Non-local elliptic equations: existence and multiplicity results

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by

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Dedicated to
My mother

Certificate

Certified that the work incorporated in the thesis entitled "*Non-local elliptic equations: existence and multiplicity results*", submitted by *Debangana Mukherjee* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Date: July 18, 2018 *Dr. Mousomi Bhakta*

Thesis Supervisor

Declaration

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Debangana Mukherjee

Contents

Abstract

The main theme of this thesis is based on non-local type elliptic equations. In particular, existence of infinitely many nontrivial solutions for a class of equations driven by non-local integro-differential operator $\mathscr{L}_K$ with concave-convex nonlinearities and homogeneous Dirichlet boundary conditions in smooth bounded domain in $\mathbb{R}^N$ is shown. Moreover, when $\mathscr{L}_K$ reduces to the fractional Laplace operator $(-\Delta)^s$, and the nonlinearity is of critical-concave type, existence of at least one sign changing solution has been established. These are then further generalized to the case of non-local equations with p-fractional Laplace operator. Existence of infinitely many nontrivial solutions for the class of equations with (p,q) fractional Laplace operator and concave-critical nonlinearities have also been studied together with existence of multiple nonnegative solutions when nonlinearity is of convex-critical type.

Also in a different project we have studied the existence/nonexistence/ qualitative properties of the positive solutions of non-local semilinear elliptic equations with critical and supercritical type nonlinearities.

Notation

We collect here a list of notation commonly used in this thesis.

$\mathbb{R}$: *the set of real numbers.*

$\mathbb{N}$: *the set of natural numbers.*

$\mathbb{R}^N$: $N-$*fold cartesian product of* $\mathbb{R}$ *with itself.*

B_r : *Ball in* $\mathbb{R}^N$ *of radius* r *centered at origin.*

$B_r(x)$: *Ball in* $\mathbb{R}^N$ *of radius* r *centered at* x.

$C(\mathbb{R}^N)$: *the set of continuous functions on* $\mathbb{R}^N$.

$C_c(\mathbb{R}^N)$: *the set of continuous functions on* $\mathbb{R}^N$ *with compact support.*

$C_0^\infty(\mathbb{R}^N)$: *the space of smooth functions from* $\mathbb{R}^N \to \mathbb{R}$ *with compact support.*

Δ : *the Laplace Operator defined by* $\Delta u = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} u$ *for any function* $u : \mathbb{R}^N \to \mathbb{R}$.

$(-\Delta)^s$: *the fractional-Laplacian Operator.*

$(-\Delta)_p^s$: *the p-fractional-Laplacian Operator.*

$\mathscr{L}_K$: *integro-differential operator.*

$\|\cdot\|_X$: *Norm in the Banach space* X.

D^α : $\alpha-$ distributional derivative.

p_s^* : the fractional critical Sobolev exponent $\frac{Np}{N-ps}$.

$\mathcal{M}(\mathbb{R}^N)$: the space of finite measures on $\mathbb{R}^N$.

$\square$: end of a proof.

Chapter 1

Introduction

A brief overview of the contents of the thesis is presented here.

The main topic of the thesis is the study of non-local elliptic equations. Fractional and non-local operators of elliptic type has caught considerable attention in the recent decades in both pure mathematics and real world applications. From physical point of view, non-local operators play fundamental role to describe several phenomena, for instance, thin obstacle problem, optimization, phase transition, material science, water wave, mathematical finance, geophysical fluid dynamics etc. To a great extent, the study of equations with integro-differential operator or non-local operator is motivated by real world application. Indeed, there are many situations in which considering a non-local operator yields significantly better model than a local operator. In mathematical finance, it is particularly important to study models involving Lévy process which is non-local in nature. Non-local operators also appear in ecology considering natural phenomena in ecology. In fluid mechanics, an example is given by surface quasi-geostropic equation which is used in oceanography to model the temperature on the surface. In elasticity, an important example is peierls-nabarro arising in crystal dislocation model. In quantum physics, fractional Schrödinger equation is also an important one to consider.

In contrast to classical differential operators, such as Δu, whose value at any point x can be computed by knowing the behavior of u in an arbitrarily small neighborhood of x, where as to define $(-\Delta)^s u$ ($s \in (0,1)$), one needs the information about u in the entire $\mathbb{R}^N$.

In this thesis, we mainly focus on the following problem with general integro-differential operator

$$(P_K)\begin{cases}\mathscr{L}_K u + \mu|u|^{q-1}u + \lambda|u|^{p-1}u = 0 & \text{in} \quad \Omega,\\ u = 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega,\end{cases}$$

where Ω is an open, bounded domain in $\mathbb{R}^N$ with smooth boundary, with parameters μ, λ and p, q will be specified later. $\mathscr{L}_K$ and the Kernel K are defined in Section 2.3.3.

The thesis is divided into four parts. In the first part, we have shown the existence of infinitely many nontrivial solutions for a class of elliptic equations driven by general integro-differential operator $\mathscr{L}_K$ and concave-convex type nonlinearities. In the second part, existence of at least one sign-changing solution is shown when $\mathscr{L}_K$ is reduced to $(-\Delta)^s$ and the nonlinearity is of concave-critical type. Also, we have generalized the results of first part in the case of p-fractional type equations. The third part consists of existence and multiplicity results of non-negative solutions for the class of (p, q) fractional Laplace equations with convex-critical nonlinearities. All these three parts are studied in bounded domains of $\mathbb{R}^N$ with homogeneous Dirichlet boundary conditions. In the last part, we have discussed various qualitative properties of the positive solution to fractional Laplace equations in $\mathbb{R}^N$ with critical and super-critical nonlinearities .

(I) **Multiplicity results of elliptic equations with operator $\mathscr{L}_K$**

An interesting problem in partial differential equations is whether one can show existence of infinitely many solutions. First, we show existence of weak solutions using variational formulation. Variational Methods (or Calculus of Variations) are useful techniques to prove existence of solutions of differential equations. The main idea is to convert the problem of solving equations into the problem of finding critical points (i.e. minimum/maximum points or saddle points) of a functional, and each critical point usually corresponds to a weak solution. However, it is sometimes very difficult to find out such critical points as we seek for critical points in an infinite-dimensional function space.

A classical topic in nonlinear analysis is the study of existence and multiplicity of solutions for nonlinear equations. There are many results on the subject of concave-convex nonlinearity involving different local and non-local operators. Elliptic problems in bounded domains involving concave and convex terms have been studied extensively since Ambrosetti, Brezis and Cerami [2] considered the following equation:

$$(E_\mu)\begin{cases} -\Delta u = & \mu u^{q-1} + u^{p-1} \quad \text{in} \quad \Omega, \\ u > & 0 \quad \text{in} \quad \Omega, \\ u = & 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $1 < q < 2 < p \leq \frac{2N}{N-2}$, $\mu > 0$ and Ω is a bounded domain in $\mathbb{R}^N$. They found existence of $\mu_0 > 0$ such that (E_μ) admits at least two positive solutions for $\mu \in (0, \mu_0)$, one positive solution for $\mu = \mu_0$ and no positive solution exists for $\mu > \mu_0$ (see also Ambrosetti, Azorero and Peral [3] for more references therein). Later on Adimurthi-Pacella-Yadava [1], Damascelli, Grossi and Pacella [36], Ouyang and Shi [67] and Tang [81] proved existence of $\mu_0 > 0$ such that for $\mu \in (0, \mu_0)$, there are exactly two positive solutions of (E_μ) when Ω is the unit ball in $\mathbb{R}^N$ and exactly one positive solution for $\mu = \mu_0$ and no positive solution exists for $\mu > \mu_0$. For the local operator we also quote [11, 21, 29, 31, 46, 87] and the references therein. In past couple of years many of these results have been generalised to the case of non-local operators, we refer a few among them [9, 22, 39, 63, 68] and the references therein. We also quote here a very important paper by Chen, Li and Ou [33], where the authors have classified all the positive solutions of the fractional Yamabe equation.

We have proved the existence of infinitely many solutions of the equation (P_K) when $0 < q < 1 < p$ and p is either critical or subcritical.

(II) **Existence of sign-changing solution**

In the last two decades, much attention has been given to the study of sign-changing solutions of nonlinear elliptic equations. There are richer structures of sign-changing solutions than that of positive and negative solutions for generic nonlinear and linear elliptic equations. To find sign-changing solutions are interesting challenges mathematically compared with positive and negative solutions because of the number and shapes of nodal domains and the measure of nodal sets. In practice, to find sign-changing solutions is an easy task for ordinary differential equations since one may count the number of zeros of solutions to select and to distinguish sign-changing solutions. One cannot implement such an idea to partial differential equations since the nodal set of a sign-changing solution of a partial differential equation may be very complicated.

In [56], the eigenvalue problem associated with $(-\Delta)_p^s$ has been studied. Some results about the existence of solutions have been considered in [48, 50, 56].

On the other hand, the non-local nonlinear problems associated with $(-\Delta)_p^s$ for $p = 2$ have been investigated by many researchers, see for example [76] for the subcritical case and [9, 16, 78] for the critical case. In [22] the authors studied the non-local equation involving a concave-convex nonlinearity in the subcritical case.

In the local case $s = 1$, equations with concave-convex nonlinearities were studied by many authors, to mention few, see [2, 3, 11, 29]. When $s = 1$ and $p = 2$, existence of sign changing solution was studied in [31].

In [47], Goyal and Sreenadh studied the existence and multiplicity of non-negative solutions of p-fractional equations with subcritical concave-convex nonlinearities. In [27], Chen and Squassina have studied the concave-critical system of equations with the $p-$fractional Laplace operator.

We have proved existence of at least one sign-changing solution for the problem (P_K) where $\mathscr{L}_K = (-\Delta)^s, p = 2^*, q$ and μ are lying in certain range of intervals.

We have further generalized this result in the case of p-fractional Laplace equations.

(III) **Multiplicity results for $(p,\, q)$ fractional Laplace equations**

In this section, we have discussed the existence of multiple nontrivial solutions of $(p,\, q)$ fractional Laplacian equations involving concave-critical type nonlinearities and existence of nonnegative solutions when nonlinearities is of convex-critical type. More precisely, we have considered equations of the type

$$(P_{\theta,\lambda}) \begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \theta V(x)|u|^{r-2}u + |u|^{p^*_{s_1}-2}u + \lambda f(x,u), & \text{in} \quad \Omega, \\ u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain, $\lambda,\, \theta > 0$, $0 < s_2 < s_1 < 1$, $1 < r < q < p < \frac{N}{s_1}$ and $p^*_s = \frac{Np}{N-sp}$ for any $s \in (0,1)$. The functions f and V satisfy certain assumptions, which have been made precise later.

For $s_1 = s_2 = 1$, the problem reduces to the $(p,\, q)$ Laplacian problem which appears in more general reaction-diffusion system

$$u_t = \operatorname{div}\Big(a(u)\nabla u\Big) + g(x,u), \tag{1.0.1}$$

where $a(u) = |\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u$. This system has a wide range of applications in Physics which include biophysics, plasma physics and chemical reaction-diffusion system, etc. In such applications, the function u describes a concentration, the first term on the right-hand side of (1.0.1) corresponds to the diffusion with a diffusion coefficient $a(u)$ and the second one is the reaction and relates to sources and loss processes. Typically, in chemical and biological applications, the reaction term $g(x,u)$ has a polynomial form with respect to the concentration u. Consequently, quasilinear elliptic boundary value problems involving this operator have been widely studied in the literature (see e.g., [12, 58, 59] and the references there-in). In particular, proving the existence and multiplicity of nontrivial solutions and nonnegative solutions were of major interest in many articles, see [28, 55, 88, 89] and the references there-in.

When $p = q$ and $s_1 = s_2$, $(P_{\theta,\lambda})$ reduces to p-fractional type equations with concave-convex nonlinearities. In recent years, existence and multiplicity result for nontrivial, positive and sign-changing solutions for the p-fractional type equations with concave-convex nonlinearities have gained considerable interest. In this regard we cite some of the related recent works [15, 21, 27, 32, 47] (also see the references there-in).

In the non-local case $s \in (0,1)$ and $p, q > 1$, equations with $(p,\ q)$ fractional Laplacian and superlinear nonlinearities have also started gaining interest very recently. In this regard, we mention some of the very recent works [5, 30, 45].

We have proved existence of infinitely many nontrivial solutions of $(P_{\theta,\lambda})$ involving concave-critical nonlinearities. Also, when the nonlinearity is of convex-critical type, we have established the multiplicity of nonnegative solutions.

(IV) **Qualitative properties of solutions**

In this section, we have studied the following problem:

$$\begin{cases}(-\Delta)^s u = u^p - u^q \quad \text{in} \quad \mathbb{R}^N, \\ \qquad u \in \dot{H}^s(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N), \\ \qquad u > 0 \quad \text{in} \quad \mathbb{R}^N,\end{cases} \tag{1.0.2}$$

where $s \in (0,1)$ is a fixed parameter, $(-\Delta)^s$ is the fractional Laplacian in $\mathbb{R}^N$, $q > p \geq \frac{N+2s}{N-2s}$ and $N > 2s$. When $s = 1$, it follows by celebrated Pohozaev identity that (1.0.2) does not have any solution when $p = 2^* - 1$ and $q > p$. In this section, we have proved this result for all $s \in (0,1)$ by establishing the Pohozaev identity in $\mathbb{R}^N$ for the equation (1.0.2). We recall that (1.0.2) has an equivalent formulation by Caffarelli-Silvestre harmonic extension method in $\mathbb{R}^{N+1}_+$. For spectral fractional laplace equation in bounded domain, some Pohozaev type identities were proved in [25, 26]. In [43], Fall and Weth have proved some nonexistence results associated with the problem $(-\Delta)^s u = f(x,u)$ in Ω and $u = 0$ in $\mathbb{R}^N \setminus \Omega$ by applying method of moving spheres.

Recently Ros-Oton and Serra [71, Theorem 1.1] have proved Pohozaev identity by direct method for the bounded solution of Dirichlet boundary value problem. More precisely they have proved the following:

Let u be a bounded solution of

$$\begin{cases}(-\Delta)^s u = f(u) \quad \text{in} \quad \Omega, \\ \qquad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega,\end{cases} \tag{1.0.3}$$

where Ω is a bounded $C^{1,1}$ domain in $\mathbb{R}^N$, f is locally Lipschitz and $\delta(x) = \text{dist}(x, \partial\Omega)$. Then u satisfies the following identity:

$$(2s - N)\int_\Omega uf(u)\ dx + 2N\int_\Omega F(u)\ dx = \Gamma(1+s)^2\int_{\partial\Omega}(\frac{u}{\delta^s})^2(x\cdot\nu)dS,$$

where $F(t) = \int_0^t f$ and ν is the unit outward normal to $\partial\Omega$ at x and Γ is the Gamma function. For nonexistence result with general integro-differential operator we cite [72].

To apply the technique of [71] in the case of $\Omega = \mathbb{R}^N$, one needs to know decay estimate of u and ∇u at infinity. In [71], Ros-Oton and Serra have remarked that assuming certain decay condition of u and ∇u, one can show that $(-\Delta)^s u = u^p$ in $\mathbb{R}^N$ does not have any nontrivial solution for $p > \frac{N+2s}{N-2s}$. In this section, for (1.0.2) we have first established decay estimate of u and ∇u at infinity and then using that we have established the Pohozaev identity for the solution of (1.0.2) for all $s \in (0,1)$ and consequently we have the nonexistence of nontrivial solution when $p = 2^* - 1$.

On the contrary to the nonexistence result for $p = 2^* - 1$, we have shown that Eq.(1.0.2) admits a positive solution when $p > 2^* - 1$. Moreover, we have studied the qualitative properties of solutions. More precisely, using Moser iteration technique we have proved that any solution, u, of (1.0.2) is in $L^\infty(\mathbb{R}^N)$ and we have established decay estimate of u and ∇u at infinity. Then using the Schauder estimate from [73] and the L^∞ bound that we have established, we have shown that $u \in C^\infty(\mathbb{R}^N)$ if both p and q are integer and $C^{2ks+2s}(\mathbb{R}^N)$, where k is the largest integer satisfying $\lfloor 2ks \rfloor < p$ if $p \notin \mathbb{N}$ and $\lfloor 2ks \rfloor < q$ if $p \in \mathbb{N}$ but $q \notin \mathbb{N}$, where $\lfloor 2ks \rfloor$ denotes the greatest integer less than equal to $2ks$. We also have proved that u is a classical solution. We further showed that solution of (1.0.2) is radially symmetric.

When Ω is a bounded domain, we have proved that our problem admits a solution for every $p \geq 2^* - 1$. For similar type of equations involving critical and supercritical exponents in the case of local operator such as $-\Delta$, we cite [19], [53], [59], [60]. For similar kind of equations with non-local operator we cite [18, 37].

Let us now explain how the work is divided and the main results in each section. The contents of the thesis mainly corresponds to a paper, or a preprint as follows: which are joint works with my supervisor Dr. Mousomi Bhakta.

- M. Bhakta and D. Mukherjee, *Multiplicity results and sign changing solutions of non-local equations with concave-convex nonlinearities*, Differential and Integral Equations. Vol 30, No. 5-6 (2017), 387–422.

- M. Bhakta and D. Mukherjee, *Semilinear non-local elliptic equations with critical and supercritical exponents*, Commun. Pure Appl. Anal. Vol. 16, No, 5, (2017).

- M. Bhakta and D. Mukherjee, *Sign changing solutions of p-fractional equations with concave-convex nonlinearities*, Topol. Methods Nonlinear Analysis. Volume 51, No. 2, (2018), 511–544.

- M. Bhakta and D. Mukherjee, *Multiplicity results for* (p, q) *fractional Laplace equations involving critical nonlinearities*, (to appear in Adv. Differential Equations), arXiv: 1801.09925

The thesis is organised as follows:

- **Chapter 2** contains the main theoretical backgrounds necessary to introduce non-local equations. We present an overview of non-local operators and non-local equations, particularly the fractional Laplacian and its definition using Fourier Transform. This chapter is written in the spririt of [38, 62].

- **Chapter 3** corresponds to the existence of infinitely many nontrivial solutions of the (P_K) with concave-convex nonlinearities and homogeneous Dirichlet boundary conditions, where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N > 2s$, $s \in (0, 1)$, $0 < q < 1 < p \leq \frac{N+2s}{N-2s}$. We mainly use Fountain and Dual Fountain Theorem to prove multiplicity results. This chapter is a part of the paper [16].

- **Chapter 4** deals with the existence of at least one sign-changing solution. When $\mathscr{L}_K$ reduces to the fractional laplacian operator $-(-\Delta)^s$, $p = \frac{N+2s}{N-2s}$, $\frac{1}{2}(\frac{N+2s}{N-2s}) < q < 1$, $N > 6s$, $\lambda = 1$, we find $\mu^* > 0$ such that for any $\mu \in (0, \mu^*)$, there exists at least one sign changing solution. We use the tools of Nehari manifold and fibering map to obtain the results of sign changing solutions on N and q. The contents of this chapter is a part of the paper [16].

- **Chapter 5** is the continuation of Chapter 4. In this chapter, we study the existence of sign changing solution of the p-fractional problem with concave-critical nonlinearities:

$$\begin{aligned}(-\Delta)^s_p u &= \mu|u|^{q-1}u + |u|^{p^*_s-2}u \quad \text{in} \quad \Omega,\\ u &= 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega,\end{aligned}$$

  where $s \in (0,1)$ and $p \geq 2$ are fixed parameters, $0 < q < p-1$, $\mu \in \mathbb{R}^+$ and $p^*_s = \frac{Np}{N-ps}$. Ω is an open, bounded domain in $\mathbb{R}^N$ with smooth boundary and $N > ps$. The contents of this chapter is borrowed from the paper [15].

- **Chapter 6** corresponds to the existence of infinitely many nontrivial solutions for the class of $(p,\, q)$ fractional elliptic equations involving concave-critical nonlinearities in bounded domains in $\mathbb{R}^N$. Further, when the nonlinearity is of convex-critical type, we have established the multiplicity of nonnegative solutions using variational methods. In particular, using Lusternik-Schinerlmann category theory, we have shown the existence of at least $cat_{\Omega}(\Omega)$ nonnegative solutions. This chapter is based on our work [14].

- **Chapter 7** is the last chapter of the dissertation. In this chapter, we have studied the existence/nonexistence/qualitative properties of the positive solutions of non-local semilinear elliptic equations with critical and supercritical type nonlinearities. This chapter is based on the paper [17].

Chapter 2

Fractional Framework

Partial Differential Equations are, in general, relations between the values of an unknown function and its derivatives of different orders. To see whether a partial differential equation is true at a particular point, one needs only the values of the function in an arbitrarily small neighborhood, so that all derivatives can be computed. In order to check whether a non-local equation holds at a point, data about the values of the function in the entire domain is required. This is because the equation involves integral operators. An example of such operator is

$$\mathscr{L}_K u(x) = P.V. \int_{\mathbb{R}^N} \Big(u(x) - u(x+y)\Big) K(y)dy \tag{2.0.1}$$

for some non-negative symmetric Kernel $K(y) = K(-y)$ satisfying

$$\int_{\mathbb{R}^N} \min\{1, |y|^2\} K(y)dy < +\infty.$$

where P.V. is a commonly used abbreviation for "in the principal value sense" in (2.0.1). When the singularity at the origin of the kernel K is not integrable, these operators are also called integro-differential operators. This is because, due to the singularity of K, the operator (2.0.1) differentiates (in some sense) the function u. The most canonical example of an elliptic integro-differential operator is the fractional Laplacian

$$(-\Delta)^s u(x) = C(N,s) P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy, s \in (0,1). \tag{2.0.2}$$

For details, see Section 2.3.

In recent years, a great deal of attention has been devoted to fractional and non-local operators of elliptic type. One of the main reasons comes from the fact that this operator naturally arises in several physical phenomenon like flames propagation and chemical reaction of liquids, population dynamics, geophysical fluid dynamics, mathematical finance etc (see [6,13,34,84,85] and the references therein). In this chapter, we will address the definition and some properties of the fractional Laplace operator. This chapter is written in the spirit of [62] and [38]. We have omitted the proofs.

2.1 Fourier transform of tempered distributions

In this section, we will briefly discuss the notion of Fourier transform of a tempered distribution. Let $\mathscr{S}$ denotes the Schwartz space of rapidly decaying $C^\infty(\mathbb{R}^N)$ functions whose topology is generated by the seminorms $\{p_j\}_{j\in\mathbb{N}}$ defined as:

$$p_j(\phi) := \sup_{x\in\mathbb{R}^N} (1+|x|)^j \sum_{|\alpha|\leq j} |D^\alpha \phi(x)|,$$

where $\phi \in \mathscr{S}(\mathbb{R}^N)$. More precisely, $\mathscr{S}$ contains the smooth functions ϕ satisfying

$$\sup_{x\in\mathbb{R}^N} |x^\alpha D^\beta \phi(x)| < +\infty,$$

for all multi-indices α and β.

We denote the Fourier transform of a function $\phi \in \mathscr{S}$ by

$$\mathscr{F}\phi(\xi) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i\xi.x} \phi(x) dx. \tag{2.1.1}$$

We note that, for every $\phi \in \mathscr{S}$, we have $\mathscr{F}\phi \in \mathscr{S}$. The inverse Fourier transform is given by

$$\mathscr{F}^{-1}\phi(x) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{ix.\xi}\phi(\xi)d\xi. \tag{2.1.2}$$

Notice that the Fourier transform (2.1.1) and the inverse Fourier transform (2.1.2) are both continuous from $\mathscr{S}(\mathbb{R}^N)$ into $\mathscr{S}(\mathbb{R}^N)$ and is an isomorphism and a homeomorphism of $\mathscr{S}(\mathbb{R}^N)$ onto $\mathscr{S}(\mathbb{R}^N)$.

Now, let $\mathscr{S}'$ be the topological dual of $\mathscr{S}$. A tempered distribution is an element of $\mathscr{S}'$. If $T \in \mathscr{S}'$, the Fourier transform of T can be defined as the tempered distribution given by

$$\langle \mathscr{F}T, \phi \rangle := \langle T, \mathscr{F}\phi \rangle ,$$

for every $\phi \in \mathscr{S}$, where $\langle \cdot, \cdot \rangle$ denotes the usual duality bracket between $\mathscr{S}$ and its dual $\mathscr{S}'$. Using (2.1.1), we have

$$u \in L^2(\mathbb{R}^N) \quad \text{if and only if} \quad \mathscr{F}u \in L^2(\mathbb{R}^N) \tag{2.1.3}$$

and

$$\|u\|_{L^2(\mathbb{R}^N)} = \|\mathscr{F}u\|_{L^2(\mathbb{R}^N)}, \tag{2.1.4}$$

for every $u \in L^2(\mathbb{R}^N)$. Formula (2.1.4) is the so-called Parseval-Plancherel formula which will be used to establish the equivalence between the fractional spaces $H^s(\mathbb{R}^N)$ and $\hat{H}^s(\mathbb{R}^N)$ (see Proposition 2.3.2).

2.2 Fractional Sobolev spaces

Let Ω be an open, smooth set in $\mathbb{R}^N$ and $p \in [1, +\infty)$. For any $s > 0$, we would define the fractional Sobolev space $W^{s,p}(\Omega)$. If $s \geq 1$ is a positive integer, $W^{s,p}(\Omega)$ denotes the classical Sobolev space equipped with the standard norm

$$\|u\|_{W^{s,p}(\Omega)} := \sum_{0 \leq |\alpha| \leq s} |D^\alpha u|_{L^p(\Omega)},$$

for every $u \in W^{s,p}(\Omega)$. We will look into the cases where $s \notin \mathbb{N}$. Now, for a fixed $s \in (0,1)$, the Sobolev space $W^{s,p}(\Omega)$ is defined as:

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{N}{p}+s}} \in L^p(\Omega \times \Omega) \right\} \tag{2.2.1}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u(x)|^p dx + \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}}, \tag{2.2.2}$$

where the term

$$[u]_{W^{s,p}(\Omega)} := \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}} \tag{2.2.3}$$

is the *Gagliardo seminorm* of u.

2.2.1 Embedding results

This subsection deals with the embeddings of fractional Sobolev spaces into Lebesgue spaces. Some basic facts are recalled briefly. For details, see [38, Sections 6 and 7], [62, Section 1].

Proposition 2.2.1. *Let $p \in [1, +\infty)$ and let Ω be an open set in $\mathbb{R}^N$. Then the following assertions hold true:*

(a) If $0 < s \leq s' < 1$, then the embedding

$$W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$$

is continuous. Hence, there exists a constant $C_1(N, s, p) \geq 1$ such that

$$\|u\|_{W^{s,p}(\Omega)} \leq C_1(N, s, p)\|u\|_{W^{s',p}(\Omega)},$$

for any $u \in W^{s',p}(\Omega)$.

(b) If $0 < s < 1$ and Ω is of class $C^{0,1}$ (that is, with the Lipschitz boundary) and with bounded boundary $\partial\Omega$, then the embedding

$$W^{1,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$$

is continuous. Hence, there exists a constant $C_2(N, s, p) \geq 1$ *such that*

$$\|u\|_{W^{s,p}(\Omega)} \leq C_2(N, s, p)\|u\|_{W^{1,p}(\Omega)},$$

for any $u \in W^{1,p}(\Omega)$.

(c) If $s' \geq s > 1$ *and* Ω *is of class* $C^{0,1}$, *then the embedding*

$$W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$$

is continuous.

Proof. For proofs, see Proposition 2.1, Proposition 2.2 and Corollary 2.3 in [38]. □

Now let us recall some basic properties about continuous (compact) embeddings of the fractional Sobolev spaces $W^{s,p}$ into Lebesgue spaces. Here, we will discuss three different cases, $sp < N$, $sp = N$ and $sp > N$. For proof, we refer [38, Sections 6-8].

Case 1: $sp < N$

Theorem 2.2.2. *Let* $s \in (0, 1)$ *and* $p \in [1, +\infty)$ *such that* $sp < N$. *Then there exists a positive constant* $C := C(N, p, s)$ *such that, for any* $u \in W^{s,p}(\mathbb{R}^N)$,

$$|u|^p_{L^{p^*_s}(\mathbb{R}^N)} \leq C \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy,$$

where the exponent

$$p^*_s := \frac{Np}{N - ps}$$

is the so-called fractional critical exponent . Consequently, the space $W^{s,p}(\mathbb{R}^N)$ *is continuously embedded in* $L^q(\mathbb{R}^N)$ *for any* $q \in [p, p^*_s]$. *Moreover, the embedding* $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q_{loc}(\mathbb{R}^N)$ *is compact for every* $q \in [p, p^*_s)$.

In an extension domain Ω, the following embedding result holds:

Theorem 2.2.3. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp < N$. Let $\Omega \subset \mathbb{R}^N$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C := C(N,p,s,\Omega)$ such that, for any $u \in W^{s,p}(\Omega)$,*

$$|u|_{L^q(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)},$$

for any $q \in [p, p_s^*]$; that is, the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [p, p_s^*]$. If, in addition, Ω is bounded, then the space $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q \in [1, p_s^*)$.

Case 2: $sp = N$

Theorem 2.2.4. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp = N$. Then there exists a positive constant $C := C(N,p,s)$ such that for any $u \in W^{s,p}(\mathbb{R}^N)$,*

$$|u|_{L^q(\mathbb{R}^N)} \leq C\|u\|_{W^{s,p}(\mathbb{R}^N)},$$

for any $q \in [p,+\infty)$; that is, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p,+\infty)$.

For an extension domain Ω, we have the following embedding result:

Theorem 2.2.5. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp = N$. Let $\Omega \subset \mathbb{R}^N$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C := C(N,p,s,\Omega)$ such that, for any $u \in W^{s,p}(\Omega)$,*

$$|u|_{L^q(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)},$$

for any $q \in [p,+\infty)$; that is, the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [p,+\infty)$. If, in addition, Ω is bounded, then the space $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q \in [1,+\infty)$.

Case 3: $sp > N$

We denote by $C^{0,\alpha}(\Omega)$ the space of Hölder continuous functions endowed with the standard norm

$$\|u\|_{C^{0,\alpha}(\Omega)} := |u|_{L^\infty(\Omega)} + \sup_{x,y\in\Omega, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

Theorem 2.2.6. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp > N$. Let Ω be a $C^{0,1}$ domain of $\mathbb{R}^N$. Then there exists a positive constant $C := C(N,p,s,\Omega)$ such that for any $u \in W^{s,p}(\Omega)$, we have,*

$$\|u\|_{C^{0,\alpha}(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)},$$

with $\alpha := (sp-N)/p$; that is, the space $W^{s,p}(\Omega)$ is continuously embedded in $C^{0,\alpha}(\Omega)$.

As a consequence of Theorem 2.2.6, we have the following result.

Corollary 2.2.7. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp > N$. Let Ω be a $C^{0,1}$ bounded domain of $\mathbb{R}^N$. Then the embedding*

$$W^{s,p}(\Omega) \hookrightarrow C^{0,\beta}(\Omega)$$

is compact for every $\beta < \alpha$, with $\alpha := (sp-N)/p$.

2.2.2 The Sobolev space $H^s(\Omega)$

This section is devoted to the case $p = 2$ where we deal its relation with the fractional Laplacian. Let Ω be an open subset of $\mathbb{R}^N$ and denote

$$H^s(\Omega) := W^{s,2}(\Omega),$$

for any $s \in (0,1)$. In this case, we note that the preceding fractional Sobolev space turns out to be a Hilbert space. The inner product on $H^s(\Omega)$ is defined by

$$\langle u, v\rangle_{H^s(\Omega)} := \int_\Omega u(x)v(x)dx + \int_{\Omega\times\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy,$$

for any $u, v \in H^s(\Omega)$ induces the norm given in (2.2.2) when $p = 2$. That is, for every $s \in (0, 1)$, we have,

$$H^s(\mathbb{R}^N) := W^{s,2}(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : [u]_{W^{s,2}(\mathbb{R}^N)} < +\infty\}, \tag{2.2.4}$$

where $[\cdot]_{W^{s,2}(\mathbb{R}^N)}$ is defined in (2.2.3).

Alternatively, we can also define the space $H^s(\mathbb{R}^N)$ via a Fourier transform, that is, we define

$$\hat{H}^s(\mathbb{R}^N) := \left\{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |y|^{2s}) |\mathscr{F}u(y)|^2 dx < +\infty\right\}, \tag{2.2.5}$$

for any $s > 0$ and

$$\hat{H}^s(\mathbb{R}^N) := \left\{u \in \mathscr{S}' : \int_{\mathbb{R}^N} (1 + |y|^2)^s |\mathscr{F}u(y)|^2 dx < +\infty\right\},$$

for every $s < 0$.

The equivalence between the space $\hat{H}^s(\mathbb{R}^N)$ defined in (2.2.5) and the one defined by the Gagliardo norm in (2.2.4) is given in Proposition 2.3.2.

2.3 The fractional Laplacian operator

A very popular non-local operator is given by the fractional Laplacian $(-\Delta)^s$ with $s \in (0, 1)$. This operator and its generalization appear in many areas of mathematics, like harmonic analysis, probability theory, potential theory, quantum machanics, statistical physics etc. This section deals with the definition of this operator and its properties.

Let $s \in (0, 1)$ and define the fractional Laplacian operator $(-\Delta)^s : \mathscr{S} \to L^2(\mathbb{R}^N)$ by

$$(-\Delta)^s u(x) := C(N, s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \; x \in \mathbb{R}^N, \tag{2.3.1}$$

where $B_\varepsilon(x)$ is the ball centred at $x \in \mathbb{R}^N$ with radius ε and $C(N, s)$ is the following (positive) normalization constant:

$$C(N, s) := \left(\int_{\mathbb{R}^N} \frac{1 - cos(\xi_1)}{|\xi|^{N+2s}} d\xi\right)^{-1} \tag{2.3.2}$$

with $\xi = (\xi_1, \xi'), \xi' \in \mathbb{R}^{N-1}$. One can also define $(-\Delta)^s$ in the principal-value sense by setting

$$P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy,$$

that is,

$$(-\Delta)^s u(x) := C(N,s) P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \; x \in \mathbb{R}^N. \tag{2.3.3}$$

The next proposition tells us that the singular integral defined in (2.3.3) can be written as a weighted second-order differential quotient.

Proposition 2.3.1. *Let $s \in (0,1)$. Then for any $u \in \mathscr{S}$,*

$$(-\Delta)^s u(x) = -\frac{1}{2} C(N,s) \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \; x \in \mathbb{R}^N. \tag{2.3.4}$$

For proof, see [62, Proposition 1.10].

Remark: Let $s \in (0, 1/2)$. Notice that for any $u \in \mathscr{S}$ and for a fixed $x \in \mathbb{R}^N$, we have that,

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy &\le C \int_{B(x,R)} \frac{|x-y|}{|x-y|^{N+2s}} dy \\ &+ |u|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N \setminus B(x,R)} \frac{1}{|x-y|^{N+2s}} dy \\ &\le C \left(\int_0^R \frac{1}{\rho^{2s}} d\rho + \int_R^{+\infty} \frac{1}{\rho^{2s+1}} d\rho \right) < +\infty, \end{aligned}$$

where C is a positive constant depending only on the dimension N and the L^∞- norm of the function u. So, in the case $s \in (0, 1/2)$, the integral

$$\int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy$$

is not singular near the point x, so one can get rid of the P.V. in (2.3.3).

Proposition 2.3.2. *Let $s \in (0,1)$ and $C(N,s)$ be the constant defined in 2.3.2. Then, for any $u \in H^s(\mathbb{R}^N)$,*

$$[u]^2_{H^s(\mathbb{R}^N)} = 2C(N,s)^{-1} \int_{\mathbb{R}^N} |\xi|^{2s} |\mathscr{F}u(\xi)|^2 d\xi. \tag{2.3.5}$$

Moreover, $H^s(\mathbb{R}^N) = \hat{H}^s(\mathbb{R}^N)$

For proof, see [62, Corollary 1.15].

2.3.1 The fractional p-Laplacian

In recent years, great attention has been devoted to a new non-local and non-linear operator, namely the *fractional* p-Laplacian operator $(-\Delta)_p^s$, for $p \in (1, +\infty)$, $s \in (0, 1)$, and u smooth enough, it is defined as,

$$\begin{aligned}(-\Delta)_p^s u(x) &= P.V. \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} dy \qquad (2.3.6)\\ &= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} dy, \quad x \in \mathbb{R}^N.\end{aligned}$$

Up to some normalization constant depending on N, p, and s, this definition is consistent with one of the fractional Laplacian $(-\Delta)^s$ in the case $p = 2$.

2.3.2 The fractional Laplacian via Fourier transform

In this section, we show that the fractional Laplacian $(-\Delta)^s$ can be viewed as a pseudo-differential operator of symbol $|\xi|^{2s}$ (see [38, Section 3]).

Proposition 2.3.3. *Let* $s \in (0, 1)$. *Then, for any* $u \in \mathscr{S}$,

$$(-\Delta)^s u(x) = \mathscr{F}^{-1}(|\xi|^{2s}(\mathscr{F}u)(\xi))(x), x \in \mathbb{R}^N, \qquad (2.3.7)$$

where $\mathscr{F}^{-1}$ *is the inverse Fourier transform defined in (2.1.2).*

For proof, (see [62, Proposition 1.17]).

The following lemma ensures the relation between the fractional Laplacian operator $(-\Delta)^s$ and the fractional Sobolev space $H^s(\mathbb{R}^N)$ (see [38]).

Proposition 2.3.4. *Let* $s \in (0, 1)$ *and* $C(N, s)$ *be the constant defined in*

(2.3.2). Then , for any $u \in H^s(\mathbb{R}^N)$,

$$[u]^2_{H^s(\mathbb{R}^N)} = 2C(N,s)^{-1}|(-\Delta)^{s/2}u|^2_{L^2(\mathbb{R}^N)}. \tag{2.3.8}$$

For proof, see [62, Proposition 1.18].

2.3.3 A generalization of $(-\Delta)^s$

In this section, we introduce a general integro-differential operator that generalizes $(-\Delta)^s$. For any fixed $s \in (0,1)$, the operator $\mathscr{L}_K$ is given by

$$\mathscr{L}_K u(x) := \int_{\mathbb{R}^N} \Big(u(x+y) + u(x-y) - 2u(x)\Big) K(y)dy, \tag{2.3.9}$$

for every $x \in \mathbb{R}^N$, where the Kernel $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is a function satisfying the following:

$$mK(x) \in L^1(\mathbb{R}^N) \quad \text{with} \quad m(x) = \min\{|x|^2, 1\}; \tag{2.3.10}$$

$$\text{there exists} \quad \theta > 0 \quad \text{such that} \quad K(x) \geq \theta |x|^{-(N+2s)} \text{ for any } x \in \mathbb{R}^N \setminus \{0\}; \tag{2.3.11}$$

$$\text{and} \quad K(x) = K(-x) \quad \text{for any} \quad x \in \mathbb{R}^N \setminus \{0\}. \tag{2.3.12}$$

A model for K is given by the singular kernel $K(x) = |x|^{-(N+sp)}$. In this case $\mathscr{L}_K$ (up to a normalization constant) reduces to the fractional p-Laplace operator $-(-\Delta)^s_p$, defined in (2.3.6) and to the fractional Laplace operator $-(-\Delta)^s$ defined in (2.3.4) when $p = 2$.

2.4 Fractional Sobolev-type space

One of the goals of this chapter is to study non-local problems driven by $(-\Delta)^s$ and its generalization and with Dirichlet boundary data via variational methods. To this purpose, we need to work in a suitable function space. For this, we consider the following functional analytical setting (see [62, Section 1.5]).

Let $s \in (0,1)$ be fixed and Ω be an open-bounded subset of $\mathbb{R}^N$ with $N > 2s$. Define the set Q as:

$$Q := (\mathbb{R}^N \times \mathbb{R}^N) \setminus (\Omega^c \times \Omega^c),$$

where $\Omega^c = \mathbb{R}^N \setminus \Omega$. Furthermore, assume $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ be a function satisfying (2.3.10) and (2.3.11). By $X(\Omega)$ we denote the linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that if $g \in X(\Omega)$ then $g|_\Omega \in L^2(\Omega)$ and

$$(g(x) - g(y))\sqrt{K(x-y)} \in L^2(Q, dxdy).$$

The space $X(\Omega)$ is endowed with the norm defined:

$$\|u\|_{X(\Omega)} = |u|_{L^2(\Omega)} + \left(\int_Q |u(x) - u(y)|^2 K(x-y)dxdy\right)^{1/2}. \tag{2.4.1}$$

Moreover, $X_{0,K}(\Omega) = \left\{u \in X(\Omega) : u = 0 \quad \text{a.e. in} \quad \mathbb{R}^N \setminus \Omega\right\}$ with the norm

$$\|u\|_{X_{0,K}(\Omega)} = \left(\int_Q |u(x) - u(y)|^2 K(x-y)dxdy\right)^{1/2}.$$

With this norm, $X_{0,K}(\Omega)$ is a Hilbert space with the scalar product

$$\langle u, v\rangle_{X_{0,K}(\Omega)} = \int_Q (u(x) - u(y))(v(x) - v(y))K(x-y)dxdy,$$

(see [75, Lemma 7]). For further details on $X(\Omega)$ and $X_{0,K}(\Omega)$ and also for their properties, we refer to [38].

In place of general K, if we have fractional p-Laplacian operator, we define

$$X_{s,p}(\Omega) := \left\{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable } \middle| u|_\Omega \in L^p(\Omega) \text{ and } \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}dxdy < \infty.\right\}$$

The space $X_{s,p}(\Omega)$ is endowed with the norm defined as

$$\|u\|_{X_{s,p}(\Omega)} = |u|_{L^p(\Omega)} + \left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{1/p}.$$

Then, we define $X_{0,s,p}(\Omega) := \left\{u \in X_{s,p}(\Omega) : u = 0 \quad \text{a.e. in} \quad \mathbb{R}^N \setminus \Omega\right\}$ or equivalently as $\overline{C_c^\infty(\Omega)}^{X_{s,p}(\Omega)}$ and for any $p > 1$, $X_{0,s,p}(\Omega)$ is a uniformly convex Banach space (see [47]) endowed with the norm

$$\|u\|_{X_{0,s,p}(\Omega)} = \left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{1/p}.$$

Since $u = 0$ in $\mathbb{R}^N \setminus \Omega$, the above integral can be extended to all of $\mathbb{R}^N$. The embedding $X_{0,s,p}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for any $r \in [1, p_s^*]$ and compact for $r \in [1, p_s^*)$. For further details on $X_{0,s,p}(\Omega)$ and it's properties we refer [38]. In the case $p = 2$, for the sake of convenience, we denote the fractional space $X_0(\Omega) = X_{0,s,2}(\Omega)$ and the norm as $\|\cdot\|_{X_0(\Omega)}$. In the next result we give some connections between the space $X_{0,K}(\Omega)$ and the usual fractional Sobolev spaces $H^s(\mathbb{R}^N)$.

Lemma 2.4.1. *The following assertions hold true.*

(a) Let $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ satisfies (2.3.10),(2.3.11) and (2.3.12). Then $X_{0,K}(\Omega) \subset H^s(\mathbb{R}^N)$ and moreover,

$$\|v\|_{H^s(\Omega)} \leq \|v\|_{H^s(\mathbb{R}^N)} \leq \|v\|_{X(\Omega)},$$

where $c(\theta) = \max\{1, \theta^{-1/2}\}$ with θ given in (2.3.11).

(b) Let $K(x) = |x|^{-(N+2s)}$. Then

$$X_{0,K}(\Omega) = \{v \in H^s(\mathbb{R}^N) : v = 0 \quad a.e \ in \quad \mathbb{R}^N \setminus \Omega\}.$$

For proof, see [[78],lemma 7]. Now, we consider the function

$$X_{0,K}(\Omega) \ni v \mapsto \|v\|_{X_{0,K}(\Omega)} = \left(\int_Q |v(x)-v(y)|^2 K(x-y)dxdy\right)^{1/2} \tag{2.4.2}$$

and we take (2.4.2) as norm on $X_{0,K}$.

Lemma 2.4.2. *Let* $s \in (0,1), N > 2s$ *and* $K : \mathbb{R}^N \setminus \{0\} \to (0,+\infty)$ *satisfy (2.3.10),(2.3.11) and (2.3.12). Then*

(a) there exists a constant $c > 1$*, depending only on* N, s, θ *and* Ω *such that for any* $v \in X_{0,K}$*,*

$$\int_Q |v(x)-v(y)|^2 K(x-y)dxdy \leq \|v\|_X^2 \leq c \int_Q |v(x)-v(y)|^2 K(x-y)dxdy,$$

that is, (2.4.2) defines a norm on $X_{0,K}$ *equivalent to the usual one given in (2.4.1).*

(b) $(X_{0,K}, \|.\|_{X_{0,K}})$ *is a Hilbert space with the scalar product*

$$\langle u, v\rangle_{X_{0,K}} = \int_Q (u(x) - u(y))(v(x) - v(y))K(x - y)dxdy.$$

For proof, see [75, Lemmas 6 and 7].

Let us look into some results related to the embeddings of the spaces $X_{0,K}$ and $H^s(\mathbb{R}^N)$ into the usual Lebesgue spaces, explained in the following results.

Lemma 2.4.3. *Let* $s \in (0,1), N > 2s$ *and* $K : \mathbb{R}^N \setminus \{0\} \to (0,+\infty)$ *satisfy (2.3.10),(2.3.11) and (2.3.12). Then the following assertions hold true:*

(a) if Ω *has a Lipschitz boundary, then the embedding* $X_{0,K} \hookrightarrow L^\gamma(\mathbb{R}^N)$ *is compact for any* $\gamma \in [1, 2^*)$*;*

(b) the embedding $X_{0,K} \hookrightarrow L^{2^*}(\mathbb{R}^N)$ *is continuous.*

For proof, see [78, Lemma 9].

Thanks to the above lemma, we can define the positive constant S_K given by

$$S_K := \inf_{u \in X_{0,K}\setminus\{0\}} S_K(u), \tag{2.4.3}$$

where, for any $u \in X_{0,K} \setminus \{0\}$,

$$S_K(u) := \frac{\displaystyle\int_{\mathbb{R}^N \times \mathbb{R}^N} |u(x) - u(y)|^2 K(x-y) dx dy}{\left(\displaystyle\int_{\Omega} |u(x)|^{2^*} dx \right)^{2/2^*}}. \tag{2.4.4}$$

Note that since in formula (2.4.4) this integral over Ω can be extended to all $\mathbb{R}^N$(being $u = 0$ a.e. in Ω^c), then the function $u \to S_K(u)$ does not depend on the domain Ω, while, in general, S_K does. The counterpart of the above lemma in the usual functional Sobolev spaces is given by the following result proved in [38, Theorem 6.5].

For $s \in (0,1)$, define

$$\dot{W}^{s,p}(\mathbb{R}^N) := \left\{ u \in L^{p_s^*}(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dx dy < \infty \right\}$$

and

$$S_{s,p} = \inf_{u \in \dot{W}^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dx dy}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{p_s^*} \right)^{\frac{p}{p_s^*}}}. \tag{2.4.5}$$

For $p = 2$, we denote $S_{s,2}$ as S_s for the sake of simplicity.

2.5 Harmonic extension to the upper half-space

In this section we recall the other useful representation of fractional laplacian $(-\Delta)^s$, which we will use to prove decay estimate of solution at infinity. Using the celebrated Caffarelli and Silvestre extension method, (see [27]), fractional laplacian $(-\Delta)^s$ can be seen as a trace class operator (see [8, 27, 47]).

Let $u \in \dot{H}^s(\mathbb{R}^N)$ be a solution of the problem

$$(-\Delta)^s u = f(u) \quad \text{in} \quad \mathbb{R}^N. \tag{2.5.1}$$

Define $w := E_s(u)$ be its s- harmonic extension to the upper half space $\mathbb{R}^{N+1}_+$, that is, there is a solution to the following problem:

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla w) = 0 & \text{in} \quad \mathbb{R}^{N+1}_+, \\ w = u & \text{on} \quad \mathbb{R}^N \times \{y = 0\}. \end{cases} \tag{2.5.2}$$

Define the space $X^{2s}(\mathbb{R}^{N+1}_+) :=$ closure of $C_0^\infty(\overline{\mathbb{R}^{N+1}_+})$ w.r.t. the following norm

$$\|w\|_{2s} = \|w\|_{X^{2s}(\mathbb{R}^{N+1}_+)} := \left(k_{2s}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\bigtriangledown w|^2 dxdy\right)^{\frac{1}{2}},$$

where $k_{2s} = \frac{\Gamma(s)}{2^{1-2s}\Gamma(1-s)}$ is a normalizing constant, chosen in such a way that the extension operator $E_s : \dot{H}^s(\mathbb{R}^N) \longrightarrow X^{2s}(\mathbb{R}^{N+1}_+)$ is an isometry (up to constants), that is, $\|E_s u\|_{2s} = \|u\|_{\dot{H}^s(\mathbb{R}^N)} = |\,(-\Delta)^s\, u|_{L^2(\mathbb{R}^N)}$. (see [39]). Conversely, for a function $w \in X^{2s}(\mathbb{R}^{N+1}_+)$, we denote its trace on $\mathbb{R}^N \times \{y = 0\}$ as:

$$\operatorname{Tr}(w) := w(x, 0).$$

This trace operator satisfies:

$$\|w(\cdot, 0)\|_{\dot{H}^s(\mathbb{R}^N)} = \|Tr(w)\|_{\dot{H}^s(\mathbb{R}^N)} \le \|w\|_{2s}\,. \tag{2.5.3}$$

Consequently,

$$\left(\int_{\mathbb{R}^N} |u(x)|^{2^*} dx\right)^{\frac{2}{2^*}} \le S(N, s)\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w(x, y)|^2 dxdy. \tag{2.5.4}$$

Inequality (2.5.4) is called the trace inequality. We note that $H^1(\mathbb{R}^{N+1}_+, y^{1-2s})$, up to a normalizing factor, is isometric to $X^{2s}(\mathbb{R}^{N+1}_+)$ (see [47]). In [27], it is shown that $E_s(u)$ satisfies the following:

$$(-\Delta)^s u(x) = \frac{\partial w}{\partial \nu^{2s}} := -k_{2s}\lim_{y\to 0^+} y^{1-2s}\frac{\partial w}{\partial y}(x, y).$$

With this above representation, (2.5.2) can be rewritten as:

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla w) = 0 \quad \text{in} \quad \mathbb{R}^{N+1}, \\ \qquad \dfrac{\partial w}{\partial \nu^{2s}} = f(w(\cdot, 0)) \quad \text{on} \quad \mathbb{R}^N. \end{cases} \tag{2.5.5}$$

Chapter 3

Multiplicity results of elliptic equations with operator $\mathscr{L}_K$

The aim of this chapter is to investigate the existence and multiplicity of weak solutions to non-local equations involving a general integro-differential operator of fractional type with concave-convex nonlinearities. This chapter is based on the paper [16]. In literature, there are many tools to obtain multiplicity results, among them are Lusternik-Schnirelmann category theory, Morse theory, minimax methods, critical point theory (to mention a few). In this chapter, we have proved existence of infinitely many solutions via "Fountain Theorem" and "Dual Fountain Theorem" due to the pioneering works of Bartsch and Willem (see [10, 11, 86]).

In this chapter, we focus our attention on the following equations driven by a non-local integro-differential operator $\mathscr{L}_K$ with concave-convex nonlinearities and homogeneous Dirichlet boundary conditions,

$$(\mathcal{P}_K)\begin{cases}\mathscr{L}_K u+\mu|u|^{q-1}u+\lambda|u|^{p-1}u=0 & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^N\setminus\Omega,\end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $s\in(0,1)$ is fixed, $N>2s$, $0<q<1<p\leq\frac{N+2s}{N-2s}$ and $\mathscr{L}_K$ is given in (2.3.9) with the Kernel

$K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is a function satisfying (2.3.10),(2.3.11) and (2.3.12).

Definition 3.0.1. *We say that $u \in X_{0,K}(\Omega)$ is a weak solution of $(\mathcal{P}_K)$ if*

$$\begin{aligned}\int_{\mathbb{R}^{2N}} (u(x)-u(y))(\phi(x)-\phi(y))K(x-y)dxdy &= \mu \int_\Omega |u(x)|^{q-1}u(x)\phi(x)dx \\ &+ \lambda \int_\Omega |u(x)|^{p-1}u(x)\phi(x)dx\end{aligned}$$

for all $\phi \in X_{0,K}(\Omega)$.

3.1 Variational formulation

The weak solutions of $(\mathcal{P}_K)$ can be found as critical points of the energy functional

$$\begin{aligned}I_\mu^\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)dxdy - \frac{\mu}{q+1}\int_\Omega |u|^{q+1}dx \\ &- \frac{\lambda}{p+1}\int_\Omega |u|^{p+1}dx. \end{aligned} \tag{3.1.1}$$

Thanks to the Sobolev embedding $X_{0,K}(\Omega) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ (see [78, Lemma 9]), I_μ^λ is well defined C^1 functional on $X_{0,K}(\Omega)$. It is well known that there exists a one-to-one correspondence between the weak solutions of $(\mathcal{P}_K)$ and the critical point of I_μ^λ on $X_{0,K}(\Omega)$. We define the best fractional critical Sobolev constant S_K as

$$S_K := \inf_{v \in X_{0,K}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2 K(x-y)dxdy}{\left(\displaystyle\int_\Omega |v(x)|^{2^*}\right)^{2/2^*}}. \tag{3.1.2}$$

3.2 Abstract Theorems

To prove infinitely many nontrivial solutions of the above stated problems, we apply the Fountain Theorem and the Dual Fountain theorem which were proved by Bartsch [10] and Bartsch-Willem [11] respectively (also see [86]).

As usual for critical point theorems, we need to study the compactness properties of the functional together with its geometric features. With respect to the compactness, we need to prove that the functional satisfies the classical Palais-Smale $(\mathrm{PS})_c$ assumption. But observe that $X_{0,K}(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact (see [78, Lemma 9-b]). Hence the $(\mathrm{PS})_c$ condition does not hold globally for all c and we have to prove that the energy level of the corresponding energy functional lies below the threshold of application of the $(\mathrm{PS})_c$ condition.

We start this section by recalling two abstract theorems namely the Fountain theorem and the Dual Fountain Theorem. For this, we need some definitions from [86].

Definition 3.2.1. *The action of a topological group G on a Banach space X is a continuous map*

$$G \times X \longrightarrow X : [g, u] \longrightarrow gu,$$

such that

$$1.u = u, \quad (gh)u = g(hu), \quad u \mapsto gu \quad \textit{is linear.}$$

The action is isometric if $||gu|| = ||u||$. The space of invariant points is defined by

$$\textit{Fix}\ (G) := \{u \in X : gu = u \quad \forall g \in G\}.$$

A set $A \subset X$ is called invariant if $gA = A$ for every $g \in G$. A functional $\varphi : X \longrightarrow \mathbb{R}$ is called invariant if $\varphi \circ g = \varphi$ for every $g \in G$. A map $f : X \longrightarrow X$ is called equivariant if $g \circ f = f \circ g$ for every $g \in G$.

Definition 3.2.2. *Let G be a compact group on Banach space X. Assume that G acts diagonally on V^k*

$$g(v_1, \cdots, v_k) := (gv_1, \cdots, gv_k),$$

where V is a finite dimensional space. The action of G is admissible if every continuous equivariant map $\partial U \longrightarrow V^{k-1}$, where U is an open bounded invariant neighborhood of 0 in V^k, $k \geq 2$, has a zero.

By *Borsuk-Ulam Theorem , the antipodal action of $G := \mathbb{Z}/2$ on $V := \mathbb{R}$ is admissible (see [86, Theorem D.17]).*

We consider the following situation:

(A1) *The compact group G acts isometrically on the Banach space $X = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$, the spaces X_j are invariant and there exists a finite dimensional space V such that, for every $j \in \mathbb{N}$, $X_j \simeq V$ and the action of G on V is admissible.*

Definition 3.2.3. *Let $\varphi \in C^1(X, \mathbb{R})$. We say that $\{u_n\}$ is a Palais-Smale sequence (in short, PS sequence) of φ at level c if $\varphi(u_n) \to c$ and $\varphi'(u_n) \to 0$ in $(X)'$, the dual space of X. Moreover, we say that φ satisfies (PS)$_c$ condition if $\{u_n\}$ is any (PS) sequence in X at level c implies $\{u_n\}$ has a convergent subsequence in X.*

Theorem 3.2.4. *[Fountain Theorem, Bartsch, 1993] Under the assumption (A1), let $\varphi \in C^1(X, \mathbb{R})$ be an invariant functional. If, for every $k \in \mathbb{N}$, there exists $0 < r_k < \rho_k$ such that*

(A2) $a_k := \max_{u\in Y_k, ||u||=\rho_k} \varphi(u) \leq 0$,

(A3) $b_k := \inf_{u\in Z_k, ||u||=r_k} \varphi(u) \to \infty \quad as \quad k \to \infty.$

(A4) φ satisfies (PS)$_c$ condition for every $c > 0$,

then φ has an unbounded sequence of critical values.

Theorem 3.2.5. *[Dual Fountain Theorem, Bartsch-Willem, 1995] Under the assumption (A1), let $\varphi \in C^1(X, \mathbb{R})$ be an invariant functional. If, for every $k \geq k_0$, there exists $0 < r_k < \rho_k$ such that*

(D1) $a_k := \inf_{u \in Z_k, ||u||=\rho_k} \varphi(u) \geq 0,$

(D2) $b_k := \max_{u \in Y_k, ||u||=r_k} \varphi(u) < 0,$

(D3) $d_k := \inf_{u \in Z_k, ||u|| \leq \rho_k} \varphi(u) \to 0 \quad as \quad k \to \infty.$

(D4) For every sequence $u_{r_j} \in X$ and $c \in [d_k, 0)$ such that

$$u_{r_j} \in Y_{r_j}, \quad \varphi(u_{r_j}) \to c \quad and \quad \varphi|'_{Y_{r_j}}(u_{r_j}) \to 0 \quad as \quad r_j \to \infty,$$

contains a subsequence converging to a critical point of φ,

then φ has a sequence of negative critical values converging to 0.

3.3 Existence of infinitely many solutions

3.3.1 Critical Case

First we study the critical case $p = 2^* - 1, \lambda = 1$, that is,

$$(\mathcal{P}'_K) \begin{cases} \mathscr{L}_K u + \mu |u|^{q-1} u + |u|^{2^*-2} u = 0 & in \quad \Omega, \\ u = 0 & in \quad \mathbb{R}^N \setminus \Omega. \end{cases}$$

Theorem 3.3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary, $N > 2s$. Then there exists $\mu^* > 0$ such that for all $\mu \in (0, \mu^*)$, problem $(\mathcal{P}'_K)$ has a sequence of non-trivial solutions $\{u_n\}_{n \geq 1}$ such that $I(u_n) < 0$ and $I(u_n) \to 0$ as $n \to \infty$ where $I(\cdot)$ is the corresponding energy functional associated with $(\mathcal{P}'_K)$.*

Remark 3.3.2. *Here we would like to mention that when $K(x) = |x|^{-(N+2s)}$, it has been proved in [9] that there exists $\Lambda > 0$ such that, $(\mathcal{P}'_K)$ has at least two positive solutions when $\mu \in (0, \Lambda)$, no positive solution when $\mu > \Lambda$ and at least one positive solution when $\mu = \Lambda$. Chen-Deng [32] have proved that*

$(\mathcal{P}'_K)$ *has at least two positive solutions when* $\mu \in (0, \mu_0)$ *for some* $\mu_0 > 0$ *under the assumption that*

$$\text{There exists } u_0 \in X_{0,K}(\Omega) \text{ with } u_0 \geq 0 \text{ a.e. in } \Omega, \text{ such that } \sup_{t\geq 0} I(tu_0) < \frac{s}{N} S_K^{\frac{N}{2s}}. \tag{3.3.1}$$

When $K(x) = |x|^{-(N+2s)}$, *condition* (3.3.1) *can be guaranteed by results of* [78].

We choose an orthonormal basis $\{e_j\}_{j=1}^{\infty}$ *of* $X_{0,K}(\Omega)$ *(see [76]). Next, we consider the antipodal action of* $G := \mathbb{Z}/2$. *Define*

$$X_j = \mathbb{R}e_j, \quad Y_k := \bigoplus_{j=1}^{k} X_j, \quad Z_k := \overline{\bigoplus_{j=k}^{\infty} X_j}. \tag{3.3.2}$$

Lemma 3.3.3. *If* $1 \leq p+1 < 2^*$, *then we have that*

$$\beta_k := \sup_{u \in Z_k, \|u\|_{X_{0,K}(\Omega)}=1} |u|_{L^{p+1}(\Omega)} \to 0 \quad as \quad k \to \infty.$$

Proof. Clearly, $0 < \beta_{k+1} \leq \beta_k$. Thus there exists $\beta \geq 0$, such that $\lim_{k\to\infty} \beta_k = \beta$. By the definition of β_k, for every $k \geq 1$, there exists $u_k \in Z_k$ such that $\|u_k\|_{X_{0,K}(\Omega)} = 1$ and $|u_k|_{L^{p+1}(\Omega)} > \frac{\beta_k}{2}$. Using the definition of Z_k, it follows $u_k \rightharpoonup 0$ in $X_{0,K}(\Omega)$. Therefore Sobolev embedding implies $u_k \to 0$ in $L^{p+1}(\Omega)$ and this completes the proof. □

Proof of Theorem 3.3.1

Proof. The energy functional associated to $(\mathcal{P}'_K)$ is the following

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)dxdy - \frac{\mu}{q+1}\int_\Omega |u|^{q+1}dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx \\ &= \frac{1}{2}||u||^2_{X_{0,K}(\Omega)} - \frac{\mu}{q+1}\int_\Omega |u|^{q+1}dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx, \end{aligned} \tag{3.3.3}$$

where $\mu > 0$. We will show that I satisfies all the assumptions of Theorem 3.2.5. X_j, Y_j, Z_j are chosen as in (3.3.2) and $G := \mathbb{Z}/2$. Therefore (A1) is satisfied.

Next to check (D1) holds, we define

$$\beta_k := \sup_{u \in Z_k, \|u\|_{X_{0,K}(\Omega)}=1} |u|_{L^{q+1}(\Omega)}, \quad c := \sup_{u \in X_{0,K}(\Omega), \|u\|_{X_{0,K}(\Omega)}=1} |u|^{2^*}_{L^{2^*}(\Omega)}$$

, and $R := \left(\frac{2^*}{4c}\right)^{\frac{1}{2^*-2}}$. It is easy to see, $||u||_{X_{0,K}(\Omega)} \leq R$ implies $\frac{c}{2^*}||u||^{2^*}_{X_{0,K}(\Omega)} \leq \frac{1}{4}||u||^2_{X_{0,K}(\Omega)}$. Therefore for $u \in Z_k,\ \|u\|_{X_{0,K}(\Omega)} \leq R$, we have

$$\begin{aligned} I(u) &\geq \frac{\|u\|^2_{X_{0,K}(\Omega)}}{2} - \frac{\mu}{q+1}\beta_k^{q+1}\|u\|^{q+1}_{X_{0,K}(\Omega)} - \frac{c}{2^*}\|u\|^{2^*}_{X_{0,K}(\Omega)} \\ &\geq \frac{\|u\|^2_{X_{0,K}(\Omega)}}{4} - \frac{\mu}{q+1}\beta_k^{q+1}\|u\|^{q+1}_{X_{0,K}(\Omega)} \end{aligned} \tag{3.3.4}$$

Choose $\rho_k := (\frac{4\mu\beta_k^{q+1}}{q+1})^{\frac{1}{1-q}}$. Using Lemma 3.3.3, we see that $\beta_k \to 0$ as $k \to \infty$. As a consequence $\rho_k \to 0$. Thus for k large, $u \in Z_k$ and $\|u\|_{X_{0,K}(\Omega)} = \rho_k$ we have $I(u) \geq 0$ and (D1) holds true.

