Seiberg-Witten theory on four manifolds

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by

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Certificate

This is to certify that this dissertation entitled Seiberg-Witten theory on four manifolds towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Eshwarat Indian Institute of Science Education and Research under the supervision of Dr. Mainak Poddar, Professor, Department of Mathematics , during the academic year 2025-2026.

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Declaration

I hereby declare that the matter embodied in the report entitled Seiberg-Witten theory on four manifolds are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Mainak Poddar and the same has not been submitted elsewhere for any other degree.

Eshwar

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Abstract

This thesis is an exposition on Seiberg-Witten theory for 4-manifolds. We begin by describing basic notions of gauge theory like bundles, connections, curvature, gauge group, configuration space etc. in the first chapter. This is followed by brief account of Spin geometry for 4-manifolds with main objective of defining Dirac operators and describing Wietzenböck formula and the Atiyah-Singer index theorem. In the third chapter, we set up the monopole moduli space based on the Seiberg-Witten equations and prove compactness and transversality. We also define the Seiberg-Witten invariants. In the final chapter, we compute the Seiberg-Witten invariants for Kahler surfaces and find exotic smooth structures on 4-manifolds.

Contents

Introduction

For any geometer or even a lay person if asked what could be the job of a geometer or the primary object of study in geometry, they would probably answer that it is to understand shapes or spaces and phenomena associated with them. And currently manifolds provide one of the most promising models for shapes or spaces we know of. Thereby in understanding spaces one could say that knowing or studying what all manifolds exist and how do we distinguish them from one another become crucial. In the later half of last century we saw remarkable developments in this regard, Surgery theory resolving classification(upto homotopy and diffeomorphism) in dimensions greater than 5, Thurston's geometrization conjecture for dimension 3 which was proved by Perelman around 2003. In particular before Milnor's work in 1960s (which later led to development of Surgery theory) there was less understanding regarding the difference between topological and smooth categories but after the said work it became clear that one had to account for the difference which is absent upto and including dimension 3 in classification. This was an important matter because all the geometric phenomena over and above the topological matters depends on the smooth structure as they are realised in Riemannian, Complex, Symplectic, Spin structures which sit over and depend on the smooth structure. To this end Surgery theory made the topological classification and later due to Kirby-Siebenmann the smoothening in dimension greater than 4 was understood(which turned out to be quite tame). But the situation in dimension 4 was not clear until the groundbreaking work of Simon Donaldson which showed that in this dimension a topological manifold could have infinitely many smooth structures. The tools employed in this work were inspired from physics and marked the beginning of a new area of study, mathematical gauge theory. Donaldson studying the solutions to Yang-Mills equations, was able to make invariants which could detect exotic smooth structures on a manifold in dimension 4. Donaldson's theory, as remarkable as it was, it was very terse and had many practical problems which didn't allow for easy engagement or working with the theory. Then around 1994-95 Edward Witten along with Nathan Seiberg was able to use different equations which recovered all the results of Donaldson's theory but with much more ease, the invariants produced from this theory are called Seiberg-Witten invariants. This led to very rapid development in study of 4-manifolds in the coming years and now have become a staple in this area of study. Clifford Taubes from 1995 to early 2000s established the strong relation of these invariants to study of Symplectic 4-manifolds. Since then these invariants have been employed to detect and understand symplectic structures, Kahler structures on 4-manifolds. In spite of these invariants and others coming from Floer theory, the study of 4 manifolds remains very active and we are far from classification in this dimension. This thesis is an exposition. I have tried to provide a careful explanation of the theory which leads to the definition of Seiberg-Witten invariants which will occupy the largest part, followed by some applications of the results from these. I will end with some remarks and directions of work.

Chapter 1

Basic objects of Gauge theory

Gauge theory(in mathematics) is the study of connections on vector bundles, principal bundles, fibre bundles. This chapter will mainly contain the definitions and basic results regarding vector bundles, connections, principal bundles but also some information on curvature, parallel transport and characteristic classes which are important objects relevant to gauge theory. The goal is to present preliminary information which is indispensable in gauge theoretic study.

1.1 Vector bundles

Before giving the formal definition of vector bundles one can imagine them to be collection of vector spaces which are parametrised by a base topological space. And this parametrisation is continuous or smooth depending of what sort of vector bundle we want to consider, which means that the collection itself has a topological or geometric structure to it. For a mental picture one could keep in mind cylinder being a vector bundle over a circle by which we mean that it is collection of one dimensional vector spaces parametrised by a circle. With that said it might not be very clear immediately from the definition as to why we construct or consider such objects. A physical way of looking comes from study of vector fields on manifolds which is crucial to dynamics, or even from something like weather where we can talk about temperature distribution, wind speed so on. But even geometrically we are aware that continuous or smooth association of vectors on manifolds can reveal a lot about the manifold itself, and vector bundle in a way encapsulate all such continuous or smooth associations one can make so they become quite important.

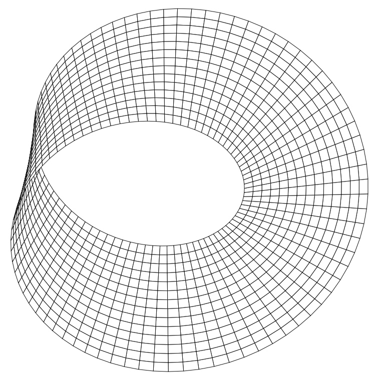

Figure 1.1: Möbius strip

Definition 1.1.1. (Real Vector Bundles of rank n and isomorphisms) Let B and E be a topological space and let $\pi : E \to B$ be a continuous map, π is called a real vector bundle of rank n if $\forall\, b \in B$, $\pi^{-1}(b)$ is a n-dimensional vector space over $\mathbb{R}$, with an open cover $\{U_\alpha\}$ on B with a collection of maps called *trivialisations* $\{h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n\}$ which are fiber preserving homeomorphisms and where restricted $h_\alpha|_{\pi^{-1}(b)} : \pi^{-1}(b) \to \{b\} \times \mathbb{R}^n$ is a $\mathbb{R}$-linear isomorphism of vector space. Here U_αs are termed as *trivialising open sets* for π. Let $\pi : E \to B$ and $\pi' : E' \to B'$ be real vector bundles then a bundle map is a pair of continuous maps ($\varphi : E \to E', \bar{\varphi} : B \to B'$ such that the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "\varphi"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\
B \arrow[r, "\bar{\varphi}"] & B'
\end{tikzcd}$$

and when we restrict $\varphi|_{\pi^{-1}(b)} : \pi^{-1}(b) \to \pi'^{-1}(\bar{\varphi}(b))$ is an $\mathbb{R}$-linear map But we consider bundle isomorphism for bundles over the same base space, we say two bundles are isomorphic if there is a bundle map with base map being identity and there exists another map ψ such that $\psi \circ \varphi = 1_E$ and $\varphi \circ \psi = 1_{E'}$.

We could make some small tweaks in this definition to get the definitions of other kinds of vector bundles, like smooth real vector bundles would require π to be a smooth map between manifolds and the trivialisations to be diffeomorphisms rather than homeomorphisms and correspondingly the bundle map would comprise of smooth maps rather than continuous. Similarly one could define complex vector bundles to be those where we replace in our definition everywhere the field from $\mathbb{R}$ to $\mathbb{C}$ and $\mathbb{R}$-linearity with $\mathbb{C}$-linearity. There is also the notion of holomorphic vector bundle which would require us to replace complex manifolds and holomorphic maps similarly.

One can note that tangent or cotangent bundle on a manifold, are some good examples. A trivial example is that of a *trivial bundle* $\pi' : B \times \mathbb{R}^n \to B$ where the open cover consists only of the whole space and there is one local trivialisation, but one can notice that this in fact gives us the local picture of a vector bundle which we will explore in the following remark about the definition of a vector bundle.

We have an alternate and equivalent definition for vector bundle which relies on gluing or patching information. This description is highly useful later on and highlights some interesting aspect which we shall describe after giving the definition. The idea is that local trivialisation gives us that a vector bundles is made of patches which look like $U_\alpha \times \mathbb{R}^n$ where $\{U_\alpha\}$ is an open cover of the base space. So we could realise E as quotient of $\coprod_\alpha U_\alpha \times \mathbb{R}^n$, where we identify using *gluing maps* or *transition maps* that are defined using the maps $h_\beta|_{\pi^{-1}(U_\alpha \cap U_\beta)} \circ h_\alpha|^{-1}_{\pi^{-1}(U_\alpha \cap U_\beta)}$ as they restrict to linear isomorphism when restricted to each fibre give $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{R})$. And we notice that they obey *co-cycle condition* $g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\gamma\alpha}$ (there is a slight abuse of notation these maps obviously don't have common domains and ranges a priori but we are referring to the appropriate restrictions). And now the idea is that if rather we are given a base space B along with an open cover $\{U_\alpha\}$ and family of maps $\{g_{\alpha\beta}\}$ which obey the co-cycle condition then we get a vector bundle over B. We can summarise this discussion in the following proposition:

Proposition 1.1.1. *Let $\pi : E \to B$ be a real vector bundle then given the open cover $\{U_\alpha\}$ of B and local trivialisations corresponding to them the* *family of gluing maps $\{g_{\alpha\beta}\}$ as defined above obey the co-cycle condition and the quotient space(given such data $\{g_{\alpha\beta}\}$ which obeys co-cycle condition) $X = (\coprod_\alpha U_\alpha \times \mathbb{R}^n)/\sim$ where $\sim$ is the equivalence relation generated by identifications from the gluing maps; $(p, v) \sim (p, g_{\alpha\beta}(p)(v))$, defines a real vector bundle $\pi' : X \to B$ which is isomorphic to our original real vector bundle π.*

1.2 Sections of vector bundle

As we had discussed in the beginning regarding the the importance of vector fields on manifolds and how one could justify defining vector bundles from that perspective there is not much to motivate here as sections are to a vector bundle what vector fields are to tangent bundles or simply serve analogous role to vector fields on manifolds.

Definition 1.2.1. (Section of a vector bundle) Given a vector bundle $\pi : E \to B$ a section of this vector bundle is defined to be a continuous map $s : B \to E$ such that $\pi \circ s = \mathbb{1}_B$. And similarly for smooth vector bundle a smooth section would such map that is smooth.

A trivial example of a section would be the *zero section* which is when you assign the zero vector to each point in the base space. One more interesting idea of a *non-vanishing section* which is nowhere zero, such a section might not exist always for example on S^2 where this manifests as the hairy-ball theorem. But these are important conceptually since an isomorphism of vector bundles will carry a non-vanishing section to a non-vanishing section. And an important criteria for finding trivial vector bundle comes through such sections which we can see in the following proposition:

Proposition 1.2.1. *A rank n real vector bundle is isomorphic to a trivial vector bundle of rank n iff it has n sections $s_1, s_2, \cdots, s_n$ such that at each point $p \in B$ $s_i(p)$ are all linearly independent. Such a collection of sections is called a* global frame*.*

[6]

The idea is that if we are given a global frame we can construct an isomorphism of bundles which is given by $f : E \to B \times \mathbb{R}^n$, defining it as $v_p = \sum_i t_i(p)s_i(p) \mapsto (p, t_i)$. And for the other way suppose we have an isomorphism given by $f : B \times \mathbb{R}^n \to E$ then we can push the global frame $\{e_i : B \to B \times \mathbb{R}^n\}_{1 \le i \le n}$ where each section is given by $p(\in B) \mapsto (p, 0, 0, \cdots, 1_{(\text{ in the ith co-ordinate})}, \cdots, 0)$. Though the first part requires some technical result which can be found in [6]. The use of global frame in the above argument and the result itself might suggest that an important object to keep an eye on is the collection of sections of the vector bundle, and even without such justifications the collection is relevant we denote it by $\Gamma(B, E)$ or simply $\Gamma(E)$ if it is evident we are taking *global sections*. We will return to this object and explore some details when we discuss connections.

1.3 Extension of operations and additional structures on vector bundles

From linear algebra we are aware of some operations on vector spaces which lets us form new vector spaces out of old. These are some operations like dual, direct sum, tensor product, quotient etc. We would like to extend these to vector bundle level as well to emulate similar theory in this context.

Definition 1.3.1. (Product of bundles) If we are given two vector bundles $\pi_1 : E_1 \to X_1$ and $\pi_2 : E_2 \to X_2$ then the product of these is defined to be $\pi_1 \times \pi_2 : E_1 \times E_2 \to X_1 \times X_2$.

The vector bundle structure is captured by the following proposition.

Proposition 1.3.1. *The product bundle $\pi : E_1 \times E_2 \to X_1 \times X_2$ of two vector bundles $\pi_1 : E_1 \to X_1$ and $\pi_2 : E_2 \to X_2$ is a real vector bundle. If the open covers on X_1 and X_2 are given by $\{U^1_\alpha\}$ and $\{U^2_\beta\}$ respectively and corresponding local trivialisations for π_1 and π_2 are given by $\{h^1_\alpha\}$ and $\{h^2_\beta\}$ respectively then for the open cover $\{U^1_\alpha \times U^2_\beta\}$ of $X_1 \times X_2$ we get that $\{h^1_\alpha \times h^2_\beta\}$ are local trivialisations for π.*

Definition 1.3.2. (Restriction of a bundle) If we are given a vector bundle $\pi : E \to X$ and a subspace A of X then restriction bundle is given by $\pi|_{\pi^{-1}(A)}$ which we could denote by $\operatorname{res}(\pi, A)$ Here it is clear that the open cover $\{U_\alpha\}$ of X yields an open cover of the subspace $\{U_\alpha \cap A\}$ and the local trivialisations are given by $\{h^{res}_\alpha = h_\alpha|_{\pi^{-1}(U_\alpha \cap A)}\}$ where $\{h_\alpha\}$ are the local trivialisations of original bundle π.

Definition 1.3.3. (Direct sum of bundles) If we are given two vector bundles $\pi_1 : E_1 \to X$ and $\pi_2 : E_2 \to X$ then the direct sum of these is defined to be $\operatorname{res}(\pi_1 \times \pi_2, \operatorname{diag}(X \times X))$ and denoted by $\pi_1 \oplus \pi_2$ or if the base space is understood then $E_1 \oplus E_2$.

Definition 1.3.4. (Tensor product of bundles) If we are given two vector bundles $\pi_1 : E_1 \to X$ and $\pi_2 : E_2 \to X$ then the tensor product of these is defined to be $\pi_1 \otimes \pi_2 : E_1 \otimes E_2 (= \coprod_{p \in X} \pi_1^{-1}(p) \otimes \pi_2^{-1}(p)) \to X$. The topology of the domain is discussed in detail in [6] To produce its local trivialisations we take the tensor product of local trivialisations $h_1 \otimes h_2 : (\pi_1 \otimes \pi_2)^{-1}(U_\alpha) \to U_\alpha \times (\mathbb{R}^m \otimes \mathbb{R}^n)$ which is given by $v_p^1 \otimes v_p^2 \mapsto (p, (\mathrm{Proj}_2^{\mathrm{Im}(h_\alpha^1)} \circ h_\alpha^1(v_p^1)) \otimes (\mathrm{Proj}_1^{\mathrm{Im}(h_\alpha^2)} \circ h_\alpha^2(v_p^2)))$ where h_α^1 and h_α^2 are local trivialisations for π_1 and π_2 and $\mathrm{Proj}_\beta^{\coprod_\alpha X_\alpha}$ are the natural projection maps of a cartesian product of sets onto their β^{th} factor, here those will be continuous considering product topology.

Remark: We could also do this in another way where we use the alternate definition of vector bundle and simply give the new gluing maps by $\{g_\alpha^1 \otimes g_\alpha^2\}$ where $\{g_\alpha^1\}$ and $\{g_\alpha^2\}$ are gluing maps for the bundles π_1 and π_2 respectively.

Definition 1.3.5. (Dual of a bundle) If we are given a vector bundle $\pi : E \to X$ whose gluing maps are given by $\{g_\alpha\}$ then the dual of this bundle is defined through gluing maps over the same open cover given by $\{g_\alpha^T\}$ where g_α^T simply takes the transpose of the outputs of g_α point wise.

Lastly we could look at pull back of a bundle which helps in giving some clarity regarding isomorphisms of vector bundle or rather the nature of vector bundles on whole. But it is also useful in describing parallel transport as we will see later on. The intuition about this concept is if we consider a continuous map $f : Y \to X$ where $\pi : E \to X$ is a vector bundle, the idea is that we replicate the vector space at each $p \in Y$ as the fibre $\pi^{-1}(f(p))$. But there is more to this, how would we know that they would be glued together properly? Continuity helps us out one can see that if $\{U_\alpha\}$ is a a trivialising open cover over for π. Then we get a cover $\{f^{-1}(U_\alpha)\}$ on Y. This means that at least for all points over each of these neighbourhoods the collection of vector spaces which we copied has a coherence or a topology(simply that of a trivial bundle). And on overlaps or in transition we borrow the same gluing functions $\{g_{\alpha\beta}\}$, we pull back those gluing functions $f^* g_{\alpha\beta}$, so the co-cycle condition is satisfied by design leading to defining a unique vector bundle which we can call the *pull back of a bundle.*

Proposition 1.3.2. *Given a vector bundle $\pi : E \to X$ and a continuous map $f : Y \to X$ there exists a vector bundle $f^*\pi : f^*E \to Y$ such that for each $p \in Y$ the fibre $(f^*\pi)^{-1}(p)$ is isomorphic to $\pi^{-1}(f(p))$ and this vector bundle*

$f^*\pi$ is unique upto isomorphism.

[6]
Definition 1.3.6. (Pull back of a bundle) Given a vector bundle $\pi : E \to X$ and a map $f : Y \to X$ the pull back bundle is defined by $f^*\pi$ as the one obtained in the proposition above.

The pull back of a bundle has very nice behaviour with respect to properties like direct sum, tensor product and composition of maps. These follow very simply by using gluing maps idea which we gave before the definition of the pull back of a bundle, which we collect here: 1. $(g \circ f)^*(\pi) = g^*(f^*(\pi))$ 2. $\mathbb{1}^*(\pi) \cong \pi$ 3. $f^*(\pi \oplus \pi') \cong f^*(\pi) \oplus f^*(\pi')$ 4. $f^*(\pi \otimes \pi') \cong f^*(\pi) \otimes f^*(\pi')$

Finally, with regard to pull back of a bundle we give the proposition and corollary which help at looking into the structure of vector bundles:
Proposition 1.3.3. *Given a vector bundle $\pi : E \to X$ where X is paracompact and homotopic maps f_0 and f_1 we have that $f_0^*(\pi) \cong f_1^*(\pi)$.*

One can find the proof in [6]. This gives us the corollary that vector bundles over any two paracompact spaces which are homotopy equivalent will be in bijection and in particular this means that for contractible paracompact spaces the vector bundles are all trivial.

With this we can move on to extension additional structures on vector bundles. On vector spaces by additional structure I mean some information which we possess regarding the vector space apart from its vector space structure, it can be having an algebra structure, having an almost complex structure, a symplectic structure, an inner product space structure, hermitian product space structure or being equipped with a representation(which can be seen as a group action as well) and so on. Most of these can be extended to the bundle level and we can enjoy a lot of these properties, and in fact for manifolds this is one of the ways that geometry can be infused above its smooth structure generally by equipping the tangent space and consequently tangent bundle with these structures. Because a lot of these come as maps going out of vector spaces or rather their direct sum with themselves we can encapsulate these using the nicely using the ideas which we have seen in this section.
Definition 1.3.7. (Inner product) If $\pi : E \to X$ is a vector bundle then an inner product on this defined to be a map $g : E \oplus E \to \mathbb{R}$ such that on each fibre of $\pi \oplus \pi$ it restricts to an inner product (or positive definite symmetric

bilinear form).

And in a similar spirit we could go on extending the others to bundle level.

1.4 Further on sheaf of sections for smooth vector bundles

Note: Reader may skip this section as this is somethings we consider in order to give a sound footing for a definition of connection and other similar matters. We will answer the questions regarding being able to restrict connections to an open neighbourhood and some other properties of sections which will be necessary. Before we move to operators between sections we have to see what information we have regarding sections in general. In the part where we discussed about sections we already saw something about frame, which makes for a good motivation to look further into sections as a collection.

Firstly we pile up a few things we have already seen. We know that a global frame exists on a vector bundle if and only if the vector bundle is trivial. We also know that vector bundles are locally trivial, so if we consider sections locally then they always have a frame. And with a frame we could locally express any section as a $C^\infty(U)$-linear combination where U is the open neighbourhood under consideration. But even without that we can at least see that sections, global or local have a vector space structure expect over $\mathcal{F} = C^\infty(X)$ the collection of smooth functions over the base manifold, which is not a field but is a ring. This leads us to the following proposition:

Proposition 1.4.1. *(Module of sections) Given a smooth vector bundle $\pi : E \to X$, the set $\Gamma(E, X)$ is a module over $C^\infty(X)$ the ring of smooth real valued maps from X, with the addition being point-wise addition and action by the ring being point-wise scalar multiplication.*

Proof-sketch: Suppose $\sigma, \tau \in \Gamma(E)$ then, so is $\sigma + \tau$ since if we take a trivialising neighbourhood U in X and we consider the map $h \circ (\sigma + \tau)$ where h is the local trivialisation, co-ordinate wise this turns out to be sum of smooth maps which is a smooth map. Similarly $f\sigma$ where $f \in C^\infty(X)$ is also smooth. The remaining properties follow similarly.

Consider the following facts regarding smooth sections:

1. If $s \in \Gamma(E)$ where $\pi : E \to M$ is a smooth vector bundle and if U is an

open subset of M then we have that $s|_U : U \to E$ is a smooth section in $\Gamma(U, E)$

2. If $s \in \Gamma(U, E)$ then we have an extension of this section, call it $\tilde{s} \in \Gamma(E)$ such that it agrees with s over an open set $W \subset U$. This is achieved by simply attaching a bump function to the section.

