# A Spin and a Twist on Conformal Field Theory

विद्या वाचस्पति की

उपाधि की अपेक्षाओं की आंशिक पूर्ति में प्रस्तुत शोध प्रबंध

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# Certificate

This is to certify that the work contained in this thesis entitled **"A Spin and a Twist on Conformal Field Theory"** submitted by **Dhruva Krishnagiri Sathyanarayanan** has been carried out by the candidate, under my supervision and that it has not been submitted elsewhere for the award of any degree or diploma from any other university or institution.

Date: June 4, 2026

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# Declaration

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The work reported in this thesis is the original work done by me under the supervision of Dr. Sachin Jain.

Date: June 4, 2026

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# Abstract

The modern scattering amplitudes program has revolutionized our understanding of quantum field theory in Minkowski spacetime, leading to insights such as novel recursion relations, uncovering hidden symmetries like dual conformal and Yangian invariance, providing a dual geometric interpretation of amplitudes such as the associahedron, amplituhedron, and much more. Beginning with the remarkably simple Parke-Taylor formula for $n-$point tree-level MHV gluon amplitudes, this program has led to powerful methods to compute higher multiplicity and loop amplitudes which are extremely difficult to obtain using the usual Lagrangian and Feynman diagram framework. A key ingredient in these pursuits was the use of the right set of kinematic variables such as spinor helicity, twistors and momentum twistors that make the simplicity manifest from the start.

Given these remarkable successes, it is natural to ask whether similar structures exist for boundary conformal correlators in Anti-de Sitter and cosmological spacetimes or more generally, for conformal field theory correlators. This forms the main subject of this thesis.

We first develop a momentum-space and spinor helicity approach to conformal field theories, particularly focusing on the physically relevant case of three dimensional conformal field theory. We also determine that $n-$point functions in the simpler, yet illuminating setting of conformal quantum mechanics take the form of Lauricella functions. We apply our formalism to Chern-Simons matter theories which unveils an anyonic form for current correlators. We also use spinor helicity to uncover the holographic dual of chiral higher spin theory in four dimensional Anti-de Sitter spacetime.

We then discuss the twistor framework for three dimensional conformal field theories which presents an advantage over the spinor framework as it makes conformal symmetry completely manifest. We derive the Penrose transform for conserved currents and extend the formalism to accommodate arbitrary conformal primaries and for supercurrents. Finally, we discuss the construction of twistor space boundary correlators in $AdS_4$ finding novel factorization properties, connections to conformal partial waves and a four point double copy relation between Yang-Mills theory and Einstein gravity.

Our results provide a starting point towards a potentially conceptually and technically rich understanding of conformal correlators. More broadly, this perspective points toward a common analytic and geometric language that could potentially bring together holography, conformal bootstrap and the S-matrix program.

# List of Publications

- The Conformal Grassmannian: A Symplectic Bi-Grassmannian for $CFT_4$ Correlators, Aswini Bala, Sachin Jain, **Dhruva K.S.**, [2605.06811]

- Super-Grassmannians for $\mathcal{N} = 2$ to 4 $SCFT_3$: From $AdS_4$ Correlators to $\mathcal{N} = 4$ SYM scattering Amplitudes, Aswini Bala, Sachin Jain, **Dhruva K.S.**, Adithya A Rao, [2604.07503]

- The $\mathcal{N} = 1$ Super-Grassmannian for $CFT_3$ and a Foray on AdS and Cosmological Correlators, Aswini Bala, Sachin Jain, **Dhruva K.S.**, Adithya A Rao, [2604.07446]

- ⋆$AdS_4$ Boundary Wightman functions in Twistor Space: Factorization, Conformal blocks and a Double Copy, Arhum Ansari, Sachin Jain, **Dhruva K.S.**, *JHEP 06 (2026) 024*, [2512.04172]

- Lectures on the Spinor and Twistor Formalism in 3D Conformal Field Theory, **Dhruva K.S.**, [2508.21633]

- ⋆An Ode to the Penrose and Witten transforms in Twistor space for 3D CFT, Aswini Bala and **Dhruva K.S.**, *JHEP 11 (2025) 056*, [2505.14082].

- A Supertwistor Formalism for $\mathcal{N} = 1, 2, 3, 4$ $SCFT_3$, Aswini Bala, Sachin Jain, **Dhruva K.S.**, Deep Mazumdar, Vibhor Singh and Brijesh Thakkar, *JHEP 05 (2026) 288*, [2503.19970]

- 3D Conformal Field Theory in Twistor Space, Aswini Bala, Sachin Jain, **Dhruva K.S.**, Deep Mazumdar and Vibhor Singh, *JHEP 12 (2025) 120*, [2502.18562]

- ⋆Hidden sectors of Chern-Simons matter theories and exact holography, Sachin Jain, **Dhruva K.S.**, Evgeny Skvortsov, *Phys. Rev. D* 111 (2025) 106017 [2405.00773]

- ⋆n-point functions in conformal quantum mechanics: a momentum space odyssey, **Dhruva K.S.**, Deep Mazumdar and Shivang Yadav, JHEP 08 (2024) 085 [2409.16947]

- A foray on $SCFT_3$ via super spinor-helicity and Grassmann twistor variables,Sachin Jain, **Dhruva K.S.**, Deep Mazumdar and Shivang Yadav, JHEP 09 (2024) 027 [2312.03059]

- A spin on the bulk locality of Slightly Broken Higher Spin theories, Sachin Jain and **Dhruva K.S.**, JHEP 12 (2024) 048 [2307.04490]

- ⋆Mapping Large N Slightly Broken Higher Spin (SBHS) theory correlators to free theory correlators, Prabhav Jain, Sachin Jain, Bibhut Sahoo, **Dhruva K.S.** and Aashna Zade, JHEP 12 (2023) 173 [2207.05101]

- Constraining momentum space CFT correlators with consistent position space OPE limit and the collider bound, Sachin Jain, Renjan Rajan John, Abhishek Mehta and **Dhruva K.S.**, JHEP 02 (2022) 084 [2111.08024]

The main references for this thesis are the ones marked with ⋆.

# Contents

# Chapter 1

# Introduction

The most successful and precise framework that we have in describing the natural world is quantum field theory (QFT) [1, 2]. It arose as an attempt to combine the special theory of relativity and quantum mechanics. Over the past century, we have understood three of the four fundamental forces in nature: electromagnetism, weak nuclear and strong nuclear, through the lens of QFT. The fourth and most elusive force to accommodate in this framework is perhaps surprisingly gravity. There are many reasons for this apparent incompatibility. The first are divergences that one encounters when understanding interactions involving gravitons which more precisely stated is the non-renormalizability of gravity [3]. Another is that unlike the standard expectation that using higher energies allows us to probe smaller distances is no longer true in gravity, since focusing sufficient energy to probe a small region of space inevitably forms a black hole. However, these facts have not stopped us from making progress to understand how QFT and gravity fit with each other. We are however, far from a complete understanding of how and if even such a unity can be achieved in a consistent manner. A fundamental insight came from ‘t Hooft and Susskind in the form of the holographic principle [4, 5]. Bekenstein had derived the entropy of black holes which turned out to be proportional to the area of the event horizon rather than the volume contained within [6]. This indicated the holographic nature of these objects perhaps hinting that the microscopic degrees of freedom describing a black hole are present on its boundary.

One can now ask a more general question. Given an arbitrary region of space which is not necessarily a black hole, can the information in the volume be described by information just at the boundary of that region? To answer this question, consider the following thought experiment inspired by Susskind [5]. Consider a region of space with some matter contained in a volume $V$. Let us for the moment assume that the entropy is proportional to $V$. Now we imagine squishing together the matter into a sufficiently small size with energy density that in a world with gravity would lead to the formation of a black hole. However, the black hole entropy is proportional to its area $A$ indicating that the original system, which required degrees of freedom in the whole of $V$ is now described by fewer degrees of freedom on $A$. Either information is destroyed by the formation of a black hole or our initial assumption about the entropy being proportional to the volume was wrong. The common opinion substantiated with evidence is that information is fundamental and cannot be destroyed and thus we do not pursue the former point of view. Therefore, we conclude that most of the bulk degrees of freedom were not independent to begin with and rather would also be described by degrees of freedom on one lower dimension: The area, rather than the volume. Extending this argument to the whole universe, the holographic principle broadly states that the entire universe can be described by degrees of freedom on a space with one lesser dimension.

A concrete realization of this idea came from Maldacena who derived an example within the framework of string theory, of a duality between a string theory in a five dimensional spacetime with a negative cosmological constant called Anti de Sitter (AdS) spacetime (also one with five additional compact dimensions) and a non gravitational *conformal* quantum field theory in four dimensional spacetime, the maximally supersymmetric Yang-Mills theory [7]. This goes by the name AdS/CFT, and although it was discovered within string theory has now been extended with many examples that are not necessarily easy to embed within string theory. Developing holography in other spacetimes such as Minkowski space and cosmological spacetimes is a subject of current research. However, coming back to the original question of the unity of QFT and gravity that we started with, holography presents one solution to harmonize them. In the general formulation of the AdS/CFT correspondence, bulk gravitational physics in $d+1$ spacetime dimensions can be described by non gravitational physics in $d$ spacetime dimensions. This lower dimensional theory "lives" on the boundary of the AdS spacetime and is a conformal field theory (CFT). This is intimately tied to the isometry group of AdS spacetime where the asymptotic bulk isometries translate to the symmetries of the boundary theory. We will now review conformal field theory, the AdS spacetime and the AdS/CFT correspondence in more detail, emphasizing their importance in the landscape of physics. The main references we follow are [8] for the general discussion on CFT, [9] for the construction of the AdS spacetime and [10] for AdS/CFT.

## Conformal Field Theory

Conformal field theory is a cornerstone of modern theoretical physics with applications ranging from quantum gravity to critical phenomena in condensed matter systems. Usual relativistic quantum field theories are invariant under the Poincare group which in $d$ spacetime dimensions consists of $d$ translations and $\frac{d(d-1)}{2}$ Lorentz transformations.

$$\begin{aligned} &x^\mu \to x^\mu + a^\mu \quad (\textbf{Translations}), \\ &x^\mu \to \Lambda^\mu_\nu x^\nu \quad (\textbf{Lorentz transformations}). \end{aligned} \tag{1.1}$$

$a^\mu$ is a constant $d-$vector and $\Lambda^\mu_\nu$ are the set of matrices that preserve the flat $d-$dimensional metric $\eta_{\mu\nu}$ [1],

$$\eta_{\mu\nu} = \Lambda^\rho_\mu \eta_{\rho\sigma} \Lambda^\sigma_\nu . \tag{1.2}$$

Conformal field theories are also invariant under scale transformations,

$$x^\mu \to x^\mu + \lambda x^\mu \quad (\textbf{Dilations}), \tag{1.3}$$

where $\lambda$ is a constant parameter. More often than not, this symmetry is enhanced to include what are known as special conformal transformations (SCTs).

$$x^\mu \to x^\mu - 2(b\cdot x)x^\mu + x^2 b^\mu \quad (\textbf{Special conformal transformations}). \tag{1.4}$$

[1]We are signature agnostic at the moment. In this thesis, we will discuss Lorentzian, Euclidean as well as split signature spacetimes.

In two dimensions, with reasonable assumptions such as unitarity, causality, a discrete spectrum etc.., the Zamolodchikov-Polchinski theorem states that scale invariance gets enhanced to conformal invariance [11]. In $d > 2$, it is still an open problem to find the assumptions that uplift scale to conformal invariance. Group theoretically, there is no reason for this to happen since Poincare+Scale forms a group by itself. However, in most theories of interest to us, the common expectation is that this enhancement will occur[2]. It is also useful to note that a special conformal transformation can be thought of as an inversion followed by a translation followed by another inversion. Together, Poincare+Scale+Special conformal form the conformal group. In $d-$dimensional Euclidean space, the conformal group is isomorphic to $SO(d+1,1)$. In $d-$dimensional Lorentzian spacetime, it is $SO(d,2)$. The Lie algebra obeyed by the generators (which is independent of spacetime signature) of this group is,

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho}),\\
&[M_{\mu\nu}, P_\alpha] = i(\eta_{\mu\alpha}P_\nu - \eta_{\nu\alpha}P_\mu), \quad [M_{\mu\nu}, K_\alpha] = i(\eta_{\mu\alpha}K_\nu - \eta_{\nu\alpha}K_\mu),\\
&[P_\mu, K_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu}), \quad [D, P_\mu] = iP_\mu, \quad [D, K_\mu] = -iK_\mu.
\end{aligned} \tag{1.5}$$

$P_\mu$ and $M_{\mu\nu}$ generate translations and Lorentz transformations while $D$ and $K_\mu$ generate scale and conformal transformations. Conformal symmetry naturally occur at the end points of renormalization group flows of quantum field theories. Usually, coupling constants in QFTs acquire a dependence on the scale at which we are probing the theory, due to quantum effects. The beta function of a theory encodes this information. At an energy scale $\mu$ we have for a coupling constant $g_1 \in \{g_i\}$,

$$\mu\frac{dg_1}{d\mu} = \beta(\{g_i\}). \tag{1.6}$$

For special values of the coupling constants where the beta function vanishes we have scale independence at these fixed points. This scale invariance, as we argued usually gets enhanced to conformal invariance. Fixed points of a QFT are thus generically CFTs. A QFT can be viewed as a renormalization group (RG) flow between conformal fixed points, one in the IR and the other in the UV. Starting from a UV CFT, one can perturb the theory by deforming it with a relevant operator to induce RG flow and potentially flow to another CFT at the IR. Thus, if one can classify all CFTs and the relevant operators that trigger RG flow, one can classify all quantum field theories thus asserting their central importance.

When one first learns quantum field theory, the observables we study are scattering amplitudes of particles created by quantum fields. The notion of scattering amplitudes depends on the existence of asymptotic free states. In CFTs however[3], correlations are long ranged due to the scale invariance and thus the notation of asymptotic free states is not well defined. The observables of interest to us in CFTs are rather correlation functions of local operators[4]. A powerful and successful framework to describe and chart out the space of CFTs goes by the name of the conformal bootstrap. The core

[2]There are interesting counterexamples to this fact, mostly in non unitary theories which are still of physical interest, see the wonderful review by Nakayama [11].

[3]And more generally in theories with massless particles that mediate long distance forces.

[4]In more recent years, there has been emphasis on the importance of correlators involving non-local

idea is to utilize symmetries and consistency conditions to determine as much as we can about the theory without resorting to a particular microscopic realization. Correlation functions are bootstrapped rather than computed using a Lagrangian like in perturbative QFT. In two dimensions, this program was initiated in the seminal paper [13]. In dimensions greater than two, there has also been progress since the initiating work [14], of which much is summarized in [15, 16].

The local operators of interest to us are called conformal primaries. To define them, we need to discuss the representation theory of the conformal algebra (1.5). As we can see from the commutation relations, the Cartan sub-algebra (maximally commuting sub-algebra) is spanned by $D$ and $M_{\mu\nu}$. We can then simultaneously diagonalize them and label operators by their eigenvalues with respect to these operators.

$$[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0), \quad [M_{\mu\nu}, \mathcal{O}(0)] = \mathcal{M}_{\mu\nu}\mathcal{O}(0). \tag{1.7}$$

The first of these equations coincides with the notation of the scaling dimension of an operator at a classical level and the second accomodates the fact that the operator can have a non-zero intrinsic spin. Further, note that the special conformal generator acts like a lowering operator for $D$ as can be seen from the algebra (1.5). It is thus natural to label an irreducible representation called a *primary* operator as one that is annihilated by $K_\mu$ at the origin.

$$[K_\mu, \mathcal{O}(0)] = 0 \implies \mathcal{O} \text{ is primary.} \tag{1.8}$$

It is then a simple exercise to see how operators transform under conformal transformations.

$$\begin{aligned}
&[D, \mathcal{O}(x)] = -i\big(x^\mu\partial_\mu + \Delta\big)\mathcal{O}(x), \\
&[M_{\mu\nu}, \mathcal{O}(x)] = -i\big(x_\mu\partial_\nu - x_\nu\partial_\mu + i\mathcal{M}_{\mu\nu}\big)\mathcal{O}(x) \\
&[P_\mu, \mathcal{O}(x)] = -i\partial_\mu\mathcal{O}(x), \\
&[K_\mu, \mathcal{O}(x)] = -i\big( - 2x^\mu x^\nu\partial_\nu + x^2\partial_\mu - 2\Delta x_\mu - 2ix^\nu\mathcal{M}_{\mu\nu}\big)\mathcal{O}(x).
\end{aligned} \tag{1.9}$$

We have suppressed possible Lorentz vector or spinor indices for the operator $\mathcal{O}$. Given a primary operator, one can form what are known as its descendants by acting with $P_\mu$ which acts as a raising operator for the dilatation operator as can be seen from the algebra (1.5). A general descendant takes the form,

$$P^{\nu_1}\cdots P^{\nu_k}(P_\mu P^\mu)^l\mathcal{O}(x), \; k, l \in \mathbb{Z}_{\geq 0}, \tag{1.10}$$

and is not annihilated by $K_\mu$ at the origin unlike the primary (1.8). Therefore, knowledge of the primary operator entails knowledge of the descendants and thus the former will be our primary focus. Conformal correlation functions obey Ward identities. Given an infinitesimal symmetry generator $G$, the Ward identities takes the form,

$$\sum_{i=1}^{n}\langle\cdots[G, \mathcal{O}_i(x_i)]\cdots\rangle = 0, \tag{1.11}$$

operators. One such example is the average null energy operator that characterizes energy flux at null infinity. Its correlators are of interest both theoretically as well as experimentally at particle accelerators, see [12] for a review of their construction.

where $G \in \{P_\mu, M_{\mu\nu}, K_\mu, D\}$ and $i$ labels a particular primary operator.

Conformal symmetry fixes one point functions to zero and is powerful enough to fix two and three point correlators up to constants[5]. Further, two point functions are non-zero only if both operators are identical[6]. For example, let us consider identical scalar operators with scaling dimension $\Delta$. We have[7],

$$\begin{aligned} \langle O_\Delta(x) \rangle &= 0, \\ \langle O_\Delta(x_1) O_\Delta(x_2) \rangle &= \frac{1}{x_{12}^{2\Delta}}, \\ \langle O_\Delta(x_1) O_\Delta(x_2) O_\Delta(x_3) \rangle &= \frac{f_{\Delta\Delta\Delta}}{x_{12}^{\Delta} x_{23}^{\Delta} x_{31}^{\Delta}}, \end{aligned} \tag{1.12}$$

where $x_{ij}$ is the magnitude of the separation between points $x_i^\mu$ and $x_j^\mu$. Higher point correlators can in principle be constructed using the operator product expansion (OPE) which is a statement about bringing together and fusing two operators into a sum of operators.

$$O_i \times O_j \sim \sum_k f_{ijk} O_k. \tag{1.13}$$

The $f_{ijk}$ are called OPE coefficients and characterize the CFT. Together with the spectrum of operators, they provide a non-perturbative definition of these theories. Iteratively using the OPE reduces higher point correlators to a sum of lower point ones. We obtain a powerful constraint that arises from performing the OPE between different pairs of operators. Since these are two distinct ways to express the same four point function, their equality thus constrains its form.

$$\langle \overbrace{O_1\ O_2}\ \overbrace{O_3\ O_4} \rangle = \langle \overbrace{O_1\ \overbrace{O_2\ O_3}\ O_4} \rangle \tag{1.14}$$

This is known as crossing symmetry and is a powerful constraint on CFTs. For scalar operators, the general four point correlator is determined up to a function of conformal cross ratios of which there are two in generic dimensions[8].

$$\langle O_\Delta(x_1) O_\Delta(x_2) O_\Delta(x_3) O_\Delta(x_4) \rangle = \frac{1}{x_{12}^{2\Delta} x_{34}^{2\Delta}} f(u, v). \tag{1.15}$$

This function is constrained by crossing symmetry,

$$f(u, v) = \left(\frac{u}{v}\right)^\Delta f(v, u), \tag{1.16}$$

[5]All correlations we consider are taken in the vacuum state which we assume is unique. It is a state that enjoys maximum symmetry and is annihilated by all the conformal generators.

[6]This statement is true when the operators are inserted at different points. Non-identical operators can have a non-zero two point function that is a contact term with a delta function $\delta^d(x_1 - x_2)$, see [17].

[7]These expressions are agnostic to what kind of correlator we are considering. It could be the Euclidean correlator which is single-valued or any Lorentzian correlator such as time-ordered or Wightman which we need to specify using an appropriate $i\epsilon$ prescription.

[8]In one dimension, there is only one independent cross ratio. The number is 2 in any other dimension.

with $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$. Further, four point functions admit an expansion in what are known as conformal blocks. A conformal block encodes the contribution to a correlator due to a particular operator. They are fixed by conformal symmetry and for external scalar operators, are known in closed form in $d = 2, 4$ for instance in terms of hypergeometric functions. The way to obtain them is as follows. Conformal blocks are eigenvalues of the quadratic Casimir of the conformal group $C_2$. The Casimir commutes with all the generators and thus depends only on the labels of the representation it acts on. To see how conformal blocks arise, let us insert in a four point function, a complete set of states (schematically),

$$G_{\Delta,s} \sim \langle O_1 O_2 | O_{\Delta,s} \rangle \langle O_{\Delta,s} | O_3 O_4 \rangle \tag{1.17}$$

The Casimir $C_2$ applied on the operators 1 and 2 translate to the casimir acting on the intermediate operator $O_{\Delta,s}$ thus resulting in the eigenvalue equation,

$$\left(C_2^{(1)} + C_2^{(2)}\right) G_{\Delta,s} = \lambda(\Delta, s) G_{\Delta,s}, \tag{1.18}$$

where $\lambda(\Delta, s)$ is the eigenvalue of the Casimir acting on the state created by $O_{\Delta,s}$. Supplementing the Casimir equation with boundary conditions allows us to solve for the conformal blocks. Thus, we can write the identical $\Delta$-scalar four point function as,

$$\frac{f(u,v)}{x_{12}^2 x_{34}^2} = \frac{1}{x_{12}^2 x_{34}^2} \sum_i f_{\Delta\Delta i}^2 G_{\Delta_i, s_i}(u,v), \tag{1.19}$$

where $f_{\Delta\Delta i}$ are OPE coefficients and $G_{\Delta_i,s_i}(u,v)$ is the conformal block associated to the contribution of the operator indexed by $i$ with scaling dimension $\Delta_i$ and spin $s_i$. Thus, crossing becomes the statement that,

$$\sum_i f_{\Delta\Delta i}^2 \left(v^\Delta G_{\Delta_i,s_i}(u,v) - u^\Delta G_{\Delta_i,s_i}(v,u)\right) = 0. \tag{1.20}$$

This is the central equation of the modern bootstrap program.

Correlators of scalar operators are important but so are correlators involving spinning operators such as the energy momentum stress tensor. For spinning correlators, there are potentially many possible independent tensor structures at four points that each come with a different function of conformal cross ratios. Crossing symmetry thus becomes a complicated constraint relating these functions. There are also various degeneracy identities that relate different tensor structures which further complicates the analysis. Further, the expressions for generic spinning conformal blocks in generic dimensions is also not known in closed form.

Thus, an important task is to find an organizing principle for spinning correlators which will form much of the subject of this thesis. However before we turn to that subject, we now return to our original motivation for discussing CFTs: The AdS/CFT correspondence and understanding quantum gravity using holography. First, we review the construction of AdS spacetime and discuss the natural physical observables in this setting. We will then connect back to our discussion on CFT and present the AdS/CFT correspondence.

## AdS spacetime

Anti-de Sitter spacetime is a maximally symmetric solution of the vacuum Einstein equations with a negative cosmological constant. To define $\text{AdS}_{d+1}$, it is convenient to embed it in a flat $d+2$ dimensional spacetime with the metric,

$$dS^2 = -(dX^0)^2 + \sum_{i=1}^{d}(dX^i)^2 - (dX^{d+1})^2. \tag{1.21}$$

We define the hyper-surface,

$$-(X^0)^2 + \sum_{i=1}^{d}(X^i)^2 - (X^{d+1})^2 = -R^2. \tag{1.22}$$

This surface is clearly preserved by $SO(d,2)$ transformations and has the topology $S^1 \times \mathbb{R}^d$ that we can see by taking the time-like coordinates to the right-hand-side of the above equation. We can use the following coordinates to chart this surface:

$$X^0 = \cosh(\rho)\sin(\tau), X^{d+1} = \cosh(\rho)\cos(\tau), X^i = n^i \sinh(\rho), i = 1, \cdots, d, \tag{1.23}$$

where $n^i, i = 1, \cdots, d$ is a unit vector on $S^d$. The induced metric on the surface then reads,

$$ds^2 = -\cosh^2(\rho)d\tau^2 + d\rho^2 + \sinh^2(\rho)d\Omega^2_{d-1}. \tag{1.24}$$

$d\Omega^2_{d-1}$ is the metric on a unit $d-1$ unit sphere. $\rho$ is a non-compact radial direction $0 \leq \rho < \infty$. $\tau$ is periodic and belongs to $[0, 2\pi)$. What we call AdS spacetime is actually the covering space of this spacetime where we take $\tau \in (-\infty, \infty)$ to avoid closed time-like curves. To understand the conformal structure of AdS, let us construct its Penrose diagram. First we define,

$$\cosh(\rho) = \frac{1}{\cos(\theta)} \implies \rho \in [0, \infty) \implies \theta \in [0, \frac{\pi}{2}]. \tag{1.25}$$

In these coordinates, the metric takes the form,

$$ds^2 = \frac{1}{\cos^2(\theta)}(-d\tau^2 + d\theta^2 + \sin^2(\theta)d\Omega^2_{d-1}) = \frac{1}{\cos(\theta)^2}(-d\tau^2 + d\Omega^2_d). \tag{1.26}$$

$\text{AdS}_{d+1}$ is therefore conformally equivalent to $\mathbb{R} \times S^d$ but with the restriction that $\theta$ goes only up to $\frac{\pi}{2}$ (rather than $\pi$ which corresponds to the Einstein static universe) which is the conformal boundary. This boundary is a time-like hypersurface with topology $\mathbb{R} \times S^{d-1}$. Suppressing the $d-1$sphere directions, $\text{AdS}_{d+1}$ is conformally equivalent to $-d\tau^2 + d\theta^2$.

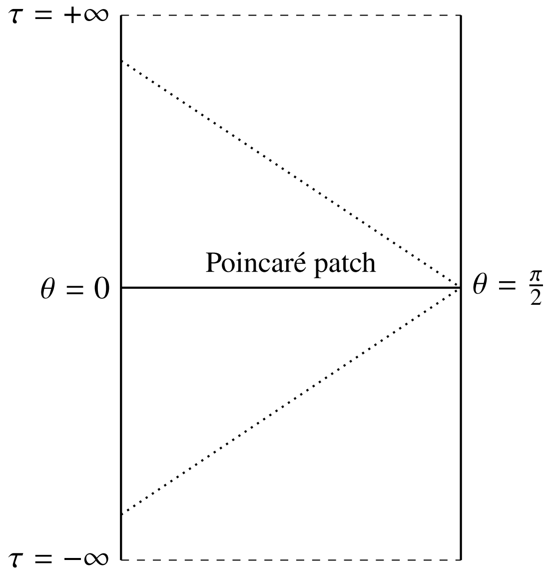


Figure 1.1: Penrose diagram of $\text{AdS}_{d+1}$ suppressing a $(d-1)$-sphere at each point. The triangular region is the Poincaré patch.

$\theta = 0$ represents the centre of AdS whereas $\theta = \frac{\pi}{2}$ is the conformal boundary. The important point to note is that at the induced metric at the $\theta = \frac{\pi}{2}$ surface is time-like. Further, one can show that lightrays reach the conformal boundary in finite coordinate time. In the AdS/CFT correspondence, the hologram lives at this conformal boundary and creates the $\text{AdS}_{d+1}$ spacetime. Before we proceed to holography, let us discuss one more coordinate system for $\text{AdS}_{d+1}$. These are called Poincare coordinates and cover part of the spacetime. These are not as obvious to define as their global counterparts from the embedding equation. We define,

$$\begin{aligned} X^{d+1} &= \frac{1}{2z}(L^2 + z^2 + \eta_{\mu\nu}x^\mu x^\nu), \\ X^d &= \frac{1}{2z}(L^2 - z^2 - \eta_{\mu\nu}x^\mu x^\nu), \\ X^\mu &= \frac{Lx^\mu}{z}, \mu \in \{0, \cdots, d-1\}. \end{aligned} \tag{1.27}$$

$\eta_{\mu\nu}$ is the flat $d$ dimensional Minkowski metric. The induced metric on the hyperrsurface is,

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu). \tag{1.28}$$

The conformal boundary is located at $z = 0$ and has the topology of $d-$dimensional Minkowski spacetime which is $\mathbb{R} \times \mathbb{R}^{d-1}$. Compared to global coordinates where we had $\mathbb{R} \times S^{d-1}$, Poincare coordinates do not cover the entire spacetime. To understand the region that they do cover, consider suppressing the $d - 1$ sphere. We have the global metric,

$$ds^2 = \frac{1}{\cos^2(\theta)}(-d\tau^2 + d\theta^2). \tag{1.29}$$

To go to Poincare coordinates define,

$$z = \frac{\cos(\theta)}{\cos(\tau) + \sin(\theta)}, t = \frac{\sin(\tau)}{\cos(\tau) + \sin(\theta)}. \tag{1.30}$$

This transformation is well defined for $\cos(\tau)+\sin(\theta) > 0$ which implies that $|\tau| < \frac{\pi}{2} - \theta$. This leads to the Poincare coordinates covering part of the AdS spacetime. In this thesis, when we work in AdS spacetime, we will mostly specialize to the Poincare patch since the conformal boundary is flat Minkowski space rather than a sphere time time and we can work with the familiar flat space CFT correlators rather than there sphere counterparts. With this discussion, let us proceed to define the AdS/CFT correspondence.

## AdS/CFT

Having discussed both AdS and CFT, let us motivate the AdS/CFT correspondence. First, if a theory in AdS spacetime is to be equivalent to a CFT in one lower dimension, the symmetries have to match. For a theory in asymptotic $\text{AdS}_{d+1}$ spacetime, the asymptotic symmetry group is $SO(d, 2)$ which precisely matches with the conformal group in $d$ dimensions. Thus, we have a match at the level of symmetries. Next, if we consider a gauge theory in $\text{AdS}_{d+1}$ such as Maxwell theory or Yang-Mills theory, their asymptotic symmetry group is $U(1)$ or $SU(N)$ and thus the CFT in question must have a global $U(1)$ or $SU(N)$ symmetry. Where does the CFT "live"? As we have seen, $\text{AdS}_{d+1}$ has a time-like boundary at $\theta = \frac{\pi}{2}$ with topology $\mathbb{R} \times S^{d-1}$ in global coordinates and at $z = 0$ with topology $\mathbb{R}^{1,d-1}$ in Poincare coordinates. This boundary shares the notion of time with the bulk and thus it makes sense to think of an ordinary non-gravitational quantum field theory "living" on this boundary generating the hologram that creates the additional bulk direction. What about observables? Well, in CFT, these are correlators of local operators. In the bulk AdS, we need to construct asymptotic observables that match with CFT correlators. Given the symmetry structure we can consider correlators of bulk quantum fields and extrapolate them to the boundary appropriately. Let $\Phi(z, x^\mu)$ be some bulk scalar field in the Poincare patch of $\text{AdS}_{d+1}$. We can construct bulk correlators[9] and appropriately take the limit to the boundary at $z = 0$ as follows:

$$\langle O_\Delta(x_1^\mu) \cdots O_\Delta(x_n^\mu)\rangle \sim \lim_{z\to 0} \prod_{i=1}^{n} z_i^{-\Delta} \langle \Phi(z, x_1^\mu) \cdots \Phi(z, x_n^\mu)\rangle, \tag{1.31}$$

where the mass of the bulk field is related to $\Delta$ (which is identified with the scaling dimension of the dual operator) via,

$$m^2 = \Delta(\Delta - d). \tag{1.32}$$

This procedure of extrapolating bulk correlators to the boundary goes by the name of the AdS/CFT extrapolate dictionary.

[9]We quantize the bulk theory and perform perturbation theory to compute bulk correlators in the vacuum state. Since the notion of time and thus positive and negative frequencies is the same in bulk and boundary, this process is unambiguous.

There is another version of the AdS/CFT correspondence which goes by the name of the GKPW dictionary. It states that the bulk path integral evaluated with boundary condition $\Phi(z, x^{\mu}) \sim \varphi(x^{\mu})$ at $z = 0$ is equivalent to the boundary generating functional for the operator dual to the bulk field viz $O_{\Delta}(x^{\mu})$.

$$Z_{\text{Bulk}}[\Phi(0, x^{\mu}) = \varphi(x^{\mu})] = Z_{\text{CFT}}[\varphi(x^{\mu})]. \tag{1.33}$$

The boundary value of the bulk field thus plays the role of the source of its dual operator. It was shown by Harlow and Stanford that these two seemingly different dictionaries are equivalent [18].

What about the bulk graviton? Well, if we consider linearized gravity about $\text{AdS}_{d+1}$ spacetime, we have the fluctuation field $h_{AB}$ which is symmetric in its indices $A, B \in \{z, \mu\}$. Further, we can work in a gauge such that $h_{zz} = h_{\mu z} = 0$ as well as use residual gauge transformations to set $h^{\mu}_{\mu} = \partial_{\mu} h^{\mu\nu} = 0$. This means the physical graviton degrees of freedom reside in a tensor $h_{\mu\nu}$ which satisfies,

$$h^{\mu}_{\mu} = 0, \partial_{\mu} h^{\mu\nu} = 0. \tag{1.34}$$

We do not have to look fat to find the dual CFT operator. The stress tensor $T_{\mu\nu}$ of a CFT is symmetric, tracelss and conserved. It is the operator dual to the bulk graviton. By similar arguments, one can show that a bulk gauge field $A_{\mu}$ is dual to a conserved current $J_{\mu}$. Using these rules, we can translate $\text{AdS}_{d+1}$ correlators into $\text{CFT}_d$ ones.

It has almost been three decades since the advent of the AdS/CFT correspondence. It has had many successes which are summarized in many review articles. One that we will highlight now is the black hole information paradox. Is information destroyed when a black hole is formed? Given the fact that the no hair theorem states that the black hole is characterized just by its mass and potentially spin and charge indicates that the information that went into forming the black hole has disappeared. However, through AdS/CFT, the black hole and what formed it are described by purely non gravitational physics of the CFT which involves no information loss thus indicating that the bulk theory should also preserve information. This does not tell us exactly how information is conserved but it at least tells us that it is.

## The Problem: Spinning correlators in AdS/CFT

Let us return to the problem of describing spinning conformal correlators. As we discussed, through holography, conserved currents are dual to massless gauge bosons such as photons, gluons and gravitons in a one higher dimensional AdS spacetime and thus are important to describe the bulk physics. These conserved current correlators are universal observables which are of interest to constrain and bootstrap. One such example are the stress tensor correlators. In even dimensions, these quantities encode trace anomalies which are RG invariants and thus constrain the space of CFTs that are connected via RG flow. Further, in any dimension there are positivity constraints on correlators involving the average null energy operator (ANEC) which is constructed out of the stress tensor. This puts bounds on the CFT data. Also, from the bootstrap point of view, it is not sufficient to obtain constraints just from applying it to four point scalar correlators. One requires the information about all other four point correlators including those involving spinning operators[10]. Since the usual position space formalism is complicated for spinning operators, a natural direction is to search for alternate descriptions that simplify the analysis. If we study theories with supersymmetry, we might be able to get away with considering scalar operators since supersymmetry relates scalar correlators to those with spin. In the more general case without supersymmetry, we must look for alternative ways to analyze spinning correlators.

Essentially the question can be restated as follows in the holographic lens. In flat space, the asymptotic observables of most interest to us are scattering amplitudes. In AdS, these are boundary conformal correlators. How much of the formalism and success of the flat space scattering amplitudes program can be extended to AdS? What lessons can we learn from the amplitudes program and import it to AdS/CFT?

In this thesis, we will mostly focus on the physically relevant case of three dimensional CFTs/ four dimensional theories in AdS and see what the study of four dimensional scattering amplitudes can teach us about AdS/CFT correlators.

## Insights from Scattering amplitudes: Spinor Helicity

Scattering amplitudes in quantum field theory were traditionally analyzed in momentum space by summing Feynman diagrams. However, even for gluon tree level amplitudes in Yang-Mills theory, the number of Feynman diagrams to compute increases factorially. At 4 points we have 4 diagrams, at 5 we have 25 and at 6 we have 220 and at 7 we have 2485 diagrams to sum up! This seemingly indicates an enormous amount of complexity in these quantities! However, Parke and Taylor, using spinor helicity variables showed that this complexity is an illusion caused by a wrong choice of variables that obscure the physical degrees of freedom [19]. In particular, using spinor helicity variables that use a pair of spinors to encode both the momentum and helicity of the gluons, they found an extremely simple expression (up to six points but conjectured for n-points which was later proven) for the n-point MHV amplitude,

$$A_n(1^-2^-3^+\cdots n^+) = g^{n-2}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1n\rangle\langle n1\rangle}. \tag{1.35}$$

The maximum helicity violating (MHV) amplitude represents a tree level interaction between two gluons with negative helicity and $n-2$ gluons with positive helicity. This simplicity indicates that there must exist a better way to understand and compute scattering amplitudes. Let us review the spinor helicity formalism and briefly discuss its advantages.

Consider a null momentum $p_\mu = (p_t, p_x, p_y, p_z)$ a massless particle in Minkowski space $\mathbb{R}^{3,1}$. Recall that the four dimensional Lorentz group $SO(3,1)$ is isomorphic to $SL(2,\mathbb{C})/\mathbb{Z}_2 \cong \frac{SU(2)\times SU(2)^*}{\mathbb{Z}_2}$. A vector representation of $SO(3,1)$ corresponds to the $(\frac{1}{2},\frac{1}{2})$ representation of $\frac{SU(2)\times SU(2)^*}{\mathbb{Z}_2}$. A null vector in particular can be written as a direct product of a $(\frac{1}{2},0)$ representation $\lambda_a$ and $(0,\frac{1}{2})$ representation $\tilde{\lambda}^{\dot{a}}$.

$$p^\mu \to p_a^{\dot{a}} = \begin{pmatrix} p_t + p_z & p_x - ip_y \\ p_x + ip_y & p_t - p_z \end{pmatrix} = \lambda_a \tilde{\lambda}^{\dot{a}}, \tag{1.36}$$

[10]An alternate might be to bootstrap $n-$point scalar correlators which contains spinning operators as intermediate states contributing to the correlator.

where $-p_t^2 + p_x^2 + p_y^2 + p_z^2 = 0$. We see that the reality condition $\lambda_a^\dagger = \tilde{\lambda}^{\dot{a}}$ ensures that the four components of the momentum vector are real. Note that this description has a redundancy,

$$\lambda \to e^{i\theta}\lambda, \tilde{\lambda} \to e^{-i\theta}\tilde{\lambda}. \tag{1.37}$$

Thus we assign helicity $-\frac{1}{2}$ to $\lambda$ and $+\frac{1}{2}$ to $\tilde{\lambda}$. Using these very same spinors, we can construct polarization vectors. For example, consider gluons which for a fixed colour, have only two degrees of freedom viz positive and negative helicity. We can construct,

$$\epsilon_+^{a\dot{a}} = \frac{\rho^a \tilde{\lambda}^{\dot{a}}}{\langle \lambda \rho \rangle}, \epsilon_-^{a\dot{a}} = \frac{\lambda^a \tilde{\rho}^{\dot{a}}}{\langle \tilde{\lambda} \tilde{\rho} \rangle}. \tag{1.38}$$

Under the little group scaling, we have,

$$\epsilon_\pm^{a\dot{a}}(e^{i\theta}\lambda, e^{-i\theta}\tilde{\lambda}) = e^{-\pm 2i\theta}\epsilon_\pm^{a\dot{a}}(\lambda, \tilde{\lambda}). \tag{1.39}$$

Thus $\epsilon_\pm$ encode the positive and negative helicity degrees of freedom of the spin-1 particle. Similarly, for gravitons which also have only two physical degrees of freedom in four dimensions, we can construct analogous polarization tensors. Thus, we can take a given momentum space scattering amplitude with arbitrary polarization vectors, choose particular helicities for the particles by picking $\epsilon_\pm$ and obtain the expression in spinor helicity variables. Now that we have obtained a description that has the physical degrees of freedom transparent, we can now ask if there are ways to directly compute scattering amplitudes in these variables. The answer is yes and there are recursion relations such as BCFW, CSW, BG and many more which have led to a lot of success such as proving the Parke Taylor formula, uncovering double copy relations and for efficiently computing scattering amplitudes at both tree level and loops.

## Insights from Scattering amplitudes: Twistors

Returning to the subject of gluon scattering amplitudes such as the Parke Taylor formula, we note that they are not only Poincare invariant but also conformally invariant due to the conformal symmetry of the Yang-Mills action[11]. Special conformal invariance is implemented on scattering amplitudes via the following second order differential operator:

$$K_{a\dot{a}} = \frac{\partial^2}{\partial \lambda^a \partial \tilde{\lambda}^{\dot{a}}}. \tag{1.40}$$

Given the fact that the momentum operator is multiplicative whereas SCTs are second order, it would be desirable to choose variables where they are both on equal footing to make manifest conformal symmetry. The problem in Minkowski space is that $\lambda$ and $\tilde{\lambda}$ are Hermitian conjugates of each other. Witten's insight [20] was that rather than working in Minkowski space, we can rather work in split signature a.k.a Klein space which we now turn to.

[11] Of course, loop effects yield a non-zero beta function but the tree level amplitudes are conformal.

Our starting point is Klein space $\mathbb{R}^{2,2}$, the space-time with two spatial and two temporal directions. Consider a null momentum $p_\mu = (p_t, p_w, p_x, p_z)$. The four dimensional Klein group $SO(2,2)$ is homomorphic to $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. A vector representation of $SO(2,2)$ corresponds to the $(\frac{1}{2},\frac{1}{2})$ representation of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. A null vector in particular can be written as a direct product of a $(\frac{1}{2},0)$ representation $\lambda_a$ and $(0,\frac{1}{2})$ representation $\tilde{\lambda}^{\dot{a}}$.

$$p^\mu \to p_a^{\dot{a}} = \begin{pmatrix} p_w + p_z & p_x - p_t \\ p_x + p_t & p_w - p_z \end{pmatrix} = \lambda_a \tilde{\lambda}^{\dot{a}}, \tag{1.41}$$

where $-p_t^2 - p_w^2 + p_x^2 + p_z^2 = 0$. Show that the reality condition $\lambda_a^* = \lambda_a, \tilde{\lambda}^{\dot{a}*} = \tilde{\lambda}^{\dot{a}}$ ensures that the four components of the momentum vector are real. There is a redundancy in the description (1.41) viz

$$\lambda_a \to \frac{1}{r}\lambda_a, \tilde{\lambda}^{\dot{a}} \to r\tilde{\lambda}^{\dot{a}}, r \in \mathbb{R} \implies p_a^{\dot{a}} \to p_a^{\dot{a}}. \tag{1.42}$$

This is the manifestation of the fact that the compact (reduced) little group $SO(2) \cong U(1)$ of massless particles in $\mathbb{R}^{3,1}$ corresponds to the non compact little group $\mathbb{R} - \{0\}$ in Klein space. In this signature, the fact that the spinors are real implies that we can Fourier transform $\tilde{\lambda}$ keeping $\lambda$ fixed to make the SCT operator first order. This yields,

$$P_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} \to -i\lambda_a \frac{\partial}{\partial \mu^{\dot{a}}}, K_{a\dot{a}} = -i\mu_{\dot{a}} \frac{\partial}{\partial \lambda^a}. \tag{1.43}$$

In fact, what happens is that all 15 conformal generators of the Klein conformal group $SO(3,3) \cong \frac{SL(4,\mathbb{R})}{\mathbb{Z}_2}$ combine together to form,

$$G_B^A = Z^A \frac{\partial}{\partial Z^B} - \frac{\delta_B^A}{4} Z^C \frac{\partial}{\partial Z^C}, \tag{1.44}$$

where we have defined the *Twistor*,

$$Z^A = (\lambda^a, \mu_{\dot{a}}). \tag{1.45}$$

The half-Fourier transform of Witten, linearizes the action of the conformal group thereby making manifest the conformal symmetry. This half-Fourier transform to the $Z^A$ variables actually takes us to Penrose's twistor space that one can see by combining the Half-Fourier and usual Fourier transforms. In these variables, gluon amplitudes become even more simple. For example, if we take the three point gluon amplitude with particles 1 and 2 having negative helicity and particle 3 having positive helicity the answer we obtain in twistor space is simply $\pm 1$! Therefore, twistors make manifest both the physical degrees of freedom (helicities) as well as the conformal symmetry of gluon amplitudes. In general, twistors are the best choice of variables for describing massless particles, as was shown by Penrose over five decades ago [21]. After the seminal work of Witten, the works [22, 23] really showed the utility of the twistor space description in unveiling the simplicity of scattering amplitudes of massless gauge bosons. The BCFW recursion relation for example, takes its most simple form in twistor space. Since then there have been many works using twistors as well as what are known as momentum twistors that have led a more geometric and structural understanding of scattering amplitudes.

Let us now return to our original question. Given the utility of spinor helicity and twistors for flat space scattering amplitudes, how much of it carries over to (A)dS spacetime and more generally, to conformal field theory correlators? That will form the main subject of this thesis.

## The Outline of the thesis

In chapter 2, we discuss the construction of spinor helicity variables and the general structure of current correlators in three dimensional conformal field theories. We detail the distinction between the Euclidean and Lorentzian settings and also discuss the analytic continuation that relates them. We also discuss higher point correlators and conformal partial waves in three dimensions. Finally, we conclude with the illustrative example of conformal quantum mechanics where we explicitly derive the general form of $n-$point momentum space correlators.

In chapter 3, we apply the momentum space and spinor helicity formalism to Chern-Simons matter theories. We show that this approach allows us to solve the higher spin equations that four point functions obey in an algebraic manner. Further, we show that the spinor variables make manifest the anyonic nature of conserved current correlators.

In chapter 4, we discuss the (anti-)chiral limit for Chern-Simons matter theories that lands us on a sub-sector that holographically describes chiral higher spin theory in four dimensional Anti-de Sitter spacetime. We show this explicitly at the level of three point functions and also make predictions at four points and discuss expectations from the bulk point of view.

In chapter 5, we switch gears from spinor helicity to twistor variables. We discuss the geometry of twistor space and derive the Penrose and Witten transforms that form a cornerstone of the subject. We also extend the twistor formalism to accommodate arbitrary primary operators. We discuss the construction of correlators involving conserved currents, scalars as well as generic operators. Further, we derive a supersymmetric version of the Penrose transform and discuss the construction of correlators of super-currents and super-scalars including contact term contributions.

In chapter 6, we take the bulk Anti-de Sitter point of view and construct boundary Wightman correlators of scalars, photons, gluons and gravitons. We uncover novel factorization kinematics where momentum space four point Wightman functions coincide with Wightman conformal partial waves. We also show that these kinematics contain sufficient information to reconstruct the entire correlator up to contact diagram contributions. Finally, we recast these results in twistor space and find a simple squaring double copy relation between the Yang-Mills and Einstein gravity four point functions.

We summarize our results and discuss interesting future directions in chapter 7.

We have several appendices to complement the material in the main-text. In appendix A, we discuss a dimensional reduction from on-shell four dimensional spinor helicity and twistors to their three dimensional off-shell counterparts. The subject of appendix B are useful projective integral identities. In appendices C and D, we discuss more about general and holographic Wightman functions respectively. We discuss the case of $\mathcal{N} = 1, 2$ super-conformal quantum mechanics in appendix E. Finally, in appendix F, we discuss the various special functions that we use throughout the text.

# Chapter 2

# Momentum space and Spinor Helicity Methods

The aim of this chapter is to study generalities about three dimensional conformal field theories in momentum space and spinor helicity variables. Over the past couple of decades there has been a lot of progress in this subject [24–58] and has a variety of applications such as cosmology [24,25,28,59–61], the study of AdS amplitudes [62–74], double copy relations [32,41,75] and connections to one higher dimensional flat space S-matrices [24,54,76].

We begin our discussion in Euclidean signature, develop off-shell spinor helicity variables, and discuss the general form of correlation functions. We then switch gears to Lorentzian CFTs where we will discuss the spinor helicity construction and focus on Wightman functions as the fundamental observables. We proceed with a discussion of conformal partial waves that are building blocks of higher point correlators. Finally, we take a case study of one dimensional conformal field theories which illustrates many of the subtleties and features of the momentum space formalism. We derive the form of $n-$point correlators which take the form of generalized hypergeometric (Lauricella) functions, discuss the reasons for multiple solutions to the conformal Ward identities and also match our results against free and interacting theory correlators.

The references for section 2.1 on Euclidean $\text{CFT}_3$ are,

- Constraining momentum space CFT correlators with consistent position space OPE limit and the collider bound, Sachin Jain, Renjan Rajan John, Abhishek Mehta and **Dhruva K.S.**, JHEP 02 (2022) 084 [2111.08024]

- ⋆Hidden sectors of Chern-Simons matter theories and exact holography, Sachin Jain, **Dhruva K.S.**, Evgeny Skvortsov, *Phys. Rev. D* 111 (2025) 106017 [2405.00773]

The main reference for section 2.2 which deals with Lorentzian $\text{CFT}_3$ is,

- 3D Conformal Field Theory in Twistor Space, Aswini Bala, Sachin Jain, **Dhruva K.S.**, Deep Mazumdar and Vibhor Singh, *JHEP 12 (2025) 120*, [2502.18562]

The reference for conformal quantum mechanics which is the subject of section 2.4 is,

- ⋆n-point functions in conformal quantum mechanics: a momentum space odyssey, **Dhruva K.S.**, Deep Mazumdar and Shivang Yadav, JHEP 08 (2024) 085 [2409.16947]

## 2.1 Euclidean Signature

We begin our study in three dimensional flat Euclidean space. We construct Euclidean $SU(2)$ spinor helicity variables, recast conserved currents in the helicity basis and discuss the general structure of their two and three point correlators.

### 2.1.1 Spinor helicity

Consider a general Euclidean three-momentum $p_\mu = (p_x, p_y, p_z)$. Since the rotation group in 3d is $SO(3)$, we can make use of its homomorphism with $SU(2)$ to represent it as a $2 \times 2$ matrix as follows:

$$p_\mu \to p_a^b = (\sigma^\mu)_a^b p_\mu = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}. \tag{2.1}$$

$(\sigma^\mu)_b^a$ are the usual Pauli matrices whose explicit forms are,

$$(\sigma^x)_b^a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, (\sigma^y)_b^a = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, (\sigma^z)_b^a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.2}$$

They obey the Euclidean Clifford algebra,

$$\{\sigma_\mu, \sigma_\nu\} = \delta_{\mu\nu}. \tag{2.3}$$

The fundamental equation of 3d spinor helicity variables is (2.1) expressed as a bi-spinor by employing a pair of $SU(2)$ fundamental spinors $\lambda$ and $\bar{\lambda}$ as follows:

$$p_a^b = \frac{(\lambda_a \bar{\lambda}^b + \bar{\lambda}_a \lambda^b)}{2}, \tag{2.4}$$

where the presence of the two terms in (2.4) ensures tracelessness as well as the fact that the matrix is full rank. There is a redundancy in this description however, since $p_{ab}$ which contains three independent components, is represented in terms of two 2-component (complex) spinors. To reduce the degrees of freedom, we must first impose reality conditions. For physical momenta we require $p_x, p_y, p_z \in \mathbb{R}$. This is satisfied provided we take the matrix in (2.1) to be Hermitian,

$$(p_a^b)^\dagger = p_b^a. \tag{2.5}$$

Using the relation to the spinor variables (2.4), we see that this is satisfied if,

$$\lambda_a^\dagger = \bar{\lambda}^a, \bar{\lambda}_a^\dagger = \lambda^a \quad (\textbf{Euclidean Reality Conditions}). \tag{2.6}$$

Therefore, $\lambda$ and $\bar{\lambda}$ are Hermitian conjugates of each other. Further, one can see in (2.4) that there is a $U(1)$ redundancy,

$$\lambda \to e^{-\frac{i\theta}{2}}\lambda, \bar{\lambda} \to e^{+\frac{i\theta}{2}}\bar{\lambda}, \tag{2.7}$$

which together with the reality condition (2.6) ensures that the degrees of freedom on both sides of (2.4) match. Before we proceed let us note that we raise and lower spinor indices using the two dimensional Levi-Civita symbol $\epsilon_{ab}$ = as follows[1]:

$$\lambda^a = \epsilon^{ab}\lambda_b, \lambda_a = \epsilon_{ba}\lambda^b. \tag{2.8}$$

[1]We work in the convention that $\epsilon_{12} = -\epsilon_{21} = 1$ and $\epsilon^{ab}\epsilon_{ac} = \delta_c^b$.

### 2.1.2 The observables of interest

Among the observables of interest to are correlation functions of conserved currents in 3d CFT which describe scattering of massless gauge bosons in $AdS_4$. Lets now define these operators more precisely. A symmetric traceless conserved current $J_s^{a_1 \cdots a_{2s}}(p)$ contains $2s$ spinor indices and is traceless with respect to any of them. In conformal field theories, it is a well known fact that such a spin-s current in three dimensions has a scaling dimension $\Delta = s + 1$ [8]. Now, the helicity basis for these currents is defined with the help of our spinors $\lambda$ and $\bar{\lambda}$ as follows[2]:

$$\zeta_-^a = \frac{\lambda^a}{\sqrt{p}}, \zeta_+^a = \frac{\bar{\lambda}^a}{\sqrt{p}}, \tag{2.9}$$

where $p = -\frac{\lambda \cdot \bar{\lambda}}{2}$ is the magnitude of the three-momentum. We assign $\lambda$ and $\bar{\lambda}$ a helicity of $-\frac{1}{2}$ and $+\frac{1}{2}$ respectively which is consistent with their little group "charge" (2.7). The positive and negative helicity components of the current which have helicity $+s$ and $-s$ respectively are then defined as,

$$J_s^{\pm}(\lambda, \bar{\lambda}) = \zeta_{\pm a_1} \cdots \zeta_{\pm a_{2s}} J_s^{a_1 \cdots a_{2s}}(p). \tag{2.10}$$

The beauty of the helicity basis is that a complicated quantity with $2s$ indices is traded for just two components. As one can easily see, the remaining components are proportional to the divergence of the current which is zero since it is conserved.

For future convenience, we define the rescaled currents with a hat,

$$\hat{J}_s^{\pm}(\lambda, \bar{\lambda}) = \frac{J_s^{\pm}(\lambda, \bar{\lambda})}{p^{s-1}}. \tag{2.11}$$

One can show that the action of the conformal generators on the rescaled helicity basis conserved currents (2.11) is as follows [24, 37]:

$$\begin{aligned} P_{ab} &= \lambda_{(a} \bar{\lambda}_{b)}, \ M_{ab} = \frac{1}{2}\big(\lambda_{(a} \frac{\partial}{\partial \lambda^{b)}} + \bar{\lambda}_{(a} \frac{\partial}{\partial \bar{\lambda}^{b)}}\big), \\ D &= \frac{1}{2}\left(\lambda^a \frac{\partial}{\partial \lambda^a} + \bar{\lambda}^a \frac{\partial}{\partial \bar{\lambda}^a} + 2\right), K_{ab} = \frac{\partial^2}{\partial \lambda^{(a} \partial \bar{\lambda}^{b)}}. \end{aligned} \tag{2.12}$$

Our task is to solve the associated conformal Ward identities for the current correlators. Translation invariance is taken care of by stripping off the momentum conserving delta function. Dilatation invariance is satisfied by making sure the correlator has the correct scaling dimension which is $3 - n$ for the stripped rescaled correlator. Rotational invariance demands that the correlator is constructed out of the following spinor contractions using the two dimensional epsilon symbol,

$$\langle ij \rangle = \lambda_{ia} \lambda_j^b \, , \langle \bar{i}\bar{j} \rangle = \bar{\lambda}_{ia} \bar{\lambda}_j^a, \langle i\bar{j} \rangle = \lambda_{ia} \bar{\lambda}_j^a, p_i = -\frac{1}{2} \lambda_{ia} \bar{\lambda}_i^a. \tag{2.13}$$

[2] We also define the polarization vectors $\epsilon_{\pm}^{ab} = \zeta_{\pm}^a \zeta_{\pm}^b$ and with $\epsilon_{\pm}^{\mu} = -\frac{1}{2}(\sigma^{\mu})_{ab} \epsilon_{\pm}^{ab}$ for future reference. We are slightly abusing notation by using $\epsilon$ to denote both the polarization vector as well as the Levi-Civita symbols but we will make it clear from context.

Special conformal transformations on the other hand act as a coupled set of second order inhomogeneous PDEs [37]:

$$\sum_{i=1}^{n} \frac{\partial^2}{\partial\lambda_i^{(a}\partial\bar{\lambda}_i^{b)}}\langle\langle\cdots \hat{J}^{\pm}_{s_i}(\lambda_i,\bar{\lambda}_i)\cdots\rangle\rangle$$
$$= \frac{1}{2}\left(\sum_{i=1}^{n}\frac{s_i(s_i+1)}{p_i^{s_i+1}}\zeta_{i\pm a}\zeta_{i\pm b}p_i^{a_1a_2}\zeta_{i\pm}^{a_3}\cdots\zeta_{i\pm}^{a_{2s}}\right)\langle\langle\cdots \hat{J}_{s_i a_1\cdots a_{2s}}(\lambda_i,\bar{\lambda}_i)\cdots\rangle\rangle. \quad (2.14)$$

Before we proceed, there is another important constraint we must impose on correlators of our conserved currents. The little group covariance. Given the fact that the currents (2.10) are contracted with $2s$ $\lambda$s for the negative helicity component and $2s$ $\bar{\lambda}$s for the positive helicity one, we require the following recaling property of our correlators:

$$\langle\cdots \hat{J}^{\pm}_{s_i}(\frac{1}{r}\lambda_i, r\bar{\lambda}_i)\cdots\rangle = r^{\pm 2s_i}\langle\cdots \hat{J}^{\pm}_{s_i}(\lambda_i,\bar{\lambda}_i)\cdots\rangle. \quad (2.15)$$

Armed with the conformal Ward identities and the helicity identity, we shall now discuss the general form of two and three point correlators of currents.

### 2.1.3 Correlation functions

We shall now discuss and derive some general facts about two and three-point correlators in three-dimensional CFT involving scalars and conserved currents in momentum space and spinor helicity variables.

#### Two-point functions

The two-point function of a primary scalar operator $O_\Delta$ is fixed by the conformal ward identities up to an overall constant. In momentum space, it is given by the following expression[3],

$$\langle\langle O_\Delta(p)O_\Delta(-p)\rangle\rangle = c_\Delta\, p^{2\Delta-3}. \quad (2.16)$$

Two-point functions of conserved currents are similarly constrained and determined up to two constants, one of which multiplies the usual parity even expression while the other, multiplies a parity odd contact term. This is an artefact of three dimensions, due to the presence of the three indexed Levi-Civita symbol. For example, the two-point function of a spin-one conserved current is given by,

$$\epsilon_{1\mu}\epsilon_{2\nu}\langle J^\mu(p)J^\nu(-p)\rangle = c_{1,\text{even}}\; p\;(\epsilon_1\cdot\epsilon_2) + c_{1,\text{odd}}\;\epsilon^{\epsilon_1\epsilon_2 p}. \quad (2.17)$$

This expression is easily generalized to the spin $s$ case:

$$\epsilon_{1\mu_1}\cdots\epsilon_{1\mu_s}\epsilon_{2\nu_1}\cdots\epsilon_{2\nu_s}\langle J_s^{\mu_1\cdots\mu_s}(p)J_s^{\nu_1\cdots\nu_s}(-p)\rangle = c_{s,\text{even}}\; p^{2s-1}\;(\epsilon_1\cdot\epsilon_2)^s + c_{s,\text{odd}}\; p^{2s-2}\;\epsilon^{\epsilon_1\epsilon_2 p}(\epsilon_1\cdot\epsilon_2)^{s-1}$$
$$= \epsilon_{1\mu_1}\cdots\epsilon_{1\mu_s}\epsilon_{2\nu_1}\cdots\epsilon_{2\nu_s}\left(c_{s,\text{even}}\langle J_s^{\mu_1\cdots\mu_s}(p)J_s^{\nu_1\cdots\nu_s}(-p)\rangle_{\text{even}} + c_{s,\text{odd}}\langle J_s^{\mu_1\cdots\mu_s}(p)J_s^{\nu_1\cdots\nu_s}(-p)\rangle_{\text{odd}}\right), \quad (2.18)$$

[3] Two point functions of non-identical operators are zero in CFTs modulo possible contact terms [17] which also play an important role in supersymmetric CFT [48].

where we used the notation $\epsilon^{\mu\nu\rho}a_\mu b_\nu c_\rho := \epsilon^{abc}$. It is interesting to note that the parity odd expression can be obtained from the parity even one via an *epsilon* transform as follows [44]:

$$\langle J_s^{\mu_1\cdots\mu_s}(p)J_s^{\nu_1\cdots\nu_s}(-p)\rangle_{\text{odd}} = \frac{1}{p}\epsilon^{p\alpha(\mu_1}\langle J_s^{\mu_2\cdots\mu_s)\alpha}(p)J_s^{\nu_1\cdots\nu_s}(-p)\rangle_{\text{even}}. \tag{2.19}$$

One may think that the $c_{s,\text{odd}}$ contribution can be removed via the addition of suitable source counter-terms as it multiplies a contact term contribution[4]. This, however, is not completely true as there could exist a scheme-independent part of the parity odd two-point functions and hence no counter-terms can remove them. This is explicitly realized in Chern-Simons theory with matter [77]. In fact, these contact terms will play a crucial role in chapter 4 while defining the chiral limit of Chern-Simons matter theory.

It turns out that (2.18) takes an interesting form in the language of three-dimensional spinor-helicity variables. In the two nonzero helicity configurations, i.e., $(-\,-)$ and $(+\,+)$, we find[5],

$$\begin{aligned}\langle J_s^-(p)J_s^-(-p)\rangle &\sim (c_{s,\text{even}} - ic_{s,\text{odd}})\frac{\langle 12\rangle^{2s}}{p},\\ \langle J_s^+(p)J_s^+(-p)\rangle &\sim (c_{s,\text{even}} + ic_{s,\text{odd}})\frac{\langle \bar{1}\bar{2}\rangle^{2s}}{p}.\end{aligned} \tag{2.20}$$

We see that in spinor-helicity variables, the even and odd terms are identical up to a helicity (but not momentum) dependent factor. This factor of $\mp i$ can be understood by looking at the epsilon transform (2.19). Effectively, the epsilon transform with respect to a conserved current $J_s$ becomes a simple multiplicative transformation in spinor-helicity variables.[6]

### Three-point functions

Three-point correlators of conserved currents can be grouped into two broad categories. Given a three-point function $\langle J_{s_1}J_{s_2}J_{s_3}\rangle$ such that $s_i + s_j \geq s_k\ \forall i,j,k \in \{1,2,3\}$, we say that the correlator is inside the (spin) triangle. If this inequality is violated, we say

[4] $p^{2(s-1)}\epsilon^{\epsilon_1\epsilon_2 p}(\epsilon_1\cdot\epsilon_2)^{s-1}$ is the Fourier transform of the following contact term: $(\epsilon_1\cdot\epsilon_2)^{s-1}\epsilon^{\epsilon_1\epsilon_2\mu}\partial_{1\mu}\Box_1^{2(s-1)}\delta^3(x_1-x_2)$.

[5] One can show using momentum conservation that the $(-+)$ and $(+-)$ helicity two point functions are zero if the $(--)$ and $(++)$ ones are non-zero.

[6] Consider the $s=1$ case for instance with a negative helicity polarization.

$$\epsilon_\mu^-(\epsilon\cdot J_s)^\mu := \epsilon_\mu^-\frac{\epsilon^{\mu p\alpha}}{p}J_\alpha. \tag{2.21}$$

By using the fact that $\epsilon^{\mu\nu\rho} = \frac{1}{2i}\text{Tr}\{\sigma^\mu\sigma^\nu\sigma^\rho\}$ and $\epsilon_\mu^- = \frac{(\sigma_\mu)^a_b\lambda_a\lambda^b}{2p}$, $p^\mu = \frac{(\sigma^\mu)^a_b\lambda_a\bar{\lambda}^b}{2}$, we find that,

$$\epsilon_\mu^-(\epsilon\cdot J_s)^\mu = -i\epsilon^{\alpha-}J_\alpha = -iJ^-. \tag{2.22}$$

For the case when the polarization has a positive helicity, we obtain a $+i$ on the RHS instead of the $-i$. Since the polarization vectors for higher-spins are formed out of products of the spin one polarization, the same results hold there. In short, if the polarization has helicity $h$ we obtain a $\text{sign}(h)i$ factor while performing the epsilon transform.

that it is outside the (spin) triangle. In this subsection, we first review some general facts about correlators both inside and outside the triangle [45, 52]. We then proceed to derive some results which will refine our understanding of these quantities in spinor helicity variables.

### Three-point functions inside the triangle

The most general three-point function of conserved currents that obeys the triangle inequality takes the following form [78]:

$$\langle J_{s_1}J_{s_2}J_{s_3}\rangle = n_f\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{FF} + n_{odd}\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{odd} + n_b\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{FB}. \tag{2.23}$$

We have used the abbreviations $FF$ =free fermion and $FB$ =free boson that we shall employ throughout this chapter. It was found that the odd part can be written in terms of the FF and FB results via an epsilon transformation [44].

$$\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{odd} = \langle\epsilon\cdot J_{s_1}J_{s_2}J_{s_3}\rangle_{FF-FB} = \langle\epsilon\cdot J_{s_1}J_{s_2}J_{s_3}\rangle_{FF} - \langle\epsilon\cdot J_{s_1}J_{s_2}J_{s_3}\rangle_{FB}. \tag{2.24}$$

For example, consider (2.24) for the stress tensor three point function. Keeping all indices explicit, it is given by,

$$\begin{aligned}\langle T_{\mu\nu}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\text{odd}} &= \langle\epsilon\cdot T_{\mu\nu}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{FF-FB}\\ &= \frac{\epsilon_{ab(\mu}p_{1a}}{p_1}\langle T_{\nu)b}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{FF-FB}.\end{aligned} \tag{2.25}$$

It is important to note that for correlators inside the triangle, one can obtain the odd correlator via the epsilon transform with respect to any of the currents in the $FF-FB$ correlator.[7] We will see that the same cannot be said for correlators that violate the triangle inequality in the next subsection.

Further, we define the homogeneous and nonhomogeneous correlators,[8]:

$$\begin{aligned}\langle J_{s_1}J_{s_2}J_{s_3}\rangle_h &= \frac{1}{2}\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{FF-FB},\\ \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{nh} &= \frac{1}{2}\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{FF+FB},\end{aligned} \tag{2.26}$$

where we used the notation $\langle\cdots\rangle_{FF\pm FB} = \langle\cdots\rangle_{FF} \pm \langle\cdots\rangle_{FB}$. These quantities satisfy[9],

$$\begin{aligned}p_{1\mu}\langle J_{s_1}^{\mu\cdots}J_{s_2}J_{s_3}\rangle_h &= 0,\\ p_{1\mu}\langle J_{s_1}^{\mu\cdots}J_{s_2}J_{s_3}\rangle_{nh} &= \text{Ward-Takahashi identity terms.}\end{aligned} \tag{2.27}$$

Since the Ward-Takahashi identity is made out of two-point functions, the coefficient of the nonhomogeneous correlator is that of the two-point function.

---

[7]This is due to the fact that the $FF-FB$ correlators are non-zero only when all their helicities have the same sign as we shall see in (2.30). Thus, the epsilon transform, which gives a factor of $i\,\text{sign}(h)$ can be performed with respect to any of the operators.

[8]The homogeneous and nonhomogeneous correlators have nice bulk interpretations. The former comes due to higher derivative interactions whereas the latter comes due to the term that also gives rise to the propagators of the involved bulk fields.

[9]Inside the triangle, the bosonic and fermionic current Ward-Takahashi identities are equal and thus the homogeneous correlator is identically conserved [45].

Now, using (2.24) and (2.26), (2.23) can be written as,

$$\langle J_{s_1}J_{s_2}J_{s_3}\rangle = (n_f+n_b)\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{nh} + (n_f-n_b)\langle J_{s_1}J_{s_2}J_{s_3}\rangle_h + 2n_{\text{odd}}\,\langle \epsilon\cdot J_{s_1}J_{s_2}J_{s_3}\rangle_h. \tag{2.28}$$

As mentioned just above this equation, the coefficient of the nonhomogeneous correlator viz $n_f+n_b$ is proportional to the two-point function coefficient.[10] $n_f-n_b$ and $n_{\text{odd}}$ on the other hand are truly three-point data. Thus we see that in contrast to the three structures appearing in (2.23), only two appear in (2.28) thanks to the epsilon transformation that related the odd part to the $FF-FB$ (homogeneous) correlator. Further, these homogeneous and nonhomogeneous correlators obey many interesting properties. However, for the best way to see these properties, we must turn to the language of three-dimensional spinor-helicity variables as we did even for the two-point functions in (2.20). For instance, (2.28) in spinor-helicity variables takes the following form (where $h_i$ is the helicity of $J_{s_i}$) in spinor-helicity variables:

$$\langle J_{s_1}^{h_1}J_{s_2}^{h_2}J_{s_3}^{h_3}\rangle = (n_f+n_b)\langle J_{s_1}^{h_1}J_{s_2}^{h_2}J_{s_3}^{h_3}\rangle_{nh} + (n_f-n_b\pm 2in_{\text{odd}})\langle J_{s_1}^{h_1}J_{s_2}^{h_2}J_{s_3}^{h_3}\rangle_h, \tag{2.29}$$

where the $\pm i$ multiplying $n_{\text{odd}}$ depends on the helicity configuration in question.

We shall now state some facts about the (non)homogeneous correlators and subsequently prove them.