To see (D2) holds we note that Y_k is finite dimensional and in finite dimensional space all the norms are equivalent. Therefore (D2) is satisfied if we choose $r_k > 0$ small enough (since $\mu > 0$) and therefore we can choose $r_k = \frac{\rho_k}{2}$.

For k large, $u \in Z_k, \|u\|_{X_{0,K}(\Omega)} \leq \rho_k$, we have from (3.3.4) that $d_k \geq \frac{-\mu}{q+1}\beta_k^{q+1}\rho_k^{q+1}$. On the other hand as $\mu > 0$ from the definition of $I(u)$ it follows $I(u) \leq \frac{\rho_k^2}{2}$. Thus $d_k \leq \frac{1}{2}\rho_k^2$. Using both upper and lower bounds of d_k and Lemma 3.3.3, we see that (D3) is also satisfied.

To check the assertion (D4), we consider a sequence $\{u_{r_j}\} \subset X_{0,K}(\Omega)$ such that as

$$\{u_{r_j}\} \in Y_{r_j}, \quad I(u_{r_j}) \to c, \quad I'|_{Y_{r_j}}(u_{r_j}) \to 0 \quad \text{as} \quad r_j \to \infty. \tag{3.3.5}$$

Claim: There exists $k > 0$ such that if $\mu > 0$ is arbitrarily chosen and

$$c < \frac{s}{N}S_K^{\frac{N}{2s}} - k\mu^{\frac{2^*}{2^*-q-1}}, \tag{3.3.6}$$

then $\{u_{r_j}\}$ contains a subsequence converging to a critical point of I, where $\{u_{r_j}\}$ is as in (3.3.5).

Assuming the claim, first let us complete the proof. Towards this, we choose $\mu^* = \left(\frac{sS_K^{\frac{N}{2s}}}{Nk}\right)^{\frac{2^*-q-1}{2^*}}$. Then $\mu \in (0,\mu^*)$ implies $\frac{s}{N}S_K^{\frac{N}{2s}} > k\mu^{\frac{2^*}{2^*-q-1}}$. Thus, if $c \in [d_k, 0)$ then we have

$$c < 0 < \frac{s}{N}S_K^{\frac{N}{2s}} - k\mu^{\frac{2^*}{2^*-q-1}}.$$

Hence applying the above claim, we see that (D4) holds true. Therefore the result follows by Theorem 3.2.5.

Here we prove the claim dividing into four steps.

Step 1: $\{u_{r_j}\}$ is bounded in $X_{0,K}(\Omega)$.
This follows by standard arguments. More precisely, since $I(u_{r_j}) = c + o(1)$ and $\left\langle I'(u_{r_j}), u_{r_j}\right\rangle = o(1)||u_{r_j}||_{X_{0,K}(\Omega)}$, computing $I(u_{r_j}) - \frac{1}{2}\left\langle I'(u_{r_j}), u_{r_j}\right\rangle$, we get $|u_{r_j}|^{2^*}_{L^{2^*}(\Omega)} \le C_1 + ||u_{r_j}||_{X_{0,K}(\Omega)}o(1) + C_2|u_{r_j}|^{q+1}_{L^{q+1}(\Omega)}$. Therefore using the definition of I along with Sobolev inequality yields

$$||u_{r_j}||^2_{X_{0,K}(\Omega)} \le C\left[1 + ||u_{r_j}||_{X_{0,K}(\Omega)}o(1) + ||u_{r_j}||^{q+1}_{X_{0,K}(\Omega)}\right]$$

and hence the boundedness follows. Therefore passing to a subsequence if necessary we may assume $u_{r_j} \rightharpoonup u$ in $X_{0,K}(\Omega)$, $u_{r_j} \to u$ in $L^\gamma(\mathbb{R}^N)$ for $1 \le \gamma < 2^*$ and point-wise.

Step 2: $\{u_{r_j}\}$ is a PS sequence in $X_{0,K}(\Omega)$ at level c, where c is as in (3.3.6).

To see this, let $v \in X_{0,K}(\Omega)$ be arbitrarily chosen. Then

$$\left\langle I'(u_{r_j}), v\right\rangle = \left\langle u_{r_j}, v\right\rangle - \int_\Omega |u_{r_j}|^{2^*-2}u_{r_j}vdx - \mu\int_\Omega |u_{r_j}|^{q-1}u_{r_j}vdx. \quad (3.3.7)$$

Therefore, using Sobolev inequality and Step 1 we have,

$$\begin{aligned}
|\left\langle I'(u_{r_j}), v\right\rangle| &\leq ||u_{r_j}||_{X_{0,K}(\Omega)}||v||_{X_{0,K}(\Omega)} + \int_\Omega |u_{r_j}|^{2^*-1}|v|dx + \mu\int_\Omega |u_{r_j}|^q|v|dx \\
&\leq ||u_{r_j}||_{X_{0,K}(\Omega)}||v||_{X_{0,K}(\Omega)} + c_1||u_{r_j}||^{2^*-1}_{X_{0,K}(\Omega)}||v||_{X_{0,K}(\Omega)} \\
&+ c_2\mu||u_{r_j}||^q_{X_{0,K}(\Omega)}||v||_{X_{0,K}(\Omega)} \\
&\leq (||u_{r_j}||_{X_{0,K}(\Omega)} + c_1||u_{r_j}||^{2^*-1}_{X_{0,K}(\Omega)} + c_2\mu||u_{r_j}||^q_{X_{0,K}(\Omega)})||v||_{X_{0,K}(\Omega)} \\
&\leq C||v||_{X_{0,K}(\Omega)},
\end{aligned}$$

which in turn implies $||I'(u_{r_j})||_{(X_{0,K}(\Omega))'} \leq M$ for all $j \geq 1$.

By the definition of Y_{r_j}, there exists a sequence $(v_{r_j}) \in Y_{r_j}$ such that $v_{r_j} \to v$ in $X_{0,K}(\Omega)$ as $r_j \to \infty$. Thus

$$\begin{aligned}
|\left\langle I'(u_{r_j}), v\right\rangle| &\leq |\left\langle I'(u_{r_j}), v_{r_j}\right\rangle| + |\left\langle I'(u_{r_j}), v - v_{r_j}\right\rangle| \\
&\leq ||I'|_{Y_{r_j}}(u_{r_j})||_{(X_{0,K}(\Omega))'}||v_{r_j}||_{X_{0,K}(\Omega)} \\
&+ ||I'(u_{r_j})||_{(X_{0,K}(\Omega))'}||v - v_{r_j}||_{X_{0,K}(\Omega)}.
\end{aligned}$$

Combining the hypothesis $I'|_{Y_{r_j}}(u_{r_j}) \to 0$ as $r_j \to \infty$ (see (3.3.5)), Step 1 and the fact that $\{I'(u_{r_j})\}$ is uniformly bounded, we have $|\left\langle I'(u_{r_j}), v\right\rangle| \to 0$ as $r_j \to \infty$. This in turn implies that $\{u_{r_j}\}$ is a PS sequence in $X_{0,K}(\Omega)$ at level c, where c is as in (3.3.6).

Step 3: u satisfies $(\mathcal{P}'_K)$.
Using Vitali's convergence theorem via Hölder inequality and Sobolev inequality, it is not difficult to check that we can pass the limit $r_j \to \infty$ in (3.3.7). Thus we obtain $\langle I'(u), v\rangle = 0$ for every v in $X_{0,K}(\Omega)$. Hence, $\mathscr{L}_K u + \mu|u|^{q-1}u + |u|^{2^*-2}u = 0 \quad \text{in} \quad \Omega.$

Step 4: Define $v_{r_j} := u_{r_j} - u$. Then it is not difficult to see that,

$$||v_{r_j}||^2_{X_{0,K}(\Omega)} = ||u_{r_j}||^2_{X_{0,K}(\Omega)} - ||u||^2_{X_{0,K}(\Omega)} + o(1). \tag{3.3.8}$$

On the other hand, by Brezis-Lieb lemma, we have

$$|u_{r_j}|^{2^*}_{L^{2^*}(\Omega)} = |v_{r_j}|^{2^*}_{L^{2^*}(\Omega)} + |u|^{2^*}_{L^{2^*}(\Omega)} + o(1). \tag{3.3.9}$$

Therefore by doing a straight forward computation and using $I(u_{r_j}) \to c$, we get

$$I(u) + \frac{1}{2}||v_{r_j}||^2_{X_{0,K}(\Omega)} - \frac{1}{2^*}|v_{r_j}|^{2^*}_{L^{2^*}(\Omega)} \to c. \tag{3.3.10}$$

Since $\left\langle I'(u_{r_j}), u_{r_j} \right\rangle \to 0$ and $\langle I'(u), u\rangle = 0$, from (3.3.8) and (3.3.9), we also have

$$||v_{r_j}||^2_{X_{0,K}(\Omega)} - |v_{r_j}|^{2^*}_{L^{2^*}(\Omega)} \to 0.$$

Therefore, we may assume that

$$||v_{r_j}||^2_{X_{0,K}(\Omega)} \to b, \;\; |v_{r_j}|^{2^*}_{L^{2^*}(\Omega)} \to b.$$

By Sobolev inequality, $||v_{r_j}||^2_{X_{0,K}(\Omega)} \geq (|v_{r_j}|^{2^*}_{L^{2^*}(\Omega)})^{2/2^*}$. As a result, we get $b \geq S_K b^{2/2^*}$. We note that if $b = 0$, then we are done since that implies $u_{r_j} \to u$ in $X_{0,K}(\Omega)$. Assume $b \neq 0$. This in turn implies $b \geq S_K^{\frac{N}{2s}}$ Then by (3.3.10), we have

$$I(u) = c - \frac{b}{2} + \frac{b}{2^*}. \tag{3.3.11}$$

It is easy to see that $\langle I'(u), u\rangle = 0$ implies

$$I(u) = \frac{s}{N}|u|^{2^*}_{L^{2^*}(\Omega)} + \left(\frac{1}{2} - \frac{1}{q+1}\right)\mu|u|^{q+1}_{L^{q+1}(\Omega)} \tag{3.3.12}$$

Combining (3.3.11) and (3.3.12) and using $q \in (0, 1)$, we obtain

$$\begin{aligned} c &= \frac{s}{N}(b + |u|^{2^*}_{L^{2^*}(\Omega)}) + \mu\left(\frac{1}{2} - \frac{1}{q+1}\right)|u|^{q+1}_{L^{q+1}(\Omega)} \\ &\geq \frac{s}{N}(S_K^{\frac{2s}{N}} + |u|^{2^*}_{L^{2^*}(\Omega)}) - \frac{1-q}{2(1+q)}\mu|\Omega|^{\frac{2^*-q-1}{2^*}}|u|^{q+1}_{L^{2^*}(\Omega)} \\ &= \frac{s}{N}S_K^{\frac{N}{2s}} + \frac{s}{N}|u|^{2^*}_{L^{2^*}(\Omega)} - a\mu|u|^{q+1}_{L^{2^*}(\Omega)}, \end{aligned} \tag{3.3.13}$$

where $a := \frac{1-q}{2(1+q)}|\Omega|^{\frac{2^*-q-1}{2^*}} > 0$. We define

$$g(t) = \frac{s}{N}t^{2^*} - a\mu t^{q+1}, \quad t \geq 0 \quad \text{and} \quad k := -\frac{1}{\mu^{\frac{2^*}{2^*-q-1}}}\min_{t\geq 0} g(t). \tag{3.3.14}$$

By elementary analysis it is easy to check that if $t_0 = (\frac{a\mu N}{s})^{\frac{1}{2^*-q-1}}$, then $g(t) < 0$ for $t \in (0, t_0)$, $g(t) \geq 0$ for $t \geq t_0$ and $g(0) = 0$. Hence, there exists

$t' \in (0, t_0)$ for which g attains minimum and $\min_{t>0} g(t) < 0$. Thus $k > 0$. Hence from (3.3.13) we have

$$c \geq \frac{s}{N} S_K^{\frac{N}{2s}} - k\mu^{\frac{2^*}{2^*-q-1}},$$

which is a contradiction to (3.3.6). Therefore, $b = 0$ and the claim follows. □

3.3.2 Subcritical case

In the succeeding theorem, we prove the existence of infinitely many nontrivial solutions in the subcritical case.

Theorem 3.3.4. *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary, $N > 2s, s \in (0,1)$. Assume $1 < p < 2^* - 1$. Then*

(a) For all $\lambda > 0$, $\mu \in \mathbb{R}$, $(\mathcal{P}_K)$ has a sequence of nontrivial solutions $\{u_k\}_{k\geq 1}$ such that $I_\mu^\lambda(u_k) \to \infty$ as $k \longrightarrow \infty$. Furthermore, if $\lambda > 0$, $\mu \geq 0$, then $\|u_k\|_{X_{0,K}(\Omega)} \to \infty$ as $k \to \infty$.

(b) For all $\mu > 0$, $\lambda \in \mathbb{R}$, $(\mathcal{P}_K)$ has a sequence of nontrivial solutions $\{v_k\}_{k\geq 1}$ such that $I_\mu^\lambda(v_k) \to 0$ as $k \to \infty$. Furthermore, if $\mu > 0$, $\lambda \leq 0$, then $\|v_k\|_{X_{0,K}(\Omega)} \to 0$ as $k \to \infty$.

Remark 3.3.5. *When $K(x) = |x|^{-(N+2s)}$, Brandle, et. al [22] have proved that there exists $\Lambda > 0$ such that, $(\mathcal{P}_K)$ has at least two positive solutions when $\mu \in (0, \Lambda)$, one positive solution when $\mu = \Lambda$ and no positive solution when $\mu > \Lambda$. For general K satisfying assumptions (2.3.10)-(2.3.12), Chen-Deng [32] have proved that there exists at least two positive solutions of $(\mathcal{P}_K)$ when $\lambda = 1$ and $\mu \in (0, \mu_0)$ for some $\mu_0 > 0$.*

Proof of Theorem 3.3.4

Before starting the proof we like to remark that when $\mu \geq 0,\ \lambda > 0$, Theorem 3.3.4 (a) also follows from [20, Theorem 1]. Here we give a proof which covers the entire range mentioned in Theorem 3.3.4.

Proof. (a)We assume $\mu \in \mathbb{R}$ and $\lambda > 0$. We prove part (a) using Fountain theorem 3.2.4. Energy functional corresponding to $(\mathcal{P}_K)$ is defined by I_μ^λ (see (5.1.1)). We need to verify that I_μ^λ satisfies (A1)-(A4) of Theorem 3.2.4. We choose X_j, Y_j, Z_j as in (3.3.2) and $G := \mathbb{Z}/2$. Therefore, (A1) is satisfied.

Next to check (A2) holds, we observe that,

$$I_\mu^\lambda(u) \leq \frac{1}{2}||u||^2_{X_{0,K}(\Omega)} + \frac{|\mu|}{q+1}|u|^{q+1}_{L^{q+1}(\Omega)} - \frac{\lambda}{p+1}|u|^{p+1}_{L^{p+1}(\Omega)}.$$

Since on the finite dimensional space Y_k all the norms are equivalent, $\lambda > 0$ and $1 < q+1 < 2 < p+1$, it is easy to see that (A2) is satisfied if we choose $\rho_k > 0$ large enough.

To see (A3) holds, we observe that

$$I_\mu^\lambda(u) \geq \frac{\|u\|^2_{X_{0,K}(\Omega)}}{2} - \frac{|\mu|}{q+1}\int_\Omega |u|^{q+1}dx - \frac{\lambda}{p+1}\int_\Omega |u|^{p+1}dx. \tag{3.3.15}$$

Applying Hölder inequality followed by Young's inequality we obtain

$$\int_\Omega |u|^{q+1}dx \leq \frac{q+1}{p+1}\int_\Omega |u|^{p+1}dx + \frac{p-q}{p+1}|\Omega|.$$

Substituting back in (3.3.15), we obtain

$$I_\mu^\lambda(u) \geq \frac{1}{2}\,\|u\|^2_{X_{0,K}(\Omega)} - \left(\frac{|\mu|}{p+1} + \frac{\lambda}{p+1}\right)\int_\Omega |u|^{p+1} - \frac{(p-q)|\mu|}{(p+1)(q+1)}|\Omega|.$$

Define

$$\beta_k := \sup_{u\in Z_k,\ \|u\|_{X_{0,K}(\Omega)}=1} |u|_{L^{p+1}(\Omega)}.$$

Hence on Z_k we have

$$I_\mu^\lambda(u) \geq \frac{1}{2}\,\|u\|^2_{X_{0,K}(\Omega)} - \frac{(\lambda+|\mu|)\beta_k^{p+1}}{p+1}\,\|u\|^{p+1} - \frac{(p-q)|\mu|}{(p+1)(q+1)}|\Omega|.$$

Choosing $r_k^{1-p} = (\lambda + |\mu|)\beta_k^{p+1}$, we have, for $u \in Z_k$ and $\|u\|_{X_{0,K}(\Omega)} = r_k$,

$$I_\mu^\lambda(u) \geq \left(\frac{1}{2} - \frac{1}{p+1}\right) r_k^2 - \frac{(p-q)|\mu|}{(p+1)(q+1)}|\Omega|.$$

Lemma 3.3.3 yields $\beta_k \to 0$ and hence $r_k \to \infty$ as $k \to \infty$. Therefore (A3) is satisfied.

In order to verify (A4), let $\{u_n\} \subset X_0$ such that

$$I_\mu^\lambda(u_n) \to c \quad \text{and} \quad (I_\mu^\lambda)'(u_n) \to 0 \quad \text{in} \quad (X_{0,K}(\Omega))',$$

where $c > 0$ and $(X_{0,K}(\Omega))'$ denotes the dual space of $X_{0,K}(\Omega)$. Following the same calculation as in Theorem 3.3.1, we get $\{u_n\}$ is bounded in $X_{0,K}(\Omega)$ and there exists $u \in X_{0,K}(\Omega)$ such that up to a subsequence $u_n \rightharpoonup u$ in $X_{0,K}(\Omega)$ and $u_n \to u$ in $L^r(\mathbb{R}^N)$ for every $r \in [1, 2^*)$. Since $\left\langle (I_\mu^\lambda)'(u_n), v \right\rangle = 0$ for every v in $X_{0,K}(\Omega)$, passing the limit using Vitali's convergence theorem, it follows $\left\langle (I_\mu^\lambda)'(u), v \right\rangle = 0$ for every v in $X_{0,K}(\Omega)$. Therefore

$$\begin{aligned} o(1) &= \left\langle (I_\mu^\lambda)'(u_n) - (I_\mu^\lambda)'(u), u_n - u \right\rangle \\ &= \|u_n - u\|^2_{X_{0,K}(\Omega)} \\ &- \mu \int_\Omega (|u_n|^{q-1}u_n - |u|^{q-1}u)(u_n - u)dx \\ &- \lambda \int_\Omega (|u_n|^{p-1}u_n - |u|^{p-1}u)(u_n - u)dx. \end{aligned}$$

Again, passing the limit by Vitali, we obtain $u_n \to u$ in $X_{0,K}(\Omega)$. Hence, (A4) is satisfied. Therefore by Theorem 3.2.4, it follows that $(\mathcal{P}_K)$ has a sequence of nontrivial solution $\{w_k\}_{k\geq 1}$ such that $I_\mu^\lambda(w_k) \to \infty$ as $k \to \infty$. Furthermore, if $\lambda > 0$, $\mu \geq 0$, then $I_\mu^\lambda(w_k) \leq ||w_k||^2_{X_{0,K}(\Omega)}$ and thus $\|w_k\|_{X_{0,K}(\Omega)} \to \infty$ as $k \to \infty$.

(b) This part follows from Theorem 3.2.5. We can proceed along the same line of proof of Theorem 3.3.1 to show (D1)-(D3) of Theorem 3.2.5 are satisfied. To check the assertion (D4), we consider a sequence $\{u_{r_j}\} \subset$

$X_{0,K}(\Omega)$ such that as

$$\{u_{r_j}\} \in Y_{r_j}, \quad I_\mu^\lambda(u_{r_j}) \to c, \quad (I_\mu^\lambda)|'_{Y_{r_j}}(u_{r_j}) \to 0 \quad \text{as} \quad r_j \to \infty.$$

We can prove exactly in the same way as in Theorem 3.3.1 that $\{u_n\}$ is a bounded PS sequence in $X_{0,K}(\Omega)$ at level c. Therefore, it is easy to conclude, as in part (a) that u_n converges strongly in $X_{0,K}(\Omega)$. Hence (D4) is also satisfied and as a result by Theorem 3.2.5, we conclude $(\mathcal{P}_K)$ has a sequence of nontrivial solutions $\{v_k\}_{k\geq 1}$ such that $c_k := I_\mu^\lambda(v_k) < 0$ and $c_k \to 0$ as $k \to \infty$. Using $\left\langle (I_\mu^\lambda)'(u_k), u_k \right\rangle = 0$ in the definition of $I_\mu^\lambda(u_k)$, we have

$$\mu\left(1-\frac{2}{q+1}\right)\int_\Omega |u|^{q+1}dx + \lambda\left(1-\frac{2}{p+1}\right)\int_\Omega |u|^{p+1}dx = 2c_k < 0.$$

Therefore, if $\mu > 0,\ \lambda \leq 0$, then

$$0 \leq -\lambda\left(1-\frac{2}{p+1}\right)\int_\Omega |u|^{p+1}dx = -2c_k + \mu\left(1-\frac{2}{q+1}\right)\int_\Omega |u|^{q+1}dx,$$

since $1 < q+1 < 2 < p+1$. This implies, $-2c_k \geq -\mu\left(1-\frac{2}{q+1}\right)\int_\Omega |u|^{q+1}dx$. Hence $\int_\Omega |u_k|^{q+1}dx \leq \frac{-2c_k q}{\mu(2-q)}$. Moreover, $\left\langle (I_\mu^\lambda)'(u_k), u_k \right\rangle = 0$ implies

$$||u_k||^2_{X_{0,K}(\Omega)} = \mu\int_\Omega |u_k|^{q+1}dx + \lambda\int_\Omega |u_k|^{p+1}dx \leq \mu\int_\Omega |u_k|^{q+1}dx \leq \frac{-2c_k q}{2-q} \to 0,$$

as $k \to \infty$. This completes the proof. □

3.3.3 A related variational problem

In this section we consider a related problem that can be solved by doing the similar type of analysis that we did in Section 3.3.1. More precisely we consider the following problem:

$$\begin{cases} (-\Delta)^s u - \frac{\alpha u}{|x|^{2s}} = \frac{|u|^{2^*(t)-2}u}{|x|^t} + \mu |u|^{q-1}u & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.3.16}$$

where $N > 2s$, Ω is an open, bounded domain in $\mathbb{R}^N$ with smooth boundary, $0 \leq t < 2s$, $0 < q < 1$, $2^*(t) = \frac{2(N-t)}{N-2s}$, $\alpha < \alpha_H := 2^{2s}\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}$ is the best fractional Hardy constant on $\mathbb{R}^N$. Thanks to the following fractional Hardy inequality :

$$\alpha_H \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2s}} dx \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx, \tag{3.3.17}$$

which was proved by Herbst [49],
$\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \alpha \int_{\Omega} \frac{|u(x)|^2}{|x|^{2s}} \right)^{\frac{1}{2}}$ is a norm equivalent to the norm (2.3.8) in $X_0(\Omega)$. Interpolating the above Hardy inequality with (4.1.1) and followed by simple calculation, we have the following fractional Hardy-Sobolev inequality

$$C \left(\int_{\Omega} \frac{|u|^{2^*(t)}}{|x|^t} dx \right)^{\frac{2}{2^*(t)}} \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \alpha \int_{\Omega} \frac{|u(x)|^2}{|x|^{2s}}.$$

Therefore we can define the quotient $S_s(\alpha) > 0$ as follows

$$S_s(\alpha) := \inf_{u \in X_0,\ u \neq 0} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \alpha \int_{\Omega} \frac{|u(x)|^2}{|x|^{2s}}}{\left(\int_{\Omega} \frac{|u|^{2^*(t)}}{|x|^t} dx \right)^{\frac{2}{2^*(t)}}}. \tag{3.3.18}$$

The following theorem regarding existence of infinitely many nontrivial solutions for fractional Hardy-Sobolev type equation can be proved in the spirit of theorem 3.3.1.

Theorem 3.3.6. *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary, $N > 2s$.Then there exists $\mu^* > 0$ such that for all $\mu \in (0, \mu^*)$, problem* (3.3.16) *has a sequence of non-trivial solutions $\{u_n\}_{n \geq 1}$ such that $I(u_n) < 0$ and $I(u_n) \to 0$ as $n \to \infty$ where $I(\cdot)$ is the corresponding energy functional associated with* (3.3.16).

In order to prove this theorem one essentially needs to verify an argument similar to (3.3.6), where RHS of (3.3.6) should be replaced by $\frac{2s-t}{2(N-t)} S_s(\alpha)^{\frac{N-t}{2s-t}} - k\mu^{\frac{2^*(t)}{2^*(t)-q-1}}$ and this would follow by the similar type of arguments as in the proof of Theorem 3.3.1.

Conclusion: In this chapter, we have established existence of infinitely many solutions using Fountain and and Dual Fountain theorem. In the local case, these results were proved by Bartsch and Bartsch-Willem (see [10, 11, 86]). We have extended these results in the non-local setting.

Chapter 4

Sign Changing Solution for fractional Laplacian type equations with concave-critical nonlinearities

In this chapter we study the existence of at least one sign-changing solution of the following problem (P). More precisely, we study

$$(P)\begin{cases}(-\Delta)^s u = \mu |u|^{q-1}u + |u|^{2^*-2}u & \text{in} \quad \Omega,\\ u = 0 & \text{in} \quad \mathbb{R}^n \setminus \Omega.\end{cases}$$

Corresponding to (P), define the energy functional I_μ as follows

$$\begin{aligned} I_\mu(u) &:= \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy - \frac{\mu}{q+1}\int_\Omega |u|^{q+1}dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx \\ &= \frac{1}{2}\|u\|^2_{X_0(\Omega)} - \frac{\mu}{q+1}\int_\Omega |u|^{q+1}dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx. \end{aligned} \tag{4.0.1}$$

We obtain existence of at least one sign-changing solution of the above problem (P) under suitable assumptions on N and q. Our method is based on the Nehari manifold technique. The main theorem of this chapter is stated below:

Theorem 4.0.1. *Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$. Assume $s \in (0,1), N > 6s$, $\frac{1}{2}\left(\frac{N+2s}{N-2s}\right) < q < 1$. Then there exists $\mu^* > 0$ such that for all $\mu \in (0, \mu^*)$ problem (P) has at least one sign changing solution.*

4.1 Sobolev minimizer

Using [78, Lemma 9], we know

$$S_s \left(\int_{\mathbb{R}^N} |v(x)|^{2^*} \right)^{2/2^*} \leq \|v\|^2_{X_0(\Omega)} \quad \forall \quad v \in X_0(\Omega), \tag{4.1.1}$$

where

$$S_s = \inf_{v \in H^s(\mathbb{R}^N), v \neq 0} \frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} dxdy}{\left(\displaystyle\int_{\mathbb{R}^N} |v(x)|^{2^*} \right)^{2/2^*}}. \tag{4.1.2}$$

It is known that (see [35]), S_s is attained by $v_\varepsilon \in H^s(\mathbb{R}^N)$, where

$$v_\varepsilon(x) := \frac{k\varepsilon^{\frac{N-2s}{4}}}{(\varepsilon + |x|^2)^{\frac{N-2s}{2}}}, \quad \text{with} \quad \varepsilon > 0, k \in \mathbb{R} \setminus \{0\}. \tag{4.1.3}$$

4.2 Cut-off technique

We note that $v_\varepsilon \notin X_0(\Omega)$. Therefore we multiply v_ε by a suitable cut-off function ψ in order to put v_ε to 0 outside Ω. For this, fix $\delta > 0$. Define $\Omega_1 = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$. We choose $\psi \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \psi \leq 1$, $\psi = 1$ in Ω_1, $\psi = 0$ in $\mathbb{R}^N \setminus \Omega$ and $\psi > 0$ in Ω. We define

$$u_\varepsilon(x) := \psi(x) v_\varepsilon(x). \tag{4.2.1}$$

In the next section, we will discuss notions of some Nehari-type sets.

4.3 Nehari type sets

To obtain sign changing solution of (P), we need to study minimization problems of I_μ over suitable Nehari-type sets. We define the following sets in the spirit of [82] (also see [31])

$$\begin{aligned}
\mathcal{N} &:= \{u \in X_0(\Omega) \setminus \{0\} : \left\langle I'_\mu(u), u \right\rangle = 0\};\\
\mathcal{N}_0 &:= \left\{u \in \mathcal{N} : (1-q)\,\|u\|^2_{X_0(\Omega)} - (2^*-q-1)|u|^{2^*}_{L^{2^*}(\Omega)} = 0\right\};\\
\mathcal{N}^+ &:= \left\{u \in \mathcal{N} : (1-q)\,\|u\|^2_{X_0(\Omega)} - (2^*-q-1)|u|^{2^*}_{L^{2^*}(\Omega)} > 0\right\};\\
\mathcal{N}^- &:= \left\{u \in \mathcal{N} : (1-q)\,\|u\|^2_{X_0(\Omega)} - (2^*-q-1)|u|^{2^*}_{L^{2^*}(\Omega)} < 0\right\}.
\end{aligned}$$

From [32], it is known that there exists $\mu_* > 0$ such that, if $\mu \in (0, \mu_*)$, then the following minimization problem:

$$\tilde{\alpha}^+_\mu := \inf_{u \in \mathcal{N}^+} J_\mu(u) \quad \text{and} \quad \tilde{\alpha}^-_\mu := \inf_{u \in \mathcal{N}^-} J_\mu(u) \tag{4.3.1}$$

achieve their minimum at w_0 and w_1 respectively, where

$$J_\mu(u) := \frac{1}{2}\|u\|^2_{X_0(\Omega)} - \frac{\mu}{q+1}\int_\Omega (u^+)^{q+1}dx - \frac{1}{2^*}\int_\Omega (u^+)^{2^*}dx. \tag{4.3.2}$$

Moreover w_0 and w_1 are critical points of J_μ. Using maximum principle [79, Proposition 2.2.8] and followed by a simple calculation , it can be checked that, if u is a critical point of J_μ, then u is strictly positive in Ω (see [9]). Thus w_0 and w_1 are positive solution of (P). Applying the Moser iteration technique it follows that any positive solution of (P) is in $L^\infty(\Omega)$ (see [9, Proposition 2.2]).

4.4 Some important lemmas

This section is devoted to some important lemmas which will be needed to prove our main result Theorem 4.0.1.

Lemma 4.4.1. *Suppose w_1 is a positive solution of (P) and u_ε is as defined in* (4.2.1)*. Then for every $\varepsilon > 0$, small enough*

(i) $A_1 := \int_\Omega w_1^{2^*-1} u_\varepsilon dx \le k_1 \varepsilon^{\frac{N-2s}{4}}$;

(ii) $A_2 := \int_\Omega w_1^q u_\varepsilon dx \leq k_2 \varepsilon^{\frac{N-2s}{4}}$;

(iii) $A_3 := \int_\Omega w_1 u_\varepsilon^q dx \leq k_3 \varepsilon^{\frac{N-2s}{4}q}$;

(iv) $A_4 := \int_\Omega w_1 u_\varepsilon^{2^*-1} dx \leq k_4 \varepsilon^{\frac{N+2s}{4}}$.

Proof. Let $R,\ M > 0$ be such that $\Omega \subset B(0,R)$ and $|w_1|_{L^\infty(\Omega)} < M$. Then

$$
\begin{aligned}
(i) \quad A_1 = \int_\Omega w_1^{2^*-1} u_\varepsilon dx &\leq M^{2^*-1} |\psi|_{L^\infty(\Omega)} k \varepsilon^{\frac{N-2s}{4}} \int_{B(0,R)} \frac{dx}{(\varepsilon + |x|^2)^{\frac{N-2s}{2}}} \\
&\leq C \varepsilon^{\frac{N}{2} - \frac{N-2s}{4}} \int_{B(0,\frac{R}{\sqrt{\varepsilon}})} \frac{dx}{(1+|x|^2)^{\frac{N-2s}{2}}} \\
&\leq k_1 \varepsilon^{\frac{N-2s}{4}}.
\end{aligned}
$$

Proof of (ii) similar to (i).

$$
\begin{aligned}
(iii) \quad A_3 = \int_\Omega w_1 u_\varepsilon^q dx &\leq M |\psi|^q_{L^\infty(\Omega)} k^q \varepsilon^{\frac{N-2s}{4}q} \int_{B(0,R)} \frac{dx}{(\varepsilon + |x|^2)^{\frac{N-2s}{2}q}} \\
&\leq C \varepsilon^{\frac{N}{2} - \frac{(N-2s)q}{4}} \int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^{N-1} dr}{(1+r^2)^{\frac{(N-2s)q}{2}}} \\
&\leq C \varepsilon^{\frac{N}{2} - \frac{(N-2s)q}{4}} \left(\frac{R}{\sqrt{\varepsilon}} \right)^{N-(N-2s)q} \\
&\leq k_3 \varepsilon^{\frac{N-2s}{4}q}.
\end{aligned}
$$

(iv) can be proved as in (iii). □

Lemma 4.4.2. *Let u_ε be as defined in* (4.2.1) *and* $0 < q < 1$. *Then for every* $\varepsilon > 0$, *small*

$$
\int_\Omega |u_\epsilon|^{q+1} dx = \begin{cases} k_5 \varepsilon^{(\frac{N-2s}{4})(q+1)} & \text{if } 0 < q < \frac{2s}{N-2s}, \\ k_6 \varepsilon^{\frac{N}{4}} |\ln \varepsilon|, & \text{if } q = \frac{2s}{N-2s}, \\ k_7 \varepsilon^{\frac{N}{2} - (\frac{N-2s}{4})(q+1)} & \text{if } \frac{2s}{N-2s} < q < 1. \end{cases}
$$

Proof. Choose $0 < R' < R$ be such that $B(0,R') \subset \Omega_1 \subset \Omega$. Then $u_\varepsilon = v_\varepsilon$ in

$B(0,R')$. Then

$$\int_\Omega |u_\varepsilon|^{q+1}dx \geqslant \int_{B(0,R')} |u_\varepsilon|^{q+1}dx = k^{q+1}\varepsilon^{\frac{(N-2s)(q+1)}{4}} \int_{B(0,R')} \frac{dx}{(\varepsilon+|x|^2)^{\frac{(N-2s)(q+1)}{2}}}.$$

Proceeding as in the proof of Lemma 4.4.1 (iii), we have

$$\begin{aligned} C\varepsilon^{\frac{N}{2}-\frac{(N-2s)(q+1)}{4}} \int_0^{\frac{R'}{\sqrt{\varepsilon}}} \frac{r^{N-1}dr}{(1+r^2)^{\frac{(N-2s)(q+1)}{2}}} &\le \int_\Omega |u_\varepsilon|^{q+1}dx \\ &\le C\varepsilon^{\frac{N}{2}-\frac{(N-2s)(q+1)}{4}} \int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^{N-1}dr}{(1+r^2)^{\frac{(N-2s)(q+1)}{2}}}. \end{aligned} \tag{4.4.1}$$

Case 1 : $0 < q < \frac{2s}{N-2s}$.

We note that

$$\int_0^{\frac{R'}{\sqrt{\varepsilon}}} \frac{r^{N-1}dr}{(1+r^2)^{\frac{(N-2s)(q+1)}{2}}} \ge C\int_{\frac{R'}{2\sqrt{\varepsilon}}}^{\frac{R'}{\sqrt{\varepsilon}}} r^{N-1-(N-2s)(q+1)}dr \ge \frac{C}{\varepsilon^{\frac{N}{2}-\frac{(N-2s)(q+1)}{2}}} \tag{4.4.2}$$

and

$$\int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^{N-1}dr}{(1+r^2)^{\frac{(N-2s)(q+1)}{2}}} \le \int_0^{\frac{R}{\sqrt{\varepsilon}}} r^{N-1-(N-2s)(q+1)}dr \le \frac{C}{\varepsilon^{\frac{N}{2}-\frac{(N-2s)(q+1)}{2}}}. \tag{4.4.3}$$

Substituting back (4.4.2) and (4.4.3) into (4.4.1), we obtain $\int_\Omega |u_\varepsilon|^{q+1}dx = k_5\varepsilon^{(\frac{N-2s}{4})(q+1)}$.

Case 2 : $q = \frac{2s}{N-2s}$.

Then

$$\int_0^{\frac{R'}{\sqrt{\varepsilon}}} \frac{r^{N-1}dr}{(1+r^2)^{\frac{(N-2s)(q+1)}{2}}} \ge C\int_1^{\frac{R'}{\sqrt{\varepsilon}}} r^{N-1-(N-2s)(q+1)}dr \ge C'|\ln\,\varepsilon|.$$

$$\begin{aligned} \int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^{N-1}dr}{(1+r^2)^{\frac{(N-2s)(q+1)}{2}}} &\le \int_0^1 r^{N-1}dr + \int_1^{\frac{R}{\sqrt{\varepsilon}}} r^{N-1-(N-2s)(q+1)}dr \\ &\le C(1+|\ln\,\varepsilon|) \le 2C|\ln\,\varepsilon|. \end{aligned}$$

Substituting back the above two expressions in (4.4.1), we have
$\int_\Omega |u_\varepsilon|^{q+1}dx = k_6\varepsilon^{\frac{N}{4}}|ln\ \varepsilon|$.

Case 3 : $\frac{2s}{N-2s} < q < 1$.

Therefore $(N-2s)(q+1) > N$ and consequently

$$\int_0^{\frac{R'}{\sqrt{\varepsilon}}} \frac{r^{N-1}dr}{(1+r^2)^{\frac{(N-2s)(q+1)}{2}}} \geq C\int_0^1 r^{N-1}dr = C,$$

$$\int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^{N-1}dr}{(1+r^2)^{\frac{(N-2s)(q+1)}{2}}} \leq \int_0^1 r^{N-1}dr + \int_1^\infty r^{N-1-(N-2s)(q+1)} \leq C,$$

Hence $\int_\Omega |u_\varepsilon|^{q+1}dx = k_7\varepsilon^{\frac{N}{2}-\frac{(N-2s)(q+1)}{4}}$. □

Set

$$\tilde{\mu} = \left(\frac{1-q}{2^*-q-1}\right)^{\frac{1-q}{2^*-2}} \frac{2^*-2}{2^*-q-1}|\Omega|^{\frac{q+1-2^*}{2^*}} S_s^{\frac{N(1-q)}{4s}+\frac{q+1}{2}}. \tag{4.4.4}$$

We prove the next three lemmas in the spirit of [82].

Lemma 4.4.3. *Let $\mu \in (0,\tilde{\mu})$. For every $u \in X_0(\Omega)$, $u \neq 0$, there exists unique*

$$0 < t^-(u) < t_0(u) = \left(\frac{(1-q)\,\|u\|^2_{X_0(\Omega)}}{(2^*-1-q)|u|^{2^*}_{L^{2^*}(\Omega)}}\right)^{\frac{N-2s}{4s}} < t^+(u),$$

such that

$$t^-(u)u \in \mathcal{N}^+ \quad \text{and} \quad I_\mu(t^-u) = \min_{t\in[0,t_0]} I_\mu(tu),$$

$$t^+(u)u \in \mathcal{N}^- \quad \text{and} \quad I_\mu(t^+u) = \max_{t\geq t_0} I_\mu(tu).$$

Proof. From (4.0.1), for $t \geq 0$,

$$I_\mu(tu) = \frac{t^2}{2}\|u\|^2_{X_0(\Omega)} - \frac{\mu t^{q+1}}{q+1}|u|^{q+1}_{L^{q+1}(\Omega)} - \frac{t^{2^*}}{2^*}|u|^{2^*}_{L^{2^*}(\Omega)}.$$

Therefore

$$\frac{\partial}{\partial t}I_\mu(tu) = t^q\left(t^{1-q}\|u\|^2_{X_0(\Omega)} - t^{2^*-q-1}|u|^{2^*}_{L^{2^*}(\Omega)} - \mu|u|^{q+1}_{L^{q+1}(\Omega)}\right).$$

Define

$$\phi(t) = t^{1-q}\|u\|^2_{X_0(\Omega)} - t^{2^*-q-1}|u|^{2^*}_{L^{2^*}(\Omega)}. \tag{4.4.5}$$

By a straight forward computation, it follows that ϕ attains maximum at the point

$$t_0 = t_0(u) = \left(\frac{(1-q)\,\|u\|^2_{X_0(\Omega)}}{(2^*-1-q)|u|^{2^*}_{L^{2^*}(\Omega)}} \right)^{\frac{1}{2^*-2}}. \tag{4.4.6}$$

Thus

$$\phi'(t_0) = 0, \quad \phi'(t) > 0 \quad \text{if} \quad t < t_0, \quad \phi'(t) < 0 \quad \text{if} \quad t > t_0. \tag{4.4.7}$$

Moreover, $\phi(t_0) = \left(\frac{1-q}{2^*-1-q}\right)^{\frac{1-q}{2^*-2}} \left(\frac{2^*-2}{2^*-1-q}\right) \left(\frac{\|u\|^{2(2^*-q-1)}_{X_0(\Omega)}}{|u|^{2^*(1-q)}_{L^{2^*}(\Omega)}} \right)^{\frac{N-2s}{4s}}$. Therefore using (4.1.1), we have

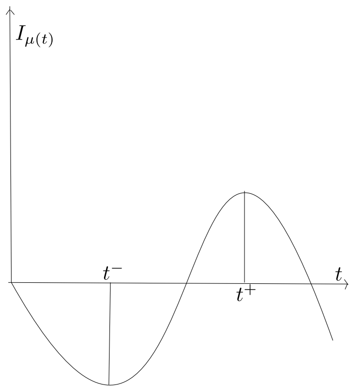
$$\phi(t_0) \geq \left(\frac{1-q}{2^*-1-q} \right)^{\frac{(1-q)(N-2s)}{4s}} \frac{2^*-2}{2^*-1-q} S_s^{\frac{N(1-q)}{4s}} \|u\|^{q+1}_{X_0(\Omega)}. \tag{4.4.8}$$

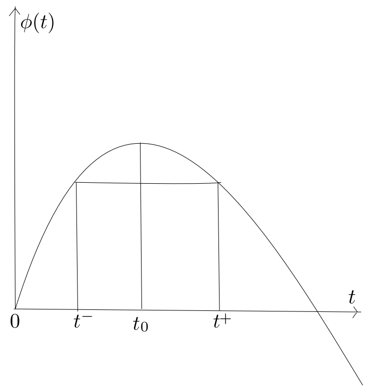
Using Hölder inequality followed by Sobolev inequality (4.1.1), and the fact that $\mu \in (0, \tilde{\mu})$, we obtain

$$\begin{aligned} \mu \int_\Omega |u|^{q+1} dx \leq \mu \|u\|^{q+1}_{X_0(\Omega)} S_s^{-(q+1)/2} |\Omega|^{\frac{2^*-q-1}{2^*}} &\leq \tilde{\mu} \|u\|^{q+1}_{X_0(\Omega)} S_s^{-(q+1)/2} |\Omega|^{\frac{2^*-q-1}{2^*}} \\ &\leq \phi(t_0), \end{aligned}$$

where in the last inequality we have used expression of $\tilde{\mu}$ (see (4.4.4) and (4.4.8)). Hence, there exists $t^+(u) > t_0 > t^-(u)$ such that

$$\phi(t^+) = \mu \int_\Omega |u|^{q+1} = \phi(t^-) \quad \text{and} \quad \phi'(t^+) < 0 < \phi'(t^-). \tag{4.4.9}$$





This in turn, implies $t^+u \in \mathcal{N}^-$ and $t^-u \in \mathcal{N}^+$. Moreover, using (4.4.7) and (4.4.9) in the expression of $\frac{\partial}{\partial t} I_\mu(tu)$, we have

$$\frac{\partial}{\partial t} I_\mu(tu) > 0 \quad \text{when} \quad t \in (t^-, t^+) \quad \text{and}$$

$$\frac{\partial}{\partial t} I_\mu(tu) < 0 \quad \text{when} \quad t \in [0, t^-) \cup (t^+, \infty),$$
$$\frac{\partial}{\partial t} I_\mu(tu) = 0 \quad \text{when} \quad t = t^\pm.$$

We note that $I_\mu(tu) = 0$ at $t = 0$ and strictly negative when $t > 0$ is small enough. Therefore it is easy to conclude that

$$\max_{t \geq t_0} I_\mu(tu) = I_\mu(t^+ u) \quad \text{and} \quad \min_{t \in [0, t_0]} I_\mu(tu) = I_\mu(t^- u).$$

□

Lemma 4.4.4. *Let $\tilde{\mu}$ be defined as in* (4.4.4). *Then $\mu \in (0, \tilde{\mu})$, implies $\mathcal{N}_0 = \emptyset$.*

Proof. Suppose not. Then there exists $w \in \mathcal{N}_0$ such that $w \neq 0$ and

$$(1 - q) \, \|w\|^2_{X_0(\Omega)} - (2^* - q - 1)|w|^{2^*}_{L^{2^*}(\Omega)} = 0. \tag{4.4.10}$$

The above expression combined with Sobolev inequality (4.1.1) yields

$$\|w\|_{X_0(\Omega)} \geq S_s^{N/4s} \left(\frac{1 - q}{2^* - 1 - q} \right)^{\frac{N-2s}{4s}}. \tag{4.4.11}$$

As $w \in \mathcal{N}_0 \subseteq \mathcal{N}$, using (4.4.10) and Hölder inequality followed by Sobolev inequality, we get

$$\begin{aligned} 0 &= \|w\|^2_{X_0(\Omega)} - |w|^{2^*}_{L^{2^*}(\Omega)} - \mu |w|^{q+1}_{L^{q+1}(\Omega)} \\ &\geq \|w\|^2_{X_0(\Omega)} - \left(\frac{1 - q}{2^* - q - 1} \right) \|w\|^2_{X_0(\Omega)} - \mu |\Omega|^{1 - \frac{q+1}{2^*}} S_s^{-(q+1)/2} \|w\|^{q+1}_{X_0(\Omega)}. \end{aligned}$$

Combining the above inequality with (4.4.11) and using $\mu < \tilde{\mu}$, we have

$$\begin{aligned} 0 \geq \|w\|^{q+1}_{X_0(\Omega)} &\left[\left(\frac{2^* - 2}{2^* - q - 1} \right) \left(\frac{1 - q}{2^* - q - 1} \right)^{\frac{(N-2s)(1-q)}{4s}} S_s^{\frac{N(1-q)}{4s}} \right. \\ &\left. - \mu |\Omega|^{1 - \frac{q+1}{2^*}} S_s^{-(q+1)/2} \right] > 0, \end{aligned} \tag{4.4.12}$$

which is a contradiction. This completes the proof. □

Lemma 4.4.5. *Let $\tilde{\mu}$ be defined as in* (4.4.4) *and* $\mu \in (0, \tilde{\mu})$. *Given* $u \in \mathcal{N}$, *there exists* $\rho_u > 0$ *and a differential function* $g_{\rho_u} : B_{\rho_u}(0) \to \mathbb{R}^+$ *satisfying the following:*

$$\begin{aligned}
&g_{\rho_u}(0) = 1,\\
&\Big(g_{\rho_u}(w)\Big)(u+w) \in \mathcal{N} \quad \forall \quad w \in B_{\rho_u}(0),\\
&\Big\langle g'_{\rho_u}(0), \phi \Big\rangle = \frac{2\langle u, \phi\rangle - 2^* \int_\Omega |u|^{2^*-2} u\phi - (q+1)\mu \int_\Omega |u|^{q-1} u\phi}{(1-q)\|u\|^2_{X_0(\Omega)} - (2^* - q - 1)|u|^{2^*}_{L^{2^*}(\Omega)}}.
\end{aligned}$$

Proof. Define $F : \mathbb{R}^+ \times X_0(\Omega) \to \mathbb{R}$ as follows:

$$F(t, w) = t^{1-q}\|u+w\|^2_{X_0(\Omega)} - t^{2^*-q-1}|u+w|^{2^*}_{L^{2^*}(\Omega)} - \mu|u+w|^{q+1}_{L^{q+1}(\Omega)}.$$

We note that $u \in \mathcal{N}$ implies

$$F(1,0) = 0, \quad \text{and} \quad \frac{\partial F}{\partial t}(1,0) = (1-q)\|u\|^2_{X_0(\Omega)} - (2^* - q - 1)|u|^{2^*}_{L^{2^*}(\Omega)} \neq 0.$$

Therefore, by Implicit function theorem, there exists neighbourhood $B_{\rho_u}(0)$ for some $\rho_u > 0$ and a C^1 function $g_{\rho_u} : B_{\rho_u}(0) \to \mathbb{R}^+$ such that

$$(i)\ g_{\rho_u}(0) = 1, \quad (ii)\ F(g_{\rho_u}(w), w) = 0,\ \forall\ w \in B_{\rho_u}(0),$$

$$(iii) F_t(g_{\rho_u}(w), w) \neq 0,\ \forall\ w \in B_{\rho_u}(0), \quad (iv)\ \Big\langle g'_{\rho_u}(0), \phi\Big\rangle = -\frac{\Big\langle \frac{\partial F}{\partial w}(1,0), \phi\Big\rangle}{\frac{\partial F}{\partial t}(1,0)}.$$

Multiplying (ii) by $(g_{\rho_u}(w))^{q+1}$, it follows that $(g_{\rho_u}(w))(u+w) \in \mathcal{N}$. In fact, simplifying (iii), we obtain

$$(1-q)(g_{\rho_u}(w))^2\|u+w\|^2_{X_0(\Omega)} - (2^*-q-1)(g_{\rho_u}(w))^{2^*}|u+w|^{2^*}_{L^{2^*}(\Omega)} \neq 0 \ \forall w \in B_{\rho_u}(0).$$

Thus $\Big(g_{\rho_u}(w)\Big)(u+w) \in \mathcal{N}^- \cup \mathcal{N}^+$ for every $w \in B_{\rho_u}(0)$. The last assertion of the lemma follows from (iv). □

4.5 Existence of sign-changing solution

In this section, we will establish existence of at least one sign-changing solution by finding sign-changing critical points of I_μ.

4.5.1 Sign changing critical points of I_μ

This subsection is very important in order to obtain the main result.

Define

$$\mathcal{N}_1^- := \{u \in \mathcal{N} : u^+ \in \mathcal{N}^-\},$$

$$\mathcal{N}_2^- := \{u \in \mathcal{N} : -u^- \in \mathcal{N}^-\},$$

We set

$$\beta_1 = \inf_{u \in \mathcal{N}_1^-} I_\mu(u) \quad \text{and} \quad \beta_2 = \inf_{u \in \mathcal{N}_2^-} I_\mu(u). \tag{4.5.1}$$

Theorem 4.5.1. *Assume* $0 < \mu < min\{\tilde{\mu}, \mu_*, \mu_1\}$, *where* μ_1 *is as in Lemma 4.6.1,* $\tilde{\mu}$ *is as in* (4.4.4) *and* μ_* *is chosen such that* $\tilde{\alpha}_\mu^-$ *is achieved in* $(0, \mu_*)$. *Let* β_1, β_2, $\tilde{\alpha}_\mu^-$ *be defined as in* (4.5.1) *and* (4.3.1) *respectively.*

(i) Let $\beta_1 < \tilde{\alpha}_\mu^-$. *Then there exists a sign changing critical point* $\tilde{w}_1$ *of* I_μ *such that* $\tilde{w}_1 \in \mathcal{N}_1^-$ *and* $I_\mu(\tilde{w}_1) = \beta_1$.

(ii) If $\beta_2 < \tilde{\alpha}_\mu^-$, *then there exists a sign changing critical point* $\tilde{w}_2$ *of* I_μ *such that in* $\tilde{w}_2 \in \mathcal{N}_1^-$ *and* $I_\mu(\tilde{w}_2) = \beta_2$.

Proof. (i) Let $\beta_1 < \tilde{\alpha}_\mu^-$.

Claim 1: $\mathcal{N}_1^-$ and $\mathcal{N}_2^-$ are closed sets.

To see this, let $\{u_n\} \subset \mathcal{N}_1^-$ such that $u_n \to u$ in $X_0(\Omega)$. It is easy to note that $|u_n|, |u| \in X_0(\Omega)$ and $|u_n| \to |u|$ in $X_0(\Omega)$. This in turn implies $u_n^+ \to u^+$ in $X_0(\Omega)$ and $L^\gamma(\mathbb{R}^N)$ for $\gamma \in [1, 2^*]$ (by (4.1.1)). Since, $u_n \in \mathcal{N}_1^-$, we have $u_n^+ \in \mathcal{N}^-$. Therefore

$$\|u_n^+\|^2_{X_0(\Omega)} - |u_n^+|^{2^*}_{L^{2^*}(\Omega)} - \mu |u_n^+|^{q+1}_{L^{q+1}(\Omega)} = 0 \tag{4.5.2}$$

and

$$(1-q)\|u_n^+\|^2_{X_0(\Omega)} - (2^* - q - 1)|u_n^+|^{2^*}_{L^{2^*}(\Omega)} < 0 \ \forall \ n \geq 1. \tag{4.5.3}$$

Passing to the limit as $n \to \infty$, we obtain $u^+ \in \mathcal{N}$ and $(1-q)\,\|u^+\|^2_{X_0(\Omega)} - (2^* - q - 1)|u^+|^{2^*}_{L^{2^*}(\Omega)} \leq 0$. But, from Lemma 4.4.4, we know $\mathcal{N}_0 = \emptyset$. Therefore $u^+ \in \mathcal{N}^-$ and hence $\mathcal{N}_1^-$ is closed. Similarly it can be shown that $\mathcal{N}_2^-$ is also closed. Hence claim 1 follows.

By Ekeland Variational Principle there exists sequence $\{u_n\} \subset \mathcal{N}_1^-$ such that

$$I_\mu(u_n) \to \beta_1 \quad \text{and} \quad I_\mu(z) \geq I_\mu(u_n) - \frac{1}{n}\,\|u_n - z\|_{X_0(\Omega)} \quad \forall \quad z \in \mathcal{N}_1^-. \tag{4.5.4}$$

Claim 2: $\{u_n\}$ is uniformly bounded in $X_0(\Omega)$.
To see this, we notice $u_n \in \mathcal{N}_1^-$ implies $u_n \in \mathcal{N}$ and this in turn implies $\left\langle I'_\mu(u_n), u_n \right\rangle = 0$, that is,

$$\|u_n\|^2_{X_0(\Omega)} = |u_n|^{2^*}_{L^{2^*}(\Omega)} + |u_n|^{q+1}_{L^{q+1}(\Omega)}.$$

Since $I_\mu(u_n) \to \beta_1$, using the above equality in the expression of $I_\mu(u_n)$, we get, for n large enough

$$\begin{aligned} \frac{s}{N}\,\|u_n\|^2_{X_0(\Omega)} &\leq \beta_1 + 1 + \left(\frac{1}{q+1} - \frac{1}{2^*}\right)|u_n|^{q+1}_{L^{q+1}(\Omega)} \\ &\leq C(1 + \|u_n\|^{q+1}_{X_0(\Omega)}). \end{aligned}$$

This implies $\{u_n\}$ is uniformly bounded in $X_0(\Omega)$.

Claim 3: There exists $b > 0$ such that $\|u_n^-\|_{X_0(\Omega)} \geq b$ for all $n \geq 1$.
Suppose the claim is not true. Then for each $k \geq 1$, there exists u_{n_k} such that

$$\|u_{n_k}^-\|_{X_0(\Omega)} < \frac{1}{k} \;\; \forall\; k \;\; \geq 1. \tag{4.5.5}$$

We note that for any $u \in X_0(\Omega)$, we have

$$\begin{aligned} \|u\|^2_{X_0(\Omega)} &= \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \\ &= \int_{\mathbb{R}^{2N}} \frac{|(u^+(x)-u^+(y))-(u^-(x)-u^-(y))|^2}{|x-y|^{N+2s}} dxdy \\ &= \|u^+\|^2_{X_0(\Omega)} + \|u^-\|^2_{X_0(\Omega)} + 2\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)+u^+(y)u^-(x)}{|x-y|^{N+2s}} dxdy \\ &\geq \|u^+\|^2_{X_0(\Omega)} + \|u^-\|^2_{X_0(\Omega)}. \end{aligned} \tag{4.5.6}$$

By a simple calculation, it follows

$$|u|^{2^*}_{L^{2^*}(\Omega)} = |u^+|^{2^*}_{L^{2^*}(\Omega)} + |u^-|^{2^*}_{L^{2^*}(\Omega)} \quad \text{and} \quad |u|^{q+1}_{L^{q+1}(\Omega)} = |u^+|^{q+1}_{L^{q+1}(\Omega)} + |u^-|^{q+1}_{L^{q+1}(\Omega)}. \tag{4.5.7}$$

Combining (4.5.6) and (4.5.7), we obtain

$$I_\mu(u) \geq I_\mu(u^+) + I_\mu(u^-) \quad \forall \quad u \in X_0(\Omega). \tag{4.5.8}$$

Moreover, (4.5.5) implies $\|u^-_{n_k}\|_{X_0(\Omega)} \to 0$ and therefore by Sobolev inequality

$$|u^-_{n_k}|_{L^{2^*}(\Omega)} \to 0, \;\; |u^-_{n_k}|_{L^{q+1}(\Omega)} \to 0, \quad \text{as} \quad k \to \infty.$$

Consequently, $I_\mu(u^-_{n_k}) \to 0$ as $k \to \infty$. As a result, we have

$$\beta_1 = I_\mu(u_{n_k}) + o(1) \geq I_\mu(u^+_{n_k}) + I_\mu(u^-_{n_k}) + o(1) = J_\mu(u^+_{n_k}) + o(1) \geq \tilde{\alpha}^-_\mu + o(1).$$

This is a contradiction to the hypothesis. Hence claim 3 follows.

Claim 4: $I'_\mu(u_n) \to 0$ in $(X_0(\Omega))'$ as $n \to \infty$.

Since $u_n \in \mathcal{N}^-_1 \subset \mathcal{N}$, by Lemma 4.4.5 applied to the element u_n, there exists

$$\rho_n := \rho_{u_n} \quad \text{and} \quad g_n := g_{\rho_{u_n}} \tag{4.5.9}$$

such that

$$g_n(0) = 1, \quad \Big(g_n(w)\Big)(u_n + w) \in N \quad \forall \quad w \in B_{\rho_n}(0). \tag{4.5.10}$$

Choose $0 < \tilde{\rho}_n < \rho_n$ such that $\tilde{\rho}_n \to 0$. Let $v \in X_0(\Omega)$ with $\|v\|_{X_0(\Omega)} = 1$.

Define

$$v_n := -\tilde{\rho}_n[v^+\chi_{\{u_n\geq 0\}} - v^-\chi_{\{u_n\leq 0\}}]$$

and

$$\begin{aligned} z_{\tilde{\rho}_n} &:= \Big(g_n(v_n^-)\Big)(u_n - v_n) \\ &=: z^1_{\tilde{\rho}_n} - z^2_{\tilde{\rho}_n}, \end{aligned}$$

where $z^1_{\tilde{\rho}_n} := \Big(g_n(v_n^-)\Big)(u_n^+ + \tilde{\rho}_n v^+\chi_{\{u_n\geq 0\}})$ and $z^2_{\tilde{\rho}_n} := \Big(g_n(v_n^-)\Big)(u_n^- + \tilde{\rho}_n v^-\chi_{\{u_n\leq 0\}})$. Note that $v_n^- = \tilde{\rho}_n v^+\chi_{\{u_n\geq 0\}}$. So, $\|v_n^-\|_{X_0(\Omega)} \leq \tilde{\rho}_n\|v\|_{X_0(\Omega)} \leq \tilde{\rho}_n$. Hence taking $w = v_n^-$ in (4.5.10) we have, $z^+_{\tilde{\rho}_n} = z^1_{\tilde{\rho}_n} \in \mathcal{N}^-$ so $z_{\tilde{\rho}_n} \in \mathcal{N}_1^-$. Hence,

$$I_\mu(z_{\tilde{\rho}_n}) \geq I_\mu(u_n) - \frac{1}{n}\|u_n - z_{\tilde{\rho}_n}\|_{X_0(\Omega)}.$$

This implies,

$$\begin{aligned} \frac{1}{n}\|u_n - z_{\tilde{\rho}_n}\|_{X_0(\Omega)} &\geq I_\mu(u_n) - I_\mu(z_{\tilde{\rho}_n}) \\ &= \Big\langle I'_\mu(u_n), u_n - z_{\tilde{\rho}_n}\Big\rangle + o(1)\|u_n - z_{\tilde{\rho}_n}\|_{X_0(\Omega)} \\ &= -\Big\langle I'_\mu(u_n), z_{\tilde{\rho}_n}\Big\rangle + o(1)\|u_n - z_{\tilde{\rho}_n}\|_{X_0(\Omega)}, \quad (4.5.11) \end{aligned}$$

as $\Big\langle I'_\mu(u_n), u_n\Big\rangle = 0$ for all n. Let $w_n = \tilde{\rho}_n v$. Then,

$$\begin{aligned} \frac{1}{n}\|u_n - z_{\tilde{\rho}_n}\|_{X_0(\Omega)} \geq -\Big\langle I'_\mu(u_n), w_n + z_{\tilde{\rho}_n}\Big\rangle + \Big\langle I'_\mu(u_n), w_n\Big\rangle \\ +o(1)\|u_n - z_{\tilde{\rho}_n}\|_{X_0(\Omega)}. \quad (4.5.12) \end{aligned}$$

Now, $\Big\langle I'_\mu(u_n), w_n\Big\rangle = \Big\langle I'_\mu(u_n), \tilde{\rho}_n v\Big\rangle = \tilde{\rho}_n\Big\langle I'_\mu(u_n), v\Big\rangle$. Define

$$\overline{v_n} := v^+\chi_{\{u_n\geq 0\}} - v^-\chi_{\{u_n\leq 0\}}.$$

So, $z_{\tilde{\rho}_n} = g_n(v_n^-)(u_n - \tilde{\rho}_n\overline{v_n})$. Hence we have,

$$\begin{aligned} \Big\langle I'_\mu(u_n), w_n + z_{\tilde{\rho}_n}\Big\rangle &= \Big\langle I'_\mu(u_n), w_n + g_n(v_n^-)(u_n - \tilde{\rho}_n\overline{v_n})\Big\rangle \\ &= \Big\langle I'_\mu(u_n), \tilde{\rho}_n v - g_n(v_n^-)\tilde{\rho}_n\overline{v_n}\Big\rangle \\ &= \tilde{\rho}_n\Big\langle I'_\mu(u_n), v - g_n(v_n^-)\overline{v_n}\Big\rangle \quad (4.5.13) \end{aligned}$$

Using (4.5.13) in (4.5.12), we have

$$\begin{aligned}\frac{1}{n}\|u_n - z_{\tilde{\rho}_n}\|_{X_0(\Omega)} \geq -\tilde{\rho}_n \left\langle I'_\mu(u_n), v - g_n(v_n^-)\overline{v_n}\right\rangle \\ +\tilde{\rho}_n \left\langle I'_\mu(u_n), v\right\rangle + o(1)\|u_n - z_{\tilde{\rho}_n}\|_{X_0(\Omega)}. \qquad (4.5.14)\end{aligned}$$

First we will estimate $\left\langle I'_\mu(u_n), v - g_n(v_n^-)\overline{v_n}\right\rangle$. For this,

$$\begin{aligned} v - g_n(v_n^-)\overline{v_n} &= v^+ - v^- - g_n(v_n^-)[v^+\chi_{\{u_n\geq 0\}} - v^-\chi_{\{u_n\leq 0\}}] \\ &= v^+[g_n(0) - g_n(v_n^-)\chi_{\{u_n\geq 0\}}] - v^-[g_n(0) - g_n(v_n^-)\chi_{\{u_n\leq 0\}}] \\ &= -v^+[\left\langle g'_n(0), v_n^-\right\rangle + o(1)\|v_n^-\|_{X_0(\Omega)}] + v^-[\left\langle g'_n(0), v_n^-\right\rangle \\ &+ o(1)\|v_n^-\|_{X_0(\Omega)}] \\ &= -v^+\tilde{\rho}_n[\left\langle g'_n(0), v^+\right\rangle + o(1)\|v^+\|_{X_0(\Omega)}] + v^-\tilde{\rho}_n[\left\langle g'_n(0), v^+\right\rangle \\ &+ o(1)\|v^+\|_{X_0(\Omega)}] \\ &= -\tilde{\rho}_n\Big[\left\langle g'_n(0), v^+\right\rangle + o(1)\|v^+\|_{X_0(\Omega)}\Big]v. \end{aligned}$$

Therefore,

$$\left\langle I'_\mu(u_n), v - g_n(v_n^-)\overline{v_n}\right\rangle = -\tilde{\rho}_n\Big(\left\langle g'_n(0), v^+\right\rangle + o(1)\left\|v^+\right\|_{X_0(\Omega)}\Big)\left\langle I'_\mu(u_n), v\right\rangle. \qquad (4.5.15)$$

Claim : $g_n(v_n^-)$ is uniformly bounded in $X_0(\Omega)$.

