we will first describe an action of a torus on a divisor, and then talk about torus invariant divisors. We will describe how torus invariant Weil divisors behave. We then discuss an important correspondence between torus invariant Cartier divisors of a toric variety X_Σ that correspond to piecewise linear functions on the support of the fan Σ.

The way we define the action of the torus T_N on a divisor of X_Σ, the toric variety obtained from the fan Σ begins with noticing that for a prime divisor $D_i \subset X_\Sigma$, the subset $t \cdot D_i$ is also a prime divisor of X_Σ. Thus, the torus action on a Weil divisor $D = \sum \lambda_i D_i$ over finitely many prime divisors D_i is given as follows:

$$t \cdot D = \sum \lambda_i (t \cdot D_i)$$

We will connect this with the prime divisors obtained from the 1 dimensional cones $\rho \in \Sigma(1)$. We know from 1.4.2.4 that the dimension of ρ is the codimension of the orbit $O(\rho)$, and hence its closure $\overline{O(\rho)} = D_\rho$. This means that D_ρ is a prime divisor of X_Σ, which is invariant under the torus action, because of the way we constructed it. We also notice that divisors of the form $D = \sum_{\rho \in \Sigma(1)} \lambda_\rho D_\rho$ will be torus invariant. We can show that all torus invariant Weil divisors are of this form, by using 1.4.2.4once again. We can describe these prime divisors for $\mathbb{P}^2$ as follows:

Example 2.1.2.1. Referring to examples 1.3.2.5 and 1.4.2.2, as well as using 1.4.2.4, we see that the torus invariant prime divisors of $\mathbb{P}^2$ are of the form:

1. $D_{\rho_1} = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 | x_1 = 0\}$
2. $D_{\rho_2} = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 | x_2 = 0\}$
3. $D_{\rho_3} = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 | x_0 = 0\}$

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We will now describe how the divisor of a character $\chi^m \in \mathbb{C}(X_\Sigma)$ can be represented as a sum of prime divisors originating from rays $\rho \in \Sigma(1)$. We have the following result:

Proposition 2.1.2.2. Consider a toric variety X_Σ originating from a fan Σ in $N_\mathbb{R}$. For cones $\rho \in \Sigma(1)$, let u_ρ be the minimal generator of ρ in $\rho \cap N$. Let D_ρ be the corresponding prime divisor of X_Σ. Then:

1. The valuation of a character χ^m at D_ρ is given by:
$$\nu_{D_\rho}(\chi^m) = \langle m, u_\rho \rangle$$
2. The divisor of the character χ^m is given by:
$$\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho$$

Sketch. To prove 1., we describe the discrete valuation ring $\mathcal{O}_{X_\Sigma, D_\rho}$ by studying the affine variety U_ρ and its intersection with the prime divisor D_ρ. We then get a discrete valuation on $\mathbb{C}(X)$, which for characters χ^m, can be represented in terms of the minimal generators u_ρ of ρ. The proof of 2. follows from 1. and the fact that D_ρ forms an open cover of X_Σ, on which χ^m is supported. □

We see that the divisor of a character is a torus invariant Weil divisor. Now, we want to describe how torus invariant Cartier divisors behave, of which $\mathrm{div}(\chi^m)$ for a character χ^m is an example. For this, we first start with affine toric varieties arising from strongly convex rational polyhedral cones $\sigma \subset N_\mathbb{R}$. For the proof of the following proposition, we refer to [CLS11].

Proposition 2.1.2.3. Let U_σ be an affine toric variety arising from a strongly convex rational polyhedral cone $\sigma \subset n_\mathbb{R}$. Then, every torus invariant Cartier divisor is of the form $\mathrm{div}(\chi^m)$ for a character χ^m of the torus T_N, and is hence a principal divisor.