**Statement 1**: For correlators inside the triangle, the homogeneous parts, $\langle J_{s_1}J_{s_2}J_{s_3}\rangle_h$ are only nonzero in the $(---)$ and $(+++)$ helicity configurations. It is important to note that the odd correlators in spinor-helicity variables are equal to the even homogeneous correlators up to factors of $\pm 2i$ that depend on the helicity[11]

$$\begin{aligned}&\langle J_{s_1}^-J_{s_2}^-J_{s_3}^-\rangle_h \neq 0\,, \qquad \langle J_{s_1}^+J_{s_2}^+J_{s_3}^+\rangle_h \neq 0\,,\\ &\langle J_{s_1}^{h_1}J_{s_2}^{h_2}J_{s_3}^{h_3}\rangle_h = 0 \text{ for all other helicities.}\end{aligned} \tag{2.30}$$

**Statement 2**: The nonhomogeneous correlators inside the triangle, $\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{nh}$ are only nonzero in the mixed helicity configurations, i.e. they are zero in the $(---)$ and $(+++)$ helicity configurations. We show that the nonhomogeneous correlators can be modified via the addition of contact terms to obey this statement[12]

$$\begin{aligned}&\langle J_{s_1}^-J_{s_2}^-J_{s_3}^-\rangle_{nh} = \langle J_{s_1}^+J_{s_2}^+J_{s_3}^+\rangle_{nh} = 0\,,\\ &\langle J_{s_1}^{h_1}J_{s_2}^{h_2}J_{s_3}^{h_3}\rangle_{nh} \neq 0 \text{ for other (net nonzero) helicities.}\end{aligned} \tag{2.31}$$

[10] For instance, consider the $\langle TT\rangle$ two-point function: $\langle TT\rangle \propto C_T$. We then have $\langle TTT\rangle_{nh}\propto C_T$ since $p_1\cdot\langle TTT\rangle \sim \langle TT\rangle$. This also implies in (2.28) that $n_f+n_b=C_T$.

[11] The fact that homogeneous correlators are dual to higher derivative interactions in the bulk supports this statement. Such interactions are nonzero only when the helicities of all three particles coincide. For example, consider $\langle TTT\rangle_h$. It can be obtained from the graviton $W^3$ (Weyl tensor cubed) interaction in the bulk which vanishes except when the gravitons all have the same helicity.

[12] Nonhomogeneous correlators arise from the $s$ derivative interactions in the bulk which have support only if one of the particles has a different helicity than the other two. For instance consider $\langle JJJ\rangle_{nh}$ (non-Abelian currents). This correlator is dual to the usual Yang-Mills interaction in the bulk which is nonzero only in the $(--+)$ configuration and those obtained via exchanges or complex conjugation.

**Statement** 3: The zero-helicity sectors (i.e. the sector where the net helicity of the correlator is zero) of the nonhomogeneous correlators is identically zero. Similar to what we discussed in the above statement, this statement too will be true modulo contact term contributions. Put another way, we will show that we can find contact terms to remove such contributions to the nonhomogeneous correlators[13]

$$\langle J_{s_1}^{h_1} J_{s_2}^{h_2} J_{s_3}^{h_3} \rangle = 0 \;\; \text{if } s_1 h_1 + s_2 h_2 + s_3 h_3 = 0. \tag{2.32}$$

Let us now present a suggestive proof for statement 1, (2.30): consistency of the momentum space correlator with the operator product expansion forces it to only have total energy poles [45]. The other type of poles which are of the form $p_i + p_j - p_k$, are inconsistent with the OPE. The momentum space expressions that give rise to the correct conformally invariant homogeneous correlators in the $(- - -)$ and $(+ + +)$ helicities can be found in [42, 45]. These expressions also yield zero in the mixed helicity configurations. Thus, in principle, any other solution that gives rise to non-zero expressions in the mixed helicities will also ruin the correct $(- - -), (+ + +)$ answers. This is further solidified by the uniqueness of the momentum space solution: Other solutions (that are consistent with the OPE) simply cannot exist except of course, modulo some contact term contributions. Also, it is interesting to note that when the spin-triangle inequality is violated, there is no homogeneous solution in any helicity configuration consistent with the OPE [45].

We now present a few examples to illustrate the validity of statements 2 (2.31) and 3 (2.32).

**Example 1** : $\langle JJJ \rangle$

This correlator in the two independent helicity configurations takes the form [42]:

$$\begin{aligned}
\langle J^{-A} J^{-B} J^{-C} \rangle &= f^{ABC} \Big( \frac{c_E + i c_O}{E^3} + \frac{c_J}{p_1 p_2 p_3} \Big) \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \,, \\
\langle J^{-A} J^{-B} J^{+C} \rangle &= f^{ABC} \frac{c_J}{p_1 p_2 p_3} \Big( 1 - \frac{2 p_3}{E} \Big) \langle 12 \rangle \langle 2\bar{3} \rangle \langle \bar{3}1 \rangle \,, \\
\text{where } E &= p_1 + p_2 + p_3 \,,
\end{aligned} \tag{2.33}$$

where $c_E$ $(i c_O)$ multiplies the parity even (parity odd) homogeneous contribution and $c_J$ is the coefficient of the parity even nonhomogeneous term. The $(+++)$ and $(++-)$ configurations can be obtained from the above expression by complex conjugation (which replaces $\langle ij \rangle \to \langle \bar{i}\bar{j} \rangle$). Finally, the remaining helicities are obtained via exchanges of labels.

Looking at (2.33), we see that the homogeneous solution is present only in the $(---)$ and $(+ + +)$ helicities which is in accordance to statement 1. However, it appears as though there is a nonhomogeneous contribution in the $(- - -)$ and $(+ + +)$ helicities

[13] As the CFT correlators give rise to S-matrices in one higher dimension when the total energy goes to zero and zero-helicity three-point S-matrices are identically zero, the CFT correlator should be free of total energy poles. Thus, it is plausible (as we shall also show) that the zero-helicity CFT correlators can be removed via suitable contact terms.

which should not be there according to statement 2 (2.31). We shall show however, that via the addition of a suitable contact term, we can cancel these contributions. Indeed, consider the following contact term contribution:

$$\langle J^A J^B J^C\rangle_{\text{contact}} = \frac{2c_J f^{ABC}}{3p_1p_2p_3}\Big((\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot p_1)(p_1+p_2)p_3 + -(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot p_1)p_2(p_1+p_3) + (\epsilon_1\cdot p_2)(\epsilon_2\cdot\epsilon_3)p_1(p_2+p_3)\Big). \tag{2.34}$$

We then redefine the correlator via the addition of (2.34) as follows: $\langle J^A J^B J^C\rangle_{new} = \langle J^A J^B J^C\rangle - \langle J^A J^B J^C\rangle_{\text{contact}}$. This results in the following expression for $\langle JJJ\rangle_{new}$ in the two independent helicity configurations:

$$\begin{aligned}\langle J^{-A} J^{-B} J^{-C}\rangle_{new} &= f^{ABC}\big(\frac{c_E + ic_O}{E^3}\big)\langle 12\rangle\langle 23\rangle\langle 31\rangle,\\ \langle J^{-A} J^{-B} J^{+C}\rangle_{new} &= f^{ABC}\frac{2c_J}{3p_1p_2p_3}\big(1-\frac{3p_3}{E}\big)\langle 12\rangle\langle 2\bar{3}\rangle\langle \bar{3}1\rangle,\end{aligned} \tag{2.35}$$

proving that the nonhomogeneous contribution to $\langle JJJ\rangle$ can be tuned to zero in the $(- - -)$ and $(+ + +)$ helicities by the addition of the contact term (2.34).[14]

**Example 2** : $\langle TTT\rangle$

Let us now consider the three-point function of the stress tensor. This correlator takes the following form in the two independent helicities [42]:

$$\begin{aligned}\langle T^-T^-T^-\rangle &= \Big(\frac{(c_E+ic_O)p_1p_2p_3}{E^6} + c_T\frac{1}{(p_1p_2p_3)^2}\big(E^3 - E(p_1p_2+p_2p_3+p_1p_3) - p_1p_2p_3\big)\Big)\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2\\ \langle T^-T^-T^+\rangle &= c_T\Big(\frac{(E-2p_3)^2(E^3 - E(p_1p_2+p_2p_3+p_3p_1) - p_1p_2p_3}{E^2p_1^2p_2^2p_3^2}\Big)\langle 12\rangle^2\langle 2\bar{3}\rangle^2\langle \bar{3}1\rangle^2.\end{aligned}$$

$c_E$ and $c_O$ are the partiy even and parity odd homogeneous coefficients whereas $c_T$ is the parity even nonhomogeneous coefficient. The remaining helicities can be obtained via exchanges or by the complex conjugation.

Just like the previous example of $\langle JJJ\rangle$, while the homogeneous contribution is in accordance with statement 1 (2.30), the nonhomogeneous part seems to violate statement 2 (2.31) as it is nonzero in the $(- - -)$ and $(+ + +)$ helicities. However, we shall show that by adding a suitable contact term, we can render such contributions zero. Writing an ansatz for the contact term and fixing the coefficients by demanding that it should cancel out the nonhomogeneous contribution in the $(- - -)$ and $(+ + +)$ helicities we find,

$$\begin{aligned}\langle TTT\rangle_{\text{contact}} = c_T\Big[&\frac{8(\epsilon_1\cdot p_2)(\epsilon_1\cdot\epsilon_2)(\epsilon_2\cdot\epsilon_3)(\epsilon_3\cdot p_1)(E+p_2)}{3} - \frac{8(\epsilon_1\cdot p_2)(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot p_1)(\epsilon_2\cdot\epsilon_3)}{3}(E+p_3)\\ &- \frac{8(\epsilon_1\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot p_1)(\epsilon_3\cdot p_1)}{3}(E+p_1) - 8(\epsilon_1\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_3)\Big(p_1^3 + 2p_1^2(p_2+p_3) + 2p_1(p_2^2+p_3^2) + (p_2+p_3)(p_2^2+p_2p_3+p_3^2)\Big)\Big].\end{aligned} \tag{2.36}$$

By redefining,

$$\langle TTT\rangle = \langle TTT\rangle - \langle TTT\rangle_{\text{contact}}, \tag{2.37}$$

[14] The same is true in the $(+ + +)$ helicity as it is the complex conjugate of the $(- - -)$ helicity.

we find,

$$\langle T^-T^-T^-\rangle = \left(\frac{(c_E + ic_O)p_1p_2p_3}{E^6}\right)\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2,$$

$$\langle T^-T^-T^+\rangle = c_T\frac{-2}{3p_1^2p_2^2p_3E^2}\Big(5p_1^4 + 3p_1^3(4p_2 + 3p_3) + p_1^2(14p_2^2 + 11p_2p_3 + 9p_3^2) + p_2(5p_2^3 + 9p_2^2p_3 + 9p_2p_3^2 + 5p_3^3)$$
$$+ p_1(12p_2^3 + 11p_2^2p_3 + 16p_2p_3^2 + 5p_3^3)\Big)\langle 12\rangle^2\langle 2\bar{3}\rangle^2\langle \bar{3}1\rangle^2,$$

proving that the contact term (2.36), renders the nonhomogeneous $\langle TTT\rangle$ zero in the $(- - -)$ and $(+ + +)$ helicities.

We now turn to an example where there also exists a zero-helicity configuration.

**Example 3** : $\langle TJJ\rangle$

The helicity decomposition of this correlator is as follows [15]:

$$\langle T^-J^-J^-\rangle = \frac{(c_1 + ic_1')p_1}{E^4}\langle 12\rangle^2\langle 31\rangle^2 - c_J\frac{\langle 12\rangle^2\langle 31\rangle^2(p_2+p_3)}{p_1^2p_2p_3},$$
$$\langle T^+J^-J^-\rangle = -c_J\frac{\langle 23\rangle^4(p_1^2 - (p_2-p_3)^2)^2(p_2+p_3)}{\langle 12\rangle^2\langle 31\rangle^2p_1^2p_2p_3} \tag{2.38}$$
$$\langle T^-J^-J^+\rangle = -c_J\frac{\langle 12\rangle^4}{3E^2\langle 23\rangle^2p_1^2p_2p_3}(E-2p_1)^2\big(3E^2(E-p_1) + 8(p_1^2 - E^2)p_2 + 8(p_1+E)p_2^2\big).$$

$c_1$ and $c_1'$ are the parity even and odd homogeneous coefficients while $c_J$ is the coefficient of the nonhomogeneous parity even contribution.

It seems that not only does $\langle TJJ\rangle$ contain a nonhomogeneous contribution in the $(- - -)$ and $(+ + +)$ helicity configurations but also a zero helicity configuration, both of which should not exist according to statements 2 (2.31) and 3 (2.32).

By writing an ansatz for contact terms to cancel these contributions, we were able to find one that simultaneously cancels out both of them:

$$\langle TJJ\rangle_{\text{contact}} = -4c_J(\epsilon_1\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_3)(p_2+p_3). \tag{2.39}$$

By redefining,

$$\langle TJJ\rangle_{new} = \langle TJJ\rangle - \langle TJJ\rangle_{\text{contact}}\,, \tag{2.40}$$

we find

$$\langle T^-J^-J^-\rangle_{new} = \frac{(c_1 + ic_1')p_1}{E^4}\langle 12\rangle^2\langle 31\rangle^2$$
$$\langle T^+J^-J^-\rangle_{new} = 0$$
$$\langle T^-J^-J^+\rangle_{new} = \frac{8c_J\langle 12\rangle^4}{3\langle 23\rangle^2p_1^2E^2}(E-2p_1)^2(E+p_1)\,. \tag{2.41}$$

Thus, the simple contact term (2.39) suffices to render the-nonhomogeneous $\langle TJJ\rangle$ zero in the $(- - -)$ and $(+ + +)$ helicities as well as the zero-helicity configurations.

[15] This expression can easily be obtained from the free theory computations using the definitions provided in (2.26).

**Example 4** : $\langle J_3 T J\rangle$

Let us now consider a correlator involving a higher-spin current to show that our statements also hold in such cases as well. Since the homogeneous part of this correlator obeys statement 1 (2.30), we focus on the apparent discrepancy between this correlators nonhomogeneous part with our statements 2 (2.31) and 3 (2.32). The nonhomogeneous part of this correlator in the $(- - -)$ and $(+ - -)$ configurations are respectively given by[16],

$$\langle J_3^- T^- J^-\rangle_{nh} = a\frac{-\langle 12\rangle^4\langle 31\rangle^2}{p_1^3 p_2^2 p_3}(6p_2^3 - p_1^2 p_3 + 4p_2(p_1+2p_2)p_3 - 7(p_1-2p_2)p_3^2 + 14p_3^3),$$
$$\langle J_3^+ T^- J^-\rangle_{nh} = a\frac{\langle 23\rangle^6(p_1+p_2-p_3)^2(p_1-p_2+p_3)^4}{\langle 12\rangle^2\langle 31\rangle^4 p_1^3 p_2^3 p_3}\big(-6p_2^3 - 14p_3^3 + (p_1^2 + 4p_1p_2 - 8p_2^2)p_3 - 7(p_1+2p_2)p_3^2\big),$$

$a$ is the coefficient of the parity even nonhomogeneous contribution. Although these expressions are quite complicated, we were able to write an ansatz and determine a contact term that can simultaneously cancel both these expressions, rendering the nonhomogeneous contribution zero in the $(- - -)$, $(+ + +)$ helicities as well as the two zero-helicity configurations. The explicit expression for the contact term is as follows:

$$\begin{aligned}\langle J_3 T J\rangle_{\text{contact}} &= 4a(\epsilon_1\cdot\epsilon_2)\Big(4(\epsilon_1\cdot p_2)\big(9(\epsilon_2\cdot p_1)(\epsilon_1\cdot\epsilon_3)\\ &- 7(\epsilon_1\cdot p_2)(\epsilon_2\cdot\epsilon_3)\big)p_3 + (\epsilon_1\cdot\epsilon_2)\big(-14(\epsilon_3\cdot p_1)(\epsilon_1\cdot p_2)p_3\\ &+ (\epsilon_1\cdot\epsilon_3)(12p_2^3 + 9p_1^2p_3 + 13p_2^2p_3 + 3p_3^3)\big)\Big).\end{aligned}$$

By redefining,

$$\langle J_3 T J\rangle_{nh,new} = \langle J_3 T J\rangle_{nh} - \langle J_3 T J\rangle_{\text{contact}}, \tag{2.42}$$

we find,

$$\langle J_3^- T^- J^-\rangle_{nh,new} = \langle J_3^+ T^- J^-\rangle_{nh,new} = 0. \tag{2.43}$$

With this, we conclude our examples and attempt to generalize our findings to arbitrary correlators inside the triangle.

**Conclusion**

In all the examples that we considered, we showed that it is always possible to add contact terms to the correlators to render the nonhomogeneous contributions zero in the $(- - -)$, $(+ + +)$ helicities as well as the zero-helicity configuration where applicable. Essentially, what this means is that the Ward-Takahashi identity for the correlators in these helicity sectors can be removed via re-defining the correlator by adding particular contact terms. If we take this as an input, i.e. the Ward-Takahashi identities being trivial in these helicity configurations, one can prove in general that the nonhomogeneous contribution to the correlator vanishes in the $(- - -)$, $(+ + +)$ and zero-helicity sector.

[16] Again, one can compute this via Wick contractions and use the definition (2.26)

This follows from the statement that there exists a unique solution (that is consistent with the operator product expansion) to the conformal Ward identities in these configurations, which are the homogeneous correlators. Given these facts, it would be nice to rigorously prove statements 2 (2.31) and 3 (2.32) but we defer such an analysis to the future.

Thus, after all the dust has settled, we see that using statements 1, 2 and 3, (2.30), (2.31) and (2.32)), the expressions for three-point functions inside the triangle for generic CFTs in the eight helicity configurations are,

$$\begin{aligned}
\langle J^-_{s_1}J^-_{s_2}J^-_{s_3}\rangle &= (n_f - n_b - 2in_{\text{odd}})\langle J^-_{s_1}J^-_{s_2}J^-_{s_3}\rangle_h, & \langle J^+_{s_1}J^+_{s_2}J^+_{s_3}\rangle &= (n_f - n_b + 2in_{\text{odd}})\langle J^+_{s_1}J^+_{s_2}J^+_{s_3}\rangle_h,\\
\langle J^-_{s_1}J^-_{s_2}J^+_{s_3}\rangle &= (n_b + n_f)\langle J^-_{s_1}J^-_{s_2}J^+_{s_3}\rangle_{nh}, & \langle J^+_{s_1}J^+_{s_2}J^-_{s_3}\rangle &= (n_b + n_f)\langle J^+_{s_1}J^+_{s_2}J^-_{s_3}\rangle_{nh},\\
\langle J^-_{s_1}J^+_{s_2}J^-_{s_3}\rangle &= (n_b + n_f)\langle J^-_{s_1}J^+_{s_2}J^-_{s_3}\rangle_{nh}, & \langle J^+_{s_1}J^-_{s_2}J^+_{s_3}\rangle &= (n_b + n_f)\langle J^+_{s_1}J^-_{s_2}J^+_{s_3}\rangle_{nh},\\
\langle J^+_{s_1}J^-_{s_2}J^-_{s_3}\rangle &= (n_b + n_f)\langle J^+_{s_1}J^-_{s_2}J^-_{s_3}\rangle_{nh}, & \langle J^-_{s_1}J^+_{s_2}J^+_{s_3}\rangle &= (n_b + n_f)\langle J^-_{s_1}J^+_{s_2}J^+_{s_3}\rangle_{nh},
\end{aligned} \tag{2.44}$$

and if there are any configurations with net zero helicity, the correlators are identically zero. We now turn to the study of three-point functions that are outside the triangle.

### Three-point functions that are outside the triangle

Three-point functions of conserved currents that violate the spin triangle inequality take the following form [78]:

$$\langle J_{s_1}J_{s_2}J_{s_3}\rangle = n_f\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{FF} + n_b\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{FB}\,, \tag{2.45}$$

i.e. in contrast to (2.23), the parity odd structure is incompatible with current conservation.[17] It is also interesting to note that in contrast to the situation inside the triangle (2.28), there is no homogeneous correlator outside the triangle. The reason for this is that the free bosonic and free fermionic Ward-Takahashi identities are different outside the triangle [45]. It is also important to note that the only non-zero Ward-Takahashi identity is only due to the highest spin current in the correlator. For example, consider a correlator with a spin four current and two spin one currents, i.e., $\langle J_4J_1J_1\rangle$. Here, the only Ward-Takahashi identity is with respect to the spin-four current.

Just as we have made for three-point functions inside the triangle, we shall make some statements for their counterparts that are outside the triangle. Without loss of generality, consider a correlator $\langle J_{s_1}J_{s_2}J_{s_3}\rangle$ with $s_1 > s_2 + s_3$, $s_2 \geq s_3$. We have then, the following statements for correlators that violate the spin triangle inequality:

[17] There can exist parity odd nonhomogeneous contributions but these are all contact term contributions. For instance, take the FF or FB Ward-Takahashi identity and replace all the two-point functions appearing there by their parity odd counterparts. This gives rise to two new structures that source these parity odd Ward-Takahashi identities. It would be interesting to see the consequences of the existence of these structures but we leave such an exercise for the future.

**Statement 4:** The $FF - FB$ correlators have support iff $J_{s_2}$ and $J_{s_3}$ have the same helicities.

$$\begin{aligned}
&\langle J_{s_1}^{h_1} J_{s_2}^- J_{s_3}^- \rangle_{FF-FB} \neq 0\,, \qquad \langle J_{s_1}^{h_1} J_{s_2}^+ J_{s_3}^+ \rangle_{FF-FB} \neq 0 \\
&\langle J_{s_1}^{h_1} J_{s_2}^- J_{s_3}^+ \rangle_{FF-FB} = \langle J_{s_1}^{h_1} J_{s_2}^+ J_{s_3}^- \rangle_{FF-FB} = 0.
\end{aligned} \tag{2.46}$$

**Statement 5:** The $FF + FB$ correlators identically vanish when the helicities of $J_{s_2}$ and $J_{s_3}$ coincide.

$$\begin{aligned}
&\langle J_{s_1}^{h_1} J_{s_2}^- J_{s_3}^- \rangle_{FF+FB} = 0\,, \qquad \langle J_{s_1}^{h_1} J_{s_2}^+ J_{s_3}^+ \rangle_{FF+FB} = 0 \\
&\langle J_{s_1}^{h_1} J_{s_2}^- J_{s_3}^+ \rangle_{FF+FB} \neq 0\,, \qquad \langle J_{s_1}^{h_1} J_{s_2}^+ J_{s_3}^- \rangle_{FF+FB} \neq 0\,.
\end{aligned} \tag{2.47}$$

We have verified statements 4 and 5 using free theory via a few examples as we will discuss below. It would be nice to find a general proof of these statements in the future. For statement 4, however, we present a different argument for its validity following as an implication of the next statement to be discussed below.

If we allow the current conservation to be "slightly" broken we instead have[18] [78],

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle = n_f \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{FF} + n_{odd} \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{odd} + n_b \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{FB}, \tag{2.48}$$

where the odd piece is the one that is not conserved but rather, slightly-broken. However, it was shown that the odd part can be given by an epsilon transform of the difference of the fermionic and bosonic correlators [45]:

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle = n_f \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{FF} + n_{odd}\, \epsilon \cdot \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{FF-FB} + n_b \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{FB}\,. \tag{2.49}$$

The question now is, epsilon transform with respect to what? The answer is provided in the below statement:

**Statement 6:** The epsilon transform required to obtain the odd correlator can be performed only with respect to the lowest two spins in the correlator. This is because they are identically conserved in the correlator and any Ward-Takahashi identity arises from the current with the highest spin. For example consider $\langle J_{s_1} J_{s_2} J_{s_3} \rangle$ with $s_1 > s_2$, $s_2 \geq s_3$. Then,

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{odd}} = \langle J_{s_1} \epsilon \cdot J_{s_2} J_{s_3} \rangle_{\text{FF-FB}} = \langle J_{s_1} J_{s_2} \epsilon \cdot J_{s_3} \rangle_{\text{FF-FB}}\,. \tag{2.50}$$

The proof of this statement relies on using the slightly broken higher spin equations which we presented in [52].

We shall now illustrate via three examples, the validity of statements 4 and 5 ((2.46) and (2.47)). Let us begin with the example of $\langle J_4 J_1 J_1 \rangle$ following [45].

[18] Slightly-broken here refers to a situation where the current non-conservation is $O\left(\frac{1}{N}\right)$ in a large-N theory.

**Example 1:** $\langle J_4 J_1 J_1 \rangle$

The $FF - FB$ correlator in the independent helicity configurations takes the following forms [45]:

$$\langle J_4^- J_1^- J_1^- \rangle_{FF-FB} = -\langle 12 \rangle^4 \langle 31 \rangle^4 \langle \bar{2}\bar{3} \rangle^2 \frac{3E^5 + 5E^4 k_1 + 8E^3 k_1^2 + 12E^2 k_1^3 + 16E k_1^4 + 16k_1^5}{524288 E^8 k_1^4},$$
$$\langle J_4^+ J_1^- J_1^- \rangle_{FF-FB} = -\langle 23 \rangle^6 \langle \bar{1}\bar{2} \rangle^4 \langle \bar{3}\bar{1} \rangle^4 \frac{3E + k_1}{524288 E^8 k_1^4}, \quad \langle J_4^- J_1^- J_1^+ \rangle_{FF-FB} = 0. \tag{2.51}$$

This statement is perfectly consistent with statement 4 (2.46), i.e. the vanishing of $FF - FB$ when the helicities of the spin one currents are different. Let us now examine the $FF + FB$ correlator. In spinor-helicity variables we have [45]

$$\langle J_4^- J_1^- J_1^- \rangle_{FB+FF} = 0, \quad \langle J_4^+ J_1^- J_1^- \rangle_{FB+FF} = 0\,,$$
$$\langle J_4^- J_1^- J_1^+ \rangle_{FB+FF} = \langle 12 \rangle^6 \langle 31 \rangle^2 \langle \bar{2}\bar{3} \rangle^4 \frac{5E^3 + 5E^2 k_1 + 4E k_1^2 + 2k_1^3}{262144 E^8 k_1^4}\,, \tag{2.52}$$

which is in perfect agreement with statement 5 (2.47) as the correlator vanishes when the helicities of the spin one currents coincide.

Let us now proceed to an example which involves half-integer spin insertions.

**Example 2:** $\langle O_{1/2} O_{1/2} T \rangle$

This correlator can be computed via Wick contractions in the free $\mathcal{N} = 1$ supersymmetric theory. For the free bosonic stress tensor we have

$$\zeta_{1a}\zeta_{2b}z_{3\alpha}z_{3\beta}\langle O^a_{1/2}(p_1) O^b_{1/2}(p_2) T^{\alpha\beta}_{FB}(p_3)\rangle = 2\zeta_{1a}\zeta_{2\alpha}z_{3\alpha}z_{3\beta}(\sigma^\rho)^{ab}\int \frac{d^3 l}{(2\pi)^3}\frac{l^\rho (l^\alpha - p_1^\alpha)(l^\beta - p_1^\beta)}{l^2 (l-p_1)^2 (l+p_2)^2}$$
$$= -\frac{\zeta_{1a}\zeta_{2b}(p_1\cdot z_3)}{32 p_1 p_2 E^3}\Bigg((p_1\cdot z_3)(E+2p_2)\Big(-p_2(\not{p}_1{}^{ab}) + p_1(\not{p}_2{}^{ab})\Big) + 2p_1 p_2 E(E+p_3)(\not{z}_3)^{ab}\Bigg). \tag{2.53}$$

A similar computation for the free fermionic stress tensor yields,

$$\zeta_{1a}\zeta_{2b}z_{3\alpha}z_{3\beta}\langle O^a_{1/2}(p_1) O^b_{1/2}(p_2) T^{\alpha\beta}_{FF}(p_3)\rangle = \zeta_{1a}\zeta_{2b}z_{3\alpha}z_{3\beta}(\sigma^\beta)^d_c(\sigma^\rho)^{cb}(\sigma^\chi)^a_d\int \frac{d^3 l}{(2\pi)^3}\frac{(l^\alpha - p_1^\alpha)(l^\rho + p_2^\rho)(l^\chi - p_1^\chi)}{l^2 (l-p_1)^2 (l+p_2)^2}$$
$$= \frac{\zeta_{1a}\zeta_{2b}}{32 p_1 p_2 p_3 E^3}\Bigg(-(p_1\cdot z_3)^2\Big((4(p_1+p_2)^3 + (p_1+p_2)(10p_1+11p_2)p_3 + 3(2p_1+3p_2)p_3^2)\not{p}_1{}^{ab} - (1\leftrightarrow 2)\Big)$$
$$+ (p_1\cdot z_3)(p_1+p_2)(2p_1+2p_2+3p_3)(p_1^2+p_2^2-p_3^2)(\not{z}_3)^{ab} - 2i(p_1\cdot z_3)(p_1+p_2)(2p_1+2p_2+3p_3)\epsilon^{ab}\epsilon^{z_3 p_1 p_2}\Bigg). \tag{2.54}$$

We now consider the difference and sum of the bosonic and fermionic contributions along with the addition of a contact term:

$$\langle O_{1/2} O_{1/2} T \rangle_{\textbf{FF-FB}} = \langle O_{1/2} O_{1/2} T_{FF} \rangle - \langle O_{1/2} O_{1/2} T_{FB} \rangle - C_{O_{\frac{1}{2}} O_{\frac{1}{2}} T}\,,$$
$$\langle O_{1/2} O_{1/2} T \rangle_{\textbf{FB+FF}} = \langle O_{1/2} O_{1/2} T_{FB} \rangle + \langle O_{1/2} O_{1/2} T_{FF} \rangle + C_{O_{\frac{1}{2}} O_{\frac{1}{2}} T}\,, \tag{2.55}$$

where the contact term is given by

$$C_{O_{\frac{1}{2}}O_{\frac{1}{2}}T} = \frac{1}{8p_3}\zeta_{1a}\zeta_{2b}(p_1\cdot z_3)(z\!\!\!/_3)^{ab}\,. \tag{2.56}$$

We now find in accordance with statement 4 (2.46) that the $FF-FB$ correlator vanishes when the $O_{1/2}$ operators have different helicities.

$$\langle O^-_{1/2}(p_1)O^-_{1/2}(p_2)T^-(p_3)\rangle_{\textbf{FF-FB}} = -\frac{\langle 23\rangle^2\langle 31\rangle^2\Big(4(p_1+p_2)^4+7(p_1+p_2)^3p_3-3(p_1+p_2)^2p_3^2-11(p_1+p_2)p_3^3+3p_3^4\Big)}{128\langle 12\rangle\sqrt{p_1}\sqrt{p_2}p_3^3E^3},$$
$$\langle O^-_{1/2}(p_1)O^+_{1/2}(p_2)T^-(p_3)\rangle_{\textbf{FF-FB}} = 0,$$
$$\langle O^+_{1/2}(p_1)O^-_{1/2}(p_2)T^-(p_3)\rangle_{\textbf{FF-FB}} = 0,$$
$$\langle O^-_{1/2}(p_1)O^-_{1/2}(p_2)T^+(p_3)\rangle_{\textbf{FF-FB}} = -\frac{\langle 12\rangle^3(4(p_1+p_2)+3p_3)((p_1-p_2)^2-p_3^2)^2}{128\langle 23\rangle^2\langle 31\rangle^2\sqrt{p_1}\sqrt{p_2}p_3^3}. \tag{2.57}$$

We also see that the FF+FB correlator obeys statement 5 (2.47):

$$\langle O^-_{1/2}(p_1)O^-_{1/2}(p_2)T^-(p_3)\rangle_{\textbf{FB+FF}} = 0,$$
$$\langle O^-_{1/2}(p_1)O^+_{1/2}(p_2)T^-(p_3)\rangle_{\textbf{FB-FF}} = -\frac{\langle 23\rangle\langle 31\rangle^3(p_1+p_2)(p_1+p_2-p_3)^2(2p_1+2p_2+3p_3)}{64\langle 12\rangle^2\sqrt{p_1}\sqrt{p_2}p_3^3E^2},$$
$$\langle O^+_{1/2}(p_1)O^-_{1/2}(p_2)T^-(p_3)\rangle_{\textbf{FB-FF}} = \frac{\langle 23\rangle^3\langle 31\rangle(p_1+p_2)(p_1+p_2-p_3)^2(2p_1+2p_2+3p_3)}{64\langle 12\rangle^2\sqrt{p_1}\sqrt{p_2}p_3^3E^2},$$
$$\langle O^-_{1/2}(p_1)O^-_{1/2}(p_2)T^+(p_3)\rangle_{\textbf{FB-FF}} = 0. \tag{2.58}$$

To conclude our examples, let us consider one more example, this time with a correlator involving a spin 4 current and two spin half operators.

**Example 3**: $\langle J_4 O_{1/2} O_{1/2}\rangle$

Computing this correlator via Wick contractions, we see that the $FF-FB$ and $FF+FB$ combinations automatically obey statements 5 and 6 ((2.47) and (2.50)). The expressions in the independent helicities are as follows:

$$\langle J_4^- O^-_{1/2}O^-_{1/2}\rangle_{FF-FB} = \frac{\langle 12\rangle^4\langle 31\rangle^4}{4096\langle 23\rangle^3E^5p_1^4\sqrt{p_2p_3}}\Big(15E^7-70E^6p_1+84E^5p_1^2-128p_1^7\Big),$$
$$\langle J_4^- O^+_{1/2}O^+_{1/2}\rangle_{FF-FB} = -\frac{\langle 12\rangle^4\langle 31\rangle^4}{4096\langle 23\rangle^5E^3p_1^4\sqrt{p_2p_3}}(E-2p_1)^5\Big(15E^2+10Ep_1+4p_1^2\Big),$$
$$\langle J_4^- O^\pm_{1/2}O^\mp_{1/2}\rangle_{FF-FB} = 0\,, \tag{2.59}$$

and

$$\langle J_4^- O^-_{1/2}O^+_{1/2}\rangle_{FF+FB} = \frac{\langle 12\rangle^5\langle 31\rangle^3}{1024\langle 23\rangle^4E^4p_1^4\sqrt{p_2p_3}}(E-2p_1)^4\Big(5E^3+5E^2p_1+4p_1^2+2p_1^3\Big),$$
$$\langle J_4^- O^+_{1/2}O^+_{1/2}\rangle = 0\,,$$
$$\langle J_4^- O^-_{1/2}O^-_{1/2}\rangle = 0\,, \tag{2.60}$$

thereby verifying the robustness of our statements.

Let us now summarize the structure of three-point correlators outside the triangle while employing (2.49) together with statements 4, 5 and 6, (2.46), (2.47) and (2.50) in mind.

**Helicity decomposition of correlators outside the triangle**

We see that the expressions for three-point functions outside the triangle for CFTs with the slightly-broken higher-spin symmetry in the eight helicity configurations are (we assume $s_1 > s_2$, $s_2 \geq s_3$ in the expression below)

$$\begin{aligned}
\langle J_{s_1}^- J_{s_2}^- J_{s_3}^- \rangle &= \tfrac{n_f - n_b - 2in_{odd}}{2} \langle J_{s_1}^- J_{s_2}^- J_{s_3}^- \rangle_{FF-FB}, & \langle J_{s_1}^+ J_{s_2}^+ J_{s_3}^+ \rangle &= \tfrac{n_f - n_b + 2in_{odd}}{2} \langle J_{s_1}^+ J_{s_2}^+ J_{s_3}^+ \rangle_{FF-FB}, \\
\langle J_{s_1}^- J_{s_2}^- J_{s_3}^+ \rangle &= \tfrac{n_f + n_b}{2} \langle J_{s_1}^- J_{s_2}^- J_{s_3}^+ \rangle_{FF+FB}, & \langle J_{s_1}^+ J_{s_2}^+ J_{s_3}^- \rangle &= \tfrac{n_f + n_b}{2} \langle J_{s_1}^+ J_{s_2}^+ J_{s_3}^- \rangle_{FF+FB}, \\
\langle J_{s_1}^- J_{s_2}^+ J_{s_3}^- \rangle &= \tfrac{n_f + n_b}{2} \langle J_{s_1}^- J_{s_2}^+ J_{s_3}^- \rangle_{FF+FB}, & \langle J_{s_1}^+ J_{s_2}^- J_{s_3}^+ \rangle &= \tfrac{n_f + n_b}{2} \langle J_{s_1}^+ J_{s_2}^- J_{s_3}^+ \rangle_{FF+FB}, \\
\langle J_{s_1}^+ J_{s_2}^- J_{s_3}^- \rangle &= \tfrac{n_f - n_b - 2in_{odd}}{2} \langle J_{s_1}^+ J_{s_2}^- J_{s_3}^- \rangle_{FF-FB}, & \langle J_{s_1}^- J_{s_2}^+ J_{s_3}^+ \rangle &= \tfrac{n_f - n_b + 2in_{odd}}{2} \langle J_{s_1}^- J_{s_2}^+ J_{s_3}^+ \rangle_{FF-FB}.
\end{aligned} \tag{2.61}$$

For the exactly conserved case there is no odd structure so we simply set $n_{\text{odd}} = 0$.

**The utility of spinor helicity**

To summarize the discussion so far, we have seen that spinor helicity provides a nice contrast and reveals the structural properties of the different three point correlation functions. In a later chapter, we shall see its utility in full glory when using this language to determine the holographic dual of chiral higher spin theory.

## 2.2 Lorentzian Signature

Having discussed Euclidean spinor helicity and the general structure of two and three point correlation functions, we turn to the Lorentzian setting now. From the get-go, the formalism is different simply because of the fact that the spinors are real rather than complex like in Euclidean signature. Thus, in this section we begin with a discussion of the Lorentzian spinor helicity construction as well as CPT properties that are important in real time physics. We then discuss the observables of interest which are Wightman functions and then present an example of a three point correlator to compare it with its Euclidean counterparts that we discussed above.

### 2.2.1 Spinor helicity

Consider a three dimensional momentum vector $p_\mu$. Since the Lorentz group in three dimensions is $SO(2,1) \cong SL(2,\mathbb{R})/\mathbb{Z}_2$, we can view it is a symmetric representation of the latter group by trading it for a $2 \times 2$ matrix as follows:

$$p_a^b = (\sigma^\mu)_a^b p_\mu. \tag{2.62}$$

$(\sigma^\mu)_b^a$ are $2 \times 2$ matrices which obey the Clifford algebra,

$$\{\sigma_\mu, \sigma_\nu\} = \eta_{\mu\nu}, \tag{2.63}$$

where $\eta_{\mu\nu} = \text{Diag}(-1, 1, 1)$ is the standard flat metric on $\mathbb{R}^{2,1}$. We can express (2.62) using a pair of fundamental $SL(2,\mathbb{R})$ spinors $\lambda$ and $\bar{\lambda}$ as follows:

$$p_a^b = \frac{(\lambda_a \bar{\lambda}^b + \lambda^b \bar{\lambda}_a)}{2}. \tag{2.64}$$

Writing out the momentum matrix explicitly we obtain,

$$p_a^b = \begin{pmatrix} p_y & p_x - p_t \\ p_x + p_t & -p_y \end{pmatrix}. \tag{2.65}$$

For $(p_t, p_x, p_z)$ to represent a valid three momentum vector we require that,

$$p_t, p_x, p_y \in \mathbb{R} \implies (p_a^b)^* = p_a^b. \tag{2.66}$$

This reality condition can be satisfied if we take the spinors in the RHS of (2.64) be purely real viz,

$$\lambda_a^* = \lambda_a, \bar{\lambda}_a^* = \bar{\lambda}_a : \ \textbf{Lorentzian Reality Conditions}. \tag{2.67}$$

The description (2.64) has a redundancy. Rescaling the two spinors by opposite amounts leaves the momentum invariant. This is called the *stabilizer* or in physics parlance, the little group:

$$\lambda \to \frac{1}{r}\lambda, \bar{\lambda} \to r\bar{\lambda}, r \in \mathbb{R} \implies p_a^b \to p_a^b. \tag{2.68}$$

### 2.2.2 CPT in spinor variables

An important feature of many of the Lorentzian field theories we study is $CPT$ invariance. Lets see how $CPT$ acts on spinor variables. Since the correlators that we are interested in that are invariant under charge conjugation, we focus on parity and time-reversal.

#### Parity and time reversal in spinor helicity variables

We work in $\mathbb{R}^{2,1}$ with the Minkowski coordinates $(t, x, z)$. A parity transformation is a flip of one of the spatial directions say,

$$P : (t, x, y) \to (t, x, -y). \tag{2.69}$$

Time-reversal flips $t \to -t$ while keeping $x, y$ unchanged.

$$T : (t, x, y) \to (-t, x, y). \tag{2.70}$$

Lets translate this statements to the spinor helicity variables. Our starting point is,

$$p_a^b = \frac{1}{2}(\lambda_a \bar{\lambda}^b + \bar{\lambda}_a \lambda^b). \tag{2.71}$$

In matrix notation,

$$p_a^b = \begin{pmatrix} p_y & p_x - p_t \\ p_x + p_t & -p_y \end{pmatrix} = \begin{pmatrix} \frac{\lambda_1\bar{\lambda}_2 + \bar{\lambda}_1\lambda_2}{2} & -\lambda_1\bar{\lambda}_1 \\ \lambda_2\bar{\lambda}_2 & -\frac{\lambda_2\bar{\lambda}_1 + \bar{\lambda}_2\lambda_1}{2} \end{pmatrix}. \tag{2.72}$$

The subscripts $1, 2$ here refer to the spinor components. We shall now derive the action of $P$ and $T$ on the spinors $\lambda$ and $\bar{\lambda}$ by considering the action of parity and time-reversal on the momentum matrix .

**Parity**

The parity transformation (2.69) clearly will act on the momentum matrix by flipping $p_y \to -p_y$:

$$P : \begin{pmatrix} p_y & p_x - p_t \\ p_x + p_t & -p_y \end{pmatrix} \to \begin{pmatrix} -p_y & p_x - p_t \\ p_x + p_t & p_y \end{pmatrix} . \tag{2.73}$$

Using the expressions of each component of this matrix in terms of the spinors (2.72), we can write this transformation as follows:

$$P : \left\{\lambda_1\bar{\lambda}_2 + \bar{\lambda}_1\lambda_2 \to -\lambda_1\bar{\lambda}_2 - \bar{\lambda}_1\lambda_2, \lambda_1\bar{\lambda}_1 \to \lambda_1\bar{\lambda}_1, \lambda_2\bar{\lambda}_2 \to \lambda_2\bar{\lambda}_2, \lambda_2\bar{\lambda}_1 + \bar{\lambda}_2\lambda_1 \to -\lambda_2\bar{\lambda}_1 - \bar{\lambda}_2\lambda_1\right\}. \tag{2.74}$$

This can be implemented by,

$$P : (\lambda_1, \;\; \lambda_2) \to (-\bar{\lambda}_1, \;\; \bar{\lambda}_2) \, , (\bar{\lambda}_1, \;\; \bar{\lambda}_2) \to (-\lambda_1, \;\; \lambda_2) \, . \tag{2.75}$$

**Time-reversal**

The time-reversal transformation (2.70) flips $p_y \to -p_y$ and $p_x \to -p_x$. Also, it is important to remember that time-reversal is an anti-unitary operator and does not flip the time component of the momentum (energy).

$$T : \begin{pmatrix} p_z & p_x - p_t \\ p_x + p_t & -p_z \end{pmatrix} \to \begin{pmatrix} -p_z & -p_x - p_t \\ -p_x + p_t & p_z \end{pmatrix} . \tag{2.76}$$

Using the expressions of each component of this matrix in terms of the spinors (2.72), we can write this transformation as follows:

$$T : \left\{\lambda_1\bar{\lambda}_2 + \bar{\lambda}_1\lambda_2 \to -\lambda_1\bar{\lambda}_2 - \bar{\lambda}_1\lambda_2, \lambda_1\bar{\lambda}_1 \to \lambda_2\bar{\lambda}_2, \lambda_2\bar{\lambda}_2 \to \lambda_1\bar{\lambda}_1, \lambda_2\bar{\lambda}_1 + \bar{\lambda}_2\lambda_1 \to -\lambda_2\bar{\lambda}_1 - \bar{\lambda}_2\lambda_1\right\}, \tag{2.77}$$

which can be implemented via,

$$T : (\lambda_1, \;\; \lambda_2) \to (\lambda_2, \;\; -\lambda_1) \, , (\bar{\lambda}_1, \;\; \bar{\lambda}_2) \to (\bar{\lambda}_2, \;\; -\bar{\lambda}_1) \, . \tag{2.78}$$

**PT**

One can now show that under parity and time-reversal, the spinor brackets transform as,

$$\begin{aligned} &P\big(\langle\lambda\chi\rangle\big) = \langle\bar{\chi}\bar{\lambda}\rangle \, , \; P\big(\langle\bar{\lambda}\bar{\chi}\rangle\big) = \langle\chi\lambda\rangle \, , \; P\big(\langle\lambda\bar{\lambda}\rangle\big) = \langle\lambda\bar{\lambda}\rangle, \\ &T\big(\langle\lambda\chi\rangle\big) = \langle\lambda\chi\rangle \, , \; T\big(\langle\bar{\lambda}\bar{\chi}\rangle\big) = \langle\bar{\lambda}\bar{\chi}\rangle \, , \; T\big(\langle\lambda\bar{\lambda}\rangle\big) = \langle\lambda\bar{\lambda}\rangle. \end{aligned} \tag{2.79}$$

Using these results, we find,

$$PT\big(\langle\lambda\chi\rangle\big) = \langle\bar{\chi}\bar{\lambda}\rangle \, , \; PT\big(\langle\bar{\lambda}\bar{\chi}\rangle\big) = \langle\chi\lambda\rangle \, , \; PT\big(\langle\lambda\bar{\lambda}\rangle\big) = \langle\lambda\bar{\lambda}\rangle. \tag{2.80}$$

For correlation functions CPT takes for instance $(+++)$ to $(---)$ as follows:

$$\mathrm{CPT}(\langle J^+_{s_1} J^+_{s_2} J^+_{s_3}\rangle) = (-1)^{s_1+s_2+s_3}\langle J^-_{s_1} J^-_{s_2} J^-_{s_3}\rangle, \tag{2.81}$$

and similarly for other helicites and higher point functions. This concludes our discussion on how $PT$ acts on the spinor helicity variables.

### 2.2.3 Wightman functions

When working in Lorentzian spacetime, we are faced with the choice of which correlator to work with. In contrast to Euclidean signature where correlation functions are single valued and there is only one type of correlator (unsurprisingly called the Euclidean correlator), there are a plethora of options to work with in Lorentzian signature. The intuitive reason for this is as follows. In Euclidean signature, correlators are naturally "time" ordered. For example, consider an n-point correlator in Euclidean signature with a particular operator ordering:

$$\langle 0|O_1(\tau_1,\vec{x}_1)\cdots O_n(\tau_n,\vec{x}_n)|0\rangle = \langle 0|O_1(0,\vec{x}_1)e^{-H(\tau_1-\tau_2)}\cdots e^{-H(\tau_{n-1}-\tau_n)}O_n(0,\vec{x}_n)|0\rangle. \tag{2.82}$$

Since we work in theories with a Hamiltonian $H$ that is bounded below, this correlator only makes sense for $\tau_1 > \tau_2 > \cdots > \tau_n$ since otherwise, the correlator would diverge due to the exponential factors. In contrast, Lorentzian correlators can be out of time order. To see that consider,

$$\langle 0|O_1(t_1,\vec{x}_1)\cdots O_n(t_n,\vec{x}_n)|0\rangle = \langle 0|O_1(0,\vec{x}_1)e^{-iH(t_1-t_2)}\cdots e^{-iH(t_{n-1}-t_n)}O_n(0,\vec{x}_n)|0\rangle. \tag{2.83}$$

The exponentials are now oscillatory which implies that there is no forced ordering of the Lorentzian times. Therefore even if we take $t_1 > t_2 > \cdots t_n$ both $\langle 0|O_1(t_1,\vec{x}_1)O_2(t_2,\vec{x}_2)\cdots O_n(t_n,\vec{x}_n)|0\rangle$ and $\langle 0|O_2(t_2,\vec{x}_2)O_1(t_1,\vec{x}_1)\cdots O_n(t_n,\vec{x}_n)|0\rangle$ and the $n!$ other operator orderings are all valid correlators. These are what are known as Wightman functions that we will make more precise momentarily.

Usually, in QFT, we work in Lorentzian signature with time-ordered correlators due to their connection with scattering amplitudes via the LSZ prescription. However, since we are working in a CFT where the observables of interest are the correlators themselves, we will rather work with the simpler Wightman functions. If desired, time-ordered and other kinds of correlators can be obtained using Wightman functions.

$$\langle 0|T\{O_1(t_1,\vec{x}_1)\cdots O_n(t_n,\vec{x}_n)\}|0\rangle = \theta(t_1 > t_2 > \cdots > t_n)\langle 0|O_1(t_1,\vec{x}_1)\cdots O_n(t_n,\vec{x}_n)|0\rangle + \cdots. \tag{2.84}$$

To illustrate one advantage of working with Wightman functions, we note that those involving a conserved current are identically conserved rather than satisfy Ward-Takahashi identities with contact terms. For example, if $J^\mu$ is a conserved $U(1)$ current and $\phi$ and $\chi$ are oppositely charged scalars we still have[19],

$$\partial_{1\mu}\langle 0|J^\mu(x_1)\phi(x_2)\chi(x_3)|0\rangle = 0. \tag{2.85}$$

Given (2.85), we can show how the familiar current conservation Ward-Takahashi identity for a time ordered correlator defined via (2.84) arises:

$$\partial_{1\mu}\langle 0|T\{J^\mu(x_1)\phi(x_2)\chi(x_3)\}|0\rangle = (q_\phi\delta^d(x_1-x_2) + q_\chi\delta^d(x_1-x_3))\langle 0|T\{\phi(x_2)\chi(x_3)\}|0\rangle. \tag{2.86}$$

[19]As a consequence of this, Wightman functions in spinor helicity obey the homogeneous conformal Ward identity, that is, the right hand side of (2.14) is zero.

Further, in more mathematical parlance, Wightman functions are tempered distributions and thus have nice properties such as having a well defined Fourier transform etc... in contrast to time ordered correlators where the presence of the Heaviside theta functions enforcing operator ordering spoils the simple distributional properties. With this motivation to study Wightman functions, let us formalize their definitions and properties.

**Wightman Functions: Formal Definitions and Properties**

A Wightman function is an ordered expectation value in a particular state, which for us is the vacuum [79]. Let $O_i$ be generic operators with $i$ shorthand for possible spinor or vector indices. A Wightman function with a particular ordering is denoted as[20],

$$\langle 0|O_1(x_1)\cdots O_n(x_n)|0\rangle. \tag{2.87}$$

$|0\rangle$ is the Poincaré invariant vacuum and $x_i = (t_i, \vec{x}_i)$ labels the spacetime location of the operator. With $n$ operators, there are $n!$ such Wightman functions which in general are distinct. However, they are all equal when every operator is space-like separated by virtue of micro-causality. Wightman functions also come equipped with a particular $i\epsilon$ prescription. To see this, let us write (2.87) in the Heisenberg picture, placing all operators at the origin in space for simplicity. Writing the $i\epsilon$ dependence explicitly we have,

$$\langle 0|O_1(0)e^{-iH(t_1-t_2)-H(\epsilon_1-\epsilon_2)}O_2(0)\cdots e^{-iH(t_{n-1}-t_n)-H(\epsilon_{n-1}-\epsilon_n)}O_n(0)|0\rangle, \tag{2.88}$$

with $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n > 0$ that ensures that the contribution of high energy states to the correlator are damped. The limit $\epsilon_i \to 0$ keeping fixed their ordering is to be taken only after smearing the Wightman functions with Schwartz functions. As such, Wightman functions are actually distributions (tempered distributions to be precise) but we shall refer to them as functions as is common practice. This $i\epsilon$ prescription also has the consequence that Wightman functions are not quite real and obey the following reality property:

$$\langle 0|O_1(x_1)\cdots O_n(x_n)|0\rangle^* = \langle 0|O_n(x_n)\cdots O_1(x_1)|0\rangle, \tag{2.89}$$

that is, complex conjugating the correlator reverses the operator ordering.

Another important property of Wightman functions is that they satisfy the spectral condition. To see this, we need to go to momentum space. The fact that the Fourier transform of a tempered distribution is also a tempered distribution makes momentum space Wightman functions well defined. Poincaré invariance in Fourier space implies momentum conservation so we can strip off this universal factor to obtain,

$$\langle 0|O_1(p_1)\cdots O_n(p_n)|0\rangle = (2\pi)^d\delta^d(p_1+\cdots+p_n)\langle\langle 0|O_1(p_1)\cdots O_n(p_n)|0\rangle\rangle, \tag{2.90}$$

that is, single bra-ket quantities in momentum space $\langle 0|\cdots|0\rangle$ including the momentum conserving delta function whereas double bra-ket quantities $\langle\langle 0|\cdots|0\rangle\rangle$ indicated that

[20] We closely follow David Duffins' TASI lecture notes on Conformal Field Theory in Lorentzian Signature available at his Caltech home page.

this factor is stripped off. The spectral condition is the statement that Wightman function are non-zero iff,

$$\left(\sum_{i=j}^{n} p_i\right)^2 < 0, \quad \sum_{i=j}^{n} p_i^0 > 0, \quad j \in \{2, 3, \cdots, n-1, n\}, \tag{2.91}$$

that is, the sum of all the momenta of the operators acting on $|0\rangle$ should be time-like and have positive energy. For example, take $j = n$. This implies that $p_n^2 < 0, p_n^0 > 0$. Taking $j = 2$ yields $(p_2^\mu + \cdots p_n^\mu)^2 < 0, (p_2^0 + \cdots p_n^0) > 0$ which by momentum conservation implies that $p_1^2 < 0, p_1^0 < 0$. Similarly, taking $j = n-1$ implies $(p_n^\mu + p_{n-1}^\mu)^2 < 0, p_n^0 + p_{n-1}^0 > 0$ and so on. Thus, for every Wightman function, the rightmost and leftmost operators have to have time-like momenta with positive and negative energy respectively. The operators in the middle can in general have time-like or space-like momenta as long as (2.91) is satisfied[21]. Finally, the reality condition (2.89) in Fourier space takes the form,

$$\langle 0|O_1(p_1)\cdots O_n(p_n)|0\rangle^* = \langle 0|O_n(-p_n)\cdots O_1(-p_1)|0\rangle. \tag{2.92}$$

Wightman functions satisfy other properties such as cluster decomposition and positivity for which we refer the reader to [46, 80] for a more detailed discussion of the Wightman axioms in the context of CFT. We also elaborate on some of the properties satisfied by Wightman functions in appendix C.

**Helicity structure of Wightman functions**

We now consider an example of a Wightman function using the Lorentzian spinor helicity variables. We suppress the spectral theta functions below.

The non-homogeneous Euclidean correlators are non-zero only in the mixed helicity configurations, i.e, it is zero in the $(---)$ and $(+++)$ configurations. The homogeneous Euclidean correlator and the parity odd correlator which is its epsilon transform on the other hand are non-zero only when all helicities coincide. In contrast, all their Wightman

[21] For example, the middle operator in a three point function can in general have time-like or space-like momenta.

counterparts are non-zero in all eight helicity configurations. For example we find,

$$\begin{aligned}
\langle 0|T^-J^-J^-|0\rangle &= (c^{(h)}_{211} - ic^{odd}_{211})\frac{\langle 12\rangle^2\langle 31\rangle^2 p_1}{E^4} + 4c^{(nh)}_{211}\frac{\langle 12\rangle^2\langle 31\rangle^2 p_1}{(E-2p_3)^2(E-2p_2)^2},\\
\langle 0|T^-J^-J^+|0\rangle &= (c^{(h)}_{211} - ic^{odd}_{211})\frac{\langle 12\rangle^2\langle \bar{3}1\rangle^2 p_1}{(E-2p_3)^4} + 4c^{(nh)}_{211}\frac{\langle 12\rangle^2\langle \bar{3}1\rangle^2 p_1}{(E-2p_1)^2E^2},\\
\langle 0|T^-J^+J^-|0\rangle &= (c^{(h)}_{211} - ic^{odd}_{211})\frac{\langle 1\bar{2}\rangle^2\langle 31\rangle^2 p_1}{(E-2p_2)^4} + 4c^{(nh)}_{211}\frac{\langle 1\bar{2}\rangle^2\langle 31\rangle^2 p_1}{(E-2p_1)^2E^2},\\
\langle 0|T^+J^-J^-|0\rangle &= (c^{(h)}_{211} - ic^{odd}_{211})\frac{\langle \bar{1}2\rangle^2\langle 3\bar{1}\rangle^2 p_1}{(E-2p_1)^4} + 4c^{(nh)}_{211}\frac{\langle \bar{1}2\rangle^2\langle 3\bar{1}\rangle^2 p_1}{(E-2p_3)^2(E-2p_2)^2},\\
\langle 0|T^+J^+J^+|0\rangle &= (c^{(h)}_{211} + ic^{odd}_{211})\frac{\langle \bar{1}\bar{2}\rangle^2\langle \bar{3}\bar{1}\rangle^2 p_1}{E^4} + 4c^{(nh)}_{211}\frac{\langle \bar{1}\bar{2}\rangle^2\langle \bar{3}\bar{1}\rangle^2 p_1}{(E-2p_3)^2(E-2p_2)^2},\\
\langle 0|T^+J^+J^-|0\rangle &= (c^{(h)}_{211} + ic^{odd}_{211})\frac{\langle \bar{1}\bar{2}\rangle^2\langle 3\bar{1}\rangle^2 p_1}{(E-2p_3)^4} + 4c^{(nh)}_{211}\frac{\langle \bar{1}\bar{2}\rangle^2\langle 3\bar{1}\rangle^2 p_1}{(E-2p_1)^2E^2},\\
\langle 0|T^+J^-J^+|0\rangle &= (c^{(h)}_{211} + ic^{odd}_{211})\frac{\langle \bar{1}2\rangle^2\langle \bar{3}\bar{1}\rangle^2 p_1}{(E-2p_2)^4} + 4c^{(nh)}_{211}\frac{\langle \bar{1}2\rangle^2\langle \bar{3}\bar{1}\rangle^2 p_1}{(E-2p_1)^2E^2},\\
\langle 0|T^-J^+J^+|0\rangle &= (c^{(h)}_{211} + ic^{odd}_{211})\frac{\langle 1\bar{2}\rangle^2\langle \bar{3}1\rangle^2 p_1}{(E-2p_1)^4} + 4c^{(nh)}_{211}\frac{\langle 1\bar{2}\rangle^2\langle \bar{3}1\rangle^2 p_1}{(E-2p_2)^2(E-2p_3)^2}.
\end{aligned} \tag{2.93}$$

These solutions follow as a consequence of the conformal Ward identities. Note that we can go from one helicity to another by using the helicity flipping operator $(\lambda, \bar{\lambda}) \to (\bar{\lambda}, \lambda)$ for any of the currents. One can also check that Wick rotation of the Euclidean correlator results in the above expression thus validating its correctness.

### 2.2.4 Analytic continuation from Euclidean space

It is useful to understand how to compute the Wightman correlator given a Euclidean space correlator since the latter is naturally computed using the standard path integral for instance. Given a Euclidean correlator

$$\langle O_1(\tau_1, \vec{x}_1)\cdots O_n(\tau_n, \vec{x}_n)\rangle, \tag{2.94}$$

one can obtain a Wightman function as follows. Analytically continue the Euclidean times $\tau_i = it_i + \epsilon_i$. We now take $\epsilon_i \to 0$ keeping fixed the ordering $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n$ to maintain the Euclidean time ordering that serves to damp contributions from high energy states. This results in the Wightman function.

$$W(x_1, \cdots, x_2) = \langle O_1(t_1 + i\epsilon_1, \vec{x}_1)\cdots O_n(t_n + i\epsilon_n, \vec{x}_n)\rangle \,, \epsilon_1 > \epsilon_2 > \cdots > \epsilon_n. \tag{2.95}$$

The corresponding process in momentum space is more involved. Let us consider a Euclidean momentum space correlator,

$$\langle\langle O_1(p_{1E}, \vec{p}_1)\cdots O_n(p_{nE}, \vec{p}_n)\rangle\rangle = \sum_{\mathcal{I}} \mathcal{T}_{\mathcal{I}}\mathcal{F}_{\mathcal{I}}, \tag{2.96}$$

where $\mathcal{T}_{\mathcal{I}}$ denotes possible tensor structures and $\mathcal{F}_{\mathcal{I}}$ are form factors that depend on the independent invariants formed out of the momenta. For example $p_i = \sqrt{p_{iE}^2 + (\vec{p}_i)^2}$ is one possible class of invariants. $s_{ij} = \sqrt{(p_{iE} + p_{jE})^2 + (\vec{p}_i + \vec{p}_j)^2}$ is another. The Wick rotation we need to perform depends on whether the momentum we want to reach to is space-like or time-like. For the space-like case $p_i^2 > 0$, we do not require an $i\epsilon$ prescription and one can simply set $p_{iE} = -ip_i^0$ resulting in,

$$p_i = \sqrt{p_{iE}^2 + (\vec{p}_i)^2} \rightarrow \sqrt{-(p_i^0)^2 + (\vec{p}_i)^2} = \sqrt{|p_i|^2} = |p_i| \quad \text{if} \quad p_i^2 > 0. \tag{2.97}$$

On the other hand, we need to specify an $i\epsilon$ prescription for time-like momenta $p_i^2 < 0$. Let us see this for the Feynman and Wightman propagators where we have two distinct $i\epsilon$ prescriptions to consider. The Feynman $i\epsilon$ prescription is,

$$p_i = \sqrt{p_{iE}^2 + (\vec{p}_i)^2} \rightarrow \sqrt{-(p_i^0)^2 + (\vec{p}_i)^2 - i\epsilon} = -i\sqrt{(p_i^0)^2 - (\vec{p}_i)^2} = -i|p_i| \quad \text{if} \quad p_i^2 < 0. \tag{2.98}$$

The Wightman $i\epsilon$ prescription on the other hand is,

$$p_i = \sqrt{p_{iE}^2 + (\vec{p}_i)^2} \rightarrow \sqrt{-(p_i^0)^2 + (\vec{p}_i)^2 + i\epsilon p_i^0} = \begin{Bmatrix} e^{-\frac{i\pi}{2}}\sqrt{|p_i|^2} = -i|p_i|, & p_i^2 < 0, p_i^0 < 0 \\ e^{+\frac{i\pi}{2}}\sqrt{|p_i|^2} = i|p_i|, & p_i^2 < 0, p_i^0 > 0 \end{Bmatrix}. \tag{2.99}$$

Further, to obtain a Wightman function we need to take particular discontinuities of the Euclidean form factors.

Let us illustrate this with a three point example. The spectral conditions require that the first and third operators have time-like momenta which we take to be $p_1^2 < 0, p_3^2 < 0$ with $p_1^0 < 0, p_3^0 > 0$. For concreteness, we consider the middle operator to have space-like momenta $p_2^2 > 0$ although the middle operator can in general also have time-like momenta. Consider the Euclidean correlator,

$$\langle\langle O_1(p_{1E}, \vec{p}_1) O_2(p_{2E}, \vec{p}_2) O_3(p_{3E}, \vec{p}_3)\rangle\rangle = \sum_{\mathcal{I}} \mathcal{T}_{\mathcal{I}} \mathcal{F}_{\mathcal{I}}(p_1, p_2, p_3). \tag{2.100}$$

To obtain the Wightman function with the middle operator having space-like momenta we need to do the following. First, we take a discontinuity with respect to $p_1^2$ and $p_3^2$. Then, we Wick rotate the momenta following the Wightman $i\epsilon$ prescription (2.99). The result is,

$$\begin{aligned}&\langle\langle 0|O_1(p_1)O_2(p_2)O_3(p_3)|0\rangle\rangle\theta(p_2^2)\\ &= \theta(-p_1^2)\theta(-p_3^2)\theta(-p_1^0)\theta(p_3^0)\theta(p_2^2)\sum_{\mathcal{I}} \mathcal{T}_{\mathcal{I}} \mathrm{Disc}_{p_1^2}\mathrm{Disc}_{p_3^2}\mathcal{F}_{\mathcal{I}}(p_1, p_2, p_3)\big|_{p_1\rightarrow -i|p_1|, p_3\rightarrow i|p_3|, p_2\rightarrow |p_2|}.\end{aligned} \tag{2.101}$$

One can also first Wick rotate and then take the discontinuities, the order of these operations does not matter. Note that the tensor structures $\mathcal{T}_{\mathcal{I}}$ are at most analytic in $p_i^2$ so one does not need to take their discontinuity. The above procedure in these kinematics can be ascertained by seeing the structure of the three point scalar Wightman function [34] and generalizing it to arbitrary spin.

For the middle operator also having time-like momenta, the analytic continuation is more complicated. Even at the level of higher point functions, the analogous procedure increases in complexity if we have more and more time-like momenta and the general procedure is not quite known. However, in chapter 6, we will directly compute these real time Wightman functions holographically and derive the procedure for analytic continuation from Euclidean signature in that setting.

## 2.3 Towards Higher point functions in momentum space

### 2.3.1 Four point conformal partial waves in general dimensions

At the level of four and higher point functions, conformal invariance is no longer strong enough to fix the functional form of correlation functions. In position space for instance, four point functions are fixed up to functions of conformal cross ratios. For example consider a correlator of identical scalar operators. It takes the form,

$$\langle O_\Delta(x_!)\cdots O_\Delta(x_4)\rangle = \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}F(u,v), \tag{2.102}$$

where the cross ratios $u, v$ are given by,

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,,\, v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \tag{2.103}$$

$n$ point functions take a similar form but with a unknown function of $n(n-3)/2$ cross ratios[22]. In momentum space, the general expression for $n$ point scalar correlators in general dimensions was discovered recently and the result takes the form of a simplex integral [81–83]. For spinning correlators, the analogous expressions remain unknown. However, we are working with a conformal field theory which in particular has a convergent operator product expansion in position space[23],

$$O_i(x_i)O_j(x_j) = \sum_k f_{ijk}(x_{ij},\partial_j)O_k(x_j), \tag{2.104}$$

The $f_{ijk}$ encode the three point function coefficient. We can see this by multiplying the above equation with a third operator. The only term which is picked out on the RHS is when the very same operator appears in the OPE since only two point functions of identical operators are non-zero in CFT up to contact term contributions. We have also suppressed the possible vector or more generally spinor indices of all operators involved. Given this formula, one can in principle construct all higher point correlators

[22]There is a subtlety in this formula. For instance in $d = 1$, there is only one independent cross ratio at the level of four points as we shall discuss in more detail subsequently. In $d = 2$ at the level of five points we have 4 rather than 5 independent cross ratios. One has to account for these and related constraints in $d$ dimensions in writing down the precise formula. The above formula $n(n-3)/2$ is only an upper bound on the possible independent cross ratios.

[23]One can perform the OPE between two operators as long as they are well separated from any other operator.

from lower point ones. For example, performing an OPE between the first two and last two operators in a four point function we obtain,

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = \sum_{k,l} f_{12k}(x_{12},\partial_2)f_{34l}(x_{34}\partial_4)\langle O_k(x_2)O_l(x_4)\rangle. \tag{2.105}$$

However, in $d > 2$ it is famously known that every four and higher point correlator receives contributions from infinitely many (primary) operators. Therefore, this sum is always an infinite one and requires detailed knowledge of the spectrum of operators and their three point function coefficients. However, in the traditional conformal bootstrap program, we also supplement the OPE with what is known as crossing symmetry. Performing the OPE say between $1,2$ and $3,4$ should yield the same result as performing it between $1,3$ and $2,4$ in a common domain of convergence. In momentum space, the issues of convergence of OPE become more complicated since the Fourier transform instructs as to integrate over all positions, including coincident ones. However, what we can still do is look at the contribution of a single exchanged operator which is called a conformal block.

Consider a general four point function,

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle. \tag{2.106}$$

Lets insert a complete set of states in the middle. In a CFT, the operator spectrum of the theory serves this purpose thanks to the operator state correspondence [8, 84]:

$$\sum_{\alpha\in O,P^\mu O,\cdots}\langle O_1(x_1)O_2(x_2)|\alpha\rangle\langle\alpha|O_3(x_3)O_4(x_4)\rangle, \tag{2.107}$$

where $\alpha$ runs over the spectrum of primary operators and their descendants in the theory. Now, what is this complete set? Well, the projector has to be dimensionless and conformally invariant. This heavily restricts the form it can take and leads us to an ansatz,

$$|\alpha\rangle\langle\alpha| = \int d^dx O_{\mu_1\cdots\mu_s}(x)|0\rangle\langle 0|\tilde{O}^{\mu_1\cdots\mu_s}(x) \tag{2.108}$$

$\tilde{O}$ is called the shadow of $O$. It has scaling dimension $d-\Delta$ and the same spin as $O$ to ensure that the integrand is invariant under Lorentz and scale transformations[24]. For scalar operators, we have,

$$\tilde{O}(x) = \int \frac{d^dy}{|x-y|^{2(d-\Delta)}}O(y). \tag{2.109}$$

The contribution to the correlator due to the exchange of a single operator with scaling dimension $\Delta$ and spin $s$ is called a conformal block. This is equivalent to (with a caveat [85])),

$$G_{\Delta,s} = \int d^dx\langle O_1(x_1)O_2(x_2)O^{\mu_1\cdots\mu_s}(x)\rangle\langle\tilde{O}_{\mu_1\cdots\mu_s}(x)O_3(x_3)O_4(x_4)\rangle. \tag{2.110}$$

[24] This discussion is valid in general dimensions so we re-instate the dimension $d$. However, when we specialize to spinor helicity variables we will set $d = 3$.