To see this, we observe that from (4.5.10) we have, $g_n(v_n^-)(u_n^+ + v_n^-) \in \mathcal{N}^- \subset \mathcal{N}$, which implies,

$$\|c_n\tilde{\psi}_n\|^2_{X_0(\Omega)} - \mu|c_n\tilde{\psi}_n|^{q+1}_{L^{q+1}(\Omega)} - |c_n\tilde{\psi}_n|^{2^*}_{L^{2^*}(\Omega)} = 0,$$

where $c_n := g_n(v_n^-)$ and $\tilde{\psi}_n := u_n^+ + v_n^-$. Dividing by $c_n^{2^*}$ we have,

$$c_n^{2-2^*}\|\tilde{\psi}_n\|^2_{X_0(\Omega)} - \mu c_n^{q+1-2^*}|\tilde{\psi}_n|^{q+1}_{L^{q+1}(\Omega)} = |\tilde{\psi}_n|^{2^*}_{L^{2^*}(\Omega)}. \qquad (4.5.16)$$

Note that $\|\tilde{\psi}_n\|_{X_0(\Omega)}$ is uniformly bounded above as $\|u_n\|_{X_0(\Omega)}$ is uniformly bounded and $\tilde{\rho}_n = o(1)$. Also, $\|\tilde{\psi}_n\|_{X_0(\Omega)} \geq \|u_n^+\|_{X_0(\Omega)} - \tilde{\rho}_n\|v\|_{X_0(\Omega)}$. Note that $\|u_n^+\|_{X_0(\Omega)} \geq \tilde{b}$ for large n. If not, then $\|u_n^+\|_{X_0(\Omega)} \to 0$ as $n \to \infty$. As $u_n \in \mathcal{N}_1^-$, so $u_n^+ \in N_\mu^-$. Now, $\mathcal{N}^-$ is a closed set and $0 \notin \mathcal{N}^-$ and therefore $\|u_n^-\|_{X_0(\Omega)} \not\to 0$ as $n \to \infty$. Thus there exists $\tilde{b} \geq 0$ such that $\|u_n^+\|_{X_0(\Omega)} \geq \tilde{b} > 0$. This in turn implies that $\|\tilde{\psi}_n\|_{X_0(\Omega)} \geq C$, for some $C > 0$ by choosing $\tilde{\rho}_n$ small enough. Consequently, if c_n is not uniformly bounded, we obtain LHS of (4.5.16) converges to 0 as $n \to \infty$.

On the other hand,

$$|\tilde{\psi}_n|_{L^{2^*}(\Omega)} \geq |u_n^+|_{L^{2^*}(\Omega)} - \tilde{\rho}_n |v|_{L^{2^*}(\Omega)} > c,$$

for some positive constant c as $\rho_n = o(1)$ and $u_n^+ \in N_\mu^-$ implies

$$(2^* - 1 - q)|u_n^+|_{L^{2^*}(\Omega)}^{2^*} > (1-q)\|u_n^+\|_{X_0(\Omega)}^2 > (1-q)\tilde{b}^2.$$

Hence, the claim follows.

Now using the fact that $g_n(0) = 1$ and the above claim we obtain

$$\begin{aligned}
\|u_n - z_{\tilde{\rho}_n}\|_{X_0(\Omega)} &\leq \|u_n\|_{X_0(\Omega)} \left|1 - g_n(v_n^-)\right| + \tilde{\rho}_n \|\overline{v_n}\|_{X_0(\Omega)} g_n(v_n^-) \\
&\leq \|u_n\|_{X_0(\Omega)} \left[| \left\langle g_n'(0), v_n^- \right\rangle | + o(1) \|\overline{v_n}\|_{X_0(\Omega)} \right] \\
&+ \tilde{\rho}_n \|v\|_{X_0(\Omega)} g_n(v_n^-) \\
&\leq \tilde{\rho}_n \Big[\|u_n\|_{X_0(\Omega)} \left\langle g_n'(0), \overline{v_n}^+ \right\rangle + o(1)\|v\|_{X_0(\Omega)} \\
&+ \|v\|_{X_0(\Omega)} g_n(v_n^-) \Big] \\
&\leq \tilde{\rho}_n C.
\end{aligned}$$

Substituting this and (4.5.15) in (4.5.14) yields

$$\tilde{\rho}_n \bigg(\left\langle g_n'(0), v^+ \right\rangle + o(1)\|v^+\|_{X_0(\Omega)} \bigg) \left\langle I_\mu'(u_n), v \right\rangle + \left\langle I_\mu'(u_n), v \right\rangle \tilde{\rho}_n + \tilde{\rho}_n o(1) \leq \tilde{\rho}_n . \frac{C}{n}.$$

This implies

$$\Big[\Big(\left\langle g_n'(0), v^+ \right\rangle + o(1)\|v^+\|_{X_0(\Omega)} \Big) + 1 \Big] \left\langle I_\mu'(u_n), v \right\rangle \leq \frac{C}{n} + o(1) \quad \text{for all} \quad n \geq n_0.$$

Since $| \langle g_n'(0), v^+ \rangle |$ is uniformly bounded (see Lemma 4.6.1 in Appendix) , letting $n \to \infty$ we have $I_\mu'(u_n) \to 0$ in $(X_0(\Omega))'$. Hence the step 4 follows.

Therefore, $\{u_n\}$ is a (PS) sequence of I_μ at level $\beta_1 < \tilde{\alpha}_\mu^-$. From [32, Proposition 4.2], it follows that

$$\tilde{\alpha}_\mu^- < \frac{s}{N} S_s^{\frac{N}{2s}} - M \mu^{\frac{2^*}{2^*-q-1}} \quad \text{for} \quad \mu \in (0, \mu_*),$$

where $M = \frac{\left(2N-(N-2s)(q+1)\right)(1-q)}{4(q+1)} \left(\frac{(1-q)(N-2s)}{4s}\right)^{\frac{q+1}{2^*-q-1}} |\Omega|$.

Therefore,

$$\beta_1 < \tilde{\alpha}_\mu^- < \frac{s}{N} S_s^{\frac{N}{2s}} - M \mu^{\frac{2^*}{2^*-q-1}}.$$

On the other hand, it follows from the proof of Theorem 3.3.1 (see (3.3.6)) that I_μ satisfies PS at level c for

$$c < \frac{s}{N} S_s^{\frac{N}{2s}} - k \mu^{\frac{2^*}{2^*-q-1}},$$

where k is as in (3.3.14). By elementary analysis, it follows $k = M$. Therefore there exists $u \in X_0(\Omega)$ such that $u_n \to u$ in $X_0(\Omega)$. By doing a simple calculation we get $u_n^- \to u^-$ in $X_0(\Omega)$. Consequently, by Claim 3 $\|u^-\|_{X_0(\Omega)} \geq b$. As $\mathcal{N}_1^-$ is a closed set and $u_n \to u$, we obtain $u \in \mathcal{N}_1^-$, that is, $u^+ \in \mathcal{N}^-$ and $u^+ \neq 0$. Therefore u is a solution of (P) with u^+ and u^- are both nonzero. Hence, u is a sign-changing solution of (P). Define $\tilde{w}_1 := u$. This completes the proof of part (i) of the theorem.

Proof of part (ii) is similar to part (i) and we omit the proof. □

Theorem 4.5.2. *Let β_1, β_2, $\tilde{\alpha}_\mu^-$ be defined as in* (4.5.1) *and* (4.3.1) *respectively. Assume $\beta_1, \beta_2 \geq \tilde{\alpha}_\mu^-$. Then there exists $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0)$, I_μ has a sign changing critical point.*

We need the following Proposition to prove the above Theorem 4.5.2.

Proposition 4.5.3. *Let $N > 6s$ and $\frac{1}{2}(\frac{N+2s}{N-2s}) < q < 1$. Assume $0 < \mu < min\{\mu_*, \tilde{\mu}\}$, where $\tilde{\mu}$ is as defined in* (4.4.4) *and $\mu_* > 0$ is chosen such that*

$\tilde{\alpha}_\mu^-$ *is achieved in* $(0, \mu_*)$. *Then for* $\varepsilon > 0$ *sufficiently small , we have*

$$\sup_{a\geq 0,\ b\in\mathbb{R}} I_\mu(aw_1 - bu_\varepsilon) < \tilde{\alpha}_\mu^- + \frac{s}{N} S_s^{\frac{N}{2s}},$$

where w_1 *and* u_ε *are as in* (4.3.1) *and* (4.2.1) *respectively.*

To prove the above proposition, we need the following lemmas.

Lemma 4.5.4. *Let* w_1 *and* μ *be as in Proposition 4.5.3. Then*

$$\sup_{s>0} I_\mu(sw_1) = \tilde{\alpha}_\mu^-.$$

Proof. By the definition of $\tilde{\alpha}_\mu^-$, we have $\tilde{\alpha}_\mu^- = \inf_{u\in\mathcal{N}^-} J_\mu(u) = J_\mu(w_1) = I_\mu(w_1)$. In the last equality we have used the fact that $w_1 > 0$. Define $g(s) := I_\mu(sw_1)$. From the proof of Lemma 4.4.3, it follows that there exists only two critical points of g, namely $t^+(w_1)$ and $t^-(w_1)$ and $\max_{s>0} g(s) = g(t^+(w_1))$. On the other hand $\langle I'_\mu(w_1), v\rangle = 0$ for every $v \in X_0(\Omega)$. Therefore $g'(1) = 0$. This in turn implies either $t^+(w_1) = 1$ or $t^-(w_1) = 1$.
Claim: $t^-(w_1) \neq 1$.
To see this, we note that $t^-(w_1) = 1$ implies $t^-(w_1)w_1 \in \mathcal{N}^-$ as $w_1 \in \mathcal{N}^-$. Also, from Lemma 4.4.3, we know $t^-(w_1)w_1 \in \mathcal{N}^+$. Thus $\mathcal{N}^+ \cap \mathcal{N}^- \neq \emptyset$, which is a contradiction. Hence the claim follows.
Therefore $t^+(w_1) = 1$ and this completes the proof. □

Lemma 4.5.5. *Let* u_ε *be as in* (4.2.1) *and* μ *be as in Proposition 4.5.3. Then for* $\varepsilon > 0$ *sufficiently small, we have*

$$\sup_{t\in\mathbb{R}} I_\mu(tu_\varepsilon) = \frac{s}{N} S_s^{\frac{N}{2s}} + C\varepsilon^{\frac{(N-2s)N}{2s}} - k_8 |u_\varepsilon|^{q+1}.$$

Proof. Define $\tilde{\phi}(t) = \frac{t^2}{2}\|u_\varepsilon\|^2_{X_0(\Omega)} - \frac{t^{2^*}}{2^*}|u_\varepsilon|^{2^*}_{L^{2^*}(\Omega)}$. Thus $I_\mu(tu_\varepsilon) = \tilde{\phi}(t) - \mu\frac{t^{q+1}}{q+1}|u_\varepsilon|^{q+1}_{L^{q+1}(\Omega)}$. On the other hand, applying the analysis done in Lemma 4.4.3 to u_ε, we obtain there exists $(t_0)_\varepsilon = \left(\frac{(1-q)\|u_\varepsilon\|^2_{X_0(\Omega)}}{(2^*-1-q)|u_\varepsilon|^{2^*}_{L^{2^*}(\Omega)}}\right)^{\frac{N-2s}{4s}} < t_\varepsilon^+$ such that

$$\begin{aligned}\sup_{t\in\mathbb{R}} I_\mu(tu_\varepsilon) = \sup_{t\geq 0} I_\mu(tu_\varepsilon) = I_\mu(t_\varepsilon^+ u_\varepsilon) &= \tilde{\phi}(t_\varepsilon^+) - \mu\frac{(t_\varepsilon^+)^{q+1}}{q+1}|u_\varepsilon|^{q+1}_{L^{q+1}(\Omega)} \\ &\leq \sup_{t\geq 0}\tilde{\phi}(t) - \mu\frac{(t_0)_\varepsilon^{q+1}}{q+1}|u_\varepsilon|^{q+1}_{L^{q+1}(\Omega)}.\end{aligned}$$

Substituting the value of $(t_0)_\varepsilon$ and using Sobolev inequality (4.1.1), we have

$$\mu\frac{(t_0)_\varepsilon^{q+1}}{q+1} \geq \frac{\mu}{q+1}\left(\frac{1-q}{2^*-q-1}S_s\right)^{\frac{(N-2s)(q+1)}{4s}} = k_8.$$

Consequently,

$$\sup_{t\in\mathbb{R}} I_\mu(tu_\varepsilon) \leq \sup_{t\geq 0}\tilde{\phi}(t) - k_8|u_\varepsilon|^{q+1}_{L^{q+1}(\Omega)}. \tag{4.5.17}$$

Using elementary analysis, it is easy to check that $\tilde{\phi}$ attains it's maximum at the point $\tilde{t}_0 = \left(\frac{\|u_\varepsilon\|^2_{X_0(\Omega)}}{|u_\varepsilon|^{2^*}_{L^{2^*}(\Omega)}}\right)^{\frac{1}{2^*-2}}$ and $\tilde{\phi}(t_0) = \frac{s}{N}\left(\frac{\|u_\varepsilon\|^2_{X_0(\Omega)}}{|u_\varepsilon|^2_{L^{2^*}(\Omega)}}\right)^{\frac{N}{2s}}$. Moreover, from Proposition 21 and Proposition 22 of [78], it follows

$$\|u_\varepsilon\|^2_{X_0(\Omega)} = S_s^{\frac{N}{2s}} + O(\varepsilon^{N-2s}), \quad \int_{\mathbb{R}^N}|u_\varepsilon|^{2^*}dx = S_s^{\frac{N}{2s}} + O(\varepsilon^N).$$

As a result,

$$\tilde{\phi}(t_0) \leq \frac{s}{N}\left[\frac{S_s^{\frac{N}{2s}} + O(\varepsilon^{N-2s})}{(S_s^{\frac{N}{2s}} + O(\varepsilon^N))^{\frac{2}{2^*}}}\right]^{\frac{N}{2s}} \leq \frac{s}{N}\left[\frac{S_s^{\frac{N}{2s}} + O(\varepsilon^{N-2s})}{(S_s^{\frac{N}{2s}})^{\frac{2}{2^*}}}\right]^{\frac{N}{2s}} \leq \frac{s}{N}S_s^{\frac{N}{2s}} + C\varepsilon^{\frac{(N-2s)N}{2s}}. \tag{4.5.18}$$

In the last inequality we have used the fact that $\varepsilon > 0$ is arbitrary small. Substituting back (4.5.18) into (4.5.17), completes the proof. □

Proof of Proposition 4.5.3: Note that, for fixed a and b, $I_\mu\Big(\eta(aw_1 - bu_\varepsilon)\Big) \to -\infty$ as $|n| \to \infty$. Therefore $\sup_{a\geq 0,\ b\in\mathbb{R}} I_\mu(aw_1 - bu_\varepsilon)$ exists and supremum will be attained in $a^2+b^2 \leq R^2$, for some large $R > 0$. Thus it is enough to estimate $I_\mu(aw_1 - bu_\varepsilon)$ in $\{(a,b)\in\mathbb{R}^+\times\mathbb{R} : a^2+b^2\leq R^2\}$. Using elementary inequality, there exists $d(m) > 0$ such that

$$|a+b|^m \geq |a|^m + |b|^m - d(|a|^{m-1}|b| + |a||b|^{m-1}) \quad \forall \quad a,\ b\in\mathbb{R},\ m > 1. \tag{4.5.19}$$

Therefore, $a^2+b^2 \le R^2$ implies

$$\begin{aligned}
I_\mu(aw_1-bu_\varepsilon) &\le \frac{1}{2}\|aw_1\|^2_{X_0(\Omega)} - ab\langle w_1,u_\varepsilon\rangle + \frac{1}{2}\|bu_\varepsilon\|^2_{X_0(\Omega)}\\
&- \frac{1}{2^*}\int_\Omega |aw_1|^{2^*}dx - \frac{1}{2^*}\int_\Omega |bu_\varepsilon|^{2^*}dx\\
&- \frac{\mu}{q+1}\int_\Omega |aw_1|^{q+1}dx - \frac{\mu}{q+1}\int_\Omega |bu_\varepsilon|^{q+1}dx\\
&+ C\left(\int_\Omega |aw_1|^{2^*-1}|bu_\varepsilon|dx + \int_\Omega |aw_1||bu_\varepsilon|^{2^*-1}dx\right)\\
&+ C\left(\int_\Omega |aw_1|^{q}|bu_\varepsilon|dx + \int_\Omega |aw_1||bu_\varepsilon|^{q}dx\right)\\
&= I_\mu(aw_1) + I_\mu(bu_\varepsilon) - ab\mu\int_\Omega |w_1|^{q-1}w_1u_\varepsilon dx\\
&- ab\int_\Omega |w_1|^{2^*-2}w_1u_\varepsilon dx\\
&+ C\left(\int_\Omega |w_1|^{2^*-1}|u_\varepsilon|dx + \int_\Omega |w_1||u_\varepsilon|^{2^*-1}dx\right)\\
&+ C\left(\int_\Omega |w_1|^{q}|u_\varepsilon|dx + \int_\Omega |w_1||u_\varepsilon|^{q}dx\right).
\end{aligned}$$

Using Lemmas 4.4.1, 4.5.4 and 4.5.5 we estimate in $a^2+b^2 \le R^2$,

$$I_\mu(aw_1-bu_\varepsilon) \le \tilde{\alpha}_\mu^- + \frac{s}{N}S_s^{\frac{N}{2s}} - k_8|u_\varepsilon|^{q+1} + C(\varepsilon^{\frac{(N-2s)N}{2s}} + \varepsilon^{\frac{N-2s}{4}} + \varepsilon^{\frac{(N-2s)q}{4}} + \varepsilon^{\frac{N+2s}{4}}).$$

Since $N > 2s$ and $q \in (0,1)$, clearly $\varepsilon^{(\frac{N-2s}{4})q}$ is the dominating term among all the terms inside the bracket. For the term $k_8|u_\varepsilon|^{q+1}$, we invoke Lemma 4.4.2. Therefore when $\frac{2s}{N-2s} < q < 1$, we have

$$\begin{aligned}
I_\mu(aw_1-bu_\varepsilon) &\le \tilde{\alpha}_\mu^- + \frac{s}{N}S_s^{\frac{N}{2s}} - k_9\varepsilon^{\frac{N}{2}-(\frac{N-2s}{4})(q+1)}\\
&\quad + C(\varepsilon^{\frac{(N-2s)N}{2s}} + \varepsilon^{\frac{N-2s}{4}} + \varepsilon^{\frac{(N-2s)q}{4}} + \varepsilon^{\frac{N+2s}{4}})
\end{aligned}$$

This in turn implies, when $\frac{1}{2}(\frac{N+2s}{N-2s}) < q < 1$ and $N > 6s$, $\varepsilon^{\frac{N}{2}-(\frac{N-2s}{4})(q+1)}$ should be the dominating one among all the ε terms and hence in this case, taking $\varepsilon > 0$ to be small enough, we obtain

$$\sup_{a\ge 0, b\in\mathbb{R}} I_\mu(aw_1-bu_\varepsilon) < \tilde{\alpha}_\mu^- + \frac{s}{N}S_s^{\frac{N}{2s}}.$$

□

Proof of Theorem 4.5.2: Define $\mu_0 := min\{\tilde{\mu}, \mu_*\}$ and

$$c_2 : \inf_{u \in \mathcal{N}_*^-} I_\mu(u), \tag{4.5.20}$$

where

$$\mathcal{N}_*^- := \mathcal{N}_1^- \cap \mathcal{N}_2^-. \tag{4.5.21}$$

Let $\mu \in (0, \mu_0)$. Using Ekland's variational principle and similar to the proof of Theorem 4.5.1, we obtain a sequence $\{u_n\} \in \mathcal{N}_*^-$ satisfying

$$I_\mu(u_n) \to c_2, \quad I'_\mu(u_n) \to 0 \quad \text{in} \quad (X_0(\Omega))'.$$

Thus $\{u_n\}$ is a (PS) sequence at level c_2. From Lemma 4.5.6, it follows that there exists $a > 0$ and $b \in R$ such that $aw_1 - bu_\varepsilon \in \mathcal{N}_*^-$. Therefore Proposition 4.5.3 yields

$$c_2 < \tilde{\alpha}_\mu^- + \frac{s}{N} S_s^{\frac{N}{2s}}. \tag{4.5.22}$$

Claim 1: There exists two positive constants c, C such that $0 < c \leq \|u_n^\pm\|_{X_0(\Omega)} \leq C$.

To see this, we note that $\{u_n\} \subset \mathcal{N}_*^- \subset \mathcal{N}_1^-$. Therefore using (4.5.6), Claim 2 and Claim 3 of the proof of Theorem 4.5.1, we have $\|u_n^\pm\|_{X_0(\Omega)} \leq C$ and $\|u_n^-\|_{X_0(\Omega)} \geq c$. To show $\|u_n^+\|_{X_0(\Omega)} \geq a$ for some $a > 0$, we use method of contradiction. Assume up to a subsequence $\|u_n^+\|_{X_0(\Omega)} \to 0$ as $n \to \infty$. This together with Sobolev embedding implies $|u_n^+|_{L^{2^*}(\Omega)} \to 0$. On the other hand, $u_n^+ \in \mathcal{N}^-$ implies $(1-q)\,\|u_n^+\|^2_{X_0(\Omega)} - (2^* - q - 1)|u_n^+|^{2^*}_{L^{2^*}(\Omega)} < 0$. Therefore by (4.1.1), we have

$$S_s \leq \frac{\|u_n^+\|^2_{X_0(\Omega)}}{|u_n^+|^2_{L^{2^*}(\Omega)}} < \frac{2^* - q - 1}{1 - q} |u_n^+|^{2^*-2}_{L^{2^*}(\Omega)},$$

which is a contradiction to the fact that $|u_n^+|_{L^{2^*}(\Omega)} \to 0$. Hence the claim follows.

Going to a subsequence if necessary we have

$$u_n^+ \rightharpoonup \eta_1,\ u_n^- \rightharpoonup \eta_2 \quad \text{in} \quad X_0(\Omega). \tag{4.5.23}$$

Claim 2: $\eta_1 \not\equiv 0,\ \eta_2 \not\equiv 0$.

Suppose not, that is $\eta_1 \equiv 0$. Then by compact embedding, $u_n^+ \to 0$ in $L^{q+1}(\Omega)$. Moreover, $u_n^+ \in \mathcal{N}^- \subset \mathcal{N}$, implies $\left\langle I'_\mu(u_n^+), u_n^+ \right\rangle = 0$. As a consequence,

$$\|u_n^+\|^2_{X_0(\Omega)} - |u_n^+|^{2^*}_{L^{2^*}(\Omega)} = \mu |u_n^+|^{q+1}_{L^{q+1}(\Omega)} = o(1).$$

So we have $|u_n^+|^{2^*}_{L^{2^*}(\Omega)} = \|u_n^+\|^2_{X_0(\Omega)} + o(1)$. This together with $\|u_n^+\|_{X_0(\Omega)} \geq c$ implies

$$\frac{|u_n^+|^{2^*}_{L^{2^*}(\Omega)}}{\|u_n^+\|^2_{X_0(\Omega)}} \geq 1 + o(1).$$

This along with Sobolev embedding gives $|u_n^+|^{2^*}_{L^{2^*}(\Omega)} \geq S_s^{N/2s} + o(1)$. Thus we have,

$$I_\mu(u_n^+) = \frac{1}{2}\|u_n^+\|^2_{X_0(\Omega)} - \frac{1}{2^*}|u_n^+|^{2^*}_{L^{2^*}(\Omega)} + o(1) \geq \frac{s}{N} S_s^{N/2s} + o(1). \tag{4.5.24}$$

Moreover, $u_n \in \mathcal{N}_*^-$ implies $-u_n^- \in \mathcal{N}^-$. Therefore using the given condition on β_2, we get

$$I_\mu(-u_n^-) \geq \beta_2 \geq \tilde{\alpha}_\mu^-. \tag{4.5.25}$$

Also it follows $I_\mu(u_n^+) + I_\mu(-u_n^-) \leq I_\mu(u_n) = c_2 + o(1)$ (see (4.5.8)). Combining this along with (4.5.25) and (4.5.22), we obtain

$$I_\mu(u_n^+) \leq c_2 - \tilde{\alpha}_\mu^- + o(1) < \frac{s}{N} S_s^{N/2s},$$

which is a contradiction to (4.5.24). Therefore, $\eta_1 \neq 0$. Similarly, $\eta_2 \neq 0$ and this proves the claim.

Set $w_2 := \eta_1 - \eta_2$.

Claim 3: $w_2^+ = \eta_1$ and $w_2^- = \eta_2$ a.e..

To see the claim we observe that $\eta_1\eta_2 = 0$ a.e. in Ω. Indeed,

$$\begin{aligned} |\int_\Omega \eta_1\eta_2 dx| &= |\int_\Omega (u_n^+ - \eta_1)u_n^- dx + \int_\Omega \eta_1(u_n^- - \eta_2)dx| \\ &\leq |u_n^+ - \eta_1|_{L^2(\Omega)}|u_n^-|_{L^2(\Omega)} + |\eta_1|_{L^2(\Omega)}|u_n^- - \eta_2|_{L^2(\Omega)} \end{aligned} \tag{4.5.26}$$

By compact embedding we have $u_n^+ \to \eta_1$ and $u_n^- \to \eta_2$ in $L^2(\Omega)$. Therefore using claim 1, we pass the limit in (4.5.26) and obtain $\int_\Omega \eta_1\eta_2 dx = 0$. Moreover by (4.5.23), $\eta_1,\ \eta_2 \geq 0$ a.e.. Hence $\eta_1\eta_2 = 0$ a.e. in Ω. We have $w_2^+ - w_2^- = w_2 = \eta_1 - \eta_2$. It is easy to check that $w_2^+ \leq \eta_1$ and $w_2^- \leq \eta_2$. To show that equality holds a.e. we apply the method of contradiction. Suppose, there exists $E \in \Omega$ such that $|E| > 0$ and $0 \leq w_2^+(x) < \eta_1(x)\ \forall\ x \in E$. Therefore $\eta_2 = 0$ a.e. in E by the observation that we made. Hence $w_2^+(x) - w_2^-(x) = \eta_1(x)$ a.e. in E. Clearly $w_2^-(x) \not> 0$ a.e., otherwise $w_2^+(x) = 0$ a.e. and that would imply $\eta_1(x) = -w_2^-(x) < 0$ a.e, which is not possible since $\eta_1 > 0$ in E. Thus $w_2^-(x) = 0$. This yields $\eta_1(x) = w_2^+(x)$ a.e. in E, which is a contradiction. Thus the claim follows.

Therefore w_2 is sign changing in Ω and $u_n \rightharpoonup w_2$ in $X_0(\Omega)$. Moreover, $I'_\mu(u_n) \to 0$ in $(X_0(\Omega))'$ implies

$$\int_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\phi(x) - \phi(y))}{|x-y|^{N+2s}} dxdy - \mu \int_\Omega |u_n|^{q-1} u_n \phi dx - \int_\Omega |u_n|^{2^*-2} u_n \phi dx = o(1),$$

for every $\phi \in X_0(\Omega)$. Passing the limit using Vitali's convergence theorem via Hölder's inequality we obtain $\left\langle I'_\mu(w_2), \phi \right\rangle = 0$. As a result, w_2 is a sign changing weak solution to (P). □

Lemma 4.5.6. *Let u_ε be as defined in* (4.2.1) *and w_1 be a positive solution of (P) for which $\tilde{\alpha}_\mu^-$ is achieved, when $\mu \in (0, \mu_*)$. Then there exists $a,\ b \in \mathbb{R},\ a \geq 0$ such that $aw_1 - bu_\varepsilon \in \mathcal{N}_*^-$, where $\mathcal{N}_*^-$ is defined as in* (4.5.21).

Proof. We will show that there exists $a > 0,\ b \in \mathbb{R}$ such that

$$a(w_1 - bu_\varepsilon)^+ \in \mathcal{N}^- \quad \text{and} \quad -a(w_1 - bu_\varepsilon)^- \in N^-.$$

Let us denote $\bar{r}_1 = \inf_{x\in\Omega} \frac{w_1(x)}{u_\varepsilon(x)},\ \bar{r}_2 = \sup_{x\in\Omega} \frac{w_1(x)}{u_\varepsilon(x)}$.
As both w_1 and u_ε are positive in Ω, we have $\bar{r}_1 \geq 0$ and $\bar{r}_2$ can be $+\infty$. Let $r \in (\bar{r}_1, \bar{r}_2)$. Then $w_1, u_\varepsilon \in X_0(\Omega)$ implies $(w_1 - ru_\varepsilon) \in X_0(\Omega)$ and

$(w_1 - ru_\varepsilon)^+ \not\equiv 0$. Otherwise, $(w_1 - ru_\varepsilon)^+ \equiv 0$ would imply $\bar{r}_2 \leq r$, which is not possible. Define $v_r := w_1 - ru_\varepsilon$. Then $0 \not\equiv v_r^+ \in X_0(\Omega)$ (since for any $u \in X_0(\Omega)$, we have $|u| \in X_0(\Omega)$). Similarly, $0 \not\equiv v_r^- \in X_0(\Omega)$. Therefore, by Lemma 4.4.3 there exists $0 < s^+(r) < s^-(r)$ such that $s^+(r)v_r^+ \in \mathcal{N}^-$, and $-s^-(r)(v_r^-) \in \mathcal{N}^-$. Let us consider the functions $s^\pm : \mathbb{R} \to (0, \infty)$ defined as above.

Claim: The functions $r \mapsto s^\pm(r)$ are continuous and

$$\lim_{r \to \bar{r}_1^+} s^+(r) = t^+(v_{\bar{r}_1}^+) \quad \text{and} \quad \lim_{r \to \bar{r}_2^-} s^+(r) = +\infty,$$

where the function t^+ is same as defined in Lemma 4.4.3.

To see the claim, choose $r_0 \in (\bar{r}_1, \bar{r}_2)$ and $\{r_n\}_{n\geq 1} \subset (\bar{r}_1, \bar{r}_2)$ such that $r_n \to r_0$ as $n \to \infty$. We need to show that $s^+(r_n) \to s^+(r_0)$ as $n \to \infty$. Corresponding to r_n and r_0, we have $v_{r_n}^+ = (w_1 - r_n u_\varepsilon)^+$ and $v_{r_0}^+ = (w_1 - r_0 u_\varepsilon)^+$. By Lemma 4.4.3. we note that $s^+(r) = t^+(v_r^+)$. Let us define the function

$$\begin{aligned} F(s, r) &:= s^{1-q}\|(w_1 - ru_\varepsilon)^+\|_{X_0(\Omega)}^2 - s^{2^*-q-1}|(w_1 - ru_\varepsilon)^+|_{L^{2^*}(\Omega)}^{2^*} \\ &- \mu|(w_1 - ru_\varepsilon)^+|_{L^{q+1}(\Omega)}^{q+1} \\ &= \phi(s, r) - \mu|(w_1 - ru_\varepsilon)^+|_{L^{q+1}(\Omega)}^{q+1}, \end{aligned}$$

where

$$\phi := s^{1-q}\|(w_1 - ru_\varepsilon)^+\|_{X_0(\Omega)}^2 - s^{2^*-q-1}|(w_1 - ru_\varepsilon)^+|_{L^{2^*}(\Omega)}^{2^*},$$

is defined similar to (4.4.5) (see Lemma 4.4.3). Doing the similar calculation as in lemma 4.4.3, we obtain that for any fixed r, the function $F(s, r)$ has only two zeros $s = t^+(v_r^+)$ and $s = t^-(v_r^+)$ (see (4.4.9)). Consequently $s^+(r)$ is the largest 0 of $F(s, r)$ for any fixed r. As $r_n \to r_0$ we have $v_{r_n}^+ \to v_{r_0}^+$ in $X_0(\Omega)$. Indeed, by straight forward computation it follows $v_{r_n} \to v_{r_0}$ in $X_0(\Omega)$. Therefore $|v_{r_n}| \to |v_{r_0}|$ in $X_0(\Omega)$. This in turn implies $v_{r_n}^+ \to v_{r_0}^+$ in $X_0(\Omega)$. Hence $\|v_{r_n}^+\|_{X_0(\Omega)} \to \|v_{r_0}^+\|_{X_0(\Omega)}$. Moreover by Sobolev inequality, we have $|v_{r_n}^+|_{L^{2^*}(\Omega)} \to |v_{r_0}^+|_{L^{2^*}(\Omega)}$ and $|v_{r_n}^+|_{L^{q+1}(\Omega)} \to |v_{r_0}^+|_{L^{q+1}(\Omega)}$. As a result, we have $F(s, r_n) \to F(s, r_0)$ uniformly. Therefore an elementary analysis yields $s^+(r_n) \to s^+(r_0)$.

Moreover, $\bar{r}_2 \geq \frac{w_1}{u_\varepsilon}$ implies $w_1 - \bar{r}_2 u_\varepsilon \leq 0$. As a consequence $r \to \bar{r}_2^-$ implies $(w_1 - ru_\varepsilon)^+ \to 0$ pointwise. Moreover, since $|(w_1 - ru_\varepsilon)^+|_{L^\infty(\Omega)} \leq |w_1|_{L^\infty(\Omega)}$, using dominated convergence theorem we have
$|(w_1 - ru_\varepsilon)^+|_{L^{2^*}(\Omega)} \to 0$. From the analysis in Lemma 4.4.3, for any r, we also have $s^+(r) > t_0(v_r^+)$, where function t_0 is defined as in (4.4.6), which is the maximum point of $\phi(\cdot, r)$. Therefore it is enough to show that $\lim_{r \to \bar{r}_2^-} t_0(v_r^+) = \infty$. Applying (4.1.1) in the definition of $t_0(v_r^+)$ we get

$$t_0(v_r^+) = \left(\frac{(1-q)\|v_r^+\|^2_{X_0(\Omega)}}{(2^* - 1 - q)|v_r^+|^{2^*}_{L^{2^*}(\Omega)}} \right)^{\frac{1}{2^*-2}} \geq \left(\frac{S_s(1-q)}{2^* - 1 - q} \right)^{\frac{1}{2^*-2}} |v_r^+|^{-1}_{L^{2^*}(\Omega)}.$$

Thus $\lim_{r \to \bar{r}_2^-} t_0(v_r^+) = \infty$.

Similarly proceeding as above we can show that if $r \to \bar{r}_1^-$ then $v_r^+ \to v_{\bar{r}_1}$ and $\lim_{r \to \bar{r}_1^+} s^+(r) = t^+(v_{\bar{r}_1}^+)$ and

$$\lim_{r \to r_1^+} s^-(r) = +\infty, \ \lim_{r \to r_2^-} s^-(r) = t^+(v_r^-) < +\infty.$$

The continuity of $s^\pm$ implies that there exists $b \in (\bar{r}_1, \bar{r}_2)$ such that $s^+(r) = s^-(r) = a > 0$. Therefore,

$$a(w_1 - bu_\varepsilon^+) \in \mathcal{N}^- \quad \text{and} \quad -a(w_1 - bu_\varepsilon^-) \in \mathcal{N}^-,$$

that is, the function $a(w_1 - bu_\varepsilon) \in \mathcal{N}_*^-$ and this completes the proof.

□

Now, we conclude the proof of our main theorem.

Proof of Theorem 4.0.1: Define $\mu^* = \min\{\mu_*, \tilde{\mu}, \mu_0, \mu_1\}$. Combining Theorem 4.5.1 and Theorem 4.5.2, we complete the proof of this theorem for $\mu \in (0, \mu^*)$

. □

4.6 Appendix

Lemma 4.6.1. *Let g_n be as in* (4.5.9) *in the Theorem 4.5.1 and $v \in X_0(\Omega)$ such that $\|v\|_{X_0(\Omega)} = 1$. Then there exists $\mu_1 > 0$ such that, $\mu \in (0, \mu_1)$ implies $\langle g_n'(0), v\rangle$ is uniformly bounded in $X_0(\Omega)$.*

Proof. In view of Lemma 4.4.5 we have,

$$\langle g_n'(0), v\rangle = \frac{2\langle u_n, v\rangle - 2^* \int_\Omega |u_n|^{2^*-2} u_n v - (q+1)\mu \int_\Omega |u_n|^{q-1} u_n v}{(1-q)\|u_n\|^2_{X_0(\Omega)} - (2^*-q-1)|u_n|^{2^*}_{L^{2^*}(\Omega)}}.$$

Using Claim 2 in theorem 4.5.1, there exists $C > 0$ such that $\|u_n\|_{X_0(\Omega)} \leq C$ for all $n \geq 1$. Therefore applying Hölder inequality followed by (4.1.1), we have

$|\langle g_n'(0), v\rangle| \leq \frac{C\|v\|_{X_0(\Omega)}}{|(1-q)\|u_n\|^2_{X_0(\Omega)} - (2^*-q-1)|u_n|^{2^*}_{L^{2^*}(\Omega)}|}$. Hence it is enough to show

$$|(1-q)\|u_n\|^2_{X_0(\Omega)} - (2^*-q-1)|u_n|^{2^*}_{L^{2^*}(\Omega)}| > C,$$

for some $C > 0$ and n large. Suppose it does not hold. Then up to a subsequence

$$(1-q)\|u_n\|^2_{X_0(\Omega)} - (2^*-q-1)|u_n|^{2^*}_{L^{2^*}(\Omega)} = o(1) \quad \text{as} \quad n \to \infty.$$

Hence,

$$\|u_n\|^2_{X_0(\Omega)} = \frac{2^*-q-1}{1-q}|u_n|^{2^*}_{L^{2^*}(\Omega)} + o(1) \quad \text{as} \quad n \to \infty. \tag{4.6.1}$$

Combining the above expression along with the fact that $u_n \in \mathcal{N}$, we obtain

$$\mu|u_n|^{q+1}_{L^{q+1}(\Omega)} = \frac{2-2^*}{1-q}|u_n|^{2^*}_{L^{2^*}(\Omega)} + o(1) = \frac{2^*-2}{2^*-1-q}\|u_n\|^2_{X_0(\Omega)} + o(1). \tag{4.6.2}$$

After applying Hölder inequality and followed by (4.1.1), expression (4.6.2) yields

$$\|u_n\|_{X_0(\Omega)} \leq \left(\mu \frac{2^*-q-1}{2^*-2}|\Omega|^{\frac{2^*-q-1}{2^*}} S_s^{-\frac{q+1}{2}}\right)^{\frac{1}{1-q}} + o(1). \tag{4.6.3}$$

Combining (4.5.6) and Claim 3 in the proof of Theorem 4.5.1, we have $\|u_n\|_{X_0(\Omega)} \geq b$, for some $b > 0$. Therefore from (4.6.1) we get

$$|u_n|^{2^*}_{L^{2^*}(\Omega)} \geq C \quad \text{for some constant } C > 0, \text{ and } n \text{ large enough.} \tag{4.6.4}$$

Define $\psi_\mu : \mathcal{N} \to \mathbb{R}$ as follows:

$$\psi_\mu(u) = k_0 \left(\frac{\|u\|^{2(2^*-1)}_{X_0(\Omega)}}{|u|^{2^*}_{L^{2^*}(\Omega)}} \right)^{\frac{1}{2^*-2}} - \mu |u|^{q+1}_{L^{q+1}(\Omega)},$$

where $k_0 = \left(\frac{1-q}{2^*-q-1}\right)^{\frac{N+2s}{4s}} \left(\frac{2^*-2}{1-q}\right)$. Simplifying $\psi_\mu(u_n)$ using (4.6.2), we obtain

$$\psi_\mu(u_n) = k_0 \left[\left(\frac{2^*-q-1}{1-q} \right)^{2^*-1} \frac{|u_n|^{(2^*-1)2^*}_{L^{2^*}(\Omega)}}{|u_n|^{2^*}_{L^{2^*}(\Omega)}} \right]^{\frac{1}{2^*-2}} - \frac{2^*-2}{1-q} |u_n|^{2^*}_{L^{2^*}(\Omega)} + o(1) = o(1). \tag{4.6.5}$$

On the other hand, using Hölder inequality in the definition of $\psi_\mu(u_n)$, we obtain

$$\begin{aligned} \psi_\mu(u_n) &= k_0 \left(\frac{\|u_n\|^{2(2^*-1)}_{X_0(\Omega)}}{|u_n|^{2^*}_{L^{2^*}(\Omega)}} \right)^{\frac{1}{2^*-2}} - \mu |u_n|^{q+1}_{L^{q+1}(\Omega)} \\ &\geq k_0 \left(\frac{\|u_n\|^{2(2^*-1)}_{X_0(\Omega)}}{|u_n|^{2^*}_{L^{2^*}(\Omega)}} \right)^{\frac{1}{2^*-2}} - \mu |\Omega|^{\frac{2^*-q-1}{2^*}} |u_n|^{q+1}_{L^{2^*}(\Omega)} \\ &= |u_n|^{q+1}_{L^{2^*}(\Omega)} \left\{ k_0 \left(\frac{\|u_n\|^{2(2^*-1)}_{X_0(\Omega)}}{|u_n|^{2^*}_{L^{2^*}(\Omega)}} \right)^{\frac{1}{2^*-2}} \frac{1}{|u_n|^{q+1}_{L^{2^*}(\Omega)}} - \mu |\Omega|^{\frac{2^*-q-1}{2^*}} \right\}. \end{aligned} \tag{4.6.6}$$

Using (4.1.1) and (4.6.3), we simplify the term $\left(\frac{\|u_n\|^{2(2^*-1)}_{X_0(\Omega)}}{|u_n|^{2^*}_{L^{2^*}(\Omega)}} \right)^{\frac{1}{2^*-2}} \frac{1}{|u_n|^{q+1}_{L^{2^*}(\Omega)}}$ and obtain

$$\begin{aligned} \left(\frac{\|u_n\|^{2(2^*-1)}_{X_0(\Omega)}}{|u_n|^{2^*}_{L^{2^*}(\Omega)}} \right)^{\frac{1}{2^*-2}} \frac{1}{|u_n|^{q+1}_{L^{2^*}(\Omega)}} &\geq S_s^{\frac{N+2s}{4s}} |u_n|^{-q}_{L^{2^*}(\Omega)} \\ &\geq S_s^{\frac{N+2s(q+1)}{4s}} \|u_n\|^{-q} \\ &\geq S_s^{\frac{N+2s(q+1)}{4s}} \left(\mu \frac{2^*-q-1}{2^*-2} |\Omega|^{\frac{2^*-q-1}{2^*}} S_s^{-\frac{q+1}{2}} \right)^{-\frac{q}{1-q}}. \end{aligned} \tag{4.6.7}$$

Substituting back (4.6.7) into (4.6.6) and using (4.6.4), we obtain

$$\begin{aligned}\psi_\mu(u_n) \geq C^{q+1}\Big[&k_0 S^{\frac{N+2s(q+1)}{4s}+(\frac{1+q}{1-q}\frac{q}{2})}\mu^{-\frac{q}{1-q}}\Big(\frac{2^*-q-1}{2^*-2}|\Omega|^{\frac{2^*-q-1}{2^*}}\Big)^{-\frac{q}{1-q}} \\ &- \mu|\Omega|^{\frac{2^*-q-1}{2^*}}\Big] \geq d_0,\end{aligned} \tag{4.6.8}$$

for some $d_0 > 0$, n large and $\mu < \mu_1$, where $\mu_1 = \mu_1(k, s, q, N, |\Omega|)$. This is a contradiction to (4.6.5). Hence the lemma follows. □

Conclusion: To be precise, this chapter deals with the existence of at least one sign-changing solution in the critical case using concave-convex nonlinearities. Since we are working in the non-local case, the computations are not straightforward. But one of the major difficulties that we have is

$$\|u\|^2_{X_0} \geq \|u^+\|^2_{X_0} + \|u^-\|^2_{X_0},$$

whereas in the classical case we have the equality. This created a lot of difficulties in getting the desired estimates and overcoming these difficulties were quiet challenging. The rectitude of our work lies in vanquishing these difficulties.

Chapter 5

Sign changing solutions for p fractional Laplacian type equations with concave-critical nonlinearities

This chapter is the generalization of the previous chapter. We have done similar kind of analysis but in the p-fractional case. This chapter is based on [15].

We consider the fractional p-Laplace equation with concave-critical non-linearities

$$(\mathcal{P}_\mu)\begin{cases}(-\Delta)^s_p u = \mu|u|^{q-1}u + |u|^{p^*_s-2}u & \text{in} \quad \Omega,\\ u = 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega,\end{cases}$$

where $s \in (0,1)$, $p > 1$ are fixed, $N > ps$, Ω is an open, bounded domain in $\mathbb{R}^N$ with smooth boundary, $0 < q < p-1$, $p^*_s = \frac{Np}{N-ps}$ and $\mu \in \mathbb{R}^+$ and the non-local operator $(-\Delta)^s_p$ is defined in Section 2.3.1.

Definition 5.0.1. *(**Weak solution**) We say that $u \in X_{0,s,p}(\Omega)$ is a weak* *solution of* $(\mathcal{P}_\mu)$ *if*

$$\begin{aligned}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}}dxdy &= \mu\int_\Omega |u|^{q-1}u\phi dx \\ &+ \int_\Omega |u|^{p_s^*-2}u\phi dx,\end{aligned}$$

for all $\phi \in X_{0,s,p}(\Omega)$.

5.1 Variational formulation of the problem

The Euler-Lagrange energy functional associated to $(\mathcal{P}_\mu)$ is

$$\begin{aligned} I_\mu(u) &= \frac{1}{p}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy - \frac{\mu}{q+1}\int_\Omega |u|^{q+1}dx - \frac{1}{p_s^*}\int_\Omega |u|^{p_s^*}dx \\ &= \frac{1}{p}\|u\|^p_{X_{0,s,p}(\Omega)} - \frac{\mu}{q+1}|u|^{q+1}_{L^{q+1}(\Omega)} - \frac{1}{p_s^*}|u|^{p_s^*}_{L^{p_s^*}(\Omega)}. \end{aligned} \tag{5.1.1}$$

We define the best fractional critical Sobolev constant $S_{s,p}$ as

$$S_{s,p} := \inf_{v\in W^{s,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}dxdy}{\left(\displaystyle\int_{\mathbb{R}^N} |v(x)|^{p_s^*}dx\right)^{p/p_s^*}}, \tag{5.1.2}$$

which is positive by fractional Sobolev inequality. Thanks to the continuous Sobolev embedding $X_{0,s,p}(\Omega) \hookrightarrow L^{p_s^*}(\mathbb{R}^N)$, I_μ is well defined C^1 functional on $X_{0,s,p}(\Omega)$. It is well known that there exists a one-to-one correspondence between the weak solutions of $(\mathcal{P}_\mu)$ and the critical points of I_μ on $X_{0,s,p}(\Omega)$.

Why studying the p-fractional case?

Since the embedding $X_{0,s,p}(\Omega) \hookrightarrow L^{p_s^*}$ is not compact, I_μ does not satisfy the Palais-Smale condition globally, but that holds true when the energy level falls inside a suitable range related to $S_{s,p}$. As it was mentioned in [27], the main difficulty dealing with critical fractional case with $p \neq 2$, is the lack of an explicit formula for minimizers of $S_{s,p}$ which is very often a key tool to handle the estimates leading to the compactness range of I_μ. This difficulty has been tactfully overcome in [27] and [64] by the optimal asymptotic behavior of minimizers, which was recently obtained in [20]. Using the same optimal asymptotic behavior of minimizer of $S_{s,p}$, we will establish suitable compactness range.

5.2 Main result

The main result of this chapter is the following:

Theorem 5.2.1. *Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$. Let $s \in (0,1)$, $p \geq 2$. Then there exist $\mu^* > 0$, $N_0 > 0$ and $q_0 \in (0,\ p-1)$ such that for all $\mu \in (0, \mu^*)$, $N > N_0$ and $q \in (q_0,\ p-1)$, problem $(\mathcal{P}_\mu)$ has at least one sign changing solution, where N_0 is given by the following relation:*

$$N_0 := \begin{cases} sp(p+1) & \text{when} \quad 2 \leq p < \frac{3+\sqrt{5}}{2}, \\ sp(p^2 - p + 1) & \text{when} \quad p \geq \frac{3+\sqrt{5}}{2}. \end{cases}$$

Define the Nehari-manifold N_μ by

$$N_\mu := \left\{ u \in X_{0,s,p}(\Omega) \setminus \{0\} \,\middle|\, \langle I'_\mu(u), u \rangle_{X_{0,s,p}(\Omega)} = 0 \right\}.$$

The Nehari manifold N_μ is closely linked to the behavior of the fibering map $\varphi_u : (0, \infty) \to \mathbb{R}$ defined by

$$\varphi_u(r) := I_\mu(ru) = \frac{r^p}{p} \|u\|^p_{X_{0,s,p}(\Omega)} - \frac{\mu r^{q+1}}{q+1} |u|^{q+1}_{L^{q+1}(\Omega)} - \frac{r^{p_s^*}}{p_s^*} |u|^{p_s^*}_{L^{p_s^*}(\Omega)},$$

which was first introduced by Drabek and Pohozaev in [41].

Lemma 5.2.2. *For any $u \in X_{0,s,p}(\Omega) \setminus \{0\}$, we have $ru \in N_\mu$ if and only if $\varphi'_u(r) = 0$.*

Proof. We note that for $r > 0$, $\varphi'_u(r) = \langle I'_\mu(ru), u \rangle_{X_{0,s,p}(\Omega)} = \frac{1}{r} \langle I'_\mu(ru), ru \rangle_{X_{0,s,p}(\Omega)}$. Hence, $\varphi'_u(r) = 0$ if and only if $ru \in N_\mu$. □

Therefore, we can conclude that the elements in N_μ corresponds to the stationary point of the maps φ_u. Observe that

$$\varphi_u'(r) = r^{p-1}\|u\|^p_{X_{0,s,p}(\Omega)} - \mu r^q |u|^{q+1}_{L^{q+1}(\Omega)} - r^{p_s^*-1}|u|^{p_s^*}_{L^{p_s^*}(\Omega)} \tag{5.2.1}$$

and

$$\varphi_u''(r) = (p-1)r^{p-2}\|u\|^p_{X_{0,s,p}(\Omega)} - q\mu r^{q-1}|u|^{q+1}_{L^{q+1}(\Omega)} - (p_s^*-1)r^{p_s^*-2}|u|^{p_s^*}_{L^{p_s^*}(\Omega)}. \tag{5.2.2}$$

By Lemma 5.2.2, we note that $u \in N_\mu$ if and only if $\varphi_u'(1) = 0$. Hence for $u \in N_\mu$, using (5.2.1) and (5.2.2), we obtain that

$$\begin{aligned} \varphi_u''(1) &= (p-1)\|u\|^p_{X_{0,s,p}(\Omega)} - q\mu|u|^{q+1}_{L^{q+1}(\Omega)} - (p_s^*-1)|u|^{p_s^*}_{L^{p_s^*}(\Omega)} \\ &= (p-p_s^*)|u|^{p_s^*}_{L^{p_s^*}(\Omega)} + (1-q)\mu|u|^{q+1}_{L^{q+1}(\Omega)} \\ &= (p-1-q)\|u\|^p_{X_{0,s,p}(\Omega)} - (p_s^*-1-q)|u|^{p_s^*}_{L^{p_s^*}(\Omega)} \\ &= (p-p_s^*)\|u\|^p_{X_{0,s,p}(\Omega)} + (p_s^*-1-q)\mu|u|^{q+1}_{L^{q+1}(\Omega)}. \end{aligned} \tag{5.2.3}$$

Therefore, we split the manifold into three parts corresponding to local minima, maxima and points of inflection

$$N_\mu^+ := \left\{ u \in N_\mu \,\middle|\, \varphi_u''(1) > 0 \right\},$$
$$N_\mu^- := \left\{ u \in N_\mu \,\middle|\, \varphi_u''(1) < 0 \right\},$$
$$N_\mu^0 := \left\{ u \in N_\mu \,\middle|\, \varphi_u''(1) = 0 \right\}.$$

In the next section, using the above Nehari type sets, we obtain existence of non-negative solutions ,thereby using maximum principle, we get at least two positive solutions of $(\mathcal{P}_\mu)$.

5.3 Existence of positive solutions

From [27], it follows that $\inf_{u\in N_\mu^+} I_\mu(u)$ and $\inf_{u\in N_\mu^-} I_\mu(u)$ are achieved and those two infimum points are two critical points of I_μ. Now if we define I_μ^+

as follows:

$$I_\mu^+(u) := \frac{1}{p}\|u\|^p_{X_{0,s,p}(\Omega)} - \frac{\mu}{q+1}|u^+|^{q+1}_{L^{q+1}(\Omega)} - \frac{1}{p_s^*}|u^+|^{p_s^*}_{L^{p_s^*}(\Omega)} \tag{5.3.1}$$

and

$$\tilde{\alpha}_\mu^+ := \inf_{u\in N_\mu^+} I_\mu^+(u) \quad \text{and} \quad \tilde{\alpha}_\mu^- := \inf_{u\in N_\mu^-} I_\mu^+(u), \tag{5.3.2}$$

then repeating the same analysis as in [27] for I_μ^+, it can be shown that there exists $\mu_* > 0$ such that for $\mu \in (0, \mu_*)$, there exists two non-trivial critical points $w_0 \in N_\mu^+$ and $w_1 \in N_\mu^-$ of I_μ^+. It is not difficult to see that w_0 and w_1 are nonnegative in $\mathbb{R}^N$. Indeed,

$$\begin{aligned}
0 &= \left\langle (I_\mu^+)'(w_0), w_0^- \right\rangle \\
&= \int_{\mathbb{R}^{2N}} \frac{|w_0(x)-w_0(y)|^{p-2}(w_0(x)-w_0(y))(w_0^-(x)-w_0^-(y))}{|x-y|^{N+sp}}dxdy \\
&= \int_{\mathbb{R}^{2N}} \frac{|w_0(x)-w_0(y)|^{p-2}((w_0^-(x)-w_0^-(y))^2+2(w_0^-(x)w_0^+(y)))}{|x-y|^{N+sp}}dxdy \\
&\geq \int_{\mathbb{R}^{2N}} \frac{|w_0^-(x)-w_0^-(y)|^p}{|x-y|^{N+sp}}dxdy = \|w_0^-\|^p_{X_{0,s,p}(\Omega)}.
\end{aligned} \tag{5.3.3}$$

Thus, $\|w_0^-\|_{X_{0,s,p}(\Omega)} = 0$ and hence, $w_0 = w_0^+$. Similarly we can show $w_1 = w_1^+$. Using maximum principle [23, Theorem A.1] we conclude that both w_0, w_1 are positive almost everywhere in Ω. Hence $(\mathcal{P}_\mu)$ has at least two positive solutions.

Set

$$\tilde{\mu} = \left(\frac{p-1-q}{p_s^*-q-1}\right)^{\frac{p-1-q}{p_s^*-p}} \frac{p_s^*-p}{p_s^*-q-1}|\Omega|^{\frac{q+1-p_s^*}{p_p^*}} S_{s,p}^{\frac{N(p-1-q)}{p^2 s}+\frac{q+1}{p}}. \tag{5.3.4}$$

5.4 Preliminary lemmas

In this section, we prove three elementary lemmas which are needed to prove the main result.

Lemma 5.4.1. *Let $\mu \in (0, \tilde{\mu})$. For every $u \in X_{0,s,p}(\Omega)$, $u \neq 0$, there exists unique*

$$t^-(u) < t_0(u) = \left(\frac{(p-1-q)\|u\|^p_{X_{0,s,p}(\Omega)}}{(p_s^*-1-q)|u|^{p_s^*}_{L^{p_s^*}(\Omega)}} \right)^{\frac{N-ps}{p^2 s}} < t^+(u),$$

such that

$$t^-(u)u \in N_\mu^+ \quad \text{and} \quad I_\mu(t^- u) = \min_{t \in [0,t_0]} I_\mu(tu),$$

$$t^+(u)u \in N_\mu^- \quad \text{and} \quad I_\mu(t^+ u) = \max_{t \geq t_0} I_\mu(tu).$$

Proof. For $t \geq 0$,

$$I_\mu(tu) = \frac{t^p}{p}\|u\|^p_{X_{0,s,p}(\Omega)} - \frac{\mu t^{q+1}}{q+1}|u|^{q+1}_{L^{q+1}(\Omega)} - \frac{t^{p_s^*}}{p_s^*}|u|^{p_s^*}_{L^{p_s^*}(\Omega)}.$$

Therefore

$$\frac{\partial}{\partial t} I_\mu(tu) = t^q \left(t^{p-1-q}\|u\|^p_{X_{0,s,p}(\Omega)} - t^{p_s^*-q-1}|u|^{p_s^*}_{L^{p_s^*}(\Omega)} - \mu |u|^{q+1}_{L^{q+1}(\Omega)} \right).$$

Define

$$\psi(t) = t^{p-1-q}\|u\|^p_{X_{0,s,p}(\Omega)} - t^{p_s^*-q-1}|u|^{p_s^*}_{L^{p_s^*}(\Omega)}. \tag{5.4.1}$$

By a straight forward computation, it follows that ψ attains maximum at the point

$$t_0 = t_0(u) = \left(\frac{(p-1-q)\|u\|^p_{X_{0,s,p}(\Omega)}}{(p_s^*-1-q)|u|^{p_s^*}_{L^{p_s^*}(\Omega)}} \right)^{\frac{1}{p_s^*-p}}. \tag{5.4.2}$$

Thus

$$\psi'(t_0) = 0, \quad \psi'(t) > 0 \quad \text{if} \quad t < t_0, \quad \psi'(t) < 0 \quad \text{if} \quad t > t_0. \tag{5.4.3}$$

Moreover, $\psi(t_0) = \left(\frac{p-1-q}{p_s^*-1-q}\right)^{\frac{p-1-q}{p_s^*-p}} \left(\frac{p_s^*-p}{p_s^*-1-q}\right) \left(\frac{\|u\|^{p(p_s^*-1-q)}_{X_{0,s,p}(\Omega)}}{|u|^{p_s^*(p-1-q)}_{L^{p_s^*}(\Omega)}} \right)^{\frac{N-ps}{p^2 s}}$. Therefore using Sobolev embedding, we have

$$\psi(t_0) \geq \left(\frac{p-1-q}{p_s^*-1-q} \right)^{\frac{(p-1-q)(N-2s)}{4s}} \left(\frac{p_s^*-p}{p_s^*-1-q} \right) S_{s,p}^{\frac{N(p-1-q)}{p^2 s}} \|u\|^{q+1}_{X_{0,s,p}(\Omega)}. \tag{5.4.4}$$

Using Hölder inequality followed by Sobolev inequality, and the fact that $\mu \in (0, \tilde{\mu})$, we obtain

$$\begin{aligned}\mu \int_\Omega |u|^{q+1} dx &\leq \mu \|u\|^{q+1}_{X_{0,s,p}(\Omega)} S_{s,p}^{-(q+1)/p} |\Omega|^{\frac{p_s^* - q - 1}{p_s^*}} \\ &\leq \tilde{\mu} \|u\|^{q+1}_{X_{0,s,p}(\Omega)} S_{s,p}^{-(q+1)/p} |\Omega|^{\frac{p_s^* - q - 1}{p_s^*}} \leq \psi(t_0),\end{aligned}$$

where in the last inequality we have used expression of $\tilde{\mu}$ (see (5.3.4)) and (5.4.4). Hence, there exists $t^+(u) > t_0 > t^-(u)$ such that

$$\psi(t^+) = \mu \int_\Omega |u|^{q+1} = \psi(t^-) \quad \text{and} \quad \psi'(t^+) < 0 < \psi'(t^-). \tag{5.4.5}$$

This in turn, implies $t^+ u \in N_\mu^-$ and $t^- u \in N_\mu^+$. Moreover, using (5.4.3) and (5.4.5) in the expression of $\frac{\partial}{\partial t} I_\mu(tu)$, we have

$$\frac{\partial}{\partial t} I_\mu(tu) > 0 \text{ when } t \in (t^-, t^+) \text{ and } \frac{\partial}{\partial t} I_\mu(tu) < 0 \text{ when } t \in [0, t^-) \cup (t^+, \infty),$$

$$\frac{\partial}{\partial t} I_\mu(tu) = 0 \quad \text{when} \quad t = t^\pm.$$

We note that $I_\mu(tu) = 0$ at $t = 0$ and strictly negative when $t > 0$ is small enough. Therefore it is easy to conclude that

$$\max_{t \geq t_0} I_\mu(tu) = I_\mu(t^+ u) \quad \text{and} \quad \min_{t \in [0, t_0]} J_\mu(tu) = I_\mu(t^- u).$$

□

Repeating the same argument as in Lemma 5.4.1, we can also prove that the following lemma holds:

Lemma 5.4.2. *Let $\mu \in (0, \tilde{\mu})$, where $\tilde{\mu}$ is defined as in* (5.3.4)*. For every $u \in X_{0,s,p}(\Omega)$, $u \neq 0$, there exist unique*

$$\tilde{t}^-(u) < \tilde{t}_0(u) = \left(\frac{(p - 1 - q) \|u\|^p_{X_{0,s,p}(\Omega)}}{(p_s^* - 1 - q) |u^+|^{p_s^*}_{L^{p_s^*}(\Omega)}} \right)^{\frac{N - ps}{p^2 s}} < \tilde{t}^+(u),$$

such that

$$\tilde{t}^-(u)u \in N_\mu^+ \quad \text{and} \quad I_\mu^+(\tilde{t}^- u) = \min_{t\in[0,t_0]} I_\mu^+(tu),$$
$$\tilde{t}^+(u)u \in N_\mu^- \quad \text{and} \quad I_\mu^+(\tilde{t}^+ u) = \max_{t\geq t_0} I_\mu^+(tu),$$

where I_μ^+ *is defined as in* (5.3.1).