This now leads us to the question as to how the operators like connection and curvature behave with these sections, as in we want their local formula. Consider connections on vector bundles. Roughly we would have a map $\nabla_X : \Gamma(M, E) \to \Gamma(M, E')$ which is at least $\mathbb{R}$-linear map. Now we wan to restrict it to the trivialising open set but we also want that when we do this we are assuming some sort of well behaved nature of the operators that is if they agree over an open set then their evaluated section must also agree, otherwise what we would have is that there are two sections which will agree over a open set will disagree post the action of the operator on them this would mean that there is no useful local theory we can form regarding these operators. But as it turns out $\mathcal{F}$-linear operators have this local consistency they have something even stronger, if they agree at a point then will agree at that point even post action of the operator. We will now try to see why this happens in detail, formally.

Definition 1.4.1. (Local and point operators) Let E, F be smooth vector bundles over M then $\alpha : \Gamma(E) \to \Gamma(F)$ is called a point operator if for any section $s \in \Gamma(E)$ is zero at some point $p \in M$ then so is $\alpha(s)$. The operator is called a local operator if whenever s vanishes on an open set U of M so does $\alpha(s)$.

Proposition 1.4.2. *(Restriction of local operators) If $\alpha : \Gamma(E) \to \Gamma(E)$ is a local operator then there exists a unique $\tilde{\alpha} : \Gamma(U, E) \to \Gamma(U, E)$ such that $\tilde{\alpha}(s|_U)(p) = \alpha(s)(p)$ for all $p \in U$.*

Proof: We define $\tilde{\alpha}(t)(p) = \alpha(\tilde{t})(p)$ where $\tilde{t}$ is the extension of the section to whole base such that they agree at p. This is well defined since say if there is some other extension t' which agree over then same neighbourhood then we have that $\alpha(\tilde{t} - t') \equiv 0$ on $W \subset U$ by virtue of being local operator. Suppose now there is some other restriction α' with the required property then it will agree in point wise evaluation as we can use the restriction of extension of a section at that point.

With a very similar argument we have the following proposition and a corollary:

Proposition 1.4.3. *If $\alpha : \Gamma(E) \to \Gamma(E)$ is a $\mathcal{F}$-linear operator then we have that the restriction which we shall denote henceforth as $\alpha|_U$ is $\mathcal{F}(U)$-linear.*

Corollary: If $\alpha : \Gamma(E) \to \Gamma(E)$ is a $\mathcal{F}$-linear operator then it is a point operator

We will now focus on bundle maps and $\mathcal{F}$-linear operators as they are closely related and curvature is one such operator, as we will see later.
Proposition 1.4.4. *(Transfer of smoothness from bundle map to bundle induced sectional operator) Let E and F be smooth vector bundles over M. A fibre preserving map $\varphi : E \to F$ which is linear over each fibre is smooth iff $\varphi_\#$ takes smooth sections to smooth sections.*

Proof: Suppose φ is smooth then it follows from definition. Now suppose that $\varphi_\#$ takes smooth sections to smooth sections then a local frame is given by $\bar{e}_i(p) = h^{-1}(p, e_i)$ where h is the trivialisation. Now $\varphi_\#(\bar{e}_i) = \sum_{j=1}^{m} b_j^i f_j$ where f_js form the local frame for F and from hypothesis we get that b_j^is are smooth. Now we evaluate $\varphi\left(\sum_{i=1}^{n} a_i \bar{e}_i\right) = \sum_{j=1}^{m}(\sum_{i=1}^{n} a_i b_j^i) f_j$ here all the coefficients are smooth so the bundle map φ is smooth.
Proposition 1.4.5. *(Restricting to a point) If $\alpha : \Gamma(E) \to \Gamma(F)$ is an $\mathcal{F}$-linear map then for each $p \in M$ there is a unique linear map $\varphi_p : E_p \to F_p$ such that for all sections $s \in \Gamma(E)$ $\varphi_p(s(p)) = \alpha(s)(p)$.*

Proof: Define the map φ_p by $\varphi_p(e_i) = \alpha(\tilde{e}_i)(p)$ where $\tilde{e}_i$ is a section which takes value e_i at p. This is possible since we can take the section from the local frame and extend it to a section of the bundle such that it agrees over some neighbourhood with the original. If we have another such extension which agree over the point then by virtue of being point operator we have the same evaluation. Now this map is unique since for any map we could write its evaluation for some vector as the evaluation of some section by α which makes it unique.

There is a one-one correspondence:

$$\{\text{Bundle maps } \varphi : E \to F\} \longleftrightarrow \{\mathcal{F}\text{-linear maps } \varphi_\# : \Gamma(E) \to \Gamma(F)\}$$
$$\varphi \longmapsto \varphi_\#$$

First surjection of this map follows from the last construction of φ_p. Then the injectivity follows from mapping some section then using the definition.

This previous proposition can be generalised to multi $\mathcal{F}$-linear operators and and this proposition too.

1.5 Connections on vector bundles

To motivate connections we can look at the definition of the directional derivative it is a map $D_{X_p} : C_p^\infty(\mathbb{R}^\ltimes) \to \mathbb{R}$ given by $D_{X_p}(f) = \lim_{t\to 0} \frac{f(p+tv)-f(p)}{t}$ where v is the co-ordinate representation of X_p in $\mathbb{R}^n$ and we use the identity from the partial derivatives section to get that $D_{X_p}(f) = X_p f$, we can now extend this definition to the vector fields on a manifold then we can define $D_X(f)(p) = D_{X_p}(f)$. In the Euclidean case we have another luxury that we have a "global frame" so we can describe our vector fields with just some fixed co-ordinate wise defined maps. So if we have a vector field $Y = \sum_{i=1}^n b_i \partial_i \in \mathfrak{X}(\mathbb{R}^n)$ we can define the directional derivative of this with respect to $X \in \mathfrak{X}(\mathbb{R}^n)$ as $D_X Y = \sum_{i=1}^n X(b_i)\partial_i$. With this we have created a map $D : \mathfrak{X}(\mathbb{R}^n) \times \mathfrak{X}(\mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n)$. This is "the example" for connection, the idea being we want to extend directional derivatives to manifold to study variation of vector fields and in general variations of sections of other bundles. The definition of connection we give subsumes the definition directional derivative. [15] **Remark**: All the results regarding vector bundles, sections and operators are collected in the previous section.

Definition 1.5.1. (Connections as covariant derivative) A connection on a smooth vector bundle $\pi : E \to M$ over a smooth manifold M is defined to be a map $\nabla^A : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ which has the following two properties (Here $\mathfrak{X}(M) = \Gamma(TM)$ and $\mathcal{F}(M)$ is the ring of smooth real valued maps on M. $f \in \mathcal{F}(M)$, $X, Y \in \mathfrak{X}$ and $\sigma, \tau \in \Gamma(E)$):

$$\nabla^A(fX + Y, \sigma) = \nabla^A_{fX+Y}(\sigma) = f\nabla^A_X(\sigma) + \nabla^A_Y(\sigma) \quad (\mathcal{F}(M)\text{-linearity in first argument})$$

$$\nabla^A(X, f\sigma + \tau) = (Xf)\sigma + f\nabla^A_X(\sigma) + \nabla^A_X(\tau) \quad (\text{Leibniz rule in second argument})$$

This particular definition of a connection is sometimes termed as the covariant derivative. The mysterious "A" appearing in the definition is merely a label for the connection, since a bundle might have many distinct connections. And as we go through other definitions and closely related maps to the covariant derivative we will use the same label to note that they are associated objects to the connection.

If we would like to look only at the action of covariant derivative on the second argument and some sense suppress the role of first argument we might do so by considering the following equation which gives the shift in perspective;

$$\nabla^A(X, f\sigma + \tau) = (df(X))\sigma + f\nabla^A(X, \sigma) + \nabla^A(X, \tau)$$

This leads to another definition of connection which takes this shift in perspective into account.

Definition 1.5.2. (Connection) A connection on a smooth vector bundle $\pi : E \to M$ over a smooth manifold is defined to be a map $d_A : \Gamma(E) \to \Gamma(T^*M, E)$ ($\cong \Gamma(\mathrm{Hom}_{\mathcal{F}(M)}(TM, E))$) which obeys the following Leibniz rule:

$$d_A(f\sigma + \tau) = df \otimes \sigma + fd_A\sigma + d_A\tau$$

This obviously allows to associate the covariant derivative associated to the connection and connection through our new definition via;

$$\nabla^A_X(\sigma) = \nabla^A(X, \sigma) = d_A(\sigma)(X)$$

There is another equivalent definition of a connection on a smooth vector bundle over a manifold which perhaps gives us a lot more information regarding the connections. The difference being that in the previous definitions the details and the heavy geometric or topological part is taken care of by use of the language of vector bundles. This new definition will flesh out the underlying topology or geometry and its interaction with this operator defined on sections of the vector bundle. Before we proceed with giving the definitions it is worth sketching the idea behind or the motivation for it as in my thought this definition and the equations and theme capture the spirit of gauge theory! Setting the foreshadowing aside let us remember the alternate definition of vector bundle (see proposition 1.1.1) which acknowledges the locally trivial nature of a vector bundle and demonstrates how it is topologically a patchwork of these pieces parts of which it borrows from the base space which could be better thought of as the parametrising space for the total space of a vector bundle. And the gluing maps exactly do the job of patching the old pieces which now come along with new piece like $\mathbb{R}^n$ or $\mathbb{C}^n$(taken as the fundamental vector spaces) as another factor and need to be accounted for in the process of patching. This trend is not unique to vector bundles but also present in any space which is defined as locally trivial in some sense this could be manifolds, varieties, locally free sheaves and so on. This geometric idea allows us to breakdown many concepts of these kind of spaces into two parts the local part and the global part which could be thought of as patching problem of the local work. With this in mind if we shift our focus back to connections on vector bundles, we have given the global definition and skipped the local description and the patch work. We would want to know how does the connection look on a trivial bundle since over there we get a neat frame and we could evaluate the connection only on these sections comprising the frame and express the action of the connection on the rest using these and the Leibniz rule in our previous definition. And then we could hope to patch this information somehow and recover the connection on the whole bundle. But there are many underlying questions, is an operator like connection even amenable to such local breakdown? Is the sections operation Γ amenable to local breakdown? Do smooth real valued maps $\mathcal{F}$ have such local breakdown? And some other questions which are integral to this process and these are answered in the part in the previous section on "sections"(no pun intended). (see section 1.2)

Now, let us look at the crucial steps and calculations involved leading to the new definition of connections. Let $\pi : E \to X$ be a smooth vector bundle over a smooth manifold X. Suppose we are given a connection d_A on π. Consider the restriction of this connection to $\Gamma(U, E)$ where U is a trivialising open set for π, denoted by d_A^U or $\mathrm{res}_U(d_A)$ or $d_A|_{\Gamma(U,E)}$. On this we would have a frame, let it be $\{\sigma_i\}_{1\leq i\leq n}$ and so this forms the basis for $\Gamma(U, E)$ (see proposition 1.2.1) and consider some section $\tau \in \Gamma(U, E)$ and its expression in terms of basis being $\tau = \sum_{i=1}^{n} f^i \sigma_i$. Then consider the following computation:

$$d_A^U(\tau) = d_A^U\left(\sum_{i=1}^{n} f^i\sigma_i\right) = \sum_{i=1}^{n} d_A^U(f^i\sigma_i)$$

$$= \sum_{i=1}^{n}(df^i \otimes \sigma_i + f^i d_A^U(\sigma_i))$$

$$= \sum_{i=1}^{n} df^i \otimes \sigma_i + \sum_{i=1}^{n} f^i d_A^U(\sigma_i)$$

This almost completes the job, the only thing to note here is that $d_A(\sigma_i)$s take fixed value in $\Gamma(U, T^*M \otimes E)$. We use the multi-linearity of tensors to split the second factor which comes from $\Gamma(U, E)$ (see proposition 1.4.3) and we can write $d_A(\sigma_i)$ as follows:

$$d_A^U(\sigma_i) = \sum_{j}^{n} \omega_{ji} \otimes \sigma_j$$

This is all neatly expressed if we use the matrix convention, combining with the previous computation we may then write it as:

$$d_A(\tau) = d_A(\begin{pmatrix} f^1 \\ f^2 \\ \vdots \\ f^n \end{pmatrix}) = \begin{pmatrix} df^1 \\ df^2 \\ \vdots \\ df^n \end{pmatrix} + \begin{pmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1n} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n1} & \omega_{n2} & \cdots & \omega_{nn} \end{pmatrix} \begin{pmatrix} f^1 \\ f^2 \\ \vdots \\ f^n \end{pmatrix}$$

Here of course, the matrices are defined with expression in terms of the frame $\{\sigma_i\}$ on both the sides of the equation. We can abbreviate this even further by setting up the following notations:

$$\tau = \begin{pmatrix} f^1 \\ f^2 \\ \vdots \\ f^n \end{pmatrix} \quad \omega = \begin{pmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1n} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n1} & \omega_{n2} & \cdots & \omega_{nn} \end{pmatrix}$$

And of course we set that d the exterior derivative acts on individual component, this leads us to the expression:

$$d_A^U(\tau) = d\tau + \omega\tau \quad \text{or} \quad d_A^U = d + \omega$$

This gives us the local expression for the connection d_A. The next step is to observe the transitions, let V be another trivialising open neighbourhood for π. and let $U \cap V$ be non-empty. So, there are two frames $Y = \{\sigma_i\}$ on U and say $X = \{\xi_i\}$ for V. Since we are restricting the same connection the evaluation will not change (see [internal-sheaf of sections]) but the two frames are related by gluing map, call it g, as $g\tau_X = \tau_Y$ and $gd_A^{U\cap V}(\tau)_X = d_A^{U\cap V}(\tau)_Y$ (see [internal-gluing map of tensors]) which leads to the following:

$$d_A^{U\cap V}(\tau)_X = g^{-1}d_A^{U\cap V}(\tau)_Y$$

$$\implies d\tau_X + \omega_X\tau_X = g^{-1}d\tau_Y + g^{-1}\omega_Y\tau_Y$$

$$\implies d(g^{-1}\tau_Y) + \omega_X(g^{-1}\tau_Y) = g^{-1}d\tau_Y + g^{-1}\omega_Y\tau_Y$$

$$\implies (dg^{-1})\tau_Y + g^{-1}d\tau_Y + \omega_X(g^{-1}\tau_Y) = g^{-1}d\tau_Y + g^{-1}\omega_Y\tau_Y$$

$$\implies dg^{-1} + \omega_X g^{-1} = g^{-1}\omega_Y$$

$$\omega_X = -(dg^{-1})g + g^{-1}\omega_Y g = g^{-1}dg + g^{-1}\omega_Y g$$

With these we have means to change the connection matrix when there is a change of frame, and this completes our job of observing the change in connection matrix when there is a transition. Now we are ready to give the new definition of connections on a vector bundle.

Definition 1.5.3. (Connections locally defined) Let $\pi : E \to X$ be a smooth vector bundle over a smooth manifold X. Then a connection on π is defined for any trivialising open cover $\{U_\alpha\}$ with transitions/gluing maps $\{g_{\alpha\beta}\}$ as a collection of (differential) operators $\{((d_A)_\alpha : \Gamma(U_\alpha, E) \to \Gamma(U_\alpha, (T^*X) \otimes E)$ such that $(d_A)_\alpha = d + \omega_\alpha$ in matrix formulation$\}$ where d is the exterior derivative from X and ω_αs are $n \times n$ matrices of 1-forms called the connection matrices which are related to each other on each $U_\alpha \cap U_\beta$ by the following relation:

$$\omega_\beta = g_{\alpha\beta} d g_{\alpha\beta}^{-1} + g_{\alpha\beta}\omega_\alpha g_{\alpha\beta}^{-1}$$

Remark: We have implicitly assumed that the matrix formulation is understood but for clarity we should understand that in $\mathrm{Dom}((d_A)_\alpha)$ we have a frame $\Sigma_\alpha = \{\sigma_1^\alpha, \sigma_2^\alpha, \cdots, \sigma_n^\alpha\}$ and in $\mathrm{Codom}((d_A)_\alpha)$ we use the same sections as the frame for the factor with respect to which we are making the matrix representation.

Naturally, we again ask how is this associated with the connection d_A we defined earlier? To which one way the answer is clear that given a connection d_A we can define the collection of operators as $\{d_A^{U_\alpha}\}$ for any given trivialising open cover. And suppose we are given a collection as mentioned in the definition $\{(d_A)_\alpha = d + \omega_\alpha\}$ for some trivialising cover, we may define a map $d_A : \Gamma(E) \to \Gamma(T^*X \otimes E)$ where we define $d_A(\sigma)(p) = (d_A)_\alpha(\sigma|_{U_\alpha})(p)$, it is well defined due to the final computation we made regarding change of frame. If obeys the Leibniz's rule since;

$$(d_A)_\alpha(f\sigma + \tau) = d(f\sigma + \tau) + \omega_\alpha(f\sigma + \tau) =$$

$$(df)\sigma + f(d\sigma) + d\tau + f\omega_\alpha\sigma + \omega_\alpha\tau =$$

$$df(\sigma) + f(d + \omega_\alpha)(\sigma) + (d + \omega_\alpha)(\tau)$$

And if we translate it back to abstract operator theoretic version this is exactly the Leibniz's rule as the second and the third summand can be seen as d_A. It would be instructive to show why the first term would translate back to $df \otimes \sigma$, so we do that in the following block:

$$d(f\sigma)_{\Sigma_\alpha} = \begin{pmatrix} d(fs^1) \\ \vdots \\ d(fs^n) \end{pmatrix} = \begin{pmatrix} s^1 df \\ \vdots \\ s^n df \end{pmatrix} + \begin{pmatrix} f ds^1 \\ \vdots \\ f ds^n \end{pmatrix} = df\sigma_{\Sigma\alpha} + fd\sigma_{\Sigma_\alpha}$$

So, it must be clear that $df(\sigma) = df\sigma_{\Sigma_\alpha}$. Then, the method to convert it to abstract convention we note that $df(\sigma)$ lies in the space $\Gamma(U_\alpha, T^*X \otimes E)$ and for which as mentioned in the previous remark the matrix representation is through second factor of the tensor product which means that the conversion map is given by,

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix} \longmapsto \sum_{i=1}^{n} \omega_i \otimes \sigma_i^\alpha$$

which means that indeed $df(\sigma)$ corresponds to $df \otimes \sigma$. So this completes the verification of the relation between the two definitions of connections.

Having seen multiple definitions for connections there should be some clarity about the object. The next step would be to see how does this notion fare with operations on vector bundles like the direct sum, tensor product, dual, pull back and so on. This will be critical to our work in all coming chapters where we aim to use connections as freely on vector bundles as possible. We will now move to curvature and parallel transport which will demonstrate the need of the idea I mentioned. After which we will introduce an idea which will help us with this process of transferring bundles and in general handling the connections much more convenient.

1.6 Parallel transport

Parallel transport is a geometric idea, where we are asking given a curve can we move a vector along it in a way that it remains parallel to the tangent to the curve at all points during the journey, the vector may come from any bundle but the comparison is with respect to the tangent vectors associated to the smooth manifold which we then restrict to the curve on it. This might seem very closely related to the idea of connection and indeed it is, over there we are making comparisons and in fact give a measures of deviation with respect to the sections of tangent bundles. Here, as one might suspect if we have a successful parallel transport then it should mean that with respect to the tangent vector field there has been no deviation which should reflect in the evaluation of connection of this section with respect to the tangent vector field as zero. This is indeed what we will find but there are again many details and questions we must answer before being able to establish such a thing and to that end, we will use pull back of the bundle to the curve and we shall define pull back of a connection in the same spirit, which will yield us the technical formulation of parallel transport.

Proposition 1.6.1. *(Pull back of a connection) Consider a smooth vector bundle $\pi : E \to X$ and a connection d_A on π. Suppose we have a smooth map $f : Y \to X$ then on the pull back bundle $f^*(\pi)$ we have a unique connection d_{f^*A} on $f^*\pi$, such that the following diagram commutes:*

$$\begin{tikzcd}
\Gamma(E) \arrow[r, "d_A"] \arrow[d, "f^*"'] & \Gamma(T^*X \otimes E) \arrow[d, "f^*"] \\
\Gamma(f^*E) \arrow[r, "d_{f^*A}"] & \Gamma(T^*Y \otimes f^*E)
\end{tikzcd}$$

To see this the local picture of a connection is very useful. We would have local connection forms $\{\omega_\alpha\}$ corresponding to some trivialising open cover $\{U_\alpha\}$ of X and the discussion on pull back of a bundle (see definition 1.3.6) would motivate us to define the connection on pull back by giving local connection forms as $\{f^*\omega_\alpha\}$ for the trivialising cover $\{f^{-1}(U_\alpha)\}$. And the gluing relation for defining the connection will hold since we can use the gluing condition for the connection d_A and pull it back, and pull backs distribute over C^∞-linear combinations, moreover they respect tensor products. These properties are proved in [6]. Also this should make clear why the diagram commutes.