The caveat is that the above expression is democratic in the operator and its shadow so one has to do some more work to extract the contribution just due to the operator. This is through a Monodromy projection which roughly speaking yields the conformal block with short distance behavior consistent with the exchange of the operator and not its shadow. Let us convert the above CPW to momentum space. Lets first define the shadow of a general spinning operator. By conformal invariance and dimensional analysis we see that it takes the form [86],

$$\tilde{O}_{\mu_1\cdots\mu_s}(x) \propto \int d^d y \frac{1}{|x-y|^{2(d-\Delta)}} \mathcal{I}_{\mu_1\cdots\mu_s;\nu_1\cdots\nu_s}(x-y) O^{\nu_1\cdots\nu_s}(y). \tag{2.111}$$

$\mathcal{I}$ is called the Inversion multi-tensor. It is made out of the following object simply called an inversion tensor[25]:

$$I_{\mu\nu}(x) = (\delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}). \tag{2.112}$$

For conserved currents, the shadow transform (2.111) drastically simplifies! The result after converting to momentum space is given by (ignoring overall proportionality constants),

$$\tilde{J}_s^{\mu_1\cdots\mu_s}(p) = \frac{1}{p^{2s+d-4}} \Pi^{\mu_1\cdots\mu_s;\nu_1\cdots\nu_s}(p) J_{s\nu_1\cdots\nu_s}(p). \tag{2.113}$$

$\Pi^{\mu_1\cdots\mu_s}$ is a transverse traceless projector. It is traceless with respect to any pair of indices and transverse with respect to every index as is appropriate for a conserved current. For example for a spin-1 current we have the two indexed object (there is no tracelessness for a spin-1 object)

$$\pi^{\mu\nu}(p) = \delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}. \tag{2.114}$$

For spin-2 we have,

$$\Pi^{\mu\nu;\rho\sigma}(p) = \pi^{\mu\nu}(p)\pi^{\rho\sigma}(p) - (\pi^{\mu\rho}(p)\pi^{\nu\sigma}(p) + \pi^{\mu\sigma}(p)\pi^{\mu\rho}(p)), \tag{2.115}$$

and so on for higher spins. Lets now use this. Now, to get the CPW (2.110) in momentum space, we perform a simple Fourier transform yielding,

$$G_{\Delta,s} = \int d^d p \langle O_1(p_1) O_2(p_2) O^{\mu_1\cdots\mu_s}(p)\rangle \langle \tilde{O}_{\mu_1\cdots\mu_s}(-p) O(p_3) O(p_4)\rangle. \tag{2.116}$$

A general four point function can be constructed using these building blocks.

[25]This is the Jacobian of the transformation $x^\mu \to \frac{x^\mu}{x^2}$ which is a conformal transformation, although one disconnected from the identity transformation.

### 2.3.2 Four point spinor helicity conformal partial waves in $d = 3$

We now consider the spinor helicity versions of the CPWs. In spinor helicity, it is easy to show that the following simple formulae for the shadow currents (we also set $d = 3$),

$$\tilde{J}_s^{\pm}(p) = \frac{1}{p^{2s-1}} J_s^{\pm}(p). \tag{2.117}$$

Lets now use this. First, we need to get the CPW (2.110) in momentum space. By a simple Fourier transform we get,

$$G_{\Delta,s} = \int d^3p \langle O_1(p_1)O_2(p_2)O^{\mu_1\cdots\mu_s}(p)\rangle\langle\tilde{O}_{\mu_1\cdots\mu_s}(-p)O(p_3)O(p_4)\rangle. \tag{2.118}$$

Specializing to exchanges of conserved currents and using the helicity basis we can use the formula (2.117) to obtain,

$$J^{\mu_1\cdots\mu_s}(p)\rangle\langle\tilde{J}_{\mu_1\cdots\mu_s}(-p) = \frac{1}{p^{2s-1}}(J_s^-(p)\rangle\langle\tilde{J}_s^+(-p) + J_s^+(p)\rangle\langle\tilde{J}_s^-(-p)). \tag{2.119}$$

Given $p^{ab} = \lambda^{(a}\bar{\lambda}^{b)}$, we can obtain the spinor decomposition of $-p^{ab}$ simply by taking $\bar{\lambda} \to -\bar{\lambda}$. Using the formula for the shadow current and using the above in the CPW, we obtain (suppressing overall numerical constants),

$$\begin{aligned} G_{\Delta,s} = \int \frac{d^3p}{p^{2s-1}}\Big(&\langle O_1(p_1)O_2(p_2)J_s^-(p)\rangle\langle J_s^+(-p)O_3(p_3)O_4(p_4)\rangle \\ &+ \langle O_1(p_1)O_2(p_2)J_s^+(p)\rangle\langle J_s^-(-p)O_3(p_3)O_4(p_4)\rangle\Big) \end{aligned} \tag{2.120}$$

However, recall that three point functions in momentum space come with a momentum conserving delta function and therefore the above integral is trivial! The result is,

$$\begin{aligned} G_{\Delta,s} = (2\pi)^3\delta^3(p_1+p_2+p_3+p_4)\frac{1}{|p_1+p_2|^{2s-1}}\Big(&\langle O_1(p_1)O_2(p_2)J_s^-(-p_1-p_2)\rangle\langle J_s^+(p_1+p_2)O_3(p_3)O_4(p_4)\rangle \\ &+ \langle O_1(p_1)O_2(p_2)J_s^+(-p_1-p_2)\rangle\langle J_s^-(p_1+p_2)O_3(p_3)O_4(p_4)\rangle\Big) \end{aligned} \tag{2.121}$$

The external operators $O_i(p_i)$ can be any operators including conserved currents but is not limited to the same. They can be scalars or even generic non conserved currents. Also, this is in the s channel with (12) and (34) paired together which is the $s-$channel. One can repeat a similar analysis in the $t$ and $u$ channels if desired. The three point functions in the above formulae can be either Euclidean correlators or Wightman functions. We shall re-discover Wightman conformal partial waves in chapter 6 naturally emerging in a region of kinematic space of $AdS_4$ boundary Wightman functions.

As we have seen, there is much progress to be made in understanding the general structure of four and higher point conformal correlators in momentum space and consequently, in spinor helicity variables. However, in the special case of one dimensional CFTs viz conformal quantum mechanics where the technical complexity is more tractable, we can actually determine the general structure of even $n-$point functions. Inside every $d-$dimensional CFT is a $SL(2,\mathbb{R})$ subgroup which corresponds to just focusing on correlation functions on a line in the $d-$dimensional space(time) and thus any results for 1d CFT form a subspace of results in d-dimensional CFTs. Further, it is of independent interest by itself as well could yield insights into how to make progress in higher dimensions. This will form the subject of the last part of this chapter.

## 2.4 Conformal Quantum Mechanics

We now specialize to the special and simplest case of conformal field theory viz conformal quantum mechanics. It will illustrate the differences between the usual position space analysis and the momentum space approach in a technically simpler setting. Compared to its higher dimensional counterparts, it provides us a path with fewer technical hurdles to obtain exact analytical results such as closed form expressions of four and higher point point functions. Historically, the study of conformal quantum mechanics dates back to 1976, with the work of de Alfaro, Fubini and Furlan (DFF) [87] who analyzed a particular quantum mechanical model (A particle moving in a inverse square potential) that possesses conformal invariance. Almost five decades after this paper, there has been a great deal of development with applications ranging from the connection of CQM to M2 branes, black holes and even to molecular physics [88–105]. A major motivation to study and bootstrap CQM stems from the AdS/CFT correspondence [7]. $\text{AdS}_2/\text{CFT}_1$ is often referred to as the runt of the correspondence as $\text{CFT}_1$ does not possess a local stress tensor that generates conformal transformations which is in sharp contrast to its higher dimensional counterparts. In the context of the DFF model, this correspondence and its subtleties were first discussed and explored in [106]. Another extremely important case is that of the SYK models and their holographic bulk duals [107–109]. To investigate such dualities, having a firm foothold on the conformal theory side of things is essential. Further, as a bonus of our analysis, the conformal Ward identities in conformal quantum mechanics coincide with those in the (anti-)holomorphic sector of two dimensional CFT [13], thereby giving a possibility of our results being applicable there too.

### 2.4.1 The generalized hypergeometric nature of $n-$point functions

In $d = 1$, the conformal group is $SO(2,1)$ which is homomorphic to $SL(2,\mathbb{R})$. Since there is no spin in one dimension, primary operators are labelled just by their scaling dimension. The action of the conformal generators are,

$$\begin{aligned}
[H, O_\Delta(t)] &= i\frac{d}{dt}O_\Delta(t),\\
[D, O_\Delta(t)] &= i\left(t\frac{d}{dt}+\Delta\right)O_\Delta(t),\\
[K, O_\Delta(t)] &= i\left(t^2\frac{d}{dt}+2t\Delta\right)O_\Delta(t).
\end{aligned} \tag{2.122}$$

In momentum space (frequency space), we obtain,

$$\begin{aligned} [H, \mathcal{O}_\Delta(\omega)] &= \omega \mathcal{O}_\Delta(\omega), \\ [D, \mathcal{O}_\Delta(\omega)] &= -i\left(\omega \frac{d}{d\omega} + (1-\Delta)\right)\mathcal{O}_\Delta(\omega), \\ [K, \mathcal{O}_\Delta(\omega)] &= -\left(\omega \frac{d^2}{d\omega^2} + 2(1-\Delta)\frac{d}{d\omega}\right)\mathcal{O}_\Delta(\omega). \end{aligned} \tag{2.123}$$

We thus obtain the following Ward identities, due to $H, D$ and $K$ respectively:

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial}{\partial t_i} f_n(t_1, \cdots, t_n) &= 0, \\ \sum_{i=1}^{n}\left(t_i \frac{\partial}{\partial t_i} + \Delta_i\right) f_n(t_1, \cdots, t_n) &= 0, \\ \sum_{i=1}^{n}\left(t_i^2 \frac{\partial}{\partial t_i} + 2t_i\Delta_i\right) f_n(t_1, \cdots, t_n) &= 0. \end{aligned} \tag{2.124}$$

The Fourier space counterparts of above equations are,

$$\sum_{i=1}^{n} \omega_i f_n(\omega_1, \cdots, \omega_n) = 0, \tag{2.125}$$

$$\sum_{i=1}^{n}\left(\omega_i \frac{\partial}{\partial \omega_i} + (1-\Delta_i)\right) f_n(\omega_1, \cdots, \omega_n) = 0, \tag{2.126}$$

$$\sum_{i=1}^{n}\left(\omega_i \frac{\partial^2}{\partial \omega_i^2} + 2(1-\Delta_i)\frac{\partial}{\partial \omega_i}\right) f_n(\omega_1, \cdots, \omega_n) = 0. \tag{2.127}$$

With the conformal ward identities, (2.124), and (2.125), (2.126), (2.127) in hand, we now proceed to solve them now.

### The time domain

In position space (which is just the temporal domain) we get,

$$\begin{aligned} \langle \mathcal{O}_{\Delta_1}(t_1)\mathcal{O}_{\Delta_2}(t_2)\rangle &= \frac{c_{12}\delta_{\Delta_1,\Delta_2}}{|t_1-t_2|^{2\Delta}}, \\ \langle \mathcal{O}_{\Delta_1}(t_1)\mathcal{O}_{\Delta_2}(t_2)\mathcal{O}_{\Delta_3}(t_3)\rangle &= \frac{f_{123}}{|t_1-t_2|^{\Delta_1+\Delta_2-\Delta_3}|t_2-t_3|^{\Delta_2+\Delta_3-\Delta_1}|t_1-t_3|^{\Delta_1+\Delta_3-\Delta_2}}, \\ \langle \mathcal{O}_{\Delta_1}(t_1)\mathcal{O}_{\Delta_2}(t_2)\mathcal{O}_{\Delta_3}(t_3)\mathcal{O}_{\Delta_4}(t_4)\rangle &= \prod_{1\le i<j\le 4}(|t_i-t_j|)^{\frac{\Delta_1+\Delta_2+\Delta_3+\Delta_4}{3}-\Delta_i-\Delta_j}\, G(\chi), \\ \langle \mathcal{O}_{\Delta_1}(t_1)\cdots\mathcal{O}_{\Delta_n}(t_n)\rangle &= \prod_{1\le i<j\le n}(|t_i-t_j|)^{2\alpha_{ij}} G_n(\chi_1, \cdots \chi_{n-3}), \end{aligned} \tag{2.128}$$

where, $\chi = \dfrac{|t_1 - t_2||t_3 - t_4|}{|t_1 - t_3||t_2 - t_4|}$ is the four point cross ratio, $\chi_1, \cdots \chi_{n-3}$ are the cross ratios for $n$-point functions which take the form $\dfrac{|t_i - t_j||t_k - t_l|}{|t_i - t_k||t_j - t_l|}$, $i, j, k, l \in \{1, \cdots n\}$ and the $\alpha_{ij}$ satisfy $\Delta_i = -\sum_{j=1}^{n} \alpha_{ij}$ , $i \in \{1, \cdots, n\}$. An artefact of the fact that we are in one dimension is that we have only a single (real) cross ratio at the level of four points in contrast to the case in higher dimensions. For $n$ point functions, we have only $n - 3$ cross ratios in contrast to the $\frac{n(n-3)}{2}$ cross ratios in sufficiently high dimensions.

### The frequency domain

We can solve the frequency space conformal Ward identities similarly. For two point functions, we obtain the solution,

$$\langle O_{\Delta_1}(\omega_1) O_{\Delta_2}(\omega_2) \rangle = \tilde{C}_{12} \delta_{\Delta_1, \Delta_2} \omega_1^{2\Delta_1 - 1} \delta(\omega_1 + \omega_2). \tag{2.129}$$

For three point functions on the other hand, we obtain two distinct solutions:

$$\begin{aligned}\langle O_{\Delta_1}(\omega_1) O_{\Delta_2}(\omega_2) O_{\Delta_3}(\omega_3) \rangle = \omega_1^{\Delta_1 + \Delta_2 + \Delta_3 - 2} \Big[ & c_1 \, {}_2F_1(2 - \Delta_t, 1 + 2\Delta_1 - \Delta_t, 2 - 2\Delta_2; -x) \\ & + c_2 \, x^{2\Delta_2 - 1} \, {}_2F_1(1 + 2\Delta_2 - \Delta_t, \Delta_t - 2\Delta_3, 2\Delta_2; -x) \Big] \delta(\omega_1 + \omega_2 + \omega_3).\end{aligned} \tag{2.130}$$

In contrast to the unique three point correlator in the time domain provided in (2.128), we have here, two linearly independent solutions. These different solutions can be traced back to the Fourier transform of the time domain correlator for different time orderings of the arguments. For example, if we take a $\Delta = -1$ operator in position space, we obtain via direct Fourier transform the results of table 2.1. Our general solution for three point functions (2.130) for $\Delta_1 = \Delta_2 = \Delta_3 = -1$ (recall that our operator $O$ has scaling dimension $-1$) is given by:

$$\tilde{G}(\omega_1, \omega_2) = \omega_1^{-5} \left[ c_1 \frac{\omega_1^2 (2\omega_1 + \omega_2)}{2(\omega_1 + \omega_2)^3} + c_2 \frac{\omega_1^2 (\omega_1 - \omega_2)}{\omega_2^3} \right] \tag{2.131}$$

$$= c_1 \frac{2\omega_1 + \omega_2}{\omega_1^3 (\omega_1 + \omega_2)^3} + c_2 \frac{(\omega_1 - \omega_2)}{\omega_1^3 \omega_2^3}. \tag{2.132}$$

The expression (2.131) covers all the cases we find in table 2.1 by appropriately choosing the constants $c_1$ and $c_2$. The true meaning of the existence of the two independent solutions in (2.130) is now clear. They correspond to the fact that the different "time orderings" give rise to two possible Fourier space expressions. A similar interpretation in higher dimensions was made in [110].

We now repeat the same drill that we carried out in the analysis of two and three point functions for the four point case.

| **Time Ordering** | **Correlator** |
|---|---|
| $t_1 > t_2 > t_3$ | $\frac{2i(2\omega_1+\omega_2)}{\omega_1^3(\omega_1+\omega_2)^3}$ |
| $t_2 > t_1 > t_3$ | $-\frac{2i(\omega_1+2\omega_2)}{\omega_2^3(\omega_1+\omega_2)^3}$ |
| $t_3 > t_2 > t_1$ | $\frac{2i(2\omega_1+\omega_2)}{\omega_1^3(\omega_1+\omega_2)^3}$ |
| $t_2 > t_3 > t_1$ | $\frac{2i(\omega_1-\omega_2)}{\omega_1^3\omega_2^3}$ |
| $t_1 > t_3 > t_2$ | $\frac{2i(\omega_1-\omega_2)}{\omega_1^3\omega_2^3}$ |
| $t_3 > t_1 > t_2$ | $-\frac{2i(\omega_1+2\omega_2)}{\omega_2^3(\omega_1+\omega_2)^3}$ |

Table 2.1: Correlators in momentum space obtained via Fourier Transform with different time orderings

Translation invariance (2.125) instills the following form for the correlator:

$$\langle O_{\Delta_1}(\omega_1)O_{\Delta_2}(\omega_2)O_{\Delta_3}(\omega_3)O_{\Delta_4}(\omega_4)\rangle = \delta(\omega_1+\omega_2+\omega_3+\omega_4)\tilde{G}(\omega_1,\omega_2,\omega_3). \tag{2.133}$$

The dilatation ward identity (2.126) yields,

$$\left(\omega_1\frac{\partial}{\partial\omega_1}+\omega_2\frac{\partial}{\partial\omega_2}+\omega_3\frac{\partial}{\partial\omega_3}+(3-\Delta_t)\right)\tilde{G}(\omega_1,\omega_2,\omega_3)=0$$
$$\implies \tilde{G}(\omega_1,\omega_2,\omega_3)=\omega_1^{\Delta_t-3}\tilde{g}(x,y), \tag{2.134}$$

where we have defined $\Delta_t = \Delta_1+\Delta_2+\Delta_3+\Delta_4$ and the ratios $x = \frac{\omega_2}{\omega_1}$ and $y = \frac{\omega_3}{\omega_1}$.

The ward identity due to special conformal transformations (2.127) gives us the following partial differential that the function $\tilde{g}(x,y)$ has to satisfy:

$$\Big(x(x+1)\frac{\partial^2}{\partial x^2}+y(y+1)\frac{\partial^2}{\partial y^2}+2xy\frac{\partial^2}{\partial x\partial y}-2\big(\Delta_2-1+x(\Delta_t-\Delta_1-3)\big)\frac{\partial}{\partial x}$$
$$-2\big(\Delta_3-1+y(\Delta_t-\Delta_1-3)\big)\frac{\partial}{\partial y}+(\Delta_t-3)(\Delta_t-2\Delta_1-2)\Big)\tilde{g}(x,y)=0. \tag{2.135}$$

Very interestingly, (2.135) is an equation obeyed by none other than Appell's generalized hypergeometric function $F_2$ [111]! The system of differential equations that

$F_2(a, b_1, b_2, c_1, c_2; x, y)$ obeys is the following:

$$x(1-x)\frac{\partial^2 F_2}{\partial x^2} - xy\frac{\partial^2 F_2}{\partial x \partial y} + (c_1 - (a+b_1+1)x)\frac{\partial F_2}{\partial x} - b_1 y\frac{\partial F_2}{\partial y} - ab_1 F_2 = 0,$$
$$y(1-y)\frac{\partial^2 F_2}{\partial y^2} - xy\frac{\partial^2 F_2}{\partial x \partial y} + (c_2 - (a+b_2+1)y)\frac{\partial F_2}{\partial y} - b_2 x\frac{\partial F_2}{\partial x} - ab_2 F_2 = 0. \tag{2.136}$$

If we add these two equations we obtain,

$$\Bigg(x(1-x)\frac{\partial^2 F_2}{\partial x^2} + (c_1 - (a+b_1+b_2+1)x)\frac{\partial F_2}{\partial x} + y(1-y)\frac{\partial^2 F_2}{\partial y^2}$$
$$+ (c_2 - (a+b_1+b_2+1)y)\frac{\partial F_2}{\partial y} - 2xy\frac{\partial^2 F_2}{\partial x \partial y} - a(b_1+b_2)F_2\Bigg) = 0. \tag{2.137}$$

Let us now perform the following re-labeling and mapping:

$$x \to -x, \quad y \to -y, \qquad c_1 = 2(1-\Delta_2), \quad c_2 = 2(1-\Delta_3),$$
$$\left\{a = (3-\Delta_t),\ \sum_{i=1}^{2} b_i = 2 + 2\Delta_1 - \Delta_t\right\} \text{ or } \left\{a = 2 + 2\Delta_1 - \Delta_t,\ \sum_{i=1}^{2} b_i = 3 - \Delta_t\right\}. \tag{2.138}$$

Thus, (2.137) becomes identical to our equation for the four point function (2.135)! We see that (2.138) fixes $a, c_1, c_2', b_1 + b_2$ in terms of the scaling dimensions of the external operators. The key point here is that the combination $b_1 + b_2$ is fixed in terms of the external operator scaling dimensions but not $b_1$ and $b_2$ individually. If we fix $b_2$ in terms of $b_1$ using (2.138), then $b_1$ is left completely undetermined. Thus, our solution to the four-point function is[26],

$$\langle O_{\Delta_1}(\omega_1)O_{\Delta_2}(\omega_2)O_{\Delta_3}(\omega_3)O_{\Delta_4}(\omega_4)\rangle = \sum_{b_1} \omega_1^{\Delta_t - 3}\delta(\omega_1+\omega_2+\omega_3+\omega_4)\Big[k_1 F_2(a, b_1, b_2, c_1, c_2; -x, -y)$$
$$+ k_2(-x)^{1-c_1}F_2(a-c_1+1, b_1-c_1+1, b_2, 2-c_1, c_2; -x, -y)$$
$$+ k_3(-y)^{1-c_2}F_2(a-c_2+1, b_1, b_2-c_2+1, c_1, 2-c_2; -x, -y)$$
$$+ k_4(-x)^{1-c_1}(-y)^{1-c_2}F_2(a-c_1-c_2+2, b_1-c_1+1, b_2-c_2+1, 2-c_1, 2-c_2; -x, -y)\Big], \tag{2.139}$$

with the parameters given in (2.138). Thus, we see that cross ratios in the time domain are traded for hypergeometric parameters in frequency space.

One can actually proceed to $n-$points this way. solving the translation and dilatation Ward identities ((2.125) and (2.126) respectively) we see that any $n$ point correlator takes the following form:

$$\langle O_{\Delta_1}(\omega_1)\cdots O_{\Delta_n}(\omega_n)\rangle = \delta(\omega_1+\cdots+\omega_n)\omega_1^{\Delta_t-(n-1)}\tilde{g}(y_1, \cdots, y_{n-2}), \tag{2.140}$$

[26]There are four linearly independent solutions to Appell's differential equation for $F_2$. Since $b_1$ is not determined via conformal invariance, we allow generic four-point functions to be a linear combination of solutions with different values of $b_1$. Even more generally, one could allow for an integral over $b_1$ rather than a sum. The constants $k_1, \cdots, k_4$ can depend on $b_1$ but for the sake of brevity, we suppress this dependence.

where we have defined,

$$\Delta_t = \sum_{i=1}^{n} \Delta_i \; , \; y_i = \frac{\omega_{i+1}}{\omega_1}. \tag{2.141}$$

The special conformal Ward identity (2.127) implies that the function $\tilde{g}(y_1, \cdots, y_{n-2})$ satisfies the following differential equation:

$$\Bigg(\sum_{i=1}^{n-2} y_i(y_i+1)\frac{\partial^2}{\partial y_i^2} + 2\sum_{1\le j<i\le n-2} y_i y_j \frac{\partial^2}{\partial y_i \partial y_j} - 2\sum_{i=1}^{n-2}\big(\Delta_{i+1} - 1 + y_i(\Delta_t - \Delta_1 - n + 1)\big)\frac{\partial}{\partial y_i} + (\Delta_t - n + 1)(\Delta_t - 2\Delta_1 - n + 2)\Bigg)\tilde{g}(y_1, \cdots, y_{n-2}) = 0. \tag{2.142}$$

The generalized Lauricella function of $n-2$ variables, $E_A^{(n-2)}(a, b_1, \cdots, b_{n-2}, c_1, \cdots, c_{n-2}; x_1, \cdots, x_{n-2})$ obeys the following system of PDEs [112]:

$$\mathcal{L}_i^{(n-2)} E_A^{(n-2)}(a, b_1, \cdots, b_{n-2}, c_1, \cdots, c_{n-2}; x_1, \cdots, x_{n-2}) = 0, \; i \in \{1, \cdots, n-2\}, \text{ where,}$$
$$\mathcal{L}_i^{(n-2)} = x_i(1-x_i)\frac{\partial^2}{\partial x_i^2} - x_i\sum_{j\neq i}^{n-2} x_j \frac{\partial^2}{\partial x_i \partial x_j} + (c_i - (a + b_i + 1)x_i)\frac{\partial}{\partial x_i} - b_i \sum_{j\neq i}^{n-2} x_j \frac{\partial}{\partial x_j}. \tag{2.143}$$

This obviously implies,

$$\sum_{i=1}^{n-2} \mathcal{L}_i^{(n-2)} E_A^{(n-2)}(a, b_1, \cdots, b_{n-2}, c_1, \cdots, c_{n-2}; x_1, \cdots, x_{n-2}) = 0. \tag{2.144}$$

If we choose,

$$x_i = -y_i, \;\; c_i = 2(1-\Delta_{i+1}),$$
$$\Big\{a = n - 1 - \Delta_t, \; \sum_{i=1}^{n-2} b_i = n - 2 + 2\Delta_1 - \Delta_t\Big\} \text{ or } \Big\{a = n - 2 + 2\Delta_1 - \Delta_t, \; \sum_{i=1}^{n-2} b_i = n - 1 - \Delta_t\Big\}. \tag{2.145}$$

Thus, (2.144) becomes identical to the n point conformal Ward identity (2.142)! The map (2.145) fixes $a$ and the $n-2$ $c_i$ but leaves $n-3$ out of the $n-2$ $b_i$ undetermined. This is exactly the number of independent conformal cross ratios in the time domain and thus we have obtained their momentum space analogues for any n point correlation function. Our result for the $n$ point function reads,

$$\langle O_{\Delta_1}(\omega_1) \cdots O_{\Delta_n}(\omega_n)\rangle = \delta(\omega_1 + \cdots + \omega_n)\omega_1^{\Delta_t-(n-1)}\Big(k_1 E_A^{(n-2)}(a, b_1, \cdots, b_{n-2}, c_1, \cdots, c_{n-2}; -x_1, \cdots, -x_{n-2}) + \cdots\Big), \tag{2.146}$$

where the parameters are given in (2.145) and the $\cdots$ stand for the remaining $2^{n-2}$ linearly independent solutions to the Lauricella $E_A^{(n-2)}$ PDEs[27]. As in the previous cases, we have suppressed a possible sum over the $b_i$ that are not determined by conformal invariance. The result (2.146) is quite pleasing and presents the first closed-form expression for a conformal $n$-point function in momentum space with analogos of cross ratios that are not variables that are integrated over. It would be interesting to see if higher dimensional conformal correlators can also be determined in terms of appropriate generalized hypergeometric functions.

### 2.4.2 A Momentum-Mellin Space representation

Motivated by the (generalized) hypergeometric structure of our correlators, we investigate their Mellin space representation. Consider the four point function that we obtained (2.139). For simplicity we focus on the first term in the expression (2.139)[28]. For the convenience of the reader we provide it here[29]:

$$\tilde{G}_4(\omega_1,\omega_2,\omega_3,\omega_4) = \sum_{b_1} k_1(b_1)\omega_1^{\Delta_t-3} F_2(a,b_1,b_2,c_1,c_2;-x,-y). \tag{2.147}$$

where,

$$c_1 = 2(1-\Delta_2),\ \ c_2 = 2(1-\Delta_3),\ \ \Big\{a = 2+2\Delta_1-\Delta_t,\ \sum_{i=1}^{2} b_i = 3-\Delta_t\Big\},\ \ \Delta_t = \Delta_1+\Delta_2+\Delta_3+\Delta_4, \tag{2.148}$$

and $k_1(b_1)$ are constants that weigh the contribution of a particular $b_1$ to the correlator. Using a Mellin-Barnes type integral representation of $F_2(a,b_1,b_2,c_1,c_2;-x,-y)$ (for instance, see section 5.8.3 in [115]), we obtain the following beautiful form for the correlator (2.147):

$$\begin{aligned}&\tilde{G}_4(\omega_1,\omega_2,\omega_3,\omega_4)\\ &= \frac{\omega_1^{\Delta_t-3}\Gamma(2-2\Delta_2)\Gamma(2-2\Delta_3)}{\Gamma(3-\Delta_t)}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\int_{-i\infty}^{i\infty}\frac{dt}{2\pi i}\frac{\Gamma(s+t+2+2\Delta_1-\Delta_t)\Gamma(-s)\Gamma(-t)}{\Gamma(s+2-2\Delta_2)\Gamma(t+2-2\Delta_3)}x^s y^t \mathcal{M}(s,t),\end{aligned} \tag{2.149}$$

where we have defined the momentum-Mellin space amplitude,

$$\mathcal{M}(s,t) = \sum_{b_1} k_1(b_1)\mathcal{M}_{b_1}(s,t) = \sum_{b_1} k_1(b_1)\frac{\Gamma(b_1+s)\Gamma(t+3-\Delta_t-b_1)}{\Gamma(b_1)\Gamma(3-\Delta_t-b_1)}. \tag{2.150}$$

where $\mathcal{M}_{b_1}(s,t)$ are momentum-Mellin space partial amplitudes which represent the contribution of a particular $b_1$ to the full amplitude. The pre-factors in (2.149) are fixed

[27] It is interesting to note that certain Lauricella functions also pop out as solutions of conformal integrals [113] as well as in solutions to the conformal Ward identity in special kinematics [114].

[28] As will be explained shortly, the different solutions represent the Fourier space counterparts of the in-equivalent time orderings.

[29] We suppress the momentum conserving delta function $\delta(\omega_1+\omega_2+\omega_3+\omega_4)$ for brevity.

by the external dimensions of the operators (kinematics) whereas $\mathcal{M}(s,t)$ represents the dynamical contributions to the correlator. Note the contrast to the usual real space Mellin amplitudes [116, 117]: The usual Mellin amplitude is an *unknown* (theory dependent) function of $s$ and $t$. Here in (2.150), the theory dependence just lies in the constants $k_1(b_1)$ which weighs the contribution of the different $b_1$ to the correlator. The functional form of the amplitude, however, is completely determined by the above ratio of gamma functions.

For example, the momentum space correlator $\langle O_1 O_1 O_1 O_1\rangle$ that we compute in (2.186) in the free bosonic theory receives contributions only from $b_1 = 4, 5$ and 6 with $k_1(b_1) = 14, 28$ and 40 respectively. Using these facts in (2.150), we obtain the following compact momentum-mellin amplitude:

$$\begin{aligned}\mathcal{M}(s,t) &= .\frac{7}{6}\Gamma(s+4)\Gamma(t+3) + \frac{7}{6}\Gamma(s+5)\Gamma(t+2) + \frac{1}{3}\Gamma(s+6)\Gamma(t+1)\\ &= \frac{\left(82+2s^2+7t(7+t)+s(25+7t)\right)}{6}\Gamma(4+s)\Gamma(1+t).\end{aligned} \tag{2.151}$$

Based on the simplicity of this representation, it would be interesting to directly bootstrap four and higher point correlators in the momentum-mellin space. We leave this analysis to a future work.

### 2.4.3 Conformal Partial Waves

In this subsection, we compute momentum space conformal partial waves specializing our general discussion earlier to $d = 1$. Conformal partial waves are eigenvectors of the quadratic conformal Casimir $C_2$ [118]. The expression for $C_2$ is as follows [95]:

$$C_2 = \frac{1}{2}(HK + KH) - D^2. \tag{2.152}$$

It is easy to see that $C_2$ commutes with $H, K$ and $D$ using the $\mathfrak{sl}(2,\mathbb{R})$ conformal algebra. The conformal partial wave $W_\Delta(t_1,t_2,t_3,t_4)$ satisfies the following differential equation:

$$C_{12}W_\Delta(t_1,t_2,t_3,t_4) = \Delta(\Delta-1)W_\Delta(t_1,t_2,t_3,t_4). \tag{2.153}$$

The conformal partial wave (in the s channel) is given by the following integral [85]:

$$W_\Delta^{(s)}(t_1,t_2,t_3,t_4) = \int dt \langle O_{\Delta_1}(t_1)O_{\Delta_2}(t_2)O_\Delta(t)\rangle\langle \tilde{O}_\Delta(t)O_{\Delta_3}(t_3)O_{\Delta_4}(t_4)\rangle, \tag{2.154}$$

where we have introduced the shadow operator, $\tilde{O}$, which is defined as,

$$\tilde{O}(t) = \int_{-\infty}^{\infty}\frac{dx}{|x-t|^{2-2\Delta}}O_\Delta(t). \tag{2.155}$$

Using (2.155) in (2.154) we obtain,

$$W_\Delta^{(s)}(t_1,t_2,t_3,t_4) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dt dx}{|x-t|^{2-2\Delta}}\langle O_{\Delta_1}(t_1)O_{\Delta_2}(t_2)O_\Delta(t)\rangle\langle O_\Delta(t)O_{\Delta_3}(t_3)O_{\Delta_4}(t_4)\rangle. \tag{2.156}$$

Fourier transforming with respect to $t_1, t_2, t_3$ and $t_4$ we obtain,

$$W_\Delta^{(s)}(\omega_1,\omega_2,\omega_3,\omega_4) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dtdx}{|x-t|^{2-2\Delta}} \langle O_{\Delta_1}(\omega_1) O_{\Delta_2}(\omega_2) O_\Delta(t)\rangle \langle O_\Delta(t) O_{\Delta_3}(\omega_3) O_{\Delta_4}(\omega_4)\rangle. \tag{2.157}$$

This equals (We ignore overall numerical factors),

$$\begin{aligned} &W_\Delta^{(s)}(\omega_1,\omega_2,\omega_3,\omega_4) \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{dtdxdq_1dq_2dq_3}{q_1^{2\Delta-1}} e^{-iq_1(x-t)-iq_2 t - iq_3 x} \langle O_{\Delta_1}(\omega_1) O_{\Delta_2}(\omega_2) O_\Delta(q_3)\rangle \langle O_\Delta(q_3) O_{\Delta_3}(\omega_3) O_{\Delta_4}(\omega_4)\rangle. \end{aligned} \tag{2.158}$$

Performing the integrals over $x$ and $t$, we obtain,

$$\begin{aligned} W_\Delta^{(s)}(\omega_1,\omega_2,\omega_3,\omega_4) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{dq_1dq_2dq_3}{q_1^{2\Delta-1}} \delta(q_1+q_3)\delta(q_1-q_2) \langle O_{\Delta_1}(\omega_1) O_{\Delta_2}(\omega_2) O_\Delta(q_3)\rangle \langle O_\Delta(q_3) O_{\Delta_3}(\omega_3) O_{\Delta_4}(\omega_4)\rangle \\ &= \int_{-\infty}^{\infty} \frac{dq_1}{q_1^{2\Delta-1}} \langle O_{\Delta_1}(\omega_1) O_{\Delta_2}(\omega_2) O_\Delta(-q_1)\rangle \langle O_\Delta(q_1) O_{\Delta_3}(\omega_3) O_{\Delta_4}(\omega_4)\rangle \\ &= \int_{-\infty}^{\infty} \frac{dq_1}{q_1^{2\Delta-1}} \delta(\omega_1+\omega_2-q_1)\delta(q_1+\omega_3+\omega_4) \langle\langle O_{\Delta_1}(\omega_1) O_{\Delta_2}(\omega_2) O_\Delta(-q_1)\rangle\rangle \langle\langle O_\Delta(q_1) O_{\Delta_3}(\omega_3) O_{\Delta_4}(\omega_4)\rangle\rangle \\ &= \frac{\delta(\omega_1+\omega_2+\omega_3+\omega_4)}{(\omega_1+\omega_2)^{2\Delta-1}} \langle\langle O_{\Delta_1}(\omega_1) O_{\Delta_2}(\omega_2) O_\Delta(-\omega_1-\omega_2)\rangle\rangle \langle\langle O_\Delta(\omega_1+\omega_2) O_{\Delta_3}(\omega_3) O_{\Delta_4}(\omega_4)\rangle\rangle, \end{aligned} \tag{2.159}$$

where we have defined the double bracket notation,

$$\langle . \rangle = \delta(\omega_1 + \cdots)\langle\langle\, . \,\rangle\rangle. \tag{2.160}$$

Having obtained the conformal partial wave in momentum space (2.159), we can now extract the conformal block. We then use the expression for generic three point functions that we obtained in (2.130) in (2.159). Recall however, that we have two, rather than a unique solution to the three point function. As we explained in the beginning of this section and as we shall elaborate that this fact owes itself to the possibility of the various possible time orderings yielding different expressions when Fourier transformed. Let us now write (2.159) explicitly. First, we define,

$$\begin{aligned} f_{\Delta_i,\Delta_j,\Delta_k}(\omega_i,\omega_j) = \omega_i^{\Delta_i+\Delta_j+\Delta_k-2}\Bigg( & c_{1,ijk}\; {}_2F_1\left(2-\Delta_i-\Delta_j-\Delta_k, 1+\Delta_i-\Delta_j-\Delta_k, 2(1-\Delta_j); \frac{-\omega_j}{\omega_i}\right) \\ & + \left(\frac{\omega_j}{\omega_i}\right)^{2\Delta_j-1} c_{2,ijk}\; {}_2F_1\left(1-\Delta_i+\Delta_j-\Delta_k, \Delta_i+\Delta_j-\Delta_k, 2\Delta_j; -\frac{\omega_j}{\omega_i}\right)\Bigg). \end{aligned} \tag{2.161}$$

We can then re-write (2.159) as,

$$W_\Delta^{(s)}(\omega_1,\omega_2,\omega_3,\omega_4) = \frac{1}{(\omega_1+\omega_2)^{2\Delta-1}} f_{\Delta_1,\Delta_2,\Delta}(\omega_1,\omega_2) f_{\Delta,\Delta_3,\Delta_4}(\omega_1+\omega_2,\omega_3)\delta(\omega_1+\omega_2+\omega_3+\omega_4), \tag{2.162}$$

which is our final expression for the $s$ channel momentum space conformal partial wave. We will put (2.162) to the test in subsection (2.4.4).

### 2.4.4 Examples from free and interacting theories

Let us begin by showing that three and four point correlators in the free bosonic theory are captured by our general formulae for correlation functions.

#### Free Bosonic Theory

The action for the $U(1)$ free bosonic theory is given by,

$$S_{FB} = \int_{-\infty}^{\infty} dt\ \partial_t \bar{\phi} \partial_t \phi. \tag{2.163}$$

$\phi$ and $\bar{\phi}$ are primary operators with scaling dimension $-\frac{1}{2}$. We also consider the following composite primary operators:

$$O(t) = \bar{\phi}(t)\phi(t),\ J_B(t) = i(\bar{\phi}\partial_t\phi - \partial_t\bar{\phi}\ \phi), \tag{2.164}$$

which have the following momentum space counterparts:

$$O(\omega) = \int_{-\infty}^{\infty} dl\ \bar{\phi}(l)\phi(\omega - l),\ J_B(\omega) = \int_{-\infty}^{\infty} dl\ (2l - \omega)\bar{\phi}(l)\phi(\omega - l). \tag{2.165}$$

The first of these is a $\Delta = -1$ scalar while the second is the conserved $U(1)$ current. Let us now compute several three and four point correlators involving $\phi, \bar{\phi}$ and these composite operators.

<u>$\langle\bar{\phi}(\omega_1)\phi(\omega_2)O(\omega_3)\rangle$</u>

Performing the Wick contractions we obtain,

$$\langle\bar{\phi}(\omega_1)\phi(\omega_2)O(\omega_3)\rangle = \frac{2}{\omega_1^2\omega_2^2}\delta(\omega_1 + \omega_2 + \omega_3). \tag{2.166}$$

We see that we can easily express (2.166) in the form of our general solution (2.130). Setting $c_1 = 0$ and $c_2 = 2$ in (2.130) we see that the result matches (2.166):

$$\langle\bar{\phi}(\omega_1)\phi(\omega_2)O(\omega_3)\rangle = \omega_1^{-4}\left(\frac{\omega_2}{\omega_1}\right)^{-2}\ {}_2F_1\left(2, 0, -1, -\frac{\omega_2}{\omega_1}\right)\delta(\omega_1 + \omega_2 + \omega_3). \tag{2.167}$$

Let us now consider a correlator involving the $U(1)$ current.

<u>$\langle\bar{\phi}(\omega_1)\phi(\omega_2)J_B(\omega_3)\rangle$</u>

The Wick contractions yield,

$$\langle\bar{\phi}(\omega_1)\phi(\omega_2)J_B(\omega_3)\rangle = \frac{\omega_1 - \omega_2}{\omega_1^2\omega_2^2}\delta(\omega_1 + \omega_2 + \omega_3). \tag{2.168}$$

Note that in $d = 1$, a conserved $U(1)$ current is a scalar with $\Delta = 0$. Using this fact, we can employ the general solution (2.130). We see that,

$$\langle\bar{\phi}(\omega_1)\phi(\omega_2)J(\omega_3)\rangle = (\omega_1)^{-3}\left(\frac{\omega_2}{\omega_1}\right)^{-2}\ {}_2F_1\left(1, -1, -1, -\frac{\omega_2}{\omega_1}\right), \tag{2.169}$$

providing another check of our result. Note that (2.168) also satisfies the charge conservation Ward Takahashi identity. We have,

$$\omega_3\langle\bar{\phi}(\omega_1)\phi(\omega_2)J(\omega_3)\rangle = \left(\frac{1}{\omega_1^2}-\frac{1}{\omega_2^2}\right)\delta(\omega_1+\omega_2+\omega_3) = \left(\langle\phi(\omega_1)\bar{\phi}(-\omega_1)\rangle - \langle\bar{\phi}(\omega_2)\phi(-\omega_2)\rangle\right)\delta(\omega_1+\omega_2+\omega_3), \tag{2.170}$$

which shows that our general solution (2.169) accommodates correlators involving conserved currents.

Let us now move on to a four point function.

<u>$\langle\bar{\phi}(\omega_1)O(\omega_2)O(\omega_3)\phi(\omega_4)\rangle$</u>

Using the definitions of the operators provided in (2.165) we obtain,

$$\begin{aligned}
&\langle\phi(\omega_1)O(\omega_2)O(\omega_3)\bar{\phi}(\omega_4)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dl dk \langle\phi(\omega_1)\bar{\phi}(l)\phi(\omega_2-l)\bar{\phi}(k)\phi(\omega_3-k)\bar{\phi}(\omega_4)\rangle \\
&= \frac{1}{(2\pi)^2\omega_1^2\omega_4^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dl dk\left[\frac{1}{k^2}\delta(\omega_1+l)\delta(\omega_2-l+k)\delta(\omega_3-k+\omega_4)+\frac{1}{l^2}\delta(l+\omega_3+k)\delta(\omega_2-l+\omega_4)\delta(\omega_1+k)\right] \\
&= \delta(\omega_1+\omega_2+\omega_3+\omega_4)\omega_1^{-6}\frac{1}{(1+x+y)^2}\left(\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}\right),
\end{aligned} \tag{2.171}$$

where $x = \frac{\omega_2}{\omega_1}$ and $y = \frac{\omega_3}{\omega_1}$ as we defined earlier.
We now define for convenience,

$$\langle\phi(\omega_1)O(\omega_2)O(\omega_3)\bar{\phi}(\omega_4)\rangle = \delta(\omega_1+\omega_2+\omega_3+\omega_4)\omega_1^{-6}\psi_{\phi OO\bar{\phi}}(x,y), \tag{2.172}$$

where,

$$\psi_{\phi OO\bar{\phi}}(x,y) = \frac{1}{(1+x+y)^2}\left(\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}\right). \tag{2.173}$$

As we saw in (2.139), the ward identities do not fix the form of the 4 point function and lead to an undetermined parameter $b_1$. Our aim now is to find value(s) of $b_1$ that reproduce this correlator.
We have $\Delta_1 = \Delta_4 = -\frac{1}{2}$ and $\Delta_2 = \Delta_3 = -1$. One of the solutions to the Ward identities (2.139) is,

$$\omega_1^{-6}f_{b_1}(x,y) = \omega_1^{-6}F_2(4,b_1,6-b_1,4,4;-x,-y). \tag{2.174}$$

Consider the following identity (see [115]):

$$F_2(a,b_1,b_2,a,a;-x,-y) = (1+x)_1^{b}(1+y)^{6-b_1}\ {}_2F_1\left(b_1,6-b_1,a,\frac{xy}{(1+x)(1+y)}\right). \tag{2.175}$$

Using this , we find that,

$$\begin{aligned}
F_2(4,2,4,4,4;-x,-y) &= \frac{1}{(1+x+y)^2(1+y)^2}, \\
F_2(4,4,2,4,4;-x,-y) &= \frac{1}{(1+x+y)^2(1+x)^2}.
\end{aligned} \tag{2.176}$$

which correspond to the choices $b_1 = 2$ and $b_2 = 4$ respectively. This gives us,

$$\omega_1^{-6}(f_2(x,y) + f_4(x,y)) = \omega_1^{-6}\frac{1}{(1+x+y)^2}\left(\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2}\right). \tag{2.177}$$

Comparing with (2.172) we find,

$$\psi_{\phi OO\phi} = \Big(F_2(4,2,4,4,4;-x,-y) + F_2(4,4,2,4,4;-x,-y)\Big). \tag{2.178}$$

Therefore, the correlator (2.171) can be written as,

$$\begin{aligned}&\langle\phi(\omega_1)O(\omega_2)O(\omega_3)\bar{\phi}(\omega_4)\rangle\\ &= \delta(\omega_1+\omega_2+\omega_3+\omega_4)\omega_1^{-6}\left(F_2\left(4,2,4,4,4;-\frac{\omega_2}{\omega_1},-\frac{\omega_3}{\omega_1}\right) + F_2\left(4,4,2,4,4;-\frac{\omega_2}{\omega_1},-\frac{\omega_3}{\omega_1}\right)\right),\end{aligned} \tag{2.179}$$

which provides a check of our general result (2.139). Let us now also reproduce this correlator using conformal partial waves. Consider the s channel conformal partial wave (2.162), set $c_{2,ijk} = 0$ in (2.161) and $\Delta_1 = \Delta_4 = -\frac{1}{2}, \Delta_2 = \Delta_3 = -1$. For the exchanged operator having $\Delta = -\frac{1}{2}$ we find,

$$W^{(s)}_{\Delta=-\frac{1}{2}} = \frac{1}{(1+x)^2(1+x+y)^2\omega_1^6}\delta(\omega_1+\omega_2+\omega_3+\omega_4). \tag{2.180}$$

We can also obtain the u channel conformal partial wave by a $(2 \leftrightarrow 3)$ exchange and add it with the s channel result which yields,

$$\begin{aligned}W^{(s)}_{\Delta=-\frac{1}{2}} + W^{(u)}_{\Delta=-\frac{1}{2}} &= \omega_1^{-6}\frac{1}{(1+x+y)^2}\left(\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2}\right)\delta(\omega_1+\omega_2+\omega_3+\omega_4)\\ &= \langle\phi(\omega_1)O(\omega_2)O(\omega_3)\bar{\phi}(\omega_4)\rangle.\end{aligned} \tag{2.181}$$

To summarize, we have obtained two distinct representations for this correlator: One in terms of the momentum space conformal partial waves and the other, in terms of the Appell $F2$ function.

$$\begin{aligned}&\langle\phi(\omega_1)O(\omega_2)O(\omega_3)\bar{\phi}(\omega_4)\rangle = W^{(s)}_{\Delta=-\frac{1}{2}} + W^{(u)}_{\Delta=-\frac{1}{2}}\\ &= \delta(\omega_1+\omega_2+\omega_3+\omega_4)\omega_1^{-6}\left(F_2\left(4,2,4,4,4;-\frac{\omega_2}{\omega_1},-\frac{\omega_3}{\omega_1}\right) + F_2\left(4,4,2,4,4;-\frac{\omega_2}{\omega_1},-\frac{\omega_3}{\omega_1}\right)\right).\end{aligned} \tag{2.182}$$

Let us now provide two more examples of correlators in the free bosonic theory. Since the details of the calculation are similar to the above example, we just provide the final results.

$\underline{\langle O_1(\omega_1)O_1(\omega_2)O_1(\omega_3)O_1(\omega_4)\rangle}$

The Fourier space expression for this correlator with the time ordering $t_1 > t_2 > t_3 > t_4$ reads[30],

$$\langle\langle O_1(\omega_1)O_1(\omega_2)O_1(\omega_3)O_1(\omega_4)\rangle\rangle = \frac{2(1+x)^2(41+7x(4+x))+2(1+x)(54+x(31+7x))y+4(10+x(5+x))y^2}{(1+x)^5(1+x+y)^3\omega_1^7}. \tag{2.183}$$

We find that this correlator receives contributions from *three* different conformal blocks corresponding to exchanges of operators with $\Delta = -2, -1$, and 0. Setting $c_{2,ijk} = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = -1$ in (2.162), we see that,

$$\langle O_1(\omega_1)O_1(\omega_2)O_1(\omega_3)O_1(\omega_4)\rangle = 16W^{(s)}_{\Delta=-1} - \frac{2}{3}W^{(s)}_{\Delta=0} + \frac{200}{3}W^{(s)}_{\Delta=-2}. \tag{2.184}$$

We also find that this result can be expressed in terms of our general four point function (2.139). For $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = -1$ and setting $k_2 = k_3 = k_4 = 0$ in (2.139) we obtain the following two possible expressions (corresponding to the two choices of $a, b_1 + b_2$ in (2.138)):

$$f_1(b_1) = \frac{1}{\omega_1^7}F_2(7, b_1, 4-b_1, 4, 4, -x, -y) \text{ and } f_2(b_1) = \frac{1}{\omega_1^7}F_2(4, b_1, 7-b_1, 4, 4, -x, -y). \tag{2.185}$$

It turns out that the second solution in (2.185) is the required one for this example. Consider the following linear combination of the $b_1 = 4$, $b_1 = 5$ and $b_1 = 6$ solutions:

$$\begin{aligned}\langle O_1(\omega_1)O_1(\omega_2)O_1(\omega_3)O_1(\omega_4)\rangle &= \delta(\omega_1+\omega_2+\omega_3+\omega_4)\big(14f_2(4)+28f_2(5)+40f_2(6)\big)\\ &= \frac{\delta(\omega_1+\omega_2+\omega_3+\omega_4)}{\omega_1^7}\big(14F_2(4,4,3,4,4,-x,-y)+28F_2(4,5,2,4,4,-x,-y)+40F_2(4,6,1,4,4,-x,-y)\big).\end{aligned} \tag{2.186}$$

Let us now compare the conformal partial wave representation (2.184) and the Appell $F_2$ representation of the correlator (2.186). In (2.184), it takes the sum of three different conformal blocks to reproduce the correlator. In (2.186), it takes three different "$b_1$ exchanges" to reproduce the correlator. In lieu of this, one might think that $b_1$ is somehow related to the dimension of the exchanged operators. This, however, is not quite true as the number of conformal blocks does not always tally up with the number of "$b_1$ exchanges".

$\underline{\langle J_B(\omega_1)J_B(\omega_2)O_1(\omega_3)O_1(\omega_4)\rangle}$

The momentum space correlator corresponding to the time ordering $t_1 > t_2 > t_3 > t_4$ is given by,

$$\langle\langle J_B(\omega_1)J_B(\omega_2)O_1(\omega_3)O(\omega_4)\rangle\rangle = \frac{1}{(1+x)(1+x+y)^3\omega_1^5}. \tag{2.187}$$

[30] One way in which this expression can be obtained is by performing the Wick contractions in the time domain and then Fourier transforming with this time ordering.

It turns out that a single conformal block suffices to reproduce this correlator. Setting $\Delta_1 = \Delta_2 = 0, \Delta_3 = \Delta_4 = -1$ and $c_{2,ijk} = 0$ in (2.162) we see that,

$$\langle J_B(\omega_1) J_B(\omega_2) O_1(\omega_3) O(\omega_4)\rangle = W^{(s)}_{\Delta=0}. \tag{2.188}$$

In terms of our general four point solution (2.139), set $\Delta_1 = \Delta_2 = 0, \Delta_3 = \Delta_4 = -1$ and $k_2 = k_3 = k_4 = 0$ to obtain the two solutions:

$$f_1(b_1) = \frac{1}{\omega_1^5} F_2(5, b_1, 4 - b_1, 2, 4; -x, -y) \text{ and } f_2(b_1) = \frac{1}{\omega_1^5} F_2(4, b_1, 5 - b_1, 2, 4; -x, -y). \tag{2.189}$$

We see that we can reproduce this correlator by a single "$b_1$ exchange":

$$\langle J_B(\omega_1) J_B(\omega_2) O_1(\omega_3) O(\omega_4)\rangle = \delta(\omega_1 + \omega_2 + \omega_3 + \omega_4) f_2(2) = \delta(\omega_1 + \omega_2 + \omega_3 + \omega_4) F_2(4, 2, 3, 2, 4; -x, -y). \tag{2.190}$$

### Free Fermionic Theory

The action for the $U(1)$ free massless Dirac fermion theory is,

$$S_{FF} = i \int dt\, \psi^\dagger \partial_t \psi. \tag{2.191}$$

$\psi$ and $\psi^\dagger$ are dimensionless operators. We also consider the conserved $U(1)$ current,

$$J_F(t) = \psi^\dagger(t)\psi(t), \tag{2.192}$$

which in momentum space is given by,

$$J_F(\omega) = \int dl\, \psi^\dagger(l)\psi(\omega - l). \tag{2.193}$$

Let us now consider the following correlator:

$\langle \bar{\psi}(\omega_1)\psi(\omega_2) J_F(\omega_3)\rangle$

Performing the Wick contractions gives,

$$\langle \bar{\psi}(\omega_1)\psi(\omega_2) O(\omega_3)\rangle = \frac{1}{\omega_1\omega_2}\delta(\omega_1 + \omega_2 + \omega_3). \tag{2.194}$$

This is reproduced by the general solution (2.130) in the following way:

$$\langle\langle \bar{\psi}(\omega_1)\psi(\omega_2) O(\omega_3)\rangle\rangle = (\omega_1)^{-2} \left(\frac{\omega_2}{\omega_1}\right)^{-1} {}_2F_1\left(1, 0, 0, \frac{-\omega_2}{\omega_1}\right). \tag{2.195}$$

We now move on to a four-point example.

$\langle\psi^\dagger(\omega)J_F(\omega_2)J_F(\omega_3)\psi(\omega_4)\rangle$

Via Wick contractions, we obtain the following expression for this correlator:

$$\langle\psi^\dagger(\omega)J_F(\omega_2)J_F(\omega_3)\psi(\omega_4)\rangle = \frac{\delta(\omega_1+\omega_2+\omega_3+\omega_4)}{\omega_1^3}\frac{2+x+y}{(1+x)(1+y)(1+x+y)}. \tag{2.196}$$

This correlator is reproduced by the sum of a single conformal block in the s and u channels (obtained by setting $\Delta_1=\Delta_2=\Delta_3=\Delta_4=-1, c_{2,ijk}=0$ in (2.162) and the u channel expression obtained via a $(2\leftrightarrow 3)$ exchange.):

$$\langle\psi^\dagger(\omega)J_F(\omega_2)J_F(\omega_3)\psi(\omega_4)\rangle = W^{(s)}_{\Delta=0}+W^{(u)}_{\Delta=0}. \tag{2.197}$$

As for the Appell function representation, set $\Delta_1=\Delta_2=\Delta_3=\Delta_4=0, k_2=k_3=k_4=0$ in (2.139). We obtain,

$$f_1(b_1)=\frac{1}{\omega_1^3}F_2(3,b_1,2-b_1,2,2;-x,-y) \text{ and } f_2(b_1)=F_2(2,b_1,3-b_1,2,2;-x,-y). \tag{2.198}$$

We find,

$$\langle\psi^\dagger(\omega)J_F(\omega_2)J_F(\omega_3)\psi(\omega_4)\rangle = \delta(\omega_1+\omega_2+\omega_3+\omega_4)f_1(1) = \delta(\omega_1+\omega_2+\omega_3+\omega_4)F_2(3,1,1,2,2;-x, \tag{2.199}$$

providing another test of our results. Let us now consider and reproduce correlation functions in the DFF model.

### The DFF model

The DFF model [87] is a one-dimensional quantum mechanical model of a particle moving in an inverse square potential. It's Lagrangian is as follows:

$$L=\int dt\,\left(\frac{1}{2}(\frac{d\phi}{dt})^2-\frac{g}{2\phi^2}\right). \tag{2.200}$$

This theory enjoys invariance under the $SL(2,\mathbb{R})$ group, and our focus will be on four point correlators in this model. The four point function in the DFF model was computed in [95] and reads,

$$\langle O^\dagger_{r_0}(t_1)\phi_\delta(t_2)\phi_\delta(t_3)O_{r_0}(t_4)\rangle = \frac{1}{(t_{24}t_{13})^{\delta-r_0}(t_{12}t_{34})^{\delta+r_0}t_{14}^{2(r_0-\delta)}}\chi^{r_0}\,{}_2F_1(\delta,\delta,2r_0;\chi), \tag{2.201}$$

where we have used the shorthand $t_{ij}=t_i-t_j$, $\chi=\dfrac{t_{12}t_{34}}{t_{13}t_{24}}$ is the cross ratio. The above expression is also in the time ordering $t_1>t_2>t_3>t_4$.

Consider the specific case where $r_0 = \delta = -1$. The Fourier space expression of the correlator reads,

$$\frac{(2+x)(2+2x+y)}{(1+x)^3(1+x+y)^3\omega_1^7}\delta(\omega_1+\omega_2+\omega_3+\omega_4). \tag{2.202}$$

This correlator (for arbitrary $\delta, r_0$) was found to receive contribution from just a single conformal block. Indeed, we see that just a single conformal block with $\Delta = -1$ exchange (obtained be setting $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = -1$, $c_{2,ijk} = 0$ in (2.162)) suffices to reproduce this correlator:

$$W^{(s)}_{\Delta=-1} = \frac{(2+x)(2+2x+y)}{(1+x)^3(1+x+y)^3\omega_1^7}\delta(\omega_1+\omega_2+\omega_3+\omega_4). \tag{2.203}$$

In terms of the Appell $F_2$ representation, set $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = -1$, $k_2 = k_3 = k_4 = 0$ in (2.139). We obtain two solutions:

$$f_1(b_1) = \frac{1}{\omega_1^7}F_2(7, b_1, 4-b_1, 4, 4; -x, -y) \text{ and } f_2(b_1) = \frac{1}{\omega_1^7}F_2(4, b_1, 7-b_1, 4, 4; -x, -y). \tag{2.204}$$

We see that,

$$\begin{aligned}\delta(\omega_1+\omega_2+\omega_3+\omega_4)\frac{(2+x)(2+2x+y)}{(1+x)^3(1+x+y)^3\omega_1^7} &= 2\delta(\omega_1+\omega_2+\omega_3+\omega_4)\big(f_2(4)+f_2(5)\big)\\ &= \frac{2\delta(\omega_1+\omega_2+\omega_3+\omega_4)}{\omega_1^7}\big(F_2(4,4,3,4,4;-x,-y)+F_2(4,5,2,4,4;-x,-y)\big).\end{aligned} \tag{2.205}$$

We can also repeat the same analysis for other values of $\delta$ and $r_0$. We find that the momentum space correlator for any $\delta, r_0$ is given in terms of exactly one conformal block in accordance with the results of [95].

Finally, we have discussed the supersymmetric extension of conformal quantum mechanics in appendix E.

## 2.5 Summary of this Chapter

In this chapter, we discussed the construction of off-shell Euclidean and Lorentzian spinor helicity variables in three dimensions. These spinor helicity variables do not satisfy any mass-shell condition in contrast to the more familiar on-shell spinor helicity construction. An alternate derivation of these formalisms from the dimensional reduction point of view from the usual on-shell spinor helicity construction for massless momenta in four dimensions is provided in appendix A.1. We analyzed Euclidean two and three point correlators of conserved currents in great detail, understanding their structure in the helicity basis. Similarly, we discussed the construction of Wightman functions in Lorentzian signature including their construction in spinor helicity variables contrasted with their Euclidean counterparts via analytic continuation. We then discussed the construction of four point functions and conformal partial waves.

Finally, we concluded with the simple, yet non-trivial example of conformal quantum mechanics which forms a sub-sector of higher dimensional CFTs and discovered a Lauricella function structure for momentum n-point correlators. One can also think of conformal quantum mechanics as a chiral sector of a two dimensional CFT. From this perspective, we have only used the global $SL(2,\mathbb{R})$ sub-group of the (chiral) Virasoro symmetry. Understanding the implications of the full Virasoro symmetry in this context presents an area of interest. It would be interesting to find out the connection between the undetermined parameters of the Lauricella functions to the conformal data of a particular theory of interest. An extension of this to higher dimensions and to spinning correlators is also desirable. Also, systematizing spinor helicity for $CFT_3$ at four points and beyond would be an important task to make progress. Finally, since the spinor helicity formalism does not rely on conformal invariance and the fact that there are interesting scale but not conformally invariant theories in three dimensions [11], understanding the utility of the formalism in such settings would be interesting.

# Chapter 3

# Chern-Simons matter theories

In this chapter, we focus on Chern-Simons theory coupled to matter in the fundamental representation of $SU(N)$. This chapter will serves as our first application of the momentum space and spinor helicity formalisms.

Over the years, there has been a lot of progress in understanding Chern-Simons matter theories in the large-$N$ limit, a significant part of which was based on the idea of the slightly-broken higher-spin symmetry [119–122]. Several remarkable achievements in these theories include exact computations of the partition function [119, 120, 123–127], understanding the Hilbert space structure [128] and the higher-spin spectrum [129–135], determination of the exact S matrix [136–138], showing the existence of BCFW recursion relations and dual superconformal invariance [139, 140].

An important set of observables in these theories are the correlation functions of gauge invariant single trace operators, aka higher-spin currents. It was found in [78] (and as we discussed in the previous chapter) that three-point functions in general three-dimensional CFTs involving higher-spin currents contain three pieces: two of which are realized by the free bosonic and free fermionic theories and a third, which is parity violating and not realized by free fields. In [121, 122] it was found that if one imposes in addition, the Ward identities due to the slightly-broken higher-spin symmetry, the relative coefficients of these structures get determined in terms of a single parameter. This parameter was then related to the Chern-Simons matter theory 't Hooft coupling in [77]. Since then, there have been many attempts to compute three- and four-point correlators in position space and Mellin space [141–155] and also in momentum space [38, 39]. It was subsequently realized by working in momentum space, that the parity odd correlators can be obtained from the difference of the free fermionic and free bosonic correlators by the so-called *epsilon* transform [44, 45, 156] as we discussed in the previous chapter. Further, by employing the technology of three-dimensional spinor-helicity variables, this relation becomes extremely simple [40–42]. Spinor-helicity variables also revealed the anyonic nature of the correlators [157]. Eventually, this culminated in the analysis of all four- and higher-point functions in the Chern-Simons matter theories [158] using the slightly-broken higher-spin symmetry which will form the subject of this chapter. Although we will not discuss it here, the implications on the bulk dual of Chern-Simons matter theory were analyzed in [159] using these results where it was found that there exists the possibility of a sub-sector of the theory that has a local bulk dual.

The main reference for this chapter is,

- ⋆Mapping Large N Slightly Broken Higher Spin (SBHS) theory correlators to free theory correlators, Prabhav Jain, Sachin Jain, Bibhut Sahoo, **Dhruva K.S.** and Aashna Zade, JHEP 12 (2023) 173 [2207.05101]

## 3.1 The Action and Spectrum of primary operators

The theory of interest is $SU(N_f)$ Chern-Simons+fundamental fermions (henceforth called the quasi-fermionic (QF) theory). The action of the QF theory is,

$$S_{FF,CS} = \int d^3x \left( \bar{\psi} \not{D} \psi + i\epsilon^{\mu\nu\rho} \frac{\kappa_f}{4\pi} \mathrm{Tr}\left\{ A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right\} \right). \tag{3.1}$$

We work in the large $N_f \to \infty$ limit and also take the limit $\kappa_f \to \infty$ keeping the t'Hooft coupling $\lambda_f = \frac{N_f}{\kappa_f}$ fixed. The spectrum of this theory includes a parity odd $\Delta = 2 + O\left(\frac{1}{N_f}\right)$ scalar $O_2$, conserved spin-1 and spin-2 currents $J$ and $T$ and weakly nonconserved currents $J_s, s = 3, 4, \cdots$ with $\Delta_s = s + 1 + O\left(\frac{1}{N_f}\right)$.

This theory enjoys a strong-weak duality with the $SU(N_b)$ critical bosonic (CB) theory, which is the theory obtained by flowing to the Wilson-Fisher fixed point starting with the following action:

$$S_{CB,CS} = \int d^3x \left( D_\mu \bar{\phi} D_\mu \phi + \frac{\lambda_4}{N_b} (\bar{\phi}\phi)^2 + i\epsilon^{\mu\nu\rho} \frac{\kappa_b}{4\pi} \mathrm{Tr}\left\{ A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right\} \right). \tag{3.2}$$

We take the limits $N_b \to \infty, \lambda_4 \to \infty$ and $\kappa_b \to \infty$ keeping fixed $\frac{\lambda_4}{N_b}$ and $\lambda_b = \frac{N_b}{\kappa_b}$. The spectrum of this theory includes a parity even $\Delta = 2 + O\left(\frac{1}{N_b}\right)$ scalar $O_2$, conserved spin-1 and spin-2 currents $J$ and $T$ and weakly nonconserved currents $J_s, s = 3, 4, \cdots$ with $\Delta = s + 1 + O\left(\frac{1}{N_b}\right)$[1].

The nonconservation is not seen at the level of two-point correlation functions, which take the form of the exactly conserved case, (2.18) with $c_{s,even} = \tilde{N}$ and $c_{s,odd} = \tilde{N}\tilde{\lambda}$. Three-point functions inside and outside the triangle respectively take the form of (2.23) and (2.48) with the coefficients are given by [122]

$$n_f = \frac{\tilde{N}}{1+\tilde{\lambda}^2}, \qquad n_{odd} = \frac{\tilde{N}\tilde{\lambda}}{1+\tilde{\lambda}^2}, \qquad n_b = \frac{\tilde{N}\tilde{\lambda}^2}{1+\tilde{\lambda}^2}. \tag{3.3}$$

We have defined,

$$\tilde{N} = 2N \frac{\sin(\pi\lambda)}{\pi\lambda}, \qquad \tilde{\lambda} = \tan\left(\frac{\pi\lambda}{2}\right), \tag{3.4}$$

[1] A CB three point correlator with a $O_2$ insertion can be obtained via a Legendre transform of a FB correlator with an $O_1$ insertion instead of the $O_2$ operator. Three point correlators without scalar insertions coincide in the FB and CB theory and thus we will use the labels interchangeably in such cases.

in terms of the 't Hooft coupling and the large $N$ parameter. If the three-point function is inside the triangle, then the correlator is conserved with respect to all the currents. However, when the spin triangle inequality is violated, the odd part of the correlator gives rise to the nonconservation. Let us now make precise what we mean by slightly broken conservation. It is the statement that the divergence of the higher spin currents take the form,

$$\partial \cdot J_s = \frac{1}{\tilde{N}}[J_{s'}J_{s''}] + \cdots , \tag{3.5}$$

where $[J_{s'}J_{s''}]$ denotes a composite operator formed out of two currents. We see that the non-conservation is suppressed by $\frac{1}{\tilde{N}}$ and thus we recover conservation at infinite $\tilde{N}$. Due to this controlled non-conservation, we can still use this to constrain the form of correlation functions of these currents. Before we present the details of the computation, we shall now summarize our results using spinor helicity variables that reveals an interesting anyonic structure of the current correlators.

## 3.2 Anyonic form of correlators

In this section we summarize our results. We first present some notation that we employ. For any correlator we define,

| Notation | Description |
|---|---|
| $\langle\ldots\rangle_{\text{QF}}$ | In quasi-fermionic theory |
| $\langle\ldots\rangle_{\text{FF/FB}}$ | In free fermionic/bosonic theory |
| $\langle\ldots\rangle_{\text{CB}}$ | In critical bosonic theory |
| $\langle\ldots\rangle_{\text{odd}}$ | Parity odd correlator |
| $\langle\ldots\rangle_{\text{FF+FB}}$ | $\langle\ldots\rangle_{\text{FF}} + \langle\ldots\rangle_{\text{FB}}$ |
| $\langle\ldots\rangle_{\text{FF-FB}}$ | $\langle\ldots\rangle_{\text{FF}} - \langle\ldots\rangle_{\text{FB}}$ |

Table 3.1: Notation for correlators

### 3.2.1 2 point function

The two point function is given by,

$$\langle J_s J_s\rangle_{\text{QF}} = \tilde{N}\langle J_s J_s\rangle_{\text{FF}} + \tilde{N}\tilde{\lambda}\langle J_s J_s\rangle_{\text{odd}}. \tag{3.6}$$

Using the epsilon transform (2.19) we find,

$$\langle J_s J_s\rangle_{\text{QF}} = \tilde{N}\langle J_s J_s\rangle_{\text{FF}} + \tilde{N}\tilde{\lambda}\langle \epsilon \cdot J_s J_s\rangle_{\text{FF}}, \tag{3.7}$$

which upon converting to spinor helicity variables and using (3.4) gives,

$$\langle J_s^- J_s^-\rangle_{QF} = \frac{N e^{i\pi\lambda_f}}{\pi\lambda_f}\langle J_s^- J_s^-\rangle_{FF}. \tag{3.8}$$

The result for the $(++)$ helicity can be found be complex conjugation.

### 3.2.2 3-point functions

It was shown in [122] that for the case of three point functions in the quasi fermionic theory we have[2]

$$
\begin{aligned}
\langle J_s O O\rangle_{\text{QF}} &= \tilde{N}(1+\tilde{\lambda}^2)\langle J_s O O\rangle_{\text{FF}}\\
\langle J_{s_1} J_{s_2} O\rangle_{\text{QF}} &= \tilde{N}\langle J_{s_1} J_{s_2} O\rangle_{\text{FF}} + \tilde{N}\tilde{\lambda}\langle J_{s_1} J_{s_2} O\rangle_{\text{CB}}\\
\langle J_{s_1} J_{s_2} J_{s_3}\rangle_{\text{QF}} &= \frac{\tilde{N}}{1+\tilde{\lambda}^2}\left[\langle J_{s_1} J_{s_2} J_{s_3}\rangle_{\text{FF}} + \tilde{\lambda}\langle J_{s_1} J_{s_2} J_{s_3}\rangle_{\text{odd}} + \tilde{\lambda}^2\langle J_{s_1} J_{s_2} J_{s_3}\rangle_{\text{FB}}\right]
\end{aligned}
\tag{3.9}
$$

In the above expression the momentum labels and the indices have been suppressed for clarity and they can be restored appropriately.