Lemma 5.4.3. *Let* $\tilde{\mu}$ *be defined as in* (5.3.4). *Then* $\mu \in (0,\tilde{\mu})$, *implies* $N_\mu^0 = \emptyset$.

Proof. Suppose not. Then there exists $w \in N_\mu^0$ such that $w \neq 0$ and

$$(p-1-q)\|w\|^p_{X_{0,s,p}(\Omega)} - (p_s^*-q-1)|w^+|^{p_s^*}_{L^{p_s^*}(\Omega)} = 0. \tag{5.4.6}$$

The above expression combined with Sobolev inequality yields

$$\|w\|_{X_{0,s,p}(\Omega)} \geq S_{s,p}^{\frac{N}{p^2 s}} \left(\frac{p-1-q}{p_s^*-1-q}\right)^{\frac{N-ps}{p^2 s}}. \tag{5.4.7}$$

As $w \in N_\mu^0 \subseteq N_\mu$, using (5.4.6) and Hölder inequality followed by Sobolev inequality, we get

$$\begin{aligned} 0 &= \|w\|^p_{X_{0,s,p}(\Omega)} - |w|^{p_s^*}_{L^{p_s^*}(\Omega)} - \mu|w|^{q+1}_{L^{q+1}(\Omega)} \\ &\geq \|w\|^p_{X_{0,s,p}(\Omega)} - \left(\frac{p-1-q}{p_s^*-q-1}\right)\|w\|^p_{X_{0,s,p}(\Omega)} - \mu|\Omega|^{1-\frac{q+1}{p_s^*}} S_{s,p}^{-(q+1)/p}\|w\|^{q+1}_{X_{0,s,p}(\Omega)}. \end{aligned}$$

Combining the above inequality with (5.4.7) and using $\mu < \tilde{\mu}$, we have

$$\begin{aligned} 0 \geq \|w\|^{q+1}_{X_{0,s,p}(\Omega)} &\Bigg[\left(\frac{p_s^*-p}{p_s^*-q-1}\right)\left(\frac{p-1-q}{p_s^*-q-1}\right)^{\frac{(N-ps)(p-1-q)}{p^2 s}} S_{s,p}^{\frac{N(p-1-q)}{p^2 s}} \\ &- \mu|\Omega|^{1-\frac{q+1}{p_s^*}} S_{s,p}^{-(q+1)/p}\Bigg] > 0, \end{aligned} \tag{5.4.8}$$

which is a contradiction. This completes the proof. □

Lemma 5.4.4. *Let* $\tilde{\mu}$ *is as defined in* (5.3.4) *and* $\mu \in (0,\tilde{\mu})$. *Given* $u \in N_\mu^-$, *there exists* $\rho_u > 0$ *and a differentiable function* $g_{\rho_u} : B_{\rho_u}(0) \to \mathbb{R}^+$ *satisfying* *the following:*

$$g_{\rho_u}(0) = 1,$$
$$\Big(g_{\rho_u}(w)\Big)(u+w) \in N_\mu^- \quad \forall \quad w \in B_{\rho_u}(0),$$
$$\Big\langle g'_{\rho_u}(0), \phi \Big\rangle = \frac{pA(u,\phi) - p_s^* \displaystyle\int_\Omega |u|^{p_s^*-2}u\phi - (q+1)\mu \int_\Omega |u|^{q-1}u\phi}{(p-1-q)\|u\|^p_{X_{0,s,p}(\Omega)} - (p_s^*-q-1)|u|^{p_s^*}_{L^{p_s^*}(\Omega)}} \quad \forall\, \phi \in B_{\rho_u}(0),$$

where

$$A(u,\phi) = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}} dxdy.$$

Proof. Define $E : \mathbb{R} \times X_{0,s,p}(\Omega) \to \mathbb{R}$ as follows:

$$E(r,w) = r^{p-1-q}\|u+w\|^p_{X_{0,s,p}(\Omega)} - r^{p_s^*-q-1}|(u+w)|^{p_s^*}_{L^{p_s^*}(\Omega)} - \mu|(u+w)|^{q+1}_{L^{q+1}(\Omega)}.$$

We note that $u \in N_\mu^- \subset N_\mu$ implies

$$E(1,0) = 0, \quad \text{and} \quad \frac{\partial E}{\partial r}(1,0) = (p-1-q)\|u\|^p_{X_{0,s,p}(\Omega)} - (p_s^*-q-1)|u|^{p_s^*}_{L^{p_s^*}(\Omega)} < 0.$$

Therefore, by implicit function theorem, there exists neighborhood $B_{\rho_u}(0) \subset N_\mu$ for some $\rho_u > 0$ and a C^1 function $g_{\rho_u} : B_{\rho_u}(0) \to \mathbb{R}^+$ such that

$$(i)\ g_{\rho_u}(0) = 1, \quad (ii)\ E(g_{\rho_u}(w), w) = 0,\ \forall\ w \in B_{\rho_u}(0),$$
$$(iii) E_r(g_{\rho_u}(w), w) < 0,\ \forall\ w \in B_{\rho_u}(0), \quad (iv)\ \Big\langle g'_{\rho_u}(0), \phi \Big\rangle = -\frac{\Big\langle \frac{\partial E}{\partial w}(1,0), \phi \Big\rangle}{\frac{\partial E}{\partial r}(1,0)}.$$

Multiplying (ii) by $(g_{\rho_u}(w))^{q+1}$, it follows that $g_{\rho_u}(w)(u+w) \in N_\mu$. In fact, simplifying (iii), we obtain

$$(p-1-q)g_{\rho_u}(w)^p\|u+w\|^p_{X_{0,s,p}(\Omega)} - (p_s^*-q-1)g_{\rho_u}(w)^{p_s^*}|(u+w)|^{p_s^*}_{L^{p_s^*}(\Omega)} < 0\ \forall w \in B_{\rho_u}(0).$$

Thus $\Big(g_{\rho_u}(w)\Big)(u+w) \in N_\mu^-$, for every $w \in B_{\rho_u}(0)$. The last assertion of the lemma follows from (iv). □

5.5 Sobolev Minimizer

Let $S_{s,p}$ be as in (5.1.2). From [20], we know that for $1 < p < \infty, s \in (0,1), N > ps$, there exists a minimizer for $S_{s,p}$, and for every minimizer U, there exist $x_0 \in \mathbb{R}^N$ and a constant sign monotone function $u : \mathbb{R} \to \mathbb{R}$ such that $U(x) = u(|x - x_0|)$. In the following, we shall fix a radially symmetric nonnegative decreasing minimizer $U = U(r)$ for $S_{s,p}$. Multiplying U by a positive constant if necessary, we may assume that

$$(-\Delta)_p^s U = U^{p_s^*-1} \quad \text{in} \quad \mathbb{R}^N. \tag{5.5.1}$$

For any $\varepsilon > 0$ we note that the function function

$$U_\varepsilon(x) = \frac{1}{\varepsilon^{\frac{(N-sp)}{p}}} U\left(\frac{|x|}{\varepsilon}\right) \tag{5.5.2}$$

is also a minimizer for $S_{s,p}$ satisfying (5.5.1). From [64], we also have the following asymptotic estimates for U.

Lemma 5.5.1. *[64] Let U be the solution of* (5.5.1). *Then, there exists $c_1, c_2 > 0$ and $\theta > 1$ such that for all $r \geq 1$,*

$$\frac{c_1}{r^{\frac{N-sp}{p-1}}} \leq U(r) \leq \frac{c_2}{r^{\frac{N-sp}{p-1}}} \tag{5.5.3}$$

and

$$\frac{U(r\theta)}{U(r)} \leq \frac{1}{2}. \tag{5.5.4}$$

Proof. See [lemma 2.2 [64]]. □

Therefore we have,

$$c_1 \frac{\varepsilon^{\frac{N-sp}{p(p-1)}}}{|x|^{\frac{N-sp}{p-1}}} \leq U_\varepsilon(x) \leq c_2 \frac{\varepsilon^{\frac{N-sp}{p(p-1)}}}{|x|^{\frac{N-sp}{p-1}}} \quad \text{for} \quad |x| > \varepsilon. \tag{5.5.5}$$

We consider a cut-off function $\psi \in C_0^\infty(\Omega)$ such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ in Ω_δ, $\psi \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, where

$$\Omega_\delta := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}.$$

Define

$$u_\varepsilon(x) = \psi(x)U_\varepsilon(x). \tag{5.5.6}$$

5.6 Some Important estimates

In this section, we will prove some important estimates in order to establish our main result.

Lemma 5.6.1. *Suppose w_1 is a positive solution of $(\mathcal{P}_\mu)$ and u_ε is as defined in (5.5.6). Then for every $\varepsilon > 0$, small enough*

(i) $A_1 := \int_\Omega w_1^{p_s^*-1} u_\varepsilon dx \leq k_1 \varepsilon^{\frac{N-ps}{p(p-1)}}$;

(ii) $A_2 := \int_\Omega w_1^q u_\varepsilon dx \leq k_2 \varepsilon^{\frac{N-ps}{p(p-1)}}$;

(iii) $A_3 := \int_\Omega w_1 u_\varepsilon^q dx \leq k_3 \varepsilon^{\frac{N-ps}{p(p-1)}q}$;

(iv) $A_4 := \int_\Omega w_1 u_\varepsilon^{p_s^*-1} dx \leq k_4 \varepsilon^{\frac{N(p-1)+ps}{p(p-1)}}$.

Proof. Applying the Moser iteration technique (see [24, Theorem 3.3]), it can be shown that any positive solution of $(\mathcal{P}_\mu)$ is in $L^\infty(\Omega)$. Let $R,\ M > 0$ be such that $\Omega \subset B(0,R)$ and $|w_1|_{L^\infty(\Omega)} < M$.

$$
\begin{aligned}
(i) \quad A_1 = \int_\Omega w_1^{p_s^*-1} u_\varepsilon dx &\leq C\left[\int_{\Omega\cap\{|x|\leq\varepsilon\}} U_\varepsilon(x)dx + \varepsilon^{\frac{N-sp}{p(p-1)}} \int_{\Omega\cap\{|x|>\varepsilon\}} \frac{dx}{|x|^{\frac{N-sp}{p-1}}}\right] \\
&\leq C\left[\varepsilon^{N-\frac{(N-sp)}{p}} \int_{\{|x|<1\}} U(x)dx \right.\\
&+ \left. \varepsilon^{\frac{N-sp}{p(p-1)}} \int_{B(0,R)} \frac{dx}{|x|^{\frac{N-sp}{p-1}}} dx\right] \\
&\leq C\left[\varepsilon^{N-\frac{(N-sp)}{p}} + \varepsilon^{\frac{N-sp}{p(p-1)}} \int_0^R r^{N-1-\frac{N-sp}{p-1}} dr\right] \\
&\leq k_1 \varepsilon^{\frac{N-sp}{p(p-1)}}.
\end{aligned}
$$

Proof of (ii) similar to (i).

$$
\begin{aligned}
(iii) \quad A_3 = \int_\Omega w_1 u_\varepsilon^q dx &\leq C\left[\int_{\Omega\cap\{|x|\leq\varepsilon\}} U_\varepsilon^q(x)dx + \varepsilon^{\frac{N-sp}{p(p-1)}q}\int_{\Omega\cap\{|x|>\varepsilon\}} \frac{dx}{|x|^{\frac{(N-sp)q}{p-1}}}\right] \\
&\leq C\left[\varepsilon^{N-\frac{(N-sp)q}{p}}\int_{\{|x|<1\}} U(x)^q dx\right. \\
&+ \left.\varepsilon^{\frac{(N-sp)q}{p(p-1)}}\int_{B(0,R)} \frac{dx}{|x|^{\frac{(N-sp)q}{p-1}}}dx\right] \\
&\leq C\left[\varepsilon^{N-\frac{(N-sp)q}{p}} + \varepsilon^{\frac{N-sp}{p(p-1)}q}\int_0^R r^{N-1-\frac{N-sp}{p-1}q}dr\right] \\
&\leq k_3\varepsilon^{\frac{N-ps}{p(p-1)}q},
\end{aligned}
$$

since $0 < q < p-1 < \frac{N(p-1)}{N-sp}$. (iv) can be proved as in (iii). □

Lemma 5.6.2. *Let u_ε be as defined in* (5.5.6), $0 < q < p-1$ *and* $N > p^2 s$. *Then for every* $\varepsilon > 0$, *small*

$$
\int_\Omega |u_\epsilon|^{q+1}dx \geq \begin{cases} k_5\varepsilon^{\frac{(N-ps)(q+1)}{p(p-1)}} & \text{if} \quad 0 < q < \frac{N(p-2)+ps}{N-ps}, \\ k_6\varepsilon^{\frac{N}{p}}|ln\ \varepsilon|, & \text{if} \quad q = \frac{N(p-2)+ps}{N-ps}, \\ k_7\varepsilon^{N-\frac{(N-ps)(q+1)}{p}} & \text{if} \quad \frac{N(p-2)+ps}{N-ps} < q < p-1. \end{cases}
$$

Proof. We recall that $R' > 0$ was chosen such that $B(0, R') \subset \Omega_\delta$. Therefore, for $\varepsilon > 0$ small, we have

$$
\begin{aligned}
\int_\Omega |u_\varepsilon|^{q+1}dx &\geq \int_{B(0,R')} |u_\varepsilon|^{q+1}dx \\
&= \int_{B(0,R')} U_\varepsilon^{q+1}(x)dx \\
&= C\varepsilon^{N-\frac{(N-sp)(q+1)}{p}}\int_{B\left(0,\frac{R'}{\varepsilon}\right)} U^{q+1}(y)dy \qquad (5.6.1) \\
&\geq C\varepsilon^{N-\frac{(N-ps)(q+1)}{p}}\int_{B\left(0,\frac{R'}{\varepsilon}\right)\setminus B(0,1)} U^{q+1}(y)dy \\
&\geq C\varepsilon^{N-\frac{(N-ps)(q+1)}{p}}\int_1^{\frac{R'}{\varepsilon}} r^{N-1-\frac{(N-ps)(q+1)}{p-1}}dr. \qquad (5.6.2)
\end{aligned}
$$

Case 1 : $0 < q \leq \frac{N(p-2)+ps}{N-ps}$.

We note that

$$\int_1^{\frac{R'}{\varepsilon}} r^{(N-1)-\frac{(N-ps)(q+1)}{p-1}} dr \geq C_1 \varepsilon^{-N+\frac{(N-ps)(q+1)}{p-1}} - C_2, \tag{5.6.3}$$

Thus substituting back in (2.17), we obtain

$$\begin{aligned} \int_\Omega |u_\varepsilon|^{q+1} dx &\geq C\varepsilon^{N-\frac{(N-ps)(q+1)}{p}} [C_1 \varepsilon^{-N+\frac{(N-ps)(q+1)}{p-1}} - C_2] \\ &= C_3 \varepsilon^{\frac{(N-ps)(q+1)}{p(p-1)}} - C_4 \varepsilon^{N-\frac{(N-ps)(q+1)}{p}} \\ &\geq k_5 \varepsilon^{\frac{(N-ps)(q+1)}{p(p-1)}}. \end{aligned} \tag{5.6.4}$$

Case 2 : $q = \frac{N(p-2)+ps}{N-ps}$.
In this case it follows

$$\int_1^{\frac{R'}{\varepsilon}} r^{N-1-\frac{(N-ps)(q+1)}{p-1}} dr \geq C|\ln \varepsilon|.$$

Plugging back in (2.17), we obtain

$$\int_\Omega |u_\varepsilon|^{q+1} dx \geq k_6 \varepsilon^{N-\frac{(N-ps)(q+1)}{p}} |\ln \varepsilon| = k_6 \varepsilon^{\frac{N}{p}} |\ln \varepsilon|.$$

Case 3 : $\frac{N(p-2)+ps}{N-ps} < q < p-1$.

$$\begin{aligned} \text{RHS of (2.16)} &\geq k_7 \varepsilon^{N-\frac{(N-sp)(q+1)}{p}} \int_{B(0,1)} U^{q+1}(x) dx \\ &\geq k_7 \varepsilon^{N-\frac{(N-sp)(q+1)}{p}}. \end{aligned} \tag{5.6.5}$$

Hence the lemma follows.

□

5.7 The Palais-Smale condition

In this section, we prove that the functional I_μ satisfies Palais-Smale condition for some c as given in the lemma below.

Let us define

$$M := \frac{(pN-(N-ps)(q+1))(p-1-q)}{p^2(q+1)}\left(\frac{(p-1-q)(N-sp)}{p^2 s}\right)^{\frac{q+1}{p_s^*-q-1}}|\Omega|. \tag{5.7.1}$$

Lemma 5.7.1. *Let M be as in (5.7.1). For any $\mu>0$, and for*

$$c < \frac{s}{N}S_{s,p}^{\frac{N}{sp}} - M\mu^{\frac{p_s^*}{p_s^*-q-1}},$$

I_μ satisfies $(PS)_c$ condition.

Proof. Let $\{u_k\}\subset X_{0,s,p}(\Omega)$ be a $(PS)_c$ sequence for I_μ, that is, we have $I_\mu(u_k)\to c$ and $I_\mu'(u_k)\to 0$ in $(X_{0,s,p}(\Omega))'$ as $k\to\infty$. By the standard method it is not difficult to see that $\{u_k\}$ is bounded in $X_{0,s,p}(\Omega)$. Then up to a subsequence, still denoted by u_k, there exists $u_\infty\in X_{0,s,p}(\Omega)$ such that

$$\begin{aligned}
&u_k \rightharpoonup u_\infty \quad \text{weakly in} \quad X_{0,s,p}(\Omega) \quad \text{as} \quad k\to\infty,\\
&u_k \rightharpoonup u_\infty \quad \text{weakly in} \quad L^{p_s^*}(\mathbb{R}^N) \quad \text{as} \quad k\to\infty,\\
&u_k \to u_\infty \quad \text{strongly in} \quad L^r(\mathbb{R}^N) \quad \text{for any} \quad 1\le r<p_s^* \quad \text{as} \quad k\to\infty,\\
&u_k \to u_\infty \quad \text{a.e. in} \quad \mathbb{R}^N \quad \text{as} \quad k\to\infty.
\end{aligned}$$

As $0<q<p-1$, we have

$$\int_\Omega |u_k|^{q+1}(x)dx \to \int_\Omega |u_\infty|^{q+1}(x)dx \quad \text{as} \quad k\to\infty.$$

Using these above properties it can be shown that $\left\langle I_\mu'(u_\infty),\varphi\right\rangle_{X_{0,s,p}(\Omega)}=0$ for any $\varphi\in X_{0,s,p}(\Omega)$.

Indeed for any $\varphi\in X_{0,s,p}(\Omega)$,

$$\begin{aligned}
\langle I_\mu'(u_k),\varphi\rangle - \langle I_\mu'(u_\infty),\varphi\rangle &= \int_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\\
&- \int_{\mathbb{R}^{2N}} \frac{|u_\infty(x)-u_\infty(y)|^{p-2}(u_\infty(x)-u_\infty(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\\
&- \mu\left(\int_\Omega |u_k|^{q-1}u_k\varphi\,dx - \int_\Omega |u_\infty|^{q-1}u_\infty\varphi\,dx\right)\\
&- \left(\int_\Omega |u_k|^{p_s^*-2}u_k\varphi\,dx - \int_\Omega |u_\infty|^{p_s^*-2}u_\infty\varphi\,dx\right).
\end{aligned}$$

As $\left\{ \frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))}{|x-y|^{\frac{N+sp}{p'}}} \right\}_{k\geq 1}$ is bounded in $L^{p'}(\mathbb{R}^{2N})$, where $p' = \frac{p}{p-1}$, upto a subsequence

$$\frac{|u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y))}{|x-y|^{\frac{N+sp}{p'}}} \rightharpoonup \frac{|u_\infty(x) - u_\infty(y)|^{p-2}(u_\infty(x) - u_\infty(y))}{|x-y|^{\frac{N+sp}{p'}}}$$

weakly in $L^{p'}(\mathbb{R}^{2N})$, $u_k \rightharpoonup u_\infty$ weakly in $L^{p_s^*}(\mathbb{R}^N)$ and $u_k \to u_\infty$ strongly in $L^{q+1}(\mathbb{R}^N)$ as $k \to \infty$.

Combining these we have $\langle I'_\mu(u_k), \varphi\rangle - \langle I'_\mu(u_\infty), \varphi\rangle \to 0$ as $k \to \infty$. But as $I'_\mu(u_k) \to 0$ in $X_{0,s,p}(\Omega)'$ as $k \to \infty$, we have $\left\langle I'_\mu(u_\infty), \varphi \right\rangle_{X_{0,s,p}(\Omega)} = 0$ for any $\varphi \in X_{0,s,p}(\Omega)$. Hence, in particular $\left\langle I'_\mu(u_\infty), u_\infty \right\rangle_{X_{0,s,p}(\Omega)} = 0$.

Furthermore, by Brezis-Lieb lemma as $k \to \infty$, we get,

$$\int_{\mathbb{R}^{2N}} \frac{|u_k(x) - u_k(y)|^p}{|x-y|^{N+sp}} dxdy = \int_{\mathbb{R}^{2N}} \frac{|u_k(x) - u_\infty(x) - u_k(y) + u_\infty(y)|^p}{|x-y|^{N+sp}} dxdy + \int_{\mathbb{R}^{2N}} \frac{|u_\infty(x) - u_\infty(y)|^p}{|x-y|^{N+sp}} dxdy + o(1)$$

and

$$\int_\Omega |u_k(x)|^{p_s^*} dx = \int_\Omega |(u_k - u_\infty)(x)|^{p_s^*} dx + \int_\Omega |u_\infty(x)|^{p_s^*} dx + o(1).$$

Now,

$$\begin{aligned} \left\langle I'_\mu(u_k), u_k \right\rangle_{X_{0,s,p}(\Omega)} &= \int_{\mathbb{R}^{2n}} \frac{|u_k(x) - u_k(y)|^p}{|x-y|^{N+sp}} dxdy \\ &- \mu \int_\Omega |u_k(x)|^{q+1} dx - \int_\Omega |u_k(x)|^{p_s^*} dx \\ &= \int_{\mathbb{R}^{2n}} \frac{|u_k(x) - u_\infty(x) - u_k(y) + u_\infty(y)|^p}{|x-y|^{N+sp}} dxdy \\ &- \int_\Omega |u_k(x) - u_\infty(x)|^{p_s^*} dx \\ &+ \left\langle I'_\mu(u_\infty), u_\infty \right\rangle_{X_{0,s,p}(\Omega)} + o(1). \end{aligned}$$

Since as $\left\langle I'_\mu(u_\infty), u_\infty \right\rangle_{X_{0,s,p}(\Omega)} = 0$ and $\left\langle I'_\mu(u_k), u_k \right\rangle_{X_{0,s,p}(\Omega)} \to 0$ as $k \to \infty$, we have that there exists $b \in \mathbb{R}$ with $b \geq 0$ such that

$$\|u_k - u_\infty\|^p_{X_{0,s,p}(\Omega)} = \int_Q \frac{|u_k(x) - u_\infty(x) - u_k(y) + u_\infty(y)|^p}{|x-y|^{N+sp}} dxdy \to b \quad (5.7.2)$$

and

$$\int_\Omega |(u_k - u_\infty)(x)|^{p_s^*} dx \to b \quad \text{as} \quad k \to \infty. \tag{5.7.3}$$

If $b = 0$, we are done. Suppose $b > 0$. Moreover, using Sobolev inequality we have,

$$\|u_k - u_\infty\|^p_{X_{0,s,p}(\Omega)} \geq S_{s,p} \left(\int_\Omega (|u_k - u_\infty)(x)|^{p_s^*} dx \right)^{p/p_s^*}.$$

Therefore, $b \geq S_{s,p} b^{p/p_s^*}$, and this implies $b \geq S_{s,p}^{N/sp}$. On the other hand, since $\left\langle I'_\mu(u_\infty), u_\infty \right\rangle_{X_{0,s,p}(\Omega)} = 0$ we obtain

$$\begin{aligned} I_\mu(u_\infty) &= I_\mu(u_\infty) - \frac{1}{p} \left\langle I'_\mu(u_\infty), u_\infty \right\rangle_{X_{0,s,p}(\Omega)} \\ &= \frac{s}{N} \int_\Omega |u_\infty(x)|^{p_s^*} dx + \mu \left(\frac{1}{p} - \frac{1}{q+1} \right) \int_\Omega |u_\infty(x)|^{q+1} dx. \end{aligned} \tag{5.7.4}$$

Using (5.7.4) and $\left\langle I'_\mu(u_k), u_k \right\rangle_{X_{0,s,p}(\Omega)} \to 0$ as $k \to \infty$, we get

$$\begin{aligned} c = \lim_{k\to\infty} I_\mu(u_k) &= \lim_{k\to\infty} [I_\mu(u_k) - \frac{1}{p} \left\langle I'_\mu(u_k), u_k \right\rangle_{X_{0,s,p}(\Omega)}] \\ &= \lim_{k\to\infty} \left[\frac{s}{N} \int_\Omega |(u_k - u_\infty)|^{p_s^*} + \frac{s}{N} \int_\Omega |u_\infty|^{p_s^*} + \mu \left(\frac{1}{p} - \frac{1}{q+1} \right) \int_\Omega |u_k|^{q+1} \right] \\ &= \frac{s}{N} b + \frac{s}{N} \int_\Omega |u_\infty(x)|^{p_s^*} dx + \mu \left(\frac{1}{p} - \frac{1}{q+1} \right) \int_\Omega |u_\infty(x)|^{q+1} dx \\ &\geq \frac{s}{N} S_{s,p}^{N/sp} + \frac{s}{N} \int_\Omega |u_\infty(x)|^{p_s^*} dx + \mu \left(\frac{1}{p} - \frac{1}{q+1} \right) \int_\Omega |u_\infty(x)|^{q+1} dx \quad (5.7.5) \\ &= \frac{s}{N} S_{s,p}^{N/sp} + I_\mu(u_\infty). \quad (5.7.6) \end{aligned}$$

Since, by assumption we have $c < \frac{s}{N} S_{s,p}^{N/sp}$, the last inequality implies $I_\mu(u_\infty) < 0$. In particular, $u_\infty \not\equiv 0$ and

$$0 < \frac{1}{p} \|u_\infty\|^p_{X_{0,s,p}(\Omega)} < \frac{\mu}{q+1} \int_\Omega (u_\infty(x))^{q+1} dx + \frac{1}{p_s^*} \int_\Omega (u_\infty(x))^{p_s^*} dx.$$

Moreover, by Hölder inequality we have,

$$\int_\Omega |u_\infty(x)|^{q+1} dx \leq |\Omega|^{\frac{p_s^* - (q+1)}{p_s^*}} \left(\int_\Omega |u_\infty(x)|^{p_s^*} dx \right)^{\frac{q+1}{p_s^*}}.$$

Thus, from (5.7.5)

$$\begin{aligned} c &\geq \frac{s}{N}S_{s,p}^{N/sp} + \frac{s}{N}\int_\Omega |u_\infty|^{p_s^*} + \mu\left(\frac{1}{p} - \frac{1}{q+1}\right)|\Omega|^{\frac{p_s^*-(q+1)}{p_s^*}}\left(\int_\Omega |u_\infty|^{p_s^*}\right)^{\frac{q+1}{p_s^*}} \\ &:= \frac{s}{N}S_{s,p}^{N/sp} + h(\eta), \end{aligned}$$

where $h(\eta) = \frac{s}{N}\eta^{p_s^*} + \mu\left(\frac{1}{p} - \frac{1}{q+1}\right)|\Omega|^{\frac{p_s^*-(q+1)}{p_s^*}}\eta^{q+1}$ with $\eta = \left(\int_\Omega |u_\infty(x)|^{p_s^*}dx\right)^{\frac{1}{p_s^*}}$. By elementary analysis, we can show that h attains its minimum at $\eta_0 = \left(\frac{\mu(p-1-q)(N-sp)}{p^2 s}\right)^{\frac{1}{p_s^*-(q+1)}}|\Omega|^{\frac{1}{p_s^*}}$ and

$$\begin{aligned} h(\eta_0) &= \frac{s}{N}\left(\frac{\mu(p-1-q)(N-sp)}{p^2 s}\right)^{\frac{p_s^*}{p_s^*-(q+1)}}|\Omega| \\ &- \frac{\mu(p-1-q)}{p(q+1)}|\Omega|^{\frac{p_s^*-(q+1)}{p_s^*}}\left(\frac{\mu(p-1-q)(N-sp)}{p^2 s}\right)^{\frac{q+1}{p_s^*-(q+1)}}|\Omega|^{\frac{q+1}{p_s^*}} \\ &= -M\mu^{\frac{p_s^*}{p_s^*-(q+1)}}, \end{aligned}$$

with M given in (5.7.1). This in turn implies $c \geq \frac{s}{N}S_{s,p}^{\frac{N}{sp}} - M\mu^{\frac{p_s^*}{p_s^*-(q+1)}}$ and that gives a contradiction to our hypothesis. Hence $b = 0$. This concludes that $u_k \to u_\infty$ strongly in $X_{0,s,p}(\Omega)$. □

5.8 Existence of sign-changing solution

Lemma 5.8.1. *Let $N \in \mathbb{N}$ be such that $N > \frac{sp}{2}[p+1+\sqrt{(p+1)^2-4}]$ and $q \in (q_1, p-1)$, where*

$$q_1 := \frac{N^2(p-1)}{(N-sp)(N-s)} - 1. \tag{5.8.1}$$

Then, there exists $\tilde{\mu}_1 > 0$ and $u_0 \in X_{0,s,p}(\Omega)$ such that

$$\sup_{t\geq 0} I_\mu^+(tu_0) < \frac{s}{N}S_{s,p}^{\frac{N}{sp}} - M\mu^{\frac{p_s^*}{p_s^*-q-1}}, \tag{5.8.2}$$

for $\mu \in (0, \tilde{\mu}_1)$. In particular,

$$\tilde{\alpha}_\mu^- < \frac{s}{N}S_{s,p}^{\frac{N}{sp}} - M\mu^{\frac{p_s^*}{p_s^*-q-1}} \tag{5.8.3}$$

where I_μ^+ is defined as in (5.3.1) *and α_μ^- and M are given as in* (5.3.2) *and (5.7.1) respectively.*

Proof. Let u_ε be as defined in (5.5.6). Then we claim

$$|u_\varepsilon^+|_{L^{p_s^*}} = |u_\varepsilon|_{L^{p_s^*}}^{p_s^*} \geq S_{s,p}^{\frac{N}{sp}} + o(\varepsilon^{\frac{N}{p-1}}). \tag{5.8.4}$$

To see this,

$$\begin{aligned} |u_\varepsilon|_{L^{p_s^*}(\Omega)}^{p_s^*} &= \int_\Omega |u_\varepsilon|^{p_s^*} dx \geq \int_{\Omega_\delta} |u_\varepsilon|^{p_s^*} dx \\ &= \int_{\Omega_\delta} |U_\varepsilon(x)|^{p_s^*} dx \\ &= \int_{\mathbb{R}^N} |U_\varepsilon(x)|^{p_s^*} dx - \int_{\mathbb{R}^N \setminus \Omega_\delta} |U_\varepsilon(x)|^{p_s^*} dx. \end{aligned} \tag{5.8.5}$$

Moreover,

$$\begin{aligned} \int_{\mathbb{R}^N \setminus \Omega_\delta} |U_\varepsilon(x)|^{p_s^*} dx \leq \int_{\mathbb{R}^N \setminus B(0,R')} |U_\varepsilon(x)|^{p_s^*} dx &= \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N \setminus B(0,R')} U^{p_s^*}(\frac{x}{\varepsilon}) dx \\ &\leq C \int_{\frac{R'}{\varepsilon}}^\infty r^{N-1-\frac{Np}{p-1}} dr \\ &\leq C\varepsilon^{\frac{N}{p-1}}. \end{aligned}$$

Therefore substituting back to (5.8.5) we obtain

$$|u_\varepsilon|_{L^{p_s^*}(\Omega)}^{p_s^*} \geq S_{s,p}^{\frac{N}{sp}} - C\varepsilon^{\frac{N}{p-1}}.$$

Furthermore, a similar analysis as in [78, Proposition 21] (see also [64, Lemma 2.7]) yields, for $\varepsilon > 0$ small ($0 < \varepsilon < \frac{\delta}{2}$) we have,

$$\|u_\varepsilon\|_{X_{0,s,p}(\Omega)}^p \leq S_{s,p}^{\frac{N}{sp}} + o(\varepsilon^{\frac{N-ps}{p-1}}). \tag{5.8.6}$$

Define,

$$J(u) := \frac{1}{p}\|u\|_{X_{0,s,p}(\Omega)}^p - \frac{1}{p_s^*}|u^+|_{L^{p_s^*}}^{p_s^*}, \quad u \in X_{0,s,p}(\Omega)$$

and choose $\varepsilon_0 > 0$ small such that (5.8.6) and (5.8.4) hold and Lemma 5.6.2 is satisfied. Let $\varepsilon \in (0, \varepsilon_0)$. Then, consider corresponding $u_0 := u_{\varepsilon_0}$. Let us consider the function $h : [0, \infty) \to \mathbb{R}$ defined by $h(t) = J(tu_0)$ for all $t \geq 0$. It can be shown that h attains its maximum at $t = t_* = \left(\frac{\|u_0\|^p_{X_{0,s,p}(\Omega)}}{|u_0^+|^{p_s^*}_{L^{p_s^*}}}\right)^{\frac{1}{p^*-p}}$ and $\sup_{t\geq 0} J(tu_0) = \frac{s}{N}\left(\frac{\|u_0\|^p_{X_{0,s,p}(\Omega)}}{|u_0^+|^p_{L^{p_s^*}}}\right)^{\frac{N}{sp}}$. Using (5.8.6) and (5.8.4) a straight forward computation yields,

$$\sup_{t\geq 0} J(tu_0) \leq \frac{s}{N} S_{s,p}^{\frac{N}{sp}} + o(\varepsilon^{\frac{N-sp}{p-1}}). \tag{5.8.7}$$

Since $I_\mu^+(tu_0) < 0$ for t small, we can find $t_0 \in (0,1)$ such that

$$\sup_{0\leq t\leq t_0} I_\mu^+(tu_0) \leq \frac{s}{N} S_{s,p}^{\frac{N}{sp}} - M\mu^{\frac{p_s^*}{p_s^*-q-1}},$$

for $\mu > 0$ small. Hence, we are left to estimate $\sup_{t_0\leq t} I_\mu^+(tu_0)$.

$$\begin{aligned}
\sup_{t\geq t_0} I_\mu^+(tu_0) &= \sup_{t\geq t_0}[J(tu_0) - \frac{t^{q+1}}{q+1}|u_0^+|^{q+1}_{L^{q+1}}] \\
&\leq \frac{s}{N} S_{s,p}^{\frac{N}{sp}} + o(\varepsilon^{\frac{N-sp}{p-1}}) - \frac{t^{q+1}}{q+1}|u_0|^{q+1}_{L^{q+1}} \\
&\leq \begin{cases}
\frac{s}{N} S_{s,p}^{\frac{N}{sp}} + c_1\varepsilon^{\frac{N-ps}{p-1}} - c_2\mu\varepsilon^{\frac{(N-ps)(q+1)}{p(p-1)}}, & 0 < q < \frac{N(p-2)+ps}{N-sp} \\
\frac{s}{N} S_{s,p}^{\frac{N}{sp}} + c_1\varepsilon^{\frac{N-ps}{p-1}} - c_2\mu\varepsilon^{\frac{N}{p}}|ln\varepsilon|, & q = \frac{N(p-2)+ps}{N-sp} \\
\frac{s}{N} S_{s,p}^{\frac{N}{sp}} + c_1\varepsilon^{\frac{N-ps}{p-1}} - c_2\mu\varepsilon^{N-\frac{(N-sp)(q+1)}{p}}, & \frac{N(p-2)+ps}{N-sp} < q < p-1.
\end{cases}
\end{aligned}$$

Choose $\varepsilon \in (0, \frac{\delta}{2})$ such that $\varepsilon^{\frac{N-sp}{p-1}} = \mu^{\frac{p_s^*}{p_s^*-q-1}}$. Then for $\frac{N(p-2)+ps}{N-sp} < q < p-1$, the term $\frac{s}{N} S_{s,p}^{\frac{N}{sp}} + c_1\varepsilon^{\frac{N-ps}{p-1}} - c_2\mu\varepsilon^{N-\frac{(N-sp)(q+1)}{p}}$ reduces to $\frac{s}{N} S_{s,p}^{\frac{N}{sp}} + c_1\mu^{\frac{p_s^*}{p_s^*-q-1}} - c_2\mu\left(\mu^{\frac{p^*}{p^*-q-1}}\right)^{(N-\frac{(N-sp)(q+1)}{p})(\frac{p-1}{N-ps})}$. Now, note that we can make

$$c_1\mu^{\frac{p_s^*}{p_s^*-q-1}} - c_2\mu\left(\mu^{\frac{p^*}{p^*-q-1}}\right)^{(N-\frac{(N-sp)(q+1)}{p})(\frac{p-1}{N-ps})} < -M\mu^{\frac{p_s^*}{p_s^*-q-1}},$$

for $\mu > 0$ small if we further choose $(\frac{p_s^*}{p_s^*-q-1})(\frac{p-1}{p})[\frac{Np}{N-ps} - (q+1)] < \frac{p_s^*}{p_s^*-q-1} - 1$ i.e., if $q+1 > \frac{N^2(p-1)}{(N-sp)(N-s)}$. This proves (5.8.2). It is easy to see that (5.8.3) follows by combining (5.8.2) along with Lemma 5.4.2 .

□

5.8.1 Sign changing critical points of I_μ

Define

$$\mathcal{N}_{\mu,1}^- := \{u \in N_\mu : u^+ \in N_\mu^-\},$$

$$\mathcal{N}_{\mu,2}^- := \{u \in N_\mu : -u^- \in N_\mu^-\},$$

We set

$$\beta_1 = \inf_{u \in \mathcal{N}_{\mu,1}^-} I_\mu(u) \quad \text{and} \quad \beta_2 = \inf_{u \in \mathcal{N}_{\mu,2}^-} I_\mu(u). \tag{5.8.8}$$

Theorem 5.8.2. *Let $p \geq 2$, $N > \frac{sp}{2}[p+1+\sqrt{(p+1)^2-4}]$ and $q_1 < q < p-1$, where q_1 is defined as in (5.8.1). Assume $0 < \mu < min\{\tilde{\mu}, \tilde{\mu}_1, \mu_*, \mu_1\}$, where $\tilde{\mu}$, $\tilde{\mu}_1$ and μ_1 are as in (5.3.4), Lemma 5.8.1 and Lemma 5.9.1 respectively. μ_* is chosen such that $\tilde{\alpha}_\mu^-$ is achieved in $(0, \mu_*)$. Let β_1, β_2, $\tilde{\alpha}_\mu^-$ be defined as in (5.8.8) and (5.3.2) respectively.*

(i) Let $\beta_1 < \tilde{\alpha}_\mu^-$. Then, there exists a sign changing critical point $\tilde{w}_1$ of I_μ such that $\tilde{w}_1 \in \mathcal{N}_{\mu,1}^-$ and $I_\mu(\tilde{w}_1) = \beta_1$.

(ii) If $\beta_2 < \tilde{\alpha}_\mu^-$, then there exists a sign changing critical point $\tilde{w}_2$ of I_μ such that $\tilde{w}_2 \in \mathcal{N}_{\mu,1}^-$ and $I_\mu(\tilde{w}_2) = \beta_2$.

Proof. (i) Let $\beta_1 < \tilde{\alpha}_\mu^-$. We prove the theorem in few steps.

Step 1: $\mathcal{N}_{\mu,1}^-$ and $\mathcal{N}_{\mu,2}^-$ are closed sets.

To see this, let $\{u_n\} \subset \mathcal{N}_{\mu,1}^-$ such that $u_n \to u$ in $X_{0,s,p}(\Omega)$. It is easy to note that $|u_n|, |u| \in X_{0,s,p}(\Omega)$ and $|u_n| \to |u|$ in $X_{0,s,p}(\Omega)$. This in turn implies $u_n^+ \to u^+$ in $X_{0,s,p}(\Omega)$ and $L^\gamma(\mathbb{R}^N)$ for $\gamma \in [1, p_s^*]$ (by Sobolev inequality). Since, $u_n \in \mathcal{N}_{\mu,1}^-$, we have $u_n^+ \in N_\mu^-$. Therefore

$$\|u_n^+\|_{X_{0,s,p}(\Omega)}^p - |u_n^+|_{L^{p_s^*}(\Omega)}^{p_s^*} - \mu |u_n^+|_{L^{q+1}(\Omega)}^{q+1} = 0 \tag{5.8.9}$$

and

$$(p-1-q)\|u_n^+\|_{X_{0,s,p}(\Omega)}^p - (p_s^* - q - 1)|u_n^+|_{L^{p_s^*}(\Omega)}^{p_s^*} < 0 \;\forall\; n \geq 1. \tag{5.8.10}$$

Passing to the limit as $n \to \infty$, we obtain $u^+ \in N_\mu$ and $(p-1-q)\|u^+\|^p_{X_{0,s,p}(\Omega)} - (p_s^* - q - 1)|u^+|^{p_s^*}_{L^{p_s^*}(\Omega)} \leq 0$. But, from Lemma 5.4.3, we know $N^0_\mu = \emptyset$. Therefore $u^+ \in N^-_\mu$ and hence $\mathcal{N}^-_{\mu,1}$ is closed. Similarly it can be shown that $\mathcal{N}^-_{\mu,2}$ is also closed. Hence step 1 follows.

By Ekeland Variational Principle there exists sequence $\{u_n\} \subset \mathcal{N}^-_{\mu,1}$ such that

$$I_\mu(u_n) \to \beta_1 \quad \text{and} \quad I_\mu(z) \geq I_\mu(u_n) - \frac{1}{n}\|u_n - z\|_{X_{0,s,p}(\Omega)} \quad \forall\, z \in \mathcal{N}^-_{\mu,1}. \tag{5.8.11}$$

Step 2: $\{u_n\}$ is uniformly bounded in $X_{0,s,p}(\Omega)$.

To see this, we notice $u_n \in \mathcal{N}^-_{\mu,1}$ implies $u_n \in N_\mu$ and this in turn implies $\left\langle I'_\mu(u_n), u_n \right\rangle = 0$, that is,

$$\|u_n\|^p_{X_{0,s,p}(\Omega)} = |u_n|^{p_s^*}_{L^{p_s^*}(\Omega)} + \mu|u_n|^{q+1}_{L^{q+1}(\Omega)}.$$

Since $I_\mu(u_n) \to \beta_1$, using the above equality in the expression of $I_\mu(u_n)$, we get, for n large enough

$$\begin{aligned} \frac{s}{N}\|u_n\|^p_{X_{0,s,p}(\Omega)} &\leq \beta_1 + 1 + \left(\frac{1}{q+1} - \frac{1}{p_s^*}\right)\mu|u_n|^{q+1}_{L^{q+1}(\Omega)} \\ &\leq C(1 + \|u_n\|^{q+1}_{X_{0,s,p}(\Omega)}). \end{aligned}$$

As $p > q+1$, the above implies $\{u_n\}$ is uniformly bounded in $X_{0,s,p}(\Omega)$.

We note that for any $u \in X_{0,s,p}(\Omega)$, we have

$$\|u\|^p_{X_{0,s,p}(\Omega)} = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy$$
$$= \int_{\mathbb{R}^{2N}} \frac{(|u(x)-u(y)|^2)^{\frac{p}{2}}}{|x-y|^{N+ps}} dxdy$$
$$= \int_{\mathbb{R}^{2N}} \frac{\left(\left|\left(u^+(x)-u^+(y)\right) - \left(u^-(x)-u^-(y)\right)\right|^2\right)^{\frac{p}{2}}}{|x-y|^{N+ps}} dxdy$$
$$= \int_{\mathbb{R}^{2N}} \frac{\left(\left(u^+(x)-u^+(y)\right)^2 + \left(u^-(x)-u^-(y)\right)^2 + 2u^+(x)u^-(y) + 2u^+(y)u^-(x)\right)^{\frac{p}{2}}}{|x-y|^{N+ps}} dxdy$$
$$\geq \int_{\mathbb{R}^{2N}} \frac{\left(\left(u^+(x)-u^+(y)\right)^2 + \left(u^-(x)-u^-(y)\right)^2\right)^{\frac{p}{2}}}{|x-y|^{N+ps}} dxdy$$
$$\geq \int_{\mathbb{R}^{2N}} \frac{\left(\left(u^+(x)-u^+(y)\right)^2\right)^{\frac{p}{2}}}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^{2N}} \frac{\left(\left(u^-(x)-u^-(y)\right)^2\right)^{\frac{p}{2}}}{|x-y|^{N+ps}} dxdy$$
$$= \|u^+\|^p_{X_{0,s,p}(\Omega)} + \|u^-\|^p_{X_{0,s,p}(\Omega)}. \tag{5.8.12}$$

By a simple calculation, it follows

$$|u|^{p_s^*}_{L^{p_s^*}(\Omega)} = |u^+|^{p_s^*}_{L^{p_s^*}(\Omega)} + |u^-|^{p_s^*}_{L^{p_s^*}(\Omega)} \quad \text{and} \quad |u|^{q+1}_{L^{q+1}(\Omega)} = |u^+|^{q+1}_{L^{q+1}(\Omega)} + |u^-|^{q+1}_{L^{q+1}(\Omega)}. \tag{5.8.13}$$

Combining (5.8.12) and (5.8.13), we obtain

$$I_\mu(u) \geq I_\mu(u^+) + I_\mu(u^-) \quad \forall \quad u \in X_{0,s,p}(\Omega). \tag{5.8.14}$$

Step 3: There exists $b > 0$ such that $\|u_n^-\|_{X_{0,s,p}(\Omega)} \geq b$ for all $n \geq 1$.

Suppose the step is not true. Then for each $k \geq 1$, there exists u_{n_k} such that

$$\|u^-_{n_k}\|_{X_{0,s,p}(\Omega)} < \frac{1}{k} \ \forall \ k \ \geq 1. \tag{5.8.15}$$

Therefore, $\|u^-_{n_k}\|_{X_{0,s,p}(\Omega)} \to 0$ as $k \to \infty$ and by Sobolev inequality

$$|u^-_{n_k}|_{L^{p_s^*}(\Omega)} \to 0, \ |u^-_{n_k}|_{L^{q+1}(\Omega)} \to 0, \quad \text{as} \quad k \to \infty.$$

Consequently, $I_\mu(u_{n_k}^-) \to 0$ as $k \to \infty$. As a result, using (5.8.14) we have

$$\beta_1 = I_\mu(u_{n_k}) + o(1) \geq I_\mu(u_{n_k}^+) + I_\mu(u_{n_k}^-) + o(1) = I_\mu^+(u_{n_k}^+) + o(1) \geq \tilde{\alpha}_\mu^- + o(1).$$

This is a contradiction to the hypothesis. Hence step 3 follows.

Step 4: $I_\mu'(u_n) \to 0$ in $(X_{0,s,p}(\Omega))'$ as $n \to \infty$.

Since $u_n \in \mathcal{N}_{\mu,1}^-$, we have $u_n^+ \in N_\mu^-$. Thus by Lemma 5.4.4 applied to the element u_n^+, there exists

$$\rho_n := \rho_{u_n^+} \quad \text{and} \quad g_n := g_{\rho_{u_n^+}}, \tag{5.8.16}$$

such that

$$g_n(0) = 1, \quad \Big(g_n(w)\Big)(u_n^+ + w) \in N_\mu^- \quad \forall \quad w \in B_{\rho_n}(0). \tag{5.8.17}$$

Choose $0 < \tilde{\rho}_n < \rho_n$ such that $\tilde{\rho}_n \to 0$. Let $v \in X_{0,s,p}(\Omega)$ with $\|v\|_{X_{0,s,p}(\Omega)} = 1$. Define

$$v_n := -\tilde{\rho}_n[v^+\chi_{\{u_n \geq 0\}} - v^-\chi_{\{u_n \leq 0\}}]$$

and

$$\begin{aligned} z_{\tilde{\rho}_n} &:= \Big(g_n(v_n^-)\Big)(u_n - v_n) \\ &=: z_{\tilde{\rho}_n}^1 - z_{\tilde{\rho}_n}^2, \end{aligned}$$

where $z_{\tilde{\rho}_n}^1 := \Big(g_n(v_n^-)\Big)(u_n^+ + \tilde{\rho}_n v^+\chi_{\{u_n \geq 0\}})$ and $z_{\tilde{\rho}_n}^2 := \Big(g_n(v_n^-)\Big)(u_n^- + \tilde{\rho}_n v^-\chi_{\{u_n \leq 0\}})$. Note that $v_n^- = \tilde{\rho}_n v^+\chi_{\{u_n \geq 0\}}$. So, $\|v_n^-\|_{X_{0,s,p}(\Omega)} \leq \tilde{\rho}_n\|v\|_{X_{0,s,p}(\Omega)} \leq \tilde{\rho}_n$. Hence taking $w = v_n^-$ in (5.8.17) we have, $z_{\tilde{\rho}_n}^+ = z_{\tilde{\rho}_n}^1 \in N_\mu^-$ so $z_{\tilde{\rho}_n} \in N_{\mu,1}^-$. Hence,

$$I_\mu(z_{\tilde{\rho}_n}) \geq I_\mu(u_n) - \frac{1}{n}\|u_n - z_{\tilde{\rho}_n}\|_{X_{0,s,p}(\Omega)}.$$

This implies,

$$\begin{aligned} \frac{1}{n}\|u_n - z_{\tilde{\rho}_n}\|_{X_{0,s,p}(\Omega)} &\geq I_\mu(u_n) - I_\mu(z_{\tilde{\rho}_n}) \\ &= \Big\langle I_\mu'(u_n), u_n - z_{\tilde{\rho}_n}\Big\rangle + o(1)\|u_n - z_{\tilde{\rho}_n}\|_{X_{0,s,p}(\Omega)} \\ &= -\Big\langle I_\mu'(u_n), z_{\tilde{\rho}_n}\Big\rangle + o(1)\|u_n - z_{\tilde{\rho}_n}\|_{X_{0,s,p}(\Omega)} \end{aligned}$$

$$, \tag{5.8.18}$$

as $\left\langle I'_\mu(u_n), u_n \right\rangle = 0$ for all n. Let $w_n = \tilde{\rho}_n v$. Then,

$$\frac{1}{n}\|u_n - z_{\tilde{\rho}_n}\|_{X_{0,s,p}(\Omega)} \geq -\left\langle I'_\mu(u_n), w_n + z_{\tilde{\rho}_n} \right\rangle + \left\langle I'_\mu(u_n), w_n \right\rangle$$
$$+o(1)\|u_n - z_{\tilde{\rho}_n}\|_{X_{0,s,p}(\Omega)}. \tag{5.8.19}$$

Now, $\left\langle I'_\mu(u_n), w_n \right\rangle = \left\langle I'_\mu(u_n), \tilde{\rho}_n v \right\rangle = \tilde{\rho}_n \left\langle I'_\mu(u_n), v \right\rangle$. Define

$$\overline{v_n} := v^+ \chi_{\{u_n \geq 0\}} - v^- \chi_{\{u_n \leq 0\}}.$$

So, $z_{\tilde{\rho}_n} = g_n(v_n^-)(u_n - \tilde{\rho}_n \overline{v_n})$. Hence we have,

$$\begin{aligned}\left\langle I'_\mu(u_n), w_n + z_{\tilde{\rho}_n} \right\rangle &= \left\langle I'_\mu(u_n), w_n + g_n(v_n^-)(u_n - \tilde{\rho}_n \overline{v_n}) \right\rangle \\ &= \left\langle I'_\mu(u_n), \tilde{\rho}_n v - g_n(v_n^-)\tilde{\rho}_n \overline{v_n} \right\rangle \\ &= \tilde{\rho}_n \left\langle I'_\mu(u_n), v - g_n(v_n^-)\overline{v_n} \right\rangle \end{aligned} \tag{5.8.20}$$

Using (5.8.20) in (5.8.19), we have

$$\frac{1}{n}\|u_n - z_{\tilde{\rho}_n}\|_{X_{0,s,p}(\Omega)} \geq -\tilde{\rho}_n \left\langle I'_\mu(u_n), v - g_n(v_n^-)\overline{v_n} \right\rangle$$
$$+\tilde{\rho}_n \left\langle I'_\mu(u_n), v \right\rangle + o(1)\|u_n - z_{\tilde{\rho}_n}\|_{X_{0,s,p}(\Omega)}. \tag{5.8.21}$$

First we will estimate $\left\langle I'_\mu(u_n), v - g_n(v_n^-)\overline{v_n} \right\rangle$. For this,

$$\begin{aligned} v - g_n(v_n^-)\overline{v_n} &= v^+ - v^- - g_n(v_n^-)[v^+ \chi_{\{u_n \geq 0\}} - v^- \chi_{\{u_n \leq 0\}}] \\ &= v^+[g_n(0) - g_n(v_n^-)\chi_{\{u_n \geq 0\}}] - v^-[g_n(0) - g_n(v_n^-)\chi_{\{u_n \leq 0\}}] \\ &= -v^+[\left\langle g'_n(0), v_n^- \right\rangle + o(1)\|v_n^-\|_{X_{0,s,p}(\Omega)}] \\ &+ v^-[\left\langle g'_n(0), v_n^- \right\rangle + o(1)\|v_n^-\|_{X_{0,s,p}(\Omega)}] \\ &= -v^+\tilde{\rho}_n[\left\langle g'_n(0), v^+ \right\rangle + o(1)\|v^+\|_{X_{0,s,p}(\Omega)}] \\ &+ v^-\tilde{\rho}_n[\left\langle g'_n(0), v^+ \right\rangle + o(1)\|v^+\|_{X_{0,s,p}(\Omega)}] \\ &= -\tilde{\rho}_n\Big[\left\langle g'_n(0), v^+ \right\rangle + o(1)\|v^+\|_{X_{0,s,p}(\Omega)}\Big]v. \end{aligned}$$

Therefore,

$$\left\langle I'_\mu(u_n), v - g_n(v_n^-)\overline{v_n} \right\rangle = -\tilde{\rho}_n\Big(\left\langle g'_n(0), v^+ \right\rangle + o(1)\|v^+\|_{X_{0,s,p}(\Omega)}\Big)\left\langle I'_\mu(u_n), v \right\rangle. \tag{5.8.22}$$

Claim : $g_n(v_n^-)$ is uniformly bounded in $X_{0,s,p}(\Omega)$.

To see this, we observe that from (5.8.17) we have, $g_n(v_n^-)(u_n^+ + v_n^-) \in N_\mu^- \subset N_\mu$, which implies,

$$\|c_n\tilde{\psi}_n\|^p_{X_{0,s,p}(\Omega)} - \mu|c_n\tilde{\psi}_n|^{q+1}_{L^{q+1}(\Omega)} - |c_n\tilde{\psi}_n|^{p_s^*}_{L^{p_s^*}(\Omega)} = 0,$$

where $c_n := g_n(v_n^-)$ and $\tilde{\psi}_n := u_n^+ + v_n^-$. Dividing by $c_n^{p^*}$ we have,

$$c_n^{p-p^*}\|\tilde{\psi}_n\|^p_{X_{0,s,p}(\Omega)} - \mu c_n^{q+1-p^*}|\tilde{\psi}_n|^{q+1}_{L^{q+1}(\Omega)} = |\tilde{\psi}_n|^{p_s^*}_{L^{p_s^*}(\Omega)}. \tag{5.8.23}$$

Note that $\|\tilde{\psi}_n\|_{X_{0,s,p}(\Omega)}$ is uniformly bounded above as $\|u_n\|_{X_{0,s,p}(\Omega)}$ is uniformly bounded and $\tilde{\rho}_n = o(1)$. Also, $\|\tilde{\psi}_n\|_{X_{0,s,p}(\Omega)} \geq \|u_n^+\|_{X_{0,s,p}(\Omega)} - \tilde{\rho}_n\|v\|_{X_{0,s,p}(\Omega)}$. Note that $\|u_n^+\|_{X_{0,s,p}(\Omega)} \geq \tilde{b}$ for large n. If not, then $\|u_n^+\|_{X_{0,s,p}(\Omega)} \to 0$ as $n \to \infty$. As $u_n \in N_{\mu,1}^-$, so $u_n^+ \in N_\mu^-$. Now, N_μ^- is a closed set and $0 \notin N_\mu^-$ and therefore $\|u_n^-\|_{X_{0,s,p}(\Omega)} \not\to 0$ as $n \to \infty$. Thus there exists $\tilde{b} \geq 0$ such that $\|u_n^+\|_{X_{0,s,p}(\Omega)} \geq \tilde{b} > 0$. This in turn implies that $\|\tilde{\psi}_n\|_{X_{0,s,p}(\Omega)} \geq C$, for some $C > 0$ by choosing $\tilde{\rho}_n$ small enough. Consequently, if c_n is not uniformly bounded, we obtain LHS of (5.8.23) converges to 0 as $n \to \infty$.

On the other hand,

$$|\tilde{\psi}_n|_{L^{p_s^*}(\Omega)} \geq |u_n^+|_{L^{p_s^*}(\Omega)} - \tilde{\rho}_n|v|_{L^{p_s^*}(\Omega)} > c,$$

for some positive constant c as $\rho_n = o(1)$ and $u_n^+ \in N_\mu^-$ implies

$$(p_s^* - 1 - q)|u_n^+|^{p_s^*}_{L^{p_s^*}(\Omega)} > (p-1-q)\|u_n^+\|^p_{X_{0,s,p}(\Omega)} > (p-1-q)\tilde{b}^p.$$

Hence, the claim follows.

Now using the fact that $g_n(0)=1$ and the above claim we obtain

$$\begin{aligned}
\|u_n - z_{\tilde\rho_n}\|_{X_{0,s,p}(\Omega)} &\leq \|u_n\|_{X_{0,s,p}(\Omega)}\Big|1-g_n(v_n^-)\Big| + \tilde\rho_n\|\overline{v_n}\|_{X_{0,s,p}(\Omega)}g_n(v_n^-)\\
&\leq \|u_n\|_{X_{0,s,p}(\Omega)}\Big[|\left\langle g_n'(0), v_n^-\right\rangle| + o(1)\|\overline{v_n}\|_{X_{0,s,p}(\Omega)}\Big]\\
&+ \tilde\rho_n\|v\|_{X_{0,s,p}(\Omega)}g_n(v_n^-)\\
&\leq \tilde\rho_n\Big[\|u_n\|_{X_{0,s,p}(\Omega)}\left\langle g_n'(0), \overline{v_n}^+\right\rangle + o(1)\|v\|_{X_{0,s,p}(\Omega)}\\
&+ \|v\|_{X_{0,s,p}(\Omega)}g_n(v_n^-)\Big]\\
&\leq \tilde\rho_n C.
\end{aligned}$$

Substituting this and (5.8.22) in (5.8.21) yields

$$\begin{aligned}
\tilde\rho_n\Bigg(\left\langle g_n'(0), v^+\right\rangle + o(1)\|v^+\|_{X_{0,s,p}(\Omega)}\Bigg)\left\langle I_\mu'(u_n), v\right\rangle &+ \left\langle I_\mu'(u_n), v\right\rangle\tilde\rho_n + \tilde\rho_n o(1)\\
&\leq \tilde\rho_n.\frac{C}{n}.
\end{aligned}$$

This implies

$$\Big[\Big(\left\langle g_n'(0), v^+\right\rangle + o(1)\|v^+\|_{X_{0,s,p}(\Omega)}\Big)+1\Big]\left\langle I_\mu'(u_n), v\right\rangle \leq \frac{C}{n}+o(1) \text{ for all } n\geq n_0.$$

Since $|\langle g_n'(0), v^+\rangle|$ is uniformly bounded (see Lemma 5.9.1 in Appendix) , letting $n\to\infty$ we have $I_\mu'(u_n)\to 0$ in $(X_{0,s,p}(\Omega))'$. Hence the step 4 follows.

Therefore $\{u_n\}$ is a (PS) sequence of I_μ at level $\beta_1 < \tilde\alpha_\mu^-$. From Lemma 5.8.1, it follows that

$$\tilde\alpha_\mu^- < \frac{s}{N}S_{s,p}^{\frac{N}{ps}} - M\mu^{\frac{p_s^*}{p_s^*-q-1}} \quad \text{for} \quad \mu\in(0,\tilde\mu_1),$$

where $M = \frac{\Big(pN-(N-ps)(q+1)\Big)(p-1-q)}{p^2(q+1)}\Big(\frac{(p-1-q)(N-ps)}{p^2 s}\Big)^{\frac{q+1}{p_s^*-q-1}}|\Omega|$. Thus,

$$\beta_1 < \tilde\alpha_\mu^- < \frac{s}{N}S_{s,p}^{\frac{N}{ps}} - M\mu^{\frac{p_s^*}{p_s^*-q-1}}.$$

On the other hand, it follows from the Lemma 5.7.1 that I_μ satisfies PS at level c for

$$c < \frac{s}{N}S_{s,p}^{\frac{N}{ps}} - M\mu^{\frac{p_s^*}{p_s^*-q-1}},$$

this yields, there exists $u \in X_{0,s,p}(\Omega)$ such that $u_n \to u$ in $X_{0,s,p}(\Omega)$. By doing a simple calculation we get $u_n^- \to u^-$ in $X_{0,s,p}(\Omega)$. Consequently, by Step 3 $\|u^-\|_{X_{0,s,p}(\Omega)} \geq b$. As $\mathcal{N}_{\mu,1}^-$ is a closed set and $u_n \to u$, we obtain $u \in \mathcal{N}_{\mu,1}^-$, that is, $u^+ \in N_\mu^-$ and $u^+ \neq 0$. Therefore u is a solution of $(\mathcal{P}_\mu)$ with u^+ and u^- are both nonzero. Hence, u is a sign-changing solution of $(\mathcal{P}_\mu)$. Define $\tilde{w}_1 := u$. This completes the proof of part (i) of the theorem.

Proof of part (ii) is similar to part (i) and we omit the proof. □

Theorem 5.8.3. *Let $\beta_1, \beta_2 \geq \tilde{\alpha}_\mu^-$ where β_1, β_2, $\tilde{\alpha}_\mu^-$ be defined as in* (5.8.8) *and* (5.3.2) *respectively. Then, there exists $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0)$, I_μ has a sign changing critical point in the following cases:*

(i) for $p \geq \frac{3+\sqrt{5}}{2}$, there exists $q_2 := \frac{Np}{N-sp} - \frac{p}{p-1}$ such that when $q > q_2$ and $N > sp(p^2 - p + 1)$,

(ii) for $2 \leq p < \frac{3+\sqrt{5}}{2}$, there exists $q_3 := \frac{N(p-1)}{N-sp} - \frac{p-1}{p}$ such that when $q > q_3$ and $N > sp(p+1)$.