Having established pull back of a connection let us consider a smooth curve $\gamma : [0, 1] \to X$ where X is the smooth manifold under consideration with a smooth vector bundle $\pi : E \to X$ over it. Suppose π is equipped with a connection d_A, so we form the pull back bundle $\gamma^*\pi$ and equip it with the pull back connection d_{γ^*A}. We note that $\gamma^*\pi$ will be a trivial bundle (we could directly use the result we discussed (see proposition 1.3.3)) using the idea that we may consider a trivialising open cover of the interval, and due to compactness we will get a finite sub-cover which is also trivialising. We may even consider them to be only connected, as this will only cause us to split the domains of some trivialisations which would be perfectly fine and then we may order them based on their infimum. Then we may patch trivialisations using gluing maps and redefining the trivialisations so they agree on the intersection, we repeat this process and reach the end of the interval and we would have one big trivialisation! (One can note that this is argument is very similar to) This might feel like sorcery and one might wonder why this might not be possible on every bundle, but we may note that the speciality here was that sequence of open sets didn't come to bite back on a previous element and if that were to be the case we would have a conflict with two gluing maps to account for, this might be clear if one visualises the Möbius strip which is also a bundle over one dimensional manifold but is not trivial like a circular strip. We also note that one-forms are top forms on one manifolds so each differ only by a coefficient smooth function each taking the form $f dt$ where $f \in C^\infty([0,1])$ and dt is the exterior derivative of the single co-ordinate function t on the interval. We may express the pull back connection as:

$$d_{\gamma^* A} = d + \gamma^* \omega_\alpha$$

And the selection of ω_α shouldn't matter as all would result in the same connection due to triviality and we may write:

$$\omega := \gamma^* \omega_\alpha = \begin{pmatrix} f_1^1 dt & f_2^1 dt & \cdots & f_n^1 dt \\ f_1^2 dt & f_2^2 dt & \cdots & f_n^2 dt \\ \vdots & \vdots & \ddots & \vdots \\ f_1^n dt & f_2^n dt & \cdots & f_n^n dt \end{pmatrix}$$

And finally we may consider the equation of parallel transport which is simply setting the evaluation of connection on a section to be zero (Let the section be given by $s = \sum_{i=1}^n s^i e_i$, where s^i are the smooth co-ordinates of the section):

$$d_{\gamma^* A}(s) = ds + \omega s = 0$$

$$\implies \frac{ds^i}{dt} + \sum_{j=1}^n s^j f_j^i = 0$$

From the above expression we get n linear ODEs which will yield us a unique solution given our choice of initial points, so this means that we may pick any vector of $(f^*\pi)^{-1}(0)$ and this would hold so this gives us isomorphism of vector spaces $(f^*\pi)^{-1}(0)$ and $(f^*\pi)^{-1}(1)$. As one can move the reverse direction starting from the other end as well, and clearly the map is linear. But one should note that this isomorphism is not any isomorphism, since that would be pointless because all finite dimensional vector spaces of the same dimension are isomorphic. This is a canonical isomorphism, for now this means that we don't refer to a basis to establish it. Also since the fibres of pull back bundle are canonically isomorphic to fibres of the original so this extends to the original bundle.

Definition 1.6.1. (Parallel transport) Given a smooth vector bundle $\pi : E \to X$ equipped with a connection d_A on it, parallel transport is an isomorphism for each smooth curve on the manifold $\gamma : [0,1] \to X$ denoted by $PT_A(\gamma) : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1))$ such that $v \in \pi^{-1}(\gamma(0))$ and $PT_A(\gamma)(1)$ are ends of section whose pull back obeys the equation $d_{\gamma^* A} s = 0$.

This also allows us to the show the theorem regarding pull backs of vector bundle via homotopic maps (see proposition 1.3.3) which we had seen topologically in the smooth case, where we may use the smooth homotopy to put the vector bundles on two ends and use smooth curves to parallel transport the fibres over the domain of this homotopy, this process gives the proof. But one may refer to [11]

1.7 Curvature

Curvature in daily life might be related to the idea of a bend in shapes or manifolds, and indeed there is some truth to it. But we would not want to account for any sort of bend, as it might simply be an effect of ambient space and not intrinsic to the manifold under consideration. For example, one would consider a cylindrical tube to be a curved surface or even simpler a circle to be a curved but these notions are due to comparisons with the ambient space which gives one additional co-ordinates or dimensions to measure the bend. But as Gauss characterised that these are extrinsic. To be more clear, consider the example of the unit circle in the Euclidean plane, this appears curved because we have two co-ordinates in the ambient space. If we were to stick to the circle itself the tangent vectors would not detect any such thing simply by the virtue that it is one dimensional. While if we now consider the ambient space then in addition to tangent there is normal direction as well and indeed the curvature would be measured thus. And the case of cylinder is similar. So, intrinsic curvature would only measure changes with respect to the tangent bundle and so one may notice that it is only possible to possess intrinsic curvature if the manifold has more than one dimension to it. Here, I borrow the idea of curvature from [Lee], where he presents it as a failure of producing a parallel vector field locally on a Riemannian surface X. The idea is to take two local co-ordinates x^1, x^2 and axes, try to spam a vector throughout but to do it smartly using parallel transport. That is first transport to each point along one axis and then transport those vectors along the second axis shifted to each point this is the best hope because we have kept them parallel at least with respect to axis. Let this vector field be ρ, and of course the connection ∇ under consideration is the Levi-Civita connection.

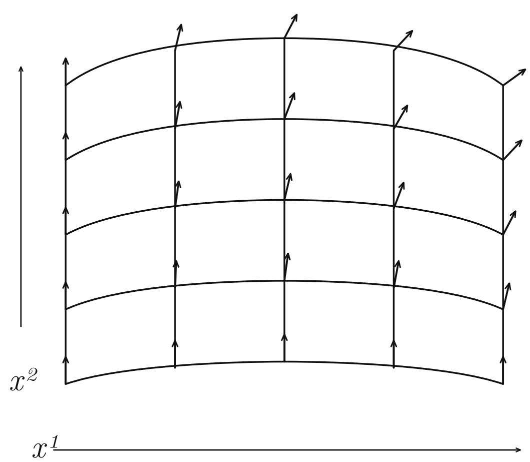

Figure 1.2: A failed parallel field of vectors

And if we were successful to produce a parallel field of vectors it would mean that $\nabla_X(\rho) = 0$ for all vector fields X which of course we could reduce to $\nabla_{\partial_{x^i}}(\rho) = 0$ for each co-ordinate x^i, and we know it holds for one vector by definition of ρ. So, we need $\nabla_{\partial_{x^1}}(\rho) = 0$, but if this is true it would mean that $\nabla_{\partial_{x^2}}\nabla_{\partial_{x^1}}(\rho) = 0$. We could have that if only the derivatives (covariant) could be swapped since we already have it for one co-ordinate, so we are asking for $\nabla_{\partial_{x^2}}\nabla_{\partial_{x^1}}(\rho) - \nabla_{\partial_{x^1}}\nabla_{\partial_{x^2}}(\rho) = 0$ which may not hold for a general Riemannian surface. One more thing to note before we move on is that we used co-ordinates and if we simply wanted to avoid it we might use the other notion of connection which does the service of suppressing the first argument of covariant derivative associated to the connection. Let us label the connection associated to this as d_{LC}.

Then we would have that $d_{LC}\rho = 0$ and then leading to $d_{LC} \circ d_{LC}(\rho) = 0$ which doesn't make sense strictly speaking as we are forcing the connection to act on an element of $\Gamma(T^*X \otimes E)$ so this leaves us with some work leading to a transition into discussing the connection and curvature more algebraically as well.

First let us look at what we would want out of the connection acting on the new space. We want it to produce an operator which gives section valued two forms taking in section valued one forms. To declutter the notation we define $\Omega^p(U, E) := \Gamma(U, \bigwedge^p T^*X \otimes E)$. Before we delve into the discussion consider the definition:

Definition 1.7.1. (Extension of connection to higher forms) Given a smooth vector bundle $\pi : E \to X$ equipped with a connection d_A there exists a unique connection $d_A^k : \Omega^k(E) \to \Omega^{k+1}(E)$ which obeys the following Leibniz rule:

$$d_A^{(k)}(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^k \omega \wedge d_A^{(0)}\sigma$$

One might wonder why such a formula appears, this will become clear if we consider the definition of a connection and it's range in order to adjust the exterior product we get the sign moreover one may ask as to why are we taking sum over tensor and demanding such a rule? This is a more subtle question and we leave this for a later discussion which touches on the part that I had mentioned before the beginning the discussion on parallel transport, how does the notion of connection behave with respect to various operations on vector bundle.

Remark: Generally we will not make the distinction in notations $d_A^{(k)}$s, we will abuse the notation just as in the case of exterior derivative and we will refer to all of them as d_A.

With these technicalities sorted we will return to our original examination of curvature which we defined to be $d_A \circ d_A$. Firstly we will note that unlike the connection curvature is C^∞- linear operator, which can be seen through the following calculation:

$$d_A \circ d_A(f\sigma + \tau) = d_A(df \otimes \sigma + f d_A \sigma + d_A \tau)$$

$$= d(df) \otimes \sigma - df \wedge d_A \sigma + df \wedge d_A \sigma + f d_A d_A \sigma + d_A d_A \tau$$

$$= f d_A d_A \sigma + d_A d_A \tau$$

So, if we consider this form locally (see [internal-local operatos]) we may express it through a matrix of two form which like in the case for connection matrix is defined with respect to a local frame:

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1n} \\ \Omega_{21} & \Omega_{22} & \cdots & \Omega_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{n1} & \Omega_{n2} & \cdots & \Omega_{nn} \end{pmatrix}$$

But we can relate this curvature matrix with the connection matrix through the following computation:

$$\Omega\sigma = d_A(d\sigma + \omega\sigma)$$

$$= d(d\sigma) + d\sigma \wedge \omega + d\omega\sigma + \omega \wedge d\sigma + \omega \wedge \omega\sigma$$

$$= d\omega\sigma + \omega \wedge \omega\sigma$$

$$\implies \Omega = d\omega + \omega \wedge \omega$$

Though it may appear simple but it is hides devious application of the Leibniz rule and may be more instructive to be done the summation form for clarity, we have avoided it in order to keep the form cleaner. This is called the structure equation.

Also, if we have the local curvature forms $\{\Omega_\alpha\}$ corresponding to a trivialising open cover $\{U_\alpha\}$ and gluing maps $\{g_{\beta\alpha}\}$ we can see that during transition these are related and the relation is made clear by the following calculation:

$$g_{\beta\alpha}(\Omega_\alpha \sigma_\alpha) = \Omega_\beta \sigma_\beta$$

$$\implies g_{\beta\alpha}\Omega_\alpha g_{\beta\alpha}^{-1}\sigma_\beta = \Omega_\beta \sigma_\beta$$

$$\implies \Omega_\beta = g_{\beta\alpha}\Omega_\alpha g_{\beta\alpha}^{-1}$$

So, like in the case of connections we have established a local description and patchwork for the curvature as well.

1.8 Characteristic classes

Characteristic classes can be described in many ways, some of these coming from algebraic topology and some from differential geometry, the way in which we will describe them here will be from differential geometry, which is generally considered part of what is called the Chern-Weil theory. This is in part because the way we will use these most of the times will be when we are working on smooth manifold with bundles and connections on them. The other more direct methods from algebraic topology will also be relevant later but we will not present that setting as it will make this part needlessly long and the essence of the theory is clearly realised through Chern-Weil theory. At times if we utilise them in slightly different light (obstruction theoretic) we will make the idea clear.

The motivation for this concept comes from trying to build diffeomorphism invariants, like how Gauss-Bonnet theorem produced for Riemannian surfaces, in fact generalising the Gauss-Bonnet theorem was one of the historical motivation behind this theory. Let us examine the formula due to the Gauss-Bonnet theorem so that we may try to replicate something of this sort:

$$\int \frac{K}{2\pi}\text{vol} = \chi(M)$$

Here Kvol is locally just the curvature form Ω^1_2, while on the other side we have the Euler characteristic of the manifold, which is a diffeomorphism invariant, independent of the Riemannian structure on the manifold. This makes us think if we can create diffeomorphism invariants using local information from Riemanninan structure like the curvature forms and such, for general higher dimensional manifolds? Or for that matter if we could do this for some bundles other than tangent bundle of the manifold?

One idea is to produce a global closed differential form from this local curvature information, which is simply the curvature matrix for us. And then to check whether it defines a genuine cohomology class, which in this case might be considered in de Rham sense as long as we are dealing with smooth vector bundles. And then we would have to show that the definition of such a class is independent of the connection we chose to define it, so that it will become a genuine diffeomorphism invariant. This seems like we are asking for a lot but the reason we might expect such a thing to be possible is because of the precedent set by the Gauss-Bonnet theorem. It will appear even more plausible if look at the resolution for the first step, which is a brilliant trick one finds to work around this problem of changing curvature matrix across different patches.

If we remember the curvature matrix varied across the patches through conjugation by gluing maps, and if we consider what are some actions which remain unchanged by conjugation on matrices we will fall upon the concept of invariant polynomials, "the examples" of which are trace of matrix and determinant of a matrix. Which leads us to define:

Definition 1.8.1. (Invariant polynomials) Let $P(X) \in \mathbb{R}[x^i_j]$ is called an polynomial on $M(n, \mathbb{R}) = \mathbb{R}^{n\times n}$ the collection of $n \times n$ matrices taking real entries if x^i_j are indeterminates we can represent as matrix. The polynomial $P(X)$ is called an *invariant polynomial* if $P(X) = P(A^{-1}XA)$ for all $A \in \mathrm{GL}(n, \mathbb{R})$.

The most important example of the invariant polynomials are $tr_i(X) = tr(X^i)$ where $tr(X)$ is the trace of the matrix, because these generate the algebra of invariant polynomials. [15]

$$\mathrm{Inv}(M(n, \mathbb{R})) = \mathbb{R}[tr_1, \cdots, tr_n]$$

This result is reassuring as now rather than being stuck with a particular examples of differential forms which may or may not work we are not left with plethora of examples coming by the way of invariant polynomials and we may be systematically produce a lot of invariants if things work out well.

We move on to the next steps of the general procedure which we had discussed, checking whether they are closed and then independent of the connection defining them, if of course something doesn't work out we may restrict to a subclass but fortunately we are able to use all of these polynomials. Just as a reminder when we do take polynomial with the matrix of 2-forms the right notion of multiplication is the wedge product and this makes for a commutative $\mathbb{R}$-algebra say for example if we take some point on the manifold $p \in M$ and we define $\mathcal{A} = \bigoplus_{i=0}^{\infty} \bigwedge_{2i} T_p^* M$ this forms a commutative algebra of co-vectors at the point p. So our form evaluated at that point will reside in such an algebra. If our curvature matrix is Ω then we will have at some point $P(\Omega_p) \in \mathcal{A}$. But we know that if we go to some other point which lies in the intersection the form evaluated there must differ by conjugation but because we are using an invariant polynomial it doesn't matter. So we are able to define a two form independent of the neighbourhood or frame we consider. Putting the discussion together we have that if we have Ω defined on some framed neighbourhood then $P(\Omega)$ is defined globally on the manifold. As we noted in the part of invariant polynomials it is sufficient to show the result of closed ness and independence for trace polynomials as they are generators of the invariant polynomials.

Suppose we have two matrices $A = [\alpha^i_j]$ and $B = [\beta^i_j]$ of forms of degree a and b respectively then we define that $A \wedge B = \left[\sum_{k=1}^{n} \alpha^i_k \wedge \beta^k_j\right]$ and $dA = [d\alpha^i_j]$. Now to show that $P(\Omega)$ is closed we will instead as mentioned before just show that $tr_k(\Omega)$ is closed i.e. $dtr_k(\Omega) = 0$.

There are some preliminary algebraic results relating trace, wedge and exterior derivative which we make note of which will be useful in showing the result:

1. $(A \wedge B)^T = (-1)^{ab} B^T \wedge A^T$
2. $tr(A \wedge B) = (-1)^{ab} tr(B \wedge A)$
3. $dtr = trd$

We can prove the above results simply by using the definition of trace and doing some basic algebraic manipulations. We also need to evaluate $d\Omega^k$, but this we are able to find from the generalised Bianchi identity:

1. $d\Omega = \Omega \wedge \omega - \omega \wedge \Omega$ (Basic Bianchi identity)
2. $d\Omega^k = \Omega^k \wedge \omega - \omega \wedge \Omega^k$(Generalised Bianchi identity)

The first one can be obtained from the differentiating the second structure equation $\Omega = d\omega + \omega \wedge \omega$ and the second general version follows from the first. With this we do the following computation which completes the proof for closedness of $P(\Omega)$:

$$\begin{aligned}
dtr(\Omega^k) &= trd\Omega^k \\
&= tr(\Omega^k \wedge \omega - \omega \wedge \Omega^k) \\
&= tr(\Omega^k \wedge \omega) - tr(\Omega^k \wedge \omega) \\
&= 0
\end{aligned}$$

To clarify what we mean by independence from connection is that suppose we have two connections ∇^0 and ∇^1 corresponding on the same bundle E over the manifold M, then we get corresponding two curvature tensors, and for our discussions we can consider them to be two matrices Ω^0 and Ω^1. So we want to show that $[tr_k(\Omega^0)] = [tr_k(\Omega^1)]$ as cohomology classes which means that $tr_k(\Omega^1) - tr_k(\Omega^0) = d\eta$. We will present the proof of independence in following steps:

1. We notice that if we have two connections on the same bundle then their convex combination $\nabla^t = (1-t)\nabla^0 + t\nabla^1$ is also a connection on our connection, and this equation allows us to define a smooth family of connections that is if we were to consider the matrix of the connection has the following form $\omega_t = (1-t)\omega^0 - t\omega^1$.
2. We can define the notion of a smoothly varying family of k-forms by, saying $\omega_t = \sum_I a_I(t)dx^I$ has smoothly varying coefficients and then we can co-ordinate wise define the derivative $\frac{d}{dt}$ and $\int$ for such family and we can work out the relation these have with $tr, \wedge$ and d.(Let us call this collection of information as Algebra of $\frac{d}{dt}$ and $\int$)
3. Now we use the algebra to compute the following $\int \frac{d}{dt}tr(\Omega_t^k)$ this definitely yields $tr_k(\Omega^1) - tr_k(\Omega^0)$ in one way but the previous way will give us another way of evaluating this and we might get what we desired for.

Indeed we do get something of this sort because the algebra we find for step 2 is conducive;

Proposition 1.8.1. *(Algebra of $\int$ and $\frac{d}{dt}$) Suppose ω and τ are smoothly varying(wrt t) matrices of forms on our manifold M. Then we have that: 1. $\frac{d}{dt}tr = tr\frac{d}{dt}$ 2. $\frac{d}{dt}d = d\frac{d}{dt}$ 3. $d\int = \int d$ 4. $\frac{d}{dt}(\omega \wedge \tau) = \dot{\omega} \wedge \tau + \omega \wedge \dot{\tau}$*

Now we make the computation which completes the proof for independence of $[P(\Omega)]$ from the connection on the bundle.

$$
\begin{aligned}
\frac{d}{dt}tr(\Omega^k) &= tr\left(\frac{d}{dt}\Omega^k\right) \\
&= tr(\dot{\Omega}\Omega\cdots\Omega + \Omega\dot{\Omega}\cdots\Omega + \cdots + \Omega\cdots\dot{\Omega}) \\
&= ktr(\Omega^{k-1}\dot{\Omega}) \\
&= ktr(\Omega^{k-1}(d\dot{\omega} + \dot{\omega}\omega + \omega\dot{\omega})) \\
&= ktr(\Omega^{k-1}d\dot{\omega} + d\Omega^{k-1}\dot{\omega}) \mid \text{This follows from generalised Bianchi identity} \\
&= ktr(d(\Omega^{k-1}\dot{\omega}))
\end{aligned}
$$

With this we have that the map $c_E : \mathrm{Inv}(\mathfrak{gl}(n, \mathbb{R})) \to H^*(M)$ given by $P(X) \longmapsto [P(\Omega)]$ is well defined diffeomorphism invariant on the manifold M for a bundle E over it. Which we will collect in the definition below:

Definition 1.8.2. (Chern-Weil homomorphism) Given a smooth vector bundle $\pi : E \to M$ equipped with a connection d_A and curvature matrix at some point on manifold given by Ω_A, there exists a map which is a homomorphism of algebras given by:

$$c_E : \mathrm{Inv}(M(n,\mathbb{R})) \to H^*(M)$$

$$P(X) \longmapsto [P(\Omega_A)]$$

1.9 Unitary connections on complex line bundles

From this section of the chapter we will try to flesh out the details of an important case, which takes the heading of this section; unitary connections on complex line bundles. In most of the previous sections we considered smooth vector bundles but most of the results we got can easily be extended to smooth complex vector bundles as well, since we don't particularly use the properties of the field anywhere. A matter where such a distinction would matter is the classification of complex vector bundles versus the real vector bundles evidently because of the difference in gluing maps. For complex vector bundles they would go into $GL(n,\mathbb{C})$ while for the real counterpart they would go into $GL(2n,\mathbb{R})$. And these have marked differences, leading to differences in orientability primarily and also in characteristic classes associated with them then. Setting that matter aside for time being we must first justify why we are considering this case or problem. There are three main reasons, first being that this is a case where we will get to define gauge transformations, gauge equivalence, gauge covariance and so on. Second reason is that this will lead us to a fresh viewpoint on Maxwell's equations and electromagnetism through formulation of Yang-Mills action functional, this will be relevant as it will give a hint for why the study of gauge theory is so crucial to physics and mathematics, which is essentially inspired from the physics. And the last reason is that in itself this work will prove relevant in later chapters due to a use of the complex line bundles (in defining Spin^c structure and consequently moduli space of monopoles). We will begin by noting a few things about unitary connection on complex line bundles. We recall that having a unitary connection on a complex vector bundle means that the connection forms $\{\omega_\alpha\}$ corresponding to a trivialising cover lie in $\mathrm{Lie}(U(n)) = \mathfrak{u}(n)$ (see **??**) which means that it is a skew hermitian matrix and in the case of complex line bundles due to dimension this will mean that it is purely imaginary. Moreover, due to the formula (see section 1.7) we find that the curvature matrix $\Omega_\alpha = d\omega_\alpha$ and again due to dimension and the formula this will yield that all the curvature matrices which are also one dimensional will be the same and so we will get a global curvature matrix Ω_A and due to it being one dimensional we can account for it by one purely imaginary 2-form. And we will also get that it is closed due to its structure. With these let move to the problem of classifying unitary connections on a complex line bundle.