However, it was realized [44, 45] that we can write the odd piece in terms of the FF and FB [3]correlators using the epsilon transform and the final answer turns out to be,

$$
\begin{aligned}
\langle J_{s_1} O O\rangle_{\text{QF}} &= \tilde{N}(1+\tilde{\lambda}^2)\langle J_{s_1} O O\rangle_{\text{FF}},\\
\langle J_{s_1} J_{s_2} O\rangle_{\text{QF}} &= \tilde{N}\langle J_{s_1} J_{s_2} O\rangle_{\text{FF}} + \tilde{N}\tilde{\lambda}\langle \epsilon\cdot J_{s_1} J_{s_2} O\rangle_{\text{FF}},\\
\langle J_{s_1} J_{s_2} J_{s_3}\rangle_{\text{QF}} &= \frac{\tilde{N}}{1+\tilde{\lambda}^2}\left[\langle J_{s_1} J_{s_2} J_{s_3}\rangle_{\text{FF}} + \tilde{\lambda}\langle \epsilon\cdot J_{s_1} J_{s_2} J_{s_3}\rangle_{\text{FF-FB}} + \tilde{\lambda}^2\langle J_{s_1} J_{s_2} J_{s_3}\rangle_{\text{FB}}\right].
\end{aligned}
\tag{3.10}
$$

We now convert the third line of (3.10) to spinor helicity variables. To illustrate the result, we consider all currents to have negative helicities. Combining the FF and FB terms, and substituting $\tilde{\lambda} = \tan\frac{\pi\lambda_f}{2}$ we get,

$$
\begin{aligned}
\langle J^-_{s_1} J^-_{s_2} J^-_{s_3}\rangle_{\text{QF}} =& \frac{\tilde{N}}{2}\left[\langle J^-_{s_1} J^-_{s_2} J^-_{s_3}\rangle_{\text{FF+FB}} + e^{-i\pi\lambda_f}\langle J^-_{s_1} J^-_{s_2} J^-_{s_3}\rangle_{\text{FF-FB}}\right]\\
=& \tilde{N}e^{-\frac{i\pi\lambda_f}{2}}\left[\cos\frac{\pi\lambda_f}{2}\langle J^-_{s_1} J^-_{s_2} J^-_{s_3}\rangle_{\text{FF}} + i\sin\frac{\pi\lambda_f}{2}\langle J^-_{s_1} J^-_{s_2} J^-_{s_3}\rangle_{\text{FB}}\right].
\end{aligned}
\tag{3.11}
$$

For, $\lambda_f = 0$, we obtain the FF correlator and for $\lambda_f = 1$, we have the FB correlator. This shows an interesting flow between these free theory correlators.

#### 3-point functions in terms of homogeneous and non-homogeneous decomposition

It is very interesting to see that the QF correlators can be written in terms of just the free theory correlators. However for the three-point case it was shown in [45] that we can make a further stronger claim by representing the correlators in terms of the homogeneous and non-homogeneous parts [45].

It was shown in [45] that when the triangle inequality,

$$
s_i + s_j \geq s_k \tag{3.12}
$$

[2]Our conventions are such that our scalar operator is related to the one in [122] as $O = \frac{O_{\text{MZ}}}{1+\tilde{\lambda}^2}$

[3]When all the spins are non-zero the critical bosonic and free bosonic correlators are identical. When we have some scalar operator then the correlation functions are legendre transforms of each other.

is satisfied, we can define the *homogeneous/non-homogeneous* parts of a correlator as we discussed in the previous chapter as,

$$\begin{aligned}\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\text{FB}} &= \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\text{nh}} + \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\text{h}},\\ \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\text{FF}} &= \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\text{nh}} - \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\text{h}}.\end{aligned} \tag{3.13}$$

We invert these relations to get

$$\begin{aligned}\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\text{nh}} &= \frac{1}{2}\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\text{FB+FF}},\\ \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\text{h}} &= \frac{1}{2}\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\text{FB-FF}}.\end{aligned} \tag{3.14}$$

Thus inside the triangle inequality we can express our result for a general spinning correlator in spinor helicity variables (3.11) as follows ,

$$\langle J^-_{s_1}J^-_{s_2}J^-_{s_3}\rangle_{\text{QF}} = \tilde{N}\langle J^-_{s_1}J^-_{s_2}J^-_{s_3}\rangle_{\text{nh}} + \tilde{N}e^{-i\pi\lambda_f}\langle J^-_{s_1}J^-_{s_2}J^-_{s_3}\rangle_{\text{h}}. \tag{3.15}$$

which is an even stronger statement than (3.10) since the homogeneous and non-homogeneous parts can be computed in just the free bosonic theory or just in the free fermionic theory [157]. When we are outside the triangle inequality, such that (3.12) does not hold, the only contribution is from the non-homogeneous parts [45], i.e. both the parity even structures and the parity odd structure are non-homogeneous. Also [45]

$$\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\text{nh,FB}} \neq \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\text{nh,FF}}. \tag{3.16}$$

In that case the distinction in (3.13) no longer holds and we can only represent in terms of the free theories as in (3.11) in spinor helicity variables.

### 3.2.3 4-point functions

Now, we turn our attention to the case of 4-point correlators.

For general 4-point correlators, we obtain the following form in momentum space [4]

$$\begin{aligned}\langle OOOO\rangle_{\text{QF}} &= \tilde{N}(1+\tilde{\lambda}^2)^2\langle OOOO\rangle_{\text{FF}},\\ \langle J_sOOO\rangle_{\text{QF}} &= \tilde{N}(1+\tilde{\lambda}^2)(\langle J_sOOO\rangle_{\text{FF}} + \tilde{\lambda}\langle J_sOOO\rangle_{\text{CB}}),\\ \langle J_{s_1}J_{s_2}OO\rangle_{QF} &= \tilde{N}\langle J_{s_1}J_{s_2}OO\rangle_{\text{FF}} + \tilde{N}\tilde{\lambda}\langle \epsilon\cdot J_{s_1}J_{s_2}OO\rangle_{\text{FF-CB}} + \tilde{N}\tilde{\lambda}^2\langle J_{s_1}J_{s_2}OO\rangle_{\text{CB}},\\ \langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{\text{QF}} &= \tilde{N}\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{\text{FF}} + \tilde{N}\tilde{\lambda}\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{\text{CB}},\\ \langle J_{s_1}J_{s_2}J_{s_3}J_{s_4}\rangle_{\text{QF}} &= \frac{\tilde{N}}{(1+\tilde{\lambda}^2)}\Big[\langle J_{s_1}J_{s_2}J_{s_3}J_{s_4}\rangle_{\text{FF}} + \tilde{\lambda}\langle \epsilon\cdot J_{s_1}J_{s_2}J_{s_3}J_{s_4}\rangle_{\text{FF-CB}} + \tilde{\lambda}^2\langle J_{s_1}J_{s_2}J_{s_3}J_{s_4}\rangle_{\text{CB}}\Big].\end{aligned} \tag{3.17}$$

[4]The results presented in this section may not be the unique solution to the SBHS equations. However, the structure of the higher spin equations are very tight as solving for say $\langle JJJJ\rangle$ does require information about $\langle JJTO\rangle$, $\langle JJOO\rangle$. To solve for $\langle JJOO\rangle$ we need to know the form of $\langle TOOO\rangle$. To solve for $\langle TOOO\rangle$ we need to know $\langle OOOO\rangle$. Thus, we see that the solutions are highly interconnected and even if there are more solutions to the SBHS equation, they will be extremely constrained.

To get a more intuitive form of these correlators, we convert the above expressions to spinor helicity variables. The general expression for an arbitrary correlator in spinor helicity looks like [5],

$$\begin{aligned}
\langle J_s OOO\rangle_{\rm QF} &= \frac{\tilde{N}}{\cos^3\frac{\pi\lambda_f}{2}}(\cos\frac{\pi\lambda_f}{2}\langle J_s OOO\rangle_{\rm FF} + \sin\frac{\pi\lambda_f}{2}\langle J_s OOO\rangle_{\rm CB}),\\
\langle J_{s_1}J_{s_2}OO\rangle_{QF} &= \frac{\tilde{N}}{\cos^2\frac{\pi\lambda_f}{2}}e^{-\frac{i\pi\lambda_f}{2}}\Big[\cos\frac{\pi\lambda_f}{2}\langle J_{s_1}J_{s_2}OO\rangle_{\rm FF} + i\sin\frac{\pi\lambda_f}{2}\langle J_{s_1}J_{s_2}OO\rangle_{\rm CB}\Big],\\
\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{\rm QF} &= \frac{\tilde{N}}{\cos\frac{\pi\lambda_f}{2}}(\cos\frac{\pi\lambda_f}{2}\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{FF} + \sin\frac{\pi\lambda_f}{2}\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{CB}),\\
\langle J_{s_1}J_{s_2}J_{s_3}J_{s_4}\rangle_{\rm QF} &= \tilde{N}e^{-\frac{i\pi\lambda_f}{2}}\Big[\cos\frac{\pi\lambda_f}{2}\langle J_{s_1}J_{s_2}J_{s_3}J_{s_4}\rangle_{\rm FF} + i\sin\frac{\pi\lambda_f}{2}\langle J_{s_1}J_{s_2}J_{s_3}J_{s_4}\rangle_{\rm CB}\Big].
\end{aligned} \tag{3.18}$$

It is clear that for $\lambda_f = 0$, we get the expression for FF correlator. For $\lambda_f = 1$ we get the CB result. For $\langle J_s OOO\rangle$ and $\langle J_{s_1}J_{s_2}J_{s_3}O\rangle$ at $\lambda_f = 1$ we need to appropriately redefine the correlator by absorbing the coupling constant dependent factor.
As was discussed in (3.15), in the case of three-point correlators we can get an even stronger statement that just the FB or just the FF theory is enough to construct QF theory correlation function. However as of yet we don't have such a statement for four point functions.

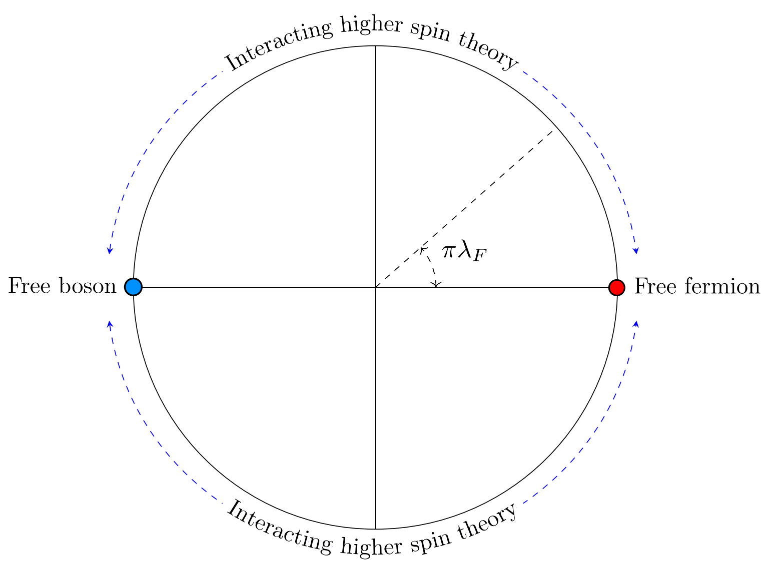


Figure 3.1: A chart visualizing the space of theories. Exactly conserved or weakly broken at large$-N$ higher spin theories lie on the circle of unit radius.

The above figure summarises our finding that correlation functions in CS matter theory are obtainable from the free theories with some anyonic phase factor [45]. For

[5] We focus on the case where all currents have negative helicity for concreteness and suppress the helicity label.

two and three point functions we can just start with the FB or FF answer and appropriately multiply with the anyonic phase factor to obtain the result in CS matter theory. For the four (and higher) point case, we do require both the FF and the FB results to get the result for the CS matter theory. We will now present the details of the computations in order to arrive at these results.

## 3.3 The Slightly Broken higher spin algebra

The existence of the higher spin currents $J_s, s > 2$ and their associated higher spin charges $Q_s \sim \int d\Sigma J_s$ lead to what are called slightly broken higher spin equations. In particular, lets focus on the spin-4 current $J_4^{\mu_1\mu_2\mu_3\mu_4}$. Its associated charge is ($n_\sigma$ is the normal to the constant time slice on which this charge is defined),

$$Q_4^{\mu\nu\rho} = \int d\Sigma\, n_\sigma J_4^{\mu\nu\rho\sigma}(x). \tag{3.19}$$

The associated Ward-Takahashi identity reads,

$$\langle [Q_4^{\mu\nu\rho}, J_{s_1} \cdots J_{s_n}] \rangle = \int d^3x \langle \partial_\sigma J_4^{\mu\nu\rho\sigma}(x) J_{s_1} \cdots J_{s_n}] \rangle. \tag{3.20}$$

If the current is conserved, the RHS vanishes (throwing away the boundary term) and we obtain the higher spin charge Ward identities. However, in the CS-matter theory, the current is non-conserved at $O\left(\frac{1}{N}\right)$. For spin-4, the divergence takes the form [158],

$$\partial_\sigma J^\sigma_{\mu\nu\rho} = r_0 \partial_\mu O(x) T_{\nu\rho}(x) + \epsilon_{\mu ab}\big(a_0 J_a(x)\partial_\rho\partial_\nu J_b(x) + b_1 J_a(x)\partial_b\partial_\nu J_\rho(x) + e_0 J_\nu(x)\partial_a\partial_\rho J_b(x)\big). \tag{3.21}$$

The constants $r_0, a_0, b_1, c_0$ all equal $\frac{1}{N}$ times a function of $\tilde{\lambda}$. Exact expressions will be given shortly. On the LHS of the Ward-Takahashi identity, we can figure out the algebra of $Q_4$ with any of the spin-s currents explicitly too. Further, given the fact that the RHS of (3.20) has a $\frac{1}{N}$ factor, only the disconnected part of the correlator appearing inside the integrand will contribute. This is called large-N factorization. This is to ensure that each of the two correlators the integrand each gives a factor of $N$ thus ensuring the expression is of $O(N)$. This equation can in principle be used to solve for the CS+matter theory correlators. However, what is interesting is that we can actually avoid explicitly solving this equation. It turns out that writing an ansatz inspired by the two and three point functions, we can actually map this interacting theory equation to the free bosonic and free fermionic equations at each order [158] as we will show in the next section. Before we conclude this section however, we present the higher spin algebras for the free fermionic and quasi fermionic theories which will be used in what is to follow. the FF theory in three dimensions has a pseudo-scalar operator $O$ with dimension $\Delta = 2$ and an infinite tower of exactly conserved currents $J_s$ with integer spin $s$ and dimension

$\Delta = s + 1$. The algebra of charges $Q_3$ and $Q_4$ with $O$ and $J_1$ is as follows :

$$[Q_{\mu\nu}, O] = \epsilon_{\mu ab}\, \partial^a \partial_\nu J^b, \tag{3.22}$$
$$[Q_{\mu\nu}, J_\rho] = \epsilon_{\sigma\rho(\mu}\partial_{\nu)}\partial_\sigma O + i\partial_{(\mu}T_{\nu)\rho} + \partial_\rho T_{\mu\nu}, \tag{3.23}$$
$$[Q_{\mu\nu}, T_{\alpha\beta}] = p_\mu p_\nu p_\alpha J_\beta + p_\mu p_\beta p_\alpha J_\nu + g_{\mu\nu} p_\alpha p_1^2 J_\beta + p_\mu J_{\nu\alpha\beta}, \tag{3.24}$$

$$[Q_{\mu\nu\rho}, O] = \partial_\mu\partial_\nu\partial_\rho O + g_{\mu\nu}\partial_\rho \Box O + \epsilon_{\mu ab}\partial_\nu\partial_a T_{b\rho}, \tag{3.25}$$
$$\begin{aligned}[Q_{\mu\nu\rho}, J_\alpha] &= a\, \partial_{(\mu}\partial_\nu\partial_{\rho)} J_\alpha + b\, \partial_\alpha \partial_{(\mu}\partial_\nu J_{\rho)} + \partial_\alpha J_{(\mu\nu\rho)} \\ &\quad + g_{(\mu\nu}\partial_{\rho)}\Box J_\alpha + g_{\mu\nu}\partial_\alpha \Box J_\rho + c g_{\alpha(\mu}\partial_\nu \Box J_{\rho)},\end{aligned} \tag{3.26}$$
$$[Q_{\mu\nu\rho}, T_{\alpha\beta}] = b_1 \partial_\mu\partial_\nu\partial_\rho T_{\alpha\beta} + b_2 \partial_\mu\partial_\nu\partial_\alpha T_{\rho\beta} + b_3 \partial_\mu J_{\nu\rho\alpha\beta} + b_4 \partial_\alpha J_{\mu\nu\rho\beta}. \tag{3.27}$$

and in the case of QF theories, the non-conservation for $J_3$ and $J_4$ are given by

$$\partial_\sigma J^\sigma_{\mu\nu} = a\partial_{(\mu} J_{\nu)} O + b J_{(\nu}\partial_{\mu)} O, \tag{3.28}$$
$$\partial_\sigma J^\sigma_{\mu\nu\rho} = r_0 \partial_\mu O(x) T_{\nu\rho}(x) + \epsilon_{\mu ab}\big(a_0 J_a(x)\partial_\rho\partial_\nu J_b(x) + b_1 J_a(x)\partial_b\partial_\nu J_\rho(x) + e_0 J_\nu(x)\partial_a\partial_\rho J_b(x)\big). \tag{3.29}$$

here the several factors $\{f_i, b_j\}$ are no longer constants but pick up a $\tilde{\lambda}$ dependence. The precise values are given as follows,

$$\begin{aligned} a &= \frac{3}{5\tilde{N}}\frac{\tilde{\lambda}}{1+\tilde{\lambda}^2}, \quad b = -\frac{2}{5\tilde{N}}\frac{\tilde{\lambda}}{1+\tilde{\lambda}^2}, \\ r_0 &= \frac{480}{7\tilde{N}}\frac{\tilde{\lambda}}{1+\tilde{\lambda}^2}, \quad b_1 = -\frac{512}{3\tilde{N}}\frac{\tilde{\lambda}}{1+\tilde{\lambda}^2}, \quad b_2 = \frac{128}{3\tilde{N}}\frac{\tilde{\lambda}}{1+\tilde{\lambda}^2} \\ f_0 &= \frac{64}{\tilde{N}}\frac{\tilde{\lambda}}{1+\tilde{\lambda}^2}, \quad f_1 = -\frac{128}{3\tilde{N}}\frac{\tilde{\lambda}}{1+\tilde{\lambda}^2}. \end{aligned} \tag{3.30}$$

The algebra for the slightly broken HS theory is also slightly modified where the coefficients may or may not pick up a dependence of the coupling constant.

$$[Q_{\mu\nu}, O] = c_1 \epsilon_{\mu ab}\, \partial^a \partial_\nu J^b, \tag{3.31}$$
$$[Q_{\mu\nu}, J_\rho] = \frac{1}{1+\tilde{\lambda}^2} c_2 \epsilon_{\sigma\rho(\mu}\partial_{\nu)}\partial_\sigma O + c_3 \partial_{(\mu}T_{\nu)\rho} + c_4 \partial_\rho T_{\mu\nu}, \tag{3.32}$$

$$[Q_{\mu\nu\rho}, O] = a_1 \partial_\mu\partial_\nu\partial_\rho O + a_2 g_{\mu\nu}\partial_\rho \Box O + a_3 \epsilon_{\mu ab}\partial_\nu\partial_a T_{b\rho}, \tag{3.33}$$
$$\begin{aligned}[Q_{\mu\nu\rho}, J_\alpha] &= b_1\, \partial_{(\mu}\partial_\nu\partial_{\rho)} J_\alpha + b_2\, \partial_\alpha \partial_{(\mu}\partial_\nu J_{\rho)} + b_3 \partial_\alpha J_{(\mu\nu\rho)} \\ &\quad + g_{(\mu\nu}\partial_{\rho)}\Box J_\alpha + g_{\mu\nu}\partial_\alpha \Box J_\rho + c g_{\alpha(\mu}\partial_\nu \Box J_{\rho)},\end{aligned} \tag{3.34}$$
$$[Q_{\mu\nu\rho}, T_{\alpha\beta}] = b_1 \partial_\mu\partial_\nu\partial_\rho T_{\alpha\beta} + b_2 \partial_\mu\partial_\nu\partial_\alpha T_{\rho\beta} + b_3 \partial_\mu J_{\nu\rho\alpha\beta} + b_4 \partial_\alpha J_{\mu\nu\rho\beta}. \tag{3.35}$$

the precise values of all the $\{a_i, b_j, c_k\}$ sometimes cannot be determined but it is usually not required. The only information we know about these coefficients is if they are modified for SBHS theories. For $\tilde{\lambda} = 0$, this algebra is that of the FF theory whereas at infinite $\tilde{\lambda}$ it becomes that of the CB theory where the only difference is that the scalar $O$ is even under parity unlike the FF theory. Our aim in this chapter is to map the correlation function in SBHS theories to the free theories which does not require explicit knowledge of these coefficients.

## 3.4 Mapping Slightly broken HS correlators to free theory correlators

In the previous section we saw that the results in the QF theory can be written in terms of the FF and FB theory results. In this section we outline the methodology which maps correlation functions in SBHS theories to the free theories. In the next section we show how our methodology works explicitly.

We make use of the SBHS equation following [122] to compute the result for correlation functions. Our method can be summarized by the following steps::

**Step 1:** Choose an appropriate charge operator and a seed correlator to write down the higher spin equation in the interacting theory, say the quasi fermionic theory.

**Step 2:** Repeat the same for the free and critical theories.

**Step 3:** Write down the ansatz for each correlator that appears in the slightly broken higher spin equation in the interacting theory.

**Step 4:** Map the equations that are at the lowest $O(\tilde{\lambda}^0)$ and highest orders in the coupling to the free fermion and critical bosonic theories respectively. This helps us identify the contributions at the lowest and highest orders as the ones from the free and critical theories, respectively.

**Step 5:** Write the pole equations which are obtained by expanding the HSE around $\tilde{\lambda} = \pm i$ to obtain the remaining unknowns in the ansatz [6].

**Step 6:** Plug back the solution in the higher spin equation and map it to a linear combination of the equations in the free and critical theories [7].
The above map allows us to identify the unknowns in the ansatz of the interacting theory correlator purely in terms of the FF and CB theory results.

## 3.5 Examples of the general principle

### 3.5.1 $\langle JJO \rangle_{\text{QF}}$

As discussed earlier in (3.9), $\langle J_\alpha J_\beta O \rangle$ in the QF theory has an odd part. In our analysis we make use of higher spin equations and follow the steps given at the end of section 3.4.

**Step 1:** We choose the charge operator and the seed correlator to be $Q_3$ and $\langle JOO \rangle$ respectively to write the following HSE in position space [39]

[6] For certain correlators such as $\langle JJJJ \rangle$ it turns out that the pole equations are not sufficient and one has to resort to the higher spin equations at intermediate orders in the coupling to extract out the remaining unknowns.

[7] For the case of four-point functions, spinor-helicity variables are extremely useful at this step since the map between parity even and odd parts of the HSE is much more transparent in these variables.

$$\langle [Q_{\mu\nu}, J_\alpha(x_1)] O(x_2) O(x_3)\rangle_{\text{QF}} + \langle J_\alpha(x_1)[Q_{\mu\nu}, O(x_2)] O(x_3)\rangle_{\text{QF}}$$
$$+ \langle J_\alpha(x_1) O(x_2)[Q_{\mu\nu}, O(x_3)]\rangle_{\text{QF}} = \int_x \langle \partial_\sigma J^\sigma_{\mu\nu}(x) J_\alpha(x_1) O(x_2) O(x_3)\rangle_{\text{QF}}. \quad (3.36)$$

We utilize the higher spin algebra (3.33) [8] and the current non conservation for $J_{\mu\nu\rho}$ (3.29) in the HSE (3.36). We then perform an integration by parts and use the large $N$ factorisation of the 5-point correlator that appears on the RHS. After a subsequent Fourier transform of the HSE we obtain the following HSE as an algebraic equation in terms of the correlators of the interacting theory

$$\begin{aligned} & p_{1(\mu}\langle T_{\nu)\alpha}(p_1) O(p_2) O(p_3)\rangle_{\text{QF}} + p_{1\alpha}\langle T_{\mu\nu}(p_1) O(p_2) O(p_3)\rangle_{\text{QF}} \\ & + \left(\epsilon_{(\mu ab} p_{2a} p_{2\nu)} \langle J_\alpha(p_1) J_b(p_2) O(p_3)\rangle_{\text{QF}} + \{2 \leftrightarrow 3\}\right) \\ & = \left(\frac{\tilde{\lambda}}{1+\tilde{\lambda}^2} p_{2(\mu} p_2 \langle J_\alpha(p_1) J_{\nu)}(p_2) O(p_3)\rangle_{\text{QF}} + \{2 \leftrightarrow 3\}\right), \end{aligned} \quad (3.37)$$

where the notation $p_{1(\mu}T_{\nu)\alpha}$ denotes $\mu, \nu$ symmetrisation of $p_{1\mu}T_{\nu\alpha}$. We note that Fourier transforming (3.37) gets rid of the integral on the RHS of (3.36) and thus makes it easier to factorise the resulting 5-point function [39].

**Step 2:** We now write down the corresponding HSEs for the FF theory

$$\begin{aligned} & p_{1(\mu}\langle T_{\nu)\alpha}(p_1) O(p_2) O(p_3)\rangle_{\text{FF}} + p_{1\alpha}\langle T_{\mu\nu}(p_1) O(p_2) O(p_3)\rangle_{\text{FF}} \\ & + \left(\epsilon_{(\mu ab} p_{2a} p_{2\nu)} \langle J_\alpha(p_1) J_b(p_2) O(p_3)\rangle_{\text{FF}} + \{2 \leftrightarrow 3\}\right) = 0. \end{aligned} \quad (3.38)$$

and the CB theory,

$$\begin{aligned} & p_{1(\mu}\langle T_{\nu)\alpha}(p_1) O(p_2) O(p_3)\rangle_{\text{CB}} + c_2 p_{1\alpha}\langle T_{\mu\nu}(p_1) O(p_2) O(p_3)\rangle_{\text{CB}} \\ & \qquad = p_{2(\mu} p_2 \langle J_\alpha(p_1) J_{\nu)}(p_2) O(p_3)\rangle_{\text{CB}} + \{2 \leftrightarrow 3\}. \end{aligned} \quad (3.39)$$

**Step 3:** We consider the following ansatz for the correlators that appear in the HSE (3.37) [122]

$$\begin{aligned} \langle T_{\nu\alpha}(p_1) O(p_2) O(p_3)\rangle_{\text{QF}} &= \tilde{N}(1+\tilde{\lambda}^2)\langle T_{\nu\alpha}(p_1) O(p_2) O(p_3)\rangle_{\text{FF}}, \\ \langle J_\alpha(p_1) J_\beta(p_2) O(p_3)\rangle_{\text{QF}} &= \tilde{N}\langle J_\alpha(p_1) J_\beta(p_2) O(p_3)\rangle_{Y_0} + \tilde{N}\tilde{\lambda}\langle J_\alpha(p_1) J_\beta(p_2) O(p_3)\rangle_{\text{odd}}. \end{aligned} \quad (3.40)$$

where $\langle J_\alpha(p_1) J_\beta(p_2) O(p_3)\rangle_{Y_0}$ and $\langle J_\alpha(p_1) J_\beta(p_2) O(p_3)\rangle_{\text{odd}}$ are the unknown parts that we wish to find. The HSE (3.37) can then be written at different orders in the coupling.

**Step 4:** At $O(\tilde{\lambda}^0)$ of (3.37) the HSE is identical to the FF theory equation (3.38) which gives

$$\langle J_\alpha(p_1) J_\beta(p_2) O(p_3)\rangle_{Y_0} = \langle J_\alpha(p_1) J_\beta(p_2) O(p_3)\rangle_{FF}. \quad (3.41)$$

[8] We keep the coefficients in the algebra arbitrary since fixing them will not affect our computation.

Similarly, the highest order equation, namely the one at $O(\tilde{\lambda}^2)$ is identical to the CB[9] equation (3.39). Thus, we identify

$$\begin{aligned} \langle J_\alpha(p_1)J_\beta(p_2)O(p_3)\rangle_{\text{odd}} &= \langle J_\alpha(p_1)J_\beta(p_2)O(p_3)\rangle_{\text{CB}}, \\ \langle T_{\nu\alpha}(p_1)O(p_2)O(p_3)\rangle_{\text{FF}} &= \langle T_{\nu\alpha}(p_1)O(p_2)O(p_3)\rangle_{\text{CB}}. \end{aligned} \tag{3.42}$$

**Step 5:** We now write the pole equation. We expand (3.37) around the pole $\tilde{\lambda} = \pm i$ to get the following pole equations

$$\begin{aligned} &\epsilon_{(\mu a b}p_{2a}p_{2\nu)}\langle J_\alpha(p_1)J_b(p_2)O(p_3)\rangle_{\text{odd}} + \{2 \leftrightarrow 3\} = p_{2(\mu}p_2\langle J_\alpha(p_1)J_{\nu)}(p_2)O(p_3)\rangle_{\text{FF}} + \{2 \leftrightarrow 3\}\,, \\ &\epsilon_{(\mu a b}p_{2a}p_{2\nu)}\langle J_\alpha(p_1)J_b(p_2)O(p_3)\rangle_{\text{FF}} + \{2 \leftrightarrow 3\} = p_{2(\mu}p_2\langle J_\alpha(p_1)J_{\nu)}(p_2)O(p_3)\rangle_{\text{odd}} + \{2 \leftrightarrow 3\}\,. \end{aligned} \tag{3.43}$$

which helps us identify the unknown correlator $\langle J_\alpha(p_1)J_\nu(p_2)O(p_3)\rangle_{\text{odd}}$ in terms of the same correlator in free theory [44, 157]

$$\langle J_\alpha(p_1)J_\nu(p_2)O(p_3)\rangle_{\text{odd}} = \frac{1}{p_2}\epsilon_{\nu a b}\, p_{2a}\, \langle J_\alpha(p_1)J_b(p_2)O(p_3)\rangle_{\text{FF}}. \tag{3.44}$$

The expression for $\langle JJO\rangle_{\text{odd}}$ obtained from (3.44) is consistent with the results obtained using perturbative techniques in special kinematic regimes [77, 129] and by solving conformal Ward identities in momentum space [44, 157]. [10]

**Step 6:** We now use our results to map the SBHS equation to the free theory HSE. To do this, we use (3.44) and substitute it back into (3.37) and see that the remaining HSE maps to the free theory equation. Thus we see that the solution for the odd piece that we obtained from the pole equation is consistent with the HSE at any order.

This confirms the result obtained for $\langle J_\alpha(p_1)J_\beta(p_2)O(p_3)\rangle_{\text{odd}}$ in (3.43). Thus we have completely determined the 3-point spinning correlator $\langle J_\alpha(p_1)J_\beta(p_2)O(p_3)\rangle_{\text{odd}}$ in the interacting theory purely in terms of free theory correlators i.e

$$\langle J_\alpha(p_1)J_\beta(p_2)O(p_3)\rangle_{\text{QF}} = \tilde{N}\langle J_\alpha(p_1)J_\beta(p_2)O(p_3)\rangle_{\text{FF}} + \tilde{N}\tilde{\lambda}\frac{\epsilon_{\beta a b}\, p_{2a}}{p_2}\langle J_\alpha(p_1)J_b(p_2)O(p_3)\rangle_{\text{FF}}. \tag{3.45}$$

Now in spinor-helicity we have $\langle \epsilon \cdot JJO\rangle \to \pm i\langle JJO\rangle$ depending on the helicity. Thus the final expression becomes,

$$\langle JJO\rangle_{\text{QF}} = \tilde{N}(1 + i\tilde{\lambda})\langle JJO\rangle_{\text{FF}} = \frac{iN(1 - e^{-i\pi\lambda_f})}{\pi\lambda_f}\langle JJO\rangle_{\text{FF}}. \tag{3.46}$$

We note that the expression of the correlator picks up an anyonic phase when we express the full correlator only in terms of the FF theory correlator.

[9] This is because the CB theory is obtained in the limit $\tilde{\lambda} \to \infty$ of the quasi fermionic theory.

[10] We note that (3.44) is one of the solutions to (3.43) where we ignore $\{2 \leftrightarrow 3\}$ exchanges. We will adopt a similar strategy while computing 4-point functions where we again ignore such permutations. However as we shall see, pole equations are not sufficient to get the odd piece in case of certain 4-point functions and we will then have to make full use of the slightly broken HS equations and provide a consistent solution to the higher spin equation.

### 3.5.2 $\langle TTT\rangle_{\mathbf{QF}}$

In this section we will make use of the HSE to obtain the odd part of $\langle TTT\rangle$ in the QF theory. As before we follow the steps presented at the end of section 3.4.

**Step 1:** We choose the charge operator and the seed correlator to be $Q_4$ and $\langle OTT\rangle$ respectively to write the following HSE in position space

$$\begin{aligned}&\langle[Q_{\mu\nu\rho},O(x_1)]T_{\alpha\beta}(x_2)T_{\gamma\theta}(x_3)\rangle_{\rm QF}+\langle O(x_1)[Q_{\mu\nu\rho},T_{\alpha\beta}(x_2)]T_{\gamma\theta}(x_3)]\rangle_{\rm QF}\\&+\langle O(x_1)T_{\alpha\beta}(x_2)[Q_{\mu\nu\rho},T_{\gamma\theta}(x_3)]\rangle_{\rm QF}=\int_x\langle\partial_\sigma J^\sigma_{\mu\nu\rho}(x)O(x_1)T_{\alpha\beta}(x_2)T_{\gamma\theta}(x_3)\rangle_{\rm QF}.\end{aligned}\tag{3.47}$$

We make use of the higher spin algebra (3.33) and (3.35) along with the current non conservation (3.29) to obtain the following HSE in momentum space,

$$\begin{aligned}&p_{1(\mu}p_{1\nu}p_{1\rho)}\langle O(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm QF}+\epsilon_{(\mu ab}p_{1a}p_{1\nu}\langle T_{b\rho)}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm QF}\\&+\Big(p_{2(\mu}p_{2\nu}p_{2\rho)}\langle O(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm QF}+p_{2(\mu}p_{2\nu}p_{2\alpha}\langle O(p_1)T_{\rho)\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm QF}+\{2\leftrightarrow3\}\Big)\\&=\frac{\tilde\lambda}{1+\tilde\lambda^2}\Bigg[p_{1(\mu}p_1\langle T_{\nu\rho)}(p_2)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm QF}+\Big(p_{2(\mu}\langle T_{\nu\rho)}T_{\alpha\beta}\rangle_{\rm QF}\langle O(p_1)O(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm QF}\\&+\{2\leftrightarrow3\}\Big)\Bigg].\end{aligned}\tag{3.48}$$

**Step 2:** We now write down the corresponding higher spin equation for the FF theory

$$\begin{aligned}&p_{1(\mu}p_{1\nu}p_{1\rho)}\langle O(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm FF}-\epsilon_{(\mu ab}p_{1a}p_{1\nu}\langle T_{b\rho)}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm FF}\\&+\big(p_{2(\mu}p_{2\nu}p_{2\rho)}\langle O(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm FF}+p_{2(\mu}p_{2\nu}p_{2\alpha}\langle O(p_1)T_{\rho)\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm FF}+\{2\leftrightarrow3\}\big)=0.\end{aligned}\tag{3.49}$$

and similarly for the CB theory,

$$\begin{aligned}&p_{1(\mu}p_{1\nu}p_{1\rho)}\langle O(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm CB}+\big(p_{2(\mu}p_{2\nu}p_{2\rho)}\langle O(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm CB}\\&\qquad+p_{2(\mu}p_{2\nu}p_{2\alpha}\langle O(p_1)T_{\rho)\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm CB}+\{2\leftrightarrow3\}\big)\\&=\Bigg[p_{1(\mu}p_1\langle T_{\nu\rho)}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm CB}+\Big(p_{2(\mu}\langle T_{\nu\rho)}T_{\alpha\beta}\rangle_{\rm QF}\langle O(p_1)O(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm CB}\\&+\{2\leftrightarrow3\}\Big)\Bigg].\end{aligned}\tag{3.50}$$

**Step 3:** We consider the following ansatz for the correlators that appear in the HSE (3.48) [122]

$$\begin{aligned}\langle O(p_1)O(-p_1)\rangle_{\rm QF}&=\tilde N(1+\tilde\lambda^2)\langle O(p_1)O(-p_1)\rangle_{\rm FF},\\\langle O(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm QF}&=\tilde N\langle O(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm FF}+\tilde N\tilde\lambda\langle O(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm CB},\\\langle T_{\rho\sigma}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\rm QF}&=\tilde N\frac{1}{1+\tilde\lambda^2}\big[\langle T_{\rho\sigma}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{Y_0}\\&\quad+\tilde\lambda^2\langle T_{\rho\sigma}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{Y_2}+\tilde\lambda\langle T_{\rho\sigma}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{odd}\big].\end{aligned}\tag{3.51}$$

Our goal is to determine the parity odd part of $\langle TTT\rangle$, viz. $\langle T_{\rho\sigma}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\text{odd}}$ in terms of the free theory correlators. We can now write the HSE at various orders in the coupling constant.

**Step 4:** At $O(\tilde{\lambda}^0)$ of (3.48), the HSE is identical to the FF HSE (3.49) which gives

$$\langle T_{\rho\sigma}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{Y_0} = \langle T_{\rho\sigma}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\text{FF}}. \tag{3.52}$$

Similarly, the highest order equation namely the $O(\tilde{\lambda}^3)$ is identical to the critical bosonic equation (3.50) since the CB theory is the $\tilde{\lambda} \to \infty$ limit of the QF theory. This happens after we identify

$$\langle T_{\rho\sigma}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{Y_2} = \langle T_{\rho\sigma}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\text{CB}}. \tag{3.53}$$

Hence we get 2 of the 3 unknowns.

**Step 5:** To find the third unknown we expand (3.48) around the pole $\tilde{\lambda} = \pm i$ to get the following pole equations

$$\begin{aligned}
&\epsilon_{(\mu ab}p_{1a}p_{1\nu}\langle T_{b\rho)}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\text{odd}} \\
&\qquad = p_{1(\mu}p_1\Big(\langle T_{\nu\rho)}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\text{FB}} - \langle T_{\nu\rho}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\text{FF}}\Big), \\
&\epsilon_{(\mu ab}p_{1a}p_{1\nu}\Big(\langle T_{b\rho)}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\text{FB}} - \langle T_{b\rho)}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\text{FF}}\Big) \\
&\qquad = p_{1(\mu}p_1\langle T_{\nu\rho)}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\text{odd}}.
\end{aligned} \tag{3.54}$$

which helps us identify the unknown correlator $\langle T_{\nu\rho}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\text{odd}}$ in terms of the same correlator in the free theories. Thus from (3.51), after contracting with $\Pi_{\mu\nu}(p_1)$ and $p_{1\rho}$ we get $\langle TTT\rangle_{\text{odd}}$ to be

$$\begin{aligned}
&\langle T_{\mu\nu}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\text{odd}} \\
&= \frac{1}{p_1}\epsilon_{\mu ab}p_{1a}\Big(\langle T_{b\nu}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\text{FB}} - \langle T_{b\rho)}(p_1)T_{\alpha\beta}(p_2)T_{\gamma\theta}(p_3)\rangle_{\text{FF}}\Big)
\end{aligned} \tag{3.55}$$

**Step 6:** We now use our results to map the SBHS equation to the free theory HSE. We use the expression for $\langle TTT\rangle_{\text{odd}}$ and substitute it back into the $O(\tilde{\lambda})$ and $O(\tilde{\lambda}^2)$ equations and see that they map to the free theory equations. Therefore we see that that our solution for $\langle TTT\rangle_{\text{odd}}$ obtained from (3.55) solves the entire higher spin equation and is also consistent with results obtained in [44, 157]. The same can also be obtained by solving conformal Ward identities in momentum space. Thus we have seen that in the QF theory the correlator $\langle TTT\rangle$ is given by

$$\begin{aligned}
\langle TTT\rangle_{\text{QF}} &= \frac{\tilde{N}}{1+\tilde{\lambda}^2}\left(\langle TTT\rangle_{\text{FF}} + \tilde{\lambda}\langle TTT\rangle_{\text{odd}} + \tilde{\lambda}^2\langle TTT\rangle_{\text{FB}}\right) \\
&= \frac{\tilde{N}}{2}\left(\langle TTT\rangle_{\text{FF+FB}} + \frac{1-\tilde{\lambda}^2}{1+\tilde{\lambda}^2}\langle TTT\rangle_{\text{FF-FB}} + \frac{2\tilde{\lambda}}{1+\tilde{\lambda}^2}\langle \epsilon\cdot TTT\rangle_{\text{FB-FF}}\right).
\end{aligned} \tag{3.56}$$

Now in spinor-helicity variables we have $\langle \epsilon \cdot TTT \rangle \to i\langle TTT \rangle$ (focussing on the case when all stress tensors have negative helicity for illustrative purposes) and thus

$$\begin{aligned}\langle TTT \rangle_{\text{QF}} &= \frac{\tilde{N}}{2}\left(\langle TTT \rangle_{\text{FF+FB}} + \left(\frac{1+i\tilde{\lambda}}{1-i\tilde{\lambda}}\right)\langle TTT \rangle_{\text{FF-FB}}\right) \\ &= \frac{\tilde{N}}{2}\left(\langle TTT \rangle_{\text{FF+FB}} + e^{-i\pi\lambda_f}\langle TTT \rangle_{\text{FF-FB}}\right). \end{aligned} \tag{3.57}$$

Thus we see the presence of an anyonic phase yet again in the expression for the correlator. There is one more representation which makes the duality manifest

$$\langle TTT \rangle_{\text{QF}} = e^{-i\pi\frac{\lambda_f}{2}}\left(\cos\frac{\pi\lambda_f}{2}\langle TTT \rangle_{\text{FF}} + i\sin\frac{\pi\lambda_f}{2}\langle TTT \rangle_{\text{FB}}\right). \tag{3.58}$$

Note that at $\lambda_f = 0$ it gives the FF and at $\lambda_f = 1$, it gives the FB answer.

### 3.5.3 $\langle TOOO \rangle_{\text{QF}}$

To start with, we look at the simple example of $\langle TOOO \rangle_{\text{QF}}$. The result of this part was obtained first in position space by [151], in momentum space in [39] and in Mellin space in [152]. Below we work in momentum space.

**Step 1:** We choose the charge operator and the seed correlator to be $Q_4$ and $\langle OOOO \rangle$ respectively. Then we make use of the higher spin algebra (3.33) and (3.35) along with the current non conservation (3.29) to obtain the following HSE in momentum space [11]

$$\begin{aligned}&p_{1\mu}p_{1\nu}p_{1\rho}\langle OOOO \rangle_{\text{QF}} + g_{(\mu\nu}p_{1\rho)}p_1^2\langle OOOO \rangle_{\text{QF}} + \epsilon_{ab(\mu}p_{1\nu}p_{1a}\langle T_{b\rho)}OOO \rangle_{\text{QF}} \\ &+ \{1 \leftrightarrow 2\} + \{1 \leftrightarrow 3\} + \{1 \leftrightarrow 4\} = \tilde{\lambda}p_{1(\mu}p_1\langle T_{\nu\rho)}OOO \rangle_{\text{QF}} + \{1 \leftrightarrow 2\} + \{1 \leftrightarrow 3\} + \{1 \leftrightarrow 4\}.\end{aligned} \tag{3.59}$$

**Step 2:** We now write down the corresponding higher spin equations for the free theory

$$\begin{aligned}&p_{1\mu}p_{1\nu}p_{1\rho}\langle OOOO \rangle_{\text{FF}} + g_{(\mu\nu}p_{1\rho)}p_1^2\langle OOOO \rangle_{\text{FF}} + \epsilon_{ab(\mu}p_{1\nu}p_{1a}\langle T_{b\rho)}OOO \rangle_{\text{FF}} \\ &+ \{1 \leftrightarrow 2\} + \{1 \leftrightarrow 3\} + \{1 \leftrightarrow 4\} = 0.\end{aligned} \tag{3.60}$$

and similarly for the CB theory,

$$\begin{aligned}&p_{1(\mu}p_{1\nu}p_{1\rho)}\langle OOOO \rangle_{\text{CB}} + g_{(\mu\nu}p_{1\rho)}p_1^2\langle OOOO \rangle_{\text{CB}} + \{1 \leftrightarrow 2\} + \{1 \leftrightarrow 3\} + \{1 \leftrightarrow 4\} \\ &= p_{1(\mu}p_1\langle T_{\nu\rho)}OOO \rangle_{\text{CB}} + \{1 \leftrightarrow 2\} + \{1 \leftrightarrow 3\} + \{1 \leftrightarrow 4\}.\end{aligned} \tag{3.61}$$

**Step 3:** We consider the following ansatz for the correlators that appear in the HSE (3.59)

$$\begin{aligned}\langle TOOO \rangle_{\text{QF}} &= \tilde{N}(1+\tilde{\lambda}^2)\Big(\langle TOOO \rangle_{Y_0} + \tilde{\lambda}\langle TOOO \rangle_{Y_1}\Big), \\ \langle OOOO \rangle_{\text{QF}} &= \tilde{N}(1+\tilde{\lambda}^2)(\langle OOOO \rangle_{\text{FF}} + \tilde{\lambda}^2\langle OOOO \rangle_{\text{CB}}).\end{aligned} \tag{3.62}$$

[11]Strictly speaking, we should have the constants from the algebra appearing in each term of the LHS but since our aim is only to *map* the SBHS HSE to the free theory HSEs, we do not need to fix these constants to their numerical values. All we had to do was fix their $\tilde{\lambda}$ dependence.

**Step 4:** At $O(\tilde{\lambda}^0)$ of (3.59) we obtain the HSE to be identical to the FF theory equation (3.60) which gives

$$\langle TOOO\rangle_{Y_0} = \langle TOOO\rangle_{\text{FF}}. \tag{3.63}$$

Similarly, the highest order equation( $O(\tilde{\lambda}^4)$ ) is identical to the CB equation (3.61) since the CB theory is the $\tilde{\lambda} \to \infty$ limit of the quasi fermionic theory. This happens after we identify

$$\langle TOOO\rangle_{Y_1} = \langle TOOO\rangle_{\text{CB}}. \tag{3.64}$$

**Step 5:** We expand the HSE (3.59) around the point $\tilde{\lambda} = \pm i$ and obtain the following pole equations

$$\begin{aligned}&\epsilon_{ab(\mu}p_{1\nu}p_{1a}\langle T_{b\rho)}OOO\rangle_{\text{FF}} + \{1 \leftrightarrow 2\} + \{1 \leftrightarrow 3\} + \{1 \leftrightarrow 4\}\\ &= p_{1(\mu}p_1\langle T_{\nu\rho)}OOO\rangle_{\text{CB}} + \{1 \leftrightarrow 2\} + \{1 \leftrightarrow 3\} + \{1 \leftrightarrow 4\},\end{aligned} \tag{3.65}$$

and

$$\begin{aligned}&\epsilon_{ab(\mu}p_{1\nu}p_{1a}\langle T_{b\rho)}OOO\rangle_{\text{CB}} + \{1 \leftrightarrow 2\} + \{1 \leftrightarrow 3\} + \{1 \leftrightarrow 4\}\\ &= p_{1(\mu}p_1\langle T_{\nu\rho)}OOO\rangle_{\text{FF}} + \{1 \leftrightarrow 2\} + \{1 \leftrightarrow 3\} + \{1 \leftrightarrow 4\}.\end{aligned} \tag{3.66}$$

One can check again that (3.66) can be mapped to FF and CB equations. In momentum space it looks complicated to map the above equation to FF and CB equations as it requires some epsilon transforms. However, going to spinor helicity variables solves this problem as in spinor helicity variables the epsilon transform becomes trivial [12].
**Step 6:** Considering different order equations of (3.59), it can be shown directly in momentum space that

$$\text{HSE at O}(\tilde{\lambda}^2) = \text{FF HSE+CB HSE}. \tag{3.68}$$

The $O(\tilde{\lambda})$ equation which is the same as the $O(\tilde{\lambda}^3)$ equation is the same as an epsilon transform of the difference between the FF and CB equations. To see this we move to spinor helicity variables where it is easily seen. Thus we get,

$$\text{HSE at O}(\tilde{\lambda}) = \text{HSE at O}(\tilde{\lambda}^3) = \pm i(\text{FF HSE-CB HSE}). \tag{3.69}$$

Thus we obtain the following form for $\langle TOOO\rangle$ in the QF theory in spinor helicity variables

$$\langle TOOO\rangle_{\text{QF}} = \tilde{N}(1+\tilde{\lambda}^2)\Big(\langle TOOO\rangle_{\text{FF}} + \tilde{\lambda}\langle TOOO\rangle_{\text{CB}}\Big) \tag{3.70}$$

---

[12] One possible solution to these pole equations is

$$\epsilon_{ab(\mu}p_{1\nu}p_{1a}\langle T_{b\nu)}OOO\rangle_{\text{FF}} = \langle T_{\mu\nu)}OOO\rangle_{\text{CB}}. \tag{3.67}$$

We chose this solution such that equation (3.65) is satisfied individually for each permutation. A direct verification of this result requires a proper analysis of the contact terms which we leave for future works. However, we note that certain HSEs demand that this individual equivalence such as that of $\langle TTTO\rangle$. It is easy to check that using naive bootstrap argument involving single trace operator only, one gets (3.67).

Using the same procedure, we can also generalize the above result for correlators with arbitrary spin as follows in spinor helicity variables.

$$\langle J_s OOO\rangle_{\text{QF}} = \tilde{N}(1+\tilde{\lambda}^2)(\langle J_s OOO\rangle_{\text{FF}} + \tilde{\lambda}\langle J_s OOO\rangle_{\text{CB}}). \tag{3.71}$$

One can check the consistency of the result using the following argument: If we consider the divergence of the correlator,

$$\langle \partial\cdot J_s OOO\rangle_{\text{QF}} = (1+\tilde{\lambda}^2)(\langle \partial\cdot J_s OOO\rangle_{\text{FF}} + \tilde{\lambda}\langle \partial\cdot J_s OOO\rangle_{\text{CB}}). \tag{3.72}$$

due to the nonconservation of $J_s$ the left hand side is nonzero while in the RHS the free theory contribution drops out but the CB term is nonzero due to its nonconservation which is exactly the same as the left hand side. One can carry out a similar analysis for more general current correlators as we have in [158] finding the results summarized earlier.

## 3.6 Summary of this Chapter

In this chapter, we have developed a methodology to systematically solve SBHS equations for spinning correlation functions. Our solution involves mapping the SBHS theory correlation functions to the free theory correlation functions. We demonstrate our procedure first with three point functions which reproduce known results. We then apply our methodology to four point functions of spinning correlators. We show that the four point functions take on a remarkably simple form and can be mapped to the free theory correlation functions. Our analysis in this chapter demonstrates the usefulness of momentum space or spinor helicity variables to deal with slightly broken HSEs. Our main strategy was to map the slightly broken HSE to exactly conserved HSEs which in turn maps the interacting correlation functions in terms of the free theory correlators[13]. In contrast, the usual position space approach is extremely complicated especially for four and higher point functions since the slightly broken higher spin equations are complicated higher order integro-differential equations [122], whereas our momentum space analysis was purely algebraic. There are many interesting future directions such as direct perturbative computations to verify these results such as the recent work [160], finding a description that makes the anyonic nature of current correlators manifest, an extension to supersymmetric Chern-Simons matter theories and understanding correlators beyond the conformal fixed point and beyond large $N$. A better understanding of their bulk duals is also desirable. Although we have not discussed it in this chapter, the coupling constants that parameterize the current correlators are constrained by boundary micro-causality and the analysis of the three point functions show that they saturate the bounds [45]. Via AdS/CFT, this indicates that the bulk dual is a causal theory. The bulk theory however, has issues of non-locality [159, 161] which gives us a natural entry point to the next chapter where the goal is to find a local tractable bulk dual to a sub-sector of Chern-Simons matter theories.

[13] We should emphasize that slightly broken HS equation can have more solutions. We checked some more possibilities but they all lead to inconsistencies. At the level of three point functions, the answers are unique as can be verified directly by using conformal symmetry [42].

# Chapter 4

# Holography of Chiral Higher Spin Theory

AdS/CFT duality [7, 162, 163] was born inside string theory and extended later with many examples that are not necessarily easy to embed into string theory. Most generally, AdS/CFT is the relation between theories of quantum gravity with negative cosmological constant and CFTs. Even more generally bulk theories can be dual to ensembles of CFTs.

AdS/CFT is often applied as a "knowledge transfer" tool to deduce some nontrivial properties on one side of the duality from another one. One can also combine pieces of information coming from both sides of duality. Nevertheless, for many reasons it would be important to have an example of a simple AdS/CFT duality where both sides are easy to define independently of each other and are simple enough to compute any relevant observable as a matter of principle. Ideally, this should lead to a direct proof of the holographic duality in such cases, which should shed more light on the mechanism of AdS/CFT and on the quantum gravity problem.

It has been surprisingly hard to establish such simple AdS/CFT pairs, if we consider the canonical stringy dualities to be beyond technical capabilities to explore them in the whole space of couplings. For example, the supergravity limit where string theory is tractable corresponds to the regime of classical tensionful strings. Even putting the string loops aside, the opposite regime of tensionless strings is surprisingly hard to understand. In the canonical example of Type-IIB strings on $AdS_5 \times S^5$ the tensionless limit corresponds to the weakly coupled $\mathcal{N} = 4$ SYM [164–169], i.e. a simple regime on the CFT side, but it has so far been difficult to take the tensionless limit directly on the string side.[1] The tensionful limit, on the other hand, corresponds to the strongly coupled SYM and can only be understood thanks to the integrability, see e.g. [171], which still gives access to a limited set of observables at present and is also hard to extend beyond the large-$N$ limit.

A natural idea to find tractable AdS/CFT pairs is to look for CFTs simpler/smaller than SYM and for bulk theories that are simpler than string theory. Among the latter, it is hard to find potentially UV-complete theories since it is related to the quantum gravity problem[2]. Some examples that have extensively been studied over the years include $3d$-gravity as Chern-Simons theory, SYK, and JT models. Here, higher-spin gravities

[1] It has been possible to directly land on the tensionless limit worldsheet theory in the case of string theory on $AdS_3$, see e.g. [170].

[2] It is possible to consider examples without a dynamical graviton, e.g. just the $\phi^4$ theory in $AdS_4$, thereby avoiding the quantum gravity problem. On the CFT side, one has to look for CFTs without the stress-tensor, which is not impossible, e.g. the long-range Ising model.

could be of some help. On the CFT side, the smallest possible conformal field theories are $3d$ vector models, in particular the $O(N)$ vector model, which describes second order phase transitions of many physical systems, e.g. the Ising critical point.

The space of $3d$ vector models can be substantially enriched by gauging some global symmetries therein and adding the Chern-Simons term, which results in Chern-Simons matter theories. One can also add supersymmetry and pass to models with bi-fundamental matter that extends to ABJ(M) theories [172, 173]. Remarkably, Chern-Simons matter theories are related to each other by a web of dualities, see e.g. [77, 119, 120, 122, 174–176], of which, perhaps, the most remarkable is the $3d$ bosonization duality [77, 119, 120] since it remains to be nontrivial in the large-$N$ limit.

Since (Chern-Simons) vector models are the smallest $3d$ CFTs it is natural to ask how the AdS/CFT dual thereof can look like [177–179]. In the large-$N$ limit, it is easy to see the signs of an infinite-dimensional extension of conformal symmetry, called higher-spin symmetry. It is immediately visible in free vector models. One manifestation of the higher-spin symmetry is the presence of infinitely many conserved higher-spin tensors, $J_{a_1...a_s}$, of which the stress tensor is a particular member. By the standard AdS/CFT dictionary a spin-$s$ conserved tensor is dual to a gauge/massless spin-$s$ field on the AdS side. Therefore, the dual theory must feature an infinite multiplet of massless fields with spins from 0 to $\infty$. Theories with massless higher-spin fields had been studied long before AdS/CFT correspondence and are usually called higher-spin gravities, see e.g. [180] for a review and e.g. [181–185] for the first results in this direction.

Many properties of higher-spin gravities can be deduced before even constructing a theory. A pro example of AdS/CFT correspondence is the Flato-Fronsdal theorem [181] that, in the modern language, states that the single trace operators in the free vector model are dual to the spectrum built from massless fields of all spins. However, constructing interactions faced a number of difficulties, the main being that higher-spin symmetry mixes spins and derivatives, thereby, rendering any generic higher-spin gravity too non-local, see e.g. [161, 186–191]. There are very few exceptions: topological $3d$ higher-spin gravities [192–198] and conformal higher-spin gravity [199–201]. The former do not have propagating degrees of freedom and the latter extends conformal gravity. The AdS/CFT itself implies that the holographic duals of vector models do not obey the usual locality assumptions in field theory [161, 187, 188, 190] and, hence, cannot be constructed by following, say, the Noether procedure. Nevertheless, certain structures that are stable under nonlocal field-redefinitions can be formulated as an $L_\infty$-algebra or as formally consistent classical equations of motion [202–210] and the dual of free/critical vector models can, in some sense, be reconstructed [211, 212]. The former leaves infinitely many coefficients unfixed and does not allow for systematic holographic calculations, while the latter does not give an independent definition of the bulk theory and cannot at present be extended to Chern-Simons matter theories.

Therefore, it turns out that simple CFTs, like free vector models, do not have simple holographic duals in the sense that the bulk theories have to be too nonlocal for the standard field theory rules to apply and new techniques are yet to be developed to construct and deal with them.

A new impulse into this puzzle came from the flat space, an unusual place for AdS/CFT duality, when it was shown that there exists a local theory with the required

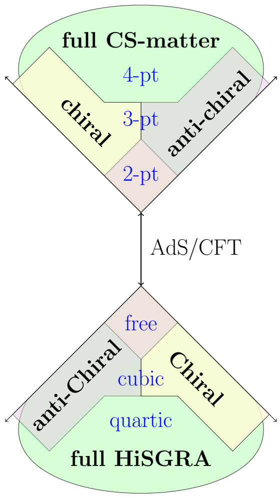


Figure 4.1: Chern-Simons vector models vs. higher-spin gravity: the structure of AdS/CFT duality. The spectrum of fields/operators is the same (free theory in the bulk/2-point correlators on the boundary). (Anti)-chiral sectors together cover all cubic vertices/three-point functions. There are non-chiral parts at higher orders.

spectrum [213–215] — chiral higher-spin gravity. It was conjectured already in [215] that there should exist a smooth deformation to $(A)dS_4$, which was constructed later in [143, 216–218]. The theory can be interpreted as a higher-spin extension of both self-dual Yang-Mills and self-dual gravity. It was shown to be at least one-loop finite [219, 220]. It is also integrable at least in flat space [189]. Therefore, there exists a tame theory in $AdS_4$ with the right spectrum to be dual to Chern-Simons matter theories. However, it cannot be dual to the full Chern-Simons matter theory. For example, the chirality of interactions is such that the sum of helicities must be positive (or negative for the anti-chiral theory) at each cubic vertex.

Given there is a well-defined (anti-)chiral higher-spin gravity in $AdS_4$ we can apply the $AdS/CFT$ correspondence to conclude that there should exist two closed hidden subsectors in Chern-Simons matter theories that are consistent on their own.

Another simple consequence of there being an $AdS$ theory is the $3d$ bosonization at the level of 3-point correlators [143]. Indeed, chiral and anti-chiral interactions form a complete basis of cubic interactions and all the couplings are completely fixed in each of the sectors. Therefore, in order to bootstrap the 3-point correlators in a theory with the same spectrum of higher-spin fields, one needs to see how to glue (anti-)chiral pieces together. The gluing depends on one phase-like parameter that gives the structure observed by Maldacena and Zhiboedov in [122]. Beyond 3-point (anti-)chiral interactions do not cover everything possible, but see [221]. The structure of the duality is illustrated on the figure.

Since chiral higher-spin gravity is a perturbatively local field theory, one can directly compute holographic correlators to define the hidden subsectors of Chern-Simons matter theories. However, it would be important to identify the hidden sectors directly on the

CFT side. This is what we attempt in this chapter. Crucial ingredients for our analysis are the epsilon transform and the usage of three-dimensional spinor-helicity variables which together manifest the anyonic nature of correlation functions. We shall show that taking appropriate limits of the 't Hooft coupling of the theory projects us into a chiral or anti-chiral sector at the level of three-point correlators. We also derive some simple constraints for the structure of higher-order correlators and loop corrections from the bulk. The loop corrections are shown to truncate at a finite order in $1/N$-expansion, which makes the latter convergent. All of this paves the way to constructing exact models of AdS/CFT correspondence. The reference for this chapter is,

- ⋆Hidden sectors of Chern-Simons matter theories and exact holography, Sachin Jain, **Dhruva K.S.**, Evgeny Skvortsov, *Phys. Rev. D* 111 (2025) 106017 [2405.00773]

## 4.1 (Anti-)chiral limit for Chern-Simons matter theory

Our goal in this section is to find a closed subsector of Chern-Simons matter theories that is dual to (anti-)Chiral higher-spin gravity in AdS$_4$. We focus on the QF theory. However, it is a simple matter to extend the analysis to follow to the other slightly broken higher spin theory, that is the theory of Chern-Simons+fundamental bosons.

The limit that we take to reach the (anti-)Chiral theory should be a nonunitary one as the (anti-)chiral bulk theory is nonunitary. To find out the correct prescription, we begin by analyzing correlation functions of the single trace primary operators in the Chern-Simons matter theory.

Before we proceed, let us set the notation straight. In contrast to [158], we normalize our single trace operators such that their two-point functions are $O(1)$, with an additional rescaling as follows:

$$\tilde{J}_s^{\pm} = \frac{e^{\mp i\theta}}{\sqrt{\tilde{N}}} J_s^{\pm}$$
$$\tilde{O}_2 = \frac{\cos\theta}{\sqrt{\tilde{N}}}\, O_2, \tag{4.1}$$

where $\theta = \frac{\pi\lambda}{2}$ and $\lambda = \frac{N}{k}$ is the 't Hooft coupling in the Chern-Simons matter theory. Note that the rescaling of the currents is different for their positive and negative components. For real $\theta$ this is required as $J_s^+$ and $J_s^-$ are complex conjugates of each other. However, if we complexify $\theta$ this is no longer true and $J_s^+$ and $J_s^-$ become independent quantities. Finally, we define for convenience,

$$\tilde{N} = 2N\frac{\sin 2\theta}{2\theta}\,, \qquad\qquad \tilde{\lambda} = \tan\theta\,. \tag{4.2}$$

We now state two limits that, as we shall show, project us into the anti-chiral or chiral sectors.

$$\textbf{anti-chiral limit: } (\tilde{N} \to \infty, \tilde{\lambda} \to -i \equiv \theta \to -i\infty) \textbf{ with } \frac{e^{i\theta}}{\sqrt{\tilde{N}}} = g_{ac}, \tag{4.3}$$

and,

$$\textbf{chiral limit: } (\tilde{N} \to \infty, \tilde{\lambda} \to i \equiv \theta \to i\infty) \textbf{ with } \frac{e^{-i\theta}}{\sqrt{\tilde{N}}} = g_c. \tag{4.4}$$

Let us begin by investigating the consequences of these limits for two-point functions.

### 4.1.1 Two-point functions

The two-point functions of the spin-$s$ currents in the QF theory are given by,

$$\langle J_s J_s \rangle_{QF} = \langle J_s J_s \rangle_{even} + \tilde{\lambda} \langle J_s J_s \rangle_{odd} \,, \tag{4.5}$$

where the odd piece is given in terms of the even one through an epsilon transformation as we discussed in chapter 2. In spinor-helicity variables, the correlator takes the following form in the two nonzero helicity configurations (note that this and all other expressions in spinor-helicity variables are after the rescaling (4.1)):

$$\begin{aligned} &\langle \tilde{J}_s^- \tilde{J}_s^- \rangle_{QF} = (1 - i\tilde{\lambda}) \langle J_s^- J_s^- \rangle_{even} \\ &, \langle \tilde{J}_s^+ \tilde{J}_s^+ \rangle_{QF} = (1 + i\tilde{\lambda}) \langle J_s^+ J_s^+ \rangle_{even}. \end{aligned} \tag{4.6}$$

Let us now take the limit $\tilde{\lambda} \to -i$ in (4.6). In this limit, only the $(--)$ configuration survives. We obtain,

$$\begin{aligned} &\lim_{\tilde{\lambda} \to -i} \langle \tilde{J}_s^- \tilde{J}_s^- \rangle_{QF} = 2 \langle J_s^- J_s^- \rangle_{even} = 2 \frac{\langle 12 \rangle^{2s}}{p} \,, \\ &\lim_{\tilde{\lambda} \to -i} \langle \tilde{J}_s^+ \tilde{J}_s^+ \rangle_{QF} = 0. \end{aligned} \tag{4.7}$$

Similarly, as $\tilde{\lambda} \to i$, we obtain only the $(++)$ helicity configuration. The story for the scalar two-point function is simple: it reads,

$$\langle \tilde{O}_2 \tilde{O}_2 \rangle_{QF} = \langle O_2 O_2 \rangle_{FF} \,. \tag{4.8}$$

Let us move on to the three-point case where we can test this limit further.

### 4.1.2 Three-point functions

We will show that (4.3) and (4.4) do indeed produce (anti-)chiral three-point functions. Let us begin with correlators that are inside the triangle.

#### Spinning correlators inside the triangle

#### Example 1: $\langle TJJ \rangle$

This correlator is given by,

$$\begin{aligned} \langle TJJ \rangle_{QF} = \frac{1}{\sqrt{\tilde{N}}(1 + \tilde{\lambda}^2)} \big( &\langle TJJ \rangle_{FF} + \tilde{\lambda}\, \epsilon \cdot \langle TJJ \rangle_{FF-FB} \\ &+ \tilde{\lambda}^2 \langle TJJ \rangle_{FB} \big). \end{aligned} \tag{4.9}$$

Using our results for correlators inside the triangle (2.44) with $n_f, n_b$ and $n_{\text{odd}}$ given in (3.3), we obtain the expression of this correlator in the eight helicity configurations:

$$\begin{aligned}
\langle \tilde{T}^-\tilde{J}^-\tilde{J}^-\rangle &= \frac{e^{i\theta}}{\sqrt{\tilde{N}}}\langle T^-J^-J^-\rangle_h, \langle \tilde{T}^+\tilde{J}^+\tilde{J}^+\rangle = \frac{e^{-i\theta}}{\sqrt{\tilde{N}}}\langle T^+J^+J^+\rangle_h \\
\langle \tilde{T}^-\tilde{J}^-\tilde{J}^+\rangle &= \frac{e^{i\theta}}{\sqrt{\tilde{N}}}\langle T^-J^-J^+\rangle_{nh}, \langle \tilde{T}^+\tilde{J}^+\tilde{J}^-\rangle = \frac{e^{-i\theta}}{\sqrt{\tilde{N}}}\langle T^+J^+J^-\rangle_{nh} \\
\langle \tilde{T}^-\tilde{J}^+\tilde{J}^-\rangle &= \frac{e^{i\theta}}{\sqrt{\tilde{N}}}\langle T^-J^+J^-\rangle_{nh}, \langle \tilde{T}^+\tilde{J}^-\tilde{J}^+\rangle = \frac{e^{-i\theta}}{\sqrt{\tilde{N}}}\langle T^+J^-J^+\rangle_{nh} \\
\langle \tilde{T}^-\tilde{J}^+\tilde{J}^+\rangle &= 0 \qquad\qquad , \langle \tilde{T}^+\tilde{J}^-\tilde{J}^-\rangle = 0.
\end{aligned} \tag{4.10}$$

Let us now take the anti-chiral limit defined in (4.3). In this limit we obtain,

$$\begin{aligned}
\langle \tilde{T}^-\tilde{J}^-\tilde{J}^-\rangle &= g_{ac}\langle T^-J^-J^-\rangle_h\,, & \langle \tilde{T}^+\tilde{J}^+\tilde{J}^+\rangle &= 0\,, \\
\langle \tilde{T}^-\tilde{J}^-\tilde{J}^+\rangle &= g_{ac}\langle T^-J^-J^+\rangle_{nh}\,, & \langle \tilde{T}^+\tilde{J}^+\tilde{J}^-\rangle &= 0\,, \\
\langle \tilde{T}^-\tilde{J}^+\tilde{J}^-\rangle &= g_{ac}\langle T^-J^+J^-\rangle_{nh}\,, & \langle \tilde{T}^+\tilde{J}^-\tilde{J}^+\rangle &= 0\,, \\
\langle \tilde{T}^-\tilde{J}^+\tilde{J}^+\rangle &= 0\,, & \langle \tilde{T}^+\tilde{J}^-\tilde{J}^-\rangle &= 0.
\end{aligned} \tag{4.11}$$

Equation (4.11) contains only the net negative helicity configurations of the correlators and is precisely the definition of the anti-chiral sector, showing that the limit (4.3) is as we desired. One point to note is that if we had not re-scaled the currents as in (4.1), we would only have been able to retain the $(- - -)$ helicity configuration. However, with the rescaling, we are able to obtain the full anti-chiral sector viz (4.11). Let us consider another example now to further illustrate and validate (4.3):

## Example 2: $\langle TTT\rangle$

This correlator is given by [3]

$$\langle TTT\rangle = \frac{1}{\sqrt{\tilde{N}}(1+\tilde{\lambda}^2)}\big(\langle TTT\rangle_{FF} + \epsilon\cdot\langle TTT\rangle_{FF-FB} + \tilde{\lambda}^2\langle TTT\rangle_{FB}\big). \tag{4.12}$$

Using (2.44) and (3.3), we see that this correlator in the eight helicity configurations reads

$$\begin{aligned}
\langle \tilde{T}^-\tilde{T}^-\tilde{T}^-\rangle &= \frac{e^{i\theta}}{\sqrt{\tilde{N}}}\langle T^-T^-T^-\rangle_h, \langle \tilde{T}^+\tilde{T}^+\tilde{T}^+\rangle = \frac{e^{-i\theta}}{\sqrt{\tilde{N}}}\langle T^+T^+T^+\rangle_h \\
\langle \tilde{T}^-\tilde{T}^-\tilde{T}^+\rangle &= \frac{e^{i\theta}}{\sqrt{\tilde{N}}}\langle T^-T^-T^+\rangle_{nh}, \langle \tilde{T}^+\tilde{T}^+\tilde{T}^-\rangle = \frac{e^{-i\theta}}{\sqrt{\tilde{N}}}\langle T^+T^+T^-\rangle_{nh} \\
\langle \tilde{T}^-\tilde{T}^+\tilde{T}^-\rangle &= \frac{e^{i\theta}}{\sqrt{\tilde{N}}}\langle T^-T^+T^-\rangle_{nh}, \langle \tilde{T}^+\tilde{T}^-\tilde{T}^+\rangle = \frac{e^{-i\theta}}{\sqrt{\tilde{N}}}\langle T^+T^-T^+\rangle_{nh} \\
\langle \tilde{T}^+\tilde{T}^-\tilde{T}^-\rangle &= \frac{e^{i\theta}}{\sqrt{\tilde{N}}}\langle T^+T^-T^-\rangle_{nh}, \langle \tilde{T}^-\tilde{T}^+\tilde{T}^+\rangle = \frac{e^{-i\theta}}{\sqrt{\tilde{N}}}\langle T^-T^+T^+\rangle_{nh}.
\end{aligned} \tag{4.13}$$

[3] There can, in principle, exist a parity odd nonhomogeneous contact term. If we allow for such contributions, $\langle TTT\rangle_{nh,\text{even}}$ can be cancelled out by $\langle TTT\rangle_{nh,\text{odd}}$ in the $(- - -)$ and $(+ + +)$ helicities as can be seen given the expressions in [42]. However, for correlators such as $\langle TJJ\rangle$, there is no nonhomogeneous parity odd contact term possible and thus our parity even contact term analysis in chapter 2 is essential.