We need the following Proposition to prove the above Theorem 5.8.3.

Proposition 5.8.4. *Assume $0 < \mu < min\{\mu_*, \tilde{\mu}, \tilde{\mu}_1\}$, where $\tilde{\mu}$ is as defined in* (5.3.4) *and $\mu_* > 0$ is chosen such that $\tilde{\alpha}_\mu^-$ is achieved in $(0, \mu_*)$ and $\tilde{\mu}_1$ is as in Lemma 5.8.1. Then, for $p \geq \frac{3+\sqrt{5}}{2}$, there exists $q_2 := \frac{Np}{N-sp} - \frac{p}{p-1}$ such that when $q > q_2$ and $N > sp(p^2 - p + 1)$ we have*

$$\sup_{a \geq 0,\ b \in \mathbb{R}} I_\mu(aw_1 - bu_\varepsilon) < \tilde{\alpha}_\mu^- + \frac{s}{N} S_{s,p}^{\frac{N}{ps}},$$

for $\varepsilon > 0$ sufficiently small , where w_1 is a positive solution of $(\mathcal{P}_\mu)$ and u_ε be as in (5.5.6).

Furthermore, when $2 \leq p < \frac{3+\sqrt{5}}{2}$, there exists $q_3 := \frac{N(p-1)}{N-sp} - \frac{p-1}{p}$ such that when $q > q_3$ and $N > sp(p+1)$, it holds

$$\sup_{a \geq 0,\ b \in \mathbb{R}} I_\mu(aw_1 - bu_\varepsilon) < \tilde{\alpha}_\mu^- + \frac{s}{N} S_{s,p}^{\frac{N}{ps}},$$

for $\varepsilon > 0$ sufficiently small .

To prove the above proposition, we need the following lemmas.

Lemma 5.8.5. *Let w_1 and μ be as in Proposition 5.8.4. Then*

$$\sup_{s>0} I_\mu(sw_1) = \tilde{\alpha}_\mu^-.$$

Proof. By the definition of $\tilde{\alpha}_\mu^-$, we have $\tilde{\alpha}_\mu^- = \inf_{u\in N_\mu^-} I_\mu^+(u) = I_\mu^+(w_1) = I_\mu(w_1)$. In the last equality we have used the fact that $w_1 > 0$. Define $g(s) := I_\mu(sw_1)$. From the proof of Lemma 5.4.1, it follows that there exists only two critical points of g, namely $t^+(w_1)$ and $t^-(w_1)$ and $\max_{s>0} g(s) = g(t^+(w_1))$. On the other hand $\langle I'{}_\mu(w_1), v\rangle = 0$ for every $v \in X_{0,s,p}(\Omega)$. Therefore $g'(1) = 0$ which implies either $t^+(w_1) = 1$ or $t^-(w_1) = 1$.
Claim: $t^-(w_1) \neq 1$.
To see this, we note that $t^-(w_1) = 1$ implies $t^-(w_1)w_1 \in N_\mu^-$ as $w_1 \in N_\mu^-$. Using Lemma 5.4.1, we know $t^-(w_1)w_1 \in N_\mu^+$. Thus $N_\mu^+ \cap N_\mu^- \neq \emptyset$, which is a contradiction. Hence we have the claim.
Therefore $t^+(w_1) = 1$ and this completes the proof. □

Lemma 5.8.6. *Let u_ε be as in* (5.5.6) *and μ be as in Proposition 5.8.4. Then for $\varepsilon > 0$ sufficiently small, we have*

$$\sup_{t\in\mathbb{R}} I_\mu(tu_\varepsilon) = \frac{s}{N} S_{s,p}^{\frac{N}{ps}} + C\varepsilon^{\frac{(N-ps)}{(p-1)}} - k_8|u_\varepsilon|_{L^{q+1}(\Omega)}^{q+1}.$$

Proof. Define $\tilde{\phi}(t) = \frac{t^p}{p}\|u_\varepsilon\|_{X_{0,s,p}(\Omega)}^p - \frac{t^{p_s^*}}{p_s^*}|u_\varepsilon|_{L^{p_s^*}(\Omega)}^{p_s^*}$. Thus $I_\mu(tu_\varepsilon) = \tilde{\phi}(t) - \mu\frac{t^{q+1}}{q+1}|u_\varepsilon|_{L^{q+1}(\Omega)}^{q+1}$. On the other hand, applying the analysis done in Lemma 5.4.1 to u_ε, we obtain there exists $(t_0)_\varepsilon = \left(\frac{(p-1-q)\|u_\varepsilon\|_{X_{0,s,p}(\Omega)}^p}{(p_s^*-1-q)|u_\varepsilon|_{L^{p_s^*}(\Omega)}^{p_s^*}}\right)^{\frac{N-ps}{p^2 s}} < t_\varepsilon^+$ such that

$$\begin{aligned}\sup_{t\in\mathbb{R}} I_\mu(tu_\varepsilon) = \sup_{t\geq 0} I_\mu(tu_\varepsilon) = I_\mu(t_\varepsilon^+ u_\varepsilon) &= \tilde{\phi}(t_\varepsilon^+) - \mu\frac{(t_\varepsilon^+)^{q+1}}{q+1}|u_\varepsilon|_{L^{q+1}(\Omega)}^{q+1} \\ &\leq \sup_{t\geq 0}\tilde{\phi}(t) - \mu\frac{(t_0)_\varepsilon^{q+1}}{q+1}|u_\varepsilon|_{L^{q+1}(\Omega)}^{q+1}.\end{aligned}$$

Substituting the value of $(t_0)_\varepsilon$ and using Sobolev inequality, we have

$$\mu\frac{(t_0)_\varepsilon^{q+1}}{q+1} \geq \frac{\mu}{q+1}\left(\frac{p-1-q}{p_s^*-q-1}S_{s,p}\right)^{\frac{(N-ps)(q+1)}{p^2 s}} = k_8.$$

Consequently,

$$\sup_{t\in\mathbb{R}} I_\mu(tu_\varepsilon) \leq \sup_{t\geq 0}\tilde{\phi}(t) - k_8|u_\varepsilon|_{L^{q+1}(\Omega)}^{q+1}. \tag{5.8.24}$$

Using elementary analysis, it is easy to check that $\tilde{\phi}$ attains it's maximum at the point $\tilde{t}_0 = \left(\frac{\|u_\varepsilon\|_{X_{0,s,p}(\Omega)}^p}{|u_\varepsilon|_{L^{p_s^*}(\Omega)}^{p_s^*}}\right)^{\frac{1}{p_s^*-p}}$ and $\tilde{\phi}(t_0) = \frac{s}{N}\left(\frac{\|u_\varepsilon\|_{X_{0,s,p}(\Omega)}^p}{|u_\varepsilon|_{L^{p_s^*}(\Omega)}^p}\right)^{\frac{N}{ps}}$.
Moreover, using (5.8.6) and (5.8.4), we can deduce as in (5.8.7) that

$$\tilde{\phi}(t_0) \leq \frac{s}{N}S_{s,p}^{\frac{N}{ps}} + C\varepsilon^{\frac{(N-ps)}{(p-1)}}. \tag{5.8.25}$$

Substituting back (5.8.25) into (5.8.24), completes the proof. □

Proof of Proposition 5.8.4: Note that, for fixed a and b, $I_\mu\Big(\eta(aw_1 - bu_{\varepsilon,\delta})\Big) \to -\infty$ as $|\eta| \to \infty$. Therefore $\sup_{a\geq 0,\ b\in\mathbb{R}} I_\mu(aw_1 - bu_{\varepsilon,\delta})$ exists and supremum will be attained in $a^2 + b^2 \leq R^2$, for some large $R > 0$. Thus it is enough to estimate $I_\mu(aw_1 - bu_{\varepsilon,\delta})$ in $\{(a,b) \in \mathbb{R}^+ \times \mathbb{R} : a^2 + b^2 \leq R^2\}$. Using elementary inequality, there exists $d(m) > 0$ such that

$$|a+b|^m \geq |a|^m + |b|^m - d(|a|^{m-1}|b| + |a||b|^{m-1}) \quad \forall \quad a,\ b \in \mathbb{R},\ m > 1. \tag{5.8.26}$$

Define, $f(v) := \|v\|_{X_{0,s,p}(\Omega)}^p$. Then using Taylor's theorem

$$\begin{aligned} &f(aw_1 - bu_{\varepsilon,\delta}) = f(aw_1) - \langle f'(aw_1), bu_\varepsilon\rangle + o(\|bu_{\varepsilon,\delta}\|_{X_{0,s,p}(\Omega)}^2) \\ &\leq \|aw_1\|_{X_{0,s,p}(\Omega)}^p \\ &- p\int_{\mathbb{R}^{2N}} \frac{|aw_1(x) - aw_1(y)|^{p-2}(aw_1(x) - aw_1(y))(bu_{\varepsilon,\delta}(x) - bu_{\varepsilon,\delta}(y))}{|x-y|^{N+ps}} \\ &+ c\|bu_{\varepsilon,\delta}\|_{X_{0,s,p}(\Omega)}^2, \end{aligned} \tag{5.8.27}$$

where $c > 0$ is small enough. We also note that from the definition of $u_{\varepsilon,\delta}$, it follows that $\|u_{\varepsilon,\delta}\|_{X_{0,s,p}(\Omega)}$ is bounded away from 0. Therefore, since $p \geq 2$ we have $c\|bu_{\varepsilon,\delta}\|^2_{X_{0,s,p}(\Omega)} \leq \|bu_{\varepsilon,\delta}\|^p_{X_{0,s,p}(\Omega)}$, for $c > 0$ small enough. Hence

$$\begin{aligned}
&\|aw_1 - bu_{\varepsilon,\delta}\|^p_{X_{0,s,p}(\Omega)} = \|aw_1\|^p_{X_{0,s,p}(\Omega)} \\
&- p\int_{\mathbb{R}^{2N}} \frac{|aw_1(x) - aw_1(y)|^{p-2}(aw_1(x) - aw_1(y))(bu_{\varepsilon,\delta}(x) - bu_{\varepsilon,\delta}(y))}{|x-y|^{N+ps}} \\
&+ \|bu_{\varepsilon,\delta}\|^p_{X_{0,s,p}(\Omega)}
\end{aligned} \tag{5.8.28}$$

Consequently, $a^2 + b^2 \leq R^2$ implies

$$\begin{aligned}
&I_\mu(aw_1 - bu_{\varepsilon,\delta}) \leq \frac{1}{p}\|aw_1\|^p_{X_{0,s,p}(\Omega)} \\
&- \int_{\mathbb{R}^{2N}} \frac{|aw_1(x) - aw_1(y)|^{p-2}(aw_1(x) - aw_1(y))(bu_{\varepsilon,\delta}(x) - bu_{\varepsilon,\delta}(y))}{|x-y|^{N+ps}}dxdy \\
&+ \frac{1}{p}\|bu_{\varepsilon,\delta}\|^p_{X_{0,s,p}(\Omega)} - \frac{1}{p_s^*}\int_\Omega |aw_1|^{p_s^*}dx - \frac{1}{p_s^*}\int_\Omega |bu_{\varepsilon,\delta}|^{p_s^*}dx \\
&- \frac{\mu}{q+1}\int_\Omega |aw_1|^{q+1}dx - \frac{\mu}{q+1}\int_\Omega |bu_{\varepsilon,\delta}|^{q+1}dx \\
&+ C\left(\int_\Omega |aw_1|^{p_s^*-1}|bu_{\varepsilon,\delta}|dx + \int_\Omega |aw_1||bu_{\varepsilon,\delta}|^{p_s^*-1}dx\right) \\
&+ C\left(\int_\Omega |aw_1|^q|bu_{\varepsilon,\delta}|dx + \int_\Omega |aw_1||bu_{\varepsilon,\delta}|^q dx\right) \\
&= I_\mu(aw_1) + I_\mu(bu_{\varepsilon,\delta}) - a^q b\mu\int_\Omega |w_1|^{q-1}w_1 u_{\varepsilon,\delta}dx \\
&- a^{p_s^*}b\int_\Omega |w_1|^{p_s^*-2}w_1 u_{\varepsilon,\delta}dx \\
&+ C\left(\int_\Omega |w_1|^{p_s^*-1}|u_{\varepsilon,\delta}|dx + \int_\Omega |w_1||u_{\varepsilon,\delta}|^{p_s^*-1}dx\right) \\
&+ C\left(\int_\Omega |w_1|^q|u_{\varepsilon,\delta}|dx + \int_\Omega |w_1||u_{\varepsilon,\delta}|^q dx\right).
\end{aligned}$$

Using Lemmas 5.6.1, 5.8.5 and 5.8.6 we estimate in $a^2 + b^2 \leq R^2$,

$$\begin{aligned}
I_\mu(aw_1 - bu_{\varepsilon,\delta}) &\leq \tilde{\alpha}_\mu^- + \frac{s}{N}S_{s,p}^{\frac{N}{ps}} - k_8|u_\varepsilon|^{q+1}_{L^{q+1}(\Omega)} \\
&+ C\left(\varepsilon^{\frac{(N-ps)}{(p-1)}} + \varepsilon^{\frac{N-ps}{p(p-1)}} + \varepsilon^{\frac{(N-ps)q}{p(p-1)}} + \varepsilon^{\frac{N(p-1)+ps}{p(p-1)}}\right).
\end{aligned}$$

For the term $k_8|u_\varepsilon|^{q+1}_{L^{q+1}(\Omega)}$, we invoke Lemma 5.6.2. Therefore when $\frac{N(p-2)+ps}{N-ps} < q < p-1$, we have

$$\begin{aligned} I_\mu(aw_1 - bu_{\varepsilon,\delta}) \leq \tilde{\alpha}_\mu^- &+ \frac{s}{N} S_{s,p}^{\frac{N}{ps}} - k_9 \varepsilon^{N - \frac{(N-ps)(q+1)}{p}} \\ &+ C\left(\varepsilon^{\frac{(N-ps)}{(p-1)}} + \varepsilon^{\frac{N-ps}{p(p-1)}} + \varepsilon^{\frac{(N-ps)q}{p(p-1)}} + \varepsilon^{\frac{N(p-1)+ps}{p(p-1)}}\right) \end{aligned} \tag{5.8.29}$$

We will choose q in such a way that the term $k_9\varepsilon^{N-\frac{(N-ps)(q+1)}{p}}$ dominates the other term involving ε. Note that among the terms in the bracket, $\varepsilon^{\frac{N-ps}{p(p-1)}}$ and $\varepsilon^{\frac{(N-ps)q}{p(p-1)}}$ dominate the others.

This in turn implies we have to choose q such that

$$N - \frac{(N-ps)(q+1)}{p} < \frac{N-ps}{p(p-1)} \tag{5.8.30}$$

and

$$N - \frac{(N-ps)(q+1)}{p} < \frac{(N-ps)q}{p(p-1)}. \tag{5.8.31}$$

(5.8.30) and (5.8.31) implies $q > q_2$ and $q > q_3$ respectively, where

$$q_2 := \frac{Np}{N-sp} - \frac{p}{p-1} \quad \text{and} \quad q_3 := \frac{N(p-1)}{N-sp} - \frac{p-1}{p}. \tag{5.8.32}$$

Case 1: $p \geq \frac{3+\sqrt{5}}{2}$

In this case by straight forward calculation it follows that $q_2 > q_3$. So in this case, we choose $q > q_2$. Moreover, since $q < p-1$, to make the interval $(q_2, p-1) \neq \emptyset$, we have to take $N > sp(p^2 - p + 1)$.

Case 2: $2 \leq p < \frac{3+\sqrt{5}}{2}$

In this case again by simple calculation it follows that $q_3 > q_2$. Thus, in this case, we choose $q > q_3$. Furthermore, as $q < p-1$, to make the interval $(q_3, p-1) \neq \emptyset$, we have to take $N > sp(p+1)$.

Hence in both the cases taking $\varepsilon > 0$ to be small enough in (5.8.29), we obtain

$$\sup_{a \geq 0, b \in \mathbb{R}} I_\mu(aw_1 - bu_{\varepsilon,\delta}) < \tilde{\alpha}_\mu^- + \frac{s}{N} S_{s,p}^{\frac{N}{ps}}.$$

□

Proof of Theorem 5.8.3: Define $\mu_0 := \min\{\tilde{\mu}, \mu_*\}$,

$$\mathcal{N}_*^- := \mathcal{N}_{\mu,1}^- \cap \mathcal{N}_{\mu,2}^-. \tag{5.8.33}$$

and

$$c_2 := \inf_{u \in \mathcal{N}_*^-} I_\mu(u), \tag{5.8.34}$$

Let $\mu \in (0, \mu_0)$. Using Ekland's variational principle and similar to the proof of Theorem 5.8.2, we obtain a sequence $\{u_n\} \in \mathcal{N}_*^-$ satisfying

$$I_\mu(u_n) \to c_2, \quad I'_\mu(u_n) \to 0 \quad \text{in} \quad (X_{0,s,p}(\Omega))'.$$

Thus $\{u_n\}$ is a (PS) sequence at level c_2. From Lemma 5.8.7, given below, it follows that there exists $a > 0$ and $b \in R$ such that $aw_1 - bu_\varepsilon \in \mathcal{N}_*^-$. Therefore Proposition 5.8.4 yields

$$c_2 < \tilde{\alpha}_\mu^- + \frac{s}{N} S_{s,p}^{\frac{N}{ps}}. \tag{5.8.35}$$

Claim 1: There exists two positive constants c, C such that $0 < c \leq \|u_n^\pm\|_{X_{0,s,p}(\Omega)} \leq C$.

To see this, we note that $\{u_n\} \subset \mathcal{N}_*^- \subset \mathcal{N}_{\mu,1}^-$. Thus using (5.8.12), Step 2 and Step 3 of the proof of Theorem 5.8.2, we have $\|u_n^\pm\|_{X_{0,s,p}(\Omega)} \leq C$ and $\|u_n^-\|_{X_{0,s,p}(\Omega)} \geq c$. To show $\|u_n^+\|_{X_{0,s,p}(\Omega)} \geq a$ for some $a > 0$, we use method of contradiction. Assume up to a subsequence $\|u_n^+\|_{X_{0,s,p}(\Omega)} \to 0$ as $n \to \infty$. This together with Sobolev embedding implies $|u_n^+|_{L^{p_s^*}(\Omega)} \to 0$. On the other hand, $u_n^+ \in N_\mu^-$ implies $(p-1-q)\|u_n^+\|^p_{X_{0,s,p}(\Omega)} - (p_s^* - q - 1)|u_n^+|^{p_s^*}_{L^{p_s^*}(\Omega)} < 0$. Therefore by Sobolev inequality, we have

$$S_{s,p} \leq \frac{\|u_n^+\|^p_{X_{0,s,p}(\Omega)}}{|u_n^+|^p_{L^{p_s^*}(\Omega)}} < \frac{p_s^* - q - 1}{p - 1 - q} |u_n^+|^{p_s^* - p}_{L^{p_s^*}(\Omega)},$$

which is a contradiction to the fact that $|u_n^+|_{L^{p_s^*}(\Omega)} \to 0$. Hence the claim follows.

Going to a subsequence if necessary we have

$$u_n^+ \rightharpoonup \eta_1, \; u_n^- \rightharpoonup \eta_2 \quad \text{in} \quad X_{0,s,p}(\Omega). \tag{5.8.36}$$

Claim 2: $\eta_1 \not\equiv 0, \; \eta_2 \not\equiv 0$.

Suppose not, that is $\eta_1 \equiv 0$. Then by compact embedding, $u_n^+ \to 0$ in $L^{q+1}(\Omega)$. Moreover, $u_n^+ \in N_\mu^- \subset N_\mu$, implies $\left\langle I'_\mu(u_n^+), u_n^+ \right\rangle = 0$. Consequently,

$$\|u_n^+\|^p_{X_{0,s,p}(\Omega)} - |u_n^+|^{p_s^*}_{L^{p_s^*}(\Omega)} = \mu |u_n^+|^{q+1}_{L^{q+1}(\Omega)} = o(1).$$

So we have $|u_n^+|^{p_s^*}_{L^{p_s^*}(\Omega)} = \|u_n^+\|^p_{X_{0,s,p}(\Omega)} + o(1)$. This together with $\|u_n^+\|_{X_{0,s,p}(\Omega)} \geq c$ implies

$$\frac{|u_n^+|^{p_s^*}_{L^{p_s^*}(\Omega)}}{\|u_n^+\|^p_{X_{0,s,p}(\Omega)}} \geq 1 + o(1).$$

This along with Sobolev embedding gives $|u_n^+|^{p_s^*}_{L^{p_s^*}(\Omega)} \geq S_{s,p}^{N/ps} + o(1)$. Thus we have,

$$I_\mu(u_n^+) = \frac{1}{p}\|u_n^+\|^p_{X_{0,s,p}(\Omega)} - \frac{1}{p_s^*}|u_n^+|^{p_s^*}_{L^{p_s^*}(\Omega)} + o(1) \geq \frac{s}{N} S_{s,p}^{N/ps} + o(1). \tag{5.8.37}$$

Moreover, $u_n \in \mathcal{N}_*^-$ implies $-u_n^- \in N_\mu^-$. Therefore using the given condition on β_2, we get

$$I_\mu(-u_n^-) \geq \beta_2 \geq \tilde{\alpha}_\mu^-. \tag{5.8.38}$$

Also it follows $I_\mu(u_n^+) + I_\mu(-u_n^-) \leq I_\mu(u_n) = c_2 + o(1)$ (see (5.8.14)). Combining this along with (5.8.38) and (5.8.35), we obtain

$$I_\mu(u_n^+) \leq c_2 - \tilde{\alpha}_\mu^- + o(1) < \frac{s}{N} S_{s,p}^{N/ps},$$

which is a contradiction to (5.8.37). Therefore $\eta_1 \neq 0$. Similarly $\eta_2 \neq 0$ and this proves the claim.

Set $w_2 := \eta_1 - \eta_2$.

Claim 3: $w_2^+ = \eta_1$ and $w_2^- = \eta_2$ a.e..
To see the claim we observe that $\eta_1\eta_2 = 0$ a.e. in Ω. Indeed,

$$\begin{aligned} \left|\int_\Omega \eta_1\eta_2 dx\right| &= \left|\int_\Omega (u_n^+ - \eta_1)u_n^- dx + \int_\Omega \eta_1(u_n^- - \eta_2)dx\right| \\ &\leq |u_n^+ - \eta_1|_{L^p(\Omega)}|u_n^-|_{L^{p'}(\Omega)} + |\eta_1|_{L^{p'}(\Omega)}|u_n^- - \eta_2|_{L^p(\Omega)} \end{aligned} \tag{5.8.39}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. By compact embedding we have $u_n^+ \to \eta_1$ and $u_n^- \to \eta_2$ in $L^p(\Omega)$. As $p \geq \frac{2N}{N+s}$, then $p' \leq p_s^*$. Therefore, using claim 1, we pass the limit in (5.8.39) and obtain $\int_\Omega \eta_1\eta_2 dx = 0$. Moreover by (5.8.36), $\eta_1,\ \eta_2 \geq 0$ a.e.. Hence $\eta_1\eta_2 = 0$ a.e. in Ω. We have $w_2^+ - w_2^- = w_2 = \eta_1 - \eta_2$. It is easy to check that $w_2^+ \leq \eta_1$ and $w_2^- \leq \eta_2$. To show that equality holds a.e. we apply method of contradiction. Suppose, there exists $E \subset \Omega$ such that $|E| > 0$ and $0 \leq w_2^+(x) < \eta_1(x)\ \forall\ x \in E$. Therefore $\eta_2 = 0$ a.e. in E by the observation that we made. Hence $w_2^+(x) - w_2^-(x) = \eta_1(x)$ a.e. in E. Clearly $w_2^-(x) \not> 0$ a.e., otherwise $w_2^+(x) = 0$ a.e. and that would imply $\eta_1(x) = -w_2^-(x) < 0$ a.e, which is not possible since $\eta_1 > 0$ in E. Thus $w_2^-(x) = 0$. Hence $\eta_1(x) = w_2^+(x)$ a.e. in E, which is a contradiction. Hence the claim follows.

Therefore w_2 is sign changing in Ω and $u_n \rightharpoonup w_2$ in $X_{0,s,p}(\Omega)$. Moreover, $I'_\mu(u_n) \to 0$ in $(X_{0,s,p}(\Omega))'$ implies

$$\begin{aligned} \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\phi(x) - \phi(y))}{|x-y|^{N+ps}}dxdy \quad &-\mu \int_\Omega |u_n|^{q-1}u_n\phi dx \\ &- \int_\Omega |u_n|^{p_s^*-2}u_n\phi dx \\ &= o(1) \end{aligned} \tag{5.8.40}$$

for every $\phi \in X_{0,s,p}(\Omega)$. Passing the limit using Vitali's convergence theorem via Hölder's inequality we obtain $\left\langle I'_\mu(w_2), \phi \right\rangle = 0$. Hence w_2 is a sign changing weak solution to $(\mathcal{P}_\mu)$. ☐

Lemma 5.8.7. *Let $u_{\varepsilon,\delta}$ be as defined in* (5.5.6) *and w_1 be a positive solution of* $(\mathcal{P}_\mu)$ *for which $\tilde{\alpha}_\mu^-$ is achieved, when $\mu \in (0, \mu_*)$. Then there exists a, $b \in \mathbb{R}$, $a \geq 0$ such that $aw_1 - bu_\varepsilon \in \mathcal{N}_*^-$, where $\mathcal{N}_*^-$ is defined as in* (5.8.33)*.*

This lemma can be proved in the spirit of [21, Lemma 4.8], for the convenience of the reader we again sketch the proof in the appendix.

We finally conclude the proof of our main result.

Proof of Theorem 5.2.1: Define $\mu^* = \min\{\mu_*, \tilde{\mu}, \tilde{\mu}_1, \mu_0, \mu_1\}$, where μ_* is chosen such that $\tilde{\alpha}_\mu^-$ is achieved in $(0, \mu_*)$. $\tilde{\mu}$, $\tilde{\mu}_1$, μ_0 and μ_1 are as in (5.3.4), Lemma 5.8.1, Theorem 5.8.3 and Lemma 5.9.1 respectively. Furthermore, define q_0 and N_0 as follows:

$$q_0 := \begin{cases} \max\{q_1, q_2\} & \text{when} \quad p \geq \frac{3+\sqrt{5}}{2}, \\ \max\{q_1, q_3\} & \text{when} \quad 2 \leq p < \frac{3+\sqrt{5}}{2}. \end{cases}$$

$$N_0 := \begin{cases} sp(p^2 - p + 1) & \text{when} \quad p \geq \frac{3+\sqrt{5}}{2}, \\ sp(p+1) & \text{when} \quad 2 \leq p < \frac{3+\sqrt{5}}{2}. \end{cases}$$

Note that $N_0 > \frac{sp}{2}[p + 1 + \sqrt{(p+1)^2 - 4}]$, where the RHS appeared in Theorem 5.8.2. Hence combining Theorem 5.8.2 and Theorem 5.8.3, we complete the proof of this theorem for $\mu \in (0, \mu^*)$, $q > q_0$ and $N > N_0$. □

5.9 Appendix

Lemma 5.9.1. *Let g_n be as in* (5.8.16) *in the Theorem 5.8.2 and $v \in X_{0,s,p}(\Omega)$ such that $\|v\|_{X_{0,s,p}(\Omega)} = 1$. Then there exists $\mu_1 > 0$ such that if $\mu \in (0, \mu_1)$ implies $\langle g_n'(0), v^+ \rangle$ is uniformly bounded in $X_{0,s,p}(\Omega)$.*

Proof. In view of Lemma 5.4.4 we have,

$$\left\langle g_n'(0), v^+ \right\rangle = \frac{pA(u_n, v^+) - p_s^* \int_\Omega |u_n|^{p_s^*-p} u_n v^+ - (q+1)\mu \int_\Omega |u_n|^{q-1} u_n v^+}{(p-1-q)\|u_n\|^p_{X_{0,s,p}(\Omega)} - (p_s^* - q - 1)|u_n|^{p_s^*}_{L^{p_s^*}(\Omega)}}.$$

Using Claim 2 in Theorem 5.8.2, there exists $C > 0$ such that $\|u_n\|_{X_{0,s,p}(\Omega)} \leq C$ for all $n \geq 1$. Therefore applying Hölder inequality followed by Sobolev inequality, we have
$|\langle g_n'(0), v^+\rangle| \leq \frac{C\|v\|_{X_{0,s,p}(\Omega)}}{\left|(p-1-q)\|u_n\|^p_{X_{0,s,p}(\Omega)} - (p_s^*-q-1)|u_n|^{p_s^*}_{L^{p_s^*}(\Omega)}\right|}$. Hence it is enough to show

$$\left|(p-1-q)\|u_n\|^p_{X_{0,s,p}(\Omega)} - (p_s^* - q - 1)|u_n|^{p_s^*}_{L^{p_s^*}(\Omega)}\right| > C,$$

for some $C > 0$ and n large. Suppose it does not hold. Then up to a subsequence

$$(p-1-q)\|u_n\|^p_{X_{0,s,p}(\Omega)} - (p_s^* - q - 1)|u_n|^{p_s^*}_{L^{p_s^*}(\Omega)} = o(1) \quad \text{as} \quad n \to \infty.$$

Hence,

$$\|u_n\|^p_{X_{0,s,p}(\Omega)} = \frac{p^* - q - 1}{p - 1 - q}|u_n|^{p_s^*}_{L^{p_s^*}(\Omega)} + o(1) \quad \text{as} \quad n \to \infty. \tag{5.9.1}$$

Combining the above expression along with the fact that $u_n \in N_\mu$, we obtain

$$\mu|u_n|^{q+1}_{L^{q+1}(\Omega)} = \frac{p_s^* - p}{p - 1 - q}|u_n|^{p_s^*}_{L^{p_s^*}(\Omega)} + o(1) = \frac{p_s^* - p}{p_s^* - 1 - q}\|u_n\|^p_{X_{0,s,p}(\Omega)} + o(1). \tag{5.9.2}$$

After applying Hölder inequality and followed by Sobolev inequality, expression (5.9.2) yields

$$\|u_n\|_{X_{0,s,p}(\Omega)} \leq \left(\mu \frac{p_s^* - q - 1}{p_s^* - p}|\Omega|^{\frac{p_s^*-q-1}{p_s^*}} S_{s,p}^{-\frac{q+1}{p}}\right)^{\frac{1}{p-1-q}} + o(1). \tag{5.9.3}$$

Combining (5.8.12) and Claim 3 in the proof of Theorem 5.8.2, we have $\|u_n\|_{X_{0,s,p}(\Omega)} \geq b$, for some $b > 0$. Therefore from (5.9.1) we get

$$|u_n|^{p_s^*}_{L^{p_s^*}(\Omega)} \geq C \quad \text{for some constant } C > 0, \text{ and } n \text{ large enough.} \tag{5.9.4}$$

Define $\psi_\mu : N_\mu \to \mathbb{R}$ as follows:

$$\psi_\mu(u) = k_0 \left(\frac{\|u\|^{p(p_s^*-1)}_{X_{0,s,p}(\Omega)}}{|u|^{p_s^*(p-1)}_{L^{p_s^*}(\Omega)}} \right)^{\frac{1}{p_s^*-p}} - \mu |u|^{q+1}_{L^{q+1}(\Omega)},$$

where $k_0 = \left(\frac{p-1-q}{p_s^*-q-1}\right)^{\frac{p_s^*-1}{p_s^*-p}} \left(\frac{p_s^*-p}{p-1-q}\right)$. Simplifying $\psi_\mu(u_n)$ using (5.9.2), we obtain

$$\psi_\mu(u_n) = k_0 \left[\left(\frac{p_s^*-q-1}{p-1-q} \right)^{p_s^*-1} \frac{|u_n|^{(p_s^*-1)p_s^*}_{L^{p_s^*}(\Omega)}}{|u_n|^{p_s^*(p-1)}_{L^{p_s^*}(\Omega)}} \right]^{\frac{1}{p_s^*-p}} - \frac{p_s^*-p}{p-1-q} |u_n|^{p_s^*}_{L^{p_s^*}(\Omega)} + o(1) = o(1). \tag{5.9.5}$$

On the other hand, using Hölder inequality in the definition of $\psi_\mu(u_n)$, we obtain

$$\begin{aligned}
\psi_\mu(u_n) &= k_0 \left(\frac{\|u_n\|^{p(p_s^*-1)}_{X_{0,s,p}(\Omega)}}{|u_n|^{p_s^*(p-1)}_{L^{p_s^*}(\Omega)}} \right)^{\frac{1}{p_s^*-p}} - \mu |u_n|^{q+1}_{L^{q+1}(\Omega)} \\
&\geq k_0 \left(\frac{\|u_n\|^{p(p_s^*-1)}_{X_{0,s,p}(\Omega)}}{|u_n|^{p_s^*(p-1)}_{L^{p_s^*}(\Omega)}} \right)^{\frac{1}{p_s^*-p}} - \mu |\Omega|^{\frac{p_s^*-q-1}{p_s^*}} |u_n|^{q+1}_{L^{p_s^*}(\Omega)} \\
&= |u_n|^{q+1}_{L^{p_s^*}(\Omega)} \left\{ k_0 \left(\frac{\|u_n\|^{p(p_s^*-1)}_{X_{0,s,p}(\Omega)}}{|u_n|^{p_s^*(p-1)}_{L^{p_s^*}(\Omega)}} \right)^{\frac{1}{p_s^*-p}} \frac{1}{|u_n|^{q+1}_{L^{p_s^*}(\Omega)}} - \mu |\Omega|^{\frac{p_s^*-q-1}{p_s^*}} \right\}.
\end{aligned} \tag{5.9.6}$$

Using Sobolev embedding and (4.6.3), we simplify the term $\left(\frac{\|u_n\|^{p(p_s^*-1)}_{X_{0,s,p}(\Omega)}}{|u_n|^{p_s^*(p-1)}_{L^{p_s^*}(\Omega)}} \right)^{\frac{1}{p_s^*-p}} \frac{1}{|u_n|^{q+1}_{L^{p_s^*}(\Omega)}}$ and obtain

$$\begin{aligned}
\left(\frac{\|u_n\|^{p(p_s^*-1)}_{X_{0,s,p}(\Omega)}}{|u_n|^{p_s^*(p-1)}_{L^{p_s^*}(\Omega)}} \right)^{\frac{1}{p_s^*-p}} \frac{1}{|u_n|^{q+1}_{L^{p_s^*}(\Omega)}} &\geq S_{s,p}^{\frac{p_s^*-1}{p_s^*-p}} |u_n|^{-q}_{L^{p_s^*}(\Omega)} \\
&\geq S_{s,p}^{\frac{p_s^*-1}{p_s^*-p}+\frac{q}{p}} \|u_n\|^{-q}_{X_{0,s,p}(\Omega)} \\
&\geq S_{s,p}^{\frac{p_s^*-1}{p_s^*-p}+\frac{q}{p}} \left(\mu \frac{p_s^*-q-1}{p_s^*-p} |\Omega|^{\frac{p^*-q-1}{p_s^*}} S_{s,p}^{-\frac{q+1}{p}} \right)^{-\frac{q}{p-1-q}}.
\end{aligned} \tag{5.9.7}$$

Substituting back (5.9.7) into (5.9.6) and using (5.9.4), we obtain

$$\psi_\mu(u_n) \geq C^{q+1}\Bigg[k_0 S_{s,p}^{\frac{p_s^*-1}{p_s^*-p}+\frac{q}{p-1-q}} \mu^{-\frac{q}{p-1-q}} \left(\frac{p_s^*-q-1}{p_s^*-p}|\Omega|^{\frac{p_s^*-q-1}{p_s^*}}\right)^{-\frac{q}{p-1-q}} - \mu|\Omega|^{\frac{p_s^*-q-1}{p_s^*}}\Bigg] \geq d_0,$$

for some $d_0 > 0$, n large and $\mu < \mu_1$, where $\mu_1 = \mu_1(k_0, s, q, N, |\Omega|)$. This is a contradiction to (5.9.5). Hence the lemma follows. □

Proof of Lemma 5.8.7

Proof. We will show that there exists $a > 0,\ b \in \mathbb{R}$ such that

$$a(w_1 - bu_\varepsilon)^+ \in N_\mu^- \quad \text{and} \quad -a(w_1 - bu_\varepsilon)^- \in N_\mu^-.$$

Let us denote $\bar{r}_1 = \inf_{x\in\Omega} \frac{w_1(x)}{u_\varepsilon(x)},\ \bar{r}_2 = \sup_{x\in\Omega} \frac{w_1(x)}{u_\varepsilon(x)}$.
As both w_1 and u_ε are positive in Ω, we have $\bar{r}_1 \geq 0$ and $\bar{r}_2$ can be $+\infty$. Let $r \in (\bar{r}_1, \bar{r}_2)$. Then $w_1, u_\varepsilon \in X_{0,s,p}(\Omega)$ implies $(w_1 - ru_\varepsilon) \in X_{0,s,p}(\Omega)$ and $(w_1 - ru_\varepsilon)^+ \not\equiv 0$. Otherwise, $(w_1 - ru_\varepsilon)^+ \equiv 0$ would imply $\bar{r}_2 \leq r$, which is not possible. Define $v_r := w_1 - ru_\varepsilon$. Hence $0 \not\equiv v_r^+ \in X_{0,s,p}(\Omega)$ (since for any $u \in X_{0,s,p}(\Omega)$, we have $|u| \in X_{0,s,p}(\Omega)$. Similarly $0 \not\equiv v_r^- \in X_{0,s,p}(\Omega)$. Therefore by lemma 5.4.1 there exists $0 < s^+(r) < s^-(r)$ such that $s^+(r)v_r^+ \in N_\mu^-$, and $-s^-(r)(v_r^-) \in N_\mu^-$. Let us consider the functions $s^\pm : \mathbb{R} \to (0, \infty)$ defined as above.
Claim: The functions $r \mapsto s^\pm(r)$ are continuous and

$$\lim_{r\to\bar{r}_1^+} s^+(r) = t^+(v_{\bar{r}_1}^+) \quad \text{and} \quad \lim_{r\to\bar{r}_2^-} s^+(r) = +\infty,$$

where the function t^+ is same as defined in lemma 5.4.1.
To see the claim, choose $r_0 \in (\bar{r}_1, \bar{r}_2)$ and $\{r_n\}_{n\geq 1} \subset (\bar{r}_1, \bar{r}_2)$ such that $r_n \to r_0$ as $n \to \infty$. We need to show that $s^+(r_n) \to s^+(r_0)$ as $n \to \infty$. Corresponding to r_n and r_0, we have $v_{r_n}^+ = (w_1 - r_n u_\varepsilon)^+$ and $v_{r_0}^+ = (w_1 - r_0 u_\varepsilon)^+$. By lemma

5.4.1. we note that $s^+(r) = t^+(v_r^+)$. Let us define the function

$$\begin{aligned} F(s,r) &:= s^{p-1-q}\|(w_1 - ru_\varepsilon)^+\|^p_{X_{0,s,p}(\Omega)} - s^{p_s^*-q-1}|(w_1 - ru_\varepsilon)^+|^{p_s^*}_{L^{p_s^*}(\Omega)} \\ &- \mu|(w_1 - ru_\varepsilon)^+|^{q+1}_{L^{q+1}(\Omega)} \\ &= \phi(s,r) - \mu|(w_1 - ru_\varepsilon)^+|^{q+1}_{L^{q+1}(\Omega)}, \end{aligned}$$

where

$$\phi(s,r) := s^{p-1-q}\|(w_1 - ru_\varepsilon)^+\|^p_{X_{0,s,p}(\Omega)} - s^{p_s^*-q-1}|(w_1 - ru_\varepsilon)^+|^{p_s^*}_{L^{p_s^*}(\Omega)}.$$

Doing the similar calculation as in lemma 5.4.1, we obtain that for any fixed r, the function $F(s,r)$ has only two zeros $s = t^+(v_r^+)$ and $s = t^-(v_r^+)$. Consequently $s^+(r)$ is the largest 0 of $F(s,r)$ for any fixed r. As $r_n \to r_0$ we have $v_{r_n}^+ \to v_{r_0}^+$ in $X_{0,s,p}(\Omega)$. Indeed, by straight forward computation it follows $v_{r_n} \to v_{r_0}$ in $X_{0,s,p}(\Omega)$. Therefore $|v_{r_n}| \to |v_{r_0}|$ in $X_{0,s,p}(\Omega)$. This in turn implies $v_{r_n}^+ \to v_{r_0}^+$ in $X_{0,s,p}(\Omega)$. Hence $\|v_{r_n}^+\|_{X_{0,s,p}(\Omega)} \to \|v_{r_0}^+\|_{X_{0,s,p}(\Omega)}$. Moreover by Sobolev inequality, we have $|v_{r_n}^+|_{L^{p_s^*}(\Omega)} \to |v_{r_0}^+|_{L^{p_s^*}(\Omega)}$ and $|v_{r_n}^+|_{L^{q+1}(\Omega)} \to |v_{r_0}^+|_{L^{q+1}(\Omega)}$. As a result, we have $F(s,r_n) \to F(s,r_0)$ uniformly. Therefore an elementary analysis yields $s^+(r_n) \to s^+(r_0)$.

Moreover, $\bar{r}_2 \geq \frac{w_1}{u_\varepsilon}$ implies $w_1 - \bar{r}_2 u_\varepsilon \leq 0$. As a consequence $r \to \bar{r}_2^-$ implies $(w_1 - ru_\varepsilon)^+ \to 0$ pointwise. Moreover, since $|(w_1 - ru_\varepsilon)^+|_{L^\infty(\Omega)} \leq |w_1|_{L^\infty(\Omega)}$, using dominated convergence theorem we have $|(w_1 - ru_\varepsilon)^+|_{L^{p_s^*}(\Omega)} \to 0$. From the analysis in Lemma 5.4.1, for any r, we also have $s^+(r) > t_0(v_r^+)$, where function t_0 is defined as in lemma 5.4.1, which is the maximum point of $\phi(\cdot, r)$. Therefore it is enough to show that $\lim_{r\to\bar{r}_2^-} t_0(v_r^+) = \infty$. Applying Sobolev inequality in the definition of $t_0(v_r^+)$ we get

$$t_0(v_r^+) = \left(\frac{(p-1-q)\|v_r^+\|^p_{X_{0,s,p}(\Omega)}}{(p_s^*-1-q)|v_r^+|^{p_s^*}_{L^{p_s^*}(\Omega)}}\right)^{\frac{1}{p_s^*-p}} \geq \left(\frac{S_{s,p}(p-1-q)}{p_s^*-1-q}\right)^{\frac{1}{p_s^*-p}} |v_r^+|^{-1}_{L^{p_s^*}(\Omega)}.$$

Hence $\lim_{r\to\bar{r}_2^-} t_0(v_r^+) = \infty$.

Proceeding similarly we can show that if $r \to \bar{r}_1^-$ then $v_r^+ \to v_{\bar{r}_1}$ and $\lim_{r\to\bar{r}_1^+} s^+(r) = t^+(v_{\bar{r}_1}^+)$ and

$$\lim_{r\to r_1^+} s^-(r) = +\infty, \ \lim_{r\to r_2^-} s^-(r) = t^+(v_r^-) < +\infty.$$

The continuity of $s^\pm$ implies that there exists $b \in (\bar{r}_1, \bar{r}_2)$ such that $s^+(r) = s^-(r) = a > 0$. Therefore,

$$a(w_1 - bu_\varepsilon)^+ \in N_\mu^- \quad \text{and} \quad - a(w_1 - bu_\varepsilon)^- \in N_\mu^-,$$

that is, the function $a(w_1 - bu_\varepsilon) \in \mathcal{N}_*^-$ and this completes the proof.

□

Conclusion: This chapter is a protraction of the previous chapter in the p-fractional case. We have used the same techniques as used in the previous chapter but in a meticulous way because of the lack of an explicit formula for Sobolev minimizer.

Chapter 6

Multiplicity results for $(p,\, q)$ fractional Laplace type equations with critical nonlinearities

In this chapter we discuss the existence of multiple nontrivial solutions of $(p,\, q)$ fractional Laplace equations involving concave-critical type nonlinearities and existence of nonnegative solutions when nonlinearities is of convex-critical type. More precisely, first we consider equations of the type

$$(P_{\theta,\lambda}) \begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \theta V(x)|u|^{r-2}u + |u|^{p^*_{s_1}-2}u + \lambda f(x,u), \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain, $\lambda,\, \theta > 0$, $0 < s_2 < s_1 < 1$, $1 < r < q < p < \frac{N}{s_1}$ and $p_s^* = \frac{Np}{N-sp}$ for any $s \in (0,1)$.

For the sake of simplicity, we use the following two notations in this chapter:

(a) $\|\cdot\|_{0,s,p}$ denotes the norm in the space $X_{0,s,p}(\Omega)$.

(b) $|\cdot|_p$ denotes the norm in the space $L^p(\Omega)$.

We assume the functions $V(\cdot)$, $f(\cdot,\cdot)$ satisfy the following:

(A1) $V \in L^\infty(\Omega)$ and there exists $\sigma > 0$, $\eta > 0$ such that $V(x) > \sigma > 0$ for all $x \in \Omega$ and

$$\int_\Omega V(x)|u|^r \, dx \le \eta \|u\|^r_{0,s_2,r}$$

for all $u \in X_{0,s_2,r}(\Omega)$.

(A2) $|f(x,t)| \le a_1|t|^{\alpha-1} + a_2|t|^{\beta-1}$ for all $x \in \Omega$, $t \in \mathbb{R}$, a_1, $a_2 > 0$, $1 < \alpha$, $\beta < p^*_{s_1}$.

(A3) There exists $a_3 > 0$ and $l \in (1,p)$ such that

$$f(x,t)t - p^*_{s_1} F(x,t) \ge -a_3|t|^l$$

for all $x \in \Omega, t \in \mathbb{R}$ where $F(x,t) = \int_0^t f(x,\tau)d\tau$.

(A4) $f(x,t) > 0$ for all $x \in \Omega, t \in \mathbb{R}^+$ and $f(x,t) = -f(x,-t)$ for all $x \in \Omega, t \in \mathbb{R}$.

Definition 6.0.1. *We say that $u \in X_{0,s_1,p}(\Omega)$ is a weak solution of $(P_{\theta,\lambda})$ if for all $\phi \in X_{0,s_1,p}(\Omega)$, we have*

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps_1}} dxdy \\
&+ \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+qs_2}} dxdy \\
&= \theta \int_\Omega V(x)|u(x)|^{r-2}u(x)\phi(x)dx + \int_\Omega |u(x)|^{p^*_{s_1}-2}u(x)\phi(x)dx + \lambda \int_\Omega f(x,u)\phi dx.
\end{aligned}$$

Here we note that, thanks to the Lemma 6.2.4, the above definition makes sense.

6.1 Main Results

Our first main result is the following:

Theorem 6.1.1. *Let* $0 < s_2 < s_1 < 1$, $1 < r < q < p < \frac{N}{s_1}$ *and assumptions (A1)-(A4) being satisfied. Then there exists* $\lambda^* > 0$ *such that for any* $\lambda \in (0, \lambda^*)$, *there exists* $\theta^* > 0$ *such that for any* $\theta \in (0, \theta^*)$, *problem* $(P_{\theta,\lambda})$ *has infinitely many nontrivial weak solutions in* $X_{0,s_1,p}(\Omega)$.

Next, for $V(x) \equiv 1$ and $\lambda = 0$, we have studied the nonnegative solutions of $(P_{\theta,\lambda})$ and obtained the following results:

Theorem 6.1.2. *Let* $0 < s_2 < s_1 < 1$ *and* $2 \le q < p < r < p^*_{s_1}$. *Then there exists* $\theta^* > 0$ *such that for any* $\theta > \theta^*$, *the problem*

$$(P)\begin{cases} (-\Delta)^{s_1}_p u + (-\Delta)^{s_2}_q u = \theta |u|^{r-2}u + |u|^{p^*_{s_1}-2}u \quad in \quad \Omega, \\ u > 0 \quad in \quad \Omega, \\ u = 0 \quad in \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{6.1.1}$$

has a nontrivial nonnegative weak solution.

To state our next theorem, we need the following definition.

Definition 6.1.3. *Let* M *be a topological space and consider a closed subset* $A \subset M$. *We say that* A *has category* k *relative to* M *(*$cat_M(A) = k$*), if* A *is covered by* k *closed sets* A_j, $1 \le j \le k$, *which are contractible in* M, *and if* k *is minimal with this property. If no such finite covering exists, we define* $cat_M(A) = \infty$. *Moreover, we define* $cat_M(\emptyset) = 0$.

Using Lusternik–Schnirelmann category theory , we prove our next result.

Theorem 6.1.4. *Let* $0 < s_2 < s_1 < 1$ *and*

$$N > p^2 s_1, \quad 2 \le q < \frac{N(p-1)}{N - s_1} < p \le \max\{p,\, p^*_{s_1} - \frac{q}{q-1}\} < r < p^*_{s_1}.$$

Then there exists $\theta_{**} > 0$ such that for any $\theta \in (0, \theta_{**})$, problem (P) has at least $cat_\Omega(\Omega)$ nontrivial nonnegative solutions in $X_{0,s_1,p}(\Omega)$.

The chapter is concluded with an appendix where we recall the statement of classical deformation lemma, general mountain pass lemma and some standard properties of genus.

6.2 Besov Spaces

6.2.1 Besov-Sobolev embeddings

In this subsection first we define Besov space of $\mathbb{R}^N$ and Ω. For $1 \leq i \leq N$ and $h \in \mathbb{R}$, let $\Delta_i^h u$ denote the difference quotient defined by $\Delta_i^h u(x) = u(x + he_i) - u(x), \quad x \in \mathbb{R}^N$.

Definition 6.2.1. *[54, pg. 415] Let $1 \leq p, q \leq \infty$ and $0 < s < 1$. A function $u \in L^1_{loc}(\mathbb{R}^N)$ belong to the Besov space $B^s_{p,q}(\mathbb{R}^N)$ if*

$$\|u\|_{B^s_{p,q}(\mathbb{R}^N)} = |u|_{L^p(\mathbb{R}^N)} + [u]_{B^s_{p,q}(\mathbb{R}^N)} < \infty,$$

where

$$[u]_{B^s_{p,q}(\mathbb{R}^N)} = \begin{cases} \sum_{i=1}^{N} \left(\int_0^\infty \|\Delta_i^h u\|^q_{L^p(\mathbb{R}^N)} \frac{dh}{h^{1+sq}} \right)^{\frac{1}{q}}, & q < \infty, \\ \sum_{i=1}^{N} \sup_{h>0} \frac{1}{h^s} \|\Delta_i^h u\|_{L^p(\mathbb{R}^N)}, & q = \infty. \end{cases} \tag{6.2.1}$$

Definition 6.2.2. *Let $\mathcal{D}'(\Omega)$ denote the set of all distributions over Ω. For $1 \leq p, q \leq \infty$, and $0 < s < 1$, we set*

$$B^s_{p,q}(\Omega) = \{u \in \mathcal{D}'(\Omega) : \exists\, g \in B^s_{p,q}(\mathbb{R}^N) \quad \text{with} \quad g|_\Omega = u\}$$

and

$$\|u\|_{B^s_{p,q}(\Omega)} = \inf_{g \in B^s_{p,q}(\mathbb{R}^N),\, g|_\Omega = u} \|g\|_{B^s_{p,q}(\mathbb{R}^N)}.$$

$B^s_{p,q}(\Omega)$ is called the Besov Space over Ω.

For more details about Besov space, we refer [54] and [83].

Lemma 6.2.3. *Let* $1 \le q \le p \le \infty$ *and* $0 < s_2 < s_1 < 1$. *Then*

$$W^{s_1,p}(\Omega) \subset W^{s_2,q}(\Omega).$$

Proof. Since $q \le p$ and $s_2 < s_1$ implies $s_2 - \frac{N}{q} < s_1 - \frac{N}{p}$, from [83, Theorem (i), pg. 196], we have

$$B^{s_1}_{p,p}(\Omega) \subset B^{s_2}_{q,q}(\Omega).$$

Further, from [83, pg. 209]), it follows that $|u|_{L^p(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}}$ is an equivalent norm for $\|u\|_{B^s_{p,p}(\Omega)}$. Therefore, $B^s_{p,p}(\Omega) = W^{s,p}(\Omega)$ for $1 \le p \le \infty$ and $0 < s < 1$. Hence the lemma follows. □

Note that, the assertion of the above Lemma fails when $s_1 = s_2$, see [61] for the counterexample.

Lemma 6.2.4. *Let* $0 < s_2 < s_1 < 1$, $1 < q \le p$ *and* Ω *be a smooth bounded domain in* $\mathbb{R}^N$, *where* $N > s_1 p$. *Then* $X_{0,s_1,p}(\Omega) \subset X_{0,s_2,q}(\Omega)$ *and there exists* $C = C(|\Omega|, N, p, q, s_1, s_2) > 0$ *such that*

$$\|u\|_{0,s_2,q} \le C\|u\|_{0,s_1,p} \quad \forall\, u \in X_{0,s_1,p}(\Omega).$$

Proof. Let $u \in X_{0,s_1,p}(\Omega)$. Then $u \in W^{s_1,p}(\mathbb{R}^N)$ with $u \equiv 0$ a.e. in $\mathbb{R}^N \setminus \Omega$. Note that, thanks to Hölder inequality and Sobolev inequality, we have

$$\begin{aligned}
\|u\|^p_{W^{s_1,p}(\Omega)} &= |u|^p_p + \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1p}} dxdy \\
&\le |u|^p_{p^*_{s_1}} |\Omega|^{1-\frac{p}{p^*_{s_1}}} + \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1p}} dxdy \\
&\le (C|\Omega|^{1-\frac{p}{p^*_{s_1}}} + 1)\|u\|^p_{0,s_1,p}.
\end{aligned}$$

This proves that $X_{0,s_1,p}(\Omega) \subset W^{s_1,p}(\Omega)$. Consequently, by Lemma 6.2.3 we also have $W^{s_1,p}(\Omega) \subset W^{s_2,q}(\Omega)$. As a result, $u \in W^{s_2,q}(\Omega)$ with $u \equiv 0$ a.e. in

$\mathbb{R}^N \setminus \Omega$. Further, as $\partial\Omega$ is smooth, the embedding $W^{s_2,q}(\Omega) \hookrightarrow W^{s_2,q}(\mathbb{R}^N)$ is continuous, that is,

$$\|u\|_{W^{s_2,q}(\mathbb{R}^N)} \leq C(|\Omega|, q, N)\|u\|_{W^{s_2,q}(\Omega)} \quad \text{for all} \quad u \in W^{s_2,q}(\Omega). \tag{6.2.2}$$

Therefore,

$$\|u\|_{W^{s_2,q}(\mathbb{R}^N)} \leq C(|\Omega|, N, p, q, s_1, s_2)\|u\|_{0,s_1,p} \quad \text{for all} \quad u \in X_{0,s_1,p}(\Omega). \tag{6.2.3}$$

Since, $\|u\|_{0,s_2,q}^q \leq \|u\|_{W^{s_2,q}(\mathbb{R}^N)}^q$, it follows

$$\|u\|_{X_{0,s_2,q}(\Omega)} \leq C(|\Omega|, N, s_1, s_2, p, q)\|u\|_{X_{0,s_1,p}(\Omega)} \quad \text{for all} \quad u \in X_{0,s_1,p}(\Omega). \tag{6.2.4}$$

Hence the lemma follows. □

6.3 Concentration-compactness Lemma

For $s \in (0, 1)$, define

$$\dot{W}^{s,p}(\mathbb{R}^N) := \left\{ u \in L^{p_s^*}(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy < \infty \right\}$$

and

$$S_{s,p} = \inf_{u \in \dot{W}^{s,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{p_s^*} \right)^{\frac{p}{p_s^*}}}. \tag{6.3.1}$$

Next, we fix some notations: $D^s u(x) := \displaystyle\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dy$. Thus, $|D^s u|_p^p = \|u\|_{0,s,p}^p$, $C_c(\mathbb{R}^N)$ denotes the set of all continuous functions with compact support. $\|\mu\| := \int_{\mathbb{R}^N} d\mu$. $\mathcal{M}(\mathbb{R}^N)$ denotes the space of finite measures on $\mathbb{R}^N$. We say a sequence (μ_n) converges weakly to μ in $\mathcal{M}(\mathbb{R}^N)$, if

$$\langle \mu_n, \phi \rangle := \int_{\mathbb{R}^N} \phi \, d\mu_n \to \langle \mu, \phi \rangle \quad \forall \quad \phi \in C_c(\mathbb{R}^N)$$

and it is denoted by $\mu_n \rightharpoonup \mu$.

Theorem 6.3.1. *Let $s \in (0,1)$ and $p > 1$. Assume $\{u_n\} \subset X_{0,s,p}(\Omega)$ is a nonnegative sequence such that $|u_n|_{p_s^*} = 1$ and $\|u_n\|_{0,s,p}^p \to S_{s,p}$ as $n \to \infty$. Then, there exists a sequence $\{y_n, \lambda_n\} \in \mathbb{R}^N \times \mathbb{R}^+$ such that*

$$v_n(x) := \lambda_n^{\frac{(N-sp)}{p}} u_n(\lambda_n x + y_n) \tag{6.3.2}$$

has a convergent subsequence (still denoted by v_n) such that $v_n \to v$ in $\dot{W}^{s,p}(\mathbb{R}^N)$ where $v(x) > 0$ in $\mathbb{R}^N$. In particular, there exists a minimizer for $S_{s,p}$. Moreover, we have, $\lambda_n \to 0$ and $y_n \to y \in \overline{\Omega}$ as $n \to \infty$.

Proof. For $p = 2$, this lemma has been proved by Palatucci-Pisante in [68, Theorem 1.3]. For general $p > 1$, using the next Lemma 6.3.2 (see the next lemma), the proof can be completed following the similar steps as in [86, Lemma 1.41](also see [57, Section I.4, Example (iii)]). We omit the details. □

Lemma 6.3.2. *Let $s \in (0,1)$ and $p > 1$. Assume $\{u_n\}$ be a sequence in $\dot{W}^{s,p}(\mathbb{R}^N)$ such that*

$$\begin{cases} u_n \rightharpoonup u \quad \text{in} \quad \dot{W}^{s,p}(\mathbb{R}^N), \\ |D^s(u_n - u)|^p \rightharpoonup \mu \quad \text{in} \quad \mathcal{M}(\mathbb{R}^N), \\ |u_n - u|^{p_s^*} \rightharpoonup \nu \quad \text{in} \quad \mathcal{M}(\mathbb{R}^N), \\ u_n \to u \quad \text{a.e. on} \quad \mathbb{R}^N, \end{cases} \tag{6.3.3}$$

and define

$$\begin{cases} \mu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |D^s u_n|^p dx, \\ \nu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |u_n|^{p_s^*} dx. \end{cases} \tag{6.3.4}$$

Then, we have

$$S_{s,p}\|\nu\|^{\frac{p}{p_s^*}} \leq \|\mu\|, \tag{6.3.5}$$

$$S_{s,p}\nu_\infty^{\frac{p}{p_s^*}} \leq \mu_\infty, \tag{6.3.6}$$

$$\limsup_{n\to\infty} |D^s u_n|_p^p = |D^s u|_p^p + \|\mu\| + \mu_\infty, \tag{6.3.7}$$

$$\limsup_{n\to\infty} |u_n|_{p_s^*}^{p_s^*} = |u|_{p_s^*}^{p_s^*} + \|\nu\| + \nu_\infty. \tag{6.3.8}$$

Moreover, if $u = 0$ and $S_{s,p}\|\nu\|^{p/p_s^*} = \|\mu\|$, then μ, ν are concentrated at a single point.

Remark 6.3.3. *(i) In the local case, Lemma 6.3.2 has been proved in [57, Lemma I.1] (see also [86, Lemma 1.40] for $s = 1, p = 2$). For the concentration-compactness result in the bounded domain, i.e., when $u_n \rightharpoonup u$ in $W_0^{s,p}(\Omega)$, we cite [65, Theorem 2.5]. Combining the ideas of [57], [65] and [86], one expects the above lemma to hold for general $s \in (0,1)$ and $p \geq 1$ (see [57, Section I.4]), but as best of our knowledge this lemma has not been proved exclusively anywhere. For $s \in (0,1)$, $p = 2$, concentration-compactness result in $\mathbb{R}^N$ has been proved in [39] using the harmonic extension method of Caffarelli-Silvestre, which clearly does not work for $p \neq 2$ case. Therefore we give here the proof for reader's convenience. Our proof is much different from [65].*

(ii) It's easy to see that for $\phi \in C_0^\infty(\mathbb{R}^N)$, $D^s\phi$ does not have compact support. Thus, when $u_n \rightharpoonup 0$ in $\dot{W}^{s,p}(\mathbb{R}^N)$, one can not just apply Rellich compactness result to

$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^p |D^s\phi|^p$ *in order to pass the limit. This makes the situation much different from the local case [57] or the nonlocal case when $u_n \rightharpoonup u$ in $W_0^{s,p}(\Omega)$, which was treated in [65].*

Proof. Let us first consider the case $u \equiv 0$.

Step 1: In this step we prove $S_{s,p}(\|\nu\|)^{p/p_s^*} \leq \|\mu\|$.