We can now use this to find a way to describe torus invariant Cartier divisors and their local data in a more convenient way:

Proposition 2.1.2.4. Let $D = \sum_{\rho\in\Sigma(1)} \lambda_\rho D_\rho$ be a torus invariant Weil divisor. D being a torus invariant Cartier divisor is equivalent to the following properties:

1. D restricts to the divisor of a character χ^m on each affine set U_σ for every cone $\sigma \in \Sigma$.

2. For every $\sigma \in \Sigma$, there exists some $m_\sigma \in M$, such that $\langle m_\sigma, u_\rho\rangle = -\lambda_\rho$ for all $\rho \in \sigma(1)$.

3. For every cone $\sigma \in \Sigma_{\max}$, i.e., for every maximal cone σ of Σ, there exists some $m_\sigma \in M$, such that $\langle m_\sigma, u_\rho\rangle = -\lambda_\rho$ for all $\rho \in \sigma(1)$.

Moreover, for each cone $\sigma \in \Sigma$, m_σ is unique modulo $\sigma^\perp \cap M$.

Sketch. We can see that being a torus invariant Cartier divisor is equivalent to 1. because of 2.1.2.3. Then, we show that 1. is equivalent to 2. using the description of the divisor of a character discussed in 2.1.2.2. The equivalence of 2. and 3. comes from the fact that every cone in Σ is a face of a maximal cone in Σ. To prove uniqueness modulo $\sigma^\perp \cap M$, we just have to show that if we have m_σ and m'_σ satisfying this condition, then since $u \in \sigma$ are spanned by u_ρ for $\rho \in \sigma(1)$, we can show that $m_\sigma - m'_\sigma \in \sigma^\perp \cap M$, because $\langle m_\sigma, u\rangle = \langle m'_\sigma, u\rangle$ for all $u \in \sigma$. □

This gives us local data of each torus invariant Cartier divisor in the form of $\{m_\sigma\}_{\sigma\in\Sigma}$. To make this easier to work with, we can view these as piecewise linear functions on the support $|\Sigma|$ of Σ. We define these functions as follows:

Definition 2.1.2.5. *A function $\phi : |\Sigma| \to \mathbb{R}$ is called a **support function** on Σ if it is linear on each cone $\sigma \in \Sigma$. Moreover, it is **integral with respect to the lattice** N, if $\phi(m) \in \mathbb{Z}$ for each $m \in |\Sigma| \cap N$.*

It is quite easy to see how once we have the local data $\{m_\sigma\}_{\sigma\in\Sigma}$ for a torus invariant Cartier divisor to see how we get an associated support function integral with respect to the lattice N, which we describe in the following proposition:

Proposition 2.1.2.6. Let $D = \sum_{\rho\in\Sigma(1)} \lambda_\rho D)\rho$ be torus invariant Cartier divisor with local data $\{m_\sigma\}_{\sigma\in\Sigma}$ of a toric variety X_Σ corresponding to a normal fan Σ. Then, we have the corresponding support function that is integral with respect to N, and is well defined:

$$\phi_D : |\Sigma| \to \mathbb{R}$$

$$u \mapsto \langle m_\sigma, u\rangle \text{for } u \in \sigma$$

Moreover, if we have a support function $\phi : |\Sigma| \to \mathbb{R}$ integral with respect to the lattice N, we have the following torus invariant Cartier divisor of X_Σ:

$$D_\phi = \sum_{\rho \in \Sigma(1)} -\phi(u_\rho) D_\rho$$

Hence, we have an identification between torus invariant Cartier divisors of X_Σ and support functions on Σ integral with respect to the lattice N.

Sketch. To prove this, we have to check that the defined support function is well defined, and that D_ϕ is a torus invariant Cartier divisor, both of which are true because of 2.1.2.4. □

Finally in this section, we describe global sections of the sheaf of $\mathcal{O}_{X_\Sigma}$-modules obtained from torus invariant Weil divisors of a toric variety X_Σ, which we do in this following proposition:

Proposition 2.1.2.7. Let $D = \sum_{\rho\in\Sigma(1)} \lambda_\rho D_\rho$ be a torus invariant Weil divisor of the toric variety X_Σ obtained from the fan Σ. Then the global sections of the sheaf $\mathcal{O}_{X_\Sigma}(D)$ can be written in the following manner:

$$\mathcal{O}_{X_\Sigma}(D)(X_\Sigma) = \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{\mathrm{div}(\chi^m)+D\geq 0} \mathbb{C}\cdot\chi^m = \bigoplus_{m\in P_D\cap M} \mathbb{C}\cdot\chi^m$$

where $P_D \subset M_\mathbb{R}$ is the polyhedron $\bigcap_{\rho\in\Sigma(1)}\{m \in M_\mathbb{R} | \langle m, u_\rho\rangle \geq -\lambda_\rho\}$.