For clarity, let us set that $\pi : E \to X$ is a smooth complex vector bundle, one dimensional. Consider a unitary connection, d_A on π. Then, as we have noted we might express it to locally look like $d + i\omega_\alpha$ for some trivialising open cover $\{U_\alpha\}$. And if we remember the gluing condition for local connection forms we have $i\omega_\beta = g_{\beta\alpha} dg_{\beta\alpha}^{-1} + g_{\beta\alpha} i\omega_\alpha g_{\beta\alpha}^{-1}$, which suggests that the first summand on the right hand side is completely dependent on the transitions of the bundle. So, if we would consider another unitary connection $d_{A'}$ on this bundle and take its local connection forms associated with the same cover we might have some different one forms but if we take the difference of both connections (defined through difference in their evaluation) we find that it will differ by a one forms say for some patch it is b and through the patches this form remains unchanged since the difference of connections transform simply by conjugation and the first parts gets cancelled out. And this means that it will be a global purely imaginary one form $b = ia$. This we may summarise in the following proposition:

Proposition 1.9.1. *(Space of unitary connections for complex line bundle) Given a smooth complex vector bundle of rank* 1*, equipped with a unitary connection d_A any other unitary connection $d_{A'}$ differs by a global purely imaginary form ia.*

So in a sense we have classified all the unitary connections over a smooth complex vector bundle of rank 1. But this is not the sort of classification we are looking for. This is where we ask are each of those connections we get simply by shifting by a one form distinct, or in other words is the equality upto evaluation the equality we want to consider for the set of connections? The answer is no and there is a very compelling reason as to why we should change the notion of equality or equivalence of connections. It is precisely the idea of gauge transformations. When we think of unitary connections on smooth complex bundle we have the action of $U(1)$ which can "twist" the frame we choose at any point or neighbourhood, in fact this may act smoothly on each fibre throughout the vector bundle and this doesn't really change the vector bundle so the connections must respect this "twisting" or to say that if they do in fact only differ by such a twist they should be treated equivalent. In this special case we may treat the action by the group $U(1)$ to be given by a smooth map $X \to S^1(= U(1))$ and this twisting is termed as gauge transformation and if two connections are related by a gauge transformation then they are considered to be gauge equivalent. We will formalise for the general set up in the last section of this chapter on principal bundles (see section 1.10), which one can say is the "right" framework to look at when we want to get rid of or handle frame change matters in one go. We will denote the affine space of connections by $\mathcal{A}$.

Definition 1.9.1. (Gauge transformations and equivalence) Given a smooth complex bundle of dimension 1 $\pi : E \to X$, the collection of maps $\mathcal{G} = \{g : X \to S^1\}$ is called gauge group and each of its element is called a gauge transformation. Two connections d_A and $d_{A'}$ area called gauge equivalent if they are related by the following formula:

$$(\omega_{A'})_\alpha = gdg^{-1} + g(\omega_A)_\alpha g^{-1} \quad \forall \alpha$$

where $(\omega_A)_\alpha$ and $(\omega_{A'})_\alpha$ are connection matrices (in this particular case 1-forms) associated to the connections d_A and $d_{A'}$ respectively, and, $g \in \mathcal{G}$.

Also then the quotient "space" of connections, consisting of equivalence class of connections with respect to gauge transformations will be denoted by $\mathcal{B} = \mathcal{A}/\mathcal{G}$. We put the word space in quotations because we haven't specified what the topology of any of these objects is. They do indeed have a notion of topology which we get by producing C^k norms or as Sobolev spaces but we will postpone this work as we will not use these properties in this particular case. Though later when we will need them we will give appropriate clarification from functional analysis. Given this new notion of equivalence how will we classify the connections? Let us recollect the fact that curvature in this case is determined by single global two form F_A, so we might be able to recover the connection using the curvature if we were to use the additional condition that they are gauge equivalent, which if we employ gives us the following calculation:

$$(\omega_{A'})_\alpha = gdg^{-1} + g(\omega_A)_\alpha g^{-1}$$

$$\implies (\omega_{A'})_\alpha = id\theta + (\omega_A)_\alpha$$

$$\implies F_{A'} = F_A$$

which shows that two gauge equivalent connections will have the same curvature form. And we also know that due to this being the single global form it defines the first Chern class of the vector bundle, so now a question arises do all 2-forms which belong to the first Chern class characterise the connections? The answer is positive:

Proposition 1.9.2. *(Classification of unitary connections over complex line bundles-1) Given a smooth complex line bundle $\pi : L \to X$ over a simply connected manifold, we have a bijection;*

$$\tau : \mathcal{B} \to C_1(L)$$

$$d_A \longmapsto \frac{iF_A}{2\pi}$$

Proof: We have seen that the map is well defined, first we will show that the map is surjective, Suppose we pick the imaginary two form Ω corresponding to an element in $C_1(L)$. We could pick any unitary connection, say d_A and we could take the curvature form associated with it, say F_A and so we may write $F_A - \Omega = da$ and we would then pick $d_A - a$ as our new connection, corresponding to which we would get the desired curvature. To show that the map is injective, consider two connections d_A and $d_{A'}$ which have the same curvature F. What we have then is that,

$$d(\omega_A)_\alpha = d(\omega_{A'})_\alpha$$

$$\implies d((\omega_A)_\alpha - (\omega_{A'})_\alpha) = 0$$

$$\implies \exists\, \varphi \in C^\infty(X): \quad (\omega_A)_\alpha - (\omega_{A'})_\alpha = id\varphi \quad \text{since } H^1(X, \mathbb{R}) = 0$$

So the gauge transformation relating these would be given by $e^{i\varphi}$, completing the proof.

We will now consider the case where instead of the manifold being simply connected we have that the $H_1(X, \mathbb{Z})$ is free abelian of rank b_1. In this case we have to account for holonomy about the curves on X which generate $H_1(X, \mathbb{Z})$, and we get the following result:

Proposition 1.9.3. *(Classification of unitary connections over complex line bundle) Given a smooth complex line bundle $\pi : L \to X$, with $H_1(X, \mathbb{Z})$ being free abelian of rank b_1 generated by curves $\gamma_1, \gamma_2, \cdots, \gamma_{b_1}$, we have a bijection;*

$$\tau : \mathcal{B} \to C_1(L) \times \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{b_1 \text{ times}}$$

$$d_A \longmapsto \left(\frac{iF_A}{2\pi}, PT_A(\gamma_1), \cdots, PT_A(\gamma_{b_1})\right)$$

[11]

As we had mentioned, one of the interesting applications of this classifications is to electromagnetism. If you recollect from physics the Maxwell's equations which are fundamental equations describing electromagnetism are usually phrased using lingo from differential geometry especially using operators curl and div which themselves can be rephrased using co-variant derivative notion. So, there is a way it turns out that those equations can be further compressed if we combine the electric and magnetic fields into one tensor called the Faraday tensor:

$$F = -E_x\, dt\wedge dx - E_y\, dt\wedge dy - E_z\, dt\wedge dz + B_x\, dy\wedge dz + B_y\, dz\wedge dx + B_z\, dx\wedge dy$$

And then using this we can rewrite Maxwell's equations as:

$$dF = 0 \qquad d\star F = \rho$$

And this may appear oddly related to our calculations regarding curvature of line bundle, which yields closed two-form. And there are more reasons than that to consider that the Faraday tensor be realised from the curvature of unitary connection over complex line bundle. Physically because $F \in C_1(L)$ it explains quantization of magnetic charge, and the holonomy can be experimentally detected as Bohm-Arahanov effect. And more importantly as we change the structure group it gives the basis or mathematical framework for other forces finally leading to the standard model.

For the relevant theorems from Hodge theory one may refer to [11]. We will end this section by setting up the Yang-Mills functional in this case we considered. From Hodge's Theorem and the classification theorem for connections presented in this section, we see that every complex line bundle with Hermitian inner product over a compact oriented simply connected Riemannian four-manifold X has a canonical connection, one with harmonic curvature, and this is captured by the Yang-Mills functional.

Definition 1.9.2. (Yang-Mills action) Given a smooth complex one dimensional vector bundle $\pi : L \to X$, the Yang-Mills action is defined as:

$$\mathcal{Y} : \mathcal{A} \to \mathbb{R}$$

$$d_A \longmapsto \int_X F_A \wedge (\star F_A)$$

The extremization of this action yields us the connections with harmonic curvature. This when generalised to quaternionic line bundle leads to non-abelian gauge theory which is the basis for Donaldson's groundbreaking work.

1.10 Principal G-bundles

There are far too many reasons for introducing the idea of principal G-bundles, it is part of the reason why this sophisticated machinery was left for the end of this chapter. Firstly when we consider the connections for vector bundles they provide the advantage of viewing them as differential operator but there are many inconveniences when it comes to inducing the connections to associated vector bundles through the simple algebraic operations on them like the direct sum, tensor product or dual. In the section on curvature especially this becomes evident as there it is not very clear as to why the Leibniz rule extends the way it does. The observation that connection forms take value in $\mathrm{Lie}(G)$ is also not quite self evident. Also as we will move ahead we will consider an additional information which was implicit in our work on vector bundles, the action of G which we saw make an appearance when we discussed gauge group, and gauge transformations. The most broad definition of the gauge group would be the best in principal bundle terminology. All of these will be useful as we move to later chapters, so we discuss this object along with the results which are useful.

When defined the connections through local differential operators we to give a rule that allowed us to relate the connection forms when we changed frames. The former definitions of connection either as covariant derivative or the one we obtain by suppressing the vector fields are rather algebraic or abstract in the sense that they don't give anything explicit like the connection matrix simply because we don't always have a global frame for bundles. But none the less if we make a choice of cover we get the connection matrices. One might wonder is there a global object or global connection matrix or connection forms of which the connection matrices are co-ordinate dependent manifestations. It would be great to have such an object since it would mean that the gauge covariance would be taken care of inherently. It would allow us to manipulate the connections more directly when it comes to inducing new connections on associated vector bundles. Through the understanding the previous section on classification of connections one might be optimistic about this idea. Reason being that in the classification we have hinted to the fact that the connections need not be distinguished when the vector bundle goes through an automorphism. This might suggest that the equivalence class of connections with respect to the gauge transformations might be the true object. But the case we considered was too special, we ended up getting a global curvature form and having U(1) as the underlying Lie group allowed us to bypass a lot of the complications. But the ease of dealing with the case comes at cost of loss of the information regarding the intricacies of gauge covariance and connections themselves.

In search of such co-ordinate free or canonical yet explicit description for connection when we turn to history we end up learning a brilliant story which goes back to study of symmetries by Cartan and how Ehresmann extends Cartan's idea to develop exactly the theory we need. We will go into this story to get broad strokes which will then allow us to appreciate and move through the technical aspects painlessly. The history is very rich and since we will only check the ideas which are directly useful to us, one may look at the more detailed accounts in [7], [13].

Lie and Klein had realised that the study of symmetries or the collection of transformations acting on a geometric space could allow one to characterise it. Lie realised that these transformations formed what we may now call as groups. And this algebraic revelation was useful as he then gave the idea of infinitesimal action of these symmetries, which yields a linearisation of these actions and suggested that one could understand the behaviour of the symmetries through these proxy object. He also established the algebraic laws which were obeyed by these infinitesimal actions. Cartan had picked up this program after Klein and Lie, firstly generalising these ideas and considering the classification of semi-simple Lie algebras, which became the term for these infinitesimal actions. Cartan unlike Lie who was more focused on Galois like result for differential equations. He wanted to realise the analogy in formal terms, the way discrete symmetries allow for the study of polynomial equations, continuous symmetries could help describe differential equations. Even though Lie was aware of applications of his ideas to geometry he did not pursue it. Cartan used these ideas in geometry and especially on the "homogenous spaces", where he was trying to formulate the method of "moving frames", which roughly speaking the idea that that co-ordinate axes or the frame could change on each point during motion in a space and could characterise the motion itself if the right moving frames were found. And it was understood that derivatives of the frame allow for such characterisation. In case of homogenous spaces the frames change through symmetries or action of a Lie group and which means that the linearisation or the derivatives of moving frames in such spaces lie in the infinitesimal actions or the Lie algebra. This led to the definition of Cartan connection which is the precursor of the connections which we know. And this idea may appear to be co-ordinate dependent due to choice of frames but it is through Ehresmann's work about defining the principal bundles that it is made abundantly clear that it is not. In fact this method reveals the underlying geometry for connections to the fullest. So we will flesh out the details leading to definition of Ehresmann connections and principal bundles.

Remark: There is more to Cartan's work on homogenous spaces which are those which are homeomorphic to $\frac{G}{H}$ where G, H are Lie groups. The general spaces or manifolds could be thought of as deformations of the homogenous spaces which is pointing to the influence of topology and curvature on them.

To begin we need to understand that Ehresmann's work rests on two aspects the local picture given by Cartan's work and accounting for change of frames which may happen when we consider two bundles which are isomorphic. And which means that we have to account for symmetries at each fibre or over each point. This means that first we will need to define this structure which is a globalisation of action of Lie groups, respecting and inheriting the topology of the space which we are using to parametrise our vector bundles which we call the base space of the vector bundle. So our structure is basically carrying symmetries over each point or more precisely "action of the group over each point". To capture this Ehresmann first considers arbitrary abstract fibres which are parametrised by base space. This basically globalises the "set" on which the group acts, we will add the bare minimum structure of manifold which we require the action to be on. Then we will define the space of symmetries or space of actions of some Lie group, say G. This is exactly the idea of principal G-bundle. And as a mental model we can keep the "homogenous spaces" in our mind.

Definition 1.10.1. (Fibre bundles) Let E, F and X be smooth manifolds. A smooth surjection $\pi : E \to X$ is called a fibre bundle with the fibre F if there exist *local trivialisations*, which are fibre preserving diffeomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ for some open cover of X, $\{U_\alpha\}$.

We will recall the definition of action of a Lie group on a manifold. It is same as how we would define actions of arbitrary groups in general but here we need to make the action smooth.

Definition 1.10.2. (G-manifolds) Let X be a smooth manifold and G a Lie group. G is said to act smoothly if we have a right action $\mu : X \times G \to X$ which is smooth, in this case X is called a G-manifold. A smooth map

$f : X \to Y$ between two right G-manifolds is called G-equivariant if $f(x \cdot g) = f(x) \cdot g \quad \forall x \in X, g \in G$.

With the above we are ready to define principal G-bundles. We are dealing with this concept in a more restricted setting so for our understanding the prime example of a principal G-bundle should be the collections of frames of a vector bundle on a manifold. We know that these transform through symmetries captured by some Lie group which is dependent on the sort of vector bundle we consider. It can be $\mathrm{GL}(n, \mathbb{R})$ for the case of real vector bundles, then if we have a metric on an oriented vector bundle we could only deal with $\mathrm{SO}(n)$ and so on. And we are trying to build an object which would intrinsically take care of all the frame changes.
Definition 1.10.3. (Principal G-bundle) A fibre bundle $\pi : P \to X$ with fibre G, a lie group, is called a principal G-bundle if P is a G-manifold where G acts freely on P and local trivialisations for π are G-equivariant. A morphism between two principal G-bundles is a pair of maps analogous to a vector bundle morphism where the upper map is G-equivariant and the square set up by base and upper map commutes, which is simply fibre preservation.

Now, we will go to the out prime application of this concept which is forming or defining the frame bundle. This is more of filling the details to the plan we had considered. But the details will be self evident if we recall the purpose of this whole exercise. The goal was to handle the change of frames or more globally speaking handling objects through automorphism of vector bundles which basically involves change of frame at each fibre. So, the fibre we want to consider become the collection of frames(which would basically be ordered basis) at each point. And the action is of whatever group we have under consideration, for simplicity let us stick to $\mathrm{GL}(n, \mathbb{R})$. Putting this together we wish to have principal $\mathrm{GL}(n, \mathbb{R})$-bundle where the fibre consists of frames. This leaves us with the task that we have to realise manifold structure on the collection of ordered bases of a finite dimensional real vector space say V, which we may denote by $\mathrm{Fr}(V)$. And the information forced on us which eliminates all choices but one is that this should be diffeomorphic to the Lie group, $\mathrm{GL}(n, \mathbb{R})$. To do this we have the key observation that the action of the general linear group on $\mathrm{Fr}(V)$ is free and transitive which means the entire collection could be thought of as orbit of any one choice of ordered basis, say B. And the orbit stabiliser theorem gives the required bijection:

$$\Phi_B : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{Fr}(V)$$

$$g \longmapsto B \cdot g$$

With this bijection we induce smooth manifold structure on $\mathrm{Fr}(V)$ so that it is diffeomorphic to the general linear group. Also it is easy to see that any other choice of basis would lead to a another smooth structure which would also be diffeomorphic. This completes the local aspect of the work, and we call $\mathrm{Fr}(V)$ to be the frame manifold of the vector space V. We move to the setting where instead of a vector space we are now given a smooth vector bundle $\pi : E \to X$ and we try forming the corresponding *frame bundle*, which has as its fibres the frame manifold of the corresponding fibres of π. As a consequence, we consider the map $\mathrm{Fr}(\pi) : \coprod_{x \in X} \mathrm{Fr}(E_x) \to X$. Here we will set the local trivialisations to be given by $\tilde{\phi}_\alpha : (\mathrm{Fr}(\pi))^{-1}(U_\alpha) \to U_\alpha \times \mathrm{GL}(n, \mathbb{R})$, this construction is in the same spirit as defining the tangent bundle of a smooth manifold. Then it is also easy to note that $\mathrm{Fr}(\pi)$ is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle. The following definition will serve as a check point for our discussion:

Definition 1.10.4. (Frame bundle of a vector bundle) Given a smooth real vector bundle $\pi : E \to X$ its frame bundle, $\mathrm{Fr}(\pi)$ (as described above) is the principal $\mathrm{GL}(n, \mathbb{R})$-bundle with fibre $\mathrm{Fr}(E_x)$ the frame manifold of the fibre of π and action inherited as it is from it.

Remark: This gives one way map from vector bundles and principal bundles, we will later see how we are able to produce vector bundles from a given principal bundle. That will complete the correspondence between these concepts. This correspondence becomes the working knowledge to use the principal bundles the way we want to.

From our analysis we can now be sure that principal bundles form the global analogue of actions of Lie groups or symmetries. We transition to the main problem which this apparatus is built to handle, the definition of connection which holds through automorphisms of the vector bundle under consideration. For this we will first try to flesh out Lie's idea of infinitesimal action of Lie groups as this holds the key to defining connections in a certain way. So, as we know the Lie group is a smooth collection of transformations or symmetries. By infinitesimal action we mean that we consider a one parameter subgroup of the Lie group, say $\phi : \mathbb{R} \to G$ such that it obeys the law, $\phi(t+s) = \phi(t)\phi(s)$ which means that if we consider take a point then this family describes its travel in the space guided by the Lie group but more importantly we note that the law implies that $\phi(0) = e$ the identity transformation so we may expect from smoothness of the family that it will yield infinitesimal action if we consider small parameter values. The derivative of the curve ϕ lies in T_eG and so the infinitesimal action is a vector field on the space given by:

$$Y(p) = \frac{d}{dt}|_{t=0}\, p \cdot \phi(t)$$

And we may note that Lie algebra associated to the Lie group G is exactly T_eG and is denoted by $\mathfrak{g}$. Moreover if we begin with an element of $\mathfrak{g}$ say $Y(e)$, then we extend it to right invariant vector field $Y(g) = (R_g)_*(Y(e))$ where R_g is right translation by g. And we solve the following equation:

$$\frac{d}{dt}\,\phi(t) = Y(\phi(t))$$

with the initial condition that $\phi(0) = e$ this has a unique solution which turns out to form a one parameter subgroup obeying the law, $\phi(s+t) = \phi(s)\phi(t)$ and this follows from the right invariant nature of the vector field on the Lie group. Which means that there is one to one correspondence between the elements of Lie algebra and one parameter subgroups obeying the law we have described. We collect all this information in the following definition:
Definition 1.10.5. (Fundamental vector fields) If P is a smooth manifold with smooth action of a Lie group G. The fundamental vector field on P for an element $Y \in \mathfrak{g}$ is denoted by $\underline{Y}$ and is defined by the following equation:

$$\underline{Y}(p) = \frac{d}{dt}|_{t=0}\, p \cdot \exp(tY)$$

Some more details about the fundamental vector fields and its integral curves can be found in [15]. Also we will be following the same reference for next few concepts in this section. We have noted some important details regarding the action of Lie group and Lie algebra. What is left, is to understand how this is related to connections on a vector bundle or for that matter on principal G-bundles. If we remember from the various descriptions of connections on vector bundles we understand a few things regarding connections. First and most important that connections give us a way (which depends on the choice of connection) to differentiate the sections of the vector bundle under consideration. Second and more structural part being that the connection can be expressed as sum of two operators first being the exterior derivative which is common for all connections while the second operator is a $\mathfrak{g}$-valued one form. The presence of the second part could be attributed to the curvature of the manifold or more generally the connection, see . And it is the second part which is non trivial and makes a connection unique and points to non flat geometry we have at hand. The local definition of connections and the example of classification of connections in the previous section shows us that we need to understand connections through automorphisms of a vector bundle rather than on a single copy, which is one of the reasons for this section to exist. First thing we are left to wonder is why does the second part lands in the lie algebra and that can be explained through the computation we make in defining the connections locally. When we have a trivial bundle at hand we pick a frame and then the connection matrix part represents how the "moving frames" themselves change, and the reason for it to land in the Lie algebra is that in a G-vector bundle the change of frame happens through the symmetries and the linearisation is the lie algebra as we have discussed. Going back to our case if we are dealing with a non trivial bundle then we would want to segregate the action of Lie group from the choice of local frames. Which we have done by defining principal G-bundles. One may note that the main issue hindering a global connection matrix was the change of frames as we moved from patch to patch. But now with the frame bundle associated to a vector bundle the matter of changing choices is itself taken care of by taking sections of the frame bundle. And we may note that the tangent space of a principal G-bundle is direct sum of two parts one the tangent space of the base manifold and the second the tangent space of the Lie group, which as we know is the Lie algebra. Moreover, since the right smooth action included in the definition of the principal G-bundles we can have a canonical choice for the Lie algebra part of the tangent space! This is the remarkable observation which hints to the part that a connection matrix might after all be global object at least when dealing with principal bundles. With this we may move to the details of realising this hunch. First we will observe that the Lie algebra part, which is considered the vertical part forms a vector sub bundle of the tangent bundle of the total space of a principal bundle. We begin by defining the **vertical tangent subspace**.