Taking the anti-chiral limit (4.3) we obtain only the anti-chiral helicity configurations as desired:

$$\begin{aligned}
\langle \tilde{T}^-\tilde{T}^-\tilde{T}^-\rangle &= g_{ac}\langle T^-T^-T^-\rangle_h\,, & \langle \tilde{T}^+\tilde{T}^+\tilde{T}^+\rangle &= 0\\
\langle \tilde{T}^-\tilde{T}^-\tilde{T}^+\rangle &= g_{ac}\langle T^-T^-T^+\rangle_{nh}\,, & \langle \tilde{T}^+\tilde{T}^+\tilde{T}^-\rangle &= 0\\
\langle \tilde{T}^-\tilde{T}^+\tilde{T}^-\rangle &= g_{ac}\langle T^-T^+T^-\rangle_{nh}\,, & \langle \tilde{T}^+\tilde{T}^-\tilde{T}^+\rangle &= 0\\
\langle \tilde{T}^+\tilde{T}^-\tilde{T}^-\rangle &= g_{ac}\langle T^+T^-T^-\rangle_{nh}\,, & \langle \tilde{T}^-\tilde{T}^+\tilde{T}^+\rangle &= 0.
\end{aligned} \tag{4.14}$$

We note that if the nonhomogeneous correlator was nonzero in the $(- - -)$ helicity configuration, we would have a contribution of the form $\frac{e^{3i\theta}}{\sqrt{\tilde{N}}}\langle T^-T^-T^-\rangle_{nh}$ which would be exponentially enhanced compared to all the other helicity configurations in the anti-chiral limit. It being zero via the addition of contact terms like (2.36) is essential to obtain a uniform scaling for all the anti-chiral helicity configurations as in (4.14).

Let us now consider a correlator outside the triangle and investigate if our findings carry over.

## Spinning correlators outside the triangle

### Example 3: $\langle J_4 J_1 J_1\rangle$

Using (2.61) and (3.3), we see that the eight helicity configurations of this correlator are,

$$\begin{aligned}
\langle \tilde{J}_4^-\tilde{J}^-\tilde{J}^-\rangle &= \frac{e^{i\theta}}{2\sqrt{\tilde{N}}}\langle J_4^-J^-J^-\rangle_{FF-FB}\,, & \langle \tilde{J}_4^+\tilde{J}^+\tilde{J}^+\rangle &= \frac{e^{-i\theta}}{2\sqrt{\tilde{N}}}\langle J_4^+J^+J^+\rangle_{FF-FB}\\
\langle \tilde{J}_4^-\tilde{J}^-\tilde{J}^+\rangle &= \frac{e^{i\theta}}{2\sqrt{\tilde{N}}}\langle J_4^-J^-J^+\rangle_{FF+FB}\,, & \langle \tilde{J}_4^+\tilde{J}^+\tilde{J}^-\rangle &= \frac{e^{-i\theta}}{2\sqrt{\tilde{N}}}\langle J_4^+J^+J^-\rangle_{FF+FB}\\
\langle \tilde{J}_4^-\tilde{J}^+\tilde{J}^-\rangle &= \frac{e^{i\theta}}{2\sqrt{\tilde{N}}}\langle J_4^-J^+J^-\rangle_{FF+FB}\,, & \langle \tilde{J}_4^+\tilde{J}^-\tilde{J}^+\rangle &= \frac{e^{-i\theta}}{2\sqrt{\tilde{N}}}\langle J_4^+J^-J^+\rangle_{FF+FB}\\
\langle \tilde{J}_4^-\tilde{J}^+\tilde{J}^+\rangle &= \frac{e^{i\theta}}{2\sqrt{\tilde{N}}}\langle J_4^-J^+J^+\rangle_{FF-FB}\,, & \langle \tilde{J}_4^+\tilde{J}^-\tilde{J}^-\rangle &= \frac{e^{-i\theta}}{2\sqrt{\tilde{N}}}\langle J_4^+J^-J^-\rangle_{FF-FB}\,.
\end{aligned} \tag{4.15}$$

Let us now take the anti-chiral limit defined in (4.3). In this limit we obtain,

$$\begin{aligned}
\langle \tilde{J}_4^-\tilde{J}^-\tilde{J}^-\rangle &= \frac{g_{ac}}{2}\langle J_4^-J^-J^-\rangle_{FF-FB}\,, & \langle \tilde{J}_4^+\tilde{J}^+\tilde{J}^+\rangle &= 0\\
\langle \tilde{J}_4^-\tilde{J}^-\tilde{J}^+\rangle &= \frac{g_{ac}}{2}\langle J_4^-J^-J^+\rangle_{FF+FB}\,, & \langle \tilde{J}_4^+\tilde{J}^+\tilde{J}^-\rangle &= 0\\
\langle \tilde{J}_4^-\tilde{J}^+\tilde{J}^-\rangle &= \frac{g_{ac}}{2}\langle J_4^-J^+J^-\rangle_{FF+FB}\,, & \langle \tilde{J}_4^+\tilde{J}^-\tilde{J}^+\rangle &= 0\\
\langle \tilde{J}_4^-\tilde{J}^+\tilde{J}^+\rangle &= \frac{g_{ac}}{2}\langle J_4^-J^+J^+\rangle_{FF-FB}\,, & \langle \tilde{J}_4^+\tilde{J}^-\tilde{J}^-\rangle &= 0\,,
\end{aligned} \tag{4.16}$$

which indeed is the anti-chiral sector. Note that we used the facts that the $FF + FB$ contribution to this correlator vanishes in the $(- - -)$ as well as the $(- + +)$ helicity configurations (as well as their complex conjugate helicities). We also used the fact that the $FF - FB$ contribution vanishes in the $(- - +)$ helicity as well as its complex conjugate and their $(2 \leftrightarrow 3)$ permutations. These facts were explicitly obtained in [45]. Some comments about parity odd contact terms are also in order. The FF and FB Ward-Takahashi identities are distinct outside the triangle. Thus, if we take the $\langle JJ \rangle$ two-point functions appearing on the RHS of these Ward-Takahashi identities and replace them with the parity odd two-point function, we will obtain two distinct parity odd correlation functions that are, of course, contact term contributions. It is not obvious what role they play but it would be interesting to further explore their consequences in the future.

## Correlators involving the scalar operator

### One scalar

Let us first consider a correlator with a single scalar insertion and two currents. In the QF theory, such correlators are given by,

$$\begin{aligned}\langle J_{s_1} J_{s_2} O_2 \rangle &= \frac{1}{\sqrt{\tilde{N}}} \big( \langle J_{s_1} J_{s_2} O_2 \rangle_{FF} + \tilde{\lambda} \langle J_{s_1} J_{s_2} O_2 \rangle_{CB} \big) \\ &= \frac{1}{\sqrt{\tilde{N}}} \big( \langle J_{s_1} J_{s_2} O_2 \rangle_{FF} + \tilde{\lambda} \epsilon \cdot \langle J_{s_1} J_{s_2} O_2 \rangle_{FF} \big) \,, \end{aligned} \tag{4.17}$$

where in the second line, the epsilon transform is performed with respect to the lower amongst the two spins. This correlator in the four helicity configurations (after the rescaling (4.1) and assuming $s_1 \geq s_2$) is given by,

$$\begin{aligned}\langle \tilde{J}^-_{s_1} \tilde{J}^-_{s_2} \tilde{O}_2 \rangle &= \frac{e^{i\theta}}{\sqrt{\tilde{N}}} \langle J^-_{s_1} J^-_{s_2} O_2 \rangle_{FF} , \\ \langle \tilde{J}^+_{s_1} \tilde{J}^+_{s_2} \tilde{O}_2 \rangle &= \frac{e^{-i\theta}}{\sqrt{\tilde{N}}} \langle J^+_{s_1} J^+_{s_2} O_2 \rangle_{FF} , \\ \langle \tilde{J}^-_{s_1} \tilde{J}^+_{s_2} \tilde{O}_2 \rangle &= \frac{e^{i\theta}}{\sqrt{\tilde{N}}} \langle J^-_{s_1} J^+_{s_2} O_2 \rangle_{FF} , \\ \langle \tilde{J}^+_{s_1} \tilde{J}^-_{s_2} \tilde{O}_2 \rangle &= \frac{e^{-i\theta}}{\sqrt{\tilde{N}}} \langle J^+_{s_1} J^-_{s_2} O_2 \rangle_{FF} \,. \end{aligned} \tag{4.18}$$

Taking the anti-chiral limit (4.3) we obtain,

$$\begin{aligned}\langle \tilde{J}^-_{s_1} \tilde{J}^-_{s_2} \tilde{O}_2 \rangle &= g_{ac} \langle J^-_{s_1} J^-_{s_2} O_2 \rangle_{FF} \,, \qquad & \langle \tilde{J}^+_{s_1} \tilde{J}^+_{s_2} \tilde{O}_2 \rangle &= 0 \\ \langle \tilde{J}^-_{s_1} \tilde{J}^+_{s_2} \tilde{O}_2 \rangle &= g_{ac} \langle J^-_{s_1} J^+_{s_2} O_2 \rangle_{FF} \,, \qquad & \langle \tilde{J}^+_{s_1} \tilde{J}^-_{s_2} \tilde{O}_2 \rangle &= 0 \,, \end{aligned} \tag{4.19}$$

which is indeed the desired result.

### Two scalars

The QF theory correlator involving two scalars is given by

$$\langle J_s O_2 O_2 \rangle = \frac{(1 + \tilde{\lambda}^2)}{\sqrt{\tilde{N}}} \langle J_s O_2 O_2 \rangle_{FF} \,. \tag{4.20}$$

In the two independent helicity configurations, it is given by,

$$\langle \tilde{J}_s^- \tilde{O}_2 \tilde{O}_2 \rangle = \frac{e^{i\theta}}{\sqrt{\tilde{N}}} \langle J_s^- \tilde{O}_2 \tilde{O}_2 \rangle_{FF}, \langle \tilde{J}_s^+ \tilde{O}_2 \tilde{O}_2 \rangle = \frac{e^{-i\theta}}{\sqrt{\tilde{N}}} \langle J_s^+ O_2 O_2 \rangle_{FF} . \tag{4.21}$$

Taking the anti-chiral limit (4.3) yet again results in the desired answer:

$$\langle \tilde{J}_s^- \tilde{O}_2 \tilde{O}_2 \rangle = g_{ac} \langle J_s^- O_2 O_2 \rangle_{FF} , \qquad \langle \tilde{J}_s^+ \tilde{O}_2 \tilde{O}_2 \rangle = 0 . \tag{4.22}$$

### Three-point Summary

Based on the examples considered above, one can easily show that the anti-chiral limit, (4.3), works for any three-point correlator that is inside or outside the triangle, also for the ones involving scalar operators, thus establishing that what we have uncovered is indeed the subsector that is dual to anti-Chiral higher-spin gravity. Indeed, at the three-point level, the main signature of (anti-)Chiral higher-spin gravity is that it features cubic interactions with the total helicity being (negative) positive.

Let us now explore the consequences of taking this limit at higher points, starting with the four-point case.

## 4.2 Beyond three-point

A general comment is that beyond three-point and at the loop level chiral and anti-chiral interactions can mix in the bulk, see section 4.3, which is not in contradiction with having them as closed subsectors on their own. However, we should not expect the simple limiting procedure we proposed above to work as it is, but it is instructive to see what happens when we take the limit.

### 4.2.1 Four-point functions

In this subsection, we investigate what happens to four-point correlators in the QF theory when we take the anti-chiral limit (4.3). Let us begin with the scalar case.

### Example 1: $\langle O_2 O_2 O_2 O_2 \rangle$

The scalar correlator in the CS+fermionic matter theory is given by [149, 150]

$$\langle \tilde{O}_2 \tilde{O}_2 \tilde{O}_2 \tilde{O}_2 \rangle = \frac{\cos^4(\theta)(1+\tilde{\lambda}^2)^2}{\tilde{N}} \langle O_2 O_2 O_2 O_2 \rangle_{FF} = \frac{1}{\tilde{N}} \langle O_2 O_2 O_2 O_2 \rangle_{FF}, \tag{4.23}$$

where we used $\tilde{\lambda} = \tan(\theta)$. Therefore, in the chiral limit and the anti-chiral limits (4.4) and (4.3), this correlator goes to zero.[4]

[4] Given the fact that the fermionic scalar four-point function is non-local [159, 161], it is a nice feature that it drops out in this limit. This is an indication that we are focusing on a local sub-sector.

### Example 2: $\langle J_s O_2 O_2 O_2 \rangle$

This correlator in the two helicity configurations is given by [151, 152]

$$\begin{aligned}\langle \tilde{J}_s^- \tilde{O}_2 \tilde{O}_2 \tilde{O}_2 \rangle &= \frac{e^{i\theta}}{\tilde{N}} \big( \cos(\theta) \langle J_s^- O_2 O_2 O_2 \rangle_{FF} + \sin(\theta) \langle J_s^- O_2 O_2 O_2 \rangle_{CB} \big) \,, \\ \langle \tilde{J}_s^+ \tilde{O}_2 \tilde{O}_2 \tilde{O}_2 \rangle &= \frac{e^{-i\theta}}{\tilde{N}} \big( \cos(\theta) \langle J_s^+ O_2 O_2 O_2 \rangle_{FF} + \sin(\theta) \langle J_s^+ O_2 O_2 O_2 \rangle_{CB} \big) \,. \end{aligned} \tag{4.24}$$

If we now take the anti-chiral limit (4.3) we see that

$$\begin{aligned}\langle \tilde{J}_s^- \tilde{O}_2 \tilde{O}_2 \tilde{O}_2 \rangle &= \frac{g_{ac}^2}{2} \big( \langle J_s^- O_2 O_2 O_2 \rangle_{FF} - i \langle J_s^- O_2 O_2 O_2 \rangle_{CB} \big) \\ \langle \tilde{J}_s^+ O_2 O_2 O_2 \rangle &= 0 \,, \end{aligned} \tag{4.25}$$

thus showing that the anti-chiral limit (4.3) is well defined and indeed picks out just the anti-chiral sector for these types of four-point functions.

In the above examples, we notice a very nice separation between the chiral (net negative helicity) and anti-chiral (net positive helicity) sectors. However, by investigating several more general four-point functions, we see that in addition to taking the (anti-)chiral limit, one needs to throw away some extra pieces to obtain finite results. However, we postpone further analysis at the four- and higher-point levels to future work.

## 4.2.2 Anomalous dimensions in the (anti-)chiral limit

The anomalous dimensions of the spin $s$ currents in the CS+fermionic matter theory at leading order in $\tilde{N}$ are given by [131]

$$\gamma_s = \frac{1}{\tilde{N}} \left( \frac{a_s \tilde{\lambda}^2}{(1+\tilde{\lambda}^2)} + \frac{b_s \tilde{\lambda}^2}{(1+\tilde{\lambda}^2)^2} \right) = \frac{\sin^2(\theta)}{\tilde{N}} \left( a_s + b_s \cos^2(\theta) \right) , \tag{4.26}$$

where we used $\tilde{\lambda} = \tan(\theta)$. Let us now take the anti-chiral limit (4.3). In this limit we have

$$\gamma_s = -\frac{a_s}{4} g_{ac}^2 - \frac{b_s}{16} e^{2i\theta} g_{ac}^2 + \cdots \,. \tag{4.27}$$

Note that $a_s$ is a rational function of $s$ that approaches constant for large spins, while $b_s \sim \log s$ for large spins. The latter signals that the CFT is a gauge theory. It seems that the chiral limit should freeze these gauge degrees of freedom and the $b_s$ term needs to be dropped.

As for the scalar operator, its anomalous dimension is given by [134]

$$\gamma_0 = -\frac{32}{3\pi^2} \frac{\tilde{\lambda}^2}{\tilde{N}(1+\tilde{\lambda}^2)} \,. \tag{4.28}$$

Using the definition $\tilde{\lambda} = \tan\theta$ we see that this equals

$$\gamma_0 = -\frac{32}{3\pi^2} \frac{(\sin\theta)^2}{\tilde{N}} \,. \tag{4.29}$$

In the anti-chiral limit, this quantity is finite and equals,

$$\gamma_0 = \frac{8g_{ac}^2}{3\pi^2} \,. \tag{4.30}$$

An important point to note here is that (4.27) and (4.30) could receive corrections from non-planar diagrams. In the usual setting, we take the 't Hooft limit $\lambda = \frac{N}{k}$ to be finite while taking $N$ and $k$ to infinity. However, the anti-chiral limit (4.3) that we take corresponds to $\lambda \to -i\infty$ which means we take $N \to \infty$ faster than we take $k \to \infty$. This can alternatively be interpreted as $N \to \infty$ while taking $k \to i$ thus indicating that the contributions of the non-planar diagrams to the anomalous dimensions need to be kept while taking this limit. Thus, it may be appropriate or even necessary to perform the re-summation of these contributions. For instance, the divergent part of the anomalous dimension of the spin $s$ currents (4.27) could be rendered finite by these contributions. We leave a detailed investigation of this to the future.

## 4.3 Holographic predictions for correlators

In this section we change the gears and move into the bulk. It is quite easy to get a number of predictions by analysing the structure of interactions in Chiral theory without having to actually compute any holographic correlators. Some of the statements below go a bit beyond what is known at present on the CFT side. Several points should be stressed before we proceed.

Firstly, as we show below, certain corrections vanish identically in the bulk because it is impossible to draw a tree/loop diagram that would contribute. These are robust predictions. However, we expect that the higher-spin symmetry implies many cancellations. For example, there have to be no loop corrections to holographic correlators for the free CFT duals. Therefore, certain combinations of loop diagrams must vanish, even though it is possible to draw them. To take this possibility into account to refer to a situation when a contribution exists, but only an explicit computation can tell us if it vanishes or not, we will use the sign $\stackrel{?}{\neq} 0$.

Secondly, we will consider the most general case of the bosonic ($U(M)$ or $O(M)$) gauged Chiral higher-spin gravity. On the CFT side, this corresponds to vector models with some leftover global symmetries, e.g. $U(M)$ or $O(M)$.[5] However, there are two standard cases in the literature:[6] (a) the spectrum contains higher-spin currents of all integer spins; (b) the spectrum has higher-spin currents with even spins only. These are obtained by simple reductions of the general statements that are made for the nonabelian Chiral theory: for (a) the sum of spins meeting at a vertex has to be even; for (b) the spins themselves have to be even. For example, for case (a) $\langle TTJ\rangle = 0$ and there is no bulk cubic vertex with spins $2-2-1$. Supersymmetric extensions are also possible, but we do not discuss them in detail.

[5] The $O(N)$-gauging relevant for Chiral higher-spin gravity was first discussed in [213] and its generalization in [222], which is very similar to [223].

[6] The two cases correspond to Chern-Simons matter theories with $U(N)$ and $O(N)$ gauged symmetry and without the leftover $U(M)$ or $O(M)$ symmetry.

Thirdly, one should be careful in comparing the bulk predictions to CFT while the chiral limit on the CFT side is not yet well-understood. Indeed, the hypothetical bulk dual of the full Chern-Simons matter theories (see also the picture in the introduction) has the same spectrum as Chiral theory. Chiral and anti-chiral interactions cover all possible cubic interactions. Starting from the quartic order this is not the case and there has to be some structures (let us call them nonchiral) not belonging to (anti-)Chiral theories. Starting from four-point functions (and loop corrections) holographic correlators receive contributions from (anti-)Chiral theories separately and from various mixtures of (anti-)chiral interactions together with the nonchiral ones.

Let us begin by giving a schematic form of the Lagrangian of Chiral theory. The Lagrangian is known in the light-cone gauge in flat space [213–215], up to the cubic order in $AdS_4$ [143, 216]. With the equations of motion [143, 217, 218] it is easy to see what kind of contact interaction vertices are present at higher orders in $AdS_4$, which can be summarized as

$$\mathcal{L} = \sum_{s\geq 0} \Phi_{-s}\Box\Phi_s + \sum_{N\geq 3} g^{N-2} \sum_{\lambda_{\text{tot}}\geq N-2} V^N_{\lambda_1,\dots,\lambda_N} \,. \tag{4.31}$$

Here, the first term is the sum of kinetic terms for all spins. Note that a massless spin-$s$, $s > 0$ field has two degrees of freedom that can be associated with helicity $+s$ and $-s$ states, which we denote $\Phi_{\pm s}$.[7] In particular, the bulk-to-bulk propagator connects helicity $\lambda$ to helicity $-\lambda$. The most important information about the interactions is that at order $N$ the total helicity $\lambda_{\text{tot}} = \sum_i \lambda_i$ entering the vertex $V^N_{\lambda_1,\dots,\lambda_N}$ must be greater or equal to $N-2$ for chiral theory or $-\lambda_{\text{tot}} > N-2$ for the anti-Chiral one. For example, all cubic interactions with $\lambda_1+\lambda_2+\lambda_3 > 0$ are present in Chiral theory and those with $\lambda_1+\lambda_2+\lambda_3 < 0$ are present in anti-Chiral one. In reality, the helicity is integer in the bosonic theory and we have $\lambda_1+\lambda_2+\lambda_3 \geq 1$ (or $\leq -1$). For simplicity, we confine ourselves to anti-Chiral theory as in the main text above.

**Three-point functions.** The main constraint for the three-point functions is the aforementioned $\lambda_1+\lambda_2+\lambda_3 < 0$. In particular, $0-0-0$ vanishes. Therefore, $\langle OOO\rangle = 0$ and all three-point correlators where the sum of the helicities greater or equal to zero vanish. This is exactly what we observed above. It is important that Chiral theory can be shown to be a unique theory under certain assumptions.[8]

**Four-point functions.** There are two types of contributions to the four-point functions at tree level: exchanges and contact, but they satisfy the same constraint $-\lambda_{\text{tot}} \geq N-2$

[7] The Lagrangian is written as if we use the light-cone gauge [216] where the helicity structure is manifest. Note that the kinetic terms do reduce to simple $\Box = \partial_m\partial^m$ in the light-cone gauge after an appropriate rescaling despite $AdS_4$ is curved, which is due to the fact that massless higher-spin fields are also conformally invariant in certain field descriptions.

[8] Assumptions can vary, for example, one can assume that there is at least one higher-spin field that has nontrivial self-interactions. The latter forces one to introduce all other spins and all other (anti-)chiral interactions, see [213–215]. It is important to note that there are two contractions of Chiral theory [189] that can be thought of as higher-spin extensions of selfdual Yang-Mills and of self-dual gravity (separately). They contain higher-spin fields but exhibit only one-derivative Yang-Mills-type interactions or two-derivative gravitational-type interactions, but no genuine higher-spin interactions. These two theories have simple actions [224].

as one can easily see. Therefore, for a four-point correlator to be nonzero we should have $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq -2$.

For example, for $0-0-0-0$ the only vertex that can contribute to the exchange is $0-0-(-s)$ and the bulk-to-bulk propagator connects $+s$ to $-s$ and $0-0-(\pm s)$ cannot be both in the same Chiral theory. Therefore, $\langle OOOO\rangle = 0$ and, in general, we expect

$$\langle J_{-s}OOO\rangle \overset{?}{\neq} 0\,, \quad s\geq 2\,, \qquad \begin{aligned} &\langle J_{-s}OOO\rangle = 0\,, \quad s=0,1\,,\\ &\langle J_{+s}OOO\rangle = 0\,, \quad s\geq 0\,, \end{aligned} \tag{4.32}$$

where the correlator on the left can still be zero as a result of some cancellation, but in general we expect that it does not vanish.

**Higher-point functions.** For higher point functions we have $-\lambda_{\text{tot}} \geq N-2$. In particular, we expect that

$$\langle J_{-s}OOOO\rangle \overset{?}{\neq} 0\,, \quad s\geq 3\,, \qquad \begin{aligned} &\langle J_{-s}OOOO\rangle = 0\,, \quad s=0,1,2\,,\\ &\langle J_{+s}OOOO\rangle = 0\,, \quad s\geq 0 \end{aligned} \tag{4.33}$$

For example, all $\langle OO...O\rangle = 0$.

**Loop corrections.** Flat space computations [219, 220] show that there are no UV divergences in the theory at least at one-loop. Any $n$-point one-loop diagram can be obtained by connecting two lines in some $(n+2)$-point tree-level diagram. Therefore, for a one-loop correction to an $n$-point correlator we get $-\lambda_{\text{tot}} \geq n$.

For example, for two-point functions, we have $\lambda_1+\lambda_2 \leq -2$. Therefore, we do not expect any one-loop correction to $\langle OO\rangle$, but there can be one to $\langle J_{-s}J_{-s}\rangle$, $s\geq 1$. For three-point functions at one-loop, we have $\lambda_1+\lambda_2+\lambda_3 \leq -3$ and so on. In particular, there should not be any loop corrections to $\langle OO...O\rangle$.

Assuming Chiral theory does not have any UV-divergences to all orders and, hence, no new counterterms need to be introduced that could change the simple conclusions here, we can estimate that an $l$-loop correction to an $n$-point function comes by gluing $2l$ lines in a $(2l+n)$-point tree level diagram. Therefore, we have to have $-\lambda_{\text{tot}} \geq 2l+n-2$. As a consequence, we see that for any given correlation function $\langle J_{\lambda_1}...J_{\lambda_n}\rangle$ loop corrections stop at a certain finite loop order, $2l \leq -\lambda_{\text{tot}}-n+2$. The latter means that the $1/N$ expansion is convergent (we can say nonuniformly because the order depends on the observable being computed).

## 4.4 Summary of this Chapter

In the chapter, we made the first step towards identifying the hidden closed subsector of Chern-Simons matter theories that is dual to Chiral higher-spin gravity. It is quite easy to do so at the level of three-point functions by taking a certain limit of the coupling constant. The main selection rule here is to keep the correlators with total helicity positive/negative. At higher orders the picture is less clear since chiral and anti-chiral sectors can mix as we discuss in more detail in the paper [52]. However, if we naively attempt to bootstrap higher-point correlators of the (anti-)chiral theory using the three-point functions as an input, we obtain simple but different results from directly taking the (anti-)chiral limit of the correlators [52]. The latter procedure should be consistent with the bulk picture. It would be interesting to systematically setup the conformal bootstrap in this setting since constraints such as crossing symmetry should still provide valuable constraints.

One should not rush to compare some of our bulk conclusions to the CFT results above because the existence of a closed subsector is not the same as existence of a limiting subsector. We see that at higher orders chiral and anti-chiral theories/subsectors mix with each other and with the nonchiral ones and a further refinement of our limiting procedure may be needed. For example, one may need to drop certain parts of correlation functions to land on the chiral subsector.

It is worth stressing that the perturbation series in the bulk seems to converge (nonuniformly in spins), which is dual to the convergence of the $1/N$ expansion in the chiral subsector. The chiral subsector can be bootstrapped with the help of the slightly-broken higher-spin symmetry starting from the three-point functions we have. Therefore, there is a good chance of establishing a pair of AdS/CFT dual theories where both sides are nontrivial, computable and are defined independently of each other.

In general, one might expect the slightly-broken higher-spin symmetry to be useful even at small $N$, given that the anomalous dimensions of higher-spin currents are small and get even smaller with spin. It would be interesting to find out if the existence of chiral subsectors can also be established directly at small $N$ via the numerical bootstrap techniques, where the main challenge seems to be to implement the notion of helicity into the structure of correlators and OPE coefficients and to be able to target nonunitary theories.

On the way towards exact models of AdS/CFT correspondence, it is interesting to identify the AdS/CFT duals of self-dual Yang-Mills and of self-dual gravity. In flat space all tree-level amplitudes in these theories vanish and the only nontrivial ones are all-helicity-plus one-loop amplitudes. However, it is not so in (anti-)de Sitter and it would be interesting to compute the corresponding AdS/CFT correlators. Already tree-level amplitudes of type $(++\ldots+-)$ do not vanish. The existence of these theories implies that there are closed subsectors of the current $\langle J\ldots J\rangle$ and stress tensor $\langle T\ldots T\rangle$ correlators. However, these two subsectors are quite small. Nevertheless, they are universal since every (local) CFT has a stress tensor and many have global symmetry currents, while the existence of the slightly-broken higher-spin currents is special to large-$N$ vector models. It also should be taken into account that both SDYM and SDGR are UV finite theories [225].

SDYM and SDGR are contained inside (anti-)Chiral theory. SDYM has only the $(--+)$ cubic vertex and SDGR also has the $(--+)$ vertex and stops there at least in the light-cone gauge [226,227]. Therefore, our expressions for $\langle T^-T^-T^+\rangle_{nh}$ and $\langle J^-J^-J^+\rangle_{nh}$ immediately give the three-point holographic functions in SDGR and SDYM, respectively. As for the four-point ones, it easy to see that $\langle J^-J^-J^-J^+\rangle$ in SDYM coincides with the one in Chiral theory with all integer spins. Similarly, $\langle T^-T^-T^-T^+\rangle$ in SDGR coincides with the one on Chiral theory restricted to even spins. In fact the same should be true for all $\langle J^-J^-\ldots J^-J^+\rangle$ and $\langle T^-T^-\ldots T^-T^+\rangle$ since exchanges via higher-spin fields are suppressed in these correlators.

In the same vein, it would be interesting to identify the chiral subsectors in ABJ theories since via triality [228] one can argue that $\mathcal{N} = 6$ $U(M)$-gauged Chiral theory should be dual to the subsector of the vector-like limit of ABJ theories. The latter indicates the existence of a closed subsector in tensionless strings on $AdS_4 \times \mathbb{CP}^3$.

Our (anti-)chiral limit can be understood as taking $N \to \infty$ while keeping the Chern-Simons level fixed at $k = \pm i$. The analysis of Chern-Simons theory with a complex level has been performed by Witten [229, 230]. It would be interesting to see if one can perform an analogous analysis when we also add matter into the mix and see for instance, if the bosonization dualities persist. In the literature, supersymmetric Chern-Simons matter theories have been analyzed in the so-called M-theory limit which corresponds to $N \to \infty$ with $k$ finite [231, 232], which is quite similar in spirit to our (anti-)chiral limit. By localization, the partition function on $S^3$ reduces to a matrix model which is then interpreted as the partition function of an ideal Fermi gas of non-interacting particles. The level $k$ is mapped to Planck's constant and, hence, one can study the theory at finite $k$ by employing the WKB approximation. This formalism also enables one to compute non-perturbative (instanton) effects. It would thus be interesting to understand the implications of the (anti-)chiral limits in this vein.

The works mentioned above were in the context of ABJM or ABJ theory with matter in the adjoint or bi-fundamental representations. There has also been some work on the supersymmetric versions of Chern-Simons theory with fundamental matter [137, 233, 234]. It would be interesting to see if we can define (anti-)chiral limits analogous to (4.4) and (4.3) to obtain a sub-sector that is dual to a supersymmetric (anti-)chiral higher-spin gravity. The formalism of super spinor-helicity variables developed in [48] for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories would provide a perfect stage for the start of such exploits.

Another avenue of interest is the connection between $AdS_4/CFT_3$ and flat space holography. $CFT_3$ correlators/ $AdS_4$ amplitudes are related to S-matrix in four dimensions. The framework of Celestial holography relates four-dimensional S-matrices to $CFT_2$ correlators. It would thus be interesting to see the implications of taking the (anti-)chiral limit on the celestial correlators.

# Chapter 5

# Deciphering $\text{CFT}_3$ using Twistor theory

Twistor theory, since it was first conceived by Penrose [21], has led to many developments such as in the context of four dimensional flat-space scattering amplitudes [20, 22, 23, 235]. More recently, there have been developments to study three dimensional conformal field theories (CFTs) in twistor space [236, 237] which through the lens of holography, also describes scattering in four dimensional (anti)de-Sitter spacetime. The study of $\text{CFT}_3$ in twistor space is complementary to the traditional position space approaches [13–16] and the more recent momentum space and spinor helicity point of view [24–45, 75]. (Real) Twistor space in some sense sits in between position space and momentum space where it is connected to the former via a Penrose transform [21] and to the latter through Witten's half-Fourier transform [20]. Thus, this presents an opportunity to import the best tools and techniques from either side and use them to set up the twistor space bootstrap. In particular, the simplicity of two and three point conserved current correlators in twistor space [236, 237] suggests that it might provide a nice stage to bootstrap spinning correlators.

In this chapter, we build on the constructed twistor space for 3D CFTs and derive several new results. One point to note is that previous works [236, 237] used an ambidextrous approach employing both twistors and their Fourier duals to represent two and three point Wightman functions involving conserved currents. On the other hand, we shall work purely with twistor variables by employing additional invariants of the conformal group which also allows us to extend the results to include twistor space Wightman functions involving $\Delta = 1$ scalar operators. An interesting observation that we make is that in order to describe correlators involving scalars with $\Delta \neq 1$, one needs to use the infinity twistor of $\mathbb{R}^{2,1}$ that breaks the natural conformal invariance in twistor space. However, we shall see that this is compensated by that fact that scalars with $\Delta \neq 1$ transform in a representation of $Sp(4)$ that involves non-local terms and with these generators, their correlators are indeed conformally invariant. The infinity twistor also features in parity odd correlators. However, the dependence is mild in the sense that it appears only in sign factors and in such a way that on the support of conformally invariant delta functions that it multiplies, the whole expression is conformally invariant as a distribution. We also derive the Penrose transform for generic non-conserved spinning operators. In all cases, the infinity twistor features. One of the main messages of this chapter is that in order to accommodate general representations of the conformal group, one must extend the space of $Sp(4)$ invariants to allow for those that involve the infinity twistor.

We then switch gears and turn towards superconformal field theories. We derive the supersymmetric Penrose transform for $\mathcal{N} = 1$ theories and discuss its relation to the supersymmetric Witten transform. An interesting feature is that the infinity twistor of $\mathbb{R}^{2,1}$ is naturally incorporated by the super-incidence relations. We then discuss the different super-conformal $\mathrm{OSp}(\mathcal{N}|4)$ invariants such as the orthosymplectic dot products and the projective super-delta functions that serve as building blocks for correlators. We explicitly determine various examples in $\mathcal{N} = 1$ theories including super-correlators involving scalar super-fields.

The reference for this chapter is,

- ⋆An Ode to the Penrose and Witten transforms in Twistor space for 3D CFT, Aswini Bala and **Dhruva K.S.**, *JHEP 11 (2025) 056*, [2505.14082].

## 5.1 The Geometry of Twistor Space

In this section, we shall discuss the essential features of real twistor space which we shall use to study $2 + 1$ dimensional CFTs. We first introduce coordinates that we use to chart the space and its connection to $\mathbb{R}^{2,1}$ through the incidence relations. The discussion here is quite reminiscent of the four dimensional case since one can interpret the twistor space for $\mathbb{R}^{2,1}$ as being derived from a dimensional reduction of the twistor space of $\mathbb{R}^{2,2}$. For more discussion, please see appendix A.2.

The projective twistor space $\mathbb{RP}^3$ is spanned by projective coordinates $Z^A$ which are in the fundamental representation of $\mathrm{Sp}(4)$, the double cover of the $2 + 1$ dimensional conformal group $SO(3, 2)$. It can be written as a direct sum of fundamental representations of $SL(2, \mathbb{R})$ which is itself the double cover of the $2 + 1$ dimensional Lorentz group $SO(2, 1)$[1]:

$$Z^A = (\lambda^a, \bar{\mu}_{a'}). \tag{5.1}$$

The connection of this twistor space to the spacetime $\mathbb{R}^{2,1}$ is through the *incidence relations*,

$$\bar{\mu}_a = -x_{ab}\lambda^b. \tag{5.2}$$

Here, $x_{ab} = (\sigma_\mu)_{ab}x^\mu$ is the usual contraction of the position vector $x^\mu$ with the Pauli matrices $\sigma_\mu$. Note that the incidence relations (5.2) after modding out by projective rescalings defines a $\mathbb{RP}^1 \subset \mathbb{RP}^3$. Thus, given any point in $x_{ab} \in \mathbb{R}^{2,1}$, we can associate a $\mathbb{RP}^1$ in twistor space which is topologically equivalent to a circle. The non-locality of this correspondence works both ways. Let us define a point in twistor space as the intersection of two lines (5.2) associated to points $x, y \in \mathbb{R}^{2,1}$. We have,

$$x_{ab}\lambda^b = y_{ab}\lambda^b \implies (x - y)_{ab}\lambda^b = 0. \tag{5.3}$$

[1]It is important to note in (5.1) that both $a$ and $a'$ are spinor indices of the same $SL(2, \mathbb{R})$ as is appropriate in $\mathbb{R}^{2,1}$. This is in contrast to the twistors associated to $\mathbb{R}^{2,2}$ which are direct sums of two independent $SL(2, \mathbb{R})$ spinors.

This implies[2],

$$(x-y)_{ab} = \lambda^a \lambda^b, \tag{5.4}$$

(5.4) corresponds to null separated points $x, y$ as can easily be seen by squaring the equation. Therefore, we see that a point in twistor space corresponds to null line in $\mathbb{R}^{2,1}$. This leads us naturally to the concept of the infinity twistor. Since null rays are invariant under conformal transformations, we see that twistor space as it stands is insensitive to the overall conformal factor of the metric in $\mathbb{R}^{2,1}$. This is where the *infinity* twistor enters the fray. It breaks the conformal invariance and encodes the structure of spacetime at infinity. First, let us see how we can construct the $\mathbb{R}^{2,1}$ metric up to an overall scale from two twistors $Z_1^A, Z_2^A$ associated to a single point $x$ using the incidence relations (5.2). We construct the skew combination,

$$X^{AB} = Z_1^{[A} Z_2^{B]} = \frac{1}{2}\begin{pmatrix} \lambda_1^a \lambda_2^b - \lambda_1^b \lambda_2^a & \lambda_1^a \bar{\mu}_{2b'} - \bar{\mu}_{1b'} \lambda_2^a \\ \bar{\mu}_{1a'} \lambda_2^b - \lambda_1^b \bar{\mu}_{2a'} & \bar{\mu}_{1a'} \bar{\mu}_{2b'} - \bar{\mu}_{1b'} \bar{\mu}_{2a'}. \end{pmatrix} = \frac{\langle 12 \rangle}{2}\begin{pmatrix} \epsilon^{ab} & -x^a_{b'} \\ x^b_{a'} & -\epsilon_{a'b'} x^2 \end{pmatrix}. \tag{5.5}$$

Given this quantity $X^{AB}$, we can construct the following natural line element

$$d\tilde{s}^2 = \frac{1}{2}\epsilon_{ABCD} dX^{AB} dX^{CD} = \frac{1}{2}\mathrm{Pf}\big(dX\big). \tag{5.6}$$

The four index Levi-Civita symbol in $\mathbb{RP}^3$ is constructed out of the Sp(4) conformally invariant tensor $\Omega$ via,

$$\epsilon_{ABCD} = -\Big(\Omega_{AB}\Omega_{CD} - \Omega_{AC}\Omega_{BD} + \Omega_{AD}\Omega_{BC}\Big), \tag{5.7}$$

where,

$$\Omega_{AB} = \begin{pmatrix} 0 & \delta_a^{b'} \\ -\delta_b^{a'} & 0 \end{pmatrix}. \tag{5.8}$$

$\mathrm{Pf}(dX)$ is the Pfaffian of the matrix $dX^{AB}$ formed using (5.5). Opening up (5.6) using the variation of (5.5), we obtain,

$$d\tilde{s}^2 = \langle 12 \rangle^2 (-dt^2 + dx^2 + dz^2) = \langle 12 \rangle^2 \eta_{\mu\nu} dx^\mu dx^\nu. \tag{5.9}$$

Note that (5.9) depends on $\langle 12 \rangle^2$ and thus reproduces the Minkowski metric up to an overall conformal factor. To obtain the flat metric we must introduce a fixed bi-twistor $I_{AB}$ to cancel out this factor. Let us define instead of (5.6),

$$ds^2 = \frac{\epsilon_{ABCD} dX^{AB} dX^{CD}}{2(I_{AB} X^{AB})^2} = \frac{\mathrm{Pf}(dX)}{2(I \cdot X)^2}. \tag{5.10}$$

[2] (5.3) tells us that $(x-y)^{ab} = \alpha^a \lambda^b$ for some $\alpha^a \in \mathbb{R}^2$. However since the Pauli matrices and thus the position bi-spinors are symmetric in their indices, one should rather say that $(x-y)^{ab} = \alpha^{(a}\lambda^{b)}$ with the constraint $\alpha \cdot \lambda = 0$ to ensure (5.3). However, in the two dimensional space spanned by these spinors this implies that $\alpha \propto \lambda$ thus yielding (5.4).

This also ensures that this metric is invariant under projective rescalings $Z \longrightarrow rZ, r \in \mathbb{R}$ as is appropriate in $\mathbb{RP}^3$. If we choose,

$$I_{AB} = \begin{pmatrix} \epsilon_{ab} & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.11}$$

we obtain,

$$(I_{AB}X^{AB})^2 = \langle 12 \rangle^2. \tag{5.12}$$

Substituting this in (5.10) and using (5.9) results in the flat Minkowski metric of $\mathbb{R}^{2,1}$ viz,

$$ds^2 = -dt^2 + dx^2 + dz^2. \tag{5.13}$$

Thus, we see that (5.11) is the infinity twistor that breaks the Sp(4) conformal invariance and picks out the $\mathbb{R}^{2,1}$ flat Minkowski metric that has only the usual $3d$ Poincare invariance. We shall return to the infinity twistor in sections 5.6 where we shall see that it is required to describe scalars with arbitrary scaling dimensions, parity odd Wightman functions as well as non-conserved currents.

It is interesting to see that the construction here is almost identical to its four dimensional counterpart [238]. One key distinction in our case however, is that $X^{AB}$ is required to satisfy,

$$X^{AB}\Omega_{AB} = 0, \tag{5.14}$$

which using (5.5) and (5.8) demands,

$$x^a_b \delta^b_a = 0 \implies x \text{ is traceless}, \tag{5.15}$$

which is guaranteed if $x^a_b = (\sigma_\mu)^a_b x^\mu$ as we indeed took below (5.2) since the Pauli matrices are individually traceless. This condition is required to ensure that the number of independent components of $X^{AB}$ is 3 as is appropriate for a quantity that describes a point in three dimensional space-time[3].

We shall now proceed to discuss the construction of conserved currents and $\Delta = 1$ scalars. First through the Penrose transform that relates twistor space to position space and second, through Witten's half Fourier transform that relates twistor space to momentum space (spinor helicity variables). To show the equivalence of these constructions, we then derive the Penrose transform from the Witten transform.

## 5.2 The Penrose Transform

Let us first introduce the Penrose transform. Just like how the spinor formalism provides an unconstrained way of describing null vectors (like in (5.4))[4], the Penrose transform provides an unconstrained way to describe conserved currents.

[3] The rescaling redundancy $X^{AB} \sim rX^{AB}, r \in \mathbb{R}$, the null condition $\text{Det}(X) \propto X^2 = 0$ and the Omega tracelessness condition (5.14) bring down the number of independent components of the anti-symmetric $4 \times 4$ matrix $X^{AB}$ from 6 to 3.

[4] Given any spinor $\lambda$, the moment we form a vector using $x_{ab} = \lambda_a \lambda_b$, the vector is automatically null. Contrast this to the standard definition of a null vector $x^\mu$. It is a vector $x^\mu$ with the constraint that $-t^2 + x^2 + z^2 = 0$. However, when we write $x_{ab} = \lambda_a \lambda_b$, there is no further constraint on $\lambda$ and hence the terminology, unconstrained.

In the foundational paper on Twistors for 3d CFT [236], the authors introduce the Penrose transform from the embedding space perspective. Here, we stick to the $2+1$ dimensional viewpoint[5]. Given a symmetric traceless conserved current $J_s^{a_1\cdots a_{2s}}(x)$ we can represent it as follows to make manifest its conservation:

$$J_s^{a_1\cdots a_{2s}}(x) = \int \langle\lambda d\lambda\rangle \lambda^{a_1}\cdots\lambda^{a_{2s}} \hat{J}_s^{+}(\lambda,\bar{\mu})\Big|_X, \tag{5.16}$$

where $X$ denotes imposing the incidence relation (5.2) and the measure is the natural one on $\mathbb{RP}^1$.

Let us check that (5.16) leads to a conserved current. Taking a divergence we obtain,

$$\frac{\partial}{\partial x^{a_1 a_2}} J_s^{a_1\cdots a_{2s}}(x) = \int \langle\lambda d\lambda\rangle \lambda^{a_1}\cdots\lambda^{a_{2s}} \frac{\partial\bar{\mu}^a}{\partial x^{a_1a_2}}\frac{\partial}{\partial\bar{\mu}^a}\hat{J}_s^{+}(\lambda,\bar{\mu})|_X \tag{5.17}$$

Using the incidence relation (5.2) we find that,

$$\frac{\partial\bar{\mu}^a}{\partial x^{a_1a_2}} = \lambda_b\big(-2\delta^b_{a_2}\delta^a_{a_1} + \epsilon_{a_1a_2}\epsilon^{ab}\big) = -2\lambda_{a_2}\delta^a_{a_1} + \epsilon_{a_1a_2}\lambda^a. \tag{5.18}$$

Substituting (5.18) in (5.17) yields,

$$\frac{\partial}{\partial x^{a_1 a_2}} J_s^{a_1\cdots a_{2s}}(x) = \int \langle\lambda d\lambda\rangle \lambda^{a_1}\cdots\lambda^{a_{2s}}\big(-2\lambda_{a_2}\delta^a_{a_1} + \epsilon_{a_1a_2}\lambda^a\big)\frac{\partial}{\partial\bar{\mu}^a}\hat{J}_s^{+}(\lambda,\bar{\mu})|_X = 0, \tag{5.19}$$

since $\lambda^{a_2}\lambda_{a_2} = 0$ and $\epsilon_{a_1a_2}\lambda^{a_1}\lambda^{a_2} = 0$. Note that this is true for any $\hat{J}_s^{+}(\lambda,\bar{\mu})$, thus showing that the Penrose transform (5.16) allows us to represent conserved currents in an unconstrained way. An important property of (5.16) is that the integrand is invariant under projective rescalings $Z^A = (\lambda^a,\bar{\mu}_{a'}) \to rZ^A = (r\lambda^a, r\bar{\mu}_{a'})$ which also requires that the twistor space current scales as,

$$\hat{J}_s^{+}(rZ) = \frac{1}{r^{2s+2}}\hat{J}_s^{+}(Z). \tag{5.20}$$

There also exists another type of Penrose transform [236] which is called the derivative based Penrose transform in contrast to its product based counterpart (5.16).

$$J_s^{a_1\cdots a_{2s}}(x) = \int \langle\bar{\lambda} d\bar{\lambda}\rangle \frac{\partial}{\partial\mu_{a_1}}\cdots\frac{\partial}{\partial\mu_{a_{2s}}}\hat{J}_s^{+}(\mu,\bar{\lambda})|_{X'}, \tag{5.21}$$

where $X'$ denotes the dual incidence relation,

$$\mu_a = x_{ab}\bar{\lambda}^b. \tag{5.22}$$

It is easy to check that this Penrose transform is also conserved just like (5.17). The current in dual-twistor variables $W_A = (\mu_a, \bar{\lambda}^{a'})$ also satisfies,

$$\hat{J}_s^{+}(rW) = \frac{1}{r^{-2s+2}}\hat{J}_s^{+}(W). \tag{5.23}$$

[5] Of course, it is easy to check that their Penrose transform when restricted to the Poincare section of the embedding space null cone gives rise to (5.16).

In this work, we stick to the product based Penrose transform (5.16) as it suffices for our purposes. Before we proceed, one important point to note is that both in (5.16) and (5.21), the twistor space current that appears inside the $\mathbb{RP}^1$ integral is the *positive* helicity component of the current which we shall define in (2.10). Recall that symmetric traceless conserved currents in three dimensions have only two independent components which we call positive and negative helicity[6]. There also exist similar Penrose transforms that make use of the negative rather than the positive helicity currents but we do not require the same for this chapter and do not present it.

Let us now discuss another aspect of twistor space: Its connection to momentum space through Witten's half Fourier transform.

## 5.3 The Witten Transform

Witten's half Fourier transform is a map from spinor helicity variables to twistor space. The essential feature is that the momentum vector, the polarization spinors are all expressed in terms of $SL(2,\mathbb{R})$ spinors $\lambda$ and $\bar{\lambda}$ which are assigned negative and positive helicity respectively. The explicit formula are given by,

$$p_{ab} = \lambda_{(a}\bar{\lambda}_{b)} \quad , \quad \zeta_-^a = \frac{\lambda^a}{|\frac{\lambda\cdot\bar{\lambda}}{2}|^{\frac{1}{2}}}, \zeta_+^a = \frac{\bar{\lambda}^a}{|\frac{\lambda\cdot\bar{\lambda}}{2}|^{\frac{1}{2}}}. \tag{5.24}$$

Given a symmetric traceless conserved current in momentum space, one can convert it into spinor helicity variables via,

$$J_s^{\pm}(\lambda,\bar{\lambda}) = \zeta_{\pm}^{a_1}\cdots\zeta_{\pm}^{a_{2s}} J_{s\ a_1\cdots a_{2s}}(p^\mu), \tag{5.25}$$

and thus $J_s^{\pm}$ denote the two independent components of the current. The remaining mixed helicity components are zero due to conservation. The operation of a half-Fourier transform to twistor space due to Witten is as follows:

$$\hat{J}_s^+(Z) = \hat{J}_s^+(\lambda,\bar{\mu}) = \int \frac{d^2\bar{\lambda}}{(2\pi)^2} e^{i\bar{\lambda}\cdot\bar{\mu}} \frac{J_s^+(\lambda,\bar{\lambda})}{|p|^{s-1}}, \tag{5.26}$$

where $p = -\frac{1}{2}\langle\lambda\bar{\lambda}\rangle$. The rescaling with the momentum magnitude ensures a natural and simple action of special conformal transformations; see [237] for more details.

There is another half-Fourier transform that takes us from spinor helicity variables to dual-twistor space[7],

$$\hat{J}_s^+(W) = \hat{J}_s^+(\mu,\bar{\lambda}) = \int \frac{d^2\lambda}{(2\pi)^2} e^{-i\lambda\cdot\mu} \frac{J_s^+(\lambda,\bar{\lambda})}{|p|^{s-1}}, \tag{5.27}$$

[6]One easy way to understand this is in the context of holography. Each spin$-s$ symmetric traceless conserved current is dual to a spin$-s$ massless gauge field in $AdS_4$. The latter have only two physical degrees of freedom which translates to the former having only two independent components.

[7]Both Witten transforms (5.26) and (5.27) are defined for space-like momenta due to the Lorentzian reality conditions which are $\lambda = \lambda^*, \bar{\lambda} = \bar{\lambda}^*$ for space-like momenta [237]. Thus we can independently Fourier transform $\lambda$ and $\bar{\lambda}$.

It is easy to see that the relation between this transform and (5.26) is as follows:

$$\hat{J}_s^+(Z) = \int \frac{d^4W}{(2\pi)^2} e^{iW\cdot Z} \tilde{J}_s^+(W). \tag{5.28}$$

Analogous formulae exist for negative helicity currents and can be found in [237].

The next important thing to check is that the twistor space obtained via a half-Fourier transform (5.26) and the twistor space discussed earlier in the context of the Penrose transform (5.16) are equivalent. As we shall see, the Witten transform (5.26) when used in tandem with the ordinary Fourier transform can be used to derive the Penrose transform (5.16). This will also make it clear that the spinor helicity currents (5.25) are indeed what appears in the Penrose transform (5.16). The Fourier transform for a spin-s symmetric traceless conserved current is given by,

$$J_s^{a_1\cdots a_{2s}}(x^\mu) = \int \frac{d^3p}{(2\pi)^3} e^{-2ip\cdot x} J_s^{a_1\cdots a_{2s}}(p^\mu). \tag{5.29}$$

Let us now express the momentum in spinor helicity variables (5.24) and re-write the integral (5.29) in spinor helicity variables using[8],

$$\int d^3p = \frac{1}{4\text{Vol}(GL(1,\mathbb{R}))} \int d^2\lambda d^2\bar{\lambda} |\lambda\cdot\bar{\lambda}|. \tag{5.30}$$

We derive this formula in appendix B. (5.29) becomes,

$$\begin{aligned} J_s^{a_1\cdots a_{2s}}(x^\mu) &= \frac{1}{4\text{Vol}(GL(1,\mathbb{R}))} \int \frac{d^2\lambda d^2\bar{\lambda}}{(2\pi)^3} |\lambda\cdot\bar{\lambda}| e^{i\bar{\lambda}_a\lambda_b x^{ab}} J_s^{a_1\cdots a_{2s}}(\lambda,\bar{\lambda}) \\ &= \frac{1}{4\text{Vol}(GL(1,\mathbb{R}))} \int \frac{d^2\lambda d^2\bar{\lambda}}{(2\pi)^3} |\lambda\cdot\bar{\lambda}| e^{i\bar{\lambda}_a\lambda_b x^{ab}} \epsilon^{a_1 b_1}\cdots\epsilon^{a_{2s}b_{2s}} J_{s b_1\cdots b_{2s}}(\lambda,\bar{\lambda}). \end{aligned} \tag{5.31}$$

Let us now use the identity,

$$\epsilon^{ab} = \frac{\lambda^a\bar{\lambda}^b - \bar{\lambda}^a\lambda^b}{\lambda\cdot\bar{\lambda}}. \tag{5.32}$$

Substituting this in (5.31) yields,

$$J_s^{a_1\cdots a_{2s}}(x) = \frac{1}{4\text{Vol}(GL(1,\mathbb{R}))} \int \frac{d^2\lambda d^2\bar{\lambda}}{(2\pi)^3} |\lambda\cdot\bar{\lambda}| e^{i\bar{\lambda}_a\lambda_b x^{ab}} \frac{(\lambda^{a_1}\bar{\lambda}^{b_1} - \bar{\lambda}^{a_1}\lambda^{b_1})\cdots(\lambda^{a_{2s}}\bar{\lambda}^{b_{2s}} - \bar{\lambda}^{a_{2s}}\lambda^{b_{2s}})}{(\lambda\cdot\bar{\lambda})^{2s}} J_{s b_1\cdots b_{2s}}(\lambda,\bar{\lambda} \tag{5.33}$$

Using the fact that a conserved current has only two helicity components viz positive and negative helicity (5.25), we can re-write the above as,

$$\begin{aligned} J_s^{a_1\cdots a_{2s}}(x) = \frac{1}{4\text{Vol}(GL(1,\mathbb{R}))} \int \frac{d^2\lambda d^2\bar{\lambda}}{(2\pi)^3} \frac{|\lambda\cdot\bar{\lambda}|}{(\lambda\cdot\bar{\lambda})^{2s}} e^{i\bar{\lambda}_a\lambda_b x^{ab}} \Big( &\lambda^{a_1}\cdots\lambda^{a_{2s}}\bar{\lambda}^{b_1}\cdots\bar{\lambda}^{b_{2s}} J_{s b_1\cdots b_{2s}}(\lambda,\bar{\lambda}) \\ &+ (-1)^{2s}\bar{\lambda}^{a_1}\cdots\bar{\lambda}^{a_{2s}}\lambda^{b_1}\cdots\lambda^{b_{2s}} J_{s b_1\cdots b_{2s}}(\lambda,\bar{\lambda})\Big). \end{aligned} \tag{5.34}$$

[8] As remarked in footnote 7, we work with space-like momenta for which $\lambda$ and $\bar{\lambda}$ are real and thus all the components of the integral in (5.30) are over the real line.

We can now re-label $\lambda \leftrightarrow \bar{\lambda}$ in the second term which yields[9],

$$
\begin{aligned}
J_s^{a_1\cdots a_{2s}}(x) &= \frac{1}{2\mathrm{Vol}(GL(1,\mathbb{R}))}\int \frac{d^2\lambda d^2\bar{\lambda}}{(2\pi)^3}\lambda^{a_1}\cdots\lambda^{a_{2s}}e^{i\bar{\lambda}_a\lambda_b x^{ab}}\frac{\bar{\lambda}^{b_1}\cdots\bar{\lambda}^{b_{2s}}}{|\lambda\cdot\bar{\lambda}|^s}\frac{J_{sb_1\cdots b_{2s}}(\lambda,\bar{\lambda})}{|\lambda\cdot\bar{\lambda}|^{s-1}} \\
&= \frac{1}{2^{2s}\mathrm{Vol}(GL(1,\mathbb{R}))}\int \frac{d^2\lambda d^2\bar{\lambda}}{(2\pi)^3}\lambda^{a_1}\cdots\lambda^{a_{2s}}e^{i\bar{\lambda}_a\lambda_b x^{ab}}\hat{J}_s^+(\lambda,\bar{\lambda}),
\end{aligned} \tag{5.35}
$$

where we have used the definition of the polarizations (5.24) and identified the rescaled positive helicity current using (5.25),

$$
\hat{J}_s^+(\lambda,\bar{\lambda}) = \zeta_+^{b_1}\cdots\zeta_+^{b_{2s}}\frac{J_{sb_1\cdots b_{2s}}}{|\frac{\lambda\cdot\bar{\lambda}}{2}|^{s-1}}, \tag{5.36}
$$

which is what features in the integrand of Witten's transform (5.26). Note that this quantity satisifes,

$$
\hat{J}_s^+(r\lambda,\frac{\bar{\lambda}}{r}) = \frac{1}{r^{2s}}\hat{J}_s^+(\lambda,\bar{\lambda}) \tag{5.37}
$$

Therefore we can use the inverse of the Witten transform (5.26),

$$
\hat{J}_s^+(\lambda,\bar{\lambda}) = \int d^2\bar{\mu}\; e^{-i\bar{\lambda}\cdot\bar{\mu}}\hat{J}_s^+(\lambda,\bar{\mu}). \tag{5.38}
$$

Using (5.37), this leads to the projective property,

$$
\hat{J}_s^+(r\lambda,r\bar{\mu}) = \frac{1}{r^{2s+2}}\hat{J}_s^+(\lambda,\bar{\mu}). \tag{5.39}
$$

Using (5.38) in (5.35) results in,

$$
\begin{aligned}
J_s^{a_1\cdots a_{2s}}(x) &= \frac{1}{2^{2s}\mathrm{Vol}(GL(1,\mathbb{R})}\int \frac{d^2\lambda d^2\bar{\mu}d^2\bar{\lambda}}{(2\pi)^3}\lambda^{a_1}\cdots\lambda^{a_{2s}}e^{-i\bar{\lambda}_a(\bar{\mu}^a-\lambda_b x^{ab})}\hat{J}_s^+(\lambda,\bar{\mu}) \\
&= \frac{1}{2^{2s}2\pi\mathrm{Vol}(GL(1,\mathbb{R})}\int d^2\lambda d^2\bar{\mu}\;\lambda^{a_1}\cdots\lambda^{a_{2s}}\delta^2(\bar{\mu}^a-x^{ab}\lambda_b)\hat{J}_s^+(\lambda,\bar{\mu}) \\
&= \frac{1}{2^{2s}2\pi\mathrm{Vol}(GL(1,\mathbb{R})}\int d^2\lambda\;\lambda^{a_1}\cdots\lambda^{a_{2s}}\hat{J}_s^+(\lambda,\bar{\mu})|_X,
\end{aligned} \tag{5.40}
$$

Thus we see that the incidence relation $X$ we discussed earlier (5.2) appears naturally in (5.35). To bring this to the form of the Penrose transform (5.16), we require one final projective integral identity. As we show in appendix,

$$
f(r\lambda,r\bar{\mu}) = \frac{1}{r^2}f(\lambda,\bar{\mu}) \implies \frac{1}{\mathrm{Vol}(GL(1,\mathbb{R}))}\int d^2\lambda f(\lambda,\bar{\mu}) = \int\langle\lambda d\lambda\rangle f(\lambda,\bar{\mu}). \tag{5.41}
$$

[9] We also use the fact that $J_{sb_1\cdots b_{2s}}(\lambda,\bar{\lambda})$ is symmetric under $\lambda\leftrightarrow\bar{\lambda}$ as it is a function of momentum (5.24). The first (second) term in (5.34) is a positive (negative) helicity component which are the two independent components of the current. However, since the spinors $\lambda,\bar{\lambda}$ are integrated over, these two components are equal inside a Penrose transform. Also, quite importantly, as is easy to check, the other terms we have thrown away are proportional to the divergence of the current. If one attempts to construct time-ordered correlators, the divergence of the current leads to contact terms in the form of Ward-Takahashi identities. However, our focus is on Wightman functions which always have zero Ward-Takahashi identity, see [237]. Therefore, we are free to set the divergence of the current to zero.

We see that the integrand in (5.40) satisfies this property by virtue of (5.39) resulting in,

$$J_s^{a_1\cdots a_{2s}}(x) = \frac{1}{2^{2s}2\pi}\int \langle\lambda d\lambda\rangle\lambda^{a_1}\cdots\lambda^{a_{2s}}\hat{J}_s^{+}(\lambda,\bar{\mu})|_X, \tag{5.42}$$

which is exactly the Penrose transform (5.16) up to an overall numerical normalization! This concludes our proof of the equivalence of the Penrose and Witten transforms.

## 5.4 The Conformal generators and Ward identities

Having obtained formulae that connect twistor space currents to their position space (5.16) and momentum space (5.26) counterparts with their equivalence proved, let us now see how the conformal generators act on twistor space currents. The space $\mathbb{RP}^3$ carries a natural action of the conformal group $Sp(4)$. Acting on conserved currents we have,

$$[T^{AB},\hat{J}_s^{\pm}(Z)] = Z^{(A}\frac{\partial}{\partial Z_{B)}}\hat{J}_s^{\pm}(Z). \tag{5.43}$$

In component language, we find,

$$T_{AB} \equiv \begin{pmatrix} -\bar{\mu}_{(a'}\frac{\partial}{\partial\lambda^{b)}} & -\bar{\mu}_{a'}\frac{\partial}{\partial\bar{\mu}_b} + \lambda^{b'}\frac{\partial}{\partial\lambda^a} \\ -\bar{\mu}_{a'}\frac{\partial}{\partial\bar{\mu}_b} + \lambda^b\frac{\partial}{\partial\lambda^{a'}} & +\lambda^{(a}\frac{\partial}{\partial\bar{\mu}_{b')}} \end{pmatrix} = \begin{pmatrix} iK_{a'b} & -iM^b_{a'} + \frac{2}{i}\delta^b_{a'}D \\ -iM^b_{a'} + \frac{2}{i}\delta^b_{a'}D & -iP^{ab'} \end{pmatrix}, \tag{5.44}$$

with,

$$\begin{aligned} P_{ab} &= i\lambda_{(a}\frac{\partial}{\partial\bar{\mu}^{b)}}, \qquad & K_{ab} &= i\bar{\mu}_{(a}\frac{\partial}{\partial\lambda^{b)}}, \\ \tilde{M}_{ab} &= i\left(\lambda_{(a}\frac{\partial}{\partial\lambda^{b)}} + \bar{\mu}_{(a}\frac{\partial}{\partial\bar{\mu}^{b)}}\right), \qquad & D &= \frac{i}{2}\left(\lambda^a\frac{\partial}{\partial\lambda^a} - \bar{\mu}^a\frac{\partial}{\partial\bar{\mu}^a}\right). \end{aligned} \tag{5.45}$$

For the dual twistor currents $\hat{J}_s^{\pm}(W)$ we find using (5.28) that the generators (5.43) translate into,

$$[T^{AB},\hat{J}_s^{\pm}(W)] = W^{(A}\frac{\partial}{\partial W_{B)}}\hat{J}_s^{\pm}(W), \tag{5.46}$$

with component expansions analogous to (5.44). Thus the conformal $n-$ point Wightman functions satisfy,

$$\sum_{i=1}^{n}\langle 0|\cdots[T_{AB},\hat{J}_{s_i}^{\pm}]\cdots|0\rangle = 0 \tag{5.47}$$

Another important operator that we must consider is the helicity operator. It reads the helicity of a symmetric traceless conserved current which can be $\pm s$. Explicitly we have,

$$\begin{aligned} h_i\langle 0|\cdots\hat{J}_{s_i}^{\pm}(Z_i)\cdots|0\rangle &= -\frac{1}{2}\big(Z_i^A\frac{\partial}{\partial Z_i^A}+2)\big)\langle 0|\cdots\hat{J}_{s_i}^{\pm}(Z_i)\cdots|0\rangle = \pm s_i\langle 0|\cdots\hat{J}_{s_i}^{\pm}(Z_i)\cdots|0\rangle, \\ h_i\langle 0|\cdots\hat{J}_{s_i}^{\pm}(W_i)\cdots|0\rangle &= \frac{1}{2}\big(W_i^A\frac{\partial}{\partial W_i^A}+2)\big)\langle 0|\cdots\hat{J}_{s_i}^{\pm}(W_i)\cdots|0\rangle = \pm s_i\langle 0|\cdots\hat{J}_{s_i}^{\pm}(W_i)\cdots|0\rangle. \end{aligned} \tag{5.48}$$

Imposing (5.48) is equivalent to imposing the projective properties such as (5.20). Now, with the stage being set, we move towards the construction of conformally invariant objects in twistor space that shall feature in correlators of conserved currents and scalars with $\Delta = 1$. More general classes of invariants involving the infinity twistor will be considered in section 5.6. Given the conformal generators (5.43),(5.46), one can construct invariants out of different twistors $Z_i$ and dual twistors $W_j$ that are annihilated by them. In [236, 237], the authors construct conformally invariant solutions by taking them to depend only on twistor dot products which belong to the set,

$$\left\{Z_i \cdot Z_j = -Z_i^A \Omega_{AB} Z_j^B \ , \ W_i \cdot W_j = W_{iA} \Omega^{AB} W_{jB} \ , \ W_i \cdot Z_j = W_{iA} Z_j^A\right\}. \tag{5.49}$$

However, there also exist another class of natural invariants: Projective delta functions. The object,

$$\delta^4(c_1 Z_1 + \cdots + c_{n-1} Z_{n-1} + Z_n), \tag{5.50}$$

is obviously an invariant of $\mathrm{Sp}(4)$. To show this, consider an $\mathrm{Sp}(4)$ transformation $Z_i \to M Z_i, M \in \mathrm{Sp}(4)$. We have,

$$\begin{aligned}
&\delta^4(c_1 Z_1 + \cdots + c_{n-1} Z_{n-1} + Z_n) \to \delta^4(c_1 M Z_1 + \cdots + c_{n-1} M Z_{n-1} + M Z_n) \\
&= \frac{1}{|\mathrm{Det}(M)|}\delta^4(c_1 Z_1 + \cdots + c_{n-1} Z_{n-1} + Z_n) = \delta^4(c_1 Z_1 + \cdots + c_{n-1} Z_{n-1} + Z_n),
\end{aligned} \tag{5.51}$$

where we used the fact that symplectic transformations preserve the volume element and hence $\mathrm{Det}(M) = 1$. To erase the arbitrariness in the parameters $c_i$, we integrate them on the support of a function $f(c_1, c_2, \cdots c_{n-1})$,

$$\mathcal{F}(Z_1, \cdots Z_n) = \int dc_1 \cdots dc_{n-1}\, f(c_1, \cdots, c_{n-1}) \delta^4(c_1 Z_1 + \cdots Z_n), \tag{5.52}$$

where the function $f$ can also depend on symplectic dot products of twistors (5.49). Since we are working in the real projective space $\mathbb{RP}^3$, we need to ensure that this quantity has good projective properties. More precisely, we demand,

$$\mathcal{F}(Z_1, \cdots, r Z_k, \cdots Z_n) = \frac{1}{r^{2\alpha_k + 2}} \mathcal{F}(Z_1, \cdots, Z_k, \cdots Z_n), \tag{5.53}$$

for some $\alpha_k \in \mathbb{R}$. When constructing Wightman functions of conserved currents this amounts to imposing the helicity counting identity (5.48) where $\alpha_k$ will be identified with the helicity of the current with argument $Z_k$.