Sketch. To prove this, we notice use 2.1.1.5 to notice that if $f \in \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$, $\mathrm{div}(f) \geq 0$ on T_N, and hence, we have that $f \in \mathbb{C}[M]$. Now we use 1.2.3.5 to prove the second equality. To prove the third equality, we use the expression of $\mathrm{div}(\chi^m)$ in terms of sums of prime divisors to represent the ms as the elements of a polyhedron P_D □

The polyhedron P_D is called the polyhedron of the divisor D. For a complete fan Σ such that $|\Sigma| = N_\mathbb{R}$, P_D for any torus invariant Weil divisor D of X_Σ ends up being a polytope.

2.2 Vector bundles over Toric Varieties

2.2.1 Introduction to Vector Bundles and Line Bundles

In this section, we will first describe vector bundles and then talk about how they correspond to locally free sheaves over varieties. A sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules over a variety X is called *locally free* of rank r, if there exists an open cover $\{U\}$ of X such that $\mathscr{F}|_U \cong \mathcal{O}_U^r$. Another kind of sheaf of interest to us are those that are generated by global sections. A sheaf $\mathscr{G}$ of $\mathcal{O}_X$-modules over a variety X is said to be generated by global sections if for every point $p \in X$, the stalk $\mathscr{G}_p$ is generated by images of $s \in \Gamma(X, \mathscr{G})$.

A vector bundle V of rank r over a variety X is the variety V with the following data:

1. A morphism $\pi : V \to X$.

2. An open cover $\{U_i\}_{i \in I}$ of X, with each open set U_i coupled with the following isomorphism:
$$\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$$
such that composing the projection $p : U_i \times \mathbb{C}^r \to U_i$ with ϕ_i is $\pi|_{\pi^{-1}(U_i)}$.

3. For every pair $i, j \in I$, there exists transition functions $g_{i,j}, g_{j,i} \in \mathrm{GL}_r(\Gamma(U_i \cap U_j, \mathcal{O}_X))$ such that $g_{i,j} = g_{j,i}^{-1}$and the following compositions are true:
$$\phi_j|_{U_i \cap U_j} = 1 \times g_{i,j} \circ \phi_i|_{U_i \cap U_j}$$
$$\phi_i|_{U_i \cap U_j} = 1 \times g_{j,i} \circ \phi_j|_{U_i \cap U_j}$$

This means that locally, a vector bundle V of rank r is of the form $U \times \mathbb{C}^r$. We study vector bundles over a variety X by studying the set of *sections* over an open set $U \subset X$, which are functions $p : U \to V$ such that $\pi \circ p = \mathrm{id}_U$. These sections form a sheaf over the variety X. This is where locally free sheaves come in. The sheaf of sections of a vector bundle or rank r over a variety X is a locally free sheaf of rank r. Moreover, every locally free sheaf of rank r over X is the sheaf of sections of vector bundle of rank r over X.

Vector bundles of special interest in this section are those of rank 1, called line bundles. These correspond to locally free sheaves of rank one, which are also called invertible sheaves. What is special about line bundles is that the sheaf of $\mathcal{O}_X$-modules over X obtained from a Cartier divisor D is locally free of rank 1. Moreover, given any invertible sheaf $\mathscr{L}$ over a variety X, we have a corresponding Cartier divisor, since we have an open cover $\{U_i\}_{i\ inI}$ of X, and for each of those U_i, a corresponding function $f_i \in \mathbb{C}(U_i)$, that gives us the isomorphism $\mathscr{L}|_{U_i} \cong \mathcal{O}_{U_i}$. This gives us the local data of Cartier divisor D, whose corresponding sheaf of modules is isomorphic to $\mathscr{L}$.