Let $\pi : P \to X$ be a principal G-bundle then we denote the vertical tangent bundle at a point $p \in P$ by $\mathcal{V}_p$ and is defined by the following short exact sequence:

$$0 \to \mathcal{V}_{\surd} \to T_pP \xrightarrow{\pi_*,p} T_{\pi(p)}X \to 0$$

And then we have the following proposition which essentially comes from the idea we described above for having a canonical vertical part:
Proposition 1.10.1. *(Canonical vertical vectors) Given an element $Y \in \mathfrak{g}$, $\underline{Y}(p) \in \mathcal{V}_p \quad \forall p \in P$.*

[15]

This means that if we choose the basis for the Lie algebra $\mathfrak{g}$ it will yield independent sections for T_pP and a frame for $\mathcal{V} = \coprod_{p\in P} \mathcal{V}_p$ which shows that it is a sub bundle. We collect this artefact:
Definition 1.10.6. (Vertical sub bundle) The vertical sub bundle of the tangent bundle of a principal G-bundle, $\pi : P \to X$ is given by the following short exact sequence of vector bundles over P:

$$0 \to \mathcal{V} \to TP \xrightarrow{\pi_*} \pi^*TX \to 0$$

The second part of the tangent space as we had noted is the tangent space of the base manifold. Unlike the vertical part we don't have any canonical choice for the horizontal part. If we take the inspiration from the example of the frame bundle associated to a vector bundle the horizontal part must be a choice of local frames we choose for our vector bundle. Before we move ahead let us define this horizontal part.
Definition 1.10.7. (Horizontal distributions) A vector sub bundle $\mathcal{H}$ of a principal G-bundle $\pi : P \to X$ is called a horizontal distribution if it is complementary to the vertical sub bundle or in other words as vector bundles $\mathcal{V} \oplus \mathcal{H} = TP$.

We are prepared to complete the hunch and look at connections through principal bundles. As noted the second part of the local description which consists of lie algebra valued one form, it appears because of the way the frames move and change with respect to the frame of vector fields. And this derivative information of moving frames can be taken care of globally first by making smooth choice of frames throughout the vector bundle which will reflect in its frame bundle. And that is exactly what gives us a horizontal distribution of the frame bundle. And this means that the characterising part of the a connection on vector bundle which is its connection matrix is completely captured by how frames move or in other words the choice of the horizontal distribution!

Remark: The choice of smooth horizontal distribution here is not any horizontal distribution but it has more qualification of being right invariant distribution, which is to say that the right action preserves the distribution. This is important because we want the action of lie group to not make a horizontal part go into vertical that would not make sense if we consider the example of choice of frames for the frame bundle. As any such action would still create a choice of frames or a horizontal distribution.

One may also notice that this is not the only way to define a connection as we may rather do it by first choosing a lie algebra valued one form on P and using it to produce the horizontal distribution. This would seem more familiar to us as this is in some sense what is happening in local description of connection but any choice of such one form would not yield us connection, the first restriction would be G-equivariance which is rather obvious and the other restriction is given by the equation:

$$\omega(\underline{Y}) = Y$$

This is suggestive of how we can pick out the horizontal part. The duality of methods of describing a connection comes from a simple observation stated in Tu's text which is that even though the vertical sub bundle is canonical the vertical component of vector field and it depends on the choice of the horizontal distribution. With this we will give the two equivalent definitions of connection on a principal bundle:

Definition 1.10.8. (Connection as a horizontal distribution) Given a principal G-bundle $\pi : P \to X$ a connection on it is defined to be a choice of horizontal distribution.

Definition 1.10.9. (Ehresmann connection) Given a principal G-bundle $\pi : P \to X$ an Ehresmann connection is defined to be a $\mathfrak{g}$-valued one form ω which is smooth, G-equivariant which means $r_g^*\omega = (\mathrm{Ad}g^{-1})\omega$ and it satisfies the following equation for all $Y \in \mathfrak{g}$:

$$\omega(\underline{Y}) = Y$$

The equivalence of the notions even though intuitive is quite a handful when all the details are considered, for reference that work may be found in Tu's text, [15].

We must first note that given a connection on the principal bundle we can lift the smooth vector fields of the base space to right invariant smooth vector fields in the principal bundle, this is in general captured by the following proposition:

Proposition 1.10.2. *(Horizontal lift) Given a connection $\mathcal{H}$ on a principal G-bundle $\pi : P \to X$, for each smooth vector field on Y there is a unique lift $\tilde{Y}$ which is a smooth right invariant horizontal vector field on P. The unique lift is due to identification $\mathcal{H}_p \cong T_{\pi(p)}X$.*

[15]

We return to the prime use case of our set up which is the case of frame bundle associated to the vector bundle under consideration. From the discussion we have had it follows naturally that to induce a connection on the frame bundle we need to use additional information we possess on TX so that we can create a horizontal distribution on the frame bundle. We will only describe the general method here and the form of the horizontal distribution on the frame bundle given a connection on the vector bundle. The idea is that we use parallel vector fields and by extension parallel frame on some curve using the notion of parallel transport. And we call the the lift of a curve to the principal bundle a **horizontal lift** if the frame at the initial point of the lift is a parallel frame, the existence and uniqueness of which is guaranteed by parallel transport. After this we have to actually show that this leads to definition of a linear injective map which splits the sequence defining vertical subspace. The map being:

$$c'(0) \longmapsto \tilde{c}'(0)$$

where $\tilde{c}$ is the horizontal lift with respect to the connection on the vector bundle under consideration. [15]

The crucial formula which leads to the proof gives us the method of relating the connection matrices to the Ehresmann connection induced on the frame bundle, we will note the formula and state the theorem below.

Proposition 1.10.3. *(Horizontal lift formula) Given a smooth real vector bundle $\pi : E \to X$ and a connection d_A on it, the expression for the horizontal lift of a vector field Y over an open neighbourhood U of X is with respect to a choice of local frame determined by the local section $s : U \to (Fr(\pi))^{-1}(U)$ is given by:*

$$\tilde{Y} = s_*(Y) - \underline{\omega_s(Y)}$$

Proposition 1.10.4. *(Connection matrices as pull back) Suppose we have a vector bundle $\pi : E \to M$ and a connection d_A on it. Over an open neighbourhood (trivialising) we choose a local frame $e : U \to (Fr(\pi))^{-1}(U)$ and let the connection matrix of d_A with respect to e be denoted by ω_e and the Ehresmann connection on the frame bundle be denoted by ω then:*

$$\omega_e = e^*\omega$$

With this we have realised our hunch and seen how Ehresmann built the global analogue of connection form. This definition or set up leads to many applications we had anticipated, first and probably the most useful to us being the induction of connection to associated bundles. After which we will see that this also allows us to define the gauge group in generality and helps in phrasing the classification problem of connections more cleanly. We begin with the idea of inducing connections on associated bundles. I had made the comment that principal G-bundles are global analogues of action of Lie groups or smooth symmetries. And by global we mean that when we leave highly symmetric spaces like the homogenous spaces and go to general curved manifolds these objects account for the topology or geometry of it. Which also means that we could also form some notion of representations of these objects and that might yield us vector bundles if we do it properly since locally it is easy to envision what that would mean and we would be left with parametrising the vector spaces which bear these representations coherently. One advantage it has is that is gives us a map other way around going from principal bundles to vector bundles. Moreover, we can extend connections to associated bundles like the tensor product, dual and so on since these concepts can be viewed through representations like tensor product of representations, dual representation and so on.

Definition 1.10.10. (Associated bundle) Given a principal G-bundle $\pi : P \to X$ and a finite dimensional representation $\rho : G \to \mathrm{GL}(V)$ on V. Then the associated bundle is denoted by $P \times_\rho V$ and is defined to be the quotient of $P \times V$ by the relation $(p, v) \sim (p, g^{-1}.v)$.

We note that the fibre of the associated bundle has vector space structure and moreover we have that:

Proposition 1.10.5. *(Fibre of associated bundle) Let $E := P \times_\rho V$ be the associated bundle as define above then for each $p \in P$ we have the following isomorphism of vector spaces,*

[15]:

$$f_p : E_x \to V$$

$$[p, v] \longmapsto v$$

Having defined the associated bundle and the canonical identification of the fibre we will now move to the notion of tensorial forms on P.

Definition 1.10.11. (Tensorial forms of type ρ) A V valued k-form φ on P is said to be right equivariant of type ρ if $r_g^*\varphi = \rho(g^{-1})\varphi$. It is called horizontal if it vanishes when one of its arguments is vertical. If it is both then it is called a tensorieal form of type ρ. The collection of all smooth V-valued tensorial k-forms is denoted by $\Omega^k_\rho(P, V)$.

Using the canonical identification f_p we are able to identify this with the k-forms of the associated bundle:

Proposition 1.10.6. *(Identification of forms) There is an isomorphism between tensorial k-forms and k-forms of the associated bundle which is given by $\varphi^\flat(v_1, v_2, \cdots, v_k) := f_p(\varphi(u_1, u_2, \cdots, u_k))$ where u_i is lift of v_i through $\pi_* : TP \to TX$. The inverse of this is denoted by $\Psi \longmapsto \Psi^\sharp$.*

[15]

Finally we are prepared to induce connections on the associated bundles. And we do this by rather defining the covariant derivative of the connection. Here the idea is that we take a horizontal k-form take its exterior derivative and then take its horizontal component.

Definition 1.10.12. (Covariant derivative) Given a connection $\mathcal{H}$ on a principal G-bundle, covariant derivative of V-valued k-forms is defined to be the map:

$$D : \Omega^k(P, V) \to \Omega^{k+1}(P, V)$$

$$D\varphi = (d\varphi)^h$$

where ^{h} represents the horizontal component of the exterior derivative with respect to the connection.

We may note that for tensorial form the following equation holds:
Proposition 1.10.7. *(Formula for covariant derivative) Given a principal G-bundle the covariant derivative on the tensorial forms satisfies the following equation:*

$$D\varphi = d\varphi + \omega \cdot \varphi$$

where

$$\tau \cdot \varphi(v_1, v_2, \cdots, v_{k+l}) :=$$
$$\frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \tau(v_{\sigma(1)}, v_{\sigma(2)}, \cdots, v_{\sigma(l)}) \varphi(v_{\sigma(l+1)}, v_{\sigma(l+2)}, \cdots, v_{\sigma(k+l)})$$

similar to how wedge product is defined using Alt on tensor product. [15]

This concludes our discussion on the first application. Next we try to understand how the language of principal bundles and connections helps in giving definitions of concepts in gauge theory. For example the gauge group is defined to be $\mathcal{G} = \Gamma(\mathrm{Ad}(P))$ where $\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} \mathfrak{g}$ where $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is the adjoint representation. And the space of connections could be defined as $\mathcal{A} = \Omega^1(X, \mathrm{Ad}(P))$. The more important part is being able to define the gauge group (in physics the total gauge group) in general.

Chapter 2

Spin geometry and Dirac operators

Spin geometry and Dirac operators, without prior information might suggest there being two topics which could be related as they appear next to each other in title. Which is true but saying they are related or even closely related might be an understatement. Before we flesh out the definitions and technical results in this chapter, we should look at the history of this subject. It will serve as a mental model and then as we will work through the details they will seem but obvious.

We will start with answering the question that where does Dirac operator come from? Even though the name points us to check Paul Dirac's work, the first appearance could be attributed further back in history to Hamilton. Since both came up with the concept for the same reason we will stick to Dirac's work to work through this idea. Paul Dirac was concerned with the formulation of quantum mechanics for the relativistic setting in mid 1920s. There was an equation called the Klein-Gordon equation which accounted for the relativistic transformations but it wouldn't lend properly to the probabilistic interpretation, so Dirac thought that the resolution of the issue was to figure out the first order version or the "linearisation" of this equation. The Klein-Gordon equation is:

$$[p^2c^2 + m^2c^4]\phi = -\hbar^2\partial_t^2\phi$$

And the way in which Dirac thought to do this is to take the "square root" of this equation, so that we would have only first order time derivative and that it would remain valid in relativistic setting. A more cleaner way of writing this would be using the box operator:

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\phi = 0$$

Where $\Box = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$ for the a given signature of metric. And now what Dirac is suggesting is that we define $\sqrt{\Box}$ so that we could write(I will for now write the scalar part as μ^2):

$$(\sqrt{\Box} + i\mu)(\sqrt{\Box} - i\mu)\phi = 0$$

And so we arrive at Dirac equation in some form:

$$(\sqrt{\Box} + i\mu)\phi = 0$$

So, now the problem seems done but what does this "square root" look like? Dirac put some coefficients and demanded that:

$$\sqrt{\Box} = A\partial_t + B\partial_x + C\partial_y + D\partial_z$$

$$\implies \Box = A^2\partial_t^2 + B^2\partial_x^2 + C^2\partial_y^2 + D^2\partial_z^2 + (AB + BA)\partial_t\partial_x + \cdots$$

So, this would mean that the coefficients obey the following rules:

If one notices these are exactly the relations which generators of Clifford algebra satisfy. And Dirac had knowledge of matrices which could serve as an example of such coefficients but what this meant is that the equation's domain would need to be amended allowing for many components. This was Dirac's proposal and it was quite bold as it wasn't based on a physical argument but a mathematical requirement. But with the promise of the equation and further advancements it became integral part of relativistic quantum mechanics. And before we move on we note that the "square root" of the box operator is called the Dirac operator.

With this history lesson we understand the origin of Dirac operator, but there are still many questions to be addressed. Firstly why would this operator be relevant to us? The answer lies in geometry. As we saw that defining Dirac operator comes along with the added requirement of appropriate coefficients which can be realised in Clifford algebra, and interestingly this links to the definition of the spin groups which were originally considered by Élie Cartan who was working on classification of representations of lie algebras. They came up as the two sheeted cover of the special orthogonal groups. The essence of the matter is that when we are considering the action of the Clifford algebra we have to globalise it when we move to the manifold set up and here the behaviour with $SO(n)$ becomes important if we are considering a Riemannian manifold. And there we will note that the right symmetries to look at are not from $SO(n)$ but rather its double cover which is the $\mathrm{Spin}(n)$, we will elaborate more on this matter when we discuss about $\mathrm{Spin}(n)$ and representation of Clifford algebra formally. But in any case this leads us to the idea that if we want to work with this operator on a manifold we have to account for these matters. But the true geometric weight of this apparatus is still sitting in the original idea of Dirac, which is to take the square root of $\Box$. We must notice that when we don't have a signature which means we are working with Riemannian metric rather than pseudo-Riemannian metric this operator simply becomes the Laplacian. And Laplacians encode crucial information of geometry of manifolds, this can be seen through Hodge theory where we have the Hodge Laplacian and in general when we have Laplacians corresponding to covariant derivatives on Riemannian manifold. And as their square roots, the Dirac operators occupy central role in geometry which can seen through the celebrated works of Lichnerowicz and of course Atiyah-Singer's index theory for elliptic operators. For us Dirac operators will take a central role in phrasing the Seiberg-Witten equations and of course we will enjoy the power of the results of Lichnerowicz and index theory when we will build the moduli space of monopoles.

2.1 The four dimensional case

When we consider complex structure on $\mathbb{R}^2$, it yields an additional algebraic structure which leads to the theory of Riemann surfaces. The nature of the additional structure is that it couples co-ordinates by which there is way to concert these co-ordinates which would in lack of such an algebraic structure not be possible. The manifestation of this perspective is the definition of an almost complex structure and the theory of almost complex and complex manifolds, one can find the details of which in [8]. Another place such algebraic features lead to remarkable result is parallelizability of spheres which is nothing but asking on which spheres do we have global frames. And we do get somewhat similar algebraic advantage in geometry of $\mathbb{R}^4$, due to it possessing the structure of quaternions, $\mathbb{H}$. Both $\mathbb{C}$ and $\mathbb{H}$ have a nice interpretation in the theory of Clifford algebras which one may find in [9]. And we will exploit this structure to our advantage.

We begin by noticing how we may view $\mathbb{R}^4$ as $\mathbb{H}$ and as $\mathbb{C}^2$ and the relations between these structures. Consider the following complex 2×2 matrices which will serve as basis for a real vector space:

$$\mathbb{H} = \mathbb{R}^4 = \{t\mathbb{1} + x\mathbb{i} + y\mathbb{j} + z\mathbb{k} | t, x, y, z \in \mathbb{R}\} \text{ where}$$

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbb{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathbb{j} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \mathbb{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

So firstly we are considering $\mathbb{R}^4$ as a 4-dimensional vector space over $\mathbb{R}$. Then we are identifying it with another 4-dimensional real vector space which is generated by the basis $\mathbb{1}, \mathbb{i}, \mathbb{j}$ and $\mathbb{k}$, which is a real subspace of $M_2(\mathbb{C})$ and we define $Q = t\mathbb{1} + x\mathbb{i} + y\mathbb{j} + z\mathbb{k} \in M_2(\mathbb{C})$. Consider the following calculation:

$$\det(t\mathbb{1} + x\mathbb{i} + y\mathbb{j} + z\mathbb{k}) = \det \begin{pmatrix} t + iz & -x + iy \\ x + iy & t - iz \end{pmatrix} = t^2 + x^2 + y^2 + z^2$$

If we denote the Euclidean dot product (on $\mathbb{R}^4$) by $\langle \cdot, \cdot \rangle$ then due the above calculation we note that $\det(Q) = \langle Q, Q \rangle$. So we now know how to find the

Euclidean norm when we are viewing $\mathbb{R}^4$ as quaternions. We also get the following formula:

$$\bar{Q}^t Q = (t^2 + x^2 + y^2 + z^2)I$$

Which means that the unit sphere S^3 consists of matrices whose inverse is their conjugate transpose which are the elements of SU(2) group, and this identification is a diffeomorphism. [9]

We note that Spin(4) group is isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2)$ which for clarity we will label as $\mathrm{SU}_+(2) \times \mathrm{SU}_-(2)$ [9]. Then we can note the representation of Spin(4) on $\mathbb{H}$ which is defined as follows:

$$\rho : \mathrm{SU}_+(2) \times \mathrm{SU}_-(2) \to \mathrm{GL}(\mathbb{H})$$

$$\rho(A_+, A_-)(Q) = A_- Q (A_+)^{-1}$$

This representation preserves the Euclidean norm since:

$$\det(A_- Q (A_+)^{-1}) = \det(Q)$$

This means that $\mathrm{Im}(\rho) \subset \mathrm{O}(4)$. In fact we note that there is more to this relation: (see [9])

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(4) \xrightarrow{\rho} \mathrm{SO}(4) \longrightarrow 0$$

We can also consider the such a short exact sequence which we get with the $\mathrm{Spin}^c(4)$ group: (see [9])

$$0 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{Spin}^c(4) \xrightarrow{\rho^c} \mathrm{SO}(4) \longrightarrow 0$$

More than the real representations of Spin(4) and analogously $\mathrm{Spin}^c(4)$ on $\mathbb{H}$, we have factor wise representations on $W_- = W_+ = \mathbb{C}^2$ (we are viewing the four dimensional Euclidean space as a complex vector space instead) given by:

$$\rho_+ : \mathrm{Spin}(4) \to \mathrm{GL}(W_+) \qquad \rho_- : \mathrm{Spin}(4) \to \mathrm{GL}(W_-)$$

$$\rho_+(A_+, A_-)(w_+) = A_+ w_+$$

$$\rho_-(A_+, A_-)(w_-) = A_- w_-$$

And analogously we could define for the $\mathrm{Spin}^c(4)$ group as:

$$\rho_+^c : \mathrm{Spin}(4) \to \mathrm{GL}(W_+) \qquad \rho_-^c : \mathrm{Spin}(4) \to \mathrm{GL}(W_-)$$

$$\rho_+^c(A_+, A_-, \lambda)(w_+) = \lambda A_+ w_+$$

$$\rho_-^c(A_+, A_-, \lambda)(w_-) = \lambda A_- w_-$$

The reason we define this is that it yields the following isomorphism of complex representations on the spaces as follows:

Proposition 2.1.1. *(Splitting the spinor representation) Consider the representations ρ, ρ_+ and ρ_- as defined earlier then there is an isomorphism of complex representations;*

$$\mathbb{H} \otimes \mathbb{C} \cong \mathrm{Hom}_{\mathbb{C}}(W_+, W_-)$$

Basically, if we complexify our representation then it can be expressed as tensor product of the basic complex representations we gave. And then we may use the identification, $\mathrm{Hom}(V, W) \cong W \otimes V^*$ for finite dimensional vector spaces and so the above holds loosely speaking. But we need to work out the dual and tensor of the representations still.