With three twistors, the unique invariant using the projective delta function that we can form is,

$$\begin{aligned}
&\delta^3(Z_1, Z_2, Z_3; \alpha_{12}, \alpha_{23}, \alpha_{31}) = (-i)^{-\alpha_{12}-\alpha_{23}-\alpha_{31}} \delta^{[-\alpha_{12}-\alpha_{23}-\alpha_{31}]}(Z_1 \cdot Z_2) \int dc_{23} dc_{31} c_{23}^{\alpha_{23}} c_{31}^{\alpha_{31}} \delta^4(c_{23} Z_1 + c_{31} Z_2 + Z_3 \\
&= \int dc_{12} dc_{23} dc_{31} c_{12}^{\alpha_{12}} c_{23}^{\alpha_{23}} c_{31}^{\alpha_{31}} \delta^4(c_{23} Z_1 + c_{31} Z_2 + c_{12} Z_3) e^{\frac{i}{3}\left(\frac{Z_1 \cdot Z_2}{c_{12}} + \frac{Z_2 \cdot Z_3}{c_{23}} + \frac{Z_3 \cdot Z_1}{c_{31}}\right)}.
\end{aligned} \tag{5.54}$$

The first line is assymmetric with respect to the three twistors and in going to the second line, we restored it by introducing an integral over $c_{12}$ on support of the four dimensional delta function. Note that the final result is very reminiscent of the $SL(4)$ invariant delta function that occurs in four dimensional scattering amplitudes . (5.54) defines the three argument $Sp(4)$ invariant projective delta function which is also three dimensional as is appropriate for a delta function in $\mathbb{RP}^3$. The important point to note about (5.54) is the projective property,

$$\delta^3(r_1 Z_1, r_2 Z_2, r_3 Z_3; \alpha_{12}, \alpha_{23}, \alpha_{31}) = \frac{1}{r_1^{-(\alpha_{12}+\alpha_{31})+2} r_2^{-(\alpha_{12}+\alpha_{23})+2} r_3^{-(\alpha_{23}+\alpha_{31})+2}} \delta^3(Z_1, Z_2, Z_3; \alpha_{12}, \alpha_{23}, \alpha_{31}). \tag{5.55}$$

For the derivation of (5.54), please see appendix. This projective delta function will allow us to construct correlation functions involving operators with definite helicity using (5.48).

At the level of four points, there exist infinitely many solutions as it should be since conformal invariance does not suffice to fix the functional form of the correlator. One example of a four point correlator involving a possible $f(c_1, c_2, c_3, c_4)$ is[10],

$$\frac{\delta^{[\alpha_{12}]}(Z_1 \cdot Z_2)\delta^{[\alpha_{34}]}(Z_3 \cdot Z_4)}{\mathrm{Vol}(GL(1,\mathbb{R}))}(\prod_{i=1}^{4} \int dc_i) c_1^{\alpha_1 - \alpha_{12}} c_2^{\alpha_2 - \alpha_{12}} c_3^{\alpha_3 - \alpha_{34}} c_4^{\alpha_4 - \alpha_{34}} \delta^4(c_1 Z_1 + c_2 Z_2 + c_3 Z_3 + c_4 Z_4), \tag{5.56}$$

with the constraint $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2(\alpha_{12} + \alpha_{34})$. Note that (5.56) satisfies the projective property (5.53) for each twistor. Moreover, we see that it is satisfied for any value of the free parameter $\alpha_{12}$ in (5.56). We have also divided by the volume of $GL(1,\mathbb{R})$ (please see (B.1) for the definition) since one of the integrals in (5.56) just provides an overall infinite factor that is canceled out by $\mathrm{Vol}(GL(1,\mathbb{R}))$. It would be very interesting to study the class of CFT correlators that enjoy more symmetry such as dual conformal and Yangian invariance. In such cases, one might be able to solve for the function $f$ in (5.52) such that it is Yangian invariant. However, we shall not attempt the same in this work. We now turn to the study of two and three point Wightman functions and how they are built using the invariants (5.49) and (5.54).

## 5.5 Wightman functions of conserved currents and $\Delta = 1$ scalars

In this section, we shall present the general forms of parity even two and three point Wightman functions of conserved currents. In contrast to previous works [236, 237] which employ an ambidextrous approach, we shall only use twistors and not dual-twistors in this work. As we shall see, this will necessitate the introduction of the projective delta function (5.54) in addition to the dot product invariants (5.49). As a bonus, this description will also allow us to accommodate $\Delta = 1$ scalar operators in twistor space.

[10] We introduced an additional auxiliary integral over $c_4$ compared to (5.52) in order to represent the expression in a more symmetric manner.

### 5.5.1 Two point functions

At the level of two point functions of currents, only the twistor dot products (5.49) feature in the expressions [236, 237]. Moreover, it is only non-zero when both currents are identical and have the same helicity [237]. We have,

$$\langle 0 \hat{J}_s^{\pm}(Z_1)\hat{J}_s^{\pm}(Z_2)|0\rangle = \frac{c_s}{(Z_1 \cdot Z_2)^{2(\pm s+1)}}. \tag{5.57}$$

This is the unique result.

### 5.5.2 Three point functions

Moving onto three points, we shall see that the projective delta function (5.54) plays an important role in contrast to previous ambidextrous works [236, 237]. Let us first review the same. At the level of three points with all three operators having non-zero spin, there exist two distinct parity even solutions: *homogeneous* and *non-homogeneous*. More explanation on the reasoning behind the terminology can be found in [237]. The expressions in the eight possible helicity configurations take the form,

$$\begin{aligned}
\langle 0|\hat{J}_{s_1}^{h_1}(T_1)\hat{J}_{s_2}^{h_2}(T_2)\hat{J}_{s_3}^{h_3}(T_3)|0\rangle_h &= c^{(h)}_{s_1s_2s_3} i^{s_1+s_2+s_3}\delta^{[s_1+s_2-s_3]}(T_1\cdot T_2)\delta^{[s_2+s_3-s_1]}(T_2\cdot T_3)\delta^{[s_3+s_1-s_2]}(T_3\cdot T_1),\\
\langle 0|\hat{J}_{s_1}^{h_1}(U_1)\hat{J}_{s_2}^{h_2}(U_2)\hat{J}_{s_3}^{h_3}(U_3)|0\rangle_{nh} &= i^{-s_1-s_2-s_3} c^{(nh)}_{s_1s_2s_3}\delta^{[-s_1-s_2+s_3]}(U_1\cdot U_2)\delta^{[-s_2-s_3+s_1]}(U_2\cdot U_3)\delta^{[-s_3-s_1+s_2]}(U_3\cdot U_1),
\end{aligned} \tag{5.58}$$

where,

$$\begin{aligned}
T_i &= Z_i \text{ if } h_i = +s_i \text{ and } T_i = W_i \text{ if } h_i = -s_i,\\
U_i &= W_i \text{ if } h_i = +s_i \text{ and } U_i = Z_i \text{ if } h_i = -s_i.
\end{aligned} \tag{5.59}$$

Note in particular that (5.58) makes use of both twistors $Z$ and dual twistors $W$ to capture all eight helicity configurations. Again, we emphasize the point that (5.58) is purely made up of elements of the set (5.49). For example, consider the $(+++)$ helicity configuration. Using (5.58), we see that the homogeneous correlator is a function of twistor variables whereas the non-homogeneous correlator depends on dual-twistor variables. However, what we are going to show now is that by using the projective delta function (5.54), we can bring both solutions on equal footing and represented in just twistor variables. Solving the helicity identities (5.48) for $(+++)$ helicity for (5.54) fixes,

$$\alpha_{ij} = -s_i - s_j + s_k \ , i \neq j \neq k, \tag{5.60}$$

and yields the result,

$$\begin{aligned}
&\langle 0|\hat{J}_{s_1}^{+}(Z_1)\hat{J}_{s_2}^{+}(Z_2)\hat{J}_{s_3}^{+}(Z_3)|0\rangle_{nh}\\
&= c^{(nh)}_{s_1s_2s_3}\int dc_{12}dc_{23}dc_{31}\, c_{12}^{-s_1-s_2+s_3}c_{23}^{-s_2-s_3+s_1}c_{31}^{-s_3-s_1+s_2}\delta^4(c_{23}Z_1+c_{31}Z_2+c_{12}Z_3)e^{\frac{i}{3}\left(c_{12}Z_1\cdot Z_2+c_{23}Z_2\cdot Z_3+c_{31}Z_3\cdot Z_1\right)}.
\end{aligned} \tag{5.61}$$

The reason for the subscript $nh$ is because this solution is the Fourier transform of the non-homogeneous $(WWW)$, $(+++)$ helicity solution (5.58) as can easily be shown;

$$\langle 0|\hat{J}^{+}_{s_1}(Z_1)\hat{J}^{+}_{s_2}(Z_2)\hat{J}^{+}_{s_3}(Z_3)|0\rangle_{nh} = \frac{1}{(2\pi)^6}\int d^4W_1 d^4W_2 d^4W_3 e^{iZ_1\cdot W_1+iZ_2\cdot W_2+iZ_3\cdot W_3}\langle 0|\hat{J}^{+}_{s_1}(W_1)\hat{J}^{+}_{s_2}(W_2)\hat{J}^{+}_{s_3}(W_3)|0\rangle_{nh}$$
$$\propto \int d^4W_1 d^4W_2 d^4W_3 e^{iZ_1\cdot W_1+iZ_2\cdot W_2+iZ_3\cdot W_3}\delta^{[-s_1-s_2+s_3]}(W_1\cdot W_2)\delta^{[-s_2-s_3+s_1]}(W_2\cdot W_1)\delta^{[-s_3-s_1+s_2]}(W_3\cdot W_1). \tag{5.62}$$

Both the delta function product dual twistor space result (5.58) and the projective delta function twistor space result (5.61) go over to the same spinor helicity variables result after Witten's half-Fourier transform (5.26), (5.27). Thus, we can represent the most general parity even solution for the $(+++)$ helicity correlator purely in the $(ZZZ)$ variables as a sum of the homogeneous solution in (5.58) (which is already given in $Z$ variables) and the non-homogeneous solution (5.61) which we can now represent in $Z$ variables thanks to the projective delta function (5.54).

$$\langle 0|\hat{J}^{+}_{s_1}(Z_1)\hat{J}^{+}_{s_2}(Z_2)\hat{J}^{+}_{s_3}(Z_3)|0\rangle = c^{(h)}_{s_1s_2s_3}\langle 0|\hat{J}^{+}_{s_1}(Z_1)\hat{J}^{+}_{s_2}(Z_2)\hat{J}^{+}_{s_3}(Z_3)|0\rangle_{h} + c^{(nh)}_{s_1s_2s_3}\langle 0|\hat{J}^{+}_{s_1}(Z_1)\hat{J}^{+}_{s_2}(Z_2)\hat{J}^{+}_{s_3}(Z_3)|0\rangle_{nh}. \tag{5.63}$$

Putting in the explicit expressions for these quantities we get,

$$\begin{aligned}&\langle 0|\hat{J}^{+}_{s_1}(Z_1)\hat{J}^{+}_{s_2}(Z_2)\hat{J}^{+}_{s_3}(Z_3)|0\rangle\\ &= c^{(h)}_{s_1s_2s_3} i^{s_1+s_2+s_3}\delta^{[\alpha_{12}]}(Z_1\cdot Z_2)\delta^{[\alpha_{23}]}(Z_2\cdot Z_3)\delta^{[\alpha_{31}]}(Z_3\cdot Z_1)\\ &\quad + c^{(nh)}_{s_1s_2s_3}\int dc_{12}\,dc_{23}\,dc_{31}\,c_{12}^{-\alpha_{12}}c_{23}^{-\alpha_{23}}c_{31}^{-\alpha_{31}}\delta^4(c_{23}Z_1+c_{31}Z_2+c_{12}Z_3)\,e^{\frac{iZ_1\cdot Z_2}{c_{12}}},\end{aligned} \tag{5.64}$$

where $\alpha_{ij} = s_i + s_j - s_k, i \neq j \neq k$.

Let us once again emphasize that this expression is purely in terms of twistor variables without using dual twistors in contrast to the earlier results (5.58). The remaining helicity expressions purely in the $Z$ twistor variables without employing dual twistors $W$ and vice versa can be found similarly. Before moving on to correlators involving $\Delta = 1$ scalars, let us comment about the CPT properties of these solutions. For a review of CPT in twistor space please see [237].

### 5.5.3 Extension to $\Delta = 1$ Scalars

Let us reconsider the Penrose transform (5.16). Setting $s = 0$ it reads,

$$O_1(x) = \int \langle\lambda d\lambda\rangle \hat{O}_1(Z)|_X. \tag{5.65}$$

The reason for the subscript 1 is due to the fact that this procedure results in a $\Delta = 1$ scalar operator. Given the representation of the conformal algebra (5.43), it is clear that the twistor space currents (which for us now includes the $s = 0$ scalar case) are dimensionless. Therefore, in (5.65), we see by dimensional analysis that the integrand has dimension 1 due to the $\mathbb{RP}^1$ measure and thus it outputs a scalar operator with $\Delta = 1$. Let us now analyze two and three point twistor space Wightman functions involving these operators.

### Two point functions

The two point result is simply obtained by setting $s = 0$ in (5.57) thus yielding,

$$\langle 0|\hat{O}_1(Z_1)\hat{O}_1(Z_2)|0\rangle = \frac{c_0}{(Z_1 \cdot Z_2)^2}. \tag{5.66}$$

It is easy to check that (5.66) goes to the correct position space two point function by performing the Penrose transform.

### Three point functions

Moving on to three points, let us consider the case with two currents and one scalar first.

$\underline{\langle 0|\hat{J}_{s_1}\hat{J}_{s_2}\hat{O}_1|0\rangle}$

Let us write down the most general ansatz involving the two classes of solutions such as those in (5.64).

$$\begin{aligned}\langle 0|\hat{J}_{s_1}(Z_1)\hat{J}_{s_2}(Z_2)\hat{O}_1(Z_3)|0\rangle &= \alpha\, i^{s_1+s_2}\delta^{[\alpha_{12}]}(Z_1 \cdot Z_2)\delta^{[\alpha_{23}]}(Z_2 \cdot Z_3)\delta^{[\alpha_{31}]}(Z_3 \cdot Z_1)\\ &+ \beta\delta^3(Z_1, Z_2, Z_3; \beta_{12}, \beta_{23}, \beta_{31}).\end{aligned} \tag{5.67}$$

The helicity identity (5.48) with $s = 0$ constrains the coefficients to take the following values:

$$\begin{aligned}&\alpha_{12} = s_1 + s_2, \alpha_{23} = s_2 - s_1, \alpha_{31} = s_1 - s_2,\\ &\beta_{12} = -s_1 - s_2, \beta_{23} = -s_2 + s_1, \beta_{31} = -s_2 + s_1.\end{aligned} \tag{5.68}$$

The result is thus,

$$\begin{aligned}\langle 0|\hat{J}_{s_1}(Z_1)\hat{J}_{s_2}(Z_2)\hat{O}_1(Z_3)|0\rangle &= \alpha\, i^{s_1+s_2}\delta^{[s_1+s_2]}(Z_1 \cdot Z_2)\delta^{[s_2-s_1]}(Z_2 \cdot Z_3)\delta^{[s_1-s_2]}(Z_3 \cdot Z_1)\\ &+ \beta\delta^3(Z_1, Z_2, Z_3; -s_1 - s_2, -s_2 + s_1, -s_2 + s_1).\end{aligned} \tag{5.69}$$

The coefficients $\alpha$ and $\beta$ are independent at this point and not fixed by conformal invariance and the helicity counting identity. However, after converting (5.69) to spinor helicity variables, we see that the value of $\beta = \alpha$ reproduces the correct Wightman functions as we can check with examples. Therefore, the functional form of a three point function involving two spinning operators is given by,

$$\begin{aligned}\langle 0|\hat{J}_{s_1}(Z_1)\hat{J}_{s_2}(Z_2)\hat{O}_1(Z_3)|0\rangle &\propto \big(i^{s_1+s_2}\delta^{[s_1+s_2]}(Z_1 \cdot Z_2)\delta^{[s_2-s_1]}(Z_2 \cdot Z_3)\delta^{[s_1-s_2]}(Z_3 \cdot Z_1)\\ &+ \delta^3(Z_1, Z_2, Z_3; -s_1 - s_2, -s_2 + s_1, -s_2 + s_1)\big).\end{aligned} \tag{5.70}$$

At this point, it is important to note that the authors of [236] did not consider the $\delta^3$ type solution in (5.70) and considered only the first term. However, they showed that just that term suffices to reproduce the correct position space result. The resolution to this apparent paradox is that both terms in (5.70) are required to reproduce the correct spinor helicity expression for the Wightman function as we have taken. However, both

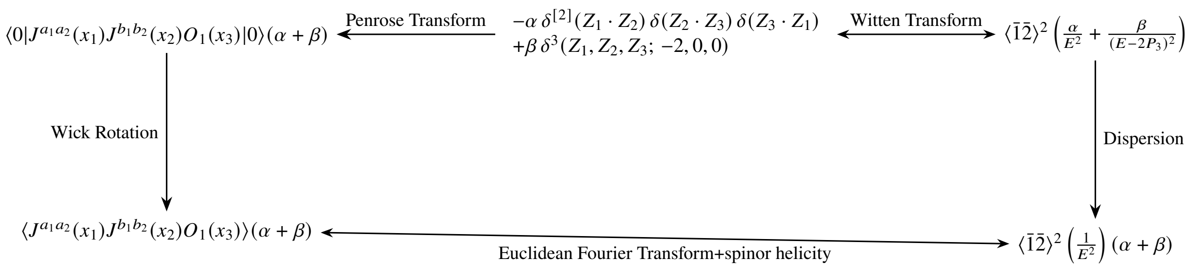


Figure 5.1: The interplay between the Penrose transform and Witten transform

solutions in (5.70) are equal inside the Penrose transform (5.16) and therefore lead to the same position space result.

The following diagram encapsulates this interplay between these transformations and also their corresponding Wick rotations to Euclidean signature.

Starting from the linear combination of the two solutions in twistor space one can perform a Penrose transform to obtain the position space Wightman function[11] (top-left in figure 5.1) which is insensitive to the twistor representative used. However, Witten's half-Fourier distinguishes these solutions and only for $\alpha = \beta$ does it produce the correct Wightman function. However, analytically continuing to Euclidean momentum space using the methods of [236, 237] yields a result that is yet again insensitive to the two solutions. This is also consistent with the Fourier transform of the Wick rotated position space Euclidean correlator (bottom left of figure 5.1). We suspect that the above features are likely due to the fact that one must restrict to space-like momentum when performing the half-Fourier transform and then analytically continue the final results to generic momenta [236, 237].

A similar analysis can be performed for Wightman functions involving two $\Delta = 1$ scalar operators and one conserved current. The result is given by,

$$\langle 0|\hat{J}_s(Z_1)\hat{O}_1(Z_2)\hat{O}_1(Z_3)|0\rangle \propto \Big( i^s\delta^{[s]}(Z_1\cdot Z_2)\delta^{[-s]}(Z_2\cdot Z_3)\delta^{[s]}(Z_3\cdot Z_1) + \delta^3(Z_1,Z_2,Z_3;-s,s,s)\Big). \tag{5.71}$$

Yet again, we verified this formula by converting to spinor helicity variables and checking that it reproduces the correct Wightman function.

Finally, the case where all three operators are $\Delta = 1$ is given by,

$$\langle 0|\hat{O}_1(Z_1)\hat{O}_1(Z_2)\hat{O}_1(Z_3)|0\rangle \propto \Big( \delta(Z_1\cdot Z_2)\delta(Z_2\cdot Z_3)\delta(Z_3\cdot Z_1) + \delta^3(Z_1,Z_2,Z_3;0,0,0)\Big). \tag{5.72}$$

Both of these after a half-Fourier transform give rise to the same result and thus we may choose to work with either of these solutions. We also note that an explicit half-Fourier transform from spinor helicity variables to Twistor space would result in the second term as the reader can easily verify.

This concludes our discussion about parity even Wightman functions involving conserved currents and also $\Delta = 1$ scalars. In the next section, we shall enlarge our

[11] The explicit expressions for the correlator $\langle JJO_1\rangle$ can be found for instance in [78]. Also recall that the Wightman function and Euclidean correlator in position space differ only by an $i\epsilon$ prescription.

space of invariants to include the infinity twistor (5.11) and as we shall see, this is required for correlators involving scalars with arbitrary scaling dimension, parity odd Wightman functions as well as general non-conserved spinning operators.

## 5.6 Conformal invariants involving the Infinity Twistor

In section 5.1, we introduced the infinity twistor as the bi-twistor that breaks the natural conformal invariance of twistor space down to its Poincare' subgroup that preserves the Minkowski metric. So far, in our discussion of parity even Wightman functions of conserved currents and $\Delta = 1$ scalars, the infinity twistor did not appear. However, we shall see in this section that whenever there are scalars with $\Delta \neq 1$ present in the correlator or for parity odd Wightman functions, the infinity twistor is a required ingredient. Although it breaks conformal invariance by itself, we shall see that it occurs in correlators in such a way that the whole expression is rendered conformally invariant. The first part of this section focuses on general scalars whereas the second part deals with parity odd Wightman functions. In the third part of this section, we derive the Penrose transform for generic primary operators with arbitrary scaling dimension and spin. We also briefly discuss some preliminary results regarding their correlation functions in twistor space.

### 5.6.1 Wightman functions of general scalar operators

As we saw in the previous section, $\Delta = 1$ scalars are naturally described in twistor space. Before discussing the construction of correlators of generic scalar operators, we start with the $\Delta = 2$ case. This case is special since a $\Delta = 2$ scalar is related to a $\Delta = 1$ scalar via a *Legendre* transform, which is a conformally covariant transformation that preserves the eigenvalues of the quadratic and quartic conformal Casimirs.

#### The Legendre transform

Given a scalar operator $O_\Delta$ in three dimensions, its Legendre or shadow transform is defined as [85],

$$O_{3-\Delta}(x) = \int \frac{d^3y}{|x-y|^{2(3-\Delta)}} O_\Delta(y). \tag{5.73}$$

Taking $\Delta = 1$ results in,

$$O_2(x) = \int \frac{d^3y}{|x-y|^4} O_1(y). \tag{5.74}$$

Using the Penrose transform (5.65) for $\Delta = 1$ scalars, let us now implement this transformation in twistor space. We have,

$$O_2(x) = \int \frac{d^3y}{|x-y|^4} \int \langle \lambda d\lambda \rangle d^2\bar{\mu}\, \delta^2(\bar{\mu}^a - y^{ab}\lambda_b)\hat{O}_1(\lambda, \bar{\mu}), \tag{5.75}$$

where we have implemented the incidence relations (5.2) using a delta function. Exponentiating the two dimensional delta function and switching the order of integrals results in,

$$\begin{aligned} O_2(x) &= \int \langle \lambda d\lambda \rangle d^2\bar{\mu} \int \frac{d^2\rho}{(2\pi)^2} \int \frac{d^3y}{|x-y|^4} e^{-i\rho_a(\bar{\mu}^a - y^{ab}\lambda_b)} \hat{O}_1(\lambda,\bar{\mu}) \\ &= \int \langle \lambda d\lambda \rangle d^2\bar{\mu} \int \frac{d^2\rho}{(2\pi)^2} e^{-i\rho_a(\bar{\mu}^a - x^{ab}\lambda_b)} \int d^3y' \frac{e^{i\rho_a y'^{ab}\lambda_b}}{|y'|^4} \hat{O}_1(\lambda,\bar{\mu}) \\ &= \int \langle \lambda d\lambda \rangle d^2\bar{\mu} \int \frac{d^2\rho}{(2\pi)^2} e^{-i\rho_a(\bar{\mu}^a - x^{ab}\lambda_b)} \frac{|\lambda\cdot\rho|}{2} \hat{O}_1(\lambda,\bar{\mu}) \\ &= -\int \langle \lambda d\lambda \rangle d^2\bar{\mu} \int \frac{dc}{4\pi\, c^2} \int \frac{d^2\rho}{(2\pi)^2} e^{-i\rho_a(\bar{\mu}^a - x^{ab}\lambda_b - c\lambda^a)} \hat{O}_1(\lambda,\bar{\mu}) \\ &= \int \langle \lambda d\lambda \rangle d^2\bar{\mu}\delta^2(\bar{\mu}^a - x^{ab}\lambda_b) \int \frac{-dc}{4\pi c^2} \hat{O}_1(\lambda,\bar{\mu}+c\lambda). \end{aligned} \tag{5.76}$$

Therefore, up to a normalization factor, we find that the twistor space Legendre transform is given by,

$$\hat{O}_2(\lambda,\bar{\mu}) = \hat{O}_2(Z) = \int_{-\infty}^{\infty} \frac{dc}{c^2} \hat{O}_1(\lambda,\bar{\mu}+c\lambda), \tag{5.77}$$

with the Penrose transform for $O_2$ given by,

$$O_2(x) = \int \langle \lambda d\lambda \rangle \hat{O}_2(\lambda,\bar{\mu})|_X. \tag{5.78}$$

Note the important difference between this Penrose transform and the one for $\Delta = 1$ scalars and conserved currents (5.16). There, the twistor space operators were dimensionless as can be seen by dimensional analysis or the action of the dilatation generator in (5.44). On the other hand, we see that in (5.78), the twistor space operator $\hat{O}_2(\lambda,\bar{\mu})$ has dimension 1 and therefore carries a representation of the conformal algebra different than (5.44). We will discuss this more when we generalize to arbitrary scaling dimensions. Now, we shall discuss two and three point correlators involving the $O_2$ operator obtained via the Legendre transform.

**A contact term and the two point function**

Consider the two point function of the $O_2$ operator with an $O_1$ operator. Generically, the two point function of an operator and its shadow is non-zero and is in fact equal to the $d-$dimensional spacetime delta function in position space [17]. Our aim now is to reproduce this in twistor space using the twistor space Legendre transform (5.77). The $\langle 0|\hat{O}_1(Z_1)\hat{O}_1(Z_2)|0\rangle$ two point function is given by (5.66). Performing a Legendre transform of the first operator results in,

$$\langle 0|\hat{O}_2(Z_1)\hat{O}_1(Z_2)|0\rangle = \int \frac{dc}{c^2}\frac{1}{(\lambda_1\cdot\bar{\mu}_2 - \lambda_2\cdot\bar{\mu}_1 + c\lambda_1\cdot\lambda_2)^2}$$
$$= \int \frac{dc}{c^2}\int dc_{12}|c_{12}|e^{-ic_{12}(\lambda_1\cdot\bar{\mu}_2-\lambda_2\cdot\bar{\mu}_1+c\lambda_1\cdot\lambda_2)} = \int dc_{12}|c_{12}|e^{-ic_{12}Z_1\cdot Z_2}\int\frac{dc}{c^2}e^{-ic_{12}c\langle 12\rangle}$$
$$= \int dc_{12}|c_{12}||c_{12}\langle 12\rangle|e^{-ic_{12}Z_1\cdot Z_2} = |\langle 12\rangle|\int dc_{12}c_{12}^2 e^{-ic_{12}Z_1\cdot Z_2} = |\langle 12\rangle|\delta^{[2]}(Z_1\cdot Z_2). \tag{5.79}$$

Recasting the result in terms of the infinity twistor (5.11) and using the fact that $\langle 12\rangle = \langle Z_1 I Z_2\rangle = Z_1^A I_{AB} Z_2^B$ with $I_{AB}$ the infinity twistor (5.11) and $Z^A$ given by (5.1) yields,

$$\langle 0|\hat{O}_2(Z_1)\hat{O}_1(Z_2)|0\rangle = |\langle Z_1 I Z_2\rangle|\delta^{[2]}(Z_1\cdot Z_2) \tag{5.80}$$

Notice that the infinity twistor has entered the fray for this correlator. Note also that (derivatives) of delta functions enforcing $Z_1\cdot Z_2 = 0$ were discarded in earlier analysis for correlators of conserved currents [236, 237]. However, we see here that it naturally occurs in this two point contact term. One can also verify that upon a Penrose transform, (5.80) results in the correct position space contact term.

Let us now consider the two point function of the $O_2$ operator. Performing the Legendre transform (5.77) for both operators on the $\langle 0|\hat{O}_1(Z_1)\hat{O}_1(Z_2)|0\rangle$ correlator (5.66) results in,

$$\langle 0|\hat{O}_2(Z_1)\hat{O}_2(Z_2)|0\rangle = \frac{\langle Z_1 I Z_2\rangle^2}{(Z_1\cdot Z_2)^4}. \tag{5.81}$$

Yet again, the infinity twistor is present in the result (5.81).

### Three point functions

The Legendre transform formula (5.77) makes it very easy to obtain three point functions with an arbitrary number of $O_2$ insertions given the corresponding correlator with $O_1$. For example, the three point function of this operator using (5.72) is given by,

$$\langle 0|\hat{O}_2(Z_1)\hat{O}_2(Z_2)\hat{O}_2(Z_3)|0\rangle =$$
$$\prod_{i=1}^{3}\left(\int\frac{dk_i}{k_i^2}\right)\delta(Z_1\cdot Z_2+(k_1+k_2)Z_1IZ_2)\delta(Z_2\cdot Z_3+(k_2+k_3)Z_2IZ_3)\delta(Z_3\cdot Z_1+(k_3+k_1)Z_3IZ_1). \tag{5.82}$$

This result can also be written as,

$$\frac{|\langle Z_1IZ_2\rangle\langle Z_2IZ_3\rangle\langle Z_3IZ_1\rangle|}{\left(\frac{Z_1\cdot Z_2}{\langle Z_1IZ_2\rangle}-\frac{Z_2\cdot Z_3}{\langle Z_2IZ_3\rangle}-\frac{Z_3\cdot Z_1}{\langle Z_3IZ_1\rangle}\right)^2\left(-\frac{Z_1\cdot Z_2}{\langle Z_1IZ_2\rangle}+\frac{Z_2\cdot Z_3}{\langle Z_2IZ_3\rangle}-\frac{Z_3\cdot Z_1}{\langle Z_3IZ_1\rangle}\right)^2\left(-\frac{Z_1\cdot Z_2}{\langle Z_1IZ_2\rangle}-\frac{Z_2\cdot Z_3}{\langle Z_2IZ_3\rangle}+\frac{Z_3\cdot Z_1}{\langle Z_3IZ_1\rangle}\right)^2}. \tag{5.83}$$

Similarly, one can obtain the expressions for cases where the $\hat{O}_2$ operator appears once or twice with conserved currents. We note that the below answers are valid only inside a Penrose transform as we ignore the second contribution in equations (5.70) and (5.71) although its easy enough to include them if required.

For two $\Delta = 2$ scalars and one conserved current we have,

$$\langle 0|\hat{J}_s^+(Z_1)\hat{O}_2(Z_2)\hat{O}_2(Z_3)|0\rangle$$
$$= \int \frac{dk_1\, dk_2}{(k_1k_2)^2}\delta^{[s]}(Z_1\cdot Z_2 + k_1Z_1IZ_2)\delta^{[-s]}(Z_2\cdot Z_3 + (k_1+k_2)Z_2IZ_3)\delta^{[s]}(Z_3\cdot Z_1 + k_2Z_3IZ_1), \tag{5.84}$$

whereas the one $\Delta = 2$ insertion yields,

$$\langle 0|\hat{J}_{s_1}^+(Z_1)\hat{J}_{s_2}^+(Z_2)\hat{O}_2(Z_3)|0\rangle$$
$$= \int \frac{dk}{k^2}\delta^{[s_1+s_2]}(Z_1\cdot Z_2)\delta^{[s_2-s_1]}(Z_2\cdot Z_3 + kZ_2IZ_3)\delta^{[s_1-s_2]}(Z_3\cdot Z_1 + kZ_3IZ_1). \tag{5.85}$$

Let us now move on to the generalization of the Penrose transform to scalars with arbitrary scaling dimension.

### The Penrose transform for arbitrary $\Delta$

Consider a scalar operator $O_\Delta(x)$. Our aim is to derive a Penrose transform for this operator. We look for an expression of the form,

$$O_\Delta(x) = \int \langle\lambda d\lambda\rangle\, \hat{O}_\Delta(Z)|_X, \tag{5.86}$$

where $X$ denotes the usual incidence relations (5.2).

By dimensional analysis, we see that $\hat{O}_\Delta(Z)$ must have scaling dimension $\Delta - 1$. The projectiveness of the integrand (5.86) also implies that $\hat{O}_\Delta(Z)$ has homogeneity $-2$. Given the fact that twistor space is connected to spinor helicity variables by a Witten transform (5.26), let us obtain $\hat{O}_\Delta(Z)$ via the same. Consider,

$$\hat{O}_\Delta(Z) = \hat{O}_\Delta(\lambda,\bar{\mu}) = \int \frac{d^2\bar{\lambda}}{(2\pi)^2}e^{i\bar{\lambda}\cdot\bar{\mu}}O_\Delta(\lambda,\bar{\lambda})\left|\frac{\lambda\cdot\bar{\lambda}}{2}\right|. \tag{5.87}$$

Note that we have rescaled the operator by $|\frac{\lambda\cdot\bar{\lambda}}{2}|$ since only then does the twistor space operator $\hat{O}_\Delta(\lambda,\bar{\mu})$ have dimension $\Delta - 1$ (The operator $O_\Delta(\lambda,\bar{\lambda})$ has scaling dimension $\Delta - 3$ due to the Fourier transform and the measure contributes $+1$ which along with the rescaling factor makes the total $\Delta - 1$). One can now follow steps very similar to our Penrose transform derivation for the conserved currents and $\Delta = 1$ scalars to see that it leads to the Penrose transform (5.86). The Witten transform (5.87) enables us to derive the action of the conformal generators on the twistor space operator $\hat{O}(\lambda,\bar{\mu})$. Translations and Lorentz transformations act exactly like in (5.44) whereas dilatations are modified to,

$$[D,\hat{O}_\Delta(\lambda,\bar{\mu})] = \frac{i}{2}\left(\lambda^a\frac{\partial}{\partial\lambda^a} - \bar{\mu}^a\frac{\partial}{\partial\bar{\mu}^a} + 2(1-\Delta)\right)\hat{O}_\Delta(\lambda,\bar{\mu}), \tag{5.88}$$

which is expected since the dimensionality of the twistor space current is $\Delta - 1$ as we saw after establishing the Penrose transform (5.86). The special conformal transformations, on the other hand, act in a very interesting *non-local* manner. We find

$$\begin{aligned}&[K_{ab},\hat{O}_\Delta(\lambda,\bar{\mu})]\\ &= 2i\left[\bar{\mu}_{(a}\frac{\partial}{\partial\lambda^{b)}} + (\Delta-1)\left((\lambda\cdot\frac{\partial}{\partial\bar{\mu}})^{-1}(\lambda_{(a}\frac{\partial}{\partial\lambda^{b)}} - \bar{\mu}_{(a}\frac{\partial}{\partial\bar{\mu}^{b)}}) - 2(\lambda\cdot\frac{\partial}{\partial\bar{\mu}})^{-2}\lambda_{(a}\frac{\partial}{\partial\bar{\mu}^{b)}}\right)\right]\hat{O}_\Delta(\lambda,\bar{\mu}).\end{aligned} \tag{5.89}$$

We define the inverse derivatives appearing in (5.89) viz,

$$(\lambda\cdot\frac{\partial}{\partial\bar{\mu}})^{-1}f(\lambda,\bar{\mu}) = -\int_0^\infty ds\, f(\lambda,\bar{\mu}+s\lambda). \tag{5.90}$$

For the inverse derivatives to make sense we require that $f(\lambda,\bar{\mu}) \notin \mathrm{Ker}(\lambda\cdot\frac{\partial}{\partial\bar{\mu}})$. For the twistor space operators $\hat{O}_\Delta(\lambda,\bar{\mu})$ which satisfy the unitary bound ($\Delta \geq \frac{d-2}{2}$), this requirement is indeed satisfied and therefore this operator is invertible on the subspace spanned by the twistor space operators. One can easily check using the definition (5.90) that it satisfies,

$$(\lambda\cdot\frac{\partial}{\partial\bar{\mu}})(\lambda\cdot\frac{\partial}{\partial\bar{\mu}})^{-1}f(\lambda,\bar{\mu}) = f(\lambda,\bar{\mu}), \tag{5.91}$$

provided $f(\lambda,\bar{\mu})$ vanishes when $\bar{\mu}\to\infty$ which is true of the examples we consider in this chapter such as the two point function to be derived soon. We also note that the quantity $\lambda\cdot\frac{\partial}{\partial\bar{\mu}}$ is the same as $Z^A I_{AB}\frac{\partial}{\partial Z_B}$ as can be seen from (5.11) and (5.1) thus showing that the infinity twistor explicitly appears in the special conformal generator (5.89). Let us remark at this point that such a non-local representation has been considered in the context of *conformal quantum mechanics* [98] where the authors show that the conformal $\mathfrak{sl}(2,\mathbb{R})$ algebra admits a representation where the special conformal transformation contains a non-local term quantified by a parameter $\rho$. We see that in our case (5.89), the analogous parameter is $\Delta - 1$. When $\Delta = 1$, both the dilatation and SCT generator collapse down to the local generators (5.44) and for $\Delta \neq 1$, we need to deal with the non-local transformations in (5.89)[12]. The $O_2$ operator discussed in the first part of this section is a special case of the general $\Delta$ case and one can check that the results (5.80), (5.81),(5.82),(5.84) and (5.85) satisfy the conformal Ward identities dictated by (5.88), (5.89) for $\Delta = 2$.

Let us now investigate two point functions involving $\hat{O}_\Delta$.

[12]In [98], the dilatation operator is unaffected by the non-local term in the SCT operator. However, in our construction one can see that the second term in the RHS of (5.89) precisely produces the additional term in (5.88) by taking the commutator of $P$ with $K$ which shows that the conformal algebra is obeyed.

### Two point functions

Consider a two point function of $\hat{O}_\Delta(Z_1)$. A general ansatz allowing for the infinity twistor (due to its presence in the generator (5.89)) takes the form,

$$\langle 0|\hat{O}_\Delta(Z_1)\hat{O}_\Delta(Z_2)|0\rangle = (Z_1 \cdot Z_2)^\alpha |\langle Z_1 I Z_2\rangle|^\beta. \tag{5.92}$$

Covariance under dilatations (5.88) fixes $\beta = 2(\Delta - 1)$ and invariance under projective rescaling in (5.86) fixes $\alpha = -2\Delta$. Thus our candidate two point function reads,

$$\langle 0|\hat{O}_\Delta(Z_1)\hat{O}_\Delta(Z_2)|0\rangle = \frac{|\langle Z_1 I Z_2\rangle|^{2(\Delta-1)}}{(Z_1 \cdot Z_2)^{2\Delta}}. \tag{5.93}$$

One can show that (5.93) is invariant under special conformal transformations (5.89). Further, one can show that it goes over to the correct position space result. We do not pursue the construction of three point functions of arbitrary scalars in this chapter. One way to proceed is to solve the conformal Ward identity (5.89) which in this case is an integro-differential equation and thus is quite involved. Another inviting possibility is to use weight-shifting operators [36,239] and the results for the correlators involving $O_1$ and $O_2$ to bootstrap higher $\Delta$ correlators. We leave such an exercise to the future. Now, we move on to another situation where the infinity twistor is an essential ingredient in the analysis: The case of parity odd correlators.

## 5.6.2 Parity odd Wightman functions

In three dimensional CFT, conformal invariance allows for the existence of parity odd contributions to two and three point functions of conserved currents. These were obtained in twistor space in [237]. For instance, at the level of three points they obtained,

$$\begin{aligned}&\langle 0|\hat{J}_{s_1}^{h_1}(T_1)\hat{J}_{s_2}^{h_2}(T_2)\hat{J}_{s_3}^{h_3}(T_3)|0\rangle_{\text{odd}}\\ &= +i\,\text{Sgn}\Big(\sum_{j=1}^{3}\frac{h_j}{s_j}\Big)c_{s_1s_2s_3}^{(\text{odd})} i^{s_1+s_2+s_3}\delta^{[s_1+s_2-s_3]}(T_1\cdot T_2)\delta^{[s_2+s_3-s_1]}(T_2\cdot T_3)\delta^{[s_3+s_1-s_2]}(T_3\cdot T_1),\end{aligned} \tag{5.94}$$

where,

$$T_i = Z_i \text{ if } h_i = +s_i \text{ and } T_i = W_i \text{ if } h_i = -s_i. \tag{5.95}$$

The result (5.94) is consistent with the correct spinor helicity results after a half-Fourier transform [237]. However, one must take caution before using it in a Penrose transform (5.16), the reason for which we shall discuss now. The point that we want to make here is that starting with (5.94) (lets say choosing the $(+ + +)$ helicity) and setting up the traditional Penrose transform (5.16), we get the same expression as the parity even homogeneous correlator in (5.58) up to an overall coefficient! As it stands (5.94) is a suitable twistor space representation for the parity odd correlator when viewed in the guise of the half-Fourier transform to spinor helicity variables for all space-like momenta. However, it is also desirable to have an expression whose Penrose transform to position space yields the parity odd correlator. That is the aim of this subsection.

### The Epsilon transform

Let us start with a discussion of the epsilon transform. It is a conformally invariant (preserves eigenvalues of quadratic and quartic Casimirs) transformation in three dimensional CFTs [44, 156]. However, its novelty is that it flips the discrete parity label of a representation. Given a current $J^{\mu_1\cdots\mu_s}(x)$, its epsilon transform is defined as,

$$(\epsilon\cdot J)^{\mu_1\cdots\mu_s}(x)=\epsilon^{\nu\sigma(\mu_1}\int\frac{d^3y}{|y-x|^2}\frac{\partial}{\partial y^\nu}J_\nu^{\mu_2\cdots\mu_s)}(y). \tag{5.96}$$

Similar to the Legendre transform, we desire to cast this transformation in twistor space. To illustrate the point, we consider a spin-1 current. We shall also employ spinor notation by contracting any vector indices with the Pauli matrices, since it is more natural in this framework. After some simple sigma matrix manipulation we get for the spin-1 case,

$$(\epsilon\cdot J)^{a_1a_2}(x)=-i\int\frac{d^3y}{|y-x|^2}\frac{\partial}{\partial y^c_{(a_2}}J^{a_1)c}(y) \tag{5.97}$$

Using the Penrose transform (5.16) for the spin-1 current in the RHS of (5.97) we obtain,

$$(\epsilon.J)_{ab}(x)=\int\langle\lambda d\lambda\rangle\lambda_{(a}\lambda_{b)}\int d^2\bar{\mu}\,\delta^2(\bar{\mu}^a-x^{ad}\lambda_d)(-i)\int\frac{d^3y}{y^2}Z^AI_{AB}\frac{\partial}{\partial Z_B}\hat{J}^+(\lambda,\bar{\mu})\Big|_{\bar{\mu}^a\to\bar{\mu}^a+y^{ad}\lambda_d}. \tag{5.98}$$

This identifies the twistor space epsilon transform as,

$$(\epsilon\cdot\hat{J})^+(Z)=-i\int\frac{d^3y}{y^2}Z^AI_{AB}\frac{\partial}{\partial Z_B}\hat{J}^+(\lambda,\bar{\mu})|_{\bar{\mu}^a\to\bar{\mu}^a+y^{ab}\lambda_b}. \tag{5.99}$$

We note that the infinity twistor (5.11) is part of the very definition of the epsilon transform (5.99)! As we will see now, it will thus feature in parity odd Wightman functions. Before we proceed, we present the generalization to arbitrary spin which is a simple generalization of (5.99).

The epsilon transform in twistor space for a spin-s current is given by,

$$(\epsilon\cdot\hat{J}_s)^+(Z)=-i\int\frac{d^3y}{y^2}Z^AI_{AB}\frac{\partial}{\partial Z_B}\hat{J}_s^+(\lambda,\bar{\mu})\Big|_{\bar{\mu}^a\to\bar{\mu}^a+y^{ab}\lambda_b}. \tag{5.100}$$

The "parity odd" Penrose transform is thus given by,

$$(\epsilon\cdot J_s)^{a_1\cdots a_{2s}}(x)=\int\langle\lambda d\lambda\rangle\lambda^{a_1}\cdots\lambda^{a_{2s}}(\epsilon\cdot\hat{J}_s)^+(Z)|_X, \tag{5.101}$$

where $(\epsilon\cdot\hat{J}_s)^+(Z)$ is given in (5.100) and $X$ are the usual incidence relations (5.2).

For convenience, we also present the epsilon transformed version of the Witten transform (5.26) that can be obtained using the Fourier transform of (5.96). The result is,

$$(\epsilon\cdot\hat{J}_s)(\lambda,\bar{\mu})=+i\int\frac{d^2\bar{\lambda}}{(2\pi)^2}e^{i\bar{\lambda}\cdot\bar{\mu}}\text{Sign}(\lambda\cdot\bar{\lambda})\frac{J_s^+(\lambda,\bar{\lambda})}{|p|^{s-1}}, \tag{5.102}$$

where as usual $p=-\frac{1}{2}\lambda\cdot\bar{\lambda}$.

### Two point functions

In three dimensions, conserved currents can have a parity odd contribution to the two point function that is a contact term. In twistor space, one can perform using (5.100) an epsilon transform with respect to the first operator of the parity even positive helicity two point function (5.57). This results in

$$\langle 0|\hat{J}_s^+(Z_1)\hat{J}_s^+(Z_2)|0\rangle_{\text{odd}} = -ic_{\text{s,odd}}\text{Sgn}(\langle Z_1 I Z_2\rangle)\delta^{[2s+1]}(Z_1\cdot Z_2). \tag{5.103}$$

Note that delta functions of twistor dot products were discarded in earlier works [236, 237] since they do not have the correct rescaling properties under rescaling by negative numbers and do not give rise to the correct spinor helicity correlators. Here, on the other hand, the Sign factor involving the infinity twistor in (5.103) takes care of both of these issues simultaneously. Under rescalings by an amount $r$ we have,

$$\text{Sgn}(\langle Z_1 I Z_2\rangle)\delta^{[2s+1]}(Z_1\cdot Z_2) \to \frac{\text{Sgn}(r)}{r^{2s+2}\text{Sgn}(r)}\text{Sgn}(\langle Z_1 I Z_2\rangle)\delta^{[2s+1]}(Z_1\cdot Z_2) = \frac{1}{r^{2s+2}}\text{Sgn}(\langle Z_1 I Z_2\rangle)\delta^{[2s+1]}(Z_1\cdot Z_2) \tag{5.104}$$

One can verify that (5.103) is conformally invariant with respect to the generators (5.43) as a *distribution* (that is, on the support of the delta function that it multiplies) despite the presence of the sign factor whose argument breaks conformal invariance.

Similarly, when converted to spinor helicity variables, $\text{Sgn}(Z_1 I Z_2)$ cancels out the $\text{Sgn}(\langle 12\rangle)$ that appears in the Witten transform (5.26) as was discussed in [237]. Moreover, it goes over to the correct conformally invariant parity odd contact term after performing the Penrose transform in (5.103) thereby validating the result.

### Three point functions

Performing an epsilon transform of the parity even homogeneous Wightman function (5.64) with respect to the first current, we obtain

$$\begin{aligned}
&\langle 0|\hat{J}_{s_1}^+(Z_1)\hat{J}_{s_2}^+(Z_2)\hat{J}_{s_3}^+(Z_3)|0\rangle_{\text{odd}}\\
&= \frac{i}{2}\int dc_{12}dc_{23}dc_{31}c_{12}^{s_1+s_2-s_3}c_{23}^{s_2+s_3-s_1}c_{31}^{s_3+s_1-s_2}\text{Sgn}\big(c_{12}\langle Z_1 I Z_2\rangle + c_{31}\langle Z_3 I Z_1\rangle\big)\\
&\qquad \times \left(e^{-ic_{12}Z_1\cdot Z_2 - ic_{23}Z_2\cdot Z_3 - ic_{31}Z_3\cdot Z_1}\right).
\end{aligned} \tag{5.105}$$

The infinity twistor similar to in the two point function (5.103) appears via sign factors. One can show similar to the two point case that (5.105) satisfies the conformal Ward identities (5.43) as a distribution.

Let us now compare our new result (5.105) with the parity odd three point function of [237] given in (5.94), in Schwinger parametrization:

$$\begin{aligned}
&\langle 0|\hat{J}_{s_1}^+(Z_1)\hat{J}_{s_2}^+(Z_2)\hat{J}_{s_3}^+(Z_3)|0\rangle_{\text{odd from [237]}}\\
&= \frac{i}{2}\int dc_{12}dc_{23}dc_{31}c_{12}^{s_1+s_2-s_3}c_{23}^{s_2+s_3-s_1}c_{31}^{s_3+s_1-s_2}e^{-ic_{12}Z_1\cdot Z_2 - ic_{23}Z_2\cdot Z_3 - ic_{31}Z_3\cdot Z_1}.
\end{aligned} \tag{5.106}$$

Note in particular that (5.106) differs from (5.105) by a factor of $\text{Sgn}\big(c_{12}\langle 12\rangle + c_{31}\langle 31\rangle\big)$ in the Schwinger parameter integrals. When converted to spinor-helicity variables via Witten's half-Fourier transform (5.26), (5.106) and (5.105) respectively yield,

$$\begin{aligned}\langle 0|\hat{J}^+_{s_1}(p_1)\hat{J}^+_{s_2}(p_2)\hat{J}^+_{s_3}(p_3)|0\rangle_{\text{odd from [237]}} &= i\frac{\langle\bar{1}\bar{2}\rangle^{s_1+s_2-s_3}\langle\bar{2}\bar{3}\rangle^{s_2+s_3-s_1}\langle\bar{3}\bar{1}\rangle^{s_3+s_1-s_2}}{E^{s_1+s_2+s_3}}\\ \langle 0|\hat{J}^+_{s_1}(p_1)\hat{J}^+_{s_2}(p_2)\hat{J}^+_{s_3}(p_3)|0\rangle_{\text{odd}} &= i\text{Sgn}\Big(\sqrt{p_1^2}\Big)\frac{\langle\bar{1}\bar{2}\rangle^{s_1+s_2-s_3}\langle\bar{2}\bar{3}\rangle^{s_2+s_3-s_1}\langle\bar{3}\bar{1}\rangle^{s_3+s_1-s_2}}{E^{s_1+s_2+s_3}}.\end{aligned} \tag{5.107}$$

Since the half-Fourier transform is performed for space-like momenta the sign factor is equal to 1 in such cases and thus both results agree. However, it is (5.105) that also gives rise to the correct position space result after a Penrose transform (5.16). Therefore, as the authors themselves commented, the results of [237] for parity-odd correlators are in the context of the half-Fourier transform for space-like momenta. Our new result (5.105) is valid in the context of both the Witten and Penrose transforms. The sign factors appearing in our formulae (5.103) and (5.105) are responsible for resulting in a parity odd expression which can be seen by using these expressions in the Penrose transform (5.16)[13]. The main result of this subsection was the twistor space epsilon transform (5.100). It allows us to define a new Penrose transform like (5.101) that involves the infinity twistor in contrast to (5.16). The generalization to other helicities is straightforward so we do not present the details here.

### 5.6.3 Non conserved currents in twistor space

In the third and final part of this section, we shall generalize the Penrose transform (5.16) to accommodate general symmetric traceless integer spin primary representations of the conformal group. We then investigate two point Wightman functions involving such operators. Let us concentrate on the spin-1 case for technical simplicity and then generalize to arbitrary spin. Consider a spin-1 operator with dimension $\Delta$. We propose the following Penrose transform for this operator:

$$O^{ab}_{\Delta}(x) = \int \langle\lambda d\lambda\rangle\Big(\lambda^a\lambda^b\hat{O}^{+1}_{\Delta}(\lambda,\bar{\mu}) - i\lambda^{(a}\frac{\partial}{\partial\bar{\mu}_{b)}}\hat{O}^{0}_{\Delta}(\lambda,\bar{\mu})\Big)|_X, \tag{5.108}$$

where $X$ denotes the usual incidence relations (5.2).

Some words about (5.108) are in order. First of all, unlike the Penrose transforms for currents (5.16) or $O_\Delta$ scalars (5.86) which output the position space operator given a single twistor space operator, the formula (5.108) requires two twistor space operators for the same process. Both these operators $\hat{O}^{+1}(\lambda,\bar{\mu})$, $\hat{O}^{0}(\lambda,\bar{\mu})$ have scaling dimension $\Delta-2$ as we can read of from (5.108). However, they have different projective properties.

[13] Recall that the parity operator in 2 + 1 dimensions takes the coordinate $z \to -z$. One can check by plugging in (5.103) and (5.105) into the Penrose transform and using the incidence relations that they pick up a sign under a parity operation, affirming that the expression is parity odd. Similarly, one can check that they are also odd under time-reversal $t \to -t, i \to -i$. This also ensures that they are invariant under $PT$.

For the integrand in (5.108) to be invariant under projective rescalings $(\lambda, \bar{\mu}) \to (r\lambda, r\bar{\mu})$ we require,

$$\hat{O}_{\Delta}^{+1}(r\lambda, r\bar{\mu}) = \frac{1}{r^4}\hat{O}_{\Delta}^{+1}(\lambda, \bar{\mu}), \;\; \hat{O}_{\Delta}^{0}(r\lambda, r\bar{\mu}) = \frac{1}{r^2}\hat{O}_{\Delta}^{0}(\lambda, \bar{\mu}). \tag{5.109}$$

These operators are obtained via the Witten transform (5.26) as follows:

$$\hat{O}_{\Delta}^{+1}(\lambda, \bar{\mu}) = \int \frac{d^2\bar{\lambda}}{(2\pi)^2} e^{i\bar{\lambda}\cdot\bar{\mu}} O_{\Delta}^{+1}(\lambda, \bar{\lambda}), \;\; \hat{O}_{\Delta}^{0}(\lambda, \bar{\mu}) = \int \frac{d^2\bar{\lambda}}{(2\pi)^2} e^{i\bar{\lambda}\cdot\bar{\mu}} O_{\Delta}^{0}(\lambda, \bar{\lambda}). \tag{5.110}$$

The spinor helicity operators appearing in the integrals in the above equation are given by (see (5.24) for the definitions of the polarizations),

$$O_{\Delta}^{+1}(\lambda, \bar{\lambda}) = \zeta_{+}^{a_1}\zeta_{+}^{a_2} O_{\Delta a_1 a_2}(\lambda, \bar{\lambda}), \;\; O_{\Delta}^{0}(\lambda, \bar{\lambda}) = \zeta_{+}^{a_1}\zeta_{-}^{a_2} O_{\Delta a_1 a_2}(\lambda, \bar{\lambda}) \tag{5.111}$$

Note that the second term in (5.111) is identically zero for conserved currents as it is simply equal to $p^{a_1 a_2} O_{\Delta a_1 a_2}(\lambda, \bar{\lambda})$ which is the divergence of the current. Let us now derive the formula (5.108) following similar steps to section 5.3. Till (5.33), the steps remain the same. The difference appears in the next step (5.34) where we threw away terms proportional to the divergence of the current. For a spin-1 non conserved current the analog of (5.33) after some simple variable-relabeling in some of the terms reads,

$$O_{\Delta}^{a_1 a_2}(x) = \frac{1}{\mathrm{Vol}(GL(1,\mathbb{R}))} \int d^2\lambda d^2\bar{\lambda} \left|\frac{\lambda\cdot\bar{\lambda}}{2}\right| \Big(\lambda^{a_1}\lambda^{a_2}\bar{\lambda}^{b_1}\bar{\lambda}^{b_2} O_{\Delta b_1 b_2}(\lambda, \bar{\lambda}) - \lambda^{(a_1}\bar{\lambda}^{a_2)}\lambda^{b_1}\bar{\lambda}^{b_2} O_{\Delta b_1 b_2}(\lambda, \bar{\lambda})\Big). \tag{5.112}$$

The second term in the above equation is the zero helicity component of the current which we defined in (5.111). Combining this formula with the Witten transform (5.110) then results in the Penrose transform (5.108). The generalization to higher spin is similar. For a spin s symmetric traceless operator with scaling dimension $\Delta$, we find a Penrose transform involving $s + 1$ terms which include the 'helicities" $+s, s-1, s-2, \cdots 1, 0$. This is due to the fact that such an operator has $2s + 1$ different independent components (which is the also the same as the number of degrees of freedom of a massive spin-s gauge boson in four dimensions) and inside the Penrose transform, the components with equal and opposite "helicities" are equivalent similar to what we saw in going from (5.34) to (5.35) . It is much simpler to write this expression after contracting with arbitrary polarization spinors $\zeta_a$. The result is,

$$O_{\Delta,s}(x, \zeta) = \int \langle\lambda d\lambda\rangle \sum_{k=0}^{s} c_k (\zeta\cdot\lambda)^{2s-k} (\zeta\cdot\frac{\partial}{\partial\bar{\mu}})^k \hat{O}_{\Delta}^{s-k}(\lambda, \bar{\mu})|_X, \tag{5.113}$$

where $X$ denotes the incidence relation (5.2) and $c_k$ are coefficients that can easily be calculated like in (5.108) and performing the Witten transform (5.115) for each component.

[14]Speaking of which, the Witten transform for each component is given by,

$$\hat{O}_{\Delta}^{+k}(\lambda,\bar{\mu}) = \int \frac{d^2\bar{\lambda}}{(2\pi)^2} e^{i\bar{\lambda}\cdot\bar{\mu}} \frac{O_{\Delta}^{+k}(\lambda,\bar{\lambda})}{|\frac{\lambda\cdot\bar{\lambda}}{2}|^{s-1}}, \tag{5.115}$$

where $O^{+k}(\lambda,\bar{\lambda})$ contains $s+k$ positive helicity polarization spinors and $s-k$ negative helicity polarization spinors (5.24) contracted with the momentum space operator $O_{\Delta a_1\cdots a_{2s}}$.

We can obtain the two point functions for each component of the spin-s operator using dimensional analysis and invariance under projective rescalings. For example, the component $\hat{O}$ has the scaling dimension $\Delta-(s+1)$ and scales with a factor of $\frac{1}{r^{2(s-k)+2}}$.

$$\langle 0|\hat{O}_{\Delta}^{+k}(Z_1)\hat{O}_{\Delta}^{+l}(Z_2)|0\rangle = c_{kl}\delta^{kl}\frac{|\langle Z_1 I Z_2\rangle|^{2(\Delta-(s+1))}}{(Z_1\cdot Z_2)^{2(\Delta-k)}}. \tag{5.116}$$

$c_{kl}$ are coefficients that are all related and are multiples of the two point function coefficient $c_{O_{\Delta,s}}$. One can check that this formula leads to the correct non-conserved current two point function by performing the Penrose transform (5.108)[15]. Similarly, it would be interesting to solve for three point functions involving at least one non-conserved current which will greatly help in setting up the conformal bootstrap in twistor space. One way is to solve the integro-differential conformal Ward identities (which is equal to (5.89)+ a contribution that depends on the spin. Alternatively, using weight shifting and spin raising operators [36,239] also seems like an interesting and viable way to proceed. In contrast to conserved currents where twistor space makes conservation and conformal invariance manifest, the above discussion indicates that the analysis of non-conserved currents in twistor space would be quite challenging. However, there are some advantages such as the fact that the helicity basis still trades a symmetric traceless tensor $O^{\mu_1\cdots\mu_s}$ for $2s+1$ scalar-like objects whose correlators we can independently consider. As mentioned above, correlators involving one non-conserved operator are extremely important to compute since a four point function even of conserved currents in general receives contributions from the exchange of non-conserved operators. However perhaps, a hybrid twistor-position space framework where we express only conserved currents in twistor space and the remaining operators in position space like remarked in [236] might also be a way forward. Perhaps an approach similar to massive spinor helicity variables for four dimensional scattering amplitudes [240] could also be a way forward to obtaining the best description of generic operators in twistor space.

We hope to return to this in the future. We now proceed to switch gears, turn on supersymmetry, and head towards super-Twistor space.

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[14]Explicitly, we have the following formula for the coefficients in (5.113):

$$c_k = \begin{cases} (-1)^k 2\binom{2s}{k}, k \neq s \\ (-1)^s\binom{2s}{s}, k = s \end{cases} \tag{5.114}$$

[15]For instance, we explicitly matched the spin-1 case with the result in [47] by converting the result of the Penrose transform (5.108) to momentum space. We hope to return to a systematic analysis of this very important problem in the future.

## 5.7 The Geometry of Super-Twistor Space

In this section, we first introduce the basics of the super-twistor space developed in [241]. We then derive the corresponding super-incidence relations and the super-Penrose transform by using the super-field expansion and the Penrose transform (5.16) for each component current. We compare this with the Super-Witten transform and show that when combined with the ordinary Fourier transform and using the Faddeev-Popov method, the super-Penrose transform can also be derived. We then discuss the form of the super-conformal generators and derive the $\mathrm{OSp}(\mathcal{N}, 4)$ invariants that serve as building blocks for super-twistor space Wightman functions.

### 5.7.1 A Brief review of super-twistor space

Let us begin with a brief review of the super-twistor space constructed in [241] for theories $\mathcal{N}-$extended supersymmetry in $2+1$ dimensions. Super twistor space is an open subset of the projective space $\mathbb{RP}^{3|\mathcal{N}}$. It is spanned by bosonic coordinates $Z^A$ and fermionic coordinates $\psi^N$ where $A = 1, 2, 3, 4$ is the $Sp(4)$ fundamental index and $N = 1, \cdots, \mathcal{N}$ is the $O(\mathcal{N})$ fundamental index. Together, they form the super twistor space coordinates $\mathcal{Z}^{\mathcal{A}} = (Z^A, \psi^N) = (\lambda^a, \bar{\mu}_{a'}, \psi^N)$ which is in the fundamental representation of the super-conformal group $OSp(\mathcal{N}|4; \mathbb{R})$. The coordinates $Z^A$ are real and $\psi^N$ satisfy $(\psi^N)^* = i\psi^N$. Similar to the non-supersymmetric case, there exist dual twistor space coordinates $\mathcal{W}_{\mathcal{A}} = (W_A, \bar{\psi}_N) = (\mu_a, \bar{\lambda}^{a'}, \bar{\psi}_N)$ with $W^A$ being real and $(\bar{\psi}^N)^* = i\bar{\psi}^N$. Indices of super-twistors are raised and lowered using the $\mathrm{OSp}(\mathcal{N}|4; \mathbb{R})$ invariant graded symplectic form $\Omega_{\mathcal{A}\mathcal{B}}$. With these ingredients and the Penrose transform (5.16), we are now in a position to make a connection to the position super-space. We focus on the $\mathcal{N} = 1$ case, but the methods and results should be easily generalizable to extended supersymmetry.

### 5.7.2 The Super-Penrose Transform

A conserved super-current in position super-space takes the following form [242]:

$$\mathbf{J}_s^{a_1 \cdots a_{2s}}(x, \theta) = J_s^{a_1 \cdots a_{2s}}(x) + \theta_a J_{s+\frac{1}{2}}^{a a_1 \cdots a_{2s}}(x) - \frac{i\theta^2}{4} \partial_a^{a_1} J_s^{a_2 \cdots a_{2s} a}(x). \tag{5.117}$$

The statement of conservation reads,

$$D_{a_1} \mathbf{J}_s^{a_1 \cdots a_{2s}} = 0, \tag{5.118}$$

where the super-covariant derivative is given by,

$$D_a = \frac{\partial}{\partial \theta^a} - \frac{i}{2} \theta_b \partial_a^b. \tag{5.119}$$

Given the super-field expansion (5.117) and the fact that we know the Penrose transform of each component current appearing there (5.16), we can re-write (5.117) as,

$$\mathbf{J}_s^{a_1\cdots a_{2s}}(x,\theta) = \int \langle\lambda d\lambda\rangle\lambda^{a_1}\cdots\lambda^{a_{2s}}\left(\hat{J}_s^+(\lambda,\bar{\mu}) + \theta_a\lambda^a \hat{J}^+_{s+\frac{1}{2}}(\lambda,\bar{\mu})\right)\Big|_X - \frac{i\theta^2}{4}\partial_a^{a_1}\int \langle\lambda d\lambda\rangle\lambda^{a_2}\cdots\lambda^{a_{2s}}\lambda^a\hat{J}_s^+(\lambda,\bar{\mu})|_X$$
$$= \int \langle\lambda d\lambda\rangle\lambda^{a_1}\cdots\lambda^{a_{2s}}\left(\hat{J}_s^+(\lambda,\bar{\mu}) + \theta_a\lambda^a \hat{J}^+_{s+\frac{1}{2}}(\lambda,\bar{\mu})\right)\Big|_X - \frac{i\theta^2}{4}\int \langle\lambda d\lambda\rangle\lambda^{a_2}\cdots\lambda^{a_{2s}}\lambda^a\frac{\partial\bar{\mu}^b}{\partial x^a_{a_1}}\frac{\partial}{\partial\bar{\mu}^b}\hat{J}_s^+(\lambda,\bar{\mu})|_X$$
$$= \int \langle\lambda d\lambda\rangle\lambda^{a_1}\cdots\lambda^{a_{2s}}\left(\hat{J}_s^+(\lambda,\bar{\mu}) + \theta_a\lambda^a \hat{J}^+_{s+\frac{1}{2}}(\lambda,\bar{\mu})\right)\Big|_X - \frac{i\theta^2}{4}\int \langle\lambda d\lambda\rangle\lambda^{a_2}\cdots\lambda^{a_{2s}}\lambda^a(-2\lambda^{a_1}\delta_a^b + \lambda^b\delta_a^{a_1})\frac{\partial}{\partial\bar{\mu}^b}\hat{J}_s^+(\lambda,\bar{\mu})|_X$$
$$= \int \langle\lambda d\lambda\rangle\lambda^{a_1}\cdots\lambda^{a_{2s}}\left(\hat{J}_s^+(\lambda,\bar{\mu}) + \theta_a\lambda^a \hat{J}^+_{s+\frac{1}{2}}(\lambda,\bar{\mu}) + \frac{i\theta^2}{4}\lambda^a\frac{\partial}{\partial\bar{\mu}^a}\hat{J}_s^+(\lambda,\bar{\mu})\right)\Big|_X. \tag{5.120}$$

where $X$ as usual denote that the incidence relation (5.2) should be imposed. We now want to rewrite this covariantly as an integral over the super-twistor space currents $\hat{\mathbf{J}}_s^+(\mathcal{Z}) = \hat{\mathbf{J}}_s^+(\lambda,\bar{\mu},\psi)$ first introduced in [241]. First of all $\mathbf{J}_s^+(\mathcal{Z})$ similar to its non-supersymmetric counterpart (5.20) satisfies [241],

$$\hat{\mathbf{J}}_s^+(r\mathcal{Z}) = \frac{1}{r^{2s+2}}\hat{\mathbf{J}}_s^+(\mathcal{Z}). \tag{5.121}$$

This leads us to the following well defined projective integral involving this current:

$$\mathbf{J}_s^{a_1\cdots a_{2s}}(x,\theta) = \int \langle\lambda d\lambda\rangle\lambda^{a_1}\cdots\lambda^{a_{2s}}\hat{\mathbf{J}}_s^+(\lambda,\bar{\mu},\psi)|_{\mathcal{X}} = \int \langle\lambda d\lambda\rangle\lambda^{a_1}\cdots\lambda^{a_{2s}}\left(\hat{J}_s^+(\lambda,\bar{\mu}) + \frac{e^{\frac{i\pi}{4}}\psi}{\sqrt{2}}\hat{J}^+_{s+\frac{1}{2}}(\lambda,\bar{\mu})\right)|_{\mathcal{X}}. \tag{5.122}$$

We have also expanded the super-twistor space super-current into its two components in (5.122) using the formula in [241]. The subscript $\mathcal{X}$ indicates the superincidence relations that we need to derive. Let us write down an ansatz for the same using dimension analysis and the projectiveness:

$$\mathcal{X} = \left\{\bar{\mu}_a = -x_{ab}\lambda^b + \alpha\theta^2\lambda_a\,,\ \psi = \beta\theta^a\lambda_a\right\}. \tag{5.123}$$

Substituting (5.123) in (5.122) and performing the Grassmann expansion yields,

$$\mathbf{J}_s^{a_1\cdots a_{2s}}(x,\theta) = \int \langle\lambda d\lambda\rangle\lambda^{a_1}\cdots\lambda^{a_{2s}}\left(\hat{J}_s^+(\lambda,\bar{\mu}) - \frac{e^{\frac{i\pi}{4}}}{\sqrt{2}}\beta\theta_a\lambda^a\ \hat{J}^+_{s+\frac{1}{2}}(\lambda,\bar{\mu}) + \alpha\theta^2\lambda^a\frac{\partial}{\partial\bar{\mu}^a}\hat{J}_s^+(\lambda,\bar{\mu})\right)\Big|_X, \tag{5.124}$$

where $X$ now denotes the usual component level usual incidence relations (5.2). Comparing the component expansion (5.120) to our ansatz (5.124) yields the values,

$$\alpha = \frac{i}{4}, \beta = -\sqrt{2}e^{-\frac{i\pi}{4}}. \tag{5.125}$$

Putting everything together, we obtain the Super-Penrose transformation:

$$\mathbf{J}_s^{a_1\cdots a_{2s}}(x,\theta) = \int \langle\lambda d\lambda\rangle\lambda^{a_1}\cdots\lambda^{a_{2s}}\hat{\mathbf{J}}_s^+(\lambda,\bar{\mu},\psi)\Big|_{\mathcal{X}}, \tag{5.126}$$

where the super-incidence relations are given by,

$$\mathcal{X} = \left\{\bar{\mu}_a = -x_{ab}\lambda^b + \frac{i}{4}\theta^2\lambda_a \ , \ \psi = -\sqrt{2}e^{-\frac{i\pi}{4}}\theta^a\lambda_a\right\}. \tag{5.127}$$

As a consistency check, it is easy to see that (5.126) along with (5.127) satisfies the super-conservation (5.118).

Let us reflect on (5.127) briefly. Given a point in position superspace $(x_{ab}, \theta_c)$ we see that (5.127) defines a projective line $\mathbb{RP}^{1|1} \in \mathbb{RP}^{3|1}$. The generalization to extended supersymmetry should be straightforward. The only difference is that the Grassmann coordinate $\psi$ becomes a vector of $SO(\mathcal{N})$ which is the R-symmetry group and thus one needs to impose incidence relations for each component. Further, for $\mathcal{N} \geq 2$, one can form contractions in different ways using the invariants of the $R-$symmetry group. We leave such an exercise for the future.

### 5.7.3 Fourier+ Super-Witten transform $\Longrightarrow$ Super-Penrose transform

In subsection 5.7.2, we have derived the Super-Penrose transform using the component level Penrose tranform (5.16) and the super-twistor currents of [241]. Similar to the non-supersymmetric exercise, we shall now derive the Super-Penrose transform starting from the Fourier transform and using the super-Witten transform developed in [241] thus proving their equivalence. A word on notation before we proceed: Our conventions for the Fourier transform (5.29) differ from those in [241] by a factor of $-2$ in the exponent of the plane wave. This is done for convenience but will result in intermediate formulae including those involving Grassmann coordinates to differ compared to [241].