We now discuss what it means for a line bundle to be basepoint free, and discuss some other properties by which this can be characterized. A subspace $W \subset \Gamma(X, \mathscr{L})$ of the global sections of a line bundle $\mathscr{L}$ is said to be basepoint free if for every point $p \in X$, there exists an $s \in W$ such that $s(p) \neq 0$. For a Cartier divisor D, the following are equivalent:

1. The sheaf $\mathcal{O}_X(D)$ is generated by global sections.
2. The space $\Gamma(X, \mathcal{O})X(D))$ associated to D is basepoint free.
3. For every point $p \in X$, there exists an $s \in \Gamma(X, \mathcal{O}_X(D))$ such that p is not in the support of the divisor of zeroes of s.

We finally discuss a map to a projective space which is the backbone of the definition of very ampleness. We first start by defining $\mathbb{P}(W^\vee)$ for a the dual $W^\vee$ of a subspace $W \subset \Gamma(X, \mathscr{L})$ for an invertible sheaf $\mathscr{L}$, as $\mathbb{P}(W^\vee) = W^\vee - \{0\}/\mathbb{C}^*$. Now to define the map $\phi_{\mathscr{L},W} : X \to \mathbb{P}(W^\vee)$, we look at each point $p \in X$, and consider a non zero point $\nu_p \in \pi^{-1}(p) \cong \mathbb{C}$, where $\pi : V_{\mathscr{L}} \to X$, is the map associated to the line bundle $V_{\mathscr{L}}$ corresponding to our invertible sheaf $\mathscr{L}$. Then, for any $s \in W$, $s(p) = \lambda_s \nu_p$, and this gives us a map $l_p(s) = \lambda_s$, which is a member of $W^\vee$ and is unique upto choices of ν_p. Therefore, the class $[l_p]$ is well defined in $\mathbb{P}(W^\vee)$ for each $p \in X$, and hence gives us our required map $\phi_{\mathscr{L},\mathscr{W}}$,which ends up being a morphism.

2.2.2 Basepoint free and Very Ample Toric Divisors

We first discuss how we can describe the property of a torus invariant Cartier divisor D of a toric variety X_Σ being basepoint free is related to the polyhedron P_D of the divisor. This is given by the following result:

Proposition 2.2.2.1. Let D be a torus invariant Cartier divisor of the toric variety X_Σ, with local data $\{m_\sigma\}_{\sigma \in \Sigma_{\max}}$. Moreover, let $\Sigma_{\max} = \Sigma(n)$, which means that the local data is unique for D. Then D is basepoint free if and only if $m_\sigma \in P_D$ for all cones $\sigma \in \Sigma(n)$.

For fans Σ such that $|\Sigma|$ is convex and has the same dimension as $N_{\mathbb{R}}$ (such fans are said to be full dimensional with convex support), the convexity of the support function ϕ_D of a torus invariant Cartier divisor D is also an equivalent characterization of D being basepoint free. First, we see that P_D for a torus invariant Cartier divisor D can be represented in terms of its support function ϕ_D as follows:

$$P_D = \{m \in M_{\mathbb{R}} | \phi_D(u) \leq \langle m, u \rangle \forall u \in |\Sigma|\}$$

A support function $\phi : |\Sigma| \to \mathbb{R}$ is said to be convex if:

$$\phi(tu + (1-t)v) \geq t\phi(u) + (1-t)\phi(v)$$

Now, we characterize the property of a torus invariant Cartier divisor D being basepoint free as follows:

Proposition 2.2.2.2. Let D be a torus invariant Cartier divisor of the toric variety X_Σ, where the fan Σ is full dimensional with convex support. Let $\{m_\sigma\}_{\sigma \in \Sigma(n)}$ be the local data for the divisor D, whose support function is ϕ_D. Then the following properties are equivalent:

1. D is basepoint free.
2. $m_\sigma \in P_D$ for all cones $\sigma \in \Sigma(n)$
3. The support function $\phi_D : |\Sigma| \to \mathbb{R}$ is convex.
4. $\phi_D(u) = \min_{\sigma \in \Sigma(n)} \langle m_\sigma, u \rangle$

Moreover, if the fan Σ is complete, the above conditions are equivalent to the following:

1. $P_D = \mathrm{Conv}(\{m_\sigma | \sigma \in \Sigma(n)\})$, and the vertices of P_D are all the m_σ.
2. $\phi_D(u) = \min_{m \in P_D \cap M} \langle m, u \rangle$.

We now describe what ample and very ample Cartier divisors are. We start with a complete normal variety X, with Cartier divisor D. Then the divisor D and the associated line bundle is said to be very ample if D is basepoint free, and for $W = \Gamma(X, \mathcal{O}_X(D))$, the morphism $\phi_{D,W} : X \to \mathbb{P}(W^\vee)$ is a closed embedding. D is said to be ample if kD is very ample for some natural number k.

We defined the projective toric variety X of a full dimensional lattice polytope P in the previous chapter. This corresponds to the toric variety X_{Σ_P} for the fan Σ_P obtained from the polytope P. Here, we will use a facet presentation of P of the following form:

$$P = \{m \in M_\mathbb{R} | \langle m, u_F \rangle \geq -a_F \text{ over all facets } F\}$$

Now, using this, we can see that the divisor $D_P = \sum_{\text{for all facets } F} a_F D_F$ is a torus invariant Cartier divisor. Furthermore, the following statements are true about D_P:

Proposition 2.2.2.3.

1. D_P is an ample divisor, which becomes very ample if and only if P is a very ample polytope.
2. For $\dim(M_\mathbb{R}) = n \geq 2$, the divisor kD is very ample for all $k \geq n - 1$.

This tells us how very ample polytopes give rise to very ample divisors in the corresponding projective toric variety.

2.2.3 Torus Equivariant Vector Bundles

In this section, we refer to chapter 2 of [Pay08] to discuss Klyachko's classification of torus equivariant vector bundles over toric varieties. We first begin with how we construct a filtration of vector spaces, given a torus equivariant vector bundle over a toric variety.

We first begin with a vector bundle V over a toric variety X_Σ, which has an action of the torus T_N on it such that the following diagram commutes for all $t \in T_N$:

$$\begin{array}{ccc} V & \xrightarrow{t\cdot} & t \cdot V \\ \downarrow & & \downarrow \\ X_\Sigma & \xrightarrow{t\cdot} & t \cdot X_\Sigma \end{array}$$

Such vector bundles are called torus equivariant vector bundles over the toric variety X_Σ. With an action of T_N on both V and X, we can define an action on the space of global sections $\Gamma(X_\Sigma, V)$ as follows, for each $s \in \Gamma(X_\Sigma, V)$:

$$\cdot : T_N \times \Gamma(X_\Sigma, V) \to \Gamma(X_\Sigma, V)$$

$$(t \cdot s)(p) = t(s(t^{-1}(p)))$$

This action on $\Gamma(X_\Sigma, V)$ leads to a decomposition of $\Gamma(X_\Sigma, V)$ into eigenspaces as follows, since $\Gamma(X_\Sigma, V)$ is a $\mathbb{C}$- vector space, by 1.1.1.3:

$$\Gamma(X_\Sigma, V) = \bigoplus_{m \in M} \Gamma(X_\Sigma, V)_m$$

Where $\Gamma(X_\Sigma, V)_m$ is the eigenspace where $t \cdot s(p) = \chi^m(t)s(p)$. Now to build the filtration of vector spaces, we first consider the fiber of the identity x_0 of the torus in X_Σ over V. Call this vector space E. Now, we have a map from $\Gamma(X_\Sigma, V)_m$ to E, by evaluation at x_0. Let the image of this map be $E_m \subset E$. Similarly, we have a map $\Gamma(U_\sigma, V)_m \hookrightarrow E$, by evaluation at x_0, whose image we denote by $E_m^\sigma \subset E$.