Proposition 2.1.2. *(Self duality) The representations ρ_+ and ρ_- are self dual.*

[11]

In the most general case, it takes some genuine work to show that a vector space is embedded in its Clifford algebra. But in our case we will may see this with a very neat trick, which is encoded in the following map:

$$\theta : \mathbb{H} \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(W)$$

$$Q \longmapsto \begin{pmatrix} 0 & -Q^{\dagger} \\ Q & 0 \end{pmatrix}$$

Where $W = W_+ \oplus W_-$. In order to see this map we will first have to look at the matrix representation of the space of endomorphisms, the following breakup can be useful:

$$\mathrm{End}_{\mathbb{C}}(W) \cong W \otimes W^* \cong (W_+ \oplus W_-) \otimes (W_+ \oplus W_-)^*$$

$$\cong (W_+ \oplus W_-) \otimes (W_+^* \oplus W_-^*) \cong (W_+ \otimes W_+^*) \oplus (W_+ \otimes W_-^*) \oplus (W_- \otimes W_+^*) \oplus (W_- \otimes W_-^*)$$

And then one may use the self duality (see proposition 2.1.2) and splitting of the representation (see proposition 2.1.1) to notice why the map makes sense. And the main reason for the mapping is the following computation:

$$(\theta(Q))^2 = \begin{pmatrix} -Q^{\dagger}Q & 0 \\ 0 & -QQ^{\dagger} \end{pmatrix} = -\det(Q)I$$

And then we may note that the map θ is injective linear (must be clear from the definition) and so we may identify it with its image. And also since $\mathrm{End}_{\mathbb{C}}(W)$ is an algebra for which there is such a subspace this means that it is isomorphic as algebra to the Clifford algebra of corresponding to $\mathbb{H} \otimes \mathbb{C}$ which follows as a direct consequence of the uniqueness theorem of Clifford algebras.

As we have been using matrices for our simplicity as a consequence of only dealing with finite dimensional case, we may also use the basis of this space to generate the basis for the Clifford algebra. And we may use a special basis (orthogonal) due to the quadratic form and further simplify our work. But now we must remember the relation between exterior algebras and Clifford algebras we have described in previous section.

Lie algebra of SU(2) realised as $\bigwedge^2_+ \mathbb{H}$. [11]

2.2 Spin and Spin^c structures, and associated bundles

In this section we will see how do we extend the structures we built in the previous section to a bundle level on a given 4-manifold. And we will base this on some ideas first being that of the structure group. When we consider an oriented Riemannian 4-manifold then the structure group reduces from $\mathrm{GL}(4,\mathbb{R})$ to SO(4) with regard to the bundles over it, primarily the tangent bundle. And if we note the work in previous sections, to realise the action of Clifford algebra we have to build a bundle which has a structure group Spin(4) rather. And for this we use the short exact sequence we had considered earlier relating the groups (see section 2.1).

Here we will take the gluing maps picture of vector bundles in order to realise the problem of forming the spinor bundles as an obstruction problem. So, suppose that we are given an oriented Riemannian 4-manifold X and suppose that we have the gluing maps corresponding tangent bundle TX for a trivialising open cover $\{U_\alpha\}$ to be given as $\{g_{\beta\alpha}\}$. Then we note that due to the additional structure we understand that $g_{\beta\alpha}$ are SO(4) valued. And we may simply lift these maps to the simply connected double sheeted covering space Spin(4). Let us denote the new collection to be given by $\{\tilde{g}_{\beta\alpha}\}$. But we do not know if they obey the co-cycle condition in order to build a vector bundle. But what we do know is that the original gluing maps did obey the co-cycle condition. Let us consider the condition over $U_\alpha \cap U_\beta \cap U_\gamma$:

$$g_{\gamma\beta} g_{\beta\alpha} = g_{\gamma\alpha} \implies g_{\gamma\beta} g_{\beta\alpha} g_{\gamma\alpha}^{-1} = \mathbb{1}$$

And we know that $\ker(\rho) = \mathbb{Z}_2$ and of course the lift respects the matrix composition or multiplication since it is defined point wise. This means we have the following relation at hand:

$$\eta_{\gamma\beta\alpha} = \tilde{g}_{\gamma\beta}\tilde{g}_{\beta\alpha}\tilde{g}_{\alpha\gamma} = \pm\mathbb{1}$$

And then we could consider the collection $\{\eta_{\alpha\beta\gamma}\}$ and we need to check that these define a co-cycle we view this in terms of the Čech cohomology terms (see [5]). Which would be clear from the following relations:

$$\eta_{\alpha\beta\gamma} = \eta_{\beta\gamma\alpha} = \eta_{\beta\alpha\gamma} = \eta^{-1}_{\beta\alpha\gamma}$$

Which means that this defines a cohomology class in $H^2(X, \mathbb{Z}_2)$ which we denote by $w_2(TX)$ and is called the second Stiefel-Whitney class (There are other ways to define them which are equivalent to this Čech theoretic approach [5]). And it would mean that if this vanishes then we would have a constant $\mathbb{Z}_2$ assignments, which we may define by $\eta_{\alpha\beta}$ which would mean that $\{\eta_{\beta\alpha}\tilde{g}_{\beta\alpha}\}$ define a vector bundle, and of course if they do form vector bundle then $w_2(TX)$ would vanish. **Remark**: Strictly speaking we would have to choose a "good cover"(see [5]) for this to work but that is not a problem on manifolds.

Definition 2.2.1. (Spin structure) Let X be a oriented Riemannian four manifold and let $\{g_{\beta\alpha}\}$ be the collection of gluing maps which define TX for some trivialising open cover and strictly landing in SO(4). X is said to have a Spin structure if the lifted maps $\{\tilde{g}_{\beta\alpha}\}$, through ρ (see section 2.1), obey the co-cycle condition or equivalently define a vector bundle.

Proposition 2.2.1. *(Obstruction to spin) An oriented Riemannian four manifold admits a Spin structure iff $w_2(TX)$ vanishes.*

The gluing maps associated to a spin structure would yield two other sets of gluing maps $\{\rho_+ \circ \tilde{g}_{\beta\alpha}\}$ and $\{\rho_- \circ \tilde{g}_{\beta\alpha}\}$ which would take values in $\mathrm{SU}_+(2)$ and $\mathrm{SU}_-(2)$ respectively which would define the smooth complex vector bundles W_+ and W_- over X respectively which would give us the bundle level analogue of our result (see proposition 2.1.1):

Proposition 2.2.2. *(Splitting of representations) Given an oriented Riemannian four manifold X admitting the associated Spin structure has the isomorphism of complex vector bundles (which fibre wise is isomorphism of representations):*

$$TX \otimes \mathbb{C} \cong \mathrm{Hom}_{\mathbb{C}}(W_+, W_-)$$

where the representations under consideration are ρ, ρ_+ and ρ_- and $\rho_{\mathbb{C}} \cong \rho_- \otimes \rho_+^*$.

Similarly we will define the Spin^c structures but it is important to note why we are invested in them. As we have noted that in order to realise Clifford algebras we require Spin structures and we may indeed go on to work through the details of Dirac operators just based on them. The only issue is that not all smooth four manifolds admit a Spin structure. One may think that it too ambitious to hope that such feature would be present on all manifolds but strikingly if we work with Spin^c structures we do have such luxury!

Definition 2.2.2. (Spin^c structure) Given an oriented Riemannian four manifold X, a Spin^c structure on it is given by $\{\tilde{g}_{\beta\alpha}\}$ gluing maps over some trivialising open cover $\{U_\alpha\}$ such that $\{g_{\alpha\beta} = \rho^c \circ g_{\alpha\beta}\}$ are the gluing maps defining TX.

Now as we had mentioned we give the main theorem which is a neat application of obstruction theoretic argument much like the one we made before but requires a bit more work.

We return to the the lifting problem again, as we had noted that for the previously defined $\{\eta_{\alpha\beta\gamma}\}$ the co-cycle condition holds iff $w_2(TX)$ vanishes. And at the core we had the issue of being able to assign $\eta_{\alpha\beta\gamma}$ the value $\pm\mathbb{1}$ coherently. So the idea we use is that we attach complex coefficients to the gluing maps $\{\tilde{g}_{\alpha\beta}\}$ smoothly by a collection $\{h_{\alpha\beta}\}$ so that they have the exact opposite obstruction to that of the original lifting. What this amounts to is that we produce that then when we attach these coefficient maps to original lifts that is we form the collection $\{h_{\alpha\beta}g_{\alpha\beta}\}$ and now when we evaluate the new $\{\tilde{\eta}_{\alpha\beta\gamma}\}$ they will obey the co-cycle condition, the only difference being that this will lead to a bundle where it is fibre wise complexified which simply means that we take tensor product with complex place. And clearly those transitions will land in Spin$^c(4)$ rather than Spin(4) group and the bundles formed with respect to the representations ρ_+^c and ρ_-^c will lead to complex bundles $W_+ \otimes L$ and $W_- \otimes L$ and the determinant bundle $L \otimes L$. And with this our focus shifts on how do we produce the collection $\{h_{\alpha\beta}\}$ which replicates the obstruction or is inverse (it doesn't matter as in $\mathbb{Z}_2$ elements are their own inverses). The trick here is done by the following short exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod}(2)} \mathbb{Z} \to 0$$

which produces Bockstein sequence which is a long exact sequence of Čech cohomology groups with the coefficient groups which appear in the short exact sequence, one may find these details in [5]. Here this yields:

$$\cdots \longrightarrow H^n(X;\mathbb{Z}) \xrightarrow{\times 2} H^n(X;\mathbb{Z}) \xrightarrow{\text{mod}(2)} H^n(X;\mathbb{Z}_2) \xrightarrow{\beta} H^{n+1}(X;\mathbb{Z}) \xrightarrow{\times 2} H^{n+1}(X;\mathbb{Z}) \longrightarrow \cdots$$

where β is the Bockstein homomorphism. To us the part of the sequence which is relevant is:

$$H^2(X;\mathbb{Z}) \xrightarrow{\times 2} H^2(X;\mathbb{Z}) \xrightarrow{\text{mod}(2)} H^2(X;\mathbb{Z}_2) \xrightarrow{\beta} H^3(X;\mathbb{Z}) \xrightarrow{\times 2} H^3(X;\mathbb{Z})$$

Here we may consider the simpler case where we assume a simply connected manifold so that $H^3(X,\mathbb{Z}) = 0$ by the Poincaré duality. Which means that when we lift the class obstruction class $w_2(TX)$ it vanishes which in Čech terms means that $\{\tilde{\eta}_{\alpha\beta\gamma}\}$ obey the co-cycle condition. Then we use the partitions of unity $\{\psi_\alpha\}$ subordinate to the cover under consideration so that we get smooth maps $\{f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}\}$ where $f_{\alpha\beta} = \sum_\gamma \psi_\gamma \tilde{\eta}_{\gamma\alpha\beta}$ and what this yields is that $f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = \tilde{\eta}_{\alpha\beta\gamma}$ which means that we could set $h_{\alpha\beta} = \exp(i\pi f_{\alpha\beta})$. And in general case we would have to show that the extension or image of the $w_2(TX)$ would vanish in the case of four manifolds which one may find in [6]. We summarise this in the following proposition:

Proposition 2.2.3. *(Admission of Spin^c structures) Every smooth oriented four manifold X, admits a Spin^c structure.*

As the existence of Spin structure lead to definition of the associated smooth complex vector bundles, when we have a Spin^c structure it leads to having smooth complex vector bundles $W_+ \otimes L$ and $W_- \otimes L$ as one would expect. Also given that manifold has Spin structure it automatically admits Spin^c structure which may be defined by attaching(tensor) any complex line bundle with hermitian metric on them. There is a one to one correspondence between them, [11]. And so unless there is an underlying Spin structure the bundles given above will not split into the factors simply because of possessing a non trivial obstruction.

Proposition 2.2.4. *(Splitting of representations) Given an oriented Riemannian four manifold X admitting the associated Spin^c structure has the* *isomorphism of complex vector bundles (which fibre wise is isomorphism of representations):*

$$TX \otimes \mathbb{C} \cong Hom_{\mathbb{C}}(W_+, W_-)$$

where the representations under consideration are ρ^c, ρ^c_+ *and* ρ^c_- *and* $\rho^c_{\mathbb{C}} \cong \rho^c_- \otimes (\rho^c_+)^*$.

2.3 Spin, Spin^c-connections and Dirac operators

In this section based on all the theory we presented in the previous sections we will get to define the Dirac operators. This will be done by first defining the Spin connection for a given Spin structure and we will look at its relation with the Levi-Civita connection associated to the underlying Riemannian structure. And then using the Clifford algebra structure we will define the Dirac operators. We will follow it up with some properties of the Dirac operator. We will end this chapter by presenting the proof for Wietzenböck formula and state Atiyah-Singer index theorem and its corollaries. When we are defining the Spin connection it will be a situation similar to when we were defining curvature and had to induce connections. We will go through with the calculations at vector bundle level as they will give a more explicit description but we will again say here that by using the language of principal G-bundles one may save a lot of work and confusion. (see section 1.10)

Definition 2.3.1. (Spin(4) connection) Let X be an oriented Riemannian four manifold with Spin structure. Let the associated basic smooth complex vector bundles be W_+, W_- and $W := W_+ \oplus W_-$. A Spin(4) connection d_A on W is defined (in local terms) by $\{d + \varphi_\alpha\}$ for a trivialising cover $\{U_\alpha\}$ such that φ_α takes value in Lie algebra of Spin(4), and obey the relations for defining a connection locally

(see definition 1.5.3).

Given a Spin(4) connection d_A on W we can induce a connection d_A^{End} on End(W), which obeys the Leibniz's rule:

$$d_A(\omega\sigma) = (d_A^{\text{End}}\omega)\sigma + \omega(d_A\sigma)$$

Remark: In general when it is understood that the connections are induced then we may not denote them by different notations as we have done above much like the case of exterior derivative or when building curvature. (see definition 1.7.1)

But now we may use the relation between exterior algebra and Clifford algebra, and we may induce a connection on $\mathrm{End}(W)$ using the Levi-Civita connection d_A^{LC} on TM. And in that case we are left with the question as to how do these connections differ from one another? And might guess again using the short exact sequence relating the groups under consideration and the tight knit relation between the Spin structure on a manifold X and its underlying Riemannian structure that they might after all turn out to be closely related or even same. Then we are not far off, in fact it is the case. We will give the formal result and the proof we will present will be by actually calculating the induced connections on $\mathrm{End}(W)$ and comparing them.

Proposition 2.3.1. *(Levi-Civita and Spin connections) Let X be an oriented Riemannian four manifold admitting a Spin structure. Then there exists a unique Spin(4) connection on it which induces the same connection on End(W) as the Levi-Civita connection. Moreover, there is a one to one correspondence between SO(4) connections and Spin(4) connections.*

Proof: We had seen in the section regarding Clifford algebras for four dimensional case as to how the quaternions are embedded in the Clifford algebra bundle and correspondingly in the exterior algebra bundle. This relation becomes relevant as it allows us to induce local frames over an open neighbourhood for both W and TX such that they are related through the θ map we had constructed, . And so if we choose locally the standard frame on W with constant coefficients it yields us a local frame for TX which we may note through the matrix representation:

$$e_1 = \begin{pmatrix} \mathbb{O} & -\mathbb{1} \\ \mathbb{1} & \mathbb{O} \end{pmatrix}, \quad e_2 = \begin{pmatrix} \mathbb{O} & -\mathbb{i} \\ \mathbb{i} & \mathbb{O} \end{pmatrix}, \quad e_3 = \begin{pmatrix} \mathbb{O} & \mathbb{j} \\ \mathbb{j} & \mathbb{O} \end{pmatrix}, \quad e_4 = \begin{pmatrix} \mathbb{O} & \mathbb{k} \\ \mathbb{k} & \mathbb{O} \end{pmatrix}$$

Where all terms are two by two matrices representing quaternions. And suppose that we note the standard frame corresponding to these on W are denoted by $\tilde{e}_i$s. And then we make the computation we wanted comparing the induced connections on the exterior algebra bundle which we know to be

End(W), where we suppose that the Spin connection under consideration is denoted by d_A:

$$d_A = d + \phi_\alpha$$

$$d_A(\tilde{e}_i) = \sum_{j,k}^{4} \phi_\alpha^{jk} e_j e_k \tilde{e}_i$$

$$d_A(e_l \tilde{e}_i) = \sum_{j,k}^{4} \phi_\alpha^{jk} e_j e_k e_l \tilde{e}_i$$

And from the Leibniz's rule for inducing this connection onto End(W) we get:

$$(d_A e_l)\tilde{e}_i = \sum_{j,k}^{4} \phi_\alpha^{jk} [e_j e_k e_l - e_l e_j e_k] \tilde{e}_i$$

Which gives us that the induced connection on TX is given by:

$$d_A e_l = -4 \sum_{i=1}^{4} \phi_\alpha^{il} e_i$$

And on the other hand if we have taken the SO(4) connection $d_{A'}$ then that would give:

$$d_{A'} e_l = \sum_{i=1}^{4} \omega_\alpha^{il} e_i$$

we may form the corresponding Spin connection by setting $\omega_\alpha^{ij} = -4\phi_\alpha^{ij}$.

Then the curvature of this connection is also related to the curvature of the SO(4) connection. One may get the formula if we follow the same route as we did in . One more thing to note here the case of Spin^c(4) connection which we may again define locally. As far as the relation is concerned there will be no practical differences from the Spin(4) counterpart except that here we will have to handle an additional connection which comes from the determinant line bundle associated with the $\text{Spin}^c(4)$ structure. And consequently is would reflect in the local formulas. One would apply the idea of inducing the connection on tensor product except that each local form would individually not be able to produce a global connection stricken by the same obstruction problem as the gluing maps don't change. As one would note even before this that W_+, W_- and L are not vector bundles unless there is an underlying Spin structure and are thus termed as virtual vector bundles. Moreover, just as we described how one would go about working with these virtual vector bundles, this is important because we note that as far as obstruction is concerned it will be fixed when we take tensor at each fibre. Even in the next chapter we will work with them as though they are genuine bundles, use their local properties as long as we are going to induce the work on the genuine vector bundles $W_+ \otimes L$, $W_- \otimes L$ and $L \otimes L$ in the end.

Since the formulas for curvature of Spin^c connections are only an extension of the idea which we used to compute the curvature of any connection we will not explicitly show the calculations.

$$\Omega = -iF_A I - \frac{1}{4}\sum_{i,j}^{4} \Omega_{ij} e_i \cdot e_j$$

where the curvature forms of the related SO(4) connection are denoted by Ω_{ij}. And $F_A = da$ where ia is the connection form associated to the complex line bundle L (virtual) associated to the Spin^c structure.

Definition 2.3.2. ($\text{Spin}^c(4)$ connection) Let X be an oriented Riemannian four manifold with Spin^c structure. Let the associated basic smooth complex vector bundles be $W_+ \otimes L$, $W_- \otimes L$ and $L \otimes L$. A $\text{Spin}^c(4)$ connection d_A on $W \otimes L$ is defined (in local terms) by $\{d + \varphi_\alpha\}$ for a trivialising cover $\{U_\alpha\}$ such that φ_α takes value in Lie algebra of $\text{Spin}^c(4)$, and obey the relations for defining a connection locally.

Proposition 2.3.2. *(Levi-Civita and Spin^c connections) Let X be an oriented Riemannian four manifold admitting a Spin^c structure. Let the associated determinant line bundle be L^2. Given a connection d_{2A} on L^2, there exists a unique $\text{Spin}^c(4)$ connection on $W \otimes L$ such that it induces the Levi-Civita connection on $End(W)$ and the connection d_{2A} on L^2.*

With this we are prepared to define the Dirac operator. If one keeps in mind the local formulation from the historical overview at the beginning of this chapter then the definition will not appear anything peculiar, in fact for a choice of local frame it will appear the same.

Definition 2.3.3. (Dirac operator with coefficients in line bundle) Let X be an oriented Riemannian four manifold with Spin^c structure, equipped with Spin^c connection d_A on the spinor bundle $W \otimes L$. Then the Dirac operator is define to be a map $D_A : \Gamma(W \otimes L) \to \Gamma(W \otimes L)$ given by:

$$D_A(\psi) = \sum_{i=1}^{4} e_i \cdot d_A(\psi)(e_i)$$

The notion of Dirac operator can be extended fruitfully to the case where rather than taking values in a line bundle it may take values in complex bundle with hermitian metric, and the idea is that if we are given a Spin^c structure with associated virtual complex line bundle L then we would want that $E \otimes L^{-1}$ be a complex vector bundle with hermitian metric and its own unitary connection then we may extend the Dirac operator to be $D_A : \Gamma(W \otimes E) \to \Gamma(W \otimes E)$. The biggest advantage of this extension is being able to realise Hodge Laplacian as a genuine Laplacian and treating $d + \delta$ Dirac operator, leading to recovery of Hodge theory with this perspective.

Example: (Recovering Hodge theory through Dirac operators) Dirac operator with coefficients in W is $D_W = d + \delta$. To realise that it is indeed the case we have to express the exterior derivative and the codifferential in terms of the covariant derivative and then use the relation between the clifford multiplication and wedge product to see that when we add them they actually yield the Dirac operator in the sense we have defined it to be.

$$d\omega = \sum_{i=1}^{n} e^i \wedge (\nabla_{e_i}\omega)$$

$$\delta\omega = -\sum_{i=1}^{n} \iota_{e_i}(\nabla_{e_i}\omega)$$

$$\implies d + \delta(\omega) = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\omega$$

If we would locally express it using a frame then as we had noted in the historical overview it would yield the Laplacian upon squaring. But this comes in the familiar form when we choose frame with constant coefficients to make this independent of the frame we define the vector bundle Laplacian where we get an additional term to make it independent of the choice of frame much like in the case for curvature described using the covariant derivative (see section 1.7).