Our starting point is the ordinary Fourier transform which is simply obtained by replacing the current by the super-current in (5.29):

$$\mathbf{J}_s^{a_1\cdots a_{2s}}(x,\theta) = \int \frac{d^3p}{(2\pi)^3} e^{-2ip\cdot x}\mathbf{J}_s^{a_1\cdots a_{2s}}(p^\mu). \tag{5.128}$$

Performing exactly the same steps as in our non-supersymmetric analysis, we land up with the analog of the expression (5.35),

$$\mathbf{J}_s^{a_1\cdots a_{2s}}(x,\theta) = \frac{1}{2^s\text{Vol}(GL(1,\mathbb{R}))}\int \frac{d^2\lambda d^2\bar{\lambda}}{(2\pi)^3}\lambda^{a_1}\cdots\lambda^{a_{2s}}e^{i\bar{\lambda}_a\lambda_b x^{ab}}\hat{\mathbf{J}}_s^{+}(\lambda,\bar{\lambda},\theta). \tag{5.129}$$

The integrand of the above expression is only in spinor helicity variables. However, for the supersymmetric scenario, it is preferable to use super-spinor helicity variables [48] by expressing $\theta$ in the basis of $\lambda$ and $\bar{\lambda}$. We do this using the Faddeev-Popov method as follows:

**Claim:**

$$(\lambda\cdot\bar{\lambda})\int d\eta\, d\bar{\eta}\, \delta^2\left(\theta^a + \frac{\bar{\eta}\lambda^a + \eta\bar{\lambda}^a}{\lambda\cdot\bar{\lambda}}\right) = 1 \tag{5.130}$$

**Proof:**

$$(\lambda\cdot\bar\lambda)\int d\eta\, d\bar\eta\, \delta^2\left(\theta^a+\frac{\bar\eta\lambda^a+\eta\bar\lambda^a}{\lambda\cdot\bar\lambda}\right)=(\lambda\cdot\bar\lambda)\int d\eta\, d\bar\eta\left(\theta^1+\frac{\bar\eta\lambda^1+\eta\bar\lambda^1}{\lambda\cdot\bar\lambda}\right)\left(\theta^2+\frac{\bar\eta\lambda^2+\eta\bar\lambda^2}{\lambda\cdot\bar\lambda}\right)$$
$$=\frac{1}{\lambda\cdot\bar\lambda}\int d\eta\, d\bar\eta\, \bar\eta\eta\left(\lambda^1\bar\lambda^2-\bar\lambda^1\lambda^2\right)=\frac{1}{\lambda\cdot\bar\lambda}\left(\lambda^1\bar\lambda^2-\bar\lambda^1\lambda^2\right)\int d\eta\, d\bar\eta\, \bar\eta\eta$$
$$=\frac{1}{\lambda\cdot\bar\lambda}\left(\lambda^1\bar\lambda^2-\bar\lambda^1\lambda^2\right)\cdot 1=1 \qquad \square \tag{5.131}$$

Inserting (5.130) in (5.129) results in,

$$\mathbf{J}_s^{a_1\cdots a_{2s}}(x,\theta)=\frac{1}{2^{2s}\mathrm{Vol}(GL(1,\mathbb{R}))}\int\frac{d^2\lambda d^2\bar\lambda}{(2\pi)^3}e^{i\bar\lambda_a\lambda_b x^{ab}}\lambda^{a_1}\cdots\lambda^{a_{2s}}(\lambda\cdot\bar\lambda)$$
$$\times\int d\eta d\bar\eta\,\delta^2\left(\theta^a+\frac{\bar\eta\lambda^a+\eta\bar\lambda^a}{\lambda\cdot\bar\lambda}\right)\hat{\mathbf{J}}_s^+(\lambda,\bar\lambda,\eta,\bar\eta), \tag{5.132}$$

where we replaced $\hat{\mathbf{J}}_s^+(\lambda,\bar\lambda,\theta)$ by $\hat{\mathbf{J}}_s^+(\lambda,\bar\lambda,\eta,\bar\eta)$ on the support of the delta function we have inserted. We now express the latter by writing it as a half-Fourier transform of its super-twistor space counterpart [241]:

The relation between the current in super-spinor helicity variables and super-twistor space is as follows:

$$\hat{\mathbf{J}}_s^+(\lambda,\bar\lambda,\eta,\bar\eta)=2\int d\chi e^{-\frac{\chi\bar\eta}{2}}\hat{\mathbf{J}}_s^+(\lambda,\bar\lambda,\eta,\chi)=2\int d\chi e^{-\frac{\chi\bar\eta}{2}}(\chi-\eta)\hat{\mathbf{J}}_s^+(\lambda,\bar\lambda,\chi+\eta)$$
$$=2\int d^2\bar\mu\, e^{-i\bar\lambda\cdot\bar\mu}\int d\chi e^{-\frac{\chi\bar\eta}{2}}(\chi-\eta)\hat{\mathbf{J}}_s^+(\lambda,\bar\mu,\chi+\eta). \tag{5.133}$$

We now substitute (5.133) back into (5.132) and change variables from $(\eta,\chi)\to(\xi_+,\xi_-)=(\chi+\eta,\chi-\eta)$[16].This yields,

$$\mathbf{J}_s^{a_1\cdots a_{2s}}(x,\theta)=-\frac{1}{2^{2s}\mathrm{Vol}(GL(1,\mathbb{R}))}\int\frac{d^2\lambda d^2\bar\lambda}{(2\pi)^3}e^{-i\bar\lambda_a(\bar\mu^a-\lambda_b x^{ab})}\lambda^{a_1}\cdots\lambda^{a_{2s}}(\lambda\cdot\bar\lambda)$$
$$\times\int d\xi_+d\xi_-d\bar\eta\delta^2\left(\theta^a+\frac{\bar\eta\lambda^a+(\frac{\xi_+-\xi_-}{2})\bar\lambda^a}{\lambda\cdot\bar\lambda}\right)\xi_-e^{-\frac{(\xi_++\xi_-)\bar\eta}{4}}\hat{\mathbf{J}}_s^+(\lambda,\bar\mu,\xi_+)$$
$$=\frac{1}{2^{2s}\mathrm{Vol}(GL(1,\mathbb{R})}\int\frac{d^2\lambda d^2\bar\mu d\xi_+}{(2\pi)^3}\lambda^{a_1}\cdots\lambda^{a_{2s}}\mathbf{J}_s^+(\lambda,\bar\mu,\xi_+)\mathcal{I}, \tag{5.134}$$

where,

$$\mathcal{I}=\int d^2\bar\lambda(\lambda\cdot\bar\lambda)d\bar\eta\delta^2\left(\theta^a+\frac{\bar\eta\lambda^a+\frac{\xi_+}{2}\bar\lambda^a}{\lambda\cdot\bar\lambda}\right)e^{-\frac{\xi_+\bar\eta}{4}}e^{-i\bar\lambda_a(\bar\mu^a-x^{ab}\lambda_b)}. \tag{5.135}$$

Let us now express the Grassmannian delta function in the basis of $\lambda$ and $\bar\lambda$. This yields,

$$\delta^2\left(\theta^a+\frac{\bar\eta\lambda^a+\frac{\xi_+}{2}\bar\lambda^a}{\lambda\cdot\bar\lambda}\right)=\frac{1}{(\lambda\cdot\bar\lambda)}\delta(\bar\eta-(\bar\lambda\cdot\theta))\delta(\xi_++2(\lambda\cdot\theta)). \tag{5.136}$$

[16]We will also later make a simple variable change from $\xi_+$ to $\psi$ where $\psi\propto\xi_+$ which is the variable we prefer to work with for the most part.

Using (5.136) in (5.134) and performing the $\bar{\eta}$ integral results in,

$$\mathcal{I} = \delta(\xi_+ + 2(\lambda\cdot\theta)) \int d^2\bar{\lambda} e^{\frac{(\lambda\cdot\theta)(\bar{\lambda}\cdot\theta)}{2}} e^{-i\bar{\lambda}_a(\bar{\mu}^a - x^{ab}\lambda_b)} = \delta(\xi_+ + 2(\lambda\cdot\theta))\delta^2(\bar{\mu}^a - x^{ab}\lambda_b - \frac{i}{4}\theta^2\lambda^a). \tag{5.137}$$

Plugging this back in (5.134), changing variables from $\xi_+ \to \psi = -\frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}\xi_+$ and performing the integrals over $\bar{\mu}$ and $\psi$ and using the projective integral formula (5.41) results in,

$$\mathbf{J}_s^{a_1\cdots a_{2s}}(x,\theta) = \int \langle\lambda d\lambda\rangle\lambda^{a_1}\cdots\lambda^{a_{2s}}\hat{\mathbf{J}}_s^+(\lambda,\bar{\mu},\psi)|_{\mathcal{X}}, \tag{5.138}$$

where $\mathcal{X}$ are precisely the super-incidence relations we derived earlier (5.127). Therefore, (5.138) is indeed the super-Penrose transform. To summarize, we have shown that the ordinary Fourier transform that takes us from position space to momentum space (5.128) coupled with the supersymmetric Witten transform (5.133) yields the super-Penrose transform (5.126)!

### 5.7.4 The super-conformal generators and invariants

So far we have discussed the super-Penrose and super-Witten transforms that respectively connect position super-space and super-spinor helicity variables to super-twistor space and discussed the relation between them. We now proceed to setup the super-conformal Ward identities in super-twistor space that we shall use to solve for Wightman functions.

Our supertwistor space carries a natural action of the super conformal group $\mathrm{OSp}(\mathcal{N}|4;\mathbb{R})$ with generators acting on super-currents as follows [241],

$$[\mathcal{T}^{\mathcal{AB}}, \hat{\mathbf{J}}_s^{\pm}(\mathcal{Z})] = \mathcal{Z}^{(\mathcal{A}}\frac{\partial}{\partial\mathcal{Z}_{\mathcal{B})}}\hat{\mathbf{J}}_s^{\pm}(\mathcal{Z}), \quad [\mathcal{T}^{\mathcal{AB}}, \hat{\mathbf{J}}_s^{\pm}(\mathcal{W})] = \mathcal{W}^{(\mathcal{A}}\frac{\partial}{\partial\mathcal{W}_{\mathcal{B})}}\hat{\mathbf{J}}_s^{\pm}(\mathcal{W}), \tag{5.139}$$

where the indices are symmetrized taking grading into account, please see [241] for more details. The $n-$ point super-conformal Ward identities take the form,

$$\sum_{i=1}^{n}\langle 0|\cdots[\mathcal{T}^{\mathcal{AB}}, \hat{\mathbf{J}}_{\mathbf{s_i}}^{\pm}]\cdots|0\rangle = 0. \tag{5.140}$$

Another important operator that we must consider is the super helicity operator. This operator ensures the correct behaviour of the super-currents under projective rescaling in the super-Penrose transform (5.126). For a $n-$point super-Wightman function, we require that for each current,

$$\begin{aligned}\mathcal{Z}_{i\mathcal{A}}\frac{\partial}{\partial\mathcal{Z}_{i\mathcal{A}}}\langle 0|\cdots\hat{\mathbf{J}}_{\mathbf{s_i}}^{+}(\mathcal{Z}_i)\cdots|0\rangle &= -2(s_i+1)\langle 0|\cdots\hat{\mathbf{J}}_{\mathbf{s_i}}^{+}(\mathcal{Z}_i)\cdots|0\rangle,\\ \mathcal{Z}_{i\mathcal{A}}\frac{\partial}{\partial\mathcal{Z}_{i\mathcal{A}}}\langle 0|\cdots\hat{\mathbf{J}}_{\mathbf{s_i}}^{-}(\mathcal{Z}_i)\cdots|0\rangle &= -2(-s_i+1-\frac{\mathcal{N}}{2})\langle 0|\cdots\hat{\mathbf{J}}_{\mathbf{s_i}}^{+}(\mathcal{Z}_i)\cdots|0\rangle,\end{aligned} \tag{5.141}$$

with similar formulae for the super-dual twistor space currents that can be obtained via the super-twistor Fourier transform that connects them viz,

$$\hat{\mathbf{J}}_s^{\pm}(\mathcal{W}) = \int \frac{d^{4|\mathcal{N}}\mathcal{Z}}{(2\pi)^{2+\frac{\mathcal{N}}{2}}} e^{i\mathcal{Z}\cdot\mathcal{W}} \hat{\mathbf{J}}_s^{\pm}(\mathcal{Z}). \tag{5.142}$$

With the aid of the above equations, one can fully fix the full functional form of two and three point correlators as was done in [241]. Given a number of Dual/twistors($\mathcal{W}/\mathcal{Z}$), natural invariants one can form are the super-symplectic dot products of $\mathcal{Z}/\mathcal{W}$ that lead to the set,

$$\{\mathcal{Z}_i \cdot \mathcal{Z}_j = -\mathcal{Z}_i^{\mathcal{A}} \Omega_{\mathcal{A}\mathcal{B}} \mathcal{Z}_j^{\mathcal{B}}, \mathcal{W}_i \cdot \mathcal{Z}_j = \mathcal{W}_{i\mathcal{A}} \mathcal{Z}_j^{\mathcal{A}}, \mathcal{W}_i \cdot \mathcal{W}_j = \mathcal{W}_{i\mathcal{A}} \Omega^{\mathcal{A}\mathcal{B}} \mathcal{W}_{j\mathcal{B}}\} \tag{5.143}$$

where i and j are indices labelling the (dual)super-twistors.

However, apart from these invariants, there is another invariant of $OSp(\mathcal{N}|4;\mathbb{R})$ which is the super-symmetrization of the projective delta function analyzed in the non-supersymmetric case in (5.50). As an illustrative example, let us obtain this additional solution by considering three twistors $\mathcal{Z}_1, \mathcal{Z}_2$ and $\mathcal{Z}_3$. Then,

$$\delta^{4|\mathcal{N}}(\mathcal{Z}_3 + c_{23}\mathcal{Z}_1 + c_{31}\mathcal{Z}_2) = \delta^4(Z_3 + c_{23}Z_1 + c_{31}Z_2)\delta^N(\psi_3 + c_{23}\psi_1 + c_{31}\psi_2) \tag{5.144}$$

where $c_{23}, c_{31}$ are arbitrary real parameters[17], is clearly also an invariant of the super-group.

In order to remove the dependence on these $c_{ij}$ parameters, we integrate it with weights $c_{ij}^{\alpha_{ij}}$ to obtain,

$$\delta^{2|\mathcal{N}}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3; \alpha_{23}, \alpha_{31}) \equiv \int dc_{23} dc_{31} c_{23}^{\alpha_{23}} c_{31}^{\alpha_{31}} \delta^{4|\mathcal{N}}(\mathcal{Z}_3 + c_{23}\mathcal{Z}_1 + c_{31}\mathcal{Z}_2), \tag{5.145}$$

where $\alpha_{ij}$ are arbitrary parameters that will be fixed by demanding the correct projective rescaling (i.e. superhelicity) (5.141). Similar to the non-supersymmetric case (5.54), one cannot obtain a function with nice homogeneity with respect to each twistor just by using $\delta^{2|\mathcal{N}}$. We need to multiply this by another OSp($4|\mathcal{N};\mathbb{R}$) invariant viz a dot product of super-twistors. Thus, we define[18],

$$\begin{aligned}
&\delta^{3|\mathcal{N}}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3; \alpha_{12}, \alpha_{23}, \alpha_{31}) \\
&= (-i)^{\alpha} \delta^{[\alpha_{12}]}(\mathcal{Z}_1 \cdot \mathcal{Z}_2) \int dc_{23} dc_{31} c_{23}^{\alpha_{23}} c_{31}^{\alpha_{31}} \delta^{4|\mathcal{N}}(\mathcal{Z}_3 + c_{23}\mathcal{Z}_1 + c_{31}\mathcal{Z}_2),
\end{aligned} \tag{5.146}$$

The super-helicity equation (5.141) can then be used to fix the values of $\alpha_{12}, \alpha_{23}$ and $\alpha_{31}$ in a super-Wightman function.

[17] We could have started with three parameters and with the weights integrated over, we can show that one can reduce down to two parameters with a volume factor similar to the non-supersymmetric case.

[18] Using one of the products $\mathcal{Z}_i \cdot \mathcal{Z}_j$ is sufficient at the support of the $\delta^{4|\mathcal{N}}$.

We can also generalize the above invariant to $n-$ points via,

$$\mathcal{F}(\mathcal{Z}_1,\cdots\mathcal{Z}_n)=\int dc_1\cdots dc_{n-1}\, f(c_1,\cdots,c_{n-1})\delta^{4|\mathcal{N}}(c_1\mathcal{Z}_1+\cdots+\mathcal{Z}_n). \quad (5.147)$$

To erase arbitrariness in the parameters $c_i$, we have integrated them on the support of a function $f(c_1,c_2,\cdots c_{n-1})$. For the projectiveness of this quantity and to be well defined in the super-Penrose transform, this function will be constrained by the super-helicity identity at each point viz (5.141).

Now, armed with these invariants, we shall discuss the construction of super-correlators in the next section.

## 5.8 Supersymmetric Wightman functions in Twistor space

We first discuss Wightman functions of super-currents and how the projective super-delta function (5.146) figures in the results. Just as in the non-supersymmetric case, where the $\delta^3$ solutions are necessary, we shall find that that their supersymmetric counterparts that we have constructed (5.146) play an analogous role. We then extend the formalism to include super-scalars. Further, we shall see the requirement of introducing one more Grassmann coordinate for super-correlators involving scalars. However, we shall see that the super-Penrose transform for super-scalars instructs us to integrate out this extra parameter, resulting in drastic simplifications. We restrict our attention to $\mathcal{N}=1$ theories although it should not be too difficult to generalize to arbitrary $\mathcal{N}$.

### 5.8.1 Two and three point functions involving only conserved currents

Starting with two point functions of super-currents we have the unique results [241],

$$\langle 0|\hat{\mathbf{J}}^+_{\mathbf{s}}(\mathcal{Z}_1)\hat{\mathbf{J}}^+_{\mathbf{s}}(\mathcal{Z}_2)|0\rangle=\frac{1}{(\mathcal{Z}_1\cdot\mathcal{Z}_2)^{2s+2}},\quad \langle 0|\hat{\mathbf{J}}^-_{\mathbf{s}}(\mathcal{Z}_1)\hat{\mathbf{J}}^-_{\mathbf{s}}(\mathcal{Z}_2)|0\rangle=\frac{1}{(\mathcal{Z}_1\cdot\mathcal{Z}_2)^{2(-s+\frac{1}{2})+2}}. \quad (5.148)$$

At the level of three points, the authors of [241] show that one obtains *homogeneous* super-correlators for net-integer spin and *non-homogeneous* super-correlators for net half-integer spin. Lets consider three integer spin supercurrents with all + helicity. Then, the correlator made out of super symplectic twistor dot products (5.143) is given by [241],

$$\langle 0|\hat{\mathbf{J}}^+_{\mathbf{s_1}}(\mathcal{Z}_1)\hat{\mathbf{J}}^+_{\mathbf{s_2}}(\mathcal{Z}_2)\hat{\mathbf{J}}^+_{\mathbf{s_3}}(\mathcal{Z}_3)|0\rangle_h=(-i)^{s_T}\delta^{[s_1+s_2-s_3]}(\mathcal{Z}_1\cdot\mathcal{Z}_2)\delta^{[s_2+s_3-s_1]}(\mathcal{Z}_2\cdot\mathcal{Z}_3)\delta^{[s_3+s_1-s_2]}(\mathcal{Z}_3\cdot\mathcal{Z}_1). \quad (5.149)$$

For three half-integer super-currents, the natural answer is given in dual super twistor variables. We have,

$$\langle 0|\hat{\mathbf{J}}^+_{\mathbf{s_1}}(\mathcal{W}_1)\hat{\mathbf{J}}^+_{\mathbf{s_2}}(\mathcal{W}_2)\hat{\mathbf{J}}^+_{\mathbf{s_3}}(\mathcal{W}_3)|0\rangle_{nh}=(-i)^{s_T}\delta^{[s_1+s_2-s_3+\frac{\mathcal{N}}{2}]}(\mathcal{W}_1\cdot\mathcal{W}_2)\delta^{[s_2+s_3-s_1+\frac{\mathcal{N}}{2}]}(\mathcal{W}_2\cdot\mathcal{W}_3)\delta^{[s_3+s_1-s_2+\frac{\mathcal{N}}{2}]}(\mathcal{W}_3\cdot\mathcal{W}_1). \quad (5.150)$$

However, it is desirable to also have the expression for this correlator in super-twistor variables. To this end, one can use (5.146) to construct this three point function. The three point function in terms of the projective super-delta function [19] is given by,

$$\langle 0|\hat{\mathbf{J}}^+_{\mathbf{s_1}}(\mathcal{Z}_1)\hat{\mathbf{J}}^+_{\mathbf{s_2}}(\mathcal{Z}_2)\hat{\mathbf{J}}^+_{\mathbf{s_3}}(\mathcal{Z}_3)|0\rangle_{nh}$$
$$= (-i)^{s_T}\delta^{[s_1+s_2+s_3]}(\mathcal{Z}_1\cdot\mathcal{Z}_2)\int dc_{23}dc_{31}c_{23}^{-s_2-s_3+s_1}c_{31}^{-s_3-s_1+s_2}\delta^{4|\mathcal{N}}(\mathcal{Z}_3^{\mathcal{A}}+c_{31}\mathcal{Z}_2^{\mathcal{A}}+c_{23}\mathcal{Z}_1^{\mathcal{A}}) \tag{5.151}$$

Therefore, (5.149) and (5.151) are the homogeneous and non-homogeneous solution respectively valid for net-integer and net-half integer super-current three point correlators. Thus, similar to the non-supersymmetric case of section 5.5, we conclude from here that if we want to express the supercorrelator using only $\mathcal{Z}$ variables, one has to use $\delta^{3|\mathcal{N}}$ invariants.

### 5.8.2 Parity odd super-conformal Wightman functions

Consider the supersymmetric generalization of the epsilon transform (5.97)[20]. For super-currents, the epsilon transform is a simple generalization of the same and reads,

$$(\epsilon\cdot\mathbf{J}_s(x,\theta))^{a_1\cdots a_{2s}} = -i\int\frac{d^3y}{|y-x|^2}\frac{\partial}{\partial y^b_{(a_1}}\mathbf{J}_s^{a_2\cdots a_{2s})b}(y,\theta) \tag{5.152}$$

The super-twistor space epsilon transform can then therefore be derived from (5.152) and the super-Penrose transform (5.126).

The Super-epsilon transform in twistor space for a spin-s current is given by,

$$(\epsilon\cdot\hat{\mathbf{J}}_s)^+(\mathcal{Z}) = -i\int\frac{d^3y}{y^2}\mathcal{Z}^A\mathcal{I}_{AB}\frac{\partial}{\partial\mathcal{Z}_B}\hat{\mathbf{J}}_s^+(\lambda,\bar{\mu},\psi)\Big|_{\bar{\mu}^a\to\bar{\mu}^a+y^{ab}\lambda_b}, \tag{5.153}$$

where $\mathcal{I}_{AB}$ is the super-infinity twistor (5.155) to be defined shortly . The parity odd super-Penrose transform is thus given by,

$$(\epsilon\cdot\mathbf{J}_s)^{a_1\cdots a_{2s}}(x,\theta) = \int\langle\lambda d\lambda\rangle\lambda^{a_1}\cdots\lambda^{a_{2s}}(\epsilon\cdot\hat{\mathbf{J}}_s)^+(\mathcal{Z})|_{\mathcal{X}}, \tag{5.154}$$

where $\mathcal{X}$ denote the super-incidence relations (5.127).

We have also defined the super-infinity twistor that is equal to,

$$\mathcal{I}_{\mathcal{AB}} = I_{AB}\oplus 0_{\mathcal{N}\times\mathcal{N}}. \tag{5.155}$$

Similarly, the parity odd super-Witten transform (5.102) is the exact analog of its non-supersymmetric counterpart (5.102) obtained using (5.133) and the momentum space version of the super-epsilon transform (5.152).

[19]One can also obtain this solution using the super Fourier transform (5.142) of (5.150).

[20]The result in (5.97)is given for spin-1 but the result is a simple generalization to arbitrary integer spin. One just needs to add extra indices to the formula and symmetrize them.

$$i\mathrm{Sign}(\sqrt{p^2})(\epsilon\cdot\hat{\mathbf{J}}_s)^+(\lambda,\bar{\lambda},\eta,\bar{\eta}) = 2\int d^2\bar{\mu}\, e^{-i\bar{\lambda}\cdot\bar{\mu}}\int d\chi e^{-\frac{\chi\bar{\eta}}{2}}(\chi-\eta)\hat{\mathbf{J}}_s^+(\lambda,\bar{\mu},\chi+\eta), \tag{5.156}$$

where the relation between the Grassmann variables $\eta,\chi$ to the ones we use more frequently is given below (5.133).

Using (5.153) and the two point even expression (5.148) we obtain,

$$\langle 0|\hat{\mathbf{J}}^+_{s_1}(\mathcal{Z}_1)\hat{\mathbf{J}}^+_{s_2}(\mathcal{Z}_2)|0\rangle = \mathrm{Sgn}(\mathcal{Z}_1\mathcal{I}\mathcal{Z}_2)\delta^{[2s+1]}(\mathcal{Z}_1\cdot\mathcal{Z}_2). \tag{5.157}$$

Similarly, using the expression of the parity even super three point function (5.149) and the super-epsilon transform (5.153) results in,

$$\begin{aligned}&\langle 0|\hat{\mathbf{J}}^+_{s_1}(\mathcal{Z}_1)\hat{\mathbf{J}}^+_{s_2}(\mathcal{Z}_2)\hat{\mathbf{J}}^+_{s_3}(\mathcal{Z}_3)|0\rangle\\ &= \frac{i}{2}\int dc_{12}dc_{23}dc_{31}c_{12}^{s_1+s_2-s_3}c_{23}^{s_2+s_3-s_1}c_{31}^{s_3+s_1-s_2}\mathrm{Sgn}\big(c_{12}\langle\mathcal{Z}_1\mathcal{I}\mathcal{Z}_2\rangle + c_{31}\langle\mathcal{Z}_3\mathcal{I}\mathcal{Z}_2\rangle\big)e^{-ic_{12}\mathcal{Z}_1\cdot\mathcal{Z}_2-ic_{23}\mathcal{Z}_2\cdot\mathcal{Z}_3-ic_{31}\mathcal{Z}_3\cdot\mathcal{Z}_1}\Big|_{\chi}.\end{aligned} \tag{5.158}$$

Very nicely, we see that these results are obtained from something as simple as replacing twistors by super-twistors in the non-supersymmetric results (5.103), (5.105). Moreover, we bring attention to the fact that these expressions also involve the super-infinity twistor (5.155). One can check that these expressions are solved by the super-conformal generators (5.139) as a distribution and moreover give rise to the correct position space and spinor helicity results.

### 5.8.3 Extensions to super scalars

The subject of the final part of this section shall be super-scalar operators with scaling dimension one. In position space, the scalar super-field is given by [242],

$$\mathbf{J}_0(x,\theta) = O_1(x) + \theta_a O^a_{1/2}(x) + \theta^2 O_2(x). \tag{5.159}$$

We have seen previously that super-twistor space for conserved super-currents was spanned by a single super-twistor coordinates $\mathcal{Z}^{\mathcal{A}} = (Z^A,\psi)$. However, for scalars, an additional Grassmann coordinate called $\psi_-$ is required to describe them. This can be understood from the structure of the scalar multiplet (which one can obtain by a Witten transform of the expression in [48] and some simple variable changes):

$$\hat{\mathbf{J}}_0(\mathcal{Z},\psi_-) = \frac{e^{\frac{i\pi}{4}}}{2\sqrt{2}}(\psi+\psi_-)\hat{O}_1(Z) + \frac{1}{2}\hat{O}^-_{1/2}(Z) + \frac{i}{2}\psi_-\psi\hat{O}^+_{1/2}(Z) - \sqrt{2}e^{\frac{i\pi}{4}}(\psi-\psi_-)\hat{O}_2(Z). \tag{5.160}$$

For conserved super-currents, the Penrose transform was given by (5.126). For the super-scalar (5.159), we propose that it is related to (5.160) as follows:

$$\mathbf{J}_0(x,\theta) = \int\langle\lambda d\lambda\rangle\int e^{-\frac{i\pi}{4}}d\psi_-\,\hat{\mathbf{J}}_0(\mathcal{Z},\psi_-)|_{\mathcal{X}}, \tag{5.161}$$

where $\mathcal{X}$ are the super-incidence relations (5.127). A few words are in order. From (5.160), we can see that $\psi, \psi_-$ both have helicity $-\frac{1}{2}$. Therefore, $\hat{\mathbf{J}}_0(\mathcal{Z})$ behaves like a $s = -\frac{1}{2}$ current (and not like a scalar with $s = 0$!) and satisfies,

$$\hat{\mathbf{J}}_0(r\mathcal{Z}, r\psi_-) = \frac{1}{r}\hat{\mathbf{J}}_0(\mathcal{Z}, \psi_-). \tag{5.162}$$

Then under $\lambda \to r\lambda, \psi_- \to r\psi_-$, (5.161), the measure transforms as $\langle\lambda d\lambda\rangle d\psi_- \to r^2\langle\lambda d\lambda\rangle d(r\psi_-) = r^2\langle\lambda d\lambda\rangle\frac{1}{r}d(\psi_-) = r\langle\lambda d\lambda\rangle d(\psi_-)$. This is precisely canceled out by the transformation of $J_0$ (5.162) and thus the integral is invariant under projective rescalings.

Another point is that the super-field (5.159) has a scaling dimension equal to one, which implies that the super-twistor space scalar is dimensionless. Note that although the super-twistor component expansion (5.160) contains both positive and negative helicity components of the spin-1/2 operator, in the context of the super-Penrose transform (5.161) only the positive helicity operator contributes. This is yet again a manifestation of the fact that equal in magnitude and opposite in sign helicities in a Penrose transform contribute equally like we discussed in the paragraph above (5.113). Next, the super-Witten transform for scalars is identical to the spinning case (5.133) with $s = 0$. We have,

$$\begin{aligned}\hat{\mathbf{J}}_0(\lambda, \bar{\lambda}, \eta, \bar{\eta}) &= 2\int d\chi e^{-\frac{\chi\bar{\eta}}{2}}\hat{\mathbf{J}}_0(\lambda, \bar{\lambda}, \eta, \chi) = 2\int d\chi e^{-\frac{\chi\bar{\eta}}{2}}\hat{\mathbf{J}}_0^+(\lambda, \bar{\lambda}, \chi+\eta, \chi-\eta)\\ &= 2\int d^2\bar{\mu}\, e^{-i\bar{\lambda}\cdot\bar{\mu}}\int d\chi e^{-\frac{\chi\bar{\eta}}{2}}\hat{\mathbf{J}}_0^+(\lambda, \bar{\mu}, \chi+\eta, \chi-\eta).\end{aligned} \tag{5.163}$$

The main difference from the spinning case (5.133) is that the scalar super-field (5.160) depends on the difference $\chi - \eta \sim \psi_-$ in a non-trivial way unlike the spinning case (5.133) where it appears as an overall multiplicative quantity.

### Two point functions

Lets start with two point functions. An ansatz consistent with projective rescaling and dimensional analysis is given by,

$$\langle 0|\hat{\mathbf{J}}_0(\mathcal{Z}_1, \psi_{1-})\hat{\mathbf{J}}_0(\mathcal{Z}_2, \psi_{2-}|0\rangle = \frac{c_{\frac{1}{2}}}{(\mathcal{Z}_1\cdot\mathcal{Z}_2)} + \frac{c_{\Delta=1}\psi_{1-}\psi_{2-}}{(\mathcal{Z}_1\cdot\mathcal{Z}_2)^2}. \tag{5.164}$$

However, the supersymmetric Penrose transform (5.161) for scalars instructs us to integrate over the $\psi_-$ coordinates thus yielding the simpler result,

$$\begin{aligned}\langle 0|\mathbf{J}_0(x_1, \theta_1)\mathbf{J}_0(x_2, \theta_2)|0\rangle &= \int\langle\lambda_1 d\lambda_1\rangle\langle\lambda_2 d\lambda_2\rangle\int d\psi_{1-}d\psi_{2-}\left(\frac{c_{\frac{1}{2}}}{(\mathcal{Z}_1\cdot\mathcal{Z}_2)} + \frac{c_{\Delta=1}\psi_{1-}\psi_{2-}}{(\mathcal{Z}_1\cdot\mathcal{Z}_2)^2}\right)\\ &= \int\langle\lambda_1 d\lambda_1\rangle\langle\lambda_2 d\lambda_2\rangle\frac{c_{\Delta=1}}{(\mathcal{Z}_1\cdot\mathcal{Z}_2)^2}.\end{aligned} \tag{5.165}$$

which is simply obtained from the spinning result (5.148) by setting $s = 0$. Using the super incidence relations (5.127) and performing a Grassmann expansion results in,

$$\begin{aligned}&\langle 0|\mathbf{J}_0(x_1,\theta_1)\mathbf{J}_0(x_2,\theta_2)|0\rangle\\&=\int\langle\lambda_1 d\lambda_1\rangle\langle\lambda_2 d\lambda_2\rangle\frac{i^{2s+2}}{(Z_1\cdot Z_2)_0^2}-4i\theta_{1a}\theta_{2b}\int\langle\lambda_1 d\lambda_1\rangle\langle\lambda_2 d\lambda_2\rangle\frac{i^2\lambda_1^a\lambda_2^b}{(Z_1\cdot Z_2)_0^3}\\&-\frac{i}{2}(\theta_1^2+\theta_2^2)\int\langle\lambda_1 d\lambda_1\rangle\langle\lambda_2 d\lambda_2\rangle\frac{\langle 12\rangle}{(Z_1\cdot Z_2)_0^3}-\frac{3}{8}\theta_1^2\theta_2^2\int\langle\lambda_1 d\lambda_1\rangle\langle\lambda_2 d\lambda_2\rangle\frac{\langle 12\rangle^2}{(Z_1\cdot Z_2)_0^4}.\end{aligned}\tag{5.166}$$

The first integral is the $\langle 0|O_1(x_1)O_1(x_2)|0\rangle$, the second one is $\langle O^a_{1/2}(x_1)O^b_{1/2}(x_2)\rangle$ whereas the fourth integral is $\langle 0|O_2O_2|0\rangle$. Remarkably, the Supersymmetric Penrose transform has reproduced for us twistor space expression for $\langle O_2(x_1)O_2(x_2)\rangle$ which we earlier had to obtain via a Legendre transform (5.82). Thus, we see that the infinity twistor is naturally accommodated by the super-incidence relations (5.127). The third integral in (5.166) is zero since it is odd under the exchange of $\lambda_1$ and $\lambda_2$. Thus, we have obtained a result that is perfectly consistent with the superfield expansion (5.159). However, we are neglecting the potential contact term contributions that arise from $\langle 0|O_1O_2|0\rangle$ as well as the parity odd contribution to $\langle 0|O_{1/2}O_{1/2}|0\rangle$. Let us now show how to obtain the supersymmetric version of the same.

### Contact terms

It is of interest to study and classify superconformal contact terms as pointed out in [17]. We already analyzed non-supersymmetric two point contact terms that occur in the $\langle O_2O_1\rangle$ two point function (5.80) and the parity odd two point functions (5.103). Given the fact that the scalar multiplet (5.160) contains $O_1, O_2$ as well as a spin-1/2 operator, let us see if supersymmetry allows for these contact term contributions. An ansatz that we can write down taking inspiration from (5.80) and (5.103) is as follows:

$$\langle 0|\hat{\mathbf{J}}_0(\mathcal{Z}_1,\psi_{1-})\hat{\mathbf{J}}_0(\mathcal{Z}_2,\psi_{2-}|0\rangle=\psi_{1-}\psi_{2-}\mathrm{Sgn}(\langle\mathcal{Z}_1\mathcal{I}\mathcal{Z}_2\rangle)\delta^{[1]}(\mathcal{Z}_1\cdot\mathcal{Z}_2).\tag{5.167}$$

The above ansatz has correct projective properties (5.162) and dimensionality. Plugging this into the super-scalar super-Penrose transform (5.161) results in,

$$\langle 0|\mathbf{J}_0(x_1,\theta_1)\mathbf{J}_0(x_2,\theta_2)|0\rangle_{\text{contact}}=\int\langle\lambda_1 d\lambda_1\rangle\langle\lambda_2 d\lambda_2\rangle\mathrm{Sgn}(\langle\mathcal{Z}_1\mathcal{I}\mathcal{Z}_2\rangle)\delta^{[1]}(\mathcal{Z}_1\cdot\mathcal{Z}_2)|_{\mathcal{X}}.\tag{5.168}$$

Using the super-incidence relations (5.127), expanding (5.168) into components like in (5.166) and matching with the position space super-field expansion obtained using (5.159) shows that the correct two point contact terms (5.80) and (5.103) are reproduced correctly. In fact, they combine very nicely thanks to supersymmetry to become the following supersymmetric contact term,

$$\langle 0|\mathbf{J}_0(x_1,\theta_1)\mathbf{J}_0(x_2,\theta_2)|0\rangle_{\text{contact}}=\delta^2(\theta_1-\theta_2)\delta^3(x_1-x_2).\tag{5.169}$$

We leave to the future, a more complete and systematic analysis of super-symmetric contact terms and their applications.

#### Three point functions

Let us now move onto three point functions involving scalar super-fields. First, consider a supercorrelator consisting of one scalar superfield and two conserved currents with positive helicity: The most general ansatz consistent with (5.126) and (5.161) is,

$$\langle 0|\hat{\mathbf{J}}^{+}_{s_1}(\mathcal{Z}_1)\hat{\mathbf{J}}^{+}_{s_2}(\mathcal{Z}_2)\hat{\mathbf{J}}_0(\mathcal{Z}_3,\psi_{3-})|0\rangle = \alpha_1\psi_{3-}\delta^{[s_1+s_2]}(\mathcal{Z}_1\cdot\mathcal{Z}_2)\delta^{[-s_1+s_2]}(\mathcal{Z}_2\cdot\mathcal{Z}_3)\delta^{[s_1-s_2]}(\mathcal{Z}_3\cdot\mathcal{Z}_1)$$
$$+\alpha_2\int dc_{12}dc_{23}dc_{31}c_{12}^{s_1+s_2}c_{23}^{s_1-s_2}c_{31}^{-s_1+s_2}e^{ic_{12}\mathcal{Z}_1\cdot\mathcal{Z}_2}\delta^{4|1}(\mathcal{Z}_3^{\mathcal{A}}+c_{23}\mathcal{Z}_1^{\mathcal{A}}+c_{31}\mathcal{Z}_2^{\mathcal{A}}), \tag{5.170}$$

where $\alpha_1$ and $\alpha_2$ are coefficients that contain information about the OPE coefficients and can be read off from the super-field expansions of the scalar and currents. The important point to note is that the we get a linear combination the $\delta\delta\delta$ and $\delta^4$ solutions which are individually super-conformal invariant. However, in the context of the super-Penrose transform (5.161), one should integrate out $\psi_{3-}$ and therefore only the first term in (5.170) suffices in that context.

The result for two scalars and one conserved super-current with positive helicity is given by,

$$\langle 0|\hat{\mathbf{J}}^{+}_{s_1}(\mathcal{Z}_1)\hat{\mathbf{J}}_0(\mathcal{Z}_2)\hat{\mathbf{J}}_0(\mathcal{Z}_3)|0\rangle = \alpha_1\delta^{[s_1]}(\mathcal{Z}_1\cdot\mathcal{Z}_2)\delta^{[-1-s_1]}(\mathcal{Z}_2\cdot\mathcal{Z}_3)\delta^{[s_1]}(\mathcal{Z}_3\cdot\mathcal{Z}_1)$$
$$+\alpha_2\psi_{2-}\psi_{3-}\delta^{[s_1]}(\mathcal{Z}_1\cdot\mathcal{Z}_2)\delta^{[-s_1]}(\mathcal{Z}_2\cdot\mathcal{Z}_3)\delta^{[s_1]}(\mathcal{Z}_3\cdot\mathcal{Z}_1)$$
$$+\alpha_3\psi_{2-}\int dc_{12}dc_{23}dc_{31}c_{12}^{s_1}c_{23}^{s_1}c_{31}^{-s_1}e^{ic_{12}\mathcal{Z}_1\cdot\mathcal{Z}_2}\delta^4(c_{23}\mathcal{Z}_1^{\mathcal{A}}+c_{31}\mathcal{Z}_2^{\mathcal{A}}+\mathcal{Z}_3^{\mathcal{A}})$$
$$+\alpha_4\psi_{3-}\int dc_{12}dc_{23}dc_{31}c_{12}^{-1-s_1}c_{23}^{s_1}c_{31}^{-s_1}e^{ic_{23}\mathcal{Z}_2\cdot\mathcal{Z}_3}\delta^4(-\mathcal{Z}_1^{\mathcal{A}}+c_{12}\mathcal{Z}_2^{\mathcal{A}}-c_{31}\mathcal{Z}_3^{\mathcal{A}}), \tag{5.171}$$

where the $\alpha_i$ are related to the OPE coefficients. However, the scalar super-Penrose transform (5.161) tells us that only the coefficient of $\alpha_2$ suffices to recover the super-position space result. Similarly, the result with three scalars can be obtained and reads,

$$\langle 0|\hat{\mathbf{J}}_0(\mathcal{Z}_1)\hat{\mathbf{J}}_0(\mathcal{Z}_2)\hat{\mathbf{J}}_0(\mathcal{Z}_3)|0\rangle = \alpha_1\int dc_{12}dc_{23}dc_{31}e^{ic_{12}\mathcal{Z}_1\cdot\mathcal{Z}_2}\delta^{4|1}(c_{23}\mathcal{Z}_2^{\mathcal{A}}-c_{31}\mathcal{Z}_1^{\mathcal{A}}+\mathcal{Z}_3^{\mathcal{A}})$$
$$+\alpha_2\psi_{1-}\delta(\mathcal{Z}_1\cdot\mathcal{Z}_2)\delta^{[-1]}(\mathcal{Z}_2\cdot\mathcal{Z}_3)\delta(\mathcal{Z}_3\cdot\mathcal{Z}_1)$$
$$+\alpha_3\psi_{2-}\delta(\mathcal{Z}_1\cdot\mathcal{Z}_2)\delta(\mathcal{Z}_2\cdot\mathcal{Z}_3)\delta^{[-1]}(\mathcal{Z}_3\cdot\mathcal{Z}_1)$$
$$+\alpha_4\psi_{3-}\delta(\mathcal{Z}_1\cdot\mathcal{Z}_2)\delta^{[-1]}(\mathcal{Z}_2\cdot\mathcal{Z}_3)\delta(\mathcal{Z}_3\cdot\mathcal{Z}_1)$$
$$+\alpha_5\psi_{1-}\psi_{2-}\int dc_{12}dc_{23}dc_{31}e^{ic_{12}\mathcal{Z}_1\cdot\mathcal{Z}_2}\delta^{4|1}(c_{23}\mathcal{Z}_1^{\mathcal{A}}+c_{31}\mathcal{Z}_2^{\mathcal{A}}+\mathcal{Z}_3^{\mathcal{A}})$$
$$+\alpha_6\psi_{1-}\psi_{3-}\int dc_{12}dc_{23}dc_{31}e^{ic_{12}\mathcal{Z}_1\cdot\mathcal{Z}_2}c_{31}^{-1}\delta^{4|1}(c_{23}\mathcal{Z}_1^{\mathcal{A}}+c_{31}\mathcal{Z}_2^{\mathcal{A}}+\mathcal{Z}_3^{\mathcal{A}})$$
$$+\alpha_7\psi_{2-}\psi_{3-}\int dc_{12}dc_{23}dc_{31}e^{ic_{12}\mathcal{Z}_1\cdot\mathcal{Z}_2}c_{23}^{-1}\delta^{4|1}(c_{23}\mathcal{Z}_1^{\mathcal{A}}+c_{31}\mathcal{Z}_2^{\mathcal{A}}+\mathcal{Z}_3^{\mathcal{A}})$$
$$+\alpha_8\psi_{1-}\psi_{2-}\psi_{3-}\delta(\mathcal{Z}_1\cdot\mathcal{Z}_2)\delta(\mathcal{Z}_2\cdot\mathcal{Z}_3)\delta(\mathcal{Z}_3\cdot\mathcal{Z}_1). \tag{5.172}$$

The eight $\alpha_i$ (which can be reduced by imposing permutation symmetry to 4) encode the information of the OPE coefficients. Similar to the previous cases, the only term that contributes to the super-Penrose transform is the coefficient of $\alpha_8$. However, one must keep in mind that the super-Witten transform (5.163) requires all eight terms to reproduce the correct Grassmann spinor helicity expression.

This ends our discussion on super-scalars. The main message we want to convey is that although the expressions for three point functions involves many terms like in (5.172), only one term contributes to the super-Penrose transform (5.161). It would however be interesting to see if any other use can be found for the other terms, perhaps to obtain different position space super-correlators. Finally, similar to the non-supersymmetric case of subsection 5.6.1 and 5.6.3, one can derive and develop the super-Penrose transform for general $\Delta$ super-fields as well as non-conserved operators. We leave such an exercise to the future.

## 5.9 Summary of this Chapter

In this chapter, we have discussed and constructed Sp(4) invariants in twistor space for 3d CFT. We find two main classes of invariants: (1) Those that are made out of symplectic dot products of twistors and projective delta functions enforcing collinearity of twistors, (2) Those that also involve the infinity twistor of $\mathbb{R}^{2,1}$. We discussed how they occur in conformal two and three point Wightman functions of conserved currents and scalars and in particular, how the latter invariants allow us to extend the twistor space construction for arbitrary $\Delta$ scalars, generic spinning primaries as well as parity odd Wightman functions. We then derived the supersymmetric Penrose transform and discussed its relation to the supersymmetric Witten transform via the Fourier transform and employing the Faddeev Popov method. Finally, we constructed the $\mathrm{OSp}(\mathcal{N}|4)$ invariants and some prototypical examples of Wightman functions involving scalar super-fields for $\mathcal{N} = 1$ theories. One of the main messages of this chapter is that to accommodate generic primary operators in (super)Twistor space, the infinity twistor of $\mathbb{R}^{2,1}$ must be incorporated into the analysis. Although this may seem paradoxical as the infinity twistor breaks conformal invariance down to its Poincare subgroup, we have shown that the representation in which generic operators transform in twistor space is non-local and naturally involves the infinity twistor and with these generators, conformal invariance is indeed present. We have presented and derived many different versions of the Penrose and Witten transforms in this chapter. They are summarized for the convenience of the reader in table 5.1.

In contrast to the position space analysis for conserved current and super-current correlators [78, 243], the expressions we have found in twistor space are extremely simple highlighting their utility as a natural language to describe these quantities.

There are a number of interesting future directions, some of which we discuss below.

### Future directions

One immediate problem is to use the formalism of this chapter to bootstrap twistor space three point correlators of involving at least one generic primary operator. This should

| **Operator** | **Penrose Transform** | **Witten Transform** |
|---|---|---|
| Conserved Currents | (5.16) | (5.26) |
| Dimension $\Delta$ Scalars | (5.86) | (5.87) |
| Epsilon transformed Current | (5.101) | (5.102) |
| Non Conserved Current | (5.113) | (5.115) |
| Conserved Super-Current | (5.126) | (5.133) |
| Epsilon transformed Super-current | (5.153) | (5.156) |
| $\Delta = 1$ Super-Scalar | (5.161) | (5.163) |

Table 5.1: The Penrose and Witten transforms for various (super)operators

also help setting up the higher point conformal bootstrap, particularly for spinning correlators since twistor space seems like an appropriate stage to study spinning correlators as they are as simple as their scalar counterparts[21]. It would be interesting to perform the twistor space version of the analysis of [244, 245] as a starting point. Developing the machinery of Weight shifting and spin raising operators [36, 239] in twistor space also presents an interesting avenue to explore. Further, as discussed earlier, it would be interesting to impose the constraints of dual conformal and Yangian symmetry given the Sp(4) and OSp($\mathcal{N}$|4) projective delta functions and study their implications. In the context of four dimensional (supersymmetric) scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory, it turns out that there is a unique Yangian invariant quantity that one can construct [246–249]. We hope to return in the future to perform an analogous analysis in the context of $CFT_3$ and study the implications of Yangian invariance. It is also desirable to extend our formalism to extended supersymmetry and also accommodate non conserved currents. The latter would also be interesting from the point of view of studying BPS states in CFT. Finally, the recent paper [250] discusses the presence of sign functions in twistor space and how they are artifacts of working in Klein space. The author shows that these factors are absent in the complexified twistor space and thus it begs the question whether the ubiquitous sign functions that occur in the results of this chapter such as in the parity odd correlators of subsection 5.6.2 are also artifacts of working in the real twistor space associated to $\mathbb{R}^{2,1}$. Finally, it would be interesting to pursue the computation of higher point functions in a holographic setting and see if twistor space yields insight into their structure. This will form the subject of the next chapter.

[21] If anything, conserved current correlators seem simpler than their scalar counterparts in twistor space!

# Chapter 6

# $AdS_4$ Boundary Wightman Functions in Spinor and Twistor space

The modern amplitudes program has revolutionized our understanding of quantum field theory by focusing on fundamental principles and consistency conditions to directly bootstrap observables. On-shell methods using techniques such as spinor helicity have led to the development of recursion relations at both tree level and loop, unveiled double copy structures that relate gauge theory and gravity amplitudes and much more, see [249, 251] for a review. The related twistor [20, 22, 23] and geometric perspective has also revealed remarkable features of amplitudes such as their interpretation as volumes of positive geometries such as the amplituhedron [252]. These perspectives also help expose previously hidden or obscure symmetries like dual conformal and Yangian symmetry [246–248, 253, 254] and shed light on interesting facts such as the Wilson-loop scattering amplitude correspondence [255].

Given these achievements, it is tempting to investigate whether these developments of the modern on-shell ideas underlying scattering amplitudes, have natural analogues for CFT correlators, particularly in Lorentzian signature. Traditionally, these quantities have been analyzed in position space [13–16]. Although this has the advantage of making manifest locality and the operator product expansion, it becomes cumbersome to deal with when we consider correlators of conserved currents due to the proliferation of tensor structures. Further, this language is also in contrast to the Fourier space description that is used for scattering amplitudes. In recent years, conformal field theory has been formulated in momentum space as we discussed in chapter 2. In the case of interest to us which is 3d CFT, spinor helicity variables lead to an advantage since they trade a spin-s symmetric traceless conserved current, (dual to a massless gauge boson in $AdS_4$), for two objects viz positive and negative helicity components [24, 41, 42, 44, 45, 48, 57]. Although this leads to simplifications at the level of three point functions, progress at four points and beyond has been limited as we saw in chapter 2. This is partly because spinor helicity variables still give rise to complicated expressions due to large degeneracies in the allowed structures, see [256] for instance.

A recent impetus came in the form of a spacetime twistor approach which focuses on Lorentzian $CFT_3$ rather than its Euclidean counterpart [57, 236, 237, 241, 257–260] which formed the subject of chapter 5. Twistor space has a long history starting with the seminal work by Penrose [21]. In the context of its application to scattering amplitudes, we refer the reader to the wonderful lecture notes [238, 261]. In Lorentzian signature, there are many types of correlators such as time-ordered, retarded etc.. The ones that we shall be interested in particular are the real-time *Wightman* functions which form the building blocks for all other correlators. As we shall make more concrete, Wightman functions (or their suitable analytic continuations) enjoy a simple description in twistor space. One reason for this fact for current correlators is that in contrast to time-ordered or retarded correlators which satisfy non trivial conservation Ward-Takahashi identities with contact terms, Wightman functions are identically conserved [237]. See however, [260], where the authors work in complex twistor space, allowing for rational and logarithmic expressions, and show that these reproduce the correct Ward-Takahashi identities for three point current correlators. In this paper, we work in the real twistor space corresponding to a Lorentzian $\text{CFT}_3$ but it would be interesting to perform an analogous analysis in the complex setting in the future. Conformal invariance fixes two and three point functions and were found in earlier works [236, 237, 257]. One of the challenging problems is to understand higher point CFT correlators in twistor space.

To obtain higher point Wightman functions, we are faced with two options. we can take an intrinsically CFT approach as we did in chapter 5 or use AdS/CFT [7, 162, 163] to perturbatively compute CFT correlators. The latter is also of independent interest as it extends the flat space amplitudes program to AdS. We take this holographic approach in this chapter. We can either Wick rotate known Euclidean results or directly compute the real time correlators. The former option although tempting is quite difficult in practice even for three point functions [34] . Thus we will focus on the latter approach for the most part to intrinsically calculate boundary Wightman functions in $\text{AdS}_4$. There are various ways to compute real time correlators in the context of AdS/CFT, mostly using the Schwinger-Keldysh formalism or its holographic dual, see [262–273] and references therein. In this chapter, we exploit the fact that Wightman functions of fundamental fields identically obey the equation of motion [274]. We start with a bulk action and equation of motion which we perturbatively solve to calculate momentum space Wightman functions of interest to us order by order in the coupling constants. This method is algebraic in nature and the calculations are straightforward.

We compute three and four point Wightman functions in theories with general scalars, photons, gluons, gravitons with a variety of interactions between them which reveal a striking simplicity compared to their time ordered counterparts. For example, we show that in special kinematics where the momentum of the middle two operators are space-like in a four point function, it factorizes and turns into a conformal partial wave associated to the operator dual to the exchanged particle in the bulk[1]. This is reminiscent of the observation previously made for time-ordered correlators when we cut the bulk to bulk propagator [275](see also [276]). We also demonstrate how our factorized Wightman functions can be analytically continued to Euclidean space recovering the correlator up to contact diagram contributions. This sidesteps the nested bulk integrals that one would usually perform in Witten diagram calculations.

Finally, returning to our main motivation for pursuing these computations, we head towards twistor space using spinor helicity variables and the half-Fourier/Witten transform [20]. In the special kinematics, we find a strikingly simple form for the four and five

[1] Conformal partial waves are related to conformal blocks by a monodromy projection, see [85].

points scalar Wightman functions with a potential generalization to the n-point case[2]. Corresponding to the same kinematics for the gluon and graviton case, we establish a simple double copy relation which is easiest to see when the twistor space correlators are expressed in Schwinger parametrization: The graviton correlator is simply the square of its gluon counterpart in these variables. This double copy relation also exists in spinor helicity and momentum space but twistor space is much more efficient in establishing this result due to its effective way of dealing with degeneracies. This double copy relation allows us to systematically compute the full graviton correlator in any kinematics.

Succinctly, the main messages of this chapter are,

1. We provide a simple and systematic method to compute Wightman functions using the bulk equations of motion.

2. Four point Wightman functions factorize in the special kinematics when the middle operators have space-like momenta resulting in a Wightman conformal partial wave.

3. This conformal partial wave can be used to construct the time-ordered or Euclidean correlator corresponding to the bulk process up to contact diagrams.

4. Wightman functions in particular enjoy a simple description in twistor space and we obtain four point functions corresponding to these special kinematics.

5. We discovered a simple double copy relation between gluon and graviton four point functions in twistor space and spinor helicity variables.

Overall, our results indicate that twistor space is a promising avenue to pursue for the study of higher point functions.

The reference for this chapter is,

- ⋆$AdS_4$ Boundary Wightman functions in Twistor Space: Factorization, Conformal blocks and a Double Copy, Arhum Ansari, Sachin Jain, **Dhruva K.S.**, [2512.04172]

## 6.1 The Anatomy of Wightman Functions in AdS/CFT

The aim of this section is to set the stage for the calculation of Wightman functions in the context of the AdS/CFT correspondence. There are a few possible approaches

[2]It is tempting to directly convert the results of [275] (for the time-ordered correlator with the bulk to bulk propagator cut) into twistor space as it is a factorized expression. However, this quantity obeys the current conservation Ward-Takahashi identities with contact terms thus preventing them from enjoying a simple representation in twistors space. One might consider performing a cut on all external legs. As one can check this would actually lead to zero! The Wightman function in the special kinematics precisely tells us what is the simplest (and non-trivial) object to consider at four points in twistor space. It is a Wightman function in the special kinematics which is obtained from the time-ordered correlator by performing three cuts: one for the exchanged momentum and two corresponding to the first and fourth operator in the Wightman function.

to compute these real-time correlators such as, (i) Solving the bulk equation of motion generalizing the methods of [274] to AdS, (ii) Through analytic continuation from Euclidean AdS such as in [34] and (iii) Via the Skenderis-van Rees construction [264, 265]. For the vacuum Wightman functions we are interested in, we choose the first option as it is straightforward, algebraic and simple to implement even when dealing with spinning particles. The second option of Wick rotation is tempting but in momentum space is actually quite involved for generic momentum configurations as we discussed in chapter 2. We shall however, use it as a check of our results whenever possible.

Our focus is on $AdS_4/CFT_3$ for the most part but the methods outlined below extend to arbitrary dimensions. In subsection 6.1.1, we discuss the AdS/CFT extrapolate dictionary which we shall use to compute tree level Wightman functions in the Poincare patch. We then present the three possible approaches discussed above to perform these calculations in subsection 6.1.2.

### 6.1.1 The AdS/CFT extrapolate dictionary

With the above Wightman axioms in mind, we proceed to discuss our objects of interest which are tree level boundary $\mathbb{R}^{2,1}$ conformal Wightman functions of scalars, photons, gluons and gravitons in the Poincaré patch of $AdS_4$. The metric is given by,

$$ds^2 = \frac{dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu}{z^2}, \tag{6.1}$$

with $\eta_{\mu\nu} = \text{diag}(-1, 1, 1)$, is the three dimensional Minkowski metric on $\mathbb{R}^{2,1}$. The $z$ coordinate lies in the range $0 < z < \infty$ with the conformal boundary located at $z = 0$.

We obtain boundary Wightman functions via the AdS/CFT extrapolate dictionary [18][3],

$$\langle 0|O_1(p_1)\cdots O_n(p_n)|0\rangle \sim \lim_{z_i \to 0} z_1^{-\Delta_1} \cdots z_n^{-\Delta_n} \langle 0|\Phi_1(z_1, p_1)\cdots \Phi_n(z_n, p_n)|0\rangle. \tag{6.2}$$

$\Phi_i(z, p)$ is the bulk dual of the CFT operator $O_i(z, p)$ which has scaling dimension $\Delta_i$ and spin $s_i$[4]. These boundary correlators obey the properties discussed in chapter 2 as will also be clear from explicit calculations in the later sections. Before we proceed to the computations, let us discuss the different types of propagators that we shall require. First of all, since we are interested in boundary correlators we require both bulk to bulk propagators representing particle exchanges in the bulk as well as bulk to boundary propagators. In Lorentzian signature AdS, there are many possibilities for propagators such as Feynman, anti-Feynman, retarded, advanced and the two Wightman propagators. The Wightman two point functions in terms of the bulk field two point function are given

[3] This equation contains an overall factor that we suppress. This is easy to compute at tree level. At loop level, this factor renormalizes non-trivially and thus (6.2) would have to be modified [277]. We thank Kostas Skenderis for this comment.

[4] The dictionary (6.2) also generalizes naturally for spinning cases. In our normalizations, we scale by $z^{-1}$ for photons and gluons and $z^{-3}$ for gravitons, when taking the boundary limit of bulk correlators involving the fields corresponding to these operators.

by,

$$W_+(z,z',x-x') = \langle 0|\Phi(z,x)\Phi(z',x')|0\rangle,$$
$$W_-(z,z',x-x') = \langle 0|\Phi(z',x')\Phi(z,x)|0\rangle. \tag{6.3}$$

These quantities as a consequence of the $i\epsilon$ prescription (which is the same as in (2.88) since it has to do only with the time coordinates which the presence of the extra bulk spatial coordinate does not affect) are also complex conjugates of each other,

$$W_+(z,z',x-x')^* = W_-(z,z',x-x'). \tag{6.4}$$

The remaining propagators such as Feynman ($G_F$), anti-Feynman ($G_{\bar{F}}$), retarded ($G_R$) and advanced ($G_A$) are determined in terms of these fundamental quantities.

$$\begin{aligned}
G_F(z,z',x-x') &= \langle 0|T\{\Phi(z,x)\Phi(z',x')\}|0\rangle = \theta(t-t')W_+(z,z',x-x') + \theta(t'-t)W_-(z,z',x-x'),\\
G_{\bar{F}}(z,z',x-x') &= \langle 0|\bar{T}\{\Phi(z,x)\Phi(z',x')\}|0\rangle = \theta(t'-t)W_+(z,z',x-x') + \theta(t-t')W_-(z,z',x-x'),\\
G_R(z,z',x-x') &= -i\theta(t-t')\langle 0|[\Phi(z,x),\Phi(z',x')]|0\rangle = -i\theta(t-t')\big(W_+(z,z',x-x') - W_-(z,z',x-x')\big),\\
G_A(z,z',x-x') &= i\theta(t'-t)\langle 0|[\Phi(z,x),\Phi(z',x')]|0\rangle = i\theta(t'-t)\big(W_+(z,z',x-x') - W_-(z,z',x-x')\big).
\end{aligned} \tag{6.5}$$

The bulk to boundary propagators are obtained by taking either $z$ or $z'$ to the boundary and rescaling to obtain a finite non-zero result. Not all these propagators are linearly independent. They satisfy what is known as the largest time equation viz,

$$G_F(z,z',x-x') + G_{\bar{F}}(z,z',x-x') = W_+(z,z',x-x') + W_-(z,z',x-x'),$$
$$i(G_R(z,z',x-x') - G_A(z,z',x-x')) = W_+(z,z',x-x') - W_-(z,z',x-x'). \tag{6.6}$$

Finally, let us note that once we calculate all the $n!$ different $n-$point Wightman functions, all other correlators of interest can be obtained. For instance the time-ordered correlator is given in terms of Wightman functions as follows,

$$\langle 0|T\{O_1(x_1)\cdots O_n(x_n)\}|0\rangle = \sum_{\pi\in S_n}\theta(t_{\pi(1)} > t_{\pi(2)} > \cdots t_{\pi(n)})\langle 0|O_{\pi(1)}(x_{\pi(1)})\cdots O_{\pi(n)}(x_{\pi(n)})|0\rangle, \tag{6.7}$$

where $S_n$ is the symmetric group on $n$ elements which is the group whose elements are all possible permutations of $n$ objects, which are $n!$ in total. Each object on the RHS of this equation are Wightman functions with different orderings. Similarly, one can derive formulae that relate other types of correlators to Wightman functions.

### 6.1.2 Wightman functions and How to Compute Them

With all the above facts in mind, let us discuss three different ways to calculate holographic Wightman functions.

### Boundary Wightman functions via Bulk EOM

The first and most straightforward way to calculate tree level holographic correlators is to perturbatively solve the non-linear bulk equation of motion. To illustrate this method, let us consider a conformally coupled scalar field in $AdS_4$ with a cubic self-interaction. The action is given by[5],

$$S = -\frac{1}{2}\int \frac{dz}{z^4} d^3x \Big( (\partial_z \Phi)^2 + (\partial_\mu \Phi)^2 - 2\Phi^2 \Big) - \frac{g}{3!}\int \frac{dz}{z^4} d^3x \Phi^3. \tag{6.8}$$

Let us perform a Weyl rescaling,

$$\Phi(z,x) = z\phi(z,x). \tag{6.9}$$

This converts (6.8) into an action in half of flat space with a $z$ dependent interaction:

$$S = -\frac{1}{2}\int dz\, d^3x \left( (\partial_z \phi)^2 + (\partial_\mu \phi)^2 \right) - \frac{g}{3!}\int \frac{dz}{z} d^3x\, \phi^3. \tag{6.10}$$

The equation of motion reads,

$$(\partial_z^2 + \Box)\phi(z,x) = \frac{g}{2z}\phi^2(z,x), \tag{6.11}$$

We perturbatively expand the field in $g$ as,

$$\phi(z,x) = \sum_{n=0}^{\infty} g^n \phi^{(n)}(z,x). \tag{6.12}$$

Substituting this expansion into the equation of motion (6.11) results in a set of equations organized by the powers of $g$.

$$\begin{aligned}
&\sum_{n=0}^{\infty} g^n (\partial_z^2 + \Box)\phi^{(n)}(z,x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{g^{n+m+1}}{2z}\phi^{(n)}(z,x)\phi^{(m)}(z,x) \\
&\implies (\partial_z^2 + \Box)\phi^{(0)}(z,x) = 0,\ (\partial_z^2 + \Box)\phi^{(1)}(z,x) = \frac{1}{2z}(\phi^{(0)}(z,x))^2, (\partial_z^2 + \Box)\phi^{(2)}(z,x) = \frac{1}{z}\phi^{(0)}(z,x)\phi^{(1)}(z,x), \cdots
\end{aligned} \tag{6.13}$$

The first equation is the free equation of motion whereas the higher order equations determine the field corrections in terms of the free field $\phi_0$. We can solve (6.13) using the technique of Green's functions as follows:

$$\begin{aligned}
\phi^{(1)}(z,x) &= \frac{i}{2}\int \frac{dz' d^3x'}{z'} \mathcal{G}(z,z',x-x')(\phi^{(0)}(z',x'))^2, \\
\phi^{(2)}(z,x) &= i\int \frac{dz' d^3x'}{z'} \mathcal{G}(z,z',x-x')\phi^{(0)}(z',x')\phi^{(1)}(z'',x''), \cdots,
\end{aligned} \tag{6.14}$$

where $\mathcal{G}(z,z',x-x')$ solves,

$$(\partial_z^2 + \Box)\mathcal{G}(z,z',x-x') = -i\delta(z-z')\delta^3(x-x'). \tag{6.15}$$

[5] Note that all integrals over $z$ run from 0 to $\infty$ even when not explicitly mentioned.

At this point a natural choice could be to take $\mathcal{G}(z,z',x-x')$ as the Feynman propagator. However, note that we can also freely add Wightman functions to these solutions since they are homogeneous quantities[6],

$$(\partial_z^2 + \Box)W_\pm(z,z',x-x') = 0. \tag{6.16}$$

These ambiguities can be fixed by demanding appropriate boundary conditions as well as reality conditions. We impose Dirichlet boundary conditions for the field so $\mathcal{G}(z,z',x-x')$ must go to zero as we take either $z$ or $z'$ to zero. Further, since our scalar field is real, $\phi^{(1)}, \phi^{(2)}$ etc.. all have to be real. Due to the overall factors of $i$ in (6.14), this implies that the propagator has to be purely imaginary to ensure that the field corrections are real. Thus, the propagator relevant for our computation of Wightman functions is the following combination of the Feynman and Wightman propagators:

$$\begin{aligned}\mathcal{G}(z,z',x-x') =& G_F(z,z',x-x') - \frac{1}{2}\big(W_+(z,z',x-x') + W_-(z,z',x-x')\big)\\ =& \, i\,\mathrm{Im}\{G_F(z,z',x-x')\}.\end{aligned} \tag{6.17}$$

Note that this propagator is purely imaginary, satisfying,

$$\mathcal{G}(z,z',x-x')^* = -\mathcal{G}(z,z',x-x'). \tag{6.18}$$

We call this propagator the EOM (equation of motion) inverter propagator. It will also ensure that the Wightman functions we construct respect the conjugation property (2.89) ensuring that our construction is correct. In the next section, we will be more explicit about the form of these propagators. Thus, (6.14) with (6.17) can be used to calculate Wightman functions perturbatively. For example, consider the $O(g)$ correction to the three point Wightman function. This receives contributions from three terms,

$$\begin{aligned}\langle 0|\phi(z_1,x_1)\phi(z_2,x_2)\phi(z_3,x_3)|0\rangle_{O(g)} =& \, g\Big(\langle 0|\phi^{(1)}(z_1,x_1)\phi^{(0)}(z_2,x_2)\phi^{(0)}(z_3,x_3)|0\rangle\\ &+ \langle 0|\phi^{(0)}(z_1,x_1)\phi^{(1)}(z_2,x_2)\phi^{(0)}(z_3,x_3)|0\rangle + \langle 0|\phi^{(0)}(z_1,x_1)\phi^{(0)}(z_2,x_2)\phi^{(1)}(z_3,x_3)|0\rangle\Big).\end{aligned} \tag{6.19}$$

After solving for $\phi^{(1)}$ in terms of $(\phi^{(0)})^2$ as in (6.14), we obtain,

$$\begin{aligned}&\langle 0|\phi(z_1,x_1)\phi(z_2,x_2)\phi(z_3,x_3)|0\rangle_{O(g)}\\ =&\frac{ig}{2}\Big(\int \frac{dz d^3x}{z}\mathcal{G}(z,z_1,x-x_1)\langle 0|(\phi^{(0)}(z,x))^2\phi^{(0)}(z_2,x_2)\phi^{(0)}(z_3,x_3)|0\rangle\\ &+\int \frac{dz d^3x}{z}\mathcal{G}(z,z_2,x-x_2)\langle 0|\phi^{(0)}(z_1,x_1)(\phi^{(0)}(z,x))^2\phi^{(0)}(z_3,x_3)|0\rangle\\ &+\int \frac{dz d^3x}{z}\mathcal{G}(z,z_3,x-x_3)\langle 0|\phi^{(0)}(z_1,x_1)\phi^{(0)}(z_2,x_2)(\phi^{(0)}(z,x))^2|0\rangle\Big).\end{aligned} \tag{6.20}$$

[6]One can easily show that the definition (6.5) implies that any of the propagators there obey (6.15) using the fact that Wightman functions identically solve the equation of motion.