Choosing $\phi \in C_0^\infty(\mathbb{R}^N)$ and applying Sobolev inequality, we have

$$\begin{aligned} S_{s,p}|u_n\phi|_{p_s^*}^{p_s^*} \leq \|u_n\phi\|_{0,s,p}^p &= |D^s(u_n\phi)|_p^p \\ &\leq (1+\theta)\int_{\mathbb{R}^N}|D^su_n|^p|\phi|^p dx + c_\theta\int_{\mathbb{R}^N}|D^s\phi|^p|u_n|^p dx, \end{aligned} \tag{6.3.9}$$

where, in the last line we have used [65, (2.1)]. Let, supp$(\phi) \in B(0,r)$ for some $r > 0$. Then for a.e. $|x| > r$,

$$|D^s\phi|^p(x) = \int_{B(0,r)} \frac{|\phi(y)|^p}{|x-y|^{N+sp}}dy \leq \int_{B(0,r)} \frac{|\phi(y)|^p}{(|x|-r)^{(N+sp)}}dy \leq \frac{|\phi|_p^p}{(|x|-r)^{N+sp}}, \tag{6.3.10}$$

Fix, $R_\theta > r$ large enough (will be chosen later) . Then,

$$\begin{aligned} c_\theta\int_{\mathbb{R}^N}|D^s\phi|^p|u_n|^p dx &= c_\theta\int_{B(0,R_\theta)}|D^s\phi|^p|u_n|^p dx + c_\theta\int_{\mathbb{R}^N\setminus B(0,R_\theta)}|D^s\phi|^p|u_n|^p dx \\ &=: J_1(n) + J_2(n), \end{aligned} \tag{6.3.11}$$

We observe that as $u_n \rightharpoonup u$ in $\dot{W}^{s,p}(\mathbb{R}^N)$ and $u \equiv 0$, it holds $u_n \to 0$ in $L^p_{loc}(\mathbb{R}^N)$. Also, $\phi \in C_0^\infty(\mathbb{R}^N)$ implies, $|D^s\phi|^p \in L^\infty(\mathbb{R}^N)$. Therefore,

$$\lim_{n\to\infty} J_1(n) = 0. \tag{6.3.12}$$

Clearly,

$$|u_n|_{p_s^*}^{p_s^*} \leq c_1 \quad \text{for all} \quad n \geq 1 \tag{6.3.13}$$

for some $c_1 > 0$. Consequently, applying Hölder inequality followed by (6.3.10) yields

$$\begin{aligned} J_2(n) &\leq c_\theta c_1^{p/p_s^*}\left(\int_{\mathbb{R}^N\setminus B(0,R_\theta)}|D^s\phi|^{\frac{N}{s}}dx\right)^{\frac{sp}{N}} \\ &\leq c_\theta c_1^{p/p_s^*}|\phi|_p\left(\omega_N\int_{R_\theta}^\infty \frac{t^{N-1}}{(t-r)^{(N+sp)\frac{N}{sp}}}dt\right)^{\frac{sp}{N}}, \end{aligned} \tag{6.3.14}$$

where ω_N denotes the surface measure of unit sphere in R^N. A straightforward computation yields,

$$\int_{R_\theta}^{\infty} \frac{t^{N-1}}{(t-r)^{(N+sp)\frac{N}{sp}}}dt = 2^{N-2}\Bigg[\frac{sp}{N^2}\frac{1}{(R_\theta-r)^{N^2/sp}} + \left(\frac{r^{N-1}sp}{N(N+sp)-sp}\right)\frac{1}{(R_\theta-r)^{\frac{N(N+sp)}{sp}-1}}\Bigg]^{\frac{sp}{N}}.$$

Choose R_θ such that

$$c_\theta c_1^{\frac{p}{p_s^*}}\omega_N^{\frac{sp}{N}}|\phi|_p 2^{N-2}\Bigg[\frac{sp}{N^2}\frac{1}{(R_\theta-r)^{N^2/sp}} + \left(\frac{r^{N-1}sp}{N(N+sp)-sp}\right)\frac{1}{(R_\theta-r)^{\frac{N(N+sp)}{sp}-1}}\Bigg]^{\frac{sp}{N}} < \theta. \tag{6.3.15}$$

As a consequence,

$$J_2(n) < \theta, \quad \forall\ n \geq 1. \tag{6.3.16}$$

Combining this with (6.3.12) and (6.3.11) yields

$$\lim_{n\to\infty} c_\theta \int_{\mathbb{R}^N} |D^s\phi|^p|u_n|^p dx < \theta.$$

Hence, taking the limit $n \to \infty$ in (6.3.9) we obtain

$$S_{s,p}\left(\int_{\mathbb{R}^N} |\phi|^{p_s^*}d\nu\right)^{p/p_s^*} \leq (1+\theta)\int_{\mathbb{R}^N} |\phi|^p d\mu + \theta. \tag{6.3.17}$$

Since $\theta > 0$ is arbitrary, so letting $\theta \to 0$ in (6.3.17) gives

$$S_{s,p}\left(\int_{\mathbb{R}^N} |\phi|^{p_s^*}d\nu\right)^{p/p_s^*} \leq \int_{\mathbb{R}^N} |\phi|^p d\mu \quad \forall \quad \phi \in C_0^\infty(\mathbb{R}^N). \tag{6.3.18}$$

Hence, taking supremum over $C_0^\infty(\mathbb{R}^N)$, we get

$$S_{s,p}(\|\nu\|)^{p/p_s^*} \leq \|\mu\|.$$

Step 2: In this step we prove $S_{s,p}\nu_\infty^{p/p_s^*} \leq \mu_\infty$.

For this first fix $R > 1$ and choose $\psi_R \in C^\infty(\mathbb{R}^N)$ be such that

$$\psi_R(x) = \begin{cases} 1, & |x| > R+1, \\ 0, & |x| < R, \end{cases} \qquad 0 \leq \psi_R \leq 1 \quad \text{in} \quad \mathbb{R}^N. \tag{6.3.19}$$

Thanks to Sobolev inequality, we have

$$S_{s,p}\bigg(\int_{\mathbb{R}^N}|\psi_R u_n|^{p_s^*}dx\bigg)^{p/p_s^*}\leq\int_{\mathbb{R}^N}|D^s(u_n\psi_R)|^p dx.$$

Therefore, as before we get

$$S_{s,p}\bigg(\int_{\mathbb{R}^N}|\psi_R|^{p_s^*}|u_n|^{p_s^*}dx\bigg)^{p/p_s^*}\leq(1+\theta)\int_{\mathbb{R}^N}|D^s u_n|^p|\psi_R|^p dx+c_\theta\int_{\mathbb{R}^N}|u_n|^p|D^s\psi_R|^p dx. \tag{6.3.20}$$

Doing an easy computation, it follows that $D^s\psi_R\in L^\infty(\mathbb{R}^N)$. Therefore, for any $\tilde{R}>R+1$,

$$\begin{aligned}c_\theta\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p|D^s\psi_R|^p dx &= c_\theta\limsup_{n\to\infty}\int_{B(0,\tilde{R})}|u_n|^p|D^s\psi_R|^p dx\\ &\quad+c_\theta\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B(0,\tilde{R})}|u_n|^p|D^s\psi_R|^p dx\\ &= c_\theta\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B(0,\tilde{R})}|u_n|^p|D^s\psi_R|^p dx.\end{aligned} \tag{6.3.21}$$

Moreover, for $x\in\overline{B(0,R+1)}^c$,

$$\begin{aligned}|D^s\psi_R(x)|=\int_{\mathbb{R}^N}\frac{|1-\psi_R(y)|^p}{|x-y|^{N+sp}}dy &\leq 2^{p-1}\int_{B(0,R+1)}\frac{1+\psi_R(y)^p}{|x-y|^{N+sp}}dy\\ &\leq \frac{2^{p-1}}{(|x|-(R+1))^{N+sp}}\int_{B(0,R+1)}(1+\psi_R(y)^p)dy\\ &\leq \frac{2^{p-1}\alpha_N}{(|x|-(R+1))^{N+sp}}\Big(2(R+1)^N-R^N\Big),\end{aligned}$$

where α_N is volume of unit ball in $\mathbb{R}^N$. Therefore, doing the similar analysis as in Step 1, we get an existence of $\tilde{R}>R+1$, for which

$$c_\theta\int_{\mathbb{R}^N\setminus B(0,\tilde{R})}|u_n|^p|D^s\psi_R|^p dx<\theta.$$

Hence, combining this along with (6.3.21) and (6.3.20) and then taking $\theta\to 0$ yields

$$\limsup_{n\to\infty}S_{s,p}\bigg(\int_{\mathbb{R}^N}|\psi_R|^{p_s^*}|u_n|^{p_s^*}dx\bigg)^{p/p_s^*}\leq\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\psi_R|^p|D^s u_n|^p dx. \tag{6.3.22}$$

On the other hand, we have

$$\int_{|x|>R+1} |D^s u_n|^p dx \leq \int_{\mathbb{R}^N} \psi_R^p |D^s u_n|^p dx \leq \int_{|x|\geq R} |D^s u_n|^p dx$$

and

$$\int_{|x|>R+1} |u_n|^{p_s^*} dx \leq \int_{\mathbb{R}^N} |u_n|^{p_s^*} \psi_R^{p_s^*} dx \leq \int_{|x|\geq R} |u_n|^{p_s^*} dx.$$

From (6.3.4) we obtain,

$$\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} \psi_R^p |D^s u_n|^p dx, \quad \nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} \psi_R^{p_s^*} |u_n|^{p_s^*} dx. \tag{6.3.23}$$

Substituting (6.3.23) into (6.3.22) yields

$$S_{s,p} \nu_\infty^{p/p_s^*} \leq \mu_\infty.$$

Step 3: Assume $S_{s,p} \|\nu\|^{p/p_s^*} = \|\mu\|$. Then following the exact similar analysis as in [86, Step 3, Lemma 1.40] we get μ and ν are concentrated at a single point.

Step 4: For the general case write $v_n = u_n - u$. Since $v_n \rightharpoonup 0$ in $\dot{W}^{s,p}(\mathbb{R}^N)$, it follows $|D^s v_n|^p \rightharpoonup \mu + |D^s u|^p$ in $\mathcal{M}(\mathbb{R}^N)$.

Using Brezis-Lieb lemma, for all $h \in C_c(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N} h|u|^{p_s^*} dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} h|u_n|^{p_s^*} dx - \int_{\mathbb{R}^N} h|v_n|^{p_s^*} dx.$$

This in turn implies

$$|u_n|^{p_s^*} \rightharpoonup |u|^{p_s^*} + \nu \quad \text{in} \quad \mathcal{M}(\mathbb{R}^N).$$

(6.3.5) follows from corresponding inequality of (v_n).

Step 5: Since ,

$$\limsup_{n\to\infty} \int_{|x|>R} |D^s v_n|^p dx = \limsup_{n\to\infty} \int_{|x|>R} |D^s u_n|^p dx - \int_{|x|>R} |D^s u|^p dx,$$

we obtain $\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |D^s v_n|^p dx$.

Similarly, applying Brezis-Lieb lemma to $\int_{|x|>R} |u|^{p_s^*} dx$ yields

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |v_n|^{p_s^*} dx.$$

Now, (6.3.6) follows from corresponding inequality for (v_n).

Step 6: For $R > 1$, we have

$$\begin{aligned}\limsup_{n\to\infty} \int_{\mathbb{R}^N} |D^s u_n|^p dx &= \limsup_{n\to\infty} \left(\int_{\mathbb{R}^N} \psi_R |D^s u_n|^p + \int_{\mathbb{R}^N} (1-\psi_R)|D^s u_n|^p \right) \\ &= \limsup_{n\to\infty} \Bigg(\int_{\mathbb{R}^N} \psi_R |D^s u_n|^p \\ &\quad + \int_{\mathbb{R}^N} (1-\psi_R) d\mu + \int_{\mathbb{R}^N} (1-\psi_R)|D^s u|^p dx \Bigg).\end{aligned}$$

Hence, taking the limit $R \to \infty$ yields

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |D^s u_n|^p dx = \mu_\infty + \|\mu\| + \|u\|_{0,s,p}^p.$$

Proof of (6.3.8) is similar. □

6.4 Proof of Theorem 6.1.1

6.4.1 Existence of infinitely many nontrivial solutions

The energy functional associated to $(P_{\theta,\lambda})$ is given by:

$$I(u) = \frac{1}{p}\|u\|_{0,s_1,p}^p + \frac{1}{q}\|u\|_{0,s_2,q}^q - \frac{\theta}{r}\int_\Omega V(x)|u|^r dx - \frac{1}{p_{s_1}^*}|u|_{p_{s_1}^*}^{p_{s_1}^*} - \lambda \int_\Omega F(x,u)dx. \tag{6.4.1}$$

We note that $I(u) = I(-u)$ for all $u \in X_{0,s_1,p}(\Omega)$ and $I \in C^1(X_{0,s_1,p}, \mathbb{R})$.

Lemma 6.4.1. *Assume (A1)-(A3) are satisfied. Then, there exists $c_1, c_2 > 0$ such that any $(PS)_c$ sequence $\{u_n\} \subset X_{0,s_1,p}(\Omega)$ of I has a convergent subsequence where*

$$c < \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}} - c_1\theta^{\frac{q}{q-r}} - c_2\lambda^{\frac{p_{s_1}^*}{p_{s_1}^*-l}}.$$

Proof. Let $\{u_n\} \subset X_{0,s_1,p}(\Omega)$ be a $(\mathrm{PS})_c$ sequence of I. Therefore,

$$I(u_n) = c + o(1),\ I'(u_n) = o(1). \tag{6.4.2}$$

Claim 1: $\|u_n\|_{0,s_1,p}$ is uniformly bounded.

We prove the Claim by method of contradiction. Thus assume the claim does not hold, that is, up to a subsequence $\|u_n\|_{0,s_1,p} \to \infty$ as $n \to \infty$. Let us define $\hat{u_n} := \frac{u_n}{\|u_n\|_{0,s_1,p}}$. Then $\|\hat{u_n}\|_{0,s_1,p} = 1$. Therefore, up to a subsequence, we may take

$$\hat{u_n} \rightharpoonup \hat{u} \quad \text{in} \quad X_{0,s_1,p}(\Omega), \quad \text{and} \quad \hat{u_n} \to \hat{u} \quad \text{in} \quad L^\gamma(\mathbb{R}^N),\ 1 \leq \gamma < p^*_{s_1} \tag{6.4.3}$$

for some $\hat{u} \in X_{0,s_1,p}(\Omega)$. From (6.4.2) using $\frac{1}{||u_n||_{0,s_1,p}} = o(1)$, we have

$$\begin{aligned}
&\frac{1}{p}\|\hat{u_n}\|^p_{0,s_1,p} + \frac{1}{q}\|u_n\|^{q-p}_{0,s_1,p}\|\hat{u_n}\|^q_{0,s_2,q} - \frac{\theta}{r}\|u_n\|^{r-p}_{0,s_1,p}\int_\Omega V(x)|\hat{u_n}|^r dx \\
&- \frac{1}{p^*_{s_1}}\|u_n\|^{p^*_{s_1}-p}_{0,s_1,p}|\hat{u_n}|^{p^*_{s_1}}_{p^*_{s_1}} - \lambda\|u_n\|^{-p}_{0,s_1,p}\int_\Omega F(x,u_n)dx \\
&= o(1),
\end{aligned} \tag{6.4.4}$$

and

$$\begin{aligned}
\|\hat{u_n}\|^p_{0,s_1,p} + \|u_n\|^{q-p}_{0,s_1,p}\|\hat{u_n}\|^q_{0,s_2,q} - \theta\|u_n\|^{r-p}_{0,s_1,p}\int_\Omega V(x)|\hat{u_n}|^r dx \\
-\|u_n\|^{p^*_{s_1}-p}_{0,s_1,p}|\hat{u_n}|^{p^*_{s_1}}_{p^*_{s_1}} - \lambda\|u_n\|^{-p}_{0,s_1,p}\int_\Omega f(x,u_n)u_n dx \\
= o(1).
\end{aligned} \tag{6.4.5}$$

As $V \in L^\infty(\Omega)$, using (6.4.3) we have

$$\int_\Omega V(x)|\hat{u_n}|^r dx \to \int_\Omega V(x)|\hat{u}|^r dx. \tag{6.4.6}$$

From (6.4.4) and (6.4.5), we obtain

$$\begin{aligned}
(\frac{p^*_{s_1}}{p} - 1)\|\hat{u_n}\|^p_{0,s_1,p} + (\frac{p^*_{s_1}}{q} - 1)\|u_n\|^{q-p}_{0,s_1,p}\|\hat{u_n}\|^q_{0,s_2,q} - \theta(\frac{p^*_{s_1}}{r} - 1)\|u_n\|^{r-p}_{0,s_1,p}\int_\Omega V(x)|\hat{u_n}|^r dx \\
-\lambda\|u_n\|^{-p}_{0,s_1,p}\left(p^*_{s_1}\int_\Omega [F(x,u_n) - f(x,u_n)u_n]dx\right) = o(1).
\end{aligned} \tag{6.4.7}$$

Using (A3), (6.4.3) and (6.4.6), we can write

$$\begin{aligned}
\left(\frac{p_{s_1}^*}{p}-1\right)\|\hat{u_n}\|_{0,s_1,p}^p &= \left(1-\frac{p_{s_1}^*}{p}\right)\|u_n\|_{0,s_1,p}^{q-p}\|\hat{u_n}\|_{0,s_2,q}^q \\
&+ \theta\left(\frac{p_{s_1}^*}{r}-1\right)\|u_n\|_{0,s_1,p}^{r-p}\int_\Omega V(x)|\hat{u_n}|^r dx \\
&+\lambda\|u_n\|_{0,s_1,p}^{-p}\left(\int_\Omega p_{s_1}^* F(x,u_n)-f(x,u_n)u_n dx\right)+o(1) \\
&\leq \left(1-\frac{p_{s_1}^*}{p}\right)\|u_n\|_{0,s_1,p}^{q-p}\|\hat{u_n}\|_{0,s_2,q}^q \\
&+ \theta\left(\frac{p_{s_1}^*}{r}-1\right)\|u_n\|_{0,s_1,p}^{r-p}\int_\Omega V(x)|\hat{u}|^r dx \\
&+\lambda a_3\|u_n\|_{0,s_1,p}^{l-p}|\hat{u}|_l^l+o(1) \\
&= o(1),
\end{aligned}$$

as $n \to \infty$. This is a contradiction as $\|\hat{u_n}\|_{0,s_1,p} = 1$ and hence Claim 1 follows.

Consequently, there exists $u \in X_{0,s_1,p}(\Omega)$ such that up to a subsequence

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{in} \quad X_{0,s_1,p}(\Omega),\\
&u_n \to u \quad \text{a.e. in} \quad \mathbb{R}^N,\\
&u_n \to u \quad \text{strongly in} \quad L^\gamma(\mathbb{R}^N) \quad \text{for} \quad 1 \leq \gamma < p_{s_1}^*.
\end{aligned}$$

Applying (A1) and (A2), we have

$$\int_\Omega f(x,u_n)u_n dx = \int_\Omega f(x,u)u dx + o(1),$$

$$\int_\Omega F(x,u_n) dx = \int_\Omega F(x,u) dx + o(1),$$

and

$$\int_\Omega V(x)|u_n|^r dx = \int_\Omega V(x)|u|^r dx + o(1).$$

Note that by Lemma 6.2.4, $\|u_n\|_{0,s_2,q}$ is also bounded. Since $u_n \to u$ a.e. in $\mathbb{R}^N$, we obtain

$$\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{(\frac{N}{p}+s_1)(p-1)}} \to \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{(\frac{N}{p}+s_1)(p-1)}}$$

a.e. $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$. On the other hand, $\|u_n\|_{0,s_1,p}$ is uniformly bounded implies there exists $C > 0$ such that

$$\int_{\mathbb{R}^{2N}} \left(\frac{|u_n(x) - u_n(y)|}{|x-y|^{\frac{N}{p}+s_1}} \right)^p dxdy \leq C \quad \text{for all} \quad n \geq 1,$$

that is,

$$\int_{\mathbb{R}^{2N}} \left| \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{(\frac{N}{p}+s_1)(p-1)}} \right|^{\frac{p}{p-1}} dxdy \leq C.$$

Therefore,

$$\frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{(\frac{N}{p}+s_1)(p-1)}} \rightharpoonup \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{(\frac{N}{p}+s_1)(p-1)}}$$

weakly in $L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$ with $p' = \frac{p}{p-1}$. Similarly, as $\|u_n\|_{0,s_2,q}$ is uniformly bounded,

$$\frac{|u_n(x) - u_n(y)|^{q-2}(u_n(x) - u_n(y))}{|x-y|^{(\frac{N}{q}+s_2)(q-1)}} \rightharpoonup \frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))}{|x-y|^{(\frac{N}{q}+s_2)(q-1)}}$$

weakly in $L^{q'}(\mathbb{R}^N \times \mathbb{R}^N)$ with $q' = \frac{q}{q-1}$. If $\phi \in X_{0,s_1,p}(\Omega)$, it follows $\frac{\phi(x)-\phi(y)}{|x-y|^{\frac{N}{p}+s_1}} \in L^p(\mathbb{R}^N \times \mathbb{R}^N)$ and $\frac{\phi(x)-\phi(y)}{|x-y|^{\frac{N}{q}+s_2}} \in L^q(\mathbb{R}^N \times \mathbb{R}^N)$. As a result,

$$\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\phi(x) - \phi(y))}{|x-y|^{(\frac{N}{p}+s_1)(p-1)}|x-y|^{\frac{N}{p}+s_1}} dxdy \longrightarrow \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x-y|^{N+s_1 p}} dxdy$$

and

$$\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{q-2}(u_n(x) - u_n(y))(\phi(x) - \phi(y))}{|x-y|^{(\frac{N}{q}+s_2)(q-1)}|x-y|^{\frac{N}{q}+s_2}} dxdy \longrightarrow \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x-y|^{N+s_2 q}} dxdy.$$

These together with (6.4.2) via Vitali's convergence theorem implies $I'(u) = 0$ that is u is weak solution of $(P_{\theta,\lambda})$.

Claim 2: $u_n \to u$ in $X_{0,s_1,p}(\Omega)$.

To prove this claim, define $v_n := u_n - u$. As $\|u_n\|_{0,s_1,p}$ and $\|u_n\|_{0,s_2,q}$ are uniformly bounded and $u_n \to u$ a.e. in $\mathbb{R}^N$, applying Brezis-Lieb lemma, we obtain

$$\begin{aligned}\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+s_1p}}dxdy &= \int_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+s_1p}}dxdy \\ &+ \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1p}}dxdy + o(1),\end{aligned}$$

i.e., $\|u_n\|^p_{0,s_1,p} = \|v_n\|^p_{0,s_1,p} + \|u\|^p_{0,s_1,p} + o(1)$.
Similarly, we have $\|u_n\|^q_{0,s_2,q} = \|v_n\|^q_{0,s_2,q} + \|u\|^q_{0,s_2,q} + o(1)$. Therefore, a straight forward computation yields

$$\begin{aligned}c + o(1) &= \frac{1}{p}\|v_n\|^p_{0,s_1,p} + \frac{1}{q}\|v_n\|^q_{0,s_2,q} - \frac{\theta}{r}\int_\Omega V(x)|u|^r dx - \frac{1}{p^*}|v_n|^{p^*_{s_1}}_{p^*_{s_1}} \\ &-\lambda\int_\Omega F(x,u)dx + \frac{1}{p}\|u\|^p_{0,s_1,p} + \frac{1}{q}\|u\|^q_{0,s_2,q} - \frac{1}{p^*}|u|^{p^*_{s_1}}_{p^*_{s_1}}. \quad (6.4.8)\end{aligned}$$

On the other hand, using $|I'(u_n)u_n| \leq o(1)\|u_n\|_{0,s_1,p} = o(1)$, we also have

$$\begin{aligned}\|v_n\|^p_{0,s_1,p} + \|v_n\|^q_{0,s_2,q} &= o(1) + \theta\int_\Omega V(x)|u|^r dx + |u|^{p^*_{s_1}}_{p^*_{s_1}} + |v_n|^{p^*_{s_1}}_{p^*_{s_1}} \\ &+\lambda\int_\Omega f(x,u)udx - \|u\|^p_{0,s_1,p} - ||u||^q_{0,s_2,q}.\end{aligned} \quad (6.4.9)$$

Combining (6.4.9) with $I'(u) = 0$ yields

$$\|v_n\|^p_{0,s_1,p} + \|v_n\|^q_{0,s_2,q} - |v_n|^{p^*_{s_1}}_{p^*_{s_1}} = o(1). \quad (6.4.10)$$

Since $\|v_n\|_{0,s_1,p}$, $\|v_n\|_{0,s_2,q}$, $|v_n|_{p^*_{s_1}}$ all are bounded sequence of real numbers, we may assume that:

$$\|v_n\|^p_{0,s_1,p} = a + o(1), \quad \|v_n\|^q_{0,s_2,q} = b + o(1), \quad |v_n|^{p^*_{s_1}}_{p^*_{s_1}} = d + o(1) \quad (6.4.11)$$

for some $a, b, d \geq 0$. Hence, (6.4.10) implies

$$a + b = d. \quad (6.4.12)$$

Thus $a \leq d$. Therefore, Sobolev inequality yields

$$a \geq S_{s_1,p} d^{p/p^*_{s_1}} \geq S_{s_1,p} a^{p/p^*_{s_1}} \tag{6.4.13}$$

If $a = 0$, we are done. If $a > 0$, then (6.4.13) implies

$$a \geq (S_{s_1,p})^{\frac{N}{s_1 p}}. \tag{6.4.14}$$

Using (6.4.8), (6.4.11), (6.4.12), (6.4.14) and the fact that $q < p < p^*_{s_1}$, taking the limit $n \to \infty$ we have

$$\begin{aligned} c &= \frac{a}{p} + \frac{b}{q} - \frac{(a+b)}{p^*_{s_1}} + \frac{1}{p}\|u\|^p_{0,s_1,p} + \frac{1}{q}\|u\|^q_{0,s_2,q} - \frac{1}{p^*_{s_1}}|u|^{p^*_{s_1}}_{p^*_{s_1}} \\ &\quad -\frac{\theta}{r}\int_\Omega V(x)|u|^r dx - \lambda\int_\Omega F(x,u)dx. \\ &\geq \frac{as_1}{N} + \frac{1}{p}\|u\|^p_{0,s_1,p} + \frac{1}{q}\|u\|^q_{0,s_2,q} - \frac{1}{p^*_{s_1}}|u|^{p^*_{s_1}}_{p^*_{s_1}} - \frac{\theta}{r}\int_\Omega V(x)|u|^r dx - \lambda\int_\Omega F(x,u)dx. \\ &\geq \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}} + \frac{1}{p}\|u\|^p_{0,s_1,p} + \frac{1}{q}\|u\|^q_{0,s_2,q} - \frac{1}{p^*_{s_1}}|u|^{p^*_{s_1}}_{p^*_{s_1}} \\ &- \frac{\theta}{r}\int_\Omega V(x)|u|^r dx - \lambda\int_\Omega F(x,u)dx. \end{aligned} \tag{6.4.15}$$

Also from $< I'(u), u >= 0$, it follows

$$\|u\|^p_{0,s_1,p} = -\|u\|^q_{0,s_2,q} + |u|^{p^*_{s_1}}_{p^*_{s_1}} + \theta\int_\Omega V(x)|u|^r\,dx + \lambda\int_\Omega f(x,u)u\,dx. \tag{6.4.16}$$

Substituting (6.4.16) into (6.4.15) and using (A1) yields

$$\begin{aligned} c &\geq \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}} + \frac{s_1}{N}|u|^{p^*_{s_1}}_{p^*_{s_1}} - \theta\left(\frac{1}{r} - \frac{1}{p}\right)\int_\Omega V(x)|u|^r dx \\ &\quad -\lambda\int_\Omega (F(x,u) - \frac{1}{p}f(x,u)u)dx + \left(\frac{1}{q} - \frac{1}{p}\right)\|u\|^q_{0,s_2,q} \\ &\geq \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}} + \frac{s_1}{N}|u|^{p^*_{s_1}}_{p^*_{s_1}} - \theta\eta\left(\frac{1}{r} - \frac{1}{p}\right)\|u\|^r_{0,s_2,r} \\ &\quad -\lambda\int_\Omega (F(x,u) - \frac{1}{p}f(x,u)u)dx + \left(\frac{1}{q} - \frac{1}{p}\right)\|u\|^q_{0,s_2,q}. \end{aligned} \tag{6.4.17}$$

Note that from (A4) it is easy to see $f(x,t)t \geq 0$ for all $t \in \mathbb{R}$, $x \in \Omega$ and from (A3), it follows that $F(x,t) \leq \frac{1}{p^*_{s_1}} f(x,t)t + \frac{a_3}{p^*_{s_1}}|t|^l$. Thus,

$$\int_\Omega \lambda\Big(F(x,u) - \frac{1}{p}f(x,u)u\Big)dx \leq \frac{\lambda a_3}{p}|u|_l^l \leq \frac{\lambda a_3}{p}|\Omega|^{1-\frac{l}{p^*_{s_1}}}|u|^l_{p^*_{s_1}} = \lambda c_0 |u|^l_{p^*_{s_1}}, \tag{6.4.18}$$

where $c_0 = \frac{a_3}{p}|\Omega|^{1-\frac{l}{p^*_{s_1}}}$. Applying Lemma 6.2.4 and Young's inequality, for any $\delta > 0$ we obtain

$$\eta\left(\frac{1}{r} - \frac{1}{p}\right)\|u\|^r_{0,s_2,r} \leq \eta\left(\frac{1}{r} - \frac{1}{p}\right)C^r\|u\|^r_{0,s_2,q} \leq \delta\|u\|^q_{0,s_2,q} + C_\delta. \tag{6.4.19}$$

Substituting (6.4.18) and (6.4.19) into (6.4.17) we have

$$c \geq \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}} + \frac{s_1}{N}|u|^{p^*_{s_1}}_{p^*_{s_1}} - \theta\delta\|u\|^q_{0,s_2,q} - \theta C_\delta - \lambda c_0|u|^l_{p^*_{s_1}} + \left(\frac{1}{q} - \frac{1}{p}\right)\|u\|^q_{0,s_2,q}. \tag{6.4.20}$$

Now choose $\delta = \frac{1}{\theta}\Big(\frac{1}{q} - \frac{1}{p}\Big)$. This implies $C_\delta = c_1\theta^{\frac{r}{q-r}}$, for some $c_1 = c_1(p,q,r,N,s_1,s_2,|\Omega|) > 0$. Substituting this in (6.4.20) yields

$$c \;\; \geq \;\; \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}} + \frac{s_1}{N}|u|^{p^*_{s_1}}_{p^*_{s_1}} - c_1\theta^{\frac{q}{q-r}} - \lambda c_0|u|^l_{p^*_{s_1}}.$$

Note that the constants c_1 and c_0 are independent of θ, λ. Let us consider the function $g : (0,\infty) \to \mathbb{R}$ by

$$g(x) = \frac{s_1}{N}x^{p^*_{s_1}} - \lambda c_0 x^l.$$

We note that g attains its minimum at $x_0 = (\frac{c_0 l N\lambda}{p^*_{s_1}})^{\frac{1}{p^*_{s_1}-l}}$. Therefore,

$$g(x) \geq g(x_0) = -c_2\lambda^{\frac{p^*_{s_1}}{p^*_{s_1}-l}},$$

where $c_2 = c_0\frac{p^*_{s_1}-l}{p^*_{s_1}}(\frac{c_0 l N}{p^*_{s_1}})^{\frac{l}{p^*_{s_1}-l}} > 0$. Consequently,

$$c \geq \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}} - c_1\theta^{\frac{q}{q-r}} - c_2\lambda^{\frac{p^*_{s_1}}{p^*_{s_1}-l}},$$

which is a contradiction to the assumption on c. Hence, $a = 0$ and this completes the proof of the lemma.

□

Using (A1) and Lemma 6.2.4, for $1 < r < p$

$$\int_\Omega V(x)|u|^r dx \leq \eta \|u\|^r_{0,s_2,r} \leq C\eta\|u\|^r_{0,s_1,p}. \tag{6.4.21}$$

Moreover, by Sobolev embedding we have $S_{s_1,p}|u|^p_{p^*_{s_1}} \leq ||u||^p_{0,s_1,p}$ and using (A2), we obtain

$$\begin{aligned}\int_\Omega F(x,u)dx &\leq \frac{a_1}{\alpha}|u|^\alpha_\alpha + \frac{a_2}{\beta}|u|^\beta_\beta \\ &\leq \frac{a_1}{\alpha}|\Omega|^{1-\frac{\alpha}{p^*_{s_1}}}|u|^\alpha_{p^*_{s_1}} + \frac{a_2}{\beta}|\Omega|^{1-\frac{\beta}{p^*_{s_1}}}|u|^\beta_{p^*_{s_1}} \\ &\leq \frac{a_1}{\alpha}|\Omega|^{1-\frac{\alpha}{p^*_{s_1}}}(S_{s_1,p})^{-\alpha/p}\|u\|^\alpha_{0,s_1,p} + \frac{a_2}{\beta}|\Omega|^{1-\frac{\beta}{p^*_{s_1}}}(S_{s_1,p})^{-\beta/p}\|u\|^\beta_{0,s_1,p}.\end{aligned}$$

This together with (6.4.21) and Sobolev embedding gives:

$$\begin{aligned}I(u) &\geq \frac{1}{p}\|u\|^p_{0,s_1,p} - \frac{(S_{s_1,p})^{-p^*_{s_1}/p}}{p^*_{s_1}}\|u\|^{p^*_{s_1}}_{0,s_1,p} - \frac{\eta C\theta}{r}\|u\|^r_{0,s_1,p} \\ &\quad -\lambda\frac{a_1}{\alpha}|\Omega|^{\frac{p^*_{s_1}-\alpha}{p^*_{s_1}}}(S_{s_1,p})^{-\alpha/p}\|u\|^\alpha_{0,s_1,p} - \lambda\frac{a_2}{\beta}|\Omega|^{\frac{p^*_{s_1}-\beta}{p^*_{s_1}}}(S_{s_1,p})^{-\beta/p}\|u\|^\beta_{0,s_1,p} \\ &= c_3\|u\|^p_{0,s_1,p} - c_4\|u\|^{p^*_{s_1}}_{0,s_1,p} - c_5\theta\|u\|^r_{0,s_1,p} - c_6\lambda\|u\|^\alpha_{0,s_1,p} - c_7\lambda\|u\|^\beta_{0,s_1,p} \\ &, \end{aligned} \tag{6.4.22}$$

where $c_3 = \frac{1}{p}$, $c_4 = \frac{(S_{s_1,p})^{-p^*_{s_1}/p}}{p^*_{s_1}}$, $c_5 = \frac{\eta}{r}C$, $c_6 = \frac{a_1}{\alpha}|\Omega|^{\frac{p^*_{s_1}-\alpha}{p^*_{s_1}}}(S_{s_1,p})^{-\alpha/p}$, $c_7 = \frac{a_2}{\beta}|\Omega|^{\frac{p^*_{s_1}-\beta}{p^*_{s_1}}}(S_{s_1,p})^{-\beta/p}$ are all positive constants. Let us define a function $h : (0,\infty) \to \mathbb{R}$ by

$$h(x) = c_3x^p - c_4x^{p^*_{s_1}} - c_5\theta x^r - c_6\lambda x^\alpha - c_7\lambda x^\beta. \tag{6.4.23}$$

As $1 < r < p$ and $1 < \alpha$, $\beta < p^*_{s_1}$, we see that there exists $\lambda_0 \geq \lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, there exists $x > 0$ such that $h(x) > 0$. Therefore, we conclude that for any $\lambda \in (0,\lambda^*)$, there exists

$$\theta^* = \theta^*(\lambda) > 0 \tag{6.4.24}$$

such that for any $\theta \in (0, \theta^*)$,

(a) $h(x)$ attains its maximum and $\max_{x\in(0,\infty)} h(x) > 0$,

(b) $\frac{s_1}{N} S^{\frac{N}{s_1 p}} - c_1\theta^{\frac{q}{q-r}} - c_2\lambda^{\frac{p^*_{s_1}}{p^*_{s_1}-l}} > 0$,

where c_1, c_2 are given in Lemma 6.4.1. From the definition of h, it is not difficult to see that h has finitely many positive roots, say $0 < r_1 < r_2 < \cdots < r_m < \infty$, where $h(r_i) = 0$.

As a result, we note that,

$$h(x) \begin{cases} < 0 & \forall\, x \in (0, r_1) \cup (r_2, r_3) \cup \cdots \cup (r_m, \infty), \\ > 0 & \forall\, x \in (r_1, r_2) \cup (r_3, r_4) \cup \cdots \cup (r_{m-1}, r_m). \end{cases} \tag{6.4.25}$$

Denote,

$$A := (0, r_1) \cup (r_2, r_3) \cup \cdots \cup (r_m, \infty), \quad B := A \setminus (r_m, \infty).$$

We choose $\tau \in C^\infty(\mathbb{R}^+; [0,1])$ such that

$$\tau(x) = \begin{cases} 1, & x \in B, \\ 0, & x \in (r_m, \infty). \end{cases} \tag{6.4.26}$$

Set $\phi(u) := \tau(\|u\|_{0,s_1,p})$ and the truncated functional

$$\begin{aligned} I_\infty(u) = \frac{1}{p}\|u\|^p_{0,s_1,p} + \frac{1}{q}\|u\|^q_{0,s_2,q} - \frac{\theta}{r}\int_\Omega V(x)|u|^r dx \\ - \frac{1}{p^*_{s_1}}\int_\Omega |u|^{p^*_{s_1}}\phi(u)dx - \lambda\int_\Omega F(x,u)\phi(u)dx. \end{aligned} \tag{6.4.27}$$

Similarly, as (6.4.23) we can consider the function $\bar{h} : (0,\infty) \to \mathbb{R}$ as

$$\bar{h}(x) = c_3 x^p - c_4 x^{p^*_{s_1}}\tau(x) - c_5\theta x^r - c_6\lambda x^\alpha \tau(x) - c_7\lambda x^\beta \tau(x), \quad \forall\, x > 0 \tag{6.4.28}$$

and have

$$I_\infty(u) \geq \bar{h}(\|u\|_{0,s_1,p}). \tag{6.4.29}$$

It is not difficult to check that from the definition of τ, A, B that

$$\bar{h}(x) \geq h(x)\ \forall\, x > 0, \quad \bar{h}(x) = h(x)\ \forall\, x \in B, \quad \bar{h}(x) \geq 0\ \forall\, x > r_m. \tag{6.4.30}$$

Therefore, we conclude

$$I(u) = I_\infty(u) \quad \text{for} \quad \|u\|_{0,s_1,p} \in B. \tag{6.4.31}$$

Also we note that $\tau \in C^\infty(\mathbb{R}^+, [0,1])$ implies $I_\infty(u) \in C^1(X_{0,s_1,p}, \mathbb{R})$.

Lemma 6.4.2. *(i) Let $I_\infty(u) < 0$. Then $\|u\|_{0,s_1,p} \in B$ and there exists a neighborhood $\mathcal{N}_u$ of u such that $I(v) = I_\infty(v)\ \ \forall\, v \in \mathcal{N}_u$.*

(ii) For any $\lambda \in (0, \lambda^*)$, there exists $\theta^* > 0$ such that for any $\theta \in (0, \theta^*)$, $I_\infty(u)$ satisfies (PS)$_c$ condition for $c < 0$.

Proof. We prove (i) by method of contradiction. Suppose $\|u\|_{0,s_1,p} \notin B$, that is, $\|u\|_{0,s_1,p} \in \mathbb{R}^+ \setminus B$ for u with $I_\infty(u) < \infty$. Now, two cases may happen.
Case 1: If $\|u\|_{0,s_1,p} \in \mathbb{R}^+ \setminus A$, then using (6.4.29), (6.4.30) and (6.4.25), we have

$$I_\infty(u) \geq \bar{h}(\|u\|_{0,s_1,p}) \geq h(\|u\|_{0,s_1,p}) > 0.$$

This contradicts $I_\infty(u) < 0$.

Case 2: If $\|u\|_{0,s_1,p} \in (r_m, \infty) = A \setminus B$. Then by (6.4.29) and (6.4.30), we have $I_\infty(u) \geq \bar{h}(\|u\|_{0,s_1,p}) \geq 0$, which again contradicts $I_\infty(u) < 0$. Hence, $\|u\|_{0,s_1,p} \in B$. Moreover as B is an open set, applying (6.4.31), we obtain there exists a neighborhood $\mathcal{N}_u$ of u such that $I(v) = I_\infty(v)\ \ \forall\, v \in \mathcal{N}_u$.

To prove (ii), let $\theta^* > 0$ be as in (6.4.24). Suppose $c < 0$ and $\{u_n\} \subseteq X_{0,s_1,p}(\Omega)$ is a (PS)$_c$ sequence of I_∞. Therefore, for n large we may take

$$I_\infty(u_n) < 0 \quad \text{and} \quad I'_\infty(u_n) = o(1).$$

Using (i) it follows that $\|u_n\|_{0,s_1,p} \in B$. Therefore, $I(u_n) = I_\infty(u_n)$ and $I'(u_n) = I'{}_\infty(u_n) = o(1)$. Since (b) holds for $\theta \in (0, \theta^*)$, applying Lemma 6.4.1, we obtain $I(u)$ satisfies (PS)$_c$ condition for $c < 0$. Therefore, $I_\infty(u)$ satisfies (PS)$_c$ condition for $c < 0$. $\square$

Define,

$$\Sigma := \{A \subset X_{0,s_1,p} \setminus \{0\} : A \quad \text{is closed}, \quad A = -A\}. \tag{6.4.32}$$

Definition 6.4.3. *Let $A \in \Sigma$. We denote by $\gamma(A)$ the genus of A which is the smallest positive integer n such that there exists an odd continuous map from A into $\mathbb{R}^n \setminus \{0\}$. We set $\gamma(\emptyset) = 0$ and if no such n exists for A, then we set $\gamma(A) = \infty$.*

Proof of Theorem 6.1.1

Proof. Define

$$c_k := \inf_{A\in\Sigma_k} \sup_A I_\infty(u),$$

where

$$\Sigma_k := \{A \in \Sigma : \gamma(A) \geq k\},$$

and Σ is as in (6.4.32) . Let,

$$K_c := \{u \in X_{0,s_1,p}(\Omega) : \ I_\infty(u) = c, \ I'_\infty(u) = 0\}$$

and θ^* be as in (6.4.24) and $\theta \in (0, \theta^*)$.

Claim: If $k, l \in \mathbb{N}$ such that $c_k = c_{k+1} = \cdots = c_{k+l} = c$, then $c < 0$ and $\gamma(K_c) \geq l + 1$.

Let us consider the set

$$I_\infty^{-\varepsilon} := \{u \in X_{0,s_1,p}(\Omega) : \ I_\infty(u) \leq -\varepsilon\}.$$

We will show that for any $k \in \mathbb{N}$, there exists $\varepsilon = \varepsilon(k) > 0$ such that $\gamma(I_\infty^{-\varepsilon}(u)) \geq k$. Fix $k \in \mathbb{N}$. Let X_k be a $k-$dimensional subspace of $X_{0,s_1,p}$. Take $u \in X_k$ with $\|u\|_{0,s_1,p} = 1$. Thus for $0 < \rho < r_1$, using (6.4.31) we have

$$\begin{aligned} I(\rho u) = I_\infty(\rho u) &= \frac{1}{p}\rho^p + \frac{\rho^q}{q}\|u\|^q_{0,s_2,q} - \frac{\theta\rho^r}{r}\int_\Omega V(x)|u|^r dx \\ &- \frac{\rho^{p^*}}{p^*}|u|^{p^*_{s_1}}_{p^*_{s_1}} - \lambda\int_\Omega F(x, \rho u)dx. \end{aligned} \tag{6.4.33}$$

As X_k is a finite dimensional subspace of $X_{0,s_1,p}(\Omega)$, all norms in X_k are equivalent and therefore

$$\alpha_k := \sup\{\|u\|^q_{0,s_2,q} : u \in X_k, \|u\|_{0,s_1,p} = 1\} < \infty, \tag{6.4.34}$$

$$\beta_k := \inf\{|u|^{p^*_{s_1}}_{p^*_{s_1}} : u \in X_k, ||u||_{0,s_1,p} = 1\} > 0, \tag{6.4.35}$$

$$\gamma_k := \inf\{|u|^r_r : u \in X_k, \|u\|_{0,s_1,p} = 1\} > 0. \tag{6.4.36}$$

Since using (A4), it follows that $F(x, \rho u) > 0$, applying (6.4.33)-(6.4.36), we obtain

$$I_\infty(\rho u) \le \frac{1}{p}\rho^p + \alpha_k \frac{\rho^q}{q} - \sigma\gamma_k \frac{\theta\rho^r}{r} - \beta_k \frac{\rho^{p^*_{s_1}}}{p^*_{s_1}}.$$

For any $\varepsilon > 0$, there exists $\rho \in (0, r_1)$ such that $I_\infty(\rho u) \le -\varepsilon$ for $u \in X_k$ with $\|u\|_{0,s_1,p} = 1$. Define, $S_\rho = \{u \in X_{0,s_1,p} : \|u\|_{0,s_1,p} = \rho\}$. Then $S_\rho \cap X_k \subseteq I^{-\varepsilon}_\infty$. By Lemma 6.7.3, it follows that

$$k = \gamma(S_\rho \cap X_k) \le \gamma(I^{-\varepsilon}_\infty).$$

Therefore, we conclude $I^{-\varepsilon}_\infty \in \Sigma_k$, since I_∞ is continuous and even. Consequently,

$$c = c_k \le \sup_{I^{-\varepsilon}_\infty} I_\infty(u) \le -\varepsilon < 0. \tag{6.4.37}$$

Note that by (6.4.29) and (6.4.30), we have $I_\infty(u) \ge h(\|u\|_{0,s_1,p})$, for all $u \in X_{0,s_1,p}$. Consequently, using (6.4.25) and (6.4.26) in the definition of I_∞, it follows that I_∞ is bounded from below. Thus $c = c_k > -\infty$. By Lemma 6.4.2, I_∞ satisfies $(\mathrm{PS})_c$ condition. We note that K_c is a compact set. To see this, let $\{u_n\}$ be a sequence in K_c. Then $I_\infty(u_n) = c$ and $I'_\infty(u_n) = 0$. Thus,

$$\lim_{n\to\infty} I_\infty(u_n) = c, \lim_{n\to\infty} I'_\infty(u_n) = 0.$$

Therefore, $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence in K_c. As $c < 0$, by Lemma 6.4.2, there exists a subsequence and $u \in X_{0,s_1,p}(\Omega)$ such that $u_{n_k} \to u$ in $X_{0,s_1,p}(\Omega)$ and $I_\infty(u) = c$, $I'_\infty(u) = 0$. As a result, $u \in K_c$, that is, $\{u_n\}$ has a convergent subsequence in K_c.

Now let us complete the proof of our claim. Suppose the claim is not true, that is, $\gamma(K_c) \leq l$. Then, by Lemma 6.7.3, there exists a neighbourhood of K_c, say $N_r(K_c)$ such that $\gamma(N_r(K_c)) \leq l$. Since $c < 0$, we may consider $N_r(K_c) \in I^0_\infty$. By Lemma 6.7.1, there exists an odd homeomorphism $\bar{\eta} : X_{0,s_1,p}(\Omega) \to X_{0,s_1,p}(\Omega)$ such that

$$\bar{\eta}(I^{c+\delta}_\infty \setminus N_r(K_c)) \subset I^{c-\delta}_\infty \quad \text{for some} \quad 0 < \delta < -c.$$

From the definition of $c = c_{k+l}$, we know there exists an $A \in \Sigma_{k+l}$ such that

$$\sup_{u \in A} I_\infty(u) < c + \delta,$$

that is, $A \subset I^{c+\delta}_\infty$ and

$$\bar{\eta}\Big(A \setminus N_r(K_c)\Big) \subset \bar{\eta}\Big(I^{c+\delta}_\infty \setminus N_r(K_c)\Big) \subset I^{c-\delta}_\infty.$$

This yields us:

$$\sup_{u \in \bar{\eta}(A \setminus N_r(K_c))} I_\infty(u) \leq c - \delta. \tag{6.4.38}$$

Again, by Lemma 6.7.3, we have,

$$\gamma(\bar{\eta}(\overline{A \setminus N_r(K_c)})) = \gamma(\overline{A \setminus N_r(K_c)}) \geq \gamma(A) - \gamma(N_r(K_c)) \geq k + l - l = k.$$

Therefore, we have $\bar{\eta}(\overline{A \setminus N_r(K_c)}) \in \Sigma_k$ and $\sup_{u \in \eta(\overline{A \setminus N_r(K_c)})} I_\infty(u) \geq c_k = c$. This is a contradiction to (6.4.38). Hence, we have the claim.

Now let us complete the proof of Theorem 6.1.1. Since $\Sigma_{k+1} \subseteq \Sigma_k$, we have $c_k \leq c_{k+1} \ \forall\ k$. If all c_k's are distinct then $\gamma(K_{c_k}) \geq 1$, since K_{c_k} is a compact set and by Lemma 6.7.3 (7), genus of a compact set is finite. Therefore, in that case I_∞ has infinitely many distinct critical points. If for some k, there exists l such that $c_k = c_{k+1} = \cdots = c_{k+l} = c$, then by the above claim, $\gamma(K_c) \geq l + 1$ and therefore K_c has infinitely many distinct elements, i.e, I_∞ has infinitely many distinct critical points. Hence combining (6.4.37) along with Lemma 6.4.2, we conclude that I has infinitely many distinct critical points. □

6.5 Proof of Theorem 6.1.2

6.5.1 Existence of nontrivial nonnegative solutions

First, we consider the problem

$$(\tilde{P}) \begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \theta (u^+)^{r-1} + (u^+)^{p_{s_1}^*-1} \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{6.5.1}$$

Definition 6.5.1. *We say that $u \in X_{0,s_1,p}(\Omega)$ is a weak solution of $(\tilde{P})$ if for all $\phi \in X_{0,s_1,p}$ we have,*

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps_1}} dxdy \\ &+ \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+qs_2}} dxdy \\ = \ & \theta \int_\Omega (u(x)^+)^{r-1}\phi(x)dx + \int_\Omega (u(x)^+)^{p_{s_1}^*-1}\phi(x)dx. \end{aligned}$$

The Euler-Lagrange energy functional associated to $(\tilde{P})$ is

$$I_\theta(u) = \frac{1}{p}\|u\|_{0,s_1,p}^p + \frac{1}{q}\|u\|_{0,s_2,q}^q - \frac{\theta}{r}\int_\Omega (u^+)^r dx - \frac{1}{p_{s_1}^*}\int_\Omega (u^+)^{p_{s_1}^*} dx. \tag{6.5.2}$$

It can be checked that $I_\theta \in C^2(X_{0,s_1,p}, \mathbb{R})$ and any critical points of I_θ is a weak solution of $(\tilde{P})$ and conversely.

We define,

$$c_\theta = \inf_{u \in N_\theta} I_\theta(u),$$

where

$$N_\theta := \{u \in X_{0,s_1,p}(\Omega) \setminus \{0\} : \langle I_\theta'(u), u\rangle = 0\}. \tag{6.5.3}$$

We will show that I_θ has the Mountain Pass (MP) Geometry.

Lemma 6.5.2. *Let $1 < q < p < r < p_{s_1}^*$. Then for any $\theta > 0$,*

(a) there exist constants $\rho, \beta > 0$ such that $I_\theta(u) > \beta$ for all $u \in X_{0,s_1,p}(\Omega)$ with $\|u\|_{0,s_1,p} = \rho$,

(b) there exist $u_0 \in X_{0,s_1,p}(\Omega)$ such that $I_\theta(u_0) < 0$ and $\|u_0\|_{0,s_1,p} > \rho$.

Proof. Using Sobolev inequality and Hölder inequality in the definition of I_θ, we obtain

$$\begin{aligned} I_\theta(u) &\geq \frac{1}{p}\|u\|_{0,s_1,p}^p + \frac{1}{q}\|u\|_{0,s_2,q}^q - \frac{\theta}{r}|\Omega|^{\frac{p_{s_1}^*-r}{p_{s_1}^*}}|u^+|_{p_{s_1}^*}^r - \frac{1}{p_{s_1}^*}|u^+|_{p_{s_1}^*}^{p_{s_1}^*} \\ &\geq \frac{1}{p}\|u\|_{0,s_1,p}^p + \frac{1}{q}\|u\|_{0,s_2,q}^q - \frac{\theta}{r}|\Omega|^{\frac{p_{s_1}^*-r}{p_{s_1}^*}} S_{s_1,p}^{\frac{-r}{p}}\|u\|_{0,s_1,p}^r - \frac{1}{p_{s_1}^*}S_{s_1,p}^{\frac{-p_{s_1}^*}{p}}\|u\|_{0,s_1,p}^{p_{s_1}^*}. \end{aligned}$$

As $1 < q < p < r < p_{s_1}^*$, there exist two constants $\rho, \beta > 0$ such that $I_\theta(u) > \beta$ for all $u \in X_{0,s_1,p}$ with $\|u\|_{0,s_1,p} = \rho$ and that proves (a).

To prove (b), we fix $u \in X_{0,s_1,p}(\Omega)$ with $u^+ \not\equiv 0$. Then it is easy to see that $\lim_{t\to+\infty} I_\theta(tu) = -\infty$. Thus we can choose $t_0 > 0$ such that $\|t_0 u\|_{0,s_1,p} > \rho$ and $I_\theta(t_0 u) < 0$. Hence (b) holds. □

Define,

$$C_\theta := \inf_{u\in X_{0,s_1,p}\setminus\{0\}} \sup_{t\geq 0} I_\theta(tu). \tag{6.5.4}$$

Lemma 6.5.3. *Let $1 < q < p < r < p_{s_1}^*$. Then for any $\theta > 0$, I_θ satisfies the $(PS)_c$ conditions for all $c \in \left(0, \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}}\right)$. Furthermore, there exists $\theta^* > 0$ such that*

$$C_\theta \in \left(0, \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}}\right) \quad \textit{for} \quad \theta > \theta^*.$$

Proof. Let $c \in \left(0, \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}}\right)$ and $\{u_n\}_{n\geq 1} \subset X_{0,s_1,p}(\Omega)$ be a $(PS)_c$ sequence of $I_\theta(\cdot)$. From Claim 1 in the proof of Lemma 6.4.1, it follows that $\{u_n\}$ is uniformly bounded in $X_{0,s_1,p}(\Omega)$. Therefore, there exists $u \in X_{0,s_1,p}(\Omega)$ such that up to a subsequence, $u_n \rightharpoonup u$ in $X_{0,s_1,p}(\Omega)$ and $u_n \to u$ in $L^\gamma(\Omega)$ for $1 \leq \gamma < p_{s_1}^*$ and $u_n \to u$ a.e. in $\mathbb{R}^N$. Also, following the same arguments as in the proof of Lemma 6.4.1, we see that u is a critical point of I_θ, that is $\langle I_\theta'(u), \phi\rangle = 0$. Next, to prove $u_n \to u$ strongly in $X_{0,s_1,p}(\Omega)$, we follow the arguments along the same line as in the proof of claim 2 of Lemma 6.4.1 and obtain either $\|u_n - u\|_{0,s_1,p} = o(1)$ or (6.4.17) holds with $\lambda = 0$. Thus in the second case,

$$\begin{aligned} c &\geq \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}} + \frac{s_1}{N}|u^+|_{p^*_{s_1}}^{p^*_{s_1}} + \theta\eta\left(\frac{1}{p} - \frac{1}{r}\right)\|u^+\|_{0,s_2,r}^r + \left(\frac{1}{q} - \frac{1}{p}\right)\|u\|_{0,s_2,q}^q \\ &\geq \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}}. \end{aligned}$$

This contradicts the fact that $c \in \left(0, \frac{s_1}{N}(S_{s_1,p}\right)^{\frac{N}{s_1 p}})$. Hence $\|u_n - u\|_{0,s_1,p} = o(1)$. Therefore, I_θ satisfies $(PS)_c$ condition for $c \in (0, \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}})$.

Next, to prove $C_\theta \in (0, \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}})$ we choose $u_0 \in X_{0,s_1,p}(\Omega)$ with $u_0^- \equiv 0$ and

$|u_0|_{p^*_{s_1}} = 1$. As $\lim_{t\to\infty} I_\theta(tu_0) = -\infty$ and $\lim_{t\to 0} I_\theta(tu_0) = 0$, there exists $t_\theta > 0$ such that $\sup_{t\geq 0} I_\theta(tu_0) = I_\theta(t_\theta u_0)$. Therefore,

$$t_\theta^{p-1}\|u_0\|_{0,s_1,p}^p + t_\theta^{q-1}\|u_0\|_{0,s_2,q}^q - \theta t_\theta^{r-1}|u_0|_r^r - t_\theta^{p^*_{s_1}-1} = 0.$$

So, we get, $t_\theta^{p-r}\|u_0\|_{0,s_1,p}^p + t_\theta^{q-r}\|u_0\|_{0,s_2,q}^q - t_\theta^{p^*_{s_1}-r} = \theta|u_0|_r^r$. As $1 < q < p < r < p^*_{s_1}$, we get $t_\theta \to 0$ as $\theta \to \infty$. Thus, there exists $\theta^* > 0$ such that for any $\theta > \theta^*$ we have,

$$\sup_{t\geq 0} I_\theta(tu_0) < \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}}.$$

Hence, $C_\theta \in \left(0, \frac{s_1}{N}(S_{s_1,p}\right)^{\frac{N}{s_1 p}})$ for $\theta > \theta^*$. □

Proof of theorem 6.1.2: Using Lemma 6.5.2, Lemma 6.5.3 and Lemma 6.7.2, we conclude that I_θ has a critical point $u \in X_{0,s_1,p}$ for $\theta > \theta^*$ where θ^* is given in (6.4.24).

Claim: $u \geq 0$ almost everywhere.

Indeed,

$$\begin{aligned} 0 = \left\langle I'_\theta(u), u^-\right\rangle &= \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+s_1 p}}dxdy \\ &+ \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+s_2 q}}dxdy \\ &:= K_1 + K_2, \end{aligned} \tag{6.5.5}$$

Note that,

$$\begin{aligned}(u(x)-u(y))(u^-(x)-u^-(y)) &= -u^+(y)u^-(x)-u^+(x)u^-(y)-(u^-(x)-u^-(y))^2\\ &\leq -(u^-(x)-u^-(y))^2\leq 0 \qquad (6.5.6)\end{aligned}$$

and

$$|u(x)-u(y)| = \left(|u(x)-u(y)|^2\right)^{\frac{1}{2}} \geq \left(|u^-(x)-u^-(y)|^2\right)^{\frac{1}{2}} = |u^-(x)-u^-(y)|. \qquad (6.5.7)$$

Since $2 \leq q < p$, using (6.5.6) and (6.5.7), we obtain

$$K_2 \leq -\int_{\mathbb{R}^{2N}} \frac{|u^-(x)-u^-(y)|^q}{|x-y|^{N+s_2q}}dxdy = -\|u^-\|^q_{0,s_2,q}.$$

Similarly, $K_1 \leq -\|u^-\|^p_{0,s_1,p}$. Therefore, (6.5.5) implies, $\|u^-\|^p_{0,s_1,p}+\|u^-\|^q_{0,s_2,q} \leq 0$ that is, $u^- = 0$ a.e and this proves the claim.

Further, we observe that $C_\theta > 0$, since I_θ satisfies the mountain pass geometry. Therefore, as u is the critical point corresponding to C_θ, u must be nontrivial. Thus, u is nontrivial nonnegative solution of $(\tilde{P})$. Consequently, u is nontrivial nonnegative solution of (P).

6.6 Proof of Theorem 6.1.4

6.6.1 Existence of $cat_\Omega(\Omega)$ nontrivial nonnegative solutions

We break the proof of Theorem 6.1.4 into several lemmas. For the rest of the section, we assume

$$N > p^2 s_1 \quad \text{and} \quad 2 \leq q < \frac{N(p-1)}{N-s} < p \leq \max\{p,\, p^*_{s_1} - \frac{q}{q-1}\} < r < p^*_{s_1}. \qquad (6.6.1)$$

Let U be a radially symmetric and decreasing minimizer for the Sobolev constant defined in (6.3.1) for $s = s_1$ and it is known from [20] that there exists constants c_1, $c_2 > 0$ and $\theta > 1$ such that

$$\frac{c_1}{|x|^{\frac{N-s_1p}{p-1}}} \leq U(|x|) \leq \frac{c_2}{|x|^{\frac{N-s_1p}{p-1}}} \quad \forall\ |x| \geq 1, \tag{6.6.2}$$

$$\frac{U(\theta r)}{U(r)} \leq \frac{1}{2} \quad \forall\ r \geq 1. \tag{6.6.3}$$

Multiplying U by a positive constant if necessary, we may assume that U satisfies the following:

$$(i)\ (-\Delta)_p^{s_1} U = U^{p^*_{s_1}-1} \quad (ii)\ \|U\|^p_{0,s_1,p} = |U|^{p^*_{s_1}}_{p^*_{s_1}} = (S_{s_1,p})^{N/s_1p}. \tag{6.6.4}$$

For any $\delta > 0$, the function

$$U_\delta(x) = \frac{1}{\delta^{\frac{N-s_1p}{p}}} U(\frac{|x|}{\delta})$$

is also a minimizer for $S_{s_1,p}$ satisfying (i) and (ii). Let θ be the universal constant defined as in (6.6.3). We may assume without loss of generality that $0 \in \Omega$. For δ, $R > 0$, we define some auxiliary functions as in [64].

$m_{\delta,R} := \frac{U_\delta(R)}{U_\delta(R) - U_\delta(\theta R)}$, and $g_{\delta,R} : [0, +\infty) \to \mathbb{R}$ by

$$g_{\delta,R}(t) = \begin{cases} 0, & 0 \leq t \leq U_\delta(\theta R) \\ m^p_{\delta,R}(t - U_\delta(\theta R)), & U_\delta(\theta R) \leq t \leq U_\delta(R) \\ t + U_\delta(R)(m^{p-1}_{\delta,R} - 1), & t \geq U_\delta(R), \end{cases} \tag{6.6.5}$$

and $G_{\delta,R} : [0, \infty) \to \mathbb{R}$ by

$$G_{\delta,R}(t) = \int_0^t (g'_{\delta,R}(\tau))^{1/p} d\tau = \begin{cases} 0, & 0 \leq t \leq U_\delta(\theta R) \\ m_{\delta,R}(t - U_\delta(\theta R)), & U_\delta(\theta R) \leq t \leq U_\delta(R) \\ t, & t \geq U_\delta(R). \end{cases} \tag{6.6.6}$$

We note that $g_{\varepsilon,\delta}$ and $G_{\delta,R}$ are non-decreasing and absolutely continuous.

Note that by definition,

$$G'_{\delta,R}(t) = \left(g'_{\delta,R}(t)\right)^{\frac{1}{p}} = \begin{cases} 0, & 0 \le t < U_\delta(\theta R) \\ m_{\delta,R}, & U_\delta(\theta R) < t < U_\delta(R) \\ 1, & t > U_\delta(R), \end{cases}$$

Therefore,

$$G'_{\delta,R}(t) \le \max\{m_{\delta,R}, 1\} \le m_{\delta,R} + 1. \tag{6.6.7}$$

Next, we estimate $m_{\delta,R}$ as follows

$$m_{\delta,R} = \frac{U_\delta(R)}{U_\delta(R) - U_\delta(\theta R)} = \frac{U(\frac{R}{\delta})}{U(\frac{R}{\delta}) - U(\frac{R\theta}{\delta})}. \tag{6.6.8}$$

Choose $\delta > 0$, small enough so that $\frac{R\theta}{\delta} > 1$ and thus $\frac{U(\frac{R\theta}{\delta})}{U(\frac{R}{\delta})} \le \frac{1}{2}$. Therefore, using (6.6.2) we have

$$m_{\delta,R} = \frac{U(\frac{R}{\delta})}{U(\frac{R}{\delta}) - U(\frac{R\theta}{\delta})} \le \frac{U(\frac{R}{\delta})}{U(\frac{R\theta}{\delta})} \le \frac{c_2}{\left(\frac{R}{\delta}\right)^{\frac{N-s_1p}{p-1}}} \times \frac{\left(\frac{R\theta}{\delta}\right)^{\frac{N-s_1p}{p-1}}}{c_1} = \frac{c_2}{c_1}\theta^{\frac{(N-s_1p)}{p-1}}. \tag{6.6.9}$$

Consider the radially symmetric non-increasing function $\bar{u}_{\delta,R} : [0, +\infty) \to \mathbb{R}$ by

$$\bar{u}_{\delta,R}(r) = G_{\delta,R}(U_\delta(r)).$$

Then we observe that, $\bar{u}_{\delta,R}$ satisfies:

$$\bar{u}_{\delta,R}(r) = \begin{cases} U_\delta(r), & r \le R \\ 0, & r \ge \theta R. \end{cases} \tag{6.6.10}$$

Therefore, we have the following estimates from [64].