Definition 2.3.4. (Vector bundle Laplacian) Let X be an oriented Riemannian four manifold with Spin^c structure, equipped with Spin^c connection d_A on the spinor bundle $W \otimes L$. Then the Laplacian operator is defined to be a map $\Delta^A : \Gamma(W \otimes L) \to \Gamma(W \otimes L)$ given by:

$$\Delta^A(\psi) = \sum_{i=1}^{4} [\nabla^A_{e_i}(\nabla^A_{e_i}(\psi)) - \nabla^A_{\nabla_{e_i} e_i}(\psi)$$

We will proceed with one of our main results, the Wietzenböck's formula. We had seen in the flat space that the square of the Dirac operator would give the Laplacian but in the case of curved spaces or manifolds we see that we have to account for the curvature and this in fact gives us some very useful information regarding the curvature of the manifold, like we had preempted in the overview. It follows from some straightforward calculation, literally squaring the Dirac operator as it should be. One may find the complete computation in [11].

Proposition 2.3.3. *(Formal adjoint) Dirac operator is formally self adjoint.*

[11]

Proposition 2.3.4. *(Wietzenböck formula) [11] For a Dirac operator with coefficients in line bundle say D_A we have:*

$$D_A^2\psi = \Delta^A\psi + \frac{1}{4}s\psi - \sum_{i<j} F_A(e_i, e_j)ie_i \cdot e_j \cdot \psi$$

Before stating Atiyah-Singer index theorem, we must note that a Dirac operator can be split into two pieces if we are able to split the Spinor bundle into two pieces, which we have in our case. And the idea is to simply note in terms of the decomposition $W = W_+ \oplus W_-$ where does Dirac operator send each part but that is somewhat straight forward from the definition of the Dirac operator as the connection preserves the chirality while the clifford multiplication as we have seen through the idea of quaternions that their action is a map from positive to negative spinors and vice versa so in combination the Dirac operator flips the chirality and we may split it into two pieces that is $D_A = D_A^+ + D_A^-$ where $D_A^+ : \Gamma(W_+ \otimes L) \to \Gamma(W_- \otimes L)$ and $D_A^- : \Gamma(W_- \otimes L) \to \Gamma(W_+ \otimes L)$. The reason to consider this splitting is that we are able to find the index of these operators like in the case of Hodge theory where it expresses as topological invariant. And Atiyah-Singer theorem does exactly this, it provides us with the index of Elliptic operators in the most general case but we will state the case of Dirac operator which are of primary interest to us.

Proposition 2.3.5. *(Atiyah Singer index theorem) Given a compact oriented Riemannian four manifold X with Spin^c structure and a Dirac operator D_A with coefficients in complex line bundle we have that:*

$$index(D_A^+) = -\frac{1}{8}\tau(X) + \frac{1}{2}\int_X c_1(L^2)$$

where $index(D_A^+) = dim(ker(D_A^+)) - dim(ker(D_A^-))$ and $\tau(X) = b_+ - b_-$ the signature of X.

For the proof one may refer to [2]. The theorem has many beautiful corollaries and applications like Lichnerowicz's theorem for manifolds with positive scalar curvature.

Chapter 3

Global Analysis

Global analysis refers to the study of differential equations on manifolds, vector bundles and other associated structures. In the last chapter we have already seen a great example of global analysis. I am referring to the Atiyah-Singer index theorem. The basic idea being that there is an underlying differential equation, but when we establish them on a manifold we have to account for the topology of the manifold. And in doing this we might gain understanding of features of manifolds. A lot of this understanding is in geometric terms and the basis for that is the smooth structure on the manifold as it allows for calculus and as an offshoot we get a tangent spaces or the tangent bundle. And as we had discussed in the first chapter before introducing inner products on vector bundles (see definition 1.3.7), the geometric features are coded in this framework by adding on to this tangent bundle or associated bundles. And when we have more than one smooth structure (non diffeomorphic) then it means that the same topological manifold exhibits different geometries and a complete study or the picture of a manifold can be prepared only when we can understand all the distinct geometries it holds. For this reasons when we consider a differential equation on a manifold it is contingent on the smooth structure which to simplify might be thought of as encoding the ways in which we can take derivatives on a manifold. In Euclidian space we generally consider only the standard smooth structure (other that four dimensional Euclidean space this statement is superfluous!) but on a general curved space, a manifold there might be many different ways of taking derivatives and so the frame work of differential equations is sensitive to this change. And so we may expect that based on the space of solutions to the differential equations we might be able to detect the changes. This is in fact what happened historically in case of Donaldson's work, [4]. And the procedure that was adopted was that one would take a promising system of partial differential equations on the manifold, by phrasing it through differential operators (basically accounting for the change of co-ordinates) and then studying the space of solutions of this system. As we have seen that a system of differential equations on a manifold acts on the space of sections of a vector bundle and takes them to space of sections of perhaps another vector bundle. So the space of solutions of such a system would be a sub collection of these sections and we would want additional structure on them to be able to work with them. For that matter it turns out that they can in fact be given the smooth manifold structure when we consider nice system of differential equations which does happen in case of Donaldson's work. But there is an additional requirement of compactness to be able to extract information from the space of solutions. And that requires some additional work. As Moore suggests in [11], perhaps this process is demonstrated with least resistance through the study of moduli space of monopoles which is the name for the space of solutions of the Seiberg-Witten equations. And we recover the results one would get through Donaldson's work, as was it was suggested by Witten, [17]. In this chapter we will try to demonstrate that process of extracting information from the space of solutions in which the major steps will be the definition of the moduli space of monopoles for Seiberg-Witten equations and their perturbed version followed by showing the transversality and compactness. Finally we will define the Seiberg-Witten invariants after we recover Donaldson's theorem.

3.1 Seiberg-Witten equations and the moduli space of monopoles

In this section we will describe the Seiberg-Witten equations and define the moduli space of monopoles for them. Like the Yang-Mills equation these come from physics and in fact Witten suggests there is a close relation between them, [17]. As much as we would like to delve into the physical origins of these equations, but lies well out of the scope of this thesis, which one may find in [17].

Definition 3.1.1. (Seiberg-Witten equations) Given an oriented Riemannian four manifold with Spin^c structure with the associated positive spinor bundle be $W_+ \otimes L$, the perturbed Seiberg-Witten equations are:

$$D_A^+(\psi) = 0$$

$$F_A^+ = \sigma(\psi) + \phi$$

where F_A^+ is the self dual part of the curvature associated with some connection d_A on the virtual line bundle L (see proposition 2.3.1) and ϕ is some self dual two form. If one sets the self dual two form to be zero then the equations are simply called the Seiberg-Witten equations. For convenience we will also abbreviate these equations as "SW equations" or "perturbed SW equations". The map σ is a quadratic map defined as follows:

$$\sigma(\psi) = 2i \text{ Trace free skew Hermitian of } \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \begin{pmatrix} \bar{\psi}_1 & \bar{\psi}_2 \end{pmatrix}$$

One might draw on some motivation for the equations by combining them with the results of the previous chapter. The first being the Wietzenböck's formula where all these terms appear together. The indication is that there is some sort of control on the scalar curvature that is achieved due to the equations. The first equation offers some restriction on the choice of spinors and the second is the restriction on the curvature of the connection on the line bundle which is in some sense the more variable part. And the second equation controls the curvature in a way that it becomes quadratic expression in terms of the spinors themselves added with some fixed perturbation. But again I must emphasise that existence of solutions to this system on any four manifold with some given Spin^c structure will lead to restrictions on the scalar curvature. The most striking feature of this theory is that the solution space turns out to be naturally compact and that is exactly due to the combination of the equations and the Wietzenböck's formula, as we shall see in the next section. One may note that apart from this difference most of the invariants we know based on gauge theory have a very similar idea that we use elliptic partial differential equations and then compactify the space of solutions, show that it is diffeomorphism invariant. We will return to this point when we define the Seiberg-Witten invariants.

We have to give a solid foundation to the space of solutions and other associated spaces in order to realise all the things we mentioned. First idea is that we want to use the theory of elliptic partial differential equations and Fredholm theory in our global setting which means that the space of solutions must have the properties of Sobolev space or some other nice function space. And we need to remember that we have to cut down on the excessive solutions which we might get due to the line bundle part just as we had seen in abelian gauge theory case. With these two guiding principles we will give the appropriate framework and lingo for the space of solutions.

Given an oriented Riemannian four manifold X with Spin^c structure for which the associated virtual line bundle is L then the configuration space is defined to be the set:

$$\mathcal{A} = \{(d_A, \psi) : \text{ where } d_A \text{ is a connection on } L, \psi \in \Gamma(W_+ \otimes L)\}$$

Another advantage of the theory is that when we are forming the moduli space we will need to account for gauge transformations only on the line bundle part, a more physical reasoning for this advantage may be found in [17]. Because of which the gauge group acting on the configuration space is essentially the same as the one we considered in abelian gauge theory, the maps from manifold to S^1, and we will denote it by $\mathcal{G}$. The action of gauge group on the configuration space is given by:

$$g \cdot (d_A, \psi) = (d_A + g d g^{-1}, g\psi)$$

And analogously we define the based gauge group to be $\mathcal{G}_0$ which acts as identity on the base point. And we will denote the quotient of the moduli space by the gauge equivalence given by these gauge groups by $\mathcal{B} = \mathcal{A}/\mathcal{G}$ and $\tilde{\mathcal{B}} = \mathcal{A}/\mathcal{G}_{\prime}$.

From the previous chapter we know that the we could define Dirac operators corresponding to the Hodge Laplacian. Even with these equations we would like to phrase it entirely in terms of Dirac operators. This is because they are elliptic differential operators, and that for us means that they will turn out to be “nice” maps when we treat the domains of those operators as infinite dimensional manifolds. The space of connections is an affine space and when we look at the total gauge group we have some insight from principal bundle descriptions . But the quotient space $\mathcal{B}$ is not as intuitive or rather we don't know what sort of structure it has. But when we consider the space $\tilde{\mathcal{B}}$ has a better description, we can identify it with a linear space, which is far better than it being just a set. We realise this by noting that the elements of this space have harmonic representatives for the imaginary one form. This follows from a neat application of Hodge theory. Since having a harmonic representative means that $\delta(a+du) = 0$ and we have to find such smooth map $u : X \to \mathbb{R}$. But this means that $\Delta u = d(\delta u) + \delta(du) = \delta(du) = -\delta a$. That is the Poisson equation. But now we consider the following computation:

$$\int_X 1 \wedge *\delta a = \int_X d(*a) = 0$$

The last step is due to Stokes' theorem. What this means is that the $\langle 1, \delta a \rangle = 0$ but this means that $\delta a \in (\ker(\Delta))^{\perp} = \operatorname{Im}(\Delta)$ which means we have the solution there are two underlying reasons for this, one that harmonic functions are simply constant valued on connected manifold since:

$$\Delta u = 0 \implies \langle \Delta u, u \rangle = 0$$

$$\langle \Delta u, u \rangle = \langle \delta du, u \rangle = \langle du, du \rangle$$

$$\implies du = 0$$

And on the other hand we have $L^2(M) = \ker(\Delta) \oplus \operatorname{Im}(\Delta)$. So this guarantees the existence of a harmonic representative. Then suppose there are two harmonic representatives which are gauge equivalent then we would have $da_1 = da_2$ which means on a simply connected manifold we would have a smooth map b such that $a_1 - a_2 = db$. But again since a_1, a_2 are harmonic we would have that $\Delta b = 0$ and we know that means $db = 0$ and we are done with uniqueness. So we have the following proposition:

Proposition 3.1.1. *(Harmonic representatives for $\tilde{\mathcal{B}}$) When the underlying manifold is simply connected then there is a one to one correspondence between $\tilde{\mathcal{B}}$ and the linear subspace $\{(d_{A_0} - ia, \psi) : a \in \Omega^1(X), \psi \in \Gamma(W_+ \otimes L), \delta a = 0\}$.*

This elucidates the structure of $\tilde{\mathcal{B}}$. But we would want to quotient by the total gauge group but that complicates the structure of this space because when we consider the action of the gauge group on configurations with zero spinors the action is not free and in fact the stabiliser is given by the group S^1. These are called the **reducible** elements or configurations while the configurations with spinors which is somewhere non zero are the **irreducible**, where the action is free. This hinders in the geometric structure of the quotient space. If we assume for now that the gauge group and the space of configurations has reasonable topology on it. Sometimes we would want to work with only the irreducible portion of the quotient which we may denote by $\mathcal{B}^*$ and $\tilde{\mathcal{B}}^*$ where we are considering the quotient of the irreducible configuration space which we may denote by $\mathcal{A}^*$. And then we may treat them as infinite dimensional manifolds modelled on Banach spaces. To do this we also need the notion of Sobolev completions. In order to work with these notions we will heavily rely on existing literature and borrow theorems rather than trying to prove them. But we will start by trying to have some conceptual understanding so that it would not hinder us while using those tools. What we are dealing with are largely space of sections their products and so on. These are vector spaces which need not be finite dimensional. So in order to put a topological structure on them we need a norm, or inner product on it. For now let us assume that we do have a meaningful norm on them. The notion of Banach space is a normed vector space for which the metric associated with its norm is complete. And similarly the gauge group is also a space of sections. Now the question shifts to how can we define meaningful norm on the space of sections. For this we visit the concept of Sobolev and C^k norms. Both the notions are built for managing the derivative data on a space. In the C^k norms we assign norm to a section based on its highest derivative values assumed by the function throughout the space. While the Sobolev norm is given by the averaging out the derivative values from throughout the space. With this we define the Sobolev norms for our case where we consider the space of sections of bundle valued tensorial fields, $\Gamma(\mathrm{Hom}(\bigotimes^k T^*X, E))$ by the following equation:

$$||\sigma||_{p,k} = \left[\int_X \sum_{i=0}^{k} ||d_A^i \sigma||^p\right]^{1/p}$$

where $d_A^j = d_A \circ d_A \circ \cdots d_A$, d_A is the connection on the vector bundle E. The completion of the space of sections $\Gamma(E)$ by this norm is denoted by $L_k^p(E)$, these spaces are called *Sobolev spaces*. Also $C^l(E)$ is the space of C^l sections of the bundle. And now that we have given the appropriate norm on these spaces using this we can define the following spaces which are of our concern:

$$\mathcal{A}^p_k = \{(d_{A_0} - ia, \psi) : a \in L^p_k(T^*X), \psi \in L^p_k(W_+ \otimes L)\}$$

$$\tilde{\mathcal{B}}^p_k = \{(d_{A_0} - ia, \psi) : a \in L^p_k(T^*X), \psi \in L^p_k(W_+ \otimes L), \delta a = 0\}$$

$$\mathcal{G}^p_{k+1} = C^0(X, S^1) \cap L^p_k(\mathrm{End}(E))$$

To make sense of the final definition we will have to take care of some theorems which relate C^k spaces and L^p_k spaces or Sobolev spaces, like the Sobolev embedding theorem which states that if $k - \frac{4}{p} > l$ then there is a continuous embedding:

$$L^p_k(E) \to C^l(E)$$

And we have due to the Rellich's theorem that the following is a compact inclusion for all p, k:

$$L^p_{k+1}(E) \to L^p_k(E)$$

We also utilise the multiplication theorems of Sobolev spaces which give us that there are continuous multiplications:

$$L^p_k(E) \times L^p_k(F) \to L^p_k(E \otimes F)$$

And we may note that when we take the product of Banach spaces, for example when we are topologizing the configurations space we may note that the product norms afford the same topology as the what comes from product topology. Here on when we refer to the configuration space, gauge group or the quotient we will assume them to be acquiring their topology from the Sobolev spaces corresponding to bundles used in definitions. And we will not write the indices until we are dealing with technical details. We may note that $\mathcal{G}$ is a infinite dimensional Lie group (Hilbert Lie group). In our case we will exploit the property of $K(\mathbb{Z},1)$ spaces since we are dealing with at least continuous maps into S^1. Which means that we have identification of its components with $H^1(X,\mathbb{Z})$ and so we may consider the component, call it U, identified with $0 \in H^1(X,\mathbb{Z})$ and for these we know that we can express them as exponential of elements of $L^2_3(X,i\mathbb{R})$ and this allows us to produce chart around identity, using *Cayley transformation* which is given by the map;

$$T : U - \{-1\} \to L^2_3(X,i\mathbb{R})$$

$$t \longmapsto \frac{1-t}{1+t}$$

Of course this is a smooth map, moreover $T^{-1} = T$, showing that it is a diffeomorphism. And then we may translate this map to other points by pulling those back to identity. And again since the map is inverse of itself it follows that the transitions are smooth. Which means that we have the smooth structure. This is now allows us to examine what sort of structure do $\mathcal{B}$ or its other iterations possess. The idea we have is that we want to extend the idea of quotient by a Lie group to the infinite dimensional case. But the reducible elements cause some trouble so as we had discussed we only deal with the irreducible part and there we have free group action at all points. For the detailed proof of this matter one may refer to [12]. The idea revolves around understanding the local structure of the quotient space. And to do that we utilise the action of the gauge group on the configuration space. Firstly, points in the quotient correspond to the orbits of the configurations traversed through the action of gauge group. So if we want to consider neighbourhood of points in the quotient we consider the notion of a local slice of a configuration which is a collection of vectors which are orthogonal to the orbit in the space of configurations. This tells us which directions in tangent space of configuration escape the orbit. So this suggestive that the tangent space of the quotient might be this local slice. Moreover, since we know in which directions to move this very suggestive of a local picture of the configuration space where we might be able to factor it or express it as a product of two pieces, one being the patch showing motion along slice and the second showing motion along orbit. And indeed this is what we get, but this requires establishing the maps which describe the motion along orbit and motion along slice and showing that they are nice enough to produce such a neighbourhood. The discussion is more clear when it comes to irreducible configurations but when we deal with reducible solutions we need to make sure that we take out the stabiliser as well. The more technical part is showing the existence of the neighbourhood corresponding to motions along of slice. But at least with the picture it becomes very plausible. We will note the operators of infinitesimal motion along orbit and slice first and then state the theorem regarding the local picture below.

Definition 3.1.2. (Infinitesimal action of the gauge group) Given the smooth action of $\mathcal{G}_{k+2}$ on the configuration space $\mathcal{A}_{k+1}$ given by the map $(g, C) = g \cdot C$ where the action is that of gauge transformation then the infinitesimal action is defined as the map $\mathfrak{L}_C : T_1\mathcal{G}_{k+2} \to T_C\mathcal{A}_{k+1}$ given by the following mapping:

$$if \longmapsto (-idf, if\psi)$$

Remark: The difference in index of the gauge group and configuration is because the action requires derivatives of the gauge transformations so we need to have that higher order derivative included in the topology. Also the definition of the configuration space and establishing the Hilbert Lie group structure on the gauge group should explain why those elements lie in the tangents of the spaces considered. This map as as clear as can be collects vectors which describe the motion along the orbit or by extension the orbit itself. As we had noted simply taking the orthogonal vectors and to do that we may regard $\mathfrak{L}_C$ as a real unbounded operator mapping from $L^2(X, i\mathbb{R})$ to $L^2(W_+ \oplus iT^*X)$. And consider its L^2 adjoint which leads to the defining relation for the map $\mathfrak{L}_C^*$ going the other way round:

$$\langle if, \mathfrak{L}_C^*(\dot{\psi} \oplus i\dot{a}) \rangle = \langle \mathfrak{L}_C(if), \dot{\psi} \oplus i\dot{a} \rangle$$

And with this we can define the transverse vectors to orbit or the local slice as follows:

Definition 3.1.3. (Local slice) The local slice for the configuration space $\mathcal{A}_2$ at the configuration C is defined to by the linear subspace $\mathcal{S}_C$ given by:

$$\mathcal{S}_C = \{\dot{C} \in L^2(W_+ \oplus iT^*X) : \mathfrak{L}_C^*(\dot{C}) = 0\}$$

And finally the theorem which establishes manifold structure of the quotient space, giving the local picture:

Proposition 3.1.2. *(Local picture of $\mathcal{B}^{k+2}$) There exists a map $\Phi : \mathcal{G}_{k+2} \times \mathcal{S}_C \to \mathcal{A}_2$ for $C \in \mathcal{A}_{k+2}$ and a neighbourhood U of $0 \in \mathcal{S}_C$ such that Φ is smooth $\mathcal{G}_{k+2}$-equivariant, $Stab(C)$-invariant, with $\Phi(1,0) = C$ and so that it induces a diffeomorphism onto a $\mathcal{G}_{k+2}$-invariant open neighbourhood:*

$$\tilde{\Phi} : \mathcal{G}_{k+2} \times U/Stab(C) \to \mathcal{A}_2$$

The map Φ is given by $\Phi(g, \dot{\psi} \oplus i\dot{a}) = (g\psi + g\dot{\psi}, A + i\dot{a} - g^{-1}dg)$.

[12]

Since the map Φ is a diffeomorphism onto gauge invariant open neighbourhood in the configuration space we may use this to define charts for the irreducible quotient space:

$$U \longrightarrow (\mathcal{B}^*_{k+1}, \delta)$$

$$\dot{\psi} \oplus i\dot{a} \longmapsto \mathcal{O}_{C+(\dot{\psi}, i\dot{a})}$$

where $\mathcal{O}_C$ is the orbit of the configuration C and δ is infimum norm(stronger norm than usual).