Now, we describe a map $\Gamma(U_\sigma, V)_m \to \Gamma(U_\sigma, V)_{m-m'}$, obtained by multiplication by $\chi^{m'}$. This is because if we have that $s \in \Gamma(U_\sigma, V)_m$, $(t \cdot s)(p) = \chi^m(t)s(p)$. Hence, we get that:

$$t \cdot (\chi^{m'}(p)s(p)) = t(\chi^{m'}(t^{-1}p)s(t^{-1}p)) = \chi^{-m'}(t)\chi^{m'}(p)\chi^m(t)s(p) = \chi^{m-m'}(t)\chi^{m'}(p)s(p)$$

Therefore, $\chi^{m'}s \in \Gamma(U_\sigma, V)_{m-m'}$. Hence, we get the following commutative diagram:

$$\begin{tikzcd}
\Gamma(U_\sigma, V)_m \arrow[r, "\chi^{m'}"] \arrow[d, "s(x_o)"] & \Gamma(U_\sigma, V)_{m-m'} \arrow[d, "s(x_0)"] \\
E \arrow[r, equal] & E
\end{tikzcd}$$

Hence we see that $E_m^\sigma \subset E_{m-m'}^\sigma$. This inclusion becomes an isomorphism for $m' \in \sigma^\perp \cap M$. We can now define the filtration of vector spaces on E by considering the rays $\rho \in \Sigma(1)$. For these cones, we define $E^\rho(i) = E_{m_i}^\rho$, where $m_i \in M$ such that $\langle m_i, u_\rho \rangle = i$. Then we get the following decreasing filtration:

$$\cdots \supset E^\rho(i-1) \supset E^\rho(i) \supset E^\rho(i+1) \supset \cdots$$

Another important aspect is that using 2.1.2.3, we can say that all torus equivariant line bundles over an affine toric variety U_σ arise as divisors of characters χ^m for $m \in M$, and are unique modulo $\sigma^\perp \cap M$. Moreover, the underlying line bundle is trivial. We then get the following result about torus equivariant vector bundles over an affine toric variety:

Proposition 2.2.3.1. Every toric vector bundle over an affine toric variety U_σ splits equivariantly as a sum of toric line bundles over U_σ. And therefore, we have a bijection between finite multisets over $M/(\sigma^\perp \cap M)$ and toric vector bundles over U_σ.

We end this section with a statement of Klyachko's classification of toric vector bundles, before which we describe a morphism ϕ between two vector spaces E and F with decreasing filtrations $\{E^\rho(i)\}$ and $\{F^\rho(i)\}$ indexed by rays $\rho \in \Sigma(1)$. ϕ is a linear map from E to F, which takes $E^\rho(i)$ into $F^\rho(i)$ for every $i \in \mathbb{Z}$ and $\rho \in \Sigma(1)$.

Proposition 2.2.3.2. The category of toric vector bundles over the toric variety X_Σ is naturally equivalent to the category of $\mathbb{C}$ vector spaces with collections of decreasing filtrations indexed by $\rho \in \Sigma(1)$, which satisfy the following compatibility criterion: For each cone $\sigma \in \Sigma$, there is a decomposition of $E = \bigoplus_{[m]\in M/\sigma^\perp \cap M} E_{[m]}$, such that:

$$E_\rho(i) = \sum_{[m] \mid \langle m, u_\rho \rangle \geq i} E_{[m]}$$

for every $\rho \in \sigma(1)$ and $i \in \mathbb{Z}$.

Bibliography

[CLS11] D.A. Cox, J.B. Little, and H.K. Schenck. *Toric Varieties*. Graduate studies in mathematics. American Mathematical Soc., 2011.

[Ful93] W. Fulton. *Introduction to Toric Varieties*. Annals of mathematics studies. Princeton University Press, 1993.

[Har77] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1977.

[Pay08] Sam Payne. Moduli of toric vector bundles. *Compositio Mathematica*, 144(5):1199–1213, 2008.