Lemma 6.6.1. *[64, Lemma 2.7] For any $R > 0$, there exists $C = C(N, p, s_1) > 0$ such that for any $\delta \le \frac{R}{2}$,*

$$\|\bar{u}_{\delta,R}\|^p_{0,s_1,p} \le (S_{s_1,p})^{N/s_1p} + C\left(\frac{\delta}{R}\right)^{\frac{N-s_1p}{p-1}}, \tag{6.6.11}$$

$$|\bar{u}_{\delta,R}|_p^p \geq \begin{cases} \frac{1}{C}\delta^{s_1 p} log(R/\delta), & N = s_1 p^2 \\ \frac{1}{C}\delta^{s_1 p}, & N > s_1 p^2 \end{cases} \tag{6.6.12}$$

and

$$|\bar{u}_{\delta,R}|_{p^*_{s_1}}^{p^*_{s_1}} \geq (S_{s_1,p})^{N/s_1 p} - C\Big(\frac{\delta}{R}\Big)^{N/(p-1)}. \tag{6.6.13}$$

Let $\varepsilon > 0$. Take $R > 0$ be fixed such that $B_{\theta R} \subset\subset \Omega$. Let us define the function $u_{\varepsilon,R} : [0, +\infty) \to \mathbb{R}$ by

$$u_{\varepsilon,R}(r) = \varepsilon^{-\frac{(N-s_1 p)}{p^2}} \bar{u}_{\delta,R}(r) \quad \text{with} \quad \delta = \varepsilon^{\frac{(p-1)}{p}}, \quad \forall\, r \geq 0. \tag{6.6.14}$$

Clearly, $u_{\varepsilon,R} \subset X_{0,s_1,p}(\Omega)$, that is, $u_{\varepsilon,R} \equiv 0$ in $\mathbb{R}^N \setminus \Omega$. Therefore, applying (6.6.11) to (6.6.14) yields

$$\|u_{\varepsilon,R}\|_{0,s_1,p}^p \leq (S_{s_1,p})^{N/s_1 p} \varepsilon^{-\frac{(N-s_1 p)}{p}} + O(1). \tag{6.6.15}$$

Lemma 6.6.2. $|u_{\varepsilon,R}|_{p^*_{s_1}}^p = (S_{s_1,p})^{\frac{N-s_1 p}{s_1 p}} \varepsilon^{-\frac{(N-s_1 p)}{p}} + O(1).$

Proof. Applying (6.6.13), it is easy to see that

$$|u_{\varepsilon,R}|_{p^*_{s_1}}^p \geq (S_{s_1,p})^{\frac{N-s_1 p}{s_1 p}} \varepsilon^{-\frac{(N-s_1 p)}{p}} + O(1).$$

To see the upper estimate, we observe that

$$\begin{aligned} |u_{\varepsilon,R}|_{p^*_{s_1}}^{p^*_{s_1}} = \int_\Omega \varepsilon^{-\frac{(N-s_1 p)p^*_{s_1}}{p^2}} |\bar{u}_{\delta,R}|^{p^*_{s_1}} dx &= \varepsilon^{-N/p} \int_\Omega |G_{\delta,R}(U_\delta(x))|^{p^*_{s_1}} dx \\ &\leq \varepsilon^{-N/p} |G'_{\delta,R}|_{L^\infty}^{p^*_{s_1}} \int_\Omega |U_\delta(x)|^{p^*_{s_1}} dx \\ &\leq \varepsilon^{-N/p} \max\{m_{\delta,R}^{p^*_{s_1}}, 1\} \int_\Omega |U_\delta(x)|^{p^*_{s_1}} dx, \end{aligned}$$

where in the last line we have used (6.6.7). Next, applying (6.6.9) to the last line, we have

$$\begin{aligned} |u_{\varepsilon,R}|_{p^*_{s_1}}^{p^*_{s_1}} \leq C\varepsilon^{-N/p} \int_{\mathbb{R}^N} |U_\delta(x)|^{p^*_{s_1}} dx &\leq C\varepsilon^{-N/p} \frac{1}{\delta^{\frac{(N-s_1 p)p^*_{s_1}}{p}}} \int_{\mathbb{R}^N} |U(\frac{x}{\delta})|^{p^*_{s_1}} dx \\ &= C\varepsilon^{-N/p} \int_{\mathbb{R}^N} |U(y)|^{p^*_{s_1}} dy \\ &= C\varepsilon^{-N/p} |U|_{p^*_{s_1}}^{p^*_{s_1}} \\ &= C\varepsilon^{-N/p} (S_{s_1,p})^{N/s_1 p}, \end{aligned}$$

where, in the last line we have used (6.6.4)(ii). Hence, we have,

$$|u_{\varepsilon,R}|^p_{p^*_{s_1}} \le \left(C(S_{s_1,p})^{N/s_1p}\varepsilon^{-N/p}\right)^{\frac{p}{p^*_{s_1}}} = C(S_{s_1,p})^{\frac{N-s_1p}{s_1p}}\varepsilon^{-\frac{N-s_1p}{p}}.$$

This completes the proof of the lemma. □

Lemma 6.6.3. *Let $u_{\varepsilon,R}$ be defined as above. Then the following estimates hold, that is, for $t \ge 1$,*

$$|u_{\varepsilon,R}|^t_t \ge \begin{cases} k\varepsilon^{\frac{N(p-1)-t(N-s_1p)}{p}} + O(1), & t > \frac{N(p-1)}{N-s_1p} \\ k|ln\,\varepsilon| + O(1), & t = \frac{N(p-1)}{N-s_1p} \\ O(1), & t < \frac{N(p-1)}{N-s_1p} \end{cases} \tag{6.6.16}$$

and

$$\|u_{\varepsilon,R}\|^t_{0,s_2,t} \le O(1), \quad 1 \le t < \frac{N(p-1)}{N-s_1}. \tag{6.6.17}$$

In particular, we have

$$|u_{\varepsilon,R}|^p_p \ge \begin{cases} k\varepsilon^{\frac{p^2s_1-N}{p}} + O(1), & N > p^2s_1 \\ k|ln\,\varepsilon| + O(1), & N = p^2s_1 \\ O(1), & N < p^2s_1 \end{cases} \tag{6.6.18}$$

where k is a positive constant independent of ε.

Proof. We have,

$$\begin{aligned}
|u_{\varepsilon,R}|_t^t = \int_\Omega |u_{\varepsilon,R}(x)|^t dx &= \int_{\mathbb{R}^N} |u_{\varepsilon,R}(x)|^t dx \geq \int_{B_R(0)} |u_{\varepsilon,R}(x)|^t dx \\
&= \varepsilon^{-\frac{(N-s_1p)t}{p^2}} \int_{B_R(0)} \left(\bar{u}_{\delta,R}(x)\right)^t dx \\
&= \varepsilon^{-\frac{(N-s_1p)t}{p^2}} \int_{B_R(0)} U_\delta^t(x) dx \\
&= \frac{\varepsilon^{-\frac{(N-s_1p)t}{p^2}}}{\delta^{\frac{(N-s_1p)t}{p}}} \int_{B_R(0)} U^t(\frac{x}{\delta}) dx \\
&= \varepsilon^{-\frac{(N-s_1p)t}{p^2}} \delta^{N-\frac{(N-s_1p)t}{p}} \int_{B_{\frac{R}{\delta}}(0)} U^t(x) dx \\
&\geq \varepsilon^{\frac{N(p-1)}{p} - t\frac{(N-s_1p)}{p}} \int_1^{\frac{R}{\delta}} U^t(r) r^{N-1} dr \\
&\geq c_1^t \varepsilon^{\frac{N(p-1)}{p} - t\frac{(N-s_1p)}{p}} \int_1^{\frac{R}{\delta}} \frac{r^{N-1}}{r^{\frac{N-s_1p}{p-1}t}} dr.
\end{aligned}$$

If $t > \frac{N(p-1)}{N-s_1p}$, then we have

$$|u_{\varepsilon,R}|_t^t \geq \frac{c_1^t \varepsilon^{\frac{N(p-1)}{p} - t\frac{(N-s_1p)}{p}}}{\frac{(N-s_1p)t}{p-1} - N} [1 - \Big(\frac{R}{\delta}\Big)^{N-\frac{(N-s_1p)t}{p-1}}].$$

Since $\delta = \varepsilon^{\frac{p-1}{p}}$, choosing $\varepsilon > 0$ small enough we can make δ suitably small so that $1 - \left(\frac{R}{\delta}\right)^{N-\frac{(N-s_1p)t}{p-1}} \geq \frac{1}{2}$. Therefore,

$$|u_{\varepsilon,R}|_t^t \geq k \varepsilon^{\frac{N(p-1)}{p} - t\frac{(N-s_1p)}{p}},$$

where $k = \frac{c_1^p}{2\left(\frac{t(N-s_1p)}{p-1} - N\right)}$.

If $\frac{t(N-s_1p)}{p-1} = N$, then

$$|u_{\varepsilon,R}|_t^t \geq c_1^t \int_1^{\frac{R}{\delta}} \frac{1}{r} dr = c_1^t (ln\, R - ln\, \varepsilon^{\frac{p-1}{p}}) \geq k |ln\, \varepsilon| + O(1).$$

On the other hand for $\frac{t(N-s_1p)}{(p-1)} < N$, we have

$$\begin{aligned}
|u_{\varepsilon,R}|_t^t &\geq c_1^t \varepsilon^{\frac{N(p-1)}{p} - \frac{t(N-s_1p)}{p}} \frac{(R/\delta)^{N-\frac{t(N-s_1p)}{p-1}} - 1}{N - \frac{t(N-s_1p)}{p-1}} \\
&= c_1^t \left[\frac{R^{N-\frac{t(N-s_1p)}{p-1}} - \varepsilon^{\frac{N(p-1)-t(N-s_1p)}{p}}}{N - \frac{t(N-s_1p)}{p-1}} \right] \\
&\geq O(1).
\end{aligned}$$

To see the proof of (6.6.17), first we note that from Lemma 6.2.4 we have

$$u_{\varepsilon,R} \in X_{0,s_1,p}(\Omega) \subset X_{0,s_2,t}(\Omega), \quad 1 \le t \le p, \quad 0 < s_2 < s_1 < 1$$

and

$$\|u_{\varepsilon,R}\|_{0,s_2,t} \le \|u_{\varepsilon,R}\|_{0,s_1,t}.$$

Therefore,

$$\begin{aligned}
&\|u_{\varepsilon,R}\|^t_{0,s_2,t} \le \|u_{\varepsilon,R}\|^t_{0,s_1,t} \\
&= \varepsilon^{-\frac{(N-s_1p)t}{p^2}} \|\bar{u}_{\delta,R}(\cdot)\|^t_{0,s_1,t} \\
&= \varepsilon^{-\frac{(N-s_1p)t}{p^2}} \|G_{\delta,R}(U_\delta(\cdot))\|^t_{0,s_1,t} \\
&= \varepsilon^{-\frac{(N-s_1p)t}{p^2}} \int_{\mathbb{R}^{2N}} \frac{|G_{\delta,R}(U_\delta(x)) - G_{\delta,R}(U_\delta(y))|^t}{|x-y|^{N+s_1t}} dxdy \\
&\le \varepsilon^{-\frac{(N-s_1p)t}{p^2}} \int_{\mathbb{R}^{2N}} \frac{|G'_{\delta,R}\Big(U_\delta(x) + \tau(U_\delta(y) - U_\delta(x)\Big)|^t |U_\delta(x) - U_\delta(y)|^t}{|x-y|^{N+s_1t}} dxdy,
\end{aligned} \tag{6.6.19}$$

for some $\tau \in (0,1)$. In the last line, we have used mean value theorem. Thus from (6.6.7), we obtain

$$G'_{\delta,R}\Big(U_\delta(x) + \tau(U_\delta(x) - U_\delta(y)\Big) \le 1 + \frac{c_2}{c_1}\theta^{\frac{N+s_1p}{p-1}} = c_3. \tag{6.6.20}$$

Substituting (6.6.20) into (6.6.19) yields

$$\begin{aligned}
\|u_{\varepsilon,R}\|^t_{0,s_2,t} &\le \varepsilon^{-\frac{(N-s_1p)t}{p^2}} c_3^t \int_{\mathbb{R}^{2N}} \frac{|U_\delta(x) - U_\delta(y)|^t}{|x-y|^{N+s_1t}} dxdy \\
&= C\varepsilon^{-\frac{(N-s_1p)t}{p^2}} \frac{\delta^{N-s_1t}}{\delta^{\frac{(N-s_1p)t}{p}}} \int_{\mathbb{R}^{2N}} \frac{|U(z) - U(w)|^t}{|z-w|^{N+s_1t}} dzdw \\
&= C\varepsilon^{-\frac{(N-s_1p)t}{p^2}} \varepsilon^{\frac{N(p-t)(p-1)}{p^2}} \|U\|^t_{0,s_1,t} \\
&= C\varepsilon^{\frac{1}{p^2}\Big(N(p-1)(p-t)-(N-s_1p)t\Big)} \|U\|^t_{0,s_1,t},
\end{aligned}$$

where we have used that $\delta = \varepsilon^{\frac{p-1}{p}}$. Note that $t < \frac{N(p-1)}{N-s_1}$ which implies,

$$N(p-1)(p-t) - (N-s_1p)t > 0.$$

Therefore, we obtain

$$\|u_{\varepsilon,R}\|_{0,s_2,t}^{t} \leq O(1) \quad \text{for} \quad 1 \leq t < \frac{N(p-1)}{N-s_1}. \tag{6.6.21}$$

This completes the proof of Lemma 6.6.3.

□

Lemma 6.6.4. *Assume* (6.6.1) *holds. Then, for any* $\theta > 0$, $C_\theta \in \left(0, \frac{s}{N}(S_{s_1,p})^{N/s_1 p}\right)$, *where* C_θ *is defined as in* (6.5.4).

Proof. As we have fixed R, we take $u_\varepsilon := u_{\varepsilon,R}$. Define

$$v_\varepsilon(x) = \frac{u_\varepsilon(x)}{|u_\varepsilon|_{p_{s_1}^*}}. \tag{6.6.22}$$

Thus $|v_\varepsilon|_{p_{s_1}^*} = 1$. Define

$$\begin{aligned} g(t) : &= I_\theta(tv_\varepsilon) \\ &= \frac{t^p}{p}\|v_\varepsilon\|_{0,s_1,p}^p + \frac{t^q}{q}\|v_\varepsilon\|_{0,s_2,q}^q - \theta\frac{t^r}{r}|v_\varepsilon|_r^r - \frac{t^{p_{s_1}^*}}{p_{s_1}^*}. \end{aligned}$$

Since g is a continuous function and $g(0) = 0$, $\lim_{t\to+\infty} g(t) = -\infty$, there exists $t_\varepsilon > 0$ such that

$$\sup_{t\geq 0} I_\theta(tv_\varepsilon) = I_\theta(t_\varepsilon v_\varepsilon).$$

Then, t_ε satisfies $g'(t_\varepsilon) = 0$ i.e.,

$$t_\varepsilon^{p-1}\|v_\varepsilon\|_{0,s_1,p}^p + t_\varepsilon^{q-1}\|v_\varepsilon\|_{0,s_2,q}^q - \theta t_\varepsilon^{r-1}|v_\varepsilon|_r^r - t_\varepsilon^{p_{s_1}^*-1} = 0. \tag{6.6.23}$$

Consequently,

$$\|v_\varepsilon\|_{0,s_1,p}^p + t_\varepsilon^{q-p}\|v_\varepsilon\|_{0,s_2,q}^q > t_\varepsilon^{p_{s_1}^*-p}. \tag{6.6.24}$$

As $q < \frac{N(p-1)}{N-s_1}$, combining (6.6.15), Lemma 6.6.2 and (6.6.17) we have

$$\|v_\varepsilon\|_{0,s_1,p}^p \leq S_{s_1,p} + O(\varepsilon^{\frac{N-s_1 p}{p}}), \quad \|v_\varepsilon\|_{0,s_2,q}^q \leq \frac{\|u_\varepsilon\|_{0,s_2,q}^q}{|u_\varepsilon|_{p_{s_1}^*}^q} = O(\varepsilon^{\frac{q(N-s_1 p)}{p^2}}). \tag{6.6.25}$$

Therefore, from (6.6.24) and (6.6.25), we see that for any $\tilde{\varepsilon} > 0$ small enough, there exists $t^0_{\tilde{\varepsilon}} > 0$ such that for all $\varepsilon \leq \tilde{\varepsilon}$ we have, $t_\varepsilon \leq t^0_{\tilde{\varepsilon}}$. Using (6.6.23) we have,

$$\|v_\varepsilon\|^p_{0,s_1,p} < \theta t_\varepsilon^{r-p}|v_\varepsilon|^r_r + t_\varepsilon^{p^*_{s_1}-p}. \tag{6.6.26}$$

Using (6.6.25)-(6.6.26) we say there exists $T > 0$ such that for any $\varepsilon > 0$, $t_\varepsilon \geq T$.

Let $h(t) = \frac{t^p}{p}\|v_\varepsilon\|^p_{0,s_1,p} - \frac{t^{p^*_{s_1}}}{p^*_{s_1}}$. Then $h(t)$ attains its maximum at $t_0 = (\|v_\varepsilon\|^p_{0,s_1,p})^{\frac{1}{p^*_{s_1}-p}}$. We note that, $N > p^2 s_1 > ps_1$ implies $N(p-1) < p(N-ps_1)$, Therefore, $\frac{N(p-1)}{N-ps_1} < p < r$. Hence, for $\varepsilon \leq \tilde{\varepsilon}$, applying Lemma 6.6.3 and Lemma 6.6.2 we obtain,

$$\begin{aligned} g(t_\varepsilon) &= h(t_\varepsilon) + \frac{t_\varepsilon^q}{q}\|v_\varepsilon\|^q_{0,s_2,q} - \theta\frac{t_\varepsilon^r}{r}|v_\varepsilon|^r_r \\ &\leq h(t_0) + \frac{(t^0_{\tilde{\varepsilon}})^q}{q}\|v_\varepsilon\|^q_{0,s_2,q} - \theta\frac{T^r}{r}|v_\varepsilon|^r_r \\ &\leq \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}} + c_1\varepsilon^{\frac{(N-s_1p)}{p}} + c_2\varepsilon^{\frac{q(N-s_1p)}{p^2}} - c_3\varepsilon^{\frac{(p-1)}{p}(N-\frac{r(N-s_1p)}{p})}, \end{aligned}$$

with $c_1, c_2, c_3 > 0$ (independent of ε.) As

$$\frac{N-s_1p}{p} > \frac{q(N-s_1p)}{p^2} > \frac{(p-1)}{p}\left(N - \frac{r(N-s_1p)}{p}\right) > 0,$$

choose $\varepsilon > 0$ small so that $g(t_\varepsilon) = \sup_{t\geq 0} I_\theta(tv_\varepsilon) < \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1p}}$.

Hence, $C_\theta \in \left(0, \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1p}}\right)$ for any $\theta > 0$. □

Lemma 6.6.5. *Assume* (6.6.1) *holds. Then for any* $\theta > 0$, $c_\theta = C_\theta$, *where* c_θ *and* C_θ *are defined as in* (6.5.1) *and* (6.5.4) *respectively.*

Proof. Using lemmas 6.5.2 and 6.5.3 we conclude that, for any $\theta > 0$ there exists $u_\theta \in X_{0,s_1,p}(\Omega)$ such that $I_\theta(u_\theta) = C_\theta$ and $I'_\theta(u_\theta) = 0$. Also for any $u \in N_\theta$, we have

$$0 = \langle I'_\theta(u), u\rangle = \|u\|^p_{0,s_1,p} + \|u\|^q_{0,s_2,q} - \theta|u^+|^r_r - |u^+|^{p^*_{s_1}}_{p^*_{s_1}}. \tag{6.6.27}$$

Therefore, if we define $f(t) := I_\theta(tu)$, where $u \in N_\theta$, then a straight forward computation yields that $f'(1) = 0$ and $f''(1) < 0$, i.e,

$$\max_{t\geq 0} I_\theta(tu) = I_\theta(u). \tag{6.6.28}$$

Observe that, from the definition of C_θ it follows $C_\theta \leq \max_{t\geq 0} I_\theta(tu)$. Consequently, we obtain $I_\theta(u) \geq C_\theta$ for all $u \in N_\theta$. Hence,

$$c_\theta = \inf_{u\in N_\theta} I_\theta(u) \geq C_\theta. \tag{6.6.29}$$

On the other hand, $u_\theta \in N_\theta$ and $I_\theta(u_\theta) = C_\theta$ implies $C_\theta \geq c_\theta$. Hence $c_\theta = C_\theta$. □

From the definition of C_θ, it is easy to see that

$$C_{\theta_1} \leq C_{\theta_2} \quad \text{if} \quad \theta_2 \leq \theta_1.$$

Therefore, using Lemma 6.6.5, we also have

$$c_{\theta_1} \leq c_{\theta_2} \quad \text{if} \quad \theta_2 \leq \theta_1,$$

which implies c_θ is non-increasing in θ. Therefore, for any $\lambda > 0$, there exists $\rho = \rho(\lambda)$ (depending on the Mountain Pass Geometry) such that $0 < \rho \leq c_\theta \leq c_0$ for all $\theta \in [0, \lambda]$, where c_0 is the MP level associated to the functional

$$I_0(u) = \frac{1}{p}\|u\|^p_{0,s_1,p} + \frac{1}{q}\|u\|^q_{0,s_2,q} - \frac{1}{p^*_{s_1}}|u^+|^{p^*_{s_1}}_{p^*_{s_1}}.$$

Lemma 6.6.6. $c_0 = \frac{s_1}{N}(S_{s_1,p})^{N/s_1p}$.

Proof. Recall $v_\varepsilon(x) = \frac{u_\varepsilon(x)}{|u_\varepsilon|_{p^*_{s_1}}}$ where $u_\varepsilon = u_{\varepsilon,R}$ is defined as in (6.6.14). Arguing as in Lemma 6.6.4, there exists $t_\varepsilon > 0$ such that $\frac{d}{dt}I_0(tv_\varepsilon)|_{t=t_\varepsilon} = 0$, that is,

$$t_\varepsilon^{p-1}\|v_\varepsilon\|^p_{0,s_1,p} + t_\varepsilon^{q-1}\|v_\varepsilon\|^q_{0,s_2,q} = t_\varepsilon^{p^*_{s_1}-1}. \tag{6.6.30}$$

Hence, $t_\varepsilon^{p^*_{s_1}-p} \geq \|v_\varepsilon\|^p_{0,s_1,p}$. Also, t_ε is bounded. Using $1 < q < p < p^*_{s_1}$, (6.6.30) and (6.6.25) we have,

$$t_\varepsilon = \left(S_{s_1,p} + O(\varepsilon^{\frac{q(N-s_1p)}{p^2}})\right)^{\frac{1}{p^*_{s_1}-p}}.$$

Therefore,

$$\begin{aligned}
c_0 \le I_0(t_\varepsilon v_\varepsilon) &= \frac{1}{p}\Big(S_{s_1,p}+O(\varepsilon^{\frac{q(N-s_1p)}{p^2}})\Big)^{\frac{p}{p^*_{s_1}-p}}\Big(S_{s_1,p}+O(\varepsilon^{\frac{(N-sp)}{p}})\Big)\\
&+\frac{1}{q}\Big(S_{s_1,p}+O(\varepsilon^{\frac{q(N-s_1p)}{p^2}})\Big)^{\frac{q}{p^*_{s_1}-p}}O(\varepsilon^{\frac{q(N-s_1p)}{p^2}})\\
&-\frac{1}{p^*_{s_1}}\Big(S_{s_1,p}+O(\varepsilon^{\frac{q(N-s_1p)}{p^2}})\Big)^{\frac{p^*_{s_1}}{p^*_{s_1}-p}}\\
&= \frac{1}{p}\Big((S_{s_1,p})^{\frac{N-s_1p}{s_1p}}+O(\varepsilon^{\frac{q(N-s_1p)}{p^2}})\Big)\Big(S_{s_1,p}+O(\varepsilon^{\frac{(N-s_1p)}{p}})\Big)\\
&+\frac{1}{q}\Big((S_{s_1,p})^{\frac{q(N-s_1p)}{p^2}}+O(\varepsilon^{\frac{q(N-s_1p)}{p^2}})\Big)O(\varepsilon^{\frac{q(N-s_1p)}{p^2}})\\
&-\frac{1}{p^*_{s_1}}\Big((S_{s_1,p})^{\frac{N}{s_1p}}+O(\varepsilon^{\frac{q(N-s_1p)}{p^2}})\Big)\\
&= \Big(\frac{1}{p}-\frac{1}{p^*_{s_1}}\Big)(S_{s_1,p})^{\frac{N}{s_1p}}+O(\varepsilon^{\frac{q(N-ss_1p)}{p^2}})+O(\varepsilon^{\frac{N-ss_1p}{p}})\\
&\to \frac{s_1}{N}(S_{s_1,p})^{N/s_1p}, \quad \text{as} \quad \varepsilon\to 0. \qquad (6.6.31)
\end{aligned}$$

Let $\{u_n\}_{n\ge 1}\subset X_{0,s_1,p}(\Omega)$ such that $I_0(u_n)\to c_0$ and $I_0'(u_n)\to 0$ in $(X_{0,s_1,p})'$ as $n\to\infty$. Arguing as in Claim 1 of Lemma 6.4.1, it follows $\{\|u_n\|_{0,s_1,p}\}_{n\ge 1}$ is bounded. Moreover, as in (6.4.12) w.l.g up to a subsequence we can assume

$$\|u_n\|^p_{0,s_1,p}=a+o(1), \quad \|u_n\|^q_{0,s_2,q}=b+o(1), \quad |u_n^+|^{p^*_{s_1}}_{p^*_{s_1}}=a+b+o(1).$$

Since $2\le q<p$, estimating $\langle I_0'(u_n),u_n^-\rangle$ as in the proof of Theorem 6.1.2, we obtain $\|u_n^-\|^p_{0,s_1,p}\to 0$ and $\|u_n^-\|^q_{0,s_2,q}\to 0$ as $n\to\infty$.

Therefore, we may assume $u_n\ge 0$. Hence, $|u_n|^{p^*_{s_1}}_{p^*_{s_1}}=a+b+o(1)$. Set $v_n(x)=\frac{u_n(x)}{|u_n|_{p^*_{s_1}}}$. Then $|v_n|_{p^*_{s_1}}=1$ and

$$S_{s_1,p}\le \|v_n\|^p_{0,s_1,p}=\frac{a+o(1)}{\Big(a+b+o(1)\Big)^{p/p^*_{s_1}}}\le (a+o(1))^{s_1p/N}.$$

Hence, we have,

$$\begin{aligned}\frac{s_1}{N}(S_{s_1,p})^{N/s_1p} &\le \frac{s_1(a+o(1))}{N}\\ &\le \frac{s_1(a+o(1))}{N}+\left(\frac{1}{q}-\frac{1}{p^*_{s_1}}\right)(b+o(1))\\ &\to c_0, \quad \text{as} \quad n\to\infty. \end{aligned} \tag{6.6.32}$$

Combining (6.6.31) and (6.6.32), we have $c_0 = \frac{s_1}{N}(S_{s_1,p})^{N/s_1p}$. Hence, proved. □

Remark:

(i) For any bounded domain $\Omega \subset \mathbb{R}^N$, the MP level of the functionals

$$I_{0,\Omega}(u) = \frac{1}{p}\|u\|^p_{0,s_1,p} + \frac{1}{q}\|u\|^q_{0,s_2,q} - \frac{1}{p^*_{s_1}}|u^+|^{p^*_{s_1}}_{p^*_{s_1}}$$

and

$$\tilde{I}_{0,\Omega}(u) = \frac{1}{p}\|u\|^p_{0,s_1,p} + \frac{1}{q}\|u\|^q_{0,s_2,q} - \frac{1}{p^*_{s_1}}|u|^{p^*_{s_1}}_{p^*_{s_1}}$$

is $\frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1p}}$, so the MP level is independent of Ω.

(ii) Using the proof of Lemma 6.6.6, we may assume that all the PS sequence of I_θ are non-negative.

Lemma 6.6.7. *Let $\theta_n \to 0$ as $n\to\infty$. Then $c_{\theta_n}\to c_0$ as $n\to\infty$.*

Proof. From the definition of c_θ, c_0 we note that

$$c_{\theta_n} \le c_0 \quad \forall \quad n\in\mathbb{N}. \tag{6.6.33}$$

Let $\{u_n\}_{n\ge1} \subset X_{0,s_1,p}(\Omega)$ such that $u_n \ge 0$ and satisfies $I_{\theta_n}(u_n) = c_{\theta_n}, I'_{\theta_n}(u_n) = 0$ and let $\{t_n\}_{n\ge1}\subset\mathbb{R}$ such that $t_nu_n \in N_0$. Hence, $c_0 \le I_0(t_nu_n) = I_{\theta_n}(t_nu_n) + \frac{\theta_n t_n^r}{r}|u_n|^r_r$. Consequently,

$$c_0 \le c_{\theta_n} + \frac{\theta_n t_n^r}{r}|u_n|^r_r. \tag{6.6.34}$$

As $c_{\theta_n} \leq c_0$, we can show as before $\{\|u_n\|_{0,s_1,p}\}_{n\geq 1}$ is bounded. We also claim that $\{t_n\}_{n\geq 1}$ is bounded. Suppose not. Then up to a subsequence, $t_n \to \infty$. Note that, $t_n u_n \in N_0$ implies

$$\|u_n\|_{0,s_1,p}^p + t_n^{q-p}\|u_n\|_{0,s_2,q}^q = t_n^{p_{s_1}^*-p}|u_n|_{p_{s_1}^*}^{p_{s_1}^*}. \tag{6.6.35}$$

Since $q < p < p_{s_1}^*$ and $\max\{\|u_n\|_{0,s_2,q}, |u_n|_{p_{s_1}^*}\} \leq C\|u_n\|_{0,s,p}$, we obtain RHS of (6.6.35) $\to \infty$ but LHS remains bounded. Hence the claim follows.

By the above claim and (6.6.34), we have

$$c_0 \leq \liminf_{n\to\infty} c_{\theta_n} \leq \limsup_{n\to\infty} c_{\theta_n} \leq c_0.$$

Hence, $c_0 = \lim_{n\to\infty} c_{\theta_n}$. This completes the proof. □

Since $\Omega \subset \mathbb{R}^N$ is a smooth domain, there exists $\delta > 0$ such that

$$\Omega_\delta^+ := \{x \in \mathbb{R}^N \,|\, \mathrm{dist}(x,\Omega) < \delta\}$$

and

$$\Omega_\delta^- := \{x \in \mathbb{R}^N \,|\, \mathrm{dist}(x,\Omega) > \delta\}$$

are homotopically equivalent to Ω. Without loss of generality, we may assume that

$B_\delta = B(0,\delta) \subset \Omega$. Define,

$$X_{0,s_1,p}^{rad}(B_\delta) := \{u \in X_{0,s_1,p}(B_\delta) \,|\, u \text{ is radial}\}.$$

Let $N_{\theta,B_\delta} := \inf\left\{u \in X_{0,s_1,p}^{rad}(B_\delta)\setminus\{0\} \middle| \left\langle I'_{\theta,B_\delta}(u), u\right\rangle = 0\right\}$ where

$$I_{\theta,B_\delta}(u) = \frac{1}{p}\|u\|_{0,s_1,p}^p + \frac{1}{q}\|u\|_{0,s_2,q}^q - \frac{\theta}{r}\int_{B_\delta}|u^+|^r\,dx - \frac{1}{p_{s_1}^*}\int_{B_\delta}|u^+|^{p_{s_1}^*}\,dx.$$

Denote $n_\theta = \inf_{u\in N_{\theta,B_\delta}} I_{\theta,B_\delta}(u)$. We note that n_θ is non-increasing in θ. Let us denote the MP level for I_{θ,B_δ} on $X_{0,s,p}(\Omega)^{rad}(B_\delta)$ by $\tilde{n_\theta}$. We also observe that $\tilde{n_\theta} > 0$ for all $\theta \geq 0$.

Lemma 6.6.8. *Assume* (6.6.1) *holds. Then, for any* $\theta > 0$, *the following holds:*

(a) I_{θ,B_δ} *satisfies the* $(PS)_c$ *condition for all* $c \in \left(0, \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}}\right)$. *Moreover,*

$$\tilde{n_\theta} \in \left(0, \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}}\right).$$

(b) $n_\theta = \tilde{n_\theta}$.

(c) $n_\theta \to \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}}$ *as* $\theta \to 0$.

Proof. Applying Brezis-Lieb lemma, it is not difficult to check that I_{θ,B_δ} in $X^{rad}_{0,s_1,p}(B_\delta)$ satisfies the (PS)$_c$ condition for all $c \in \left(0, \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}}\right)$. By a similar argument as in Lemma 6.5.3, we also obtain $\tilde{n_\theta} \in \left(0, \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}}\right)$. Further, following the same argument as in Lemma 6.6.6 and Lemma 6.6.7, it yields $n_\theta \to \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}}$ and $\theta \to 0$ respectively. □

Let us define a map $\tau : N_\theta \to \mathbb{R}^N$ by

$$\tau(u) := (S_{s_1,p})^{-\frac{N}{s_1 p}} \int_\Omega |u^+(x)|^{p^*_{s_1}} x \, dx.$$

Let us denote $I_\theta^{n_\theta} = \{u \in X_{0,s_1,p}(\Omega) : I_\theta \le n_\theta\}$.

Lemma 6.6.9. *There exists* $\theta^* > 0$ *such that for any* $\theta \in (0, \theta^*)$ *and* $u \in N_\theta \cap I_\theta^{n_\theta}$, *it holds* $\tau(u) \in \Omega_\delta^+$.

Proof. We will prove this by contradiction. Let us suppose $\theta_n \to 0$ and $u_n \in N_{\theta_n} \cap I_{\theta_n}^{n_{\theta_n}}$ but $\tau(u_n) \notin \Omega_\delta^+$. We observe that

$$c_{\theta_n} \le I_{\theta_n}(u_n) = \frac{1}{p}\|u_n\|^p_{0,s_1,p} + \frac{1}{q}\|u_n\|^q_{0,s_2,q} - \frac{\theta_n}{r}|u_n^+|^r_r - \frac{1}{p^*_{s_1}}|u_n^+|^{p^*_{s_1}}_{p^*_{s_1}} \le n_{\theta_n}$$

and

$$\|u_n\|^p_{0,s_1,p} + \|u_n\|^q_{0,s_2,q} - \theta_n|u_n^+|^r_r - |u_n^+|^{p^*_{s_1}}_{p^*_{s_1}} = \langle I'_\theta(u_n), u_n\rangle = 0.$$

It can be shown as before that $\|u_n\|_{0,s_1,p}$ is bounded. Therefore, we have,

$$c_{\theta_n} \le I_{\theta_n}(u_n) = \frac{1}{p}\|u_n\|^p_{0,s_1,p} + \frac{1}{q}\|u_n\|^q_{0,s_2,q} - \frac{1}{p^*_{s_1}}|u_n^+|^{p^*_{s_1}}_{p^*_{s_1}} + o(1) \le n_{\theta_n} + o(1) \tag{6.6.36}$$

and

$$\|u_n\|_{0,s_1,p}^p + \|u_n\|_{0,s_2,q}^q - |u_n^+|_{p_{s_1}^*}^{p_{s_1}^*} = o(1). \tag{6.6.37}$$

Using (6.6.36) and (6.6.37) we have,

$$\frac{s_1}{N}\|u_n\|_{0,s_1,p}^p \leq \left(\frac{1}{p} - \frac{1}{p_{s_1}^*}\right)\|u_n\|_{0,s_1,p}^p + \left(\frac{1}{q} - \frac{1}{p_{s_1}^*}\right)\|u_n\|_{0,s_2,q}^q \leq n_{\theta_n} + o(1).$$

Consequently, applying Lemma 6.6.8(c) it yields

$$\|u_n\|_{0,s_1,p}^p \leq (S_{s_1,p})^{\frac{N}{s_1 p}} + o(1). \tag{6.6.38}$$

From (6.6.37), it follows

$$\|u_n\|_{0,s_1,p}^p \leq |u_n^+|_{p_{s_1}^*}^{p_{s_1}^*} + o(1). \tag{6.6.39}$$

Define $w_n = \frac{u_n}{|u_n^+|_{p_{s_1}^*}}$, which implies $|w_n^+|_{p_{s_1}^*} = 1$. Using (6.6.38) and (6.6.39), we obtain

$$S_{s_1,p} \leq \|w_n\|_{0,s_1,p}^p \leq \frac{\|u_n\|_{0,s_1,p}^p}{|u_n^+|_{p_{s_1}^*}^p} \leq \|u_n\|_{0,s_1,p}^{p-\frac{p^2}{p_{s_1}^*}} + o(1) \leq S_{s_1,p} + o(1). \tag{6.6.40}$$

Hence, the function $\tilde{w_n}(x) := w_n^+(x)$ satisfies

$$|\tilde{w_n}|_{p_{s_1}^*} = 1 \quad \text{and} \quad \|\tilde{w_n}\|_{0,s_1,p}^p \to S_{s_1,p} \quad \text{as} \quad n \to \infty.$$

Using Theorem 6.3.1, there exists a sequence $(y_n, \lambda_n) \in \mathbb{R}^N \times \mathbb{R}^+$ such that the sequence v_n defined by

$$v_n(x) = \lambda_n^{\frac{(N-ps_1)}{p}} \tilde{w_n}(\lambda_n x + y_n),$$

converges strongly to some $v \in W^{s_1,p}(\mathbb{R}^N)$. Combining (6.6.40) and (6.6.39), we get

$$S_{s_1,p}|u_n^+|_{p_{s_1}^*}^p + o(1) = \|u_n\|_{0,s_1,p}^p \leq |u_n^+|_{p_{s_1}^*}^{p_{s_1}^*} + o(1).$$

Hence,

$$|u_n^+|_{p_{s_1}^*}^{p_{s_1}^*} \geq (S_{s_1,p})^{\frac{N}{s_1 p}} + o(1), \quad n \to \infty. \tag{6.6.41}$$

Further, from (6.6.40) and (6.6.38) it follows

$$S_{s_1,p}|u_n^+|_{p^*}^p + o(1) = \|u_n\|_{0,s,p}^p \leq (S_{s_1,p})^{\frac{N}{s_1 p}} + o(1).$$

Hence,

$$|u_n^+|_{p_{s_1}^*}^{p_{s_1}^*} \leq (S_{s_1,p})^{\frac{N}{s_1 p}} + o(1). \tag{6.6.42}$$

Using (6.6.41) and (6.6.42) we conclude that,

$$|u_n^+|_{p_{s_1}^*}^{p_{s_1}^*} \to (S_{s_1,p})^{\frac{N}{s_1 p}} \quad \text{as} \quad n \to \infty. \tag{6.6.43}$$

Now,

$$\begin{aligned}
\tau(u_n) &= (S_{s_1,p})^{-\frac{N}{s_1 p}} \int_\Omega |u_n^+(x)|^{p_{s_1}^*} x\, dx \\
&= (S_{s_1,p})^{-\frac{N}{s_1 p}} |u_n^+|_{p_{s_1}^*}^{p_{s_1}^*} \int_\Omega \tilde{w_n}^{p_{s_1}^*}(x) x\, dx \\
&= (S_{s_1,p})^{-\frac{N}{s_1 p}} |u_n^+|_{p_{s_1}^*}^{p_{s_1}^*} \int_\Omega x \lambda_n^{-N} v_n^{p_{s_1}^*}\left(\frac{x - y_n}{\lambda_n}\right) dx \\
&= (S_{s_1,p})^{-\frac{N}{s_1 p}} |u_n^+|_{p_{s_1}^*}^{p_{s_1}^*} \int_{\frac{\Omega - y_n}{\lambda_n}} (\lambda_n z + y_n) v_n^{p_{s_1}^*}(z)\, dz.
\end{aligned}$$

Applying dominated convergence theorem via (6.6.43) and Theorem 6.3.1 to the last line of the above expression we obtain

$$\tau(u_n) \to y \int_{\mathbb{R}^N} |v|^{p_{s_1}^*}\, dz = y \in \bar{\Omega},$$

which is a contradiction to the assumption. Hence the lemma follows.

□

Using Lemma 6.6.8, we can find a non-negative radial function $v_\theta \in N_{\theta,B_\delta}$ such that $I_\theta(v_\theta) = I_{\theta,B_\delta}(v_\theta) = n_\theta$. Let us define a map $\gamma : \Omega_\delta^- \to I_\theta^{n_\theta}$ by $\gamma(y) = \psi_y$, where ψ_y is defined as follows

$$\psi_y(x) = \begin{cases} v_\theta(x - y), & \text{if} \quad x \in B_\delta(y), \\ 0, & \text{otherwise.} \end{cases} \tag{6.6.44}$$

Now, for each $y \in \Omega_\delta^-$ we have,

$$
\begin{aligned}
(\tau \circ \gamma)(y) = \tau \circ \psi_y &= (S_{s_1,p})^{-\frac{N}{s_1 p}} \int_{B_\delta(y)} v_\theta(x-y)^{p^*_{s_1}} x\, dx \\
&= (S_{s_1,p})^{-\frac{N}{s_1 p}} \int_{B_\delta(0)} v_\theta(z)^{p^*_{s_1}} (z+y) dz \\
&= y (S_{s_1,p})^{-\frac{N}{s_1 p}} \int_{B_\delta(0)} v_\theta(z)^{p^*_{s_1}} dz + (S_{s_1,p})^{-\frac{N}{s_1 p}} \int_{B_\delta(0)} v_\theta(z)^{p^*_{s_1}} z\, dz.
\end{aligned} \tag{6.6.45}
$$

Further, using the fact that v_θ is radial, it is easy to check that

$$
\int_{B_\delta(0)} v_\theta(z)^{p^*_{s_1}} z\, dz = 0. \tag{6.6.46}
$$

Substitution of (6.6.46) into (6.6.45) yields

$$
(\tau \circ \gamma)(y) = \alpha_\theta y, \tag{6.6.47}
$$

where, $\alpha_\theta = (S_{s_1,p})^{-\frac{N}{s_1 p}} \int_{B_\delta(0)} v_\theta(z)^{p^*_{s_1}} dz.$

Lemma 6.6.10. *$\alpha_\theta \to 1$ if $\theta \to 0$.*

Proof. From Lemma 6.6.8, we observe that

$$
n_\theta = I_{\theta, B_\delta}(v_\theta) = \frac{1}{p}\|v_\theta\|^p_{0,s_1,p} + \frac{1}{q}\|v_\theta\|^q_{0,s_2,q} - \frac{\theta}{r}\int_{B_\delta} |v_\theta|^r - \frac{1}{p^*_{s_1}} \int_{B_\delta} |v_\theta|^{p^*_{s_1}} \le \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}}
$$

and

$$
\|v_\theta\|^p_{0,s_1,p} + \|v_\theta\|^q_{0,s_2,q} - \theta |v_\theta|^r_r - |v_\theta|^{p^*_{s_1}}_{p^*_{s_1}} = 0.
$$

By similar argument as in Lemma 6.6.9 we have, $|v_\theta|^{p^*_{s_1}}_{p^*_{s_1}} \to (S_{s_1,p})^{\frac{N}{s_1 p}}$ as $\theta \to 0$. Hence the lemma follows.

∎

Let us define a map $H_\theta : [0,1] \times (N_\theta \cap I_\theta^{n_\theta}) \to \mathbb{R}^N$ by

$$
H_\theta(t,u) = \left(t + \frac{1-t}{\alpha_\theta} \right) \tau(u). \tag{6.6.48}
$$

Lemma 6.6.11. *There exists $\theta_* > 0$ such that for any $\theta \in (0, \theta_*)$, it holds*

$$
H_\theta([0,1] \times (N_\theta \cap I_\theta^{n_\theta})) \subset \Omega_\delta^+.
$$

Proof. We will prove it by method of contradiction. Suppose there exists sequence $\theta_n \to 0$ and $(t_n, u_n) \in [0,1] \times (N_\theta \cap I_\theta^{n_\theta})$ such that

$$H_{\theta_n}(t_n, u_n) \notin \Omega_\delta^+ \quad \forall n \in \mathbb{N}. \tag{6.6.49}$$

As $t_n \in [0,1]$, up to a subsequence, we assume $t_n \to t_0 \in [0,1]$. Moreover, by Lemma 6.6.10 and from the proof of the Lemma 6.6.9, we have $\alpha_{\theta_n} \to 1$ and $\tau(u_n) \to y \in \overline{\Omega}$. Hence, $H_{\theta_n}(t_n, u_n) = \Big(t_n + \frac{1-t_n}{\alpha_{\theta_n}}\Big)\tau(u_n) \to y \in \overline{\Omega}$. This is a contradiction to (6.6.49). Hence the lemma follows. □

Lemma 6.6.12. *Let u_θ be a critical point of I_θ on N_θ. Then, u_θ is a critical point of I_θ on $X_{0,s_1,p}(\Omega)$.*

Proof. Suppose, u_θ is a critical point of I_θ on N_θ. Therefore,

$$\langle I_\theta'(u_\theta), u_\theta \rangle = 0. \tag{6.6.50}$$

Using Lagrange multiplier method, there exists $\mu \in \mathbb{R}$ such that

$$I_\theta'(u_\theta) = \mu J_\theta'(u_\theta), \tag{6.6.51}$$

where

$$J_\theta(u) := \|u\|_{0,s_1,p}^p + \|u\|_{0,s_2,q}^q - \theta |u^+|_r^r - |u^+|_{p_{s_1}^*}^{p_{s_1}^*}. \tag{6.6.52}$$

Therefore,

$$\mu \, \langle J_\theta'(u_\theta), u_\theta \rangle = 0. \tag{6.6.53}$$

Observe that,

$$\begin{aligned} \langle J_\theta'(u_\theta), u_\theta \rangle &= p\|u_\theta\|_{0,s_1,p}^p + q\|u_\theta\|_{0,s_2,q}^q - r\theta |u_\theta^+|_r^r - p_{s_1}^* |u_\theta^+|_{p_{s_1}^*}^{p_{s_1}^*} \\ &= (p-r)\|u_\theta\|_{0,s_1,p}^p + (q-r)\|u_\theta\|_{0,s_2,q}^q - (p_{s_1}^* - r)|u_\theta^+|_{p_{s_1}^*}^{p_{s_1}^*} < 0. \end{aligned} \tag{6.6.54}$$

Consequently, from (6.6.53) we conclude that $\mu = 0$ and therefore by (6.6.51) we have $I_\theta'(u_\theta) = 0$ and this completes the proof. □

In the next two lemmas, we denote $I_{N_\theta} := I_\theta|_{N_\theta}$(restriction of I_θ on N_θ.)

Lemma 6.6.13. *Assume* (6.6.1) *holds and* $\theta > 0$ *is fixed. Then for any sequence* $\{u_n\} \subset N_\theta$ *such that*

$$I_\theta(u_n) \to c < \frac{s_1}{N}\Big(S_{s_1,p}\Big)^{\frac{N}{s_1 p}}, \quad I'_{N_\theta}(u_n) \to 0,$$

there exists $u \in N_\theta$ *such that up to a subsequence,* $u_n \to u$ *as* $n \to \infty$.

Proof. From the given assumption, we get there exists a sequence $\{\mu_n\} \subset \mathbb{R}$ such that

$$\|I'_\theta(u_n) - \mu_n J'_\theta(u_n)\| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence,

$$I'_\theta(u_n) = \mu_n J'_\theta(u_n) + o(1). \tag{6.6.55}$$

By (6.6.54), we have $\langle J'_\theta(u_n), u_n\rangle < 0$ for every $n \geq 1$. Note that, up to a subsequence, $\langle J'_\theta(u_n), u_n\rangle \to l < 0$ as $n \to \infty$. Otherwise, if $\langle J'_\theta(u_n), u_n\rangle \to 0$ as $n \to \infty$, then

$$\|u_n\|_{0,s_1,p} \to 0, \ \|u_n\|_{0,s_2,q} \to 0, \ |u_n^+|_{p^*_{s_1}} \to 0 \quad \text{as} \quad n \to \infty.$$

On the other hand, as $u_n \in N_\theta$ using Sobolev embedding theorem, there exists $C > 0$ such that

$$\|u_n\|^p_{0,s_1,p} \leq \|u_n\|^p_{0,s_1,p} + \|u_n\|^q_{0,s_2,q} = \theta|u_n^+|^r_r + |u_n|^{p^*_{s_1}}_{p^*_{s_1}} \leq C(\theta\|u_n\|^r_{0,s_1,p} + \|u_n\|^{p^*_{s_1}}_{0,s_1,p}).$$

This in turn implies

$$1 \leq C(\theta\|u_n\|^{r-p}_{0,s_1,p} + \|u_n\|^{p^*_{s_1}-p}_{0,s_1,p}),$$

which is a contradiction. Hence, up to a subsequence, we have,

$$\langle J'_\theta(u_n), u_n\rangle \to l < 0 \quad \text{as} \quad n \to \infty.$$

Moreover, $u_n \in N_\theta$ for all $n \geq 1$, implies $\langle I'_\theta(u_n), u_n\rangle = 0$ for all $n \geq 1$. As a consequence, from (6.6.55) we have, $\mu_n \to 0$ as $n \to \infty$. Therefore,

$I'_\theta(u_n) \to 0$ as $n \to \infty$. As $I_\theta(u_n) \to c < \frac{s_1}{N}\Big(S_{s_1,p}\Big)^{\frac{N}{s_1 p}}$, using Lemma 6.5.3 we conclude the result. □

Define,

$$\theta_{**} = \min\{\theta^*, \theta_*\}, \tag{6.6.56}$$

where θ^* is same as in Lemma 6.6.9 and θ_* is as found in Lemma 6.6.11 .

Lemma 6.6.14. *Assume* (6.6.1) *holds and* $\theta \in (0, \theta_{**})$*, where* θ_{**} *is as defined in* (6.6.56)*. Then*

$$cat_{I^{n_\theta}_{N_\theta}}(I^{n_\theta}_{N_\theta}) \geq cat_\Omega(\Omega).$$

This follows exactly by the same argument as in [89, Lemma 4.4]. For the convenience of the reader, we briefly sketch the proof below.

Proof. Let, $cat_{I^{n_\theta}_{N_\theta}}(I^{n_\theta}_{N_\theta}) = n$. By the definition of $cat_{I^{n_\theta}_{N_\theta}}(I^{n_\theta}_{N_\theta})$, we can write $I^{n_\theta}_{N_\theta} = A_1 \cup A_2 \cup \cdots \cup A_n$ where $\{A_j\}_{j=1}^n$ are closed and contractible in $I^{n_\theta}_{N_\theta}$, that is, there exists $h_j \in C([0,1] \times A_j; I^{n_\theta}_{N_\theta})$ such that

$$h_j(0, u) = u, \quad h_j(1, u) = u_0 \quad \forall \quad u \in A_j,$$

where $u_0 \in A_j$ is fixed. Let γ be as defined in (6.6.44). Define, $B_j := \gamma^{-1}(A_j), 1 \leq j \leq n$. Then, B_j is closed for $1 \leq j \leq n$ and $\cup_{j=1}^n B_j = \Omega_\delta^-$. Set, $g_j : [0,1] \times B_j \to \Omega_\delta^+$ by

$$g_j(t, y) = H_\theta(t, h_j(t, \gamma(y))), \quad \text{for} \quad \theta \in (0, \theta_{**}),$$

where H_θ is as defined in (6.6.48). Therefore,

$$g_j(0, y) = H_\theta(0, h_j(0, \gamma(y))) = \frac{\tau(h_j(0, \gamma(y)))}{\alpha_\theta} = \frac{(\tau \circ \gamma)(y)}{\alpha_\theta} = \frac{\alpha_\theta y}{\alpha_\theta} = y \ \forall \, y \in B_j,$$

here we have have used (6.6.47). Further,

$$g_j(1, y) = H_\theta(1, h_j(1, \gamma(y))) = \tau(h_j(1, \gamma(y))) = \tau(u_0) \in \Omega_\delta^+,$$

which follows from Lemma 6.6.9. Therefore, the sets $\{B_j\}_{j=1}^n$ are contractible in Ω_δ^+. Hence,

$$cat_\Omega(\Omega) = cat_{\Omega_\delta^+}(\Omega_\delta^-) \le n.$$

This proves the lemma. □

Proof of Theorem 6.1.4: Using Lemma 6.5.3 and Lemma 6.6.8, we have for all $\theta > 0$,

$$c_\theta,\, n_\theta < \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}}.$$

By Lemma 6.6.13, I_{N_θ} satisfies the $(\mathrm{PS})_c$ condition for all $c \in \left(0, \frac{s_1}{N}(S_{s_1,p})^{\frac{N}{s_1 p}}\right)$. Hence, by Lemma 6.6.14, a standard deformation argument implies that, for $\theta \in (0, \theta_*)$, $I_{N_\theta}^{n_\theta}$ contains at least $cat_\Omega(\Omega)$ critical points of the restriction of I_θ on N_θ. Now, Lemma 6.6.12 implies that I_θ has at least $cat_\Omega(\Omega)$ critical points on $X_{0,s_1,p}(\Omega)$. Now, following the same argument as in Theorem 6.1.2, it follows (P) has at least $cat_\Omega(\Omega)$ nontrivial nonnegative solutions.

6.7 Appendix

Here we first recall the classical deformation lemma from [4, Lemma 1.3].

Lemma 6.7.1. *Let $J \in C^1(X, \mathbb{R})$ satisfy (PS)-condition. If $c \in \mathbb{R}$ and N is any neighborhood of $K_c = \{u \in X : J(u) = c,\ J'(u) = 0\}$, then there exists $\eta(t, x) \equiv \eta_t(x) \in C([0, 1] \times X, X)$ and constants $0 < \varepsilon < \bar{\varepsilon}$ such that*

(1) $\eta_0(x) = x$ for all $x \in X$.

(2) $\eta_t(x) = x$ for all $x \in J^{-1}[c - \bar{\varepsilon}, c + \bar{\varepsilon}]$.

(3) $\eta_t(x)$ is a homeomorphism of X onto X for all $t \in [0, 1]$.

(4) $J(\eta_t(x)) \le J(x)$ for all $x \in X, t \in [0, 1]$.

(5) $\eta_t(A_{c+\varepsilon} - N) \subset A_{c-\varepsilon}$ where $A_c = \{x \in X : J(x) \le c\}$ for any $c \in \mathbb{R}$.

(6) *If* $K_c = \emptyset, \eta_t(A_{c+\varepsilon}) \subset A_{c-\varepsilon}$.

(7) *If* J *is even,* η_t *is odd in* x.

Note that the above lemma is also true if J satisfies $(\mathrm{PS})_c$ condition for $c < c_0$ for some $c_0 \in \mathbb{R}$. Next, recall the general version of Mountain Pass Lemma (see [7]).

Lemma 6.7.2. *Let* X *be a Banach space. Let* $I \in C^1(X, \mathbb{R})$. *Let us assume for some* $\beta, \rho > 0$, *we have,*

(i) $I(u) > \beta$ *for all* $u \in X$ *with* $\|u\|_X = \rho$.

(ii) $I(0) = 0$ *and* $I(v_0) < \beta$ *for some* $v_0 \in X$ *with* $\|v\|_X > \rho$.

Then there exists a sequence $\{u_n\} \subset X$ *such that* $I(u_n) \to \alpha$ *and* $I'(u_n) \to 0$ *in* X' *as* $n \to \infty$, *where* α *is given by:*

$$\alpha := \inf_{u \in X \setminus \{0\}} \max_{t \geq 0} I(tu).$$

The next lemma is regarding the elementary properties of Krasnoselskii genus.

Lemma 6.7.3. *Let* $A, B \in \Sigma$. *Then,*

(1) *if there exists* $f \in C(A, B)$, *odd, then* $\gamma(A) \leq \gamma(B)$.

(2) *if* $A \subset B$, *then* $\gamma(A) \leq \gamma(B)$.

(3) *if there exists an odd homeomorphism between* A *and* B, *then* $\gamma(A) = \gamma(B)$.

(4) *if* S^{N-1} *denotes the unit sphere in* $\mathbb{R}^N$, *then* $\gamma(S^{N-1}) = N$.

(5) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$,

(6) *If* $\gamma(A) < \infty$, *then* $\gamma(\overline{A \cup B}) \geq \gamma(A) - \gamma(B)$.

(7) If A is compact, then $\gamma(A) < \infty$ and there exists $\delta > 0$ such that $\gamma(A) = \gamma(N_\delta(A))$ where $N_\delta(A) = \{x \in X : d(x, A) \leq \delta\}$.

(8) If X_0 is a subspace of X with codimension k and $\gamma(A) > k$, then $A \cap X_0 \neq \emptyset$.

Proof. See [4, Lemma 1.2] . □

Remark 6.7.4. *It's easy to observe that if A contains finitely many antipodal points u_i, $-u_i$ $u_i \neq 0$, then $\gamma(A) = 1$.*

Conclusion: In this chapter, we have studied the existence of multiple nontrivial solutions of (p, q) fractional Laplacian equations involving concave-critical type nonlinearities and existence of nonnegative solutions when nonlinearities is of convex-critical type.

There are two major difficulties which we had faced in obtaining the results, first to get the right function space to look for the solution, where we used Besov-Sobolev embedding to obtain Lemma 6.2.4 and secondly, one variant of Concentration Compactness result which is Lemma 6.3.2. (mentioned in Remark 6.3.3). Nobility of our work lies here.

Chapter 7

Equations involving fractional Laplacian with critical and supercritical exponents

The aim of this chapter is to study the following problem

$$\begin{cases} (-\Delta)^s u = u^p - u^q \quad \text{in} \quad \mathbb{R}^N, \\ \qquad u \in H^s(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N), \\ \qquad u > 0 \quad \text{in} \quad \mathbb{R}^N, \end{cases} \tag{7.0.1}$$

and

$$\begin{cases} (-\Delta)^s u = u^p - u^q \quad \text{in} \quad \Omega, \\ \qquad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \\ \qquad u > 0 \quad \text{in} \quad \Omega, \\ \qquad u \in H^s(\Omega) \cap L^{q+1}(\Omega), \end{cases} \tag{7.0.2}$$

where $s \in (0,1)$ is fixed, $(-\Delta)^s$ denotes the fractional Laplace operator defined, up to a normalization factors,

$$-(-\Delta)^s u(x) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{N+2s}} dy, \quad x \in \mathbb{R}^N. \tag{7.0.3}$$

In (7.0.1) and (7.0.2), $q > p \geq 2^* - 1 = \frac{N+2s}{N-2s}$ and $N > 2s$. In (7.0.2), Ω is a bounded subset of $\mathbb{R}^N$ with smooth boundary.

7.1 Preliminaries: Schauder type estimates

Recalling Section 2.5, we note that for $u \in \dot{H}^s(\mathbb{R}^N)$ to be a solution of (7.0.1),we define $w := E_s(u)$ be its $s-$ harmonic extension to the upper half space $\mathbb{R}^{N+1}_+$, that is, there is a solution to the following problem:

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla w) = 0 & \text{in} \quad \mathbb{R}^{N+1}_+, \\ w = u & \text{on} \quad \mathbb{R}^N \times \{y = 0\}. \end{cases} \tag{7.1.1}$$

Hence, (7.1.1) can be rewritten as:

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla w) = 0 \quad \text{in} \quad \mathbb{R}^{N+1}, \\ \qquad \dfrac{\partial w}{\partial \nu^{2s}} = w^p(.,0) - w^q(.,0) \quad \text{on} \quad \mathbb{R}^N. \end{cases} \tag{7.1.2}$$

A function $w \in X^{2s}(\mathbb{R}^{N+1}_+)$ is said to be a weak solution to (7.1.2) if for all $\varphi \in X^{2s}(\mathbb{R}^{N+1}_+)$, we have

$$k_{2s} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla w \nabla \varphi \, dxdy = \int_{\mathbb{R}^N} w^p(x,0)\varphi(x,0)\, dx - \int_{\mathbb{R}^N} w^q(x,0)\varphi(x,0)\, dx. \tag{7.1.3}$$

Note that for any weak solution $w \in X^{2s}(\mathbb{R}^{N+1}_+)$ to (7.1.2), the function $u := \operatorname{Tr}(w) = w(.,0) \in \dot{H}^s(\mathbb{R}^N)$ is a weak solution to (7.0.1).

Next, we recall Schauder estimate for the nonlocal equation by Ros-Oton and Serra [73].

Theorem 7.1.1. *[Ros-Oton and Serra, [73]] Let $s \in (0,1)$ and u be any bounded weak solution to*

$$(-\Delta)^s u = f \quad \textit{in} \quad B_1(0).$$

Then,

(a) If $u \in L^\infty(\mathbb{R}^N)$ *and* $f \in L^\infty(B_1(0))$,

$$\|u\|_{C^{2s}(B_{\frac{1}{2}}(0))} \leq C(|u|_{L^\infty(\mathbb{R}^N)} + |f|_{L^\infty(B_1(0))}) \quad \text{if} \quad s \neq \frac{1}{2}$$

and

$$\|u\|_{C^{2s-\varepsilon}(B_{\frac{1}{2}}(0))} \leq C(|u|_{L^\infty(\mathbb{R}^N)} + |f|_{L^\infty(B_1(0))}) \quad \text{if} \quad s = \frac{1}{2},$$

for all $\varepsilon > 0$.

(b) If $f \in C^\alpha(B_1(0))$ *and* $u \in C^\alpha(\mathbb{R}^N)$ *for some* $\alpha > 0$, *then*

$$\|u\|_{C^{\alpha+2s}(B_{\frac{1}{2}}(0))} \leq C(\|u\|_{C^\alpha(\mathbb{R}^N)} + \|f\|_{C^\alpha(B_1(0))}),$$

whenever $\alpha + 2s$ *is not an integer. The constant* C *depends only on* $N, s, \alpha, \varepsilon$.

We conclude this section by recalling some weighted embedding results from Tan and Xiong [80]. For this, we introduce the following notations

$$Q_R = B_R \times [0, R) \subset \mathbb{R}^{N+1},$$

where B_R is a ball in $\mathbb{R}^N$ with radius R and centered at origin. Note that, $B_R \times \{0\} \subset Q_R$. We define,

$$H(Q_R, y^{1-2s}) := \left\{ U \in H^1(Q_R) : \int_{Q_R} y^{1-2s}(U^2 + |\nabla U|^2) dxdy < \infty \right\}$$

and $X_0^{2s}(Q_R)$ is the closure of $C_0^\infty(Q_R)$ with respect to the norm

$$\|w\|_{X_0^{2s}(Q_R)} = \left(\int_{Q_R} y^{1-2s} |\nabla w|^2 dxdy \right)^{\frac{1}{2}}.$$

We note that, $s \in (0,1)$ implies the weight y^{1-2s} belongs to the Muckenhoupt class A_2 (see [66]) which consists of all non-negative functions w on $\mathbb{R}^{N+1}$ satisfying for some constant C, the estimate

$$\sup_B \left(\frac{1}{|B|} \int_B w dx \right) \left(\frac{1}{|B|} \int_B w^{-1} dx \right) \leq C,$$

where the supremum is taken over all balls B in $\mathbb{R}^{N+1}$.