This finally reveals the local picture of the quotient space. Before we define the moduli space of monopoles we will first note some global topological feature of the quotient space which should now be more clear as we have established the manifold structure on it. This has to do the with difference between the action of the based gauge group and usual total gauge group. We first note that $\mathcal{A}$ is topologically same as an infinite dimensional sphere since it is an affine space, which can be viewed as a linear space with a point removed. And this means that all its homotopy groups vanish. If we pick a simply connected base manifold then the gauge group also becomes contractible leading to the following exact homotopy sequence of bundles:

$$S^1 \to \tilde{\mathcal{B}}^* \to \mathcal{B}^*$$

This makes the space $\mathcal{B}^*$ homotopy equivalent to $P^\infty\mathbb{C}$. Then we may regard $\tilde{\mathcal{B}}^*$ as the bundle of unit length vectors in E a complex line bundle over $\mathcal{B}^*$. This gives us some topological understanding of the quotient spaces and difference in the action of based and usual gauge group.

With all this set up, we are finally prepared to define the moduli space of monopoles:

Definition 3.1.4. (Moduli space of monopoles) Given an oriented Riemannian four manifold X with Spin^c structure and associated virtual line bundle L and the self dual two form used to perturb the Seiberg-Witten equations be ϕ. Then the moduli space of monopoles is defined to be the space:

$$\mathcal{M}_{X,L,\phi} = \{[d_A, \psi] \in \mathcal{B} : \text{ they satisfy the perturbed SW equations}\}$$

And this may be viewed as a quotient of the following space, using the proposition :

$$\begin{aligned}\tilde{\mathcal{M}}_{X,L,\phi} = \{(d_{A_0} + ia, \psi) \in \Omega^1(X) \times \Gamma(W_+ \otimes L) :\ \delta a = 0 \\ \text{and the pair satisfies perturbed SW equations}\}\end{aligned}$$

Remark: Before we move on to the next section, the notion of infinite dimensional manifolds and Lie groups modelled on Banach space or Hilbert spaces are used implicitly throughout this section and even in the next. A very neat presentation of the basics of these may be found in [10]. And similarly for more analytical details or for that matter details of the arguments presented here one may refer [11] and [12], which have been closely followed for this thesis.

3.2 Compactness and transversality

In this section we will go over the proofs for two very remarkable properties of the monopole moduli spaces which is that they are compact and for perturbation by a generic choice of two form they are smooth manifolds (oriented). To appreciate these properties one may take a quick look into the section on Seiberg-Witten invariants and return.

Compactness for the monopole moduli space follows from the idea we had hinted at in the previous section before we defined the SW equations. It is that the combination of Wietzenböck's formula and SW equations leads to bound on the length of spinors and then this eventually leads to the result.

We follow the argument presented in [11]. Consider a configuration (A, ψ) which is a solution to the perturbed SW equations. We start by considering the Wietzenböck's formula:

$$D_A^2\psi = \Delta^A\psi + \frac{s}{4}\psi - \sum_{i<j}^{4} F_A^+(e_i, e_j)(ie_i \cdot e_j \cdot \psi)$$

We use the first equation, now we must remember that we are selecting the positive spinor so the square of the Dirac operator acting on it becomes $D_A^- D_A^+\psi = 0$ and so the left hand side of the equation vanishes. Since we are trying to bound lengths we take the inner product of sections on each side with ψ. Combined with the definition of the quadratic map σ we may rewrite the equation as follows:

$$0 = \langle \Delta^A\psi, \psi\rangle + \frac{s}{4}\langle\psi, \psi\rangle + 2\langle F_A^+, \sigma(\psi)\rangle$$

Now we may use the second of the equations to obtain,

$$0 = \langle \Delta^A\psi, \psi\rangle + \frac{s}{4}\langle\psi, \psi\rangle + 2\langle \sigma(\psi) + \phi, \sigma(\psi)\rangle$$

When a smooth function acquires its local maxima on the manifold we have the relation that the Hodge Laplacian must be non negative i.e. $\Delta f = -\frac{1}{\sqrt{g}}\sum_{i,j=1}^{4} \partial_{x^i}(g^{ij}\partial_{x^j})$ where g_{ij} are the coefficients of the metric g expressed in terms of the co-ordinates x^i and $g^{ij} = g_{ij}^{-1}$ and we use the following relation for the Hodge Laplacian:

$$\frac{1}{2}\Delta|\psi|^2 = \langle \Delta^A\psi, \psi\rangle - |d_A\psi|^2$$

Due to this we get:

$$\frac{s}{4}\langle\psi,\psi\rangle + 2\langle\sigma(\psi)+\phi,\sigma(\psi)\rangle \leq 0$$

And we may use the relation that $2|\sigma(\psi)|^2 = |\psi|^4$ which gives us a bound for the maxima if we have one. Leading to the following proposition, [11]:
Proposition 3.2.1. *(Bound on length of spinors) If the pair (d_A,ψ) where ψ is non zero spinor is a solution to the perturbed SW equations then we have that:*

$$|\psi|^2(p) \leq -\frac{s(p)}{4} + \sqrt{2}|\phi(p)|$$

This means that the range of the positive spinors appearing in the moduli space of monopoles lie in a compact subset of $W_+ \otimes L$. But now we are to deal with the moduli space itself and now we will rely on theory of PDEs to get a bound using the Dirac operators with coefficients. Suppose D is a Dirac operator with coefficients then the following inequality holds:

$$||\psi||_{p,k+1} \leq c(||D\psi||_{p,k} + ||\psi||_{p,k})$$

where c is some constant. The term $||\psi||_{p,k}$ can be omitted if the kernel of the Dirac operator is trivial.

Here, we can note that using the proposition regarding harmonic representatives we can introduce the Dirac operator $\delta \oplus d^+$ in the equations. The advantage is we would be dealing with the line bundle part using Dirac operators and we would be able to use the inequality, to generate bounds. Then using the Sobolev embedding and Rellich's theorem we are able to show bounds for all norms and thus completing the proof. What happens is that when we demand harmonic representative we end up fixing a base connection. So with respect to this we get the SW equations in the following form:

$$D_{A_0}\psi = -ia\psi$$

$$(da)^+ = \sigma(\psi) + \phi - F^+_{A_0}$$

$$\delta a = 0$$

If we take the four manifold under consideration to be simply connected then the kernel of the Hodge Dirac operator becomes trivial. Then using the inequality we would get that a is bounded for all L^p_1. And the spinor parts are bounded anyways in C^0 and therefore in all L^p. After this if we use a fact of Banach algebra then it leads to cascading effect and we get that these are bounded in L^p_k for all k. Which leads to being bounded in C^l for all l. Thus showing compactness.

Proposition 3.2.2. *(Compactness) If X is a simply connected smooth four manifold then for any perturbation ϕ to SW equations $\mathcal{M}_{X,\phi}$ is compact.*

[11]

We now move on to the second technical part, transversality. The term might be familiar from differential topology, where it serves as the right criteria for identifying when intersections of submanifolds are submanifolds and preimage of smooth maps are submanifolds. A very important important theorem in that context being the Sard's theorem which suggests that almost all points are regular for a smooth map. This theorem was extended to the context of infinite dimensional manifolds by Smale, [14]. Which proves to be crucial in the context of gauge theory as it provides structure of manifold to the moduli space and in combination with Atiyah-Singer index theorem it gives finite dimensional smooth manifold structure to the moduli space. Another tool we need comes from the theory of elliptic PDEs, which is called the Unique continuation theorem which one can find in [1].

Proposition 3.2.3. *(Transversality-1) Consider a simply connected compact smooth four manifold X with Spin^c structure. Additionally when $b_+ = 0$ we require that index$(D^+_A) > 0$. Then for a generic choice ϕ for perturbation of SW equations we have that $\tilde{\mathcal{M}}_{X,\phi}$ is an oriented smooth manifold with dim$(\tilde{\mathcal{M}}_{X,\phi}) = 2$ index$(D^+_A) - b_+$.*

[11]

To see how this works we will break it down into a few steps and carefully go through each of those. Before we begin we note an important detail. The reason why we use perturbed SW equations is that it gives us an opportunity to skip dealing with reducible configurations coming in the monopole moduli space altogether. And that depends on how we choose our perturbation. In order to use this perturbation first we consider the product space, $\tilde{\mathcal{B}} \times \Omega^2_+(X)$. This allows us to deal with all perturbations altogether. First what we try to do is to realise the space of solutions in this larger space as a submanifold. To do this we have to show that the map associated to the perturbed SW equations, call it $\mathfrak{T}$ has $0 \in \Gamma(W_- \otimes L) \times \Omega^2_+(X)$ is a regular point. Of course for this first we will have to figure out that it is a smooth map. Then using implicit function theorem, which holds for infinite dimensional case we can get that the larger solution space $\mathfrak{T}^{-1}(0)$ is a submanifold. Then next step is to work with this space and project it onto $\Omega^2_+(X)$, and this serves the task of choosing the perturbation and the fibre of that perturbation must correspond to the monopole moduli space. Again we are to show that the choice of perturbation is a regular point. And this is where we will use Smale's version of Sard's theorem to get the fibre as finite dimensional smooth manifold. Of course the additional requirement is to make sure that the map under consideration is Fredholm.

So we begin with the first step by defining the map $\mathfrak{T} : \tilde{\mathcal{B}} \times \Omega^2_+(X) \rightarrow \Gamma(W_- \otimes L)$. It is given along with its linearisation:

$$(A, \psi, \phi) \longmapsto (D^+_A\psi, F^+_A - \sigma(\psi) - \phi)$$

$$d\mathfrak{T}_{(A,\psi,\phi)}(a, \dot{\psi}, \dot{\phi}) = (D^+_A\dot{\psi} - ia \cdot \psi, (da)^+ - \sigma(\psi, \dot{\psi}) - \dot{\phi})$$

The linearisation is quite easy to check we have to note what happens when we vary the parameters, for example when we change the connection from A to $A - ia$ the Dirac operator changes, it gets the additional ia part to it. And similarly we get a change in other parameters, the method used here for linearisation is through an apparatus called the *Landau symbol*, details of which can be found in [10]. Also, in the linearisation the map σ is defined as follows:

$$\sigma(\psi, \dot{\psi}) = 2i \text{ Trace free Hermitian part of } \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \begin{pmatrix} \dot{\bar{\psi}}_1 & \dot{\bar{\psi}}_2 \end{pmatrix}$$

Now we need to see why $d\mathfrak{T}$ is surjective at solutions. First we may note that no matter what comes from the first part we can adjust $\dot{\phi}$ to get the last factor to be any element of $\Omega^2_+(X)$. So we need to worry about the first factor, and that means if we bother only with the case when $\dot{\phi} = 0$ we are fine. So let us take an element in the orthogonal complement of the image of our linearisation $(\eta, 0)$, it must lie in the $\ker((D_A^+)^*)$. But we know that $(D_A^+)^* = D_A^-$. So we get that $D_A^-\eta = 0$, now the unique continuation theorem suggests that it is non zero on a dense open set. Similarly due to the same theorem we have that if $\psi \neq 0$ then must vanish on an open dense set U. Now the map $a \longmapsto a \cdot \psi$ is injective and hence an isomorphism point wise on U. But since η is L^2 orthogonal to $\mathrm{Im}(a \longmapsto a \cdot \psi)$ at sections level it means that it has to vanish on the whole open neighbourhood and then it means that $\eta = 0$. And this means that the map is surjective. Then this gives us that $\mathfrak{T}^{-1}(0)$ is a submanifold and its tangent space can be described by:

$$T_{(A,\psi,\phi)}\mathfrak{T}^{-1}(0) = \{(a, \dot{\psi}, \dot{\phi}) : L(a, \dot{\psi}) = (0, 0, \dot{\phi})\}$$

where the map L is given by:

$$L : \Gamma(W_+ \otimes L) \oplus \Omega^1(X) \to \Gamma(W_- \otimes L) \oplus \tilde{\Omega}^0(X) \oplus \Omega^2_+(X)$$

$$L(\dot{\psi}, a) = (D_A\dot{\psi} - ia\psi, \delta a, (da)^+ - 2\sigma(\psi, \dot{\psi}))$$

This happens to be an elliptic operator. Now as we remember we wanted the projection $\pi : \mathfrak{T}^{-1}(0) \to \Omega^2_+(X)$ to be Fredholm. We know that the kernel matches with L. Also their cokernels turn out to have the same dimension. This follows from the same idea as before that we try to track the adjoint of the Dirac operator and then use its ellipticity and unique continuation theorem to get the desired result. The dimension of the space is determined by the index of the operator L which comes from the Atiyah Singer index theorem.

To see that it is an oriented manifold one may refer to [11] or [12].

Proposition 3.2.4. *(Transversality-2) Consider a simply connected compact smooth four manifold X with Spin^c structure with $b_+ > 0$. Then for a generic choice ϕ for perturbation of SW equations we have that $\mathcal{M}_{X,\phi}$ is an oriented smooth manifold with dim$(\mathcal{M}_{X,\phi}) = 2$ index$(D_A^+) - b_+ - 1$.*

This simply follows from the first transversality as S^1 acts freely on $\tilde{\mathcal{M}}_{X,\phi}$ and we can quotient it out. Therefore the dimension drops by one.

3.3 Seiberg-Witten invariants

In the previous sections we have been able to realise a lot of geometric structure on monopole moduli space. And finally we are at the stage where we can extract useful information from this object. One very important application is that at times it allows us to detect non diffeomorphic smooth structures on a topological manifold. This is carried out by defining invariant out of the moduli space. Before we even consider this we would have to ensure that it is not dependent on the additional geometric structures which we adopt on the manifold like the Spin^c structure and of course it shouldn't depend on the perturbation. But that is not exactly the case, indeed the Moduli spaces depend on the Riemannian or Spin^c structure and on the perturbation of the SW equations. But this is manageable as for different choice of perturbation the moduli spaces turn out to be cobordant which means that they share some amount of topological information. And similarly the case of different Riemannian metrics the moduli spaces are cobordant. And it is in producing these cobordisms that our compactness helps us out unlike other theories where one has to compactify the moduli space before going ahead with such investigation.

Definition 3.3.1. (Seiberg-Witten invariants) Given for a smooth four manifold with $b_+ \geq 2$ and that $\mathcal{M}_{L,\phi}$ is even dimensional then we define the Seiberg-Witten invariant of a complex line bundle L as:

$$\mathrm{SW}(L) = \langle c_1^d, [\mathcal{M}_{L,\phi}] \rangle$$

where c_1 is the first Chern class of the complex line bundle associated with the S^1 bundle $\tilde{\mathcal{B}}^* \to \mathcal{B}^*$ and $d = \frac{1}{2}\mathrm{dim}(\mathcal{M}_{L,\phi})$

Proposition 3.3.1. *(Witten) Given a smooth four manifold X which admits Riemannian metric with positive scalar curvature all its Seiberg-Witten* *invariants vanish.*

This follows from the proposition where we produced bounds on the lengths of the spinors which solve for the SW equations. If they admit positive scalar curvature their length will be bound by a negative number which means there are no solutions.

It is due to Gromov and Lawson, that if two manifolds admit metric with positive scalar curvature then their connected sum will also admit a metric with positive scalar curvature. This will be useful to us in the next part.

Chapter 4

Application to Kahler surfaces

In this part we will focus on two theorems which demonstrate the use for Seiberg-Witten invariants. First is to produce examples of manifolds which have more than one smooth structure. The second one is a more fancier version of the first where we produce infinitely many non diffeomorphic smooth structures on a manifold. In order to do this we will assume a lot of theory of Kähler surfaces and some theory of complex algebraic surfaces. The reason we use such setting is because the nice properties of these geometries reflect in the nature of the Dirac operator, existence of special line bundles and other restrictions.

For the first one consider the following proposition:
Proposition 4.0.1. *For a simply connected Kahler surface with $b^+ \geq 2$ we have $SW(K^{-1}) = \pm 1$.*

[11]

The proof of this is carried out in two parts. First we show that the for the given hypothesis we can choose a perturbation such that we get an immediate solution to the perturbed SW equations. Then the more cumbersome part is to show that in fact it is the only solution at least upto gauge equivalence. Meaning that the monopole moduli space consists of one point and thus the theorem holds.

The guess is that we set the perturbation as $\phi = F_A^+ + \omega$ where ω comes from the Kähler structure. Then it turns out that we get the solution to be:

$$\psi = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}, A = A_0$$

where A_0 is the connection determined by the Kähler structure, where the connection form is given by $a = \frac{1}{2}(\omega_{12} + \omega_{34})$ where ω is the Kähler form. And the relation $\sigma(\psi) = -\omega$ holds.

Now in order to show that these are the only connections we use transformations given by Witten which take solutions to other solutions. Given we consider the spinor as follows:

$$\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

and the transformations are given by $A \mapsto A, \quad \alpha \mapsto \alpha, \quad \beta \mapsto -\beta$. In order to see that these take solutions to solutions we may note that they leave the following action whose extrema correspond to solutions of perturbed SW equations:

$$S[A, \psi] = \int_X (|D_A^+\psi|^2 + |F_A^+ - \sigma(\psi) - \phi|^2)\text{vol}$$

To actually see that we will have to use the Wietzenbock's formula, expand out terms in terms of α and β. But we get normed terms and that shows that it remains unchanged by the transformation. Due the invariance we also note that each component of the assumed solution must be a solution to the first equation and we also get that $\alpha\bar{\beta} = 0$.

Using the fact that the curvature of the connections A, A_0 lie in the same de Rham class we may deduce using the following equation:

$$0 = \int_X F_{A_0} - F_A \wedge \omega = \int (da)^+ \wedge \omega = \int 1 + \frac{1}{2}(|\beta|^2 - |\alpha|^2)\omega \wedge \omega$$

Then using this and the earlier equation which said one of these components must be zero we get that β must be zero. Then we are able to recover that the Dirac operator must act on α as $\bar{\partial}\alpha = 0$. Also from the same computation we get that $(da)^+ = 1 - \frac{1}{2}|\alpha|^2\omega$ also with gauge fixing we may take $\delta a = 0$. Then we can solve this equations by expressing $\alpha = he_1$ where h is holomorphic and e_1 is unit length section. This leads to the solution that α is constant $\sqrt{2}$.

This exercise exhibits the ease of operations on these special surfaces. We have equipped very restrictive structures, like the spinor bundle being isomorphic to direct sum of trivial and anti canonical bundles is what allows for these many computational advantages. But the consequence of this theorem is that we have Kahler surfaces with non trivial invariant value. While due to Freedman's work we get that manifolds with odd intersection forms can be decomposed into sums of complex projective surfaces. And we know that they admit metrics with positive scalar curvature thus having vanishing invariants. Which means that there are non diffeomorphic smooth structures on the same topological manifold.

With the aid of theory of complex algebraic surfaces and some special transformations called logarithmic transformations we are able to preserve the intersection form while the Seiberg Witten invariant varies with these transformations and this procedure leads to the following theorem.
Proposition 4.0.2. *The topological manifold $P^2\mathbb{C}\sharp 9\overline{P^2\mathbb{C}}$ admits infinitely many non diffeomorphic smooth structures.*

[11]

Chapter 5

Bibliography

[1] Aronszajn, Nachman. "A Unique Continuation Theorem for Solutions of Elliptic Partial Differential Equations or Inequalities." *Journal de Mathématiques Pures et Appliquées* 36 (1957): 235–249.

[2] Berline, Nicole, Ezra Getzler, and Michèle Vergne. *Heat Kernels and Dirac Operators*. Grundlehren der Mathematischen Wissenschaften 298. Berlin: Springer-Verlag, 1992.

[3] Chevalley, Claude. *The Algebraic Theory of Spinors*. New York: Columbia University Press, 1954.

[4] Donaldson, Simon K. "An Application of Gauge Theory to Four-Dimensional Topology." *Journal of Differential Geometry* 18, no. 2 (1983): 279–315.

[5] Griffiths, Phillip, and Joseph Harris. *Principles of Algebraic Geometry*. New York: John Wiley & Sons, 1978.

[6] Hatcher, Allen. *Vector Bundles and K-Theory*. Version 2.2. Ithaca, NY: Cornell University, 2017.

[7] Hawkins, Thomas. *The Emergence of the Theory of Lie Groups: An Essay in the History of Mathematics 1869–1926*. New York: Springer, 2000.

[8] Huybrechts, Daniel. *Complex Geometry: An Introduction*. Berlin: Springer, 2005.

[9] Lawson, H. Blaine, Jr., and Marie-Louise Michelsohn. *Spin Geometry.* Princeton Mathematical Series 38. Princeton, NJ: Princeton University Press, 1989.

[10] Marsden, Jerrold E., Tudor S. Ratiu, and Ralph Abraham. *Manifolds, Tensor Analysis, and Applications.* 2nd ed. New York: Springer-Verlag, 1988.

[11] Moore, John Douglas. *Lectures on Seiberg–Witten Invariants.* Lecture notes, University of California, Santa Barbara, 1996.

[12] Nicolaescu, Liviu I. *Notes on Seiberg–Witten Theory.* Graduate Studies in Mathematics 28. Providence, RI: American Mathematical Society, 2000.

[13] Rowe, David E. *Felix Klein: A Mathematical Biography.* New York: Springer, 1989.

[14] Smale, Stephen. "An Infinite Dimensional Version of Sard's Theorem." *American Journal of Mathematics* 87, no. 4 (1965): 861–866.

[15] Tu, Loring W. *Differential Geometry: Connections, Curvature, and Characteristic Classes.* Cham: Springer, 2017.

[16] Warner, Frank W. *Foundations of Differentiable Manifolds and Lie Groups.* New York: Springer, 1983.

[17] Witten, Edward. "Monopoles and Four-Manifolds." *Mathematical Research Letters* 1, no. 6 (1994): 769–796.