Lemma 7.1.2. *Let $f \in X_0^{2s}(Q_R)$. Then there exists constant C and $\delta > 0$ depending only on N and s such that for any $1 \leq k \leq \frac{n+1}{n} + \delta$,*

$$\left(\int_{Q_R} y^{1-2s} |f|^{2k} dxdy \right)^{\frac{1}{2k}} \leq C(R) \left(\int_{Q_R} y^{1-2s} |\nabla f|^2 dxdy \right)^{\frac{1}{2}}.$$

Proof. It is known from [80, Lemma 2.1] that the lemma holds for $f \in C_c^1(Q_R)$ (also see [42]). For general f, the lemma can be easily proved applying density argument and Fatou's lemma. □

Lemma 7.1.3. *Let $f \in X_0^{2s}(Q_R)$. Then there exists a positive constant δ depending only on N and s such that*

$$\int_{B_R \times \{y=0\}} |f|^2 dx \leq \varepsilon \int_{Q_R} y^{1-2s} |\nabla f|^2 dxdy + \frac{C(R)}{\varepsilon^\delta} \int_{Q_R} y^{1-2s} |f|^2 dxdy,$$

for any $\varepsilon > 0$.

Proof. If $f \in C_c^1(Q_R)$, then the lemma holds (see [80, Lemma 2.3]). For $f \in X_0^{2s}(Q_R)$, there exists $f_n \in C_0^\infty(Q_R)$ such that $f_n \to f$ in $\|.\|_{X_0^{2s}(Q_R)}$ and for f_n, we have

$$\int_{B_R \times \{y=0\}} |f_n|^2 dx \leq \varepsilon \int_{Q_R} y^{1-2s} |\nabla f_n|^2 dxdy + \frac{C(R)}{\varepsilon^\delta} \int_{Q_R} y^{1-2s} |f_n|^2 dxdy, \tag{7.1.4}$$

for any $\varepsilon > 0$. Clearly the 1st integral on RHS converges to $\int_{Q_R} y^{1-2s} |\nabla f|^2 dxdy$. Thanks to Lemma 7.1.2, it follows that the embedding $X_0^{2s}(Q_R) \hookrightarrow L^2(Q_R, y^{1-2s})$ is continuous. Therefore, we can also pass to the limit in the 2nd integral of the RHS. On the other hand, using the trace embedding result, we can also pass to the limit on LHS. Hence, the lemma follows. □

In the next section, we will recall some basic definitions.

7.2 Definitions

Definition 7.2.1. *(**Weak solution**) We say that $u \in \dot{H}^s(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$ is a weak solution of Eq.* (7.0.1)*, if $u > 0$ in $\mathbb{R}^N$ and for every $\varphi \in \dot{H}^s(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} dxdy = \int_{\mathbb{R}^N} u^p \varphi \; dx - \int_{\mathbb{R}^N} u^q \varphi \; dx$$

or equivalently,

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi \; dx = \int_{\mathbb{R}^N} u^p \varphi \; dx - \int_{\mathbb{R}^N} u^q \varphi \; dx.$$

Similarly, when Ω is a bounded domain, we say $u \in X_0 \cap L^{q+1}(\Omega)$ is a weak solution of Eq. (7.0.2) *if $u > 0$ in Ω and for every $\varphi \in X_0$, the above integral expression holds.*

Definition 7.2.2. *(**Classical solution**) A positive function $u \in C^{2s+\alpha}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, \frac{dx}{(1+|x|)^{N+2s}})$ is said to be a classical solution of*

$$(-\Delta)^s u = f(u) \quad in \quad \mathbb{R}^N, \tag{7.2.1}$$

if $(-\Delta)^s u$ can be written as (7.0.3) *and* (7.2.1) *is satisfied pointwise in all $\mathbb{R}^N$.*

7.3 Main results

We turn now to a brief description of the main theorems presented below.

Theorem 7.3.1. *Let $s \in (0,1)$, $p \geq 2^* - 1$ and $q > (p-1)\frac{N}{2s} - 1$. If u is any weak solution of Eq.*(7.0.1) *or Eq.*(7.0.2)*, then $u \in L^\infty(\mathbb{R}^N)$. Moreover, if $\Omega = \mathbb{R}^N$, then there exist two positive constants C_1, C_2 such that*

$$C_1|x|^{-(N-2s)} \leq u(x) \leq C_2|x|^{-(N-2s)}, \quad |x| > R_0, \tag{7.3.1}$$

for some $R_0 > 0$.

Theorem 7.3.2. *Let s, p, q are as in Theorem 7.3.1.*
(i) If u is a weak solution of Eq. (7.0.1), then $u \in C^{\infty}(\mathbb{R}^N)$ if both p and q are integer and $u \in C^{2ks+2s}(\mathbb{R}^N)$, where k is the largest integer satisfying $\lfloor 2ks \rfloor < p$ if $p \notin \mathbb{N}$ and $\lfloor 2ks \rfloor < q$ if $p \in \mathbb{N}$ but $q \notin \mathbb{N}$, where $\lfloor 2ks \rfloor$ denotes the greatest integer less than equal to $2ks$.
(ii) If u is a weak solution of Eq.(7.0.2), then $u \in C^s(\mathbb{R}^N) \cap C^{2s+\alpha}_{loc}(\Omega)$, for some $\alpha \in (0,1)$.

Theorem 7.3.3. *Let s, p, q are as in Theorem 7.3.1. If u is a solution of Eq.(7.0.1), then*

$$|\nabla u(x)| \leq C|x|^{-(N-2s+1)}, \quad |x| > R', \tag{7.3.2}$$

for some positive constants C and R'.

Theorem 7.3.4. *Let $s \in (0,1)$ and $p = 2^* - 1$ and $q > p$. Then (7.0.1) does not have any solution.*

We define the functional

$$F(v, \Omega) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy + \frac{1}{q+1}\int_{\Omega}|v|^{q+1}dx. \tag{7.3.3}$$

Define,

$$\mathcal{K} := \inf\left\{F(v, \mathbb{R}^N) : v \in \dot{H}^s(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N), \int_{\mathbb{R}^N}|v|^{p+1}dx = 1\right\}. \tag{7.3.4}$$

Theorem 7.3.5. *Let $s \in (0,1)$ and $q > p > 2^* - 1$. Then $\mathcal{K}$ in (7.3.4) is achieved by a radially decreasing function $u \in \dot{H}^s(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$ and Eq.(7.0.1) admits a nonnegative solution. Furthermore, if $q > (p-1)\frac{N}{2s} - 1$, then Eq. (7.0.1) admits a positive solution.*

When Ω is a smooth bounded domain, we define

$$S_{\Omega} := \inf\left\{F(v, \Omega) : v \in X_0(\Omega) \cap L^{q+1}(\Omega), \int_{\Omega}|v|^{p+1}dx = 1\right\}. \tag{7.3.5}$$

Theorem 7.3.6. *Let $s \in (0,1)$ and $q > p \geq 2^* - 1$. Then $\mathcal{S}_\Omega$ in (7.3.5) is achieved by a function $u \in X_0(\Omega) \cap L^{q+1}(\Omega)$. Furthermore, there exists a constant $\lambda > 0$, such that u satisfies*

$$\begin{cases} (-\Delta)^s u = \lambda |u|^{p-1}u - |u|^{q-1}u \quad in \quad \Omega, \\ \qquad u = 0 \quad in \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{7.3.6}$$

Furthermore, if $p \geq 2^* - 1$ and $q > (p-1)\frac{N}{2s} - 1$, then Eq.(7.3.6) admits a positive solution.

Note that the scaled function $U = \lambda^{\frac{1}{p-1}} u$ satisfies the equation

$$(-\Delta)^s U = U^p - c^* U^q, \quad c^* = \lambda^{-\frac{q-1}{p-1}}. \tag{7.3.7}$$

Few notations:

We use the notation $C^\beta(\mathbb{R}^N)$, with $\beta > 0$ to refer the space $C^{k,\beta'}(\mathbb{R}^N)$, where k is the greatest integer such that $k < \beta$ and $\beta' = \beta - k$. According to this, $[\cdot]_{C^\beta(\mathbb{R}^N)}$ denotes the following seminorm

$$[u]_{C^\beta(\mathbb{R}^N)} = [u]_{C^{k,\beta'}(\mathbb{R}^N)} = \sup_{x,\ y \in \mathbb{R}^N, x \neq y} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^{\beta'}}.$$

Throughout this paper, C denotes the generic constant, which may vary from line to line and $\mathbf{n}$ denotes the unit outward normal.

7.4 Decay estimates and Regularity results

In this section we prove Theorem 7.3.1, Theorem 7.3.2 and Theorem 7.3.3.

Proof of Theorem 7.3.1

Proof. **Case 1:** Suppose $\Omega = \mathbb{R}^N$.

Let u be an arbitrary weak solution of Eq.(7.0.1). We first prove that $u \in L^\infty_{loc}(\mathbb{R}^N)$ by Moser iterative technique (see, for example [52, 80]). From

Section-2, we know that $w(x,y)$, the $s-$harmonic extension of u, is a solution of (7.1.2).

Let B_r denote the ball in $\mathbb{R}^N$ of radius r and centered at origin. We define

$$Q_r = B_r \times [0,r).$$

Set $\bar{w} = w^+ + 1$ and for $L > 1$, define

$$w_L = \begin{cases} \bar{w} & \text{if} \quad w < L \\ 1+L & \text{if} \quad w \geq L. \end{cases}$$

For $t > 1$, we choose the test function φ in (7.1.3) as follows:

$$\varphi(x,y) = \eta^2(x,y)\Big(\bar{w}(x,y)w_L^{2(t-1)}(x,y) - 1\Big), \tag{7.4.1}$$

where $\eta \in C_0^\infty(Q_R)$ with $0 \leq \eta \leq 1$, $\eta = 1$ in Q_r, $0 < r < R \leq 1$ and $|\nabla\eta| \leq \frac{2}{R-r}$. Note that $\varphi \in X^{2s}(\mathbb{R}_+^{N+1})$. Using this test function φ, we obtain from (7.1.3)

$$\begin{aligned} k_{2s}\int_{\mathbb{R}_+^{N+1}} y^{1-2s}\nabla w(x,y)\nabla\left(\eta^2(x,y)\Big(\bar{w}(x,y)w_L^{2(t-1)}(x,y)-1\Big)\right)dxdy \\ = \int_{\mathbb{R}^N}\Big(w^p(x,0)-w^q(x,0)\Big)\eta^2(x,0)\Big(\bar{w}(x,0)w_L^{2(t-1)}(x,0)-1\Big)dx. \end{aligned} \tag{7.4.2}$$

Direct calculation yields

$$\begin{aligned} \nabla\Big(\eta^2(\bar{w}w_L^{2(t-1)}-1)\Big) &= 2\eta(\bar{w}w_L^{2(t-1)}-1)\nabla\eta \\ &\quad + \eta^2 w_L^{2(t-1)}\nabla\bar{w} + 2(t-1)\eta^2\bar{w}w_L^{2(t-1)-1}\nabla w_L. \end{aligned} \tag{7.4.3}$$

Here we observe that on the set $\{w < 0\}$, we have $\varphi = 0$ and $\nabla\varphi = 0$. Thus (7.4.2) remains same if we change the domain of integration to $\{w \geq 0\}$. Therefore, in the support of the integrand $\nabla w = \nabla\bar{w}$. As a result, substituting (7.4.3) into (7.4.2), it follows

$$\begin{aligned} k_{2s}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\Big(& 2\eta(\bar{w}w_L^{2(t-1)}-1)\nabla\eta\nabla\bar{w} \\ &+\eta^2 w_L^{2(t-1))}\nabla\bar{w}\nabla w + 2(t-1)\eta^2 w_L^{2(t-1)-1}\bar{w}\nabla w_L\nabla w\Big)(x,y)dxdy \\ &\leq \int_{\mathbb{R}^N}\eta^2(x,0)w^p(x,0)\bar{w}(x,0)w_L^{2(t-1)}(x,0)dx. \end{aligned}$$

Notice that in the support of the integrand of second integral on the LHS $\nabla\bar{w}=\nabla w$ and in the third integral $w_L=\bar{w}$, $\nabla w_L=\nabla w$. Hence the above expression reduces to

$$\begin{aligned} k_{2s}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\Big(& 2\eta(\bar{w}w_L^{2(t-1)}-1)\nabla\eta\nabla\bar{w} \\ &+\eta^2 w_L^{2(t-1))}|\nabla\bar{w}|^2 + 2(t-1)\eta^2 w_L^{2(t-1)}|\nabla w_L|^2\Big)(x,y)dxdy \\ &\leq \int_{\mathbb{R}^N}\eta^2(x,0)\bar{w}^{p+1}(x,0)w_L^{2(t-1)}(x,0)dx, \end{aligned} \tag{7.4.4}$$

where for the RHS, we have used the fact that $w\leq\bar{w}$.

Using Young's inequality we have,

$$\left|2\eta(\bar{w}w_L^{2(t-1)}-1)\nabla\eta\nabla\bar{w}\right| \leq \frac{1}{2}\eta^2 w_L^{2(t-1)}|\nabla\bar{w}|^2 + 2\bar{w}^2 w_L^{2(t-1)}|\nabla\eta|^2. \tag{7.4.5}$$

Using (7.4.5), from (7.4.4) we obtain,

$$\begin{aligned} \frac{k_{2s}}{2}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\Big(|\nabla\bar{w}|^2+(t-1)|\nabla w_L|^2\Big)\eta^2 w_L^{2(t-1)}(x,y)dxdy \\ \leq 2k_{2s}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\bar{w}^2 w_L^{2(t-1)}|\nabla\eta|^2(x,y)dxdy \\ +\int_{\mathbb{R}^N}\bar{w}^{p+1}w_L^{2(t-1)}\eta^2(x,0)dx. \end{aligned} \tag{7.4.6}$$

As $t > 1$ and $\nabla w_L = 0$ for $w \geq L$, it is not difficult to observe that,

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla(\eta\bar{w}w_L^{t-1})|^2 dxdy \\
&\leq 3\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\Bigg(\bar{w}^2 w_L^{2(t-1)}|\nabla\eta|^2 + \eta^2 w_L^{2(t-1)}|\nabla\bar{w}|^2 \\
&+ (t-1)^2\eta^2 w_L^{2(t-1)}|\nabla w_L|^2\Bigg)dxdy \\
&\leq 3t\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\bar{w}^2 w_L^{2(t-1)}|\nabla\eta|^2 dxdy \\
&+ 3t\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\Bigg(|\nabla\bar{w}|^2 + (t-1)|\nabla w_L|^2\Bigg)\eta^2 w_L^{2(t-1)} dxdy. \qquad (7.4.7)
\end{aligned}$$

Combining (7.4.7) and (7.4.6), we have

$$\begin{aligned}
&k_{2s}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla(\eta\bar{w}w_L^{t-1})|^2 dxdy \\
&\leq 3tk_{2s}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\bar{w}^2 w_L^{2(t-1)}|\nabla\eta|^2 dxdy \\
&+3t\Bigg\{4k_{2s}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\bar{w}^2 w_L^{2(t-1)}|\nabla\eta|^2(x,y)dxdy \\
&\qquad +2\int_{\mathbb{R}^N} \bar{w}^{p+1} w_L^{2(t-1)}\eta^2(x,0)dx\Bigg\}.
\end{aligned} \qquad (7.4.8)$$

For $p \geq 2^* - 1$, choose $\alpha > 1$ as follows:

$$\frac{N}{2s} < \alpha < \frac{q+1}{p-1}. \qquad (7.4.9)$$

Note that for $p = 2^* - 1$ the interval $(\frac{N}{2s}, \frac{q+1}{p-1})$ is always a nonempty set. On the other hand, as $q > (p-1)\frac{N}{2} - 1$, it follows $(\frac{N}{2s}, \frac{q+1}{p-1}) \neq \emptyset$, when $p > 2^* - 1$. From (7.4.9) we have,

$$(p-1)\alpha < q+1 \quad \text{and} \quad 2 < \frac{2\alpha}{\alpha - 1} < 2^*.$$

As $\text{supp}(\eta(\cdot,0)) \subset B_R$ and $w(x,0) = u \in L^{q+1}(\mathbb{R}^N)$, it follows $\bar{w}(\cdot,0) = w^+(x,0)+1 = u+1 \in L^{q+1}(B_1)$. This along with the fact that supp $\eta \subset Q_R$, where $R < 1$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \bar{w}^{p+1} w_L^{2(t-1)} \eta^2(x,0)dx \\
&= \int_{B_1} \bar{w}^{p+1} w_L^{2(t-1)} \eta^2(x,0)dx \\
&= \int_{B_1} |\eta \bar{w} w_L^{(t-1)}(x,0)|^2 \bar{w}^{p-1}(x,0)dx \\
&\leq \left(\int_{B_1} \bar{w}^{\alpha(p-1)}(x,0)dx \right)^{\frac{1}{\alpha}} \left(\int_{B_R} |\eta \bar{w} w_L^{(t-1)}|^{\frac{2\alpha}{\alpha-1}}(x,0)dx \right)^{\frac{\alpha-1}{\alpha}} \\
&\leq C|\eta \bar{w} w_L^{(t-1)}|^2_{L^{\frac{2\alpha}{\alpha-1}}(B_R)}. \qquad (7.4.10)
\end{aligned}$$

By interpolation inequality,

$$|\eta \bar{w} w_L^{(t-1)}|^2_{L^{\frac{2\alpha}{\alpha-1}}(B_R)} \leq |\eta \bar{w} w_L^{(t-1)}|^{2\theta}_{L^2(B_R)} |\eta \bar{w} w_L^{(t-1)}|^{2(1-\theta)}_{L^{2^*}(B_R)}, \qquad (7.4.11)$$

where θ is determined by

$$\frac{\alpha-1}{2\alpha} = \frac{\theta}{2} + \frac{1-\theta}{2^*}. \qquad (7.4.12)$$

Applying Young's inequality, (7.4.11) yields

$$\begin{aligned}
|\eta \bar{w} w_L^{(t-1)}|^2_{L^{\frac{2\alpha}{\alpha-1}}(B_R)} &\leq C(s,\alpha,N)\varepsilon^2 |\eta \bar{w} w_L^{(t-1)}|^2_{L^{2^*}(\mathbb{R}^N)} \\
&\quad + C(\alpha,s,N)\varepsilon^{-\frac{2(1-\theta)}{\theta}} |\eta \bar{w} w_L^{(t-1)}|^2_{L^2(B_R)}. \qquad (7.4.13)
\end{aligned}$$

Therefore, using Sobolev Trace inequality (2.5.3) and the value of θ from (7.4.12), we have

$$\begin{aligned}
|\eta \bar{w} w_L^{(t-1)}|^2_{L^{\frac{2\alpha}{\alpha-1}}(B_R)} &\leq C(s,\alpha,N)\varepsilon^2 \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \left(\eta \bar{w} w_L^{(t-1)} \right)|^2 dxdy \\
&\quad + C(\alpha,s,N)\varepsilon^{-\frac{2N}{2\alpha s-N}} \int_{B_R} |\eta \bar{w} w_L^{(t-1)}(x,0)|^2 dx.
\end{aligned} \qquad (7.4.14)$$

Thanks to Lemma 7.1.3, for $\delta > 0$ we have

$$\begin{aligned}
\int_{B_R} |\eta \bar{w} w_L^{(t-1)}(x,0)|^2 dx &= \int_{B_1} |\eta w w_L^{(t-1)}(x,0)|^2 dx \\
&\leq \delta \int_{Q_1} y^{1-2s} |\nabla \left(\eta \bar{w} w_L^{(t-1)} \right)|^2 dxdy \\
&+ \frac{C}{\delta^\beta} \int_{Q_1} y^{1-2s} |\eta \bar{w} w_L^{(t-1)}|^2 dxdy, \qquad (7.4.15)
\end{aligned}$$

where $\beta = \frac{s'+1}{s'-1}$, with some $1 < s' < \frac{1}{1-s}$. Substituting (7.4.15) in (7.4.14) and then (7.4.14) in (7.4.10) yields

$$\begin{aligned}
&\int_{\mathbb{R}^N} \bar{w}^{p+1} w_L^{2(t-1)} \eta^2(x,0)dx \\
&\leq C(s,\alpha,N)\varepsilon^2 \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \left(\eta \bar{w} w_L^{(t-1)}\right)|^2 dxdy \\
&+ C(\alpha,s,N)\varepsilon^{-\frac{2N}{2\alpha s-N}} \delta \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \left(\eta \bar{w} w_L^{(t-1)}\right)|^2 dxdy \\
&+ C(\alpha,s,N)\varepsilon^{-\frac{2N}{2rs-N}} \frac{1}{\delta^\beta} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\eta \bar{w} w_L^{(t-1)}|^2 dxdy. \qquad (7.4.16)
\end{aligned}$$

Consequently, substituting (7.4.16) in (7.4.8), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla (\eta \bar{w} w_L^{t-1})|^2 dxdy \\
&\leq Ct \int_{\mathbb{R}^{N+1}} y^{1-2s} \bar{w}^2 w_L^{2(t-1)} |\nabla \eta|^2 dxdy \\
&+ Ct \left(\varepsilon^2 + \varepsilon^{-\frac{2N}{2\alpha s-N}} \delta\right) \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \left(\eta \bar{w} w_L^{(t-1)}\right)|^2 dxdy \\
&+ Ct\varepsilon^{-\frac{2N}{2\alpha s-N}} \delta^{-\beta} \int_{\mathbb{R}^{N+1}} y^{1-2s} |\eta \bar{w} w_L^{(t-1)}|^2 dxdy. \qquad (7.4.17)
\end{aligned}$$

Choose

$$\varepsilon = \frac{1}{2\sqrt{Ct}} \quad \text{and} \quad \delta = \frac{\varepsilon^{\frac{2N}{2\alpha s-N}}}{4Ct}.$$

Hence, from (7.4.17), a direct calculation yields

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla (\eta \bar{w} w_L^{t-1})|^2 dxdy \\
&\leq Ct \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \bar{w}^2 w_L^{2(t-1)} |\nabla \eta|^2 dxdy \\
&+ Ct^{\frac{2\alpha s(\beta+1)}{2\alpha s-N}} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\eta \bar{w} w_L^{(t-1)}|^2 dxdy \\
&\leq Ct^\gamma \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \left(\eta^2 + |\nabla \eta|^2\right) \bar{w}^2 w_L^{2(t-1)} dxdy.
\end{aligned} \qquad (7.4.18)$$

where $\gamma = \frac{2\alpha s(\beta+1)}{2\alpha s-N}$. Applying Sobolev inequality (see Lemma 7.1.2), we obtain from (7.4.18)

$$\begin{aligned}
&\left(\int_{Q_1} y^{1-2s}|\eta\bar{w}w_L^{t-1}|^{2\chi}dxdy\right)^{\frac{1}{\chi}}\\
&\leq C\int_{Q_1} y^{1-2s}|\nabla(\eta\bar{w}w_L^{t-1})|^2dxdy\\
&\leq Ct^{\gamma}\int_{Q_1} y^{1-2s}\left(\eta^2+|\nabla\eta|^2\right)\bar{w}^2w_L^{2(t-1)}dxdy,
\end{aligned}$$

where $\chi = \frac{N+1}{N} > 1$. Now using the fact that $0 < r < R < 1$, $\eta = 1$ in Q_r, $|\nabla\eta| \leq \frac{2}{R-r}$ and supp $\eta = Q_R$, we get

$$\left(\int_{Q_r} y^{1-2s}\bar{w}^{2\chi}w_L^{2(t-1)\chi}dxdy\right)^{\frac{1}{\chi}} \leq \frac{Ct^{\gamma}}{(R-r)^2}\int_{Q_R} y^{1-2s}\bar{w}^2w_L^{2(t-1)}dxdy.$$

As $w_L \leq \bar{w}$, the above expression yields,

$$\left(\int_{Q_r} y^{1-2s}w_L^{2t\chi}dxdy\right)^{\frac{1}{\chi}} \leq \frac{Ct^{\gamma}}{(R-r)^2}\int_{Q_R} y^{1-2s}\bar{w}^{2t}dxdy,$$

provided the right-hand side is bounded. Passing to the limit $L \to \infty$ via Fatou's lemma we obtain

$$\left(\int_{Q_r} y^{1-2s}\bar{w}^{2t\chi}dxdy\right)^{\frac{1}{\chi}} \leq \frac{Ct^{\gamma}}{(R-r)^2}\int_{Q_R} y^{1-2s}\bar{w}^{2t}dxdy,$$

that is,

$$\left(\int_{Q_r} y^{1-2s}\bar{w}^{2t\chi}dxdy\right)^{\frac{1}{2\chi t}} \leq \left(\frac{Ct^{\gamma}}{(R-r)^2}\right)^{\frac{1}{2t}}\left(\int_{Q_R} y^{1-2s}\bar{w}^{2t}dxdy\right)^{\frac{1}{2t}}. \quad (7.4.19)$$

Now we iterate the above relation. We take $t_i = \chi^i$ and $r_i = \frac{1}{2} + \frac{1}{2^{i+1}}$ for $i = 0, 1, 2, \ldots$ Note that $t_i = \chi t_{i-1}$, $r_{i-1} - r_i = \frac{1}{2^{i+1}}$. Hence from (7.4.19), with $t = t_i$, $r = r_i$, $R = r_{i-1}$, we have

$$\left(\int_{Q_{r_i}} y^{1-2s}\bar{w}^{2t_{i+1}}dxdy\right)^{\frac{1}{2t_{i+1}}} \leq C^{\frac{i}{\chi^i}}\left(\int_{Q_{r_{i-1}}} y^{1-2s}\bar{w}^{2t_i}dxdy\right)^{\frac{1}{2t_i}}, \quad i = 0, 1, 2, \cdots,$$

where C depend only on N, s, p, q. Hence, by iteration we have

$$\left(\int_{Q_{r_i}} y^{1-2s}\bar{w}^{2t_{i+1}}dxdy\right)^{\frac{1}{2t_{i+1}}} \leq C^{\sum\frac{i}{\chi^i}}\left(\int_{Q_{r_0}} y^{1-2s}\bar{w}^{2t_0}dxdy\right)^{\frac{1}{2t_0}}, \quad i = 0, 1, 2, \cdots,$$

Letting $i \to \infty$ we have

$$\sup_{Q_{\frac{1}{2}}} \bar{w} \leq C|\bar{w}|_{L^2(Q_1, y^{1-2s})},$$

which in turn implies

$$\sup_{B_{\frac{1}{2}}} u = \sup_{B_{\frac{1}{2}}} w^+ \leq \sup_{Q_{\frac{1}{2}}} w^+ \leq C|w|_{L^2(Q_1, y^{1-2s})}.$$

Hence, $u \in L^\infty(B_{\frac{1}{2}}(0))$. Translating the equation, similarly it follows that $u \in L^\infty_{loc}(\mathbb{R}^N)$.

To show the L^∞ bound at infinity, we define the Kelvin transform of u by the function $\tilde{u}$ as follows:

$$\tilde{u}(x) = \frac{1}{|x|^{N-2s}} u(\frac{x}{|x|^2}), \quad x \in \mathbb{R}^N \setminus \{0\}.$$

It follows from [70, Proposition A.1],

$$(-\Delta)^s \tilde{u}(x) = \frac{1}{|x|^{N+2s}} (-\Delta)^s u\bigg(\frac{x}{|x|^2}\bigg). \tag{7.4.20}$$

Thus

$$\begin{aligned}(-\Delta)^s \tilde{u}(x) &= \frac{1}{|x|^{N+2s}} \left(u^p(\frac{x}{|x|^2}) - u^q(\frac{x}{|x|^2}) \right) \\ &= \frac{1}{|x|^{N+2s}} \left(|x|^{p(N-2s)} \tilde{u}^p(x) - |x|^{q(N-2s)} \tilde{u}^q(x) \right).\end{aligned}$$

This implies $\tilde{u}$ satisfies the following equation

$$\begin{cases} (-\Delta)^s \tilde{u} = |x|^{p(N-2s)-(N+2s)} \tilde{u}^p - |x|^{q(N-2s)-(N+2s)} \tilde{u}^q \quad \text{in} \quad \mathbb{R}^N, \\ \tilde{u} \in \dot{H}^s(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N, |x|^{(N-2s)(q+1)-2N}), \\ \tilde{u} > 0 \quad \mathbb{R}^N. \end{cases} \tag{7.4.21}$$

That is,

$$(-\Delta)^s \tilde{u} = f(x, \tilde{u}) \quad \text{in} \quad \mathbb{R}^N, \tag{7.4.22}$$

where

$$f(x, \tilde{u}) := |x|^{p(N-2s)-(N+2s)} \tilde{u}^p - |x|^{q(N-2s)-(N+2s)} \tilde{u}^q. \tag{7.4.23}$$

Since $q > p \geq \frac{N+2s}{N-2s}$, we get $(-\Delta)^s \tilde{u} \leq \tilde{u}^p$ in $(B_1(0))$. Applying the Moser iteration technique along the same line of arguments as above with a suitable modification, we get $\sup_{B_\rho(0)} \tilde{u} \leq C$, for some $\rho > 0$ and C is a positive constant. This in turn implies,

$$u(x) \leq \frac{C}{|x|^{N-2s}}, \quad |x| > R_0, \tag{7.4.24}$$

for some large R_0. Hence, $u \in L^\infty(\mathbb{R}^N)$. As a consequence $\tilde{u} \in L^\infty(\mathbb{R}^N)$ and therefore $(-\Delta)^s \tilde{u} \in L^\infty(B_1(0))$. Applying Theorem 7.1.1, it follows that $\tilde{u} \in C(B_{\frac{1}{2}}(0))$. Thus there exists $C_1 > 0$ such that $\tilde{u} > C_1$ in $(B_{\frac{1}{2}}(0))$, which in turn implies $u(x) > \frac{C_1}{|x|^{N-2s}}$, for $|x| > 2$. This along with (7.4.24), yields (7.3.1) .

Case 2: Ω is a bounded domain.

Arguing along the same line with minor modifications, it can be shown that $u \in L^\infty(\Omega)$. Therefore the conclusion follows as $u = 0$ in $\mathbb{R}^N \setminus \Omega$. □

Proof of Theorem 7.3.2:

Proof. (i) From Theorem 7.3.1, we know any solution u of Eq.(7.0.1) is in $L^\infty(\mathbb{R}^N)$. Therefore, we have

$$(-\Delta)^s u = f(u), \quad f(u) := u^p - u^q \in L^\infty(\mathbb{R}^N). \tag{7.4.25}$$

As a result, applying Theorem 7.1.1(a) , we obtain

$$\begin{aligned} \|u\|_{C^{2s}(B_{\frac{1}{2}}(0))} &\leq C(|u|_{L^\infty(\mathbb{R}^N)} + |f(u)|_{L^\infty(B_1(0))}) \\ &\leq C(|u|_{L^\infty(\mathbb{R}^N)} + |f(u)|_{L^\infty(\mathbb{R}^N)}) \quad \text{if} \quad s \neq \frac{1}{2}, \end{aligned} \tag{7.4.26}$$

$$\begin{aligned} \|u\|_{C^{2s-\varepsilon}(B_{\frac{1}{2}}(0))} &\leq C(|u|_{L^\infty(\mathbb{R}^N)} + |f(u)|_{L^\infty(B_1(0))}) \\ &\leq C(|u|_{L^\infty(\mathbb{R}^N)} + |f(u)|_{L^\infty(\mathbb{R}^N)}) \quad \text{if} \quad s = \frac{1}{2}, \end{aligned} \tag{7.4.27}$$

for all $\varepsilon > 0$. Here the constants C are independent of u, but may depend on radius $\frac{1}{2}$ and centre 0. Since the equation is invariant under translation, translating the equation, we obtain

$$\begin{aligned} \|u\|_{C^{2s}(B_{\frac{1}{2}}(y))} &\leq C(|u|_{L^\infty(\mathbb{R}^N)} + |f(u)|_{L^\infty(\mathbb{R}^N)}) \\ &\leq C(1 + |u|_{L^\infty(\mathbb{R}^N)})^q \quad \text{when} \quad s \neq \frac{1}{2}, \end{aligned} \tag{7.4.28}$$

$$\|u\|_{C^{2s-\varepsilon}(B_{\frac{1}{2}}(y))} \leq C(1 + |u|_{L^\infty(\mathbb{R}^N)})^q \quad \text{when} \quad s = \frac{1}{2}, \tag{7.4.29}$$

Note that in (7.4.28) and (7.4.29) constants C are same as in (7.4.26) and (7.4.27) respectively. Thus, in (7.4.28) and (7.4.29) constants do not depend on y. This implies $u \in C^{2s}(\mathbb{R}^N)$ when $s \neq \frac{1}{2}$ and in $C^{2s-\varepsilon}(\mathbb{R}^N)$, when $s = \frac{1}{2}$. Hence, $f(u) \in C^{2s}(\mathbb{R}^N)$ when $s \neq \frac{1}{2}$ and in $C^{2s-\varepsilon}(\mathbb{R}^N)$, when $s = \frac{1}{2}$. Therefore, applying Theorem 7.1.1(b), we have

$$\begin{aligned} \|u\|_{C^{4s}(B_{\frac{1}{2}}(0))} &\leq C(\|u\|_{C^{2s}(\mathbb{R}^N)} + \|f(u)\|_{C^{2s}(B_1(0))}) \\ &\leq C(\|u\|_{C^{2s}(\mathbb{R}^N)} + \|f(u)\|_{C^{2s}(\mathbb{R}^N)}) \\ &\leq C(1 + |u|_{L^\infty(\mathbb{R}^N)})^{2q} \quad \text{if} \quad s \neq \frac{1}{4}, \frac{1}{2}, \frac{3}{4}. \end{aligned} \tag{7.4.30}$$

Similarly,

$$\begin{aligned} \|u\|_{C^{4s-\varepsilon}(B_{\frac{1}{2}}(0))} &\leq C(\|u\|_{C^{2s-\varepsilon}(\mathbb{R}^N)} + \|f(u)\|_{C^{2s-\varepsilon}(B_1(0))}) \\ &\leq C(1 + |u|_{L^\infty(\mathbb{R}^N)})^{2q} \quad \text{if} \quad s = \frac{1}{2} \text{ and } 4s - \varepsilon \notin \mathbb{N}. \end{aligned} \tag{7.4.31}$$

Arguing as before, we can show that $u \in C^{4s}(\mathbb{R}^N)$ when $s \neq \frac{1}{2}$ and in $C^{4s-\varepsilon}(\mathbb{R}^N)$, when $s = \frac{1}{2}$. We can repeat this argument to improve the regularity $C^\infty(\mathbb{R}^N)$ if both p and q are integer and $C^{2ks+2s}(\mathbb{R}^N)$, where k is the largest integer satisfying $\lfloor 2ks \rfloor < p$ if $p \notin \mathbb{N}$ and $\lfloor 2ks \rfloor < q$ if $p \in \mathbb{N}$ but $q \notin \mathbb{N}$, where $\lfloor 2ks \rfloor$ denotes the greatest integer less than equal to $2ks$.

(ii) Suppose, u is an arbitrary solution of (7.0.2), then by Theorem 7.3.1, $u \in L^\infty(\mathbb{R}^N)$ and thus $f(u) = u^p - u^q \in L^\infty(\mathbb{R}^N)$. Consequently, by [70, Proposition 1.1], it follows $u \in C^s(\mathbb{R}^N)$. Since $q, p > 1$, we have $f(u) \in C^s_{loc}(\mathbb{R}^N)$. Therefore by Theorem 7.1.1(ii), $u \in C^{2s+\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$. □

Proposition 7.4.1. *Let* p, q, s *are as in Theorem 7.3.1. If* u *is any non-negative weak solution of Eq.*(7.0.1) *or* (7.0.2), *then* u *is a classical solution.*

Proof. **Case 1**: Let u be a weak solution of (7.0.1).

First, we show that $(-\Delta)^s u(x)$ can be defined as in (7.0.3). Using $u \in L^\infty(\mathbb{R}^N)$, we see that

$$\left| \int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}(0)} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{N+2s}} dy \right| \leq C \int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}(0)} \frac{dy}{|y|^{N+2s}} < \infty.$$

On the other hand, since by Theorem 7.3.2, $u \in C^{2s+\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$, it follows that $\left| \int_{B_{\frac{1}{2}}(0)} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{N+2s}} dy \right| < \infty$. Hence $(-\Delta)^s u(x)$ is defined pointwise.

Next, we show that the Eq. (7.0.1) is satisfied in pointwise sense. u is a weak solution implies

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi \, dx = \int_{\mathbb{R}^N} u^p \varphi \, dx - \int_{\mathbb{R}^N} u^q \varphi \, dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

This in turn implies

$$\int_{\mathbb{R}^N} \varphi (-\Delta)^s u \, dx = \int_{\mathbb{R}^N} u^p \varphi \, dx - \int_{\mathbb{R}^N} u^q \varphi \, dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

Therefore, $(-\Delta)^s u = u^p - u^q$ in $\mathbb{R}^N$ almost everywhere and $u \in C^{2s+\alpha}$ implies

$$(-\Delta)^s u(x) = u^p(x) - u^q(x) \quad \forall x \in \mathbb{R}^N.$$

Hence, u is a classical solution of (7.0.1).

Case 2: Suppose u is a weak solution of (7.0.2). Then applying Theorem 7.3.1 and Theorem 7.3.2, we can show as in Case 1 that $(-\Delta)^s u(x)$ can be defined in pointwise sense.

Now we are left to show that (7.0.2) is satisfied in pointwise sense. Towards this goal, we define

$$f(u) = u^p - u^q, \quad u_\varepsilon := u * \rho_\varepsilon \quad \text{and} \quad f_\varepsilon := f(u) * \rho_\varepsilon,$$

where ρ_ε is the standard molifier. Namely, we take $\rho_\varepsilon = \varepsilon^{-N}\rho(\frac{x}{\varepsilon})$ where $\rho \in C_0^\infty(\mathbb{R}^N)$ with $0 \le \rho \le 1$, supp $\rho \subseteq \{|x| \le 1\}$ and $\displaystyle\int_{\mathbb{R}^N} \rho \, dx = 1$.

Then $u_\varepsilon,\ f_\varepsilon \in C^\infty$. Proceeding along the same line as in the proof of [77, Proposition 5], we can show that, for $\varepsilon > 0$ small enough it holds

$$(-\Delta)^s u_\varepsilon = f_\varepsilon \quad \text{in} \quad U, \tag{7.4.32}$$

in the classical sense, where U is any arbitrary subset of Ω with $U \subset\subset \Omega$. Moreover, it is easy to note that $u_\varepsilon \to u$ and $f_\varepsilon \to f(u)$ locally uniformly and

$$|u_\varepsilon|_{L^\infty(B_1(0))} \le |u|_{L^\infty(\mathbb{R}^N)} \quad \text{and} \quad |f_\varepsilon|_{L^\infty(B_1(0))} \le C|u|_{L^\infty(\mathbb{R}^N)}.$$

Taking the limit $\varepsilon \to 0$ on both the sides of (7.4.32) and using the regularity estimate of u_ε from Theorem 7.3.2, we obtain,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \frac{u_\varepsilon(x+y) - 2u_\varepsilon(x) + u_\varepsilon(x-y)}{|y|^{N+2s}} dy = f(u).$$

Using the arguments used before, it is not difficult to check that LHS of above relation converges to $(-\Delta)^s u$ as $\varepsilon \to 0$ and hence the result follows.

□

Proof of Theorem 7.3.3. First, we observe that from Theorem 7.3.2, it follows u is differentiable as $p > 1$. Let R_0 be as in Theorem 7.3.1. For $R > R_0$, define $v(x) = R^{N-2s}u(Rx)$. Then

$$\begin{aligned}(-\Delta)^s v(x) =& R^N \Big((-\Delta)^s u\Big)(Rx) \\ =& R^N (u^p(Rx) - u^q(Rx)) \\ =& R^{N-p(N-2s)} v^p - R^{N-q(N-2s)} v^q.\end{aligned} \tag{7.4.33}$$

From Theorem 7.3.1, we have $|u(x)| \leq \frac{C}{|x|^{N-2s}}$ for $|x| > R_0$. Consequently, we get

$$|v(x)| \leq \frac{C}{|x|^{N-2s}} \quad \text{for} \quad |x| > \frac{R_0}{R}, \tag{7.4.34}$$

where C is independent of R. Let $A_1 := \{1 < |x| < 2\}$ and $x_0 \in A_1$. Suppose $r > 0$ is such that $B_{2r}(x_0) \subset A_1$. We choose $\eta \in C_0^\infty(\mathbb{R}^N)$ such that $\eta = 1$ in $B_r(x_0)$ and supp $\eta \subset B_{2r}(x_0)$. Clearly $v\eta \in L^\infty(\mathbb{R}^N)$ and $||\eta v||_{L^\infty(\mathbb{R}^N)} \leq C_1$, where C_1 is independent of R. Moreover,

$$(-\Delta)^s(v\eta) = (-\Delta)^s v + (-\Delta)^s\Big((\eta-1)v\Big). \tag{7.4.35}$$

Note that, for $z \in B_r(x_0)$ we have

$$(-\Delta)^s\Big((\eta-1)v\Big)(z) = c_{N,s}\int_{\mathbb{R}^N\setminus B_r(x_0)} \frac{-\Big((\eta-1)v\Big)(y)}{|z-y|^{N+2s}}dy.$$

From this expression we obtain

$$\begin{aligned}||(-\Delta)^s\Big((\eta-1)v\Big)||_{L^\infty(B_r(x_0))} &\leq C\int_{\mathbb{R}^N} \frac{v(y)}{(1+|y|)^{N+2s}}dy\\ =&C\int_{B_{\frac{R_0}{R}}(0)} \frac{v(y)}{(1+|y|)^{N+2s}}dy + C\int_{|y|>\frac{R_0}{R}} \frac{v(y)}{(1+|y|)^{N+2s}}dy.\end{aligned} \tag{7.4.36}$$

Now, using the definition of v and the fact that $u \in L^\infty(\mathbb{R}^N)$, we get

$$\begin{aligned}\int_{B_{\frac{R_0}{R}}(0)} \frac{v(y)}{(1+|y|)^{N+2s}}dy =&R^{N-2s}\int_{B_{\frac{R_0}{R}}(0)} \frac{u(Ry)}{(1+|y|)^{N+2s}}dy\\ =&CR^N\int_{B_{R_0}(0)} \frac{u(x)dx}{(R+|x|)^{N+2s}}\\ \leq&C\frac{R^N}{R^{N+2s}}|B_{R_0}(0)| < C',\end{aligned} \tag{7.4.37}$$

where C' is independent of R (since, $R^{-2s} < 1$). On the other hand, using

(7.4.34) we have

$$\begin{aligned}\int_{|y|>\frac{R_0}{R}} \frac{v(y)}{(1+|y|)^{N+2s}}dy =& C\int_{|y|>\frac{R_0}{R}} \frac{dy}{|y|^{N-2s}(1+|y|)^{N+2s}}\\ \leq& C\int_{\mathbb{R}^N} \frac{dy}{|y|^{N-2s}(1+|y|)^{N+2s}}\\ \leq& C\int_{B_1(0)} \frac{dy}{|y|^{N-2s}} + \int_{|y|>1}\frac{dy}{|y|^{2N}}\\ \leq& C, \end{aligned} \tag{7.4.38}$$

for some constant $C > 0$, which does not depend on R. Plugging (7.4.37) and (7.4.38) into (7.4.36) we have

$$||(-\Delta)^s((\eta-1)v)||_{L^\infty(B_r(x_0))} < C, \tag{7.4.39}$$

where C depends only on N, s, p, q, R_0. Furthermore, we observe that if $z \in B_r(x_0) \subset A_1$ then $|Rz| > R > R_0$ and thus $|u(Rz)| < \frac{C}{|Rz|^{N-2s}}$. Consequently, from (7.4.33), it follows that

$$|(-\Delta)^s v(z)| \leq R^N\Big(u^p(Rz)+u^q(Rz)\Big) \leq R^{N-p(N-2s)} + R^{N-q(N-2s)} < C.$$

In the last inequality we have use the fact that $N - p(N-2s) < 0$ and $N - q(N-2s) < 0$, as $q, p \geq 2^* - 1$. Hence,

$$||(-\Delta)^s v||_{L^\infty(B_r(x_0))} \leq C, \tag{7.4.40}$$

where C is independent of R. Combining (7.4.39) and (7.4.40) along with (7.4.35) yields $||(-\Delta)^s(\eta v)||_{L^\infty(B_r(x_0))} < C$, where C depends only on N, s, p, q, R_0. Consequently, using [70, Proposition 2.3] (see also [73]), we obtain

$$||(\eta v)||_{C^\beta(\overline{B_{\frac{r}{2}}(x_0)})} \leq C \quad \forall\, \beta \in (0, 2s),$$

where C depends only on N, s, p, q, R_0. As a consequence,

$$||v||_{C^\beta(\overline{B_{\frac{r}{2}}(x_0)})} \leq C.$$

Thus, thanks to [70, Corollary 2.4] we have

$$||v||_{C^{\beta+2s}(\overline{B_{\frac{r}{8}}(x_0)})} \leq C.$$

We continue to apply this bootstrap argument and after a finitely many steps we have $||v||_{C^{\beta+ks}(\overline{B_{r_0}(x_0)})} \leq C$. for some $r_0 > 0$ and $\beta + ks > 1$. This in turn implies $||\nabla v||_{L^\infty(\overline{B_{r_0}(x_0)})} \leq C$. This further yields to

$$||\nabla v||_{L^\infty(A_1)} \leq C,$$

where C depends only on N, s, p, q, R_0. Therefore, using the definition of v, we obtain

$$|\nabla u(Rx)| \leq \frac{C}{R^{N-2s+1}} \quad \text{for } 1 < |x| < 2.$$

From the above expression, it is easy to deduce that

$$|\nabla u(y)| \leq \frac{C}{|y|^{N-2s+1}} \quad \text{for } R < |y| < 2R.$$

As $R > R_0$ was arbitrary we get

$$|\nabla u(y)| \leq \frac{C}{|y|^{N-2s+1}} \quad \text{for } |y| > R,$$

for some R large. □

7.5 Existence and nonexistence results

Proof of Theorem 7.3.4. We prove this theorem by establishing Pohozaev identity in the spirit of Ros-Oton and Serra [71]. For $\lambda > 0$, define $u_\lambda(x) = u(\lambda x)$. Multiplying the equation (7.0.1) by u_λ yields,

$$\begin{aligned}\int_{\mathbb{R}^N} (u^p - u^q)u_\lambda dx =& \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} u_\lambda dx \\ =&\lambda^s \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u(x) \Big((-\Delta)^{\frac{s}{2}} u\Big)(\lambda x) dx \\ =&\lambda^s \int_{\mathbb{R}^N} w w_\lambda dx, \end{aligned} \tag{7.5.1}$$

where, $w(x) := (-\Delta)^{\frac{s}{2}}u(x)$ and $w_\lambda(x) = w(\lambda x)$. With the change of variable $x = \sqrt{\lambda}y$, we have

$$\lambda^s \int_{\mathbb{R}^N} w w_\lambda dx = \lambda^s \int_{\mathbb{R}^N} w(x)w(\lambda x)dx = \lambda^{-\frac{N-2s}{2}} \int_{\mathbb{R}^N} w_{\sqrt{\lambda}} w_{\frac{1}{\sqrt{\lambda}}} dy. \tag{7.5.2}$$

Therefore,

$$\int_{\mathbb{R}^N} (u^p - u^q)u_\lambda dx = \lambda^{-\frac{N-2s}{2}} \int_{\mathbb{R}^N} w_{\sqrt{\lambda}} w_{\frac{1}{\sqrt{\lambda}}} dy. \tag{7.5.3}$$

Observe that using the decay estimate at infinity of u and ∇u from Theorem 7.3.1 and Theorem 7.3.3 , we get $\int_{\mathbb{R}^N} (u^p - u^q)(x \cdot \nabla u)dx$ is well defined and that integral can be written as $\int_{\mathbb{R}^N} x \cdot \nabla \left(\frac{u^{p+1}}{p+1} - \frac{u^{q+1}}{q+1} \right) dx$. Again using the decay estimate of u from Theorem 7.3.1, we justify the following integration by parts

$$-\frac{N}{p+1} \int_{\mathbb{R}^N} u^{p+1}dx + \frac{N}{q+1} \int_{\mathbb{R}^N} u^{q+1}dx = \int_{\mathbb{R}^N} x \cdot \nabla \left(\frac{u^{p+1}}{p+1} - \frac{u^{q+1}}{q+1} \right) dx. \tag{7.5.4}$$

Thus, using (7.5.3) we simplify the LHS of above expression as follows:

$$\begin{aligned}
\text{LHS of (7.5.4)} &= \int_{\mathbb{R}^N} (u^p - u^q)(x \cdot \nabla u)dx \\
&= \frac{d}{d\lambda}\Big|_{\lambda=1} \int_{\mathbb{R}^N} (u^p - u^q)u_\lambda dx \\
&= \frac{d}{d\lambda}\Big|_{\lambda=1} \left(\lambda^{-\frac{N-2s}{2}} \int_{\mathbb{R}^N} w_{\sqrt{\lambda}} w_{\frac{1}{\sqrt{\lambda}}} \right) dx. \\
&= -\left(\frac{N-2s}{2} \right) \int_{\mathbb{R}^N} w^2 dx + \frac{d}{d\lambda}\Big|_{\lambda=1} \int_{\mathbb{R}^N} w_{\sqrt{\lambda}} w_{\frac{1}{\sqrt{\lambda}}} dy \\
&= -\left(\frac{N-2s}{2} \right) ||u||^2_{\dot{H}^s(\mathbb{R}^N)}.
\end{aligned}$$

On the other hand, multiplying (7.0.1) by u we have,

$$||u||^2_{\dot{H}^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (u^{p+1} - u^{q+1})dx.$$

Combining the above two expressions, we obtain the Pohozaev identity

$$\left(\frac{N-2s}{2} - \frac{N}{p+1} \right) \int_{\mathbb{R}^N} u^{p+1}dx = \left(\frac{N-2s}{2} - \frac{N}{q+1} \right) \int_{\mathbb{R}^N} u^{q+1}dx.$$

Clearly, from the above identity, it follows that (7.0.1) does not admit any solution when $p = 2^* - 1$ and $q > p$. This completes the theorem. □

7.5.1 Symmetry and monotonically decreasing property

Theorem 7.5.1. *Let p, q, s are as in Theorem 7.3.1 and u be any solution of Eq.(7.0.1). Then u is radially symmetric and strictly decreasing about some point in $\mathbb{R}^N$.*

Proof. By Proposition 7.4.1, u is a classical solution of (7.0.1). Define $f(u) = u^p - u^q$. Then clearly f is locally Lipschitz.

Claim: There exists $s_0, \gamma, C > 0$ such that

$$\frac{f(v) - f(u)}{v - u} \leq C(u+v)^\gamma \quad \text{for all} \quad 0 < u < v < s_0.$$

To see the claim,

$$\begin{aligned} f(v) - f(u) =& (v^p - u^p) - (v^q - u^q) \\ =& p\Big(\theta_1 v + (1-\theta_1)u\Big)^{p-1}(v-u) - q\Big(\theta_2 v + (1-\theta_2)u\Big)^{q-1}(v-u), \end{aligned}$$

for some $\theta_1, \theta_2 \in (0,1)$. Thus, for $0 < u < v$

$$\begin{aligned} \frac{f(v) - f(u)}{v - u} =& p\Big(\theta_1 v + (1-\theta_1)u\Big)^{p-1} - q\Big(\theta_2 v + (1-\theta_2)u\Big)^{q-1} \\ \leq& p\Big(\theta_1 v + (1-\theta_1)u\Big)^{p-1} \\ \leq& p(u+v)^{p-1}. \end{aligned}$$

Therefore, the claim holds with $C = p$ and $\gamma = p - 1$ and for any positive s_0.

Moreover, from Theorem 7.3.2, we have

$$u(x) = O\Big(\frac{1}{|x|^{N-2s}}\Big) \quad \text{as} \quad |x| \to \infty.$$

Since $p \geq \frac{N+2s}{N-2s}$, it is easy to check that

$$N - 2s > \max\left(\frac{2s}{\gamma}, \frac{N}{\gamma+2}\right),$$

where $\gamma = p - 1$, as found in the above claim. Hence, the theorem follows from [44, Theorem 1.2]. □

Theorem 7.5.2. *Suppose Ω is a smooth bounded convex domain, p, q, s are as in Theorem 7.3.1 . Assume further that Ω is convex in x_1 direction and symmetric w.r.t. to the hyperplane $x_1 = 0$. Let $s \in (0,1)$ and u be any solution of Eq.(7.0.2). Then u is symmetric w.r.t. x_1 and strictly decreasing in x_1 direction for $x = (x_1, x') \in \Omega$, $x_1 > 0$.*

Proof. Follows from [43, Theorem 3.1] (also see [51, Cor. 1.2]). □

Existence results

Lemma 7.5.3. *Let $s \in (0,1)$. If u is any radially symmetric decreasing function in $\dot{H}^s(\mathbb{R}^N)$, then*

$$u(|x|) \le \frac{C}{|x|^{\frac{N-2s}{2}}}.$$

Proof. It is enough to show that if $u \in \dot{H}^s(\mathbb{R}^N)$ with $u(x) = u(|x|)$ and $u(r_1) \le u(r_2)$, when $r_1 \ge r_2$, then it holds $u(R) \le \frac{C}{R^{\frac{N-2s}{2}}}$ for any $R > 0$. To see this, we note that by Sobolev inequality we can write,

$$\begin{aligned}
\frac{1}{S_s}||(-\Delta)^{\frac{s}{2}}u||_{L^2(\mathbb{R}^N)} \ge & \left(\int_{\mathbb{R}^N} |u(x)|^{2^*} dx\right)^{\frac{1}{2^*}} \\
\ge & \left(\int_0^R \int_{\partial B_r} |u(r)|^{2^*} dS dr\right)^{\frac{1}{2^*}} \\
\ge & u(R)\left(\int_0^R \omega_n r^{N-1} dr\right)^{\frac{1}{2^*}} \\
= & \left(\frac{\omega_N}{N}\right)^{\frac{1}{2^*}} u(R) R^{\frac{N}{2^*}}. \qquad (7.5.5)
\end{aligned}$$

As $u \in \dot{H}^s(\mathbb{R}^N)$ implies LHS is bounded above, the above inequality yields

$$u(R) \le \left(\frac{N}{\omega_N}\right)^{\frac{1}{2^*}} \frac{1}{S_s} ||(-\Delta)^{\frac{s}{2}}u||_{L^2(\mathbb{R}^N)} R^{-\frac{N-2s}{2}} \le C R^{-\frac{N-2s}{2}}.$$

□

Proof of Theorem 7.3.5. We are going to work on the manifold

$$\mathcal{N} = \left\{u \in \dot{H}^s(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^{p+1} dx = 1\right\},$$

and $F(\cdot)$ on $\mathcal{N}$ reduces as

$$F(u) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy + \frac{1}{q+1}\int_{\mathbb{R}^N}|u|^{q+1}dx.$$

Let u_n be a minimizing sequence in $\mathcal{N}$ such that

$$F(u_n) \to \mathcal{K} \text{ with } \int_{\mathbb{R}^N}|u_n|^{p+1}dx = 1.$$

Thus, $\{u_n\}$ is a bounded sequence in $\dot{H}^s(\mathbb{R}^N)$ and $L^{q+1}(\mathbb{R}^N)$. Therefore, there exists $u \in \dot{H}^s(\mathbb{R}^N)$ and $L^{q+1}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $\dot{H}^s(\mathbb{R}^N)$ and $L^{q+1}(\mathbb{R}^N)$. Consequently $u_n \to u$ pointwise almost everywhere.

Using symmetric rearrangement technique, without loss of generality, we can assume that u_n is radially symmetric and decreasing (see [69]). We claim that $u_n \to u$ in $L^{p+1}(\mathbb{R}^N)$.

To see the claim, we note that $u_n^{p+1} \to u^{p+1}$ pointwise almost everywhere. Since $\{u_n\}$ is uniformly bounded in $L^{q+1}(\mathbb{R}^N)$, using Vitali's convergence theorem, it is easy to check that $\int_K |u_n|^{p+1}dx \to \int_K |u|^{p+1}dx$ for any compact set K in $\mathbb{R}^N$ containing the origin. Furthermore, applying Lemma 7.5.3 it follows, $\int_{\mathbb{R}^N\setminus K}|u_n|^{p+1}dx$ is very small and hence we have strong convergence. Moreover, $\int_{\mathbb{R}^N}|u_n|^{p+1}dx = 1$ implies $\int_{\mathbb{R}^N}|u|^{p+1}dx = 1$.

Now we show that $\mathcal{K} = F(u)$.

We note that $u \mapsto ||u||^2$ is weakly lower semicontinuous. Using this fact along with Fatou's lemma, we have

$$\begin{aligned}
\mathcal{K} &= \lim_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dxdy + \frac{1}{q+1}\int_{\mathbb{R}^N}|u_n|^{q+1}dx\right]\\
&= \lim_{n\to\infty}\left[\frac{1}{2}||u_n||^2 + \frac{1}{q+1}\int_{\mathbb{R}^N}|u_n|^{q+1}dx\right]\\
&\geq \frac{1}{2}||u||^2 + \frac{1}{q+1}\int_{\mathbb{R}^N}|u|^{q+1}dx\bigg]\\
&\geq F(u).
\end{aligned}$$

This proves $F(u) = \mathcal{K}$. Moreover, using the symmetric rearrangement technique via. Polya-Szego inequality (see [69]), it is easy to check that u is nonnegative, radially symmetric and radially decreasing Applying the Lagrange multiplier rule, we obtain u satisfies

$$-\Delta u + u^q = \lambda u^p,$$

for some $\lambda > 0$. This in turn implies

$$(-\Delta)^s u = \lambda u^p - u^q \quad \text{ in } \quad \mathbb{R}^N.$$

Finally, if $q > (p-1)\frac{N}{2s} - 1$, then we know that u is a classical solution. Therefore, if there exists $x_0 \in \mathbb{R}^N$ such that $u(x_0) = 0$, that that would imply $(-\Delta)^s u(x_0) < 0$ (since, u is a nontrivial solution). On the other hand, $(\lambda u^p - u^q)(x_0) = 0$ and that yields a contradiction. Hence $u > 0$ in $\mathbb{R}^N$.

Furthermore, we observe that by setting $v(x) = \lambda^{-\frac{1}{q-p}} u(\lambda^{-\frac{q-1}{2s(q-p)}} x)$, it holds

$$(-\Delta)^s v = v^p - v^q \quad \text{in} \quad \mathbb{R}^N.$$

Hence the theorem follows. □

Proof of Theorem 7.3.6. We are going to work on the manifold

$$\tilde{\mathcal{N}} = \left\{ u \in X_0(\Omega) \cap L^{q+1}(\Omega) : \int_\Omega |u|^{p+1} = 1 \right\}.$$

Then F_Ω reduces to

$$F_\Omega(u) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy + \frac{1}{q+1} \int_\Omega |u|^{q+1} dx.$$

Let u_n be a minimizing sequence in $\tilde{\mathcal{N}}$ such that $F_\Omega(u_n) \to S_\Omega$, then

$$F(u_n) \to S_\Omega \text{ with } \int_\Omega |u_n|^{p+1} dx = 1.$$

Then u_n is bounded in $X_0(\Omega) \cap L^{q+1}(\Omega)$. Consequently, $u_n \rightharpoonup u$ on $H^s(\Omega)$ and $u_n \to u$ on $L^2(\Omega)$. As a result, $u_n \to u$ pointwise almost everywhere. By the interpolation inequality, we must have $u_n \to u$ on $L^{p+1}(\Omega)$. Hence, $\int_\Omega |u|^{p+1} dx = 1$.

Now we show that $S_\Omega = F_\Omega(u)$. Using Fatou's Lemma and the fact that $u \mapsto ||u||^2$ is weakly lower semicontinuous ,

$$\begin{aligned}
S_\Omega =& \lim_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dxdy + \frac{1}{q+1}\int_\Omega |u_n|^{q+1}dx\right]\\
\geq& \left[\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy + \frac{1}{q+1}\int_\Omega |u|^{q+1}dx\right]\\
\geq& F_\Omega(u).
\end{aligned}$$

By the Lagrange multiplier rule, we obtain u satisfies

$$(-\Delta)^s u + |u|^{q-1}u = \lambda |u|^{p-1}u.$$

Now we replace $\tilde{\mathcal{N}}$ by $\tilde{\mathcal{N}}_+ := \left\{u \in X_0(\Omega)\cap L^{q+1}(\Omega) : \int_\Omega (u^+)^{p+1} = 1\right\}$, the functional $F_\Omega(\cdot)$ by $\tilde{F}_\Omega(\cdot)$ defined as follows

$$\tilde{F}_\Omega(u) := \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy + \frac{1}{q+1}\int_\Omega (u^+)^{q+1}dx,$$

and S_Ω by $\tilde{S}_\Omega := \inf\left\{F(v,\Omega) : v \in \tilde{\mathcal{N}}_+\right\}$. Repeating the same argument as before (with a little modification), it can be easily shown that there exists $u \in X_0(\Omega)\cap L^{q+1}(\Omega)$ which satisfies

$$(-\Delta)^s u + (u^+)^q = \lambda (u^+)^p \quad \text{in} \quad \Omega. \tag{7.5.6}$$

Taking u^- as the test function for (7.5.6) we obtain from Definition 7.2.1 that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+2s}}dxdy = 0. \tag{7.5.7}$$

Furthermore,

$$\begin{aligned}
&\text{LHS of (7.5.7)}\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+2s}}dxdy\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Big((u^+(x)-u^+(y))-(u^-(x)-u^-(y))\Big)(u^-(x)-u^-(y))}{|x-y|^{N+2s}}dxdy\\
&= -u^-(x)u^+(y) - u^+(x)u^-(y) - ||u^-||^2_{X_0(\Omega)}\\
&\leq -||u^-||^2_{X_0(\Omega)} \qquad (7.5.8)
\end{aligned}$$

Hence, from (7.5.7) we obtain $u^- = 0$, i.e, $u \geq 0$. Moreover, since for $p \geq 2^* - 1$ and $q > (p-1)\frac{N}{2s} - 1$, Proposition 7.4.1 implies u is a classical solution, applying maximum principle as in Theorem 7.3.5, we conclude $u > 0$ in Ω. This completes the proof. □

Conclusion: In this chapter, we have discussed qualitative properties of solutions and obtained decay of u and ∇u at infinity but the computations are not effortless as we are in the non-local case. Probity of our result lies in overcoming pitfall of the computations.

Some open-problems and Remarks

- To characterize the properties of Sobolev minimizer like symmetry, asymptotic property etc. under some additional conditions on K will be a good topic for future research.

- With the weight V used in Chapter 6, one can try to find sign-changing solutions and deduce the results obtained in Chapter 4 and 5.

- With the following K, (see [74])

$$K(y) = \frac{a(\frac{y}{|y|})}{|y|^{N+2s}}, \text{ where } a \in L^1(S^{N-1}) \text{ is nonnegative and even,}$$

  and $K(x,y) \sim \frac{a(x,y)}{|y|^{N+2s}}$, where $a(x,y)$ is homogeneous in y of order zero and $a(x,y)$ and derivatives of $a(x,y)$ w.r.t y are uniformly continuous in x, (see [40]), one could try to establish the results obtained in the thesis.

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