As the correlators in the RHS of the above expressions are free theory Wightman functions, we can perform Wick contractions using Wightman propagators[7]. It is also important to note that when we encounter composite operators, we follow the standard practice of normal ordering them to avoid self contractions. Thus, we obtain,

$$\begin{aligned}
&\langle 0|\phi(z_1,x_1)\phi(z_2,x_2)\phi(z_3,x_3)|0\rangle_{O(g)}\\
=&ig\bigg(\int\frac{dzd^3x}{z}\mathcal{G}(z,z_1,x-x_1)W_+(z,z_2,x-x_2)W_+(z,z_3,x-x_3)\\
&+\int\frac{dzd^3x}{z}\mathcal{G}(z,z_2,x-x_2)W_-(z,z_1,x-x_1)W_+(z,z_3,x-x_3)\\
&+\int\frac{dzd^3x}{z}\mathcal{G}(z,z_3,x-x_3)W_-(z,z_1,x-x_1)W_-(z,z_2,x-x_2)\bigg).
\end{aligned}\tag{6.21}$$

Note that the choice of using the propagator (6.17) and its reality condition (6.18), the Wightman propagator conjugation property (6.4), ensures the Wightman conjugation property (2.89). Given this expression, one can then obtain the dual boundary conformal correlator via the extrapolate dictionary,

$$\langle 0|O_\Delta(x_1)O_\Delta(x_2)O_\Delta(x_3)|0\rangle_{O(g)}=\lim_{z_i\to 0}(z_1z_2z_3)^{1-\Delta}\langle 0|\phi(z_1,x_1)\phi(z_2,x_2)\phi(z_3,x_3)|0\rangle_{O(g)},\tag{6.22}$$

where $\Delta=1,2$ depending on whether we impose Neumann or Dirichlet boundary conditions for $\phi_0(z,x)$ at $z=0$ which will also determine the exact forms of the propagators in the above expression. The extra 1 in the rescaling exponents is due to the fact that we are working with Weyl rescaled fields (6.9). The analysis for higher point functions and those involving operators with arbitrary scaling dimension and spin is analogous. Also, as mentioned earlier, we will compute these objects in momentum space. This is obtained via a Fourier transform of the above equations but it is more convenient to directly calculate momentum space Wightman functions as we shall illustrate in the subsequent sections. Now, we proceed to another method to obtain Wightman functions: by Wick rotating their Euclidean AdS counterparts.

### From Euclidean correlators to Wightman functions via Wick rotation

The conformal boundary of Euclidean $\text{AdS}_4$ is $\mathbb{R}^3$. Conformal correlators in Euclidean space are single-valued and thus there exists only one kind of correlator which is unsurprisingly called the Euclidean correlator. We have already discussed how one can Wick rotate these quantities appropriately to obtain Wightman functions in an earlier section.

As we discussed then, the analytic continuation procedure at the level of momentum space higher point functions is not known in general. However, the direct computation using the equation of motion discussed earlier allows us to still obtain these results in a straightforward manner. In the special (but as we shall see, important!) case when

[7]Note that this is in contrast to what we usually do in standard QFT for time-ordered correlators, which involve Feynman, rather than Wightman propagators in Wick's theorem.

all the middle operators all have space-like momenta, we will show that the procedure is identical to (2.101). We simply take discontinuities with respect to the squares of all the momenta we want to take to be time-like (including the exchanged momenta such as $s^\mu = p_1^\mu + p_2^\mu$) and then Wick rotate following the $i\epsilon$ prescription (2.99). We will discuss this in much more detail in section 6.3.3 as well as other cases where we have more operators in the Wightman function with time-like momenta.

#### The Skenderis-Van Rees prescription

The standard way to calculate non time-ordered correlators such as Wightman functions in QFT is using the Schwinger-Keldysh formalism. In the context of AdS/CFT, Skenderis and Van Rees developed a formalism to compute these real time quantities holographically [264, 265]. Essentially, their idea is to map every segment of the Schwinger Keldysh contour to a bulk geometry. For instance, let us consider the process required to calculate a two point Wightman function in the vacuum state. In the Schwinger Keldysh formalism, we need a contour with two segments, one going forward in time and the other backwards, with one operator insertion in each. First we prepare the vacuum state by a Euclidean path integral. At some reference time, we join it to the first Lorentzian segment and insert one operator at time $t_1$. We then move till $t = \infty$ and then fold and evolve back to $t = -\infty$ inserting the second operator at some time $t_2$ on the way. We then connect this segment to another Euclidean one to get back the vacuum state. The holographic prescription is to identify the Euclidean segments with (half of) Euclidean AdS where we prepare the vacuum state and the Lorentzian segments with the Lorentzian AdS. First, we solve the field equations in each of these segments. We then apply matching conditions wherever these geometries join and obtain a solution with sources inserted at the conformal boundary of both Lorentzian AdS spacetimes. Plugging these solutions into the on-shell action and taking one functional derivative with respect to each source results in the Wightman function. For three and four point Wightman functions, we need to use multi-fold contours which entails joining four Lorentzian AdS spacetimes with two Euclidean caps to prepare the vacuum state. Although we shall not take this approach in this work, it is essentially equivalent to our approach of using the equation motion. We choose the latter as it is completely algebraic and sufficient for our purposes.

Armed with all these tools and techniques, let us set the stage for the theories of interest to us in this work.

## 6.2 Setting the Stage: Propagators in the Theories of Interest

Having outlined a formalism for the computation of AdS boundary Wightman functions and their relation via analytic continuation to EAdS correlators, we turn our attention towards specific theories. These include theories with scalars, photons, gluons and gravitons with a variety of interactions between them as we shall discuss case by case in the sections to follow. Here, our aim is to calculate the Wightman, Feynman, and the propagators we use to invert the free equations of motion (6.17).

### 6.2.1 Scalars

The free action for a massive scalar field in the Poincare patch of AdS$_4$ is given by,

$$S_{KG} = \int \frac{dzd^3x}{z^4}\Big( -\frac{1}{2}g^{AB}\partial_A\Phi\partial_B\Phi - \frac{m^2}{2}\Phi^2\Big), \tag{6.23}$$

in units where the $AdS_4$ radius is set to unity. Let us perform the following Weyl transformation:

$$\Phi(z,x) = z\phi(z,x). \tag{6.24}$$

The resulting action for $\phi$ is simply,

$$S_{KG,\text{massive}} = \int dzd^3x\,\Big( -\frac{1}{2}\eta^{AB}\partial_A\phi\partial_B\phi - \frac{(m^2+2)}{2z^2}\phi^2\Big). \tag{6.25}$$

The free equation of motion is,

$$(\partial_z^2 + \Box - \frac{(m^2+2)}{z^2})\phi(z,x) = 0. \tag{6.26}$$

The above equation is solved by $\phi(z,x) = \frac{z^{\Delta-1}}{(z^2+x^2)^\Delta}$ with $m^2 = \Delta(\Delta-3)$ which is the usual AdS/CFT relation. We impose Dirichlet boundary conditions for the general scalar field at the $z = 0$ conformal boundary.

The Feynman propagator is a Green's function of the operator $(\partial_z^2 + \Box - \frac{(m^2+2)}{z^2})$ which appears above. In particular, it satisfies,

$$(\partial_z^2 + \Box - \frac{(m^2+2)}{z^2})G_{F,\Delta}(z,z',x-x') = -i\delta(z-z')\delta^3(x-x'). \tag{6.27}$$

The Wightman propagators on the other hand are homogeneous solution to (6.27).

$$(\partial_z^2 + \Box - \frac{(m^2+2)}{z^2})W_{\Delta,\pm}(z,z',x-x') = 0, \tag{6.28}$$

Converting these equations to momentum space, imposing Dirichlet boundary conditions, the spectral conditions for the Wightman functions and matching the normalization via Fourier transforming the position space results yields the expressions in table 6.1. Apart from the bulk to bulk propagators, we have also listed the bulk to boundary propagators obtained by taking one of the bulk points to the boundary and rescaling to obtain a finite result. Further, we have presented the expressions for the propagator used to invert the equation of motion obtained via its definition viz (6.17). For conformally coupled scalars which have $m^2 = -2$, we provide the explicit details of computation in our paper [278].

| | |
|---|---|
| **Feynman BtB** | $G_{F,\Delta}(z,z',p) = \frac{\sqrt{\pi}(zz')^{\frac{1}{2}}\Gamma(\Delta-\frac{1}{2})}{\Gamma(\Delta)}\Big[\theta(-p^2)\Big(H_\nu^{(1)}(\lvert p\rvert z)J_\nu(\lvert p\rvert z')\theta(z-z')+ (z\leftrightarrow z')\Big) - \frac{2i}{\pi}\theta(p^2)\Big(K_\nu(\lvert p\rvert z)I_\nu(\lvert p\rvert z')\theta(z-z') + (z\leftrightarrow z')\Big)\Big]$ |
| **Feynman Btb** | $G_{F,\Delta}(z,p) = \frac{2^{\frac{3}{2}-\Delta}\sqrt{z}\lvert p\rvert^\nu}{\sqrt{\pi}\Gamma(\Delta)}\Big(\theta(-p^2)\pi H_\nu^{(1)}(\lvert p\rvert z) - 2i\theta(p^2)K_\nu(\lvert p\rvert z)\Big)$ |
| **Wightman BtB** | $W_{\Delta,\pm}(z,z',p) = \frac{2\sqrt{\pi}(zz')^{\frac{1}{2}}\Gamma(\Delta-\frac{1}{2})}{\Gamma(\Delta)}J_\nu(\lvert p\rvert z)J_\nu(\lvert p\rvert z')\theta(-p^2)\theta(\mp p^0)$ |
| **Wightman Btb** | $W_{\Delta,\pm}(z,p) = \frac{2^{\frac{5}{2}-\Delta}\sqrt{\pi}\sqrt{z}\lvert p\rvert^\nu}{\Gamma(\Delta)}J_\nu(\lvert p\rvert z)\theta(-p^2)\theta(\mp p^0)$ |
| **EOM Inverter BtB** | $\mathcal{G}_\Delta(z,z',p) = \frac{i\sqrt{\pi}(zz')^{\frac{1}{2}}\Gamma(\Delta-\frac{1}{2})}{\Gamma(\Delta)}\Big[\theta(-p^2)\Big(Y_\nu(\lvert p\rvert z)J_\nu(\lvert p\rvert z')\theta(z-z') + (z\leftrightarrow z')\Big) - \frac{2}{\pi}\theta(p^2)\Big(K_\nu(\lvert p\rvert z)I_\nu(\lvert p\rvert z')\theta(z-z') + (z\leftrightarrow z')\Big)\Big]$ |
| **EOM Inverter Btb** | $\mathcal{G}_\Delta(z,p) = \frac{i2^{\frac{3}{2}-\Delta}\sqrt{z}\lvert p\rvert^\nu}{\sqrt{\pi}\Gamma(\Delta)}\Big(\theta(-p^2)\pi Y_\nu(\lvert p\rvert z) - 2\theta(p^2)K_\nu(\lvert p\rvert z)\Big)$ |

Table 6.1: Momentum space Scalar BtB (Bulk to Bulk) and Btb (Bulk to boundary) propagators for generic $\Delta = \nu + \frac{3}{2}$ scalars. We use the notation $|p| = \sqrt{|p^2|}$. $H_\nu^{(1)}$ is the Hankel function of the first kind, $K_\nu, J_\nu$ and $Y_\nu$ are the BesselK, BesselJ and BesselY functions. Gauge and gravity Wightman propagators (6.36),(6.46) can be obtained by multiplying $\Delta = 2$ and $\Delta = 3$ propagators by appropriate projectors respectively (with an extra $zz'$ rescaling for the latter case) and adding the longitudinal parts for the Feynman and EOM Inverter propagators (6.38), (6.50).

### 6.2.2 Photons and Gluons

The next two theories of interest to use are those involving Photons and gluons. The free Maxwell action is given by,

$$S_{EM} = -\frac{1}{4}\int \frac{dz}{z^4}\, d^3xz^4\Big((F_{\mu\nu}F^{\mu\nu} + 2F_{\mu z}F^{\mu z}\Big). \tag{6.29}$$

where the field strength is given in terms of the gauge field $A_\mu$ as follows:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\mu, F_{\mu z} = \partial_\mu A_z - \partial_z A_\mu. \tag{6.30}$$

The Maxwell action has the gauge redundancy,

$$(A_z, A_\mu) \to (A_z - \partial_z\alpha, A_\mu - \partial_\mu\alpha). \tag{6.31}$$

Thanks to the fact that Maxwell theory in four dimensions enjoys conformal invariance, the action is the same as in half of flat space:

$$S_{EM} = -\frac{1}{2}\int dzd^3x\,\Big((\partial_\mu A_\nu)^2 - (\partial_\mu A_\nu)(\partial^\nu A^\mu) + (\partial_\mu A_z - \partial_z A_\mu)^2)\Big). \tag{6.32}$$

To fix the gauge redundancy (6.31) we work in the gauge $A_z = 0$. In the action, this corresponds to adding a gauge fixing term $\zeta A_z^2$ and then taking $\zeta \to -\infty$ thus effectively freezing the value $A_z = 0$. This still leaves behind residual gauge transformations with $z-$independent gauge parameters $\alpha$. To fix this, we then set $\partial_\mu A^\mu = 0$, thus fully fixing the gauge. Thus, the action in axial gauge up to boundary terms is given by,

$$S_{EM,axial} = -\frac{1}{2}\int dz d^3x \left((\partial_z A_\mu)^2 + (\partial_\nu A_\mu)^2 - \partial_\nu A_\mu \partial^\mu A^\nu\right). \tag{6.33}$$

The equation of motion with the constraint equation are respectively given by,

$$(\partial_z^2 + \Box)A_\mu(z,x) = 0, \tag{6.34}$$

and,

$$\partial_z(\partial_\mu A^\mu(z,x)) = 0. \tag{6.35}$$

First, note that the constraint equation (6.35) which arises due to the equation of motion of $A_z$ is automatically satisfied in our choice $\partial_\mu A^\mu = 0$. The equation of motion (6.34) is simply that of a free conformally coupled scalar obtained by setting $m^2 = -2$ in (6.26). The Wightman propagators and the EOM inverter propagators are thus obtained by dressing $\Delta = 2$ scalar propagators obtained from table 6.1 by projectors that take into account the tensor structures and gauge constraint.

$$W_{\mu\nu,\pm}(z,z',p) = \pi_{\mu\nu}(p)W_{\Delta=2}(z,z',p), W_{\mu\nu,\pm}(z,z',p) = \pi_{\mu\nu}(p)W_{\Delta=2}(z,p), \tag{6.36}$$

with the transverse projector in the given by,

$$\pi_{\mu\nu}(p) = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}. \tag{6.37}$$

The EOM inverter propagator found using its definition (6.17) on the other hand is given by,

$$\begin{aligned}\mathcal{G}_{\mu\nu}(z,z',p) &= \pi_{\mu\nu}(p)\mathcal{G}_{\Delta=2}(z,z',p) - i\frac{p_\mu p_\nu}{p^2}\Big(\theta(z-z')z' + \theta(z'-z)z\Big),\\ \mathcal{G}_{\mu\nu}(z,p) &= \pi_{\mu\nu}(p)\mathcal{G}_{\Delta=2}(z,p) - i\frac{p_\mu p_\nu}{p^2}.\end{aligned} \tag{6.38}$$

The expression for the Feynman propagator can be found in [74]. Note in particular that although the Wightman propagators are identically transverse whereas the EOM inverter propagators have a longitudinal contribution[8].

Finally, for the gluon case, we just need to dress the gauge fields with colour indices which we take as the adjoint indices of $SU(N)$. This is because the free $SU(N)$ Yang-Mills theory is simply $N^2 - 1$ copies of the Maxwell action. Thus, the gauge fixing remains identical to the above discussion. The propagators are also the same as in Maxwell's theory with a simple additional contribution taking account of the colour indices.

$$\begin{aligned}W^{AB}_{\mu\nu,\pm}(z,z',p) &= \delta^{AB}W_{\mu\nu,\pm}(z,z',p), W^{AB}_{\mu\nu,\pm}(z,p) = \delta^{AB}W_{\mu\nu,\pm}(z,p)\\ \mathcal{G}^{AB}_{\mu\nu}(z,z',p) &= \delta^{AB}\mathcal{G}_{\mu\nu}(z,z',p), \mathcal{G}^{AB}_{\mu\nu}(z,p) = \delta^{AB}\mathcal{G}_{\mu\nu}(z,p).\end{aligned} \tag{6.39}$$

[8]These are terms that arise due to the Feynman propagator in $\mathcal{G}$ via (6.17). This fact can be traced back to the Heaviside theta functions that enforce time-ordering in a Feynman propagator (6.5).

### 6.2.3 Gravity

Finally, we turn towards gravity. The Einstein Hilbert action is given by,

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R-2\Lambda), \tag{6.40}$$

with $\kappa = 8\pi G$ defined for future use and with the metric expanded about the $AdS_4$ Poincare patch metric as follows,

$$g_{\mu\nu} = \frac{(\eta_{\mu\nu}+h_{\mu\nu})}{z^2}, g_{\mu z} = \frac{(\eta_{\mu z}+h_{\mu z})}{z^2}, g_{zz} = \frac{(\eta_{zz}+h_{zz})}{z^2}. \tag{6.41}$$

We follow exactly the conventions of [62]. We work in the axial gauge,

$$h_{\mu z} = 0. \tag{6.42}$$

The equations of motion are,

$$\left(\partial_z^2(h_{\mu\nu}-\eta_{\mu\nu}h^\rho_\rho) - \frac{2}{z}\partial_z(h_{\mu\nu}-\eta_{\mu\nu}h^\rho_\rho) + \Box\tilde{h}_{\mu\nu} - \partial_\rho\partial_\mu\tilde{h}^\rho_\nu - \partial_\rho\partial_\nu\tilde{h}^\rho_\mu + \eta_{\mu\nu}\partial_\rho\partial_\alpha\tilde{h}^{\alpha\rho}\right) = 0, \tag{6.43}$$

with $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\rho_\rho$. The constraint equations due to the $h_{zz}$ and $h_{z\mu}$ equations of motion are,

$$\partial_z(\partial_\nu h^\nu_\mu - \partial_\mu h^\rho_\rho) = 0, -\partial_\mu\partial_\nu h^{\mu\nu} + \Box h^\rho_\rho - \frac{2}{z}\partial_z h^\rho_\rho = 0. \tag{6.44}$$

If we use the residual gauge transformations to set $\partial_\mu h^{\mu\nu} = 0$ and $h^\mu_\mu = 0$, the constraint equations are automatically satisfied and the equation of motion becomes extremely simple viz,

$$\partial_z^2 h_{\mu\nu} - \frac{2}{z}\partial_z h_{\mu\nu} + \Box h_{\mu\nu} = 0 \implies \partial_z^2 h_{\mu\nu} - \frac{2}{z}\partial_z h_{\mu\nu} - p^2 h_{\mu\nu} = 0. \tag{6.45}$$

The Wightman propagators solve the equation of motion and satisfy all the gauge constraints identically and are given in terms of $\Delta = 3$ scalar propagators found in table 6.1 as follows:

$$W_{\mu\nu\rho\sigma,\pm}(z,z',p) = \Pi_{\mu\nu\rho\sigma}(p)zz'W_{\Delta=3,\pm}(z,z',p), W_{\mu\nu\rho\sigma}(z,p) = \Pi_{\mu\nu\rho\sigma}(p)zW_{\Delta=3,\pm}(z,p), \tag{6.46}$$

The transverse traceless projector is given by,

$$\Pi_{\mu\nu\rho\sigma}(p) = \pi_{\mu\nu}(p)\pi_{\rho\sigma}(p) - \pi_{\mu\rho}(p)\pi_{\nu\sigma}(p) - \pi_{\mu\sigma}(p)\pi_{\nu\rho}(p), \tag{6.47}$$

with the transverse projector itself given in (6.37). The Feynman propagator on the other hand is given by [63][9],

$$G_F^{\mu\nu\rho\sigma}(z,z',p) = \int \frac{-idk^2}{2}\frac{(zz')^{3/2}J_{3/2}(kz)J_{3/2}(kz')}{p^2+k^2-i\epsilon}\mathcal{T}^{\mu\nu\rho\sigma}, \tag{6.48}$$

[9]Note that our normalization differs from [63] by a factor of $\frac{1}{z^2(z')^2}$.

with,

$$\mathcal{T}^{\mu\nu\rho\sigma} = (\tilde{\pi}^{\mu\rho}\tilde{\pi}^{\nu\sigma} + \tilde{\pi}^{\mu\sigma}\tilde{\pi}^{\nu\rho} - \tilde{\pi}^{\mu\nu}\tilde{\pi}^{\rho\sigma}), \tilde{\pi}^{\mu\nu} = \eta^{\mu\nu} + \frac{p^\mu p^\nu}{k^2}. \tag{6.49}$$

The Feynman propagator contains both transverse and longitudinal contributions which are picked up by enclosing the different poles in the above integral. This fact can again be traced back to the relation that relates Feynman propagators to Wightman functions (6.5). Finally, the EOM inverter propagator can be found using its definition (6.17),

$$\mathcal{G}_{\mu\nu\rho\sigma}(z,z',p) = G_{F\mu\nu\rho\sigma}(z,z',p) - \frac{1}{2}\bigg(W_{\mu\nu\rho\sigma,+}(z,z',p) + W_{\mu\nu\rho\sigma,-}(z,z',p)\bigg). \tag{6.50}$$

The bulk to boundary propagators can be found by rescaling by $\frac{1}{(z')^3}$ and taking the limit $z' \to 0$.

## 6.3 Wightman functions in momentum space

Having discussed the theories of interest to us, we calculate several examples of two, three and four point Wightman functions in this section to illustrate our formalism.

### 6.3.1 Two point functions

Let us begin with the simplest case of two point Wightman functions. These quantities can be obtained by extrapolating the bulk point in the bulk to boundary propagators to the boundary. First for scalars, consider the Wightman plus propagator in table 6.1. Taking $z \to 0$ along with rescaling results in,

$$\begin{aligned}\langle\langle 0|O_\Delta(p_1)O_\Delta(p_2)|0\rangle\rangle &= \lim_{z\to 0}\frac{1}{z^{\Delta-1}}W_{\Delta,+}(z,p_1)\\ &= \lim_{z\to 0}\frac{1}{z^{\Delta-1}}\frac{2^{\frac{5}{2}-\Delta}\sqrt{\pi\, z}}{\Gamma(\Delta)}|p_1|^{\Delta-\frac{3}{2}}J_{\Delta-\frac{3}{2}}(|p_1|z)\theta(-p_1^2)\theta(-p_1^0)\\ &= \frac{4}{\Gamma(2\Delta-1)}|p_1|^{2\Delta-3}\theta(-p_1^2)\theta(-p_1^0),\end{aligned} \tag{6.51}$$

which is indeed the correct scalar two point Wightman plus function, spectral theta functions and all. Similarly, we can do the same for the Wightman minus propagator which can also be found in table 6.1.

$$\begin{aligned}\langle\langle 0|O_\Delta(p_2)O_\Delta(p_1)|0\rangle\rangle &= \lim_{z\to 0}\frac{1}{z^{\Delta-1}}W_{\Delta,-}(z,p_1)\\ &= \lim_{z\to 0}\frac{1}{z^{\Delta-1}}\frac{2^{\frac{5}{2}-\Delta}\sqrt{\pi\, z}}{\Gamma(\Delta)}|p_1|^{\Delta-\frac{3}{2}}J_{\Delta-\frac{3}{2}}(|p_1|z)\theta(-p_1^2)\theta(p_1^0)\\ &= \frac{4}{\Gamma(2\Delta-1)}|p_1|^{2\Delta-3}\theta(-p_1^2)\theta(p_1^0).\end{aligned} \tag{6.52}$$

Similarly, for the spin-1 case we find using (6.36),

$$\langle\langle 0|J^{\mu}(p_1)J^{\nu}(p_2)|0\rangle\rangle = \lim_{z\to 0}\frac{1}{z}\pi^{\mu\nu}(p_1)2\sin(|p_1|z)\theta(-p_1^2)\theta(-p_1^0) = 2|p_1|\pi^{\mu\nu}(p_1)\theta(-p_1^2)\theta(-p_1^0),$$
$$\langle\langle 0|J^{\mu}(p_2)J^{\nu}(p_1)|0\rangle\rangle = \lim_{z\to 0}\frac{1}{z}\pi^{\mu\nu}(p_1)2\sin(|p_1|z)\theta(-p_1^2)\theta(p_1^0) = 2|p_1|\pi^{\mu\nu}(p_1)\theta(-p_1^2)\theta(p_1^0), \tag{6.53}$$

which are the correct results. For gluons the result is the above with an extra factor of $\delta^{AB}$, demanding orthogonality in colour space.

Finally, let us consider gravitons. We find using the bulk to boundary propagator (6.46),

$$\begin{aligned}\langle\langle 0|T_{\mu\nu}(p_1)T_{\rho\sigma}(p_2)|0\rangle\rangle &= \lim_{z\to 0}\frac{1}{z^3}\Pi_{\mu\nu\rho\sigma}(p_1)\big(\sin(|p_1|z) - |p_1|z\cos(|p_1|z)\big)\theta(-p_1^2)\theta(-p_1^0)\\ &= \frac{\Pi_{\mu\nu\rho\sigma}(p_1)}{6}|p_1|^3\theta(-p_1^2)\theta(-p_1^0),\\ \langle\langle 0|T_{\mu\nu}(p_2)T_{\rho\sigma}(p_1)|0\rangle\rangle &= \frac{\Pi_{\mu\nu\rho\sigma}(p_1)}{6}|p_1|^3\theta(-p_1^2)\theta(p_1^0),\end{aligned} \tag{6.54}$$

as desired.

### 6.3.2 Three point functions

Moving on to three points, we begin with three illustrative examples in detail. Generic scalars, scalar QED and Yang-Mills. We then provide the results for more examples including Wightman functions involving gravitons. To evaluate some of the integrals, the following nice integral is useful (see 6.578.2 in [279]),

$$\begin{aligned}&\int_0^\infty dz\, z^{\rho-1}J_\lambda(az)J_\mu(bz)K_\nu(cz)\\ &= \frac{2^{\rho-2}a^\lambda b^\mu c^{-\rho-\lambda-\mu}}{\Gamma(\lambda+1)\Gamma(\mu+1)}\Gamma(\frac{\rho+\lambda+\mu-\nu}{2})\Gamma(\frac{\rho+\lambda+\mu+\nu}{2})F_4(\frac{\rho+\lambda+\mu-\nu}{2},\frac{\rho+\lambda+\mu+\nu}{2};\lambda+1,\mu+1;-\frac{a^2}{c^2},-\frac{b^2}{c^2}),\end{aligned} \tag{6.55}$$

which also requires $\rho+\lambda+\mu-\nu > 0, c > 0$. $F_4(\alpha,\beta;\gamma,\gamma';x,y)$ is the generalized hypergeometric Appell $F_4$ function. For $\sqrt{|x|}+\sqrt{|y|} < 1$, it admits the series expansion,

$$F_4(\alpha,\beta;\gamma,\gamma';x,y) = \sum_{m,n=0}^{\infty}\frac{(\alpha)_{n+m}(\beta)_{n+m}}{(\gamma)_m(\gamma')_n m!n!}x^m y^n, \tag{6.56}$$

where $(\alpha)_m$ denotes the rising Pochhammer symbol.

We perform all computations using the equation of motion as well as analytic continuation when the middle operator has spacelike momenta which also serves to verify our results.

#### Scalars

Let us calculate the three point function of arbitrary scalar operators dual to bulk fields with any mass. We do so in the two ways we discussed previously. First, via the bulk equation of motion and second, via analytic continuation.

### Via the Equation of Motion

The relevant action reads,

$$S=\sum_{i=1}^{3}\int dzd^3x\Big(-\frac{1}{2}(\partial_z\phi_i)^2-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i-\frac{(m_i^2+2)}{2z^2}\phi_i^2\Big)-\lambda\int\frac{dz}{z}d^3x\,\phi_1\phi_2\phi_3, \tag{6.57}$$

with $m_i^2=\Delta_i(\Delta_i-3)$. The equations of motion are given by,

$$\begin{aligned}
(\partial_z^2+\Box-\frac{(m_1^2+2)}{z^2})\phi_1(z,x)&=\frac{\lambda}{z}\phi_2(z,x)\phi_3(z,x),\\
(\partial_z^2+\Box-\frac{(m_2^2+2)}{z^2})\phi_2(z,x)&=\frac{\lambda}{z}\phi_1(z,x)\phi_3(z,x),\\
(\partial_z^2+\Box-\frac{(m_3^2+2)}{z^2})\phi_3(z,x)&=\frac{\lambda}{z}\phi_1(z,x)\phi_2(z,x).
\end{aligned} \tag{6.58}$$

We solve these equations perturbatively in the coupling $\lambda$ via the expansions,

$$\phi_i(z,x)=\sum_{n=0}^{\infty}\lambda^n\phi_i^{(n)}(z,x). \tag{6.59}$$

For the tree level three point function only the zeroth order and $O(\lambda)$ corrections contribute. Solving the equations of motion (6.58) at this order we get,

$$\begin{aligned}
\phi_i^{(1)}(z_i,x_i)&=i\int_0^\infty\frac{dz}{z}\int d^3x\mathcal{G}(z_i,z,x_i-x)\phi_j^{(0)}(z,x)\phi_k^{(0)}(z,x)\\
\implies\ \phi_i(z_i,p_i)&=i\int_0^\infty\frac{dz}{z}\mathcal{G}(z_i,z,p_i)(\phi_j^{(0)}\phi_k^{(0)})(z,p_i),
\end{aligned} \tag{6.60}$$

where $i\neq j\neq k\in\{1,2,3\}$ and $\mathcal{G}$ is the EOM inverter propagator defined in (6.17) with its explicit expression given in table 6.1. Further, the free theory composite operators are defined by (with an implicit normal ordering),

$$(\phi_i^{(0)}\phi_j^{(0)})(z,p_k)=\int\frac{d^3l}{(2\pi)^3}\,\phi_i^{(0)}(z,l)\phi_j^{(0)}(z,p_k-l). \tag{6.61}$$

Thus, we obtain for the boundary three point function the following expression:

$$\begin{aligned}
&\langle 0|O_{\Delta_1}(p_1)O_{\Delta_2}(p_2)O_{\Delta_3}(p_3)|0\rangle_{O(\lambda)}=\lim_{z_1,z_2,z_3\to0}z_1^{1-\Delta_1}z_2^{1-\Delta_2}z_3^{1-\Delta_3}\langle 0|\phi_1(z_1,p_1)\phi_2(z_2,p_2)\phi_3(z_3,p_3)|0\rangle_{O(\lambda)}\\
&=\lambda\lim_{z_1,z_2,z_3\to0}z_1^{1-\Delta_1}z_2^{1-\Delta_2}z_3^{1-\Delta_3}\Big(\langle 0|\phi_1^{(1)}(z_1,p_1)\phi_2^{(0)}(z_2,p_2)\phi_3^{(0)}(z_3,p_3)|0\rangle+\langle 0|\phi_1^{(0)}(z_1,p_1)\phi_2^{(1)}(z_2,p_2)\phi_3^{(0)}(z_3,p_3)|0\rangle\\
&\qquad+\langle 0|\phi_1^{(0)}(z_1,p_1)\phi_2^{(0)}(z_2,p_2)\phi_3^{(1)}(z_3,p_3)|0\rangle\Big)\\
&=i\lambda\lim_{z_1,z_2,z_3\to0}z_1^{1-\Delta_1}z_2^{1-\Delta_2}z_3^{1-\Delta_3}\int_0^\infty\frac{dz}{z}\Big(\mathcal{G}_{\Delta_1}(z,z_1,p_1)\langle 0|(\phi_2^{(0)}\phi_3^{(0)})(z,p_1)\phi_2^{(0)}(z_2,p_2)\phi_3^{(0)}(z_3,p_3)|0\rangle\\
&\qquad+\mathcal{G}_{\Delta_2}(z,z_2,p_2)\langle 0|\phi_1^{(0)}(z_1,p_1)(\phi_1^{(0)}\phi_3^{(0)})(z,p_2)\phi_3^{(0)}(z_3,p_3)|0\rangle\\
&\qquad+\mathcal{G}_{\Delta_3}(z,z_3,p_3)\langle 0|\phi_1^{(0)}(z_1,p_1)\phi_2^{(0)}(z_2,p_2)(\phi_1^{(0)}\phi_2^{(0)})(z,p_3)|0\rangle\Big),
\end{aligned} \tag{6.62}$$

Performing the Wick contractions we obtain (stripping off the momentum conserving delta function),

$$i \lim_{z_1,z_2,z_3\to 0} z_1^{1-\Delta_1} z_2^{1-\Delta_2} z_3^{1-\Delta_3} \int \frac{dz}{z}\Big(\mathcal{G}_{\Delta_1}(z,z_1,p_1)W_{\Delta_2,+}(z,z_2,-p_2)W_{\Delta_3,+}(z,z_3,-p_3)$$
$$+ W_{\Delta_1,-}(z,z_1,-p_1)\mathcal{G}_{\Delta_2}(z,z_2,p_2)W_{\Delta_3,+}(z,z_3,-p_3) + W_{\Delta_1,-}(z,z_1,-p_1)W_{\Delta_2,-}(z,z_2,-p_2)\mathcal{G}_{\Delta_3}(z,z_3,p_3)\Big)$$
$$= i\lambda \int_0^\infty \frac{dz}{z}\Big(\mathcal{G}_{\Delta_1}(z,p_1)W_{\Delta_2,+}(z,-p_2)W_{\Delta_3,+}(z,-p_3) + W_{\Delta_1,-}(z,-p_1)\mathcal{G}_{\Delta_2}(z,p_2)W_{\Delta_3,+}(z,-p_3)$$
$$+ W_{\Delta_1,-}(z,-p_1)W_{\Delta_2,-}(z,-p_2)\mathcal{G}_{\Delta_3}(z,p_3)\Big). \tag{6.63}$$

Before plugging in the explicit forms of the propagators, let us note a few important points. In the first term of the above equation, we have Wightman plus propagators with momentum $-p_2^\mu$ and $-p_3^\mu$ which imply that $p_2^2 < 0, p_3^2 < 0, p_2^0 > 0, p_3^0 > 0$. By momentum conservation this implies that $p_1^0 < 0$ as well as $p_1^2 < 0$. This implies that only the time-like part of $\mathcal{G}_{\Delta_1}(z,p_1)$ with negative energy contributes to this correlator. Similarly, in the third term, we find that $p_1^2 < 0, p_2^2 < 0, p_1^0 < 0, p_2^0 < 0$ implies that $p_3^2 < 0, p_3^0 > 0$ which tells us that only the time-like part of $\mathcal{G}_{\Delta_3}(z,p_3)$ with positive energy contributes. For the second term, which involves $\mathcal{G}_{\Delta_2}(z,p_2)$, we can make no such kinematic statement and thus both its time-like (positive and negative energy components) and space-like contributions are present. Using these facts we obtain,

$$\langle\langle 0|O_{\Delta_1}(p_1)O_{\Delta_2}(p_2)O_{\Delta_3}(p_3)|0\rangle\rangle$$
$$= \frac{\lambda\sqrt{\pi}}{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)} 2^{\frac{13}{2}-\Delta_1-\Delta_2-\Delta_3}\theta(-p_1^2)\theta(-p_3^2)\theta(-p_1^0)\theta(p_3^0)\mathcal{A}_{\Delta_1,\Delta_2,\Delta_3}, \tag{6.64}$$

with,

$$\mathcal{A}_{\Delta_1,\Delta_2,\Delta_3} = |p_1|^{\nu_1}|p_2|^{\nu_2}|p_3|^{\nu_3} \int_0^\infty dz\sqrt{z}\Big(2\,\theta(p_2^2)J_{\nu_1}(z|p_1|)K_{\nu_2}(z|p_2|)J_{\nu_3}(z|p_3|)$$
$$- \pi\theta(-p_2^2)\theta(-p_2^0)J_{\nu_1}(z|p_1|)\Big(J_{\nu_2}(z|p_2|)Y_{\nu_3}(z|p_3|) + Y_{\nu_2}(z|p_2|)J_{\nu_3}(z|p_3|)\Big)$$
$$- \pi\theta(-p_2^2)\theta(p_2^0)\Big(J_{\nu_1}(z|p_1|)Y_{\nu_2}(z|p_2|) + Y_{\nu_1}(z|p_1|)J_{\nu_2}(z|p_2|)\Big)J_{\nu_3}(z|p_3|), \tag{6.65}$$

and $\nu_i = \Delta_i - \frac{3}{2}$. These integrals can be evaluated in terms of Appell functions using the formulae found in GR such as (6.55) if desired.

### Via analytic continuation

Let us now compare our answer (6.65) with the results of [34] where the authors obtained the CFT three point function of generic scalar operators via analytic continuation from

Euclidean space. The Euclidean correlator is given by,

$$\langle\langle O_{\Delta_1}(p_1)O_{\Delta_2}(p_2)O_{\Delta_3}(p_3)\rangle\rangle = c_{123}\frac{2^{\frac{17}{2}-(\Delta_1+\Delta_2+\Delta_3)}\pi^3}{\Gamma(\frac{\Delta_1+\Delta_2-\Delta_3}{2})\Gamma(\frac{\Delta_1-\Delta_2+\Delta_3}{2})\Gamma(\frac{-\Delta_1+\Delta_2+\Delta_3}{2})\Gamma(\frac{\Delta_1+\Delta_2+\Delta_3-3}{2})}$$
$$\times p_1^{\nu_1}p_2^{\nu_2}p_3^{\nu_3}\int dz\sqrt{z}K_{\nu_1}(zp_1)K_{\nu_2}(zp_2)K_{\nu_3}(zp_3). \tag{6.66}$$

One then translates the Wightman $i\epsilon$ prescription in position space (2.87) to momentum space using the Fourier transform and carefully analytically continue the result taking care of the branch cuts in the momentum space correlator. Their result matches exactly with ours (6.64) with our coefficient $g$ being determined in terms of theirs through the relation,

$$g = 2c_{123}\pi^{\frac{9}{2}}\frac{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)}{\Gamma(\frac{\Delta_1+\Delta_2-\Delta_3}{2})\Gamma(\frac{\Delta_1-\Delta_2+\Delta_3}{2})\Gamma(\frac{-\Delta_1+\Delta_2+\Delta_3}{2})\Gamma(\frac{\Delta_1+\Delta_2+\Delta_3-3}{2})}. \tag{6.67}$$

This comparision serves as a rigorous check on our method as well as highlight its utility. In contrast to performing involved analytic continuations from the Euclidean CFT correlator to obtain the Wightman function, we are able to directly calculate the real-time correlator and arrive at the correct answer quite easily.

**Photon-Scalar-Scalar**

We move on to an example involving a photon in the scalar QED theory. Again, we perform the computation in the two distinct ways.

**Via the Equation of Motion**

We consider the scalar QED action in the axial gauge $A_z = 0$.

$$S_{\text{scalar-QED}} = -\int dzd^3x\Big((\partial_z\phi)(\partial_z\phi^*) + \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi^* + \frac{(m^2+2)}{z^2}\phi^*\phi + \frac{1}{2}(\partial_z A_\mu)^2 + \frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu - \frac{1}{2}\partial_\mu A_\nu\partial^\nu A^\mu\Big)$$
$$+\int dzd^3x\Big(-ieA_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) + e^2A_\mu A^\mu\phi\phi^*\Big). \tag{6.68}$$

The first line of the above equation represents the quadratic portion of the action while the second line encodes the interactions. Note that there is no $z$ dependence in the interaction terms which is a consequence of massless scalar QED being classically conformally invariant. The equations of motion are,

$$(\partial_z^2 + \Box - \frac{(m^2+2)}{z^2})\phi = ieA^\mu\partial_\mu\phi - e^2A_\mu A^\mu\phi,$$
$$(\partial_z^2 + \Box - \frac{(m^2+2)}{z^2})\phi^* = -ieA^\mu\partial_\mu\phi^* - e^2A_\mu A^\mu\phi^*,$$
$$(\partial_z^2 + \Box)A_\mu = ie(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*) - 2e^2A_\mu\phi^*\phi. \tag{6.69}$$

We expand all the fields perturbatively in $e$ as follows:

$$\phi = \sum_{n=0}^{\infty} e^n \phi^{(n)}, \phi^* = \sum_{n=0}^{\infty} e^n \phi^{*(n)}, A_\mu = \sum_{n=0}^{\infty} e^n A_\mu^{(n)}. \tag{6.70}$$

We are interested in the $O(e)$ tree level contribution to the boundary three point function,

$$\langle 0|J^\mu(p_1) O_\Delta(p_2) O_\Delta^*(p_3)|0\rangle_{O(e)} = \lim_{z_1,z_2,z_3\to 0} \frac{(z_2 z_3)^{1-\Delta}}{z_1} \Big( \langle 0|A^{\mu(1)}(z_1,p_1)\phi^{(0)}(z_2,p_2)\phi^{*(0)}(z_3,p_3)|0\rangle$$
$$\langle 0|A^{\mu(0)}(z_1,p_1)\phi^{(1)}(z_2,p_2)\phi^{*(0)}(z_3,p_3)|0\rangle + \langle 0|A^{\mu(0)}(z_1,p_1)\phi^{(0)}(z_2,p_2)\phi^{*(1)}(z_3,p_3)|0\rangle \Big). \tag{6.71}$$

For concreteness, we focus on the case where $p_2^2 > 0$ although one can do the analysis for timelike $p_{2\mu}$ similarly. Following the steps we did for the scalar three point function, we solve the equations of motion at first order to obtain the analog of (6.60) ultimately leading to (6.63). From there we see the only contributions with $p_{2\mu}$ space-like, occur due to the EOM inverter propagator $\mathcal{G}$. Any term without it will vanish since it contains only Wightman propagators which have support only for time-like momenta. Both these facts can be seen from table 6.1. The same is true for this spinning three point function or any other one for that matter. Thus, only one term contributes to this correlator which is where the middle operator $\phi$ receives the $O(e)$ correction and thus results in $\mathcal{G}_\Delta(z,p_2)$ in the expression. Thus we have,

$$\begin{aligned}
&\langle 0|J^\mu(p_1) O_\Delta(p_2) O_\Delta^*(p_3|0\rangle \theta(p_2^2) \\
&= ie\theta(p_2^2) \lim_{z_1,z_2,z_3\to 0} \frac{(z_2z_3)^{1-\Delta}}{z_1} \int_0^\infty dz \int \frac{d^3 l}{(2\pi)^3} \mathcal{G}_\Delta(z,z_2,p_2) \langle 0|A^\mu(z_1,p_1)(A^\nu(z,p_2-l) l_\nu \phi(z,l))\phi^*(z_3,p_3)|0\rangle \\
&= ie\theta(p_2^2)\, \pi^{\mu\nu}(p_1)(p_1+p_2)_\nu \int_0^\infty dz\, \mathcal{G}_\Delta(z,p_2) W_{\Delta=2,-}(z,-p_1) W_{\Delta,+}(z,-p_3) \\
&= \frac{e4^{3-\Delta}}{\Gamma(\Delta)^2} \theta(-p_1^2)\theta(p_2^2)\theta(-p_3^2)\theta(-p_1^0)\theta(p_3^0) \mathcal{A}_{\gamma\phi_\Delta\phi_\Delta^*},
\end{aligned} \tag{6.72}$$

where,

$$\begin{aligned}
\mathcal{A}_{\gamma\phi_\Delta\phi_\Delta^*} &= \pi^{\mu\nu}(p_1) p_{2\nu} |p_2|^{\Delta-\frac{3}{2}} |p_3|^{\Delta-\frac{3}{2}} \int_0^\infty dz\, z \sin(|p_1|z) K_{\Delta-\frac{3}{2}}(|p_2|z) J_{\Delta-\frac{3}{2}}(|p_3|z) \\
&= \pi^{\mu\nu}(p_1) p_{2\nu} \frac{\sqrt{\pi}\Gamma(\Delta)}{\Gamma(\Delta-\frac{1}{2})} \frac{|p_1||p_3|^{2\Delta-3}}{|p_2|^3} F_4(\frac{3}{2},\Delta;\frac{3}{2},\Delta-\frac{1}{2};-\frac{|p_1|^2}{|p_2|^2},-\frac{|p_3|^2}{|p_2|^2}).
\end{aligned} \tag{6.73}$$

For example , let us take $\Delta = 1$ and $\Delta = 2$. Explicitly evaluating the Appell $F_4$ for these arguments using (6.56) yields,

$$\begin{aligned}
\mathcal{A}_{\gamma\phi_{\Delta=1}\phi_{\Delta=1}^*} &= \frac{|p_1|(|p_1|^2+|p_2|^2-|p_3|^2)}{|p_2||p_3|(|p_1|^4+2|p_1|^2(|p_2|^2-|p_3|^2)+(|p_2|^2+|p_3|^2)^2)}, \\
\mathcal{A}_{\gamma\phi_{\Delta=2}\phi_{\Delta=2}^*} &= \frac{2|p_1||p_2||p_3|}{|p_1|^4+2|p_1|^2(|p_2|^2-|p_3|^2)+(|p_2|^2+|p_3|^2)^2}.
\end{aligned} \tag{6.74}$$

Let us write the results in terms of $p = \sqrt{p^2}$ rather than $|p| = \sqrt{|p|^2}$. First of all, since $p_2^2 > 0$, $|p_2| = p_2$. However, since $p_1^2 < 0$ and $p_3^2 < 0$ we need to be a bit more careful taking into account the Wightman $i\epsilon$ prescription (2.99). Using the facts that $p_1^0 < 0, p_3^0 > 0$, we have $|p_1| = ip_1, |p_3| = -ip_3$. This results in,

$$\begin{aligned}
\mathcal{A}_{\gamma\phi_{\Delta=1}\phi^*_{\Delta=1}} &= \frac{1}{4p_2p_3}\left(\frac{1}{E} - \frac{1}{E-2p_1} + \frac{1}{E-2p_2} + \frac{1}{E-2p_3}\right),\\
\mathcal{A}_{\gamma\phi_{\Delta=2}\phi^*_{\Delta=2}} &= \frac{1}{4}\left(\frac{1}{E} - \frac{1}{E-2p_1} - \frac{1}{E-2p_2} - \frac{1}{E-2p_3}\right),
\end{aligned} \tag{6.75}$$

where $E = p_1 + p_2 + p_3$.

### Via analytic continuation

Let us start with the Euclidean correlator. Computing the Witten diagram for this interaction results in,

$$\langle\langle J^\mu(p_1)O_\Delta(p_2)O_\Delta(p_3)\rangle\rangle \propto \pi^{\mu\nu}(p_1)p_{2\nu}p_2^{\Delta-\frac{3}{2}}p_3^{\Delta-\frac{3}{2}}\int_0^\infty dz\, z\, e^{-p_1 z}K_{\Delta-\frac{3}{2}}(p_2 z)K_{\Delta-\frac{3}{2}}(p_3 z). \tag{6.76}$$

We want to Wick rotate this result to a Wightman function with $p_1^2 < 0, p_1^0 < 0, p_3^2 < 0, p_3^0 > 0$ and $p_2^2 > 0$. For such a configuration, we need to take first take discontinuities with respect to the squares of the momenta we want to be time-like.

$$\text{Disc}_{p_1^2}\text{Disc}^2_{p_3}\langle\langle J^\mu(p_1)O_\Delta(p_2)O_\Delta(p_3)\rangle\rangle \propto \pi^{\mu\nu}(p_1)p_{2\nu}p_2^{\Delta-\frac{3}{2}}p_3^{\Delta-\frac{3}{2}}\int_0^\infty dz\, z\,(-2\sinh(p_1 z))K_{\Delta-\frac{3}{2}}(p_2 z)((-1)^{\Delta-1}\pi I_{\Delta-\frac{3}{2}}(p_3 z)) \tag{6.77}$$

Wick rotating following the Wightman $i\epsilon$ prescription (2.99) results in,

$$2\pi\pi^{\mu\nu}(p_1)p_{2\nu}|p_2|^{\Delta-\frac{3}{2}}|p_3|^{\Delta-\frac{3}{2}}\theta(-p_1^2)\theta(-p_1^0)\theta(p_2^2)\theta(-p_3^2)\theta(p_3^0)\int_0^\infty z\sin(|p_1|z)K_{\Delta-\frac{3}{2}}(|p_2|z)J_{\Delta-\frac{3}{2}}(|p_3|z), \tag{6.78}$$

which is the same result as we obtained earlier via the EOM viz (6.72), (6.73).

### Yang-Mills theory

We now move on to the gluon three point function in Yang-Mills theory with the gauge group $SU(N)$.

### Via the Equation of Motion

The Yang-Mills action in the axial gauge is given by,

$$\begin{aligned}
S = -\frac{1}{2}\int dz d^3x\Big(&(\partial_z A^A_\mu)^2 + (\partial_\nu A^A_\mu)^2 - \partial_\nu A^A_\mu \partial^\mu A^{\nu A} + g f^{ABC}(\partial_\mu A^A_\nu - \partial_\nu A^A_\mu)A^{\mu B}A^{\nu C}\\
&+ g^2 f^{ABC}f^{ADE}A^B_\mu A^C_\nu A^{\mu D}A^{\nu E}\Big).
\end{aligned} \tag{6.79}$$

The equation of motion for $A^{\alpha A}$ is,

$$(\partial_z^2 + \Box)A^{\alpha A} = -gf^{ABC}(2A^{\nu B}\partial_\nu A^{\alpha C} + A^{\nu C}\partial^\alpha A^B_\nu) + g^2 f^{ABC} f^{CDE} A^E_\mu A^{\mu B} A^{\alpha D}. \tag{6.80}$$

we expand the gauge field in $g$ as follows:

$$A^{\mu A} = \sum_{n=0}^{\infty} g^n A^{\mu A(n)}. \tag{6.81}$$

Let us compute the three point function of the non-abelian conserved currents dual to this gauge field. Again, we focus on the case where the middle operator has space-like momenta. The result of the calculation after contracting with transverse polarization vectors is,

$$\begin{aligned}
&\epsilon_{1\mu}\epsilon_{2\nu}\epsilon_{3\rho}\langle\langle 0|J^{\mu A}(p_1)J^{\mu_2 B}(p_2)J^{\rho C}(p_3)|0\rangle\rangle\theta(p_2^2) \\
&= \frac{ig}{2}\theta(p_2^2)\theta(-p_1^2)\theta(-p_3^2)\theta(-p_1^0)\theta(p_3^0)f^{ABC}V_{3,YM}\int_0^\infty dz\ \sin(|p_1|z)e^{-|p_2|z}\sin(|p_3|z) \\
&= \frac{ig}{2}\theta(p_2^2)\theta(-p_1^2)\theta(-p_3^2)\theta(-p_1^0)\theta(p_3^0)f^{ABC}V_{3,YM}\left(\frac{1}{E} - \frac{1}{E-2p_1} - \frac{1}{E-2p_2} - \frac{1}{E-2p_3}\right) \\
&= 4ig\theta(p_2^2)\theta(-p_1^2)\theta(-p_3^2)\theta(-p_1^0)\theta(p_3^0)f^{ABC}V_{3,YM}\frac{p_1p_2p_3}{E(E-2p_1)(E-2p_2)(E-2p_3)},
\end{aligned} \tag{6.82}$$

where the Yang-Mills three point factor is,

$$V_{3,YM} = \Big((\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot p_1) + (\epsilon_2\cdot\epsilon_3)(\epsilon_1\cdot p_2) + (\epsilon_3\cdot\epsilon_1)(\epsilon_2\cdot p_3)\Big), \tag{6.83}$$

which is also the flat space scattering amplitude for three gluons in YM theory.

**Via Analytic Continuation**

Consider the Euclidean AdS Yang-Mills three point correlator. It is given by,

$$\epsilon_{1\mu}\epsilon_{2\nu}\epsilon_{3\rho}\langle\langle J^{A\mu}(p_1)J^{B\nu}(p_2)J^{C\rho}(p_3)\rangle\rangle = \frac{ig}{2}V_3^{YM}f^{ABC}\frac{1}{E}. \tag{6.84}$$

To obtain the Wightman function with $p1$ and $p3$ timelike and past and future pointing and $p2$ spacelike, we take discontinuities with respect to $p_3^2$ and $p_1^2$. This yields,

$$4igV_3^{YM}f^{ABC}\frac{p_1p_2p_3}{E(E-2p_1)(E-2p_3)(E-2p_4)}. \tag{6.85}$$

Inserting the $\theta$ functions and taking care of the Wightman $i\epsilon$ prescription, we obtain the same answer as above (6.82).

**Examples involving gravitons**

To conclude our section on three point functions, we present some results for Wightman functions involving gravitons. We also focus on the case $p_2^2 > 0$ for simplicity.

**Minimal coupling of scalars to gravitons: $\langle T O_\Delta O_\Delta \rangle$**

Consider the minimal coupling of scalar fields to gravity via their kinetic term. Calculating the resulting Wightman function with $p_2^2 > 0$ yields,

$$\langle\langle 0|T(p_1,\epsilon_1)O_\Delta(p_2)O_\Delta(p_3)|0\rangle\rangle\theta(p_2^2) = \kappa(\epsilon_1\cdot p_2)^2\theta(p_2^2)\int_0^\infty dz W_{\Delta=3,-}(z,-p_1)\mathcal{G}_\Delta(z,p_2)W_{\Delta,+}(z,-p_3)$$
$$= \kappa\theta(-p_1^2)\theta(-p_1^0)\theta(p_2^2)\theta(-p_3^2)\theta(p_3^0)(\epsilon_1\cdot p_2)^2\frac{4^{2-\Delta}}{\Gamma(\Delta)^2}|p_2|^\nu|p_3|^\nu\int_0^\infty(\sin(z|p_1|)-z|p_1|\cos(z|p_1|))K_\nu(z|p_2|)J_\nu(|p_3|z)dz,$$
$$= \kappa\theta(-p_1^2)\theta(-p_1^0)\theta(p_2^2)\theta(-p_3^2)\theta(p_3^0)(\epsilon_1\cdot p_2)^2\frac{2\sqrt{2}|p_1|^3|p_3|^{2\nu}}{|p_2|^5\Gamma(\nu+1)}F_4(\frac{5}{2},\frac{5}{2}+\nu;1+\nu,\frac{5}{2};-\frac{|p_3|^2}{|p_2|^2},-\frac{|p_1|^2}{|p_2|^2}). \tag{6.86}$$

where $\nu=\Delta-\frac{3}{2}$. For example, the $\Delta=1$ result is given by,

$$\kappa\theta(-p_1^2)\theta(-p_1^0)\theta(p_2^2)\theta(-p_3^2)\theta(p_3^0)(\epsilon_1\cdot p_2)^2\frac{8p_1^3(p_1^4+p_2^4+6p_2^2p_3^2+p_3^4-2p_1^2(p_2^2+p_3^2))}{p_2p_3E^2(E-2p_1)^2(E-2p_2)^2(E-2p_3)^2}. \tag{6.87}$$

One can also easily confirm that this expression matches with the Wick rotation of the Euclidean correlator following the discontinuity procedure (2.101).

**Minimal coupling of photons to gravity: $\langle TJJ\rangle$**

We now consider the minimal coupling of photons to gravity. Following similar steps to the previous examples we obtain,

$$\langle 0|T(p_1,\epsilon_1)J(p_2,\epsilon_2)J(p_3,\epsilon_3)|0\rangle\theta(p_2^2) = \theta(-p_1^2)\theta(-p_1^0)\theta(p_2^2)\theta(-p_3^2)\theta(p_3^0)\mathcal{A}_{3,h\gamma\gamma}, \tag{6.88}$$

where,

$$\mathcal{A}_{3,h\gamma\gamma} = \frac{16p_1^3p_2p_3}{(p_1^4+(p_2^2-p_3^2)^2-2p_1^2(p_2^2+p_3^2))^2}\Big(-(\epsilon_1\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_3)(p_1^4+(p_2^2-p_3^2)^2-2p_1^2(p_2^2+p_3^2))$$
$$+4(p_1^2-p_2^2-p_3^2)\big((\epsilon_1\cdot p_2)^2(\epsilon_2\cdot\epsilon_3)+(\epsilon_1\cdot p_2)(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot p_1)-(\epsilon_1\cdot p_2)(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot p_1)\big)\Big). \tag{6.89}$$

**Einstein Gravity three point function: $\langle TTT\rangle$**

We consider the $O(\kappa)$ contribution to the three point function where the middle operator is spacelike. Following the same steps as in the previous examples and performing some algebra results in,

$$\langle\langle 0|T(p_1,\epsilon_1)T(p_2,\epsilon_2)T(p_3,\epsilon_3)|0\rangle\rangle\theta(p_2^2) = \kappa\theta(-p_1^2)\theta(p_2^2)\theta(-p_3^2)\theta(-p_1^0)\theta(p_3^0)\mathcal{A}_{3,\mathrm{GR}},$$
$$\mathcal{A}_{3,\mathrm{GR}} = \sqrt{\frac{\pi}{2}}V_{3,GR}(\epsilon_1,\epsilon_2,\epsilon_3)|p_1|^{\frac{3}{2}}|p_2|^{\frac{3}{2}}|p_3|^{\frac{3}{2}}\int_0^\infty dz\, z^{\frac{5}{2}}\,J_{\frac{3}{2}}(z|p_1|)K_{\frac{3}{2}}(z|p_2|)J_{\frac{3}{2}}(z|p_3|)$$
$$= \frac{16p_1^3p_2^3p_3^3}{E^2(E-2p_1)^2(E-2p_2)^2(E-2p_3)^2}\Big((\epsilon_1\cdot p_2)(\epsilon_2\cdot\epsilon_3)+(\epsilon_2\cdot p_3)(\epsilon_3\cdot\epsilon_1)+(\epsilon_3\cdot p_1)(\epsilon_1\cdot\epsilon_2)\Big)^2, \tag{6.90}$$

where we identified the Einstein gravity three point vertex as the square of its Yang-Mills counterpart (Double copy!). Wick rotation from the Euclidean result yields the same answer confirming the result.

### 6.3.3 Four point functions

In this subsection and the next, we proceed to the more complicated case of four point Wightman functions. We calculate explicitly examples of scalar, photon, gluon and graviton Wightman functions involving a variety of contact and exchange interactions. The general expression for the Wightman functions are quite complicated since the spectral condition (2.91) allows for a lot more possibilities of momenta configurations. However, we shall find that when we take the middle two operators to have space-like momenta, it leads to dramatic simplifications. For scalars, we will do the general kinematics computation to illustrate this in this subsection and then for the spinning case in the next subsection, we present only the results in these special (but as we shall see, interesting!) kinematics.

**Quartic contact interactions**

We begin with the simple example of four distinct scalar fields interacting with a quartic contact interaction. As usual, we shall obtain this result both by the EOM as well as Wick rotation from Euclidean space for various kinematic regimes when possible.

**Via Equation of Motion**

The action we work with is,

$$S = -\frac{1}{2}\sum_{i=1}^{4}\int dzd^3x\big((\partial_z\phi_i)^2 + (\partial_\mu\phi_i)^2 + \frac{(m_i^2+2)}{z^2}\phi_i^2\big) - \lambda\int dzd^3x\phi_1\phi_2\phi_3\phi_4, \tag{6.91}$$

with $m_i^2 = \Delta_i(\Delta_i - 3)$. The equations of motion read,

$$(\partial_z^2 + \Box - \frac{(m_i^2+2)}{z^2})\phi_i = \lambda\phi_j\phi_k\phi_l\ , i \neq j \neq k \neq l \in \{1,2,3,4\}. \tag{6.92}$$

Let us expand every field in $\lambda$ as follows:

$$\phi_i(z,x) = \sum_{n=0}^{\infty}\lambda^n\phi_i^{(n)}(z,x). \tag{6.93}$$

For the four point contact Wightman function we need the $O(\lambda)$ corrections to the fields. Inverting the above equation of motion at $O(\lambda)$ yields,

$$\begin{aligned}
&\phi_i^{(1)}(z_i,x_i) = i\int_0^\infty dz\int d^3x\mathcal{G}_{\Delta_i}(z_i,z,x_i-x)\phi_j^{(0)}(z,x)\phi_k^{(0)}(z,x)\phi_l^{(0)}(z,x)\\
&\implies \phi_i^{(1)}(z_i,p_i) = i\int_0^\infty dz\,\mathcal{G}_{\Delta_i}(z_i,z,p_i)\int\frac{d^3q_1d^3q_2}{(2\pi)^6}\phi_j^{(0)}(z,q_1)\phi_k^{(0)}(z,q_2)\phi_l^{(0)}(z,p_i-q_1-q_2),
\end{aligned} \tag{6.94}$$

with $i \neq j \neq k \neq l \in \{1,2,3,4\}$ and $\mathcal{G}$ is the EOM inverter propagator that can be found in table 6.1. We use this to compute,

$$\begin{aligned}
&\langle 0|O_{\Delta_1}(p_1)O_{\Delta_2}(p_2)O_{\Delta_3}(p_3)O_{\Delta_4}(p_4)|0\rangle_{O(\lambda)} = \lambda\Bigg(\prod_{i=1}^{4}\lim_{z_i\to 0} z_i^{1-\Delta_i}\Bigg)\Bigg(\langle\phi_1^{(1)}(z_1,p_1)\phi_2^{(0)}(z_2,p_2)\phi_3^{(0)}(z_3,p_3)\phi_4^{(0)}(z_4,p_4)|0\rangle \\
&+\langle\phi_1^{(0)}(z_1,p_1)\phi_2^{(1)}(z_2,p_2)\phi_3^{(0)}(z_3,p_3)\phi_4^{(0)}(z_4,p_4)|0\rangle+\langle\phi_1^{(0)}(z_1,p_1)\phi_2^{(0)}(z_2,p_2)\phi_3^{(1)}(z_3,p_3)\phi_4^{(0)}(z_4,p_4)|0\rangle \\
&+\langle\phi_1^{(0)}(z_1,p_1)\phi_2^{(0)}(z_2,p_2)\phi_3^{(0)}(z_3,p_3)\phi_4^{(1)}(z_4,p_4)|0\rangle\Bigg) \\
&=\lambda\Bigg(\int_0^\infty dz\,\mathcal{G}_{\Delta_1}(z,p_1)W_{\Delta_2,+}(z,-p_2)W_{\Delta_3,+}(z,-p_3)W_{\Delta_4,+}(z,-p_4) \\
&\quad+\int_0^\infty dz\,W_{\Delta_1,-}(z,-p_1)\mathcal{G}_{\Delta_2}(z,p_2)W_{\Delta_3,+}(z,-p_3)W_{\Delta_4,+}(z,-p_4) \\
&\quad+\int_0^\infty dz\,W_{\Delta_1,-}(z,-p_1)W_{\Delta_2,-}(z,-p_2)\mathcal{G}_{\Delta_3}(z,p_3)W_{\Delta_4,-}(z,-p_4) \\
&\quad+\int_0^\infty dz\,W_{\Delta_1,-}(z,-p_1)W_{\Delta_2,-}(z,-p_2)W_{\Delta_3,-}(z,-p_3)\mathcal{G}_{\Delta_4}(z,p_4)\Bigg).
\end{aligned} \tag{6.95}$$

Using the explicit forms of the propagators from table 6.1 results in,

$$\langle\langle 0|O_{\Delta_1}(p_1)O_{\Delta_2}(p_2)O_{\Delta_3}(p_3)O_{\Delta_4}(p_4)|0\rangle\rangle_{O(\lambda)} = \frac{\pi\lambda 2^{4-\nu_1-\nu_2-\nu_3-\nu_4}}{\prod_{i=1}^4\Gamma(\nu_i+\frac{3}{2})}\theta(-p_1^2)\theta(-p_1^0)\theta(-p_4^2)\theta(p_4^0)\mathcal{A}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}, \tag{6.96}$$

where[10],

$$\begin{aligned}
&\mathcal{A}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4} = |p_1|^{\nu_1}|p_2|^{\nu_2}|p_3|^{\nu_3}|p_4|^{\nu_4}\Bigg(\theta(p_2^2)\theta(-p_3^2)\theta(p_3^0)\int_0^\infty dz\,z^2 J_{\nu_1}(z|p_1|)K_{\nu_2}(z|p_2|)J_{\nu_3}(z|p_3|)J_{\nu_4}(z|p_4|) \\
&+\theta(p_3^2)\theta(-p_2^2)\theta(-p_2^0)\int_0^\infty dz\,z^2 J_{\nu_1}(z|p_1|)J_{\nu_2}(z|p_2|)K_{\nu_3}(z|p_3|)J_{\nu_4}(z|p_4|) \\
&-\frac{\pi}{2}\theta(-p_2^2)\theta(-p_3^2)\theta(-p_2^0)\theta(-p_3^0)\int_0^\infty dz\,z^2 J_{\nu_1}(z|p_1|)J_{\nu_2}(z|p_2|)\big(J_{\nu_3}(z|p_3|)Y_{\nu_4}(z|p_4|)+Y_{\nu_3}(z|p_3|)J_{\nu_4}(z|p_4|)\big) \\
&-\frac{\pi}{2}\theta(-p_2^2)\theta(-p_3^2)\theta(-p_2^0)\theta(p_3^0)\int_0^\infty dz\,z^2 J_{\nu_1}(z|p_1|)J_{\nu_4}(z|p_4|)\big(J_{\nu_2}(z|p_2|)Y_{\nu_3}(z|p_3|)+Y_{\nu_2}(z|p_2|)J_{\nu_3}(z|p_3|)\big) \\
&-\frac{\pi}{2}\theta(-p_2^2)\theta(-p_3^2)\theta(p_2^0)\theta(p_3^0)\int_0^\infty dz\,z^2 J_{\nu_3}(z|p_3|)J_{\nu_4}(z|p_4|)\big(J_{\nu_1}(z|p_1|)Y_{\nu_2}(z|p_2|)+Y_{\nu_1}(z|p_1|)J_{\nu_2}(z|p_2|)\big)\Bigg).
\end{aligned} \tag{6.97}$$

Note that in (6.96), $p_1^\mu$ and $p_4^\mu$ are time-like and have negative and positive energies respectively respecting the spectral condition (2.91) whereas we see from (6.97) that $p_2^\mu$ and $p_3^\mu$ can be either space-like or time-like[11]. Let us evaluate this expression for the

[10] Comparing this result with its three point counterpart (6.65) shows that it is a natural generalization and gives an interesting pattern as to how this result could potentially generalize to higher point contact interactions.

[11] Interestingly, there is no term with support when both $p_2^\mu$ and $p_3^\mu$ are space-like. Although this is clear mathematically, it would be interesting to see if there is a physical reason for this fact.

simple case of $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 1$ with $p_2^2 > 0$. The result is,

$$\mathcal{A}_{1,1,1,1}\theta(p_2^2) = \frac{1}{|p_1||p_2||p_3||p_4|}\left(\frac{|p_2|}{|p_2|^2 + (|p_1| + |p_3| - |p_4|)^2} + \frac{|p_2|}{|p_2|^2 + (|p_1| - |p_3| + |p_4|)^2}\right.$$
$$\left. + \frac{|p_2|}{|p_2|^2 + (-|p_1| + |p_3| + |p_4|)} + \frac{|p_2|}{|p_2|^2 + (|p_1| + |p_3| + |p_4|)^2}\right)$$
$$= \frac{1}{2p_1p_2p_3p_4}\left(\frac{1}{E} + \frac{1}{E - 2p_1} - \frac{1}{E - 2p_2} + \frac{1}{E - 2p_3} + \frac{1}{E - 2p_4} + \frac{1}{E - 2p_1 - 2p_3} + \frac{1}{E - 2p_1 - 2p_4} + \frac{1}{E - 2p_3 - 2p_4}\right), \tag{6.98}$$

where we have re-expressed the results in terms of the $p_i$ following the Wightman $i\epsilon$ prescription (2.99). We have also defined the four point total energy $E = p_1+p_2+p_3+p_4$. For another case, let us consider the same scaling dimensions but take all the momenta to be time-like with $p_2$ and $p_3$ also having negative energy.

$$\mathcal{A}_{1,1,1,1}\theta(-p_2^2)\theta(-p_3^2)\theta(-p_2^0)\theta(-p_3^0) = -\frac{i\pi}{2p_1p_2p_3p_4}\left(\frac{1}{E - 2p_2 - 2p_4} + \frac{1}{E - 2p_1 - 2p_4} + \frac{1}{E - 2p_4} - \frac{1}{E - 2p_3}\right). \tag{6.99}$$

One can similarly obtain the other cases by evaluating the integrals in (6.97).

### Via Analytic Continuation

To obtain the entire expression (6.97) via Wick rotation from Euclidean space would entail extending the careful analysis of [34] to four points and we do not attempt it here although it should be straightforward, but tedious to check. What we do is focus on the analytic continuation required to obtain the Wightman function in the two kinematic cases we discussed above. We also look at example when all operators have $\Delta = 1$ for simplicity though one can perform a similar analysis for generic $\Delta$.

In the first case, We want $p_1, p_3$ and $p_4$ to be time-like and $p_2$ spacelike. The Euclidean correlator found using standard Witten diagramatics is,

$$\langle\langle O_\Delta(p_1)O_\Delta(p_2)O_\Delta(p_3)O_\Delta(p_4)\rangle\rangle = \frac{1}{p_1p_2p_3p_4}\int_0^\infty dz\, e^{-(p_1+p_2+p_3+p_4)z} = \frac{1}{p_1p_2p_3p_4(p_1 + p_2 + p_3 + p_4} \tag{6.100}$$

In this case, taking its discontinuity with respect to the three time-like momenta results in,

$$\text{Disc}_{p_1^2}\text{Disc}_{p_3^2}\text{Disc}^2_{p_4}\left(\frac{1}{p_1p_2p_3p_4(p_1 + p_2 + p_3 + p_4)}\right)$$
$$= \frac{1}{2p_1p_2p_3p_4}\left(\frac{1}{E} + \frac{1}{E - 2p_1} - \frac{1}{E - 2p_2} + \frac{1}{E - 2p_3} + \frac{1}{E - 2p_4}\right.$$
$$\left. + \frac{1}{E - 2p_1 - 2p_3} + \frac{1}{E - 2p_1 - 2p_4} + \frac{1}{E - 2p_3 - 2p_4}\right), \tag{6.101}$$

which is exactly (6.98) thus confirming our result.

Next, let us Wick rotate to the configuration where all momenta are time-like with $p_2$ and $p_3$ past pointing. This is already a more complicated Wick rotation. We need to perform a combination of discontinuities to obtain the Wightman function:

$$\text{Disc}_{p_1^2}\text{Disc}_{p_2^2}\left(\text{Disc}_{p_3^2} - \text{Disc}_{p_4^2}\right)\left(\frac{1}{p_1p_2p_3p_4(p_1+p_2+p_3+p_4)}\right). \tag{6.102}$$

This results in the correct answer (6.99). Thus, we see that there is no obvious pattern on the required procedure to obtain Wightman functions from Euclidean correlators. This highlights the utility of the equation of motion method with the result (6.97) directly giving the full answer avoiding these non obvious Wick rotations.

### $\phi^3$ theory exchange

We proceed to the case of exchange interactions now. We consider a scalar field $\phi$ with scaling dimension $\Delta$ exchanging a scalar $\chi$ with scaling dimension $\Delta'$. The results are easily generalizable (albeit with a lot more algebra and more coupling constants to deal with) to the case with non-identical scalars.

### Via the Equation of Motion

The action is given by,

$$S = -\frac{1}{2}\int dzd^3x\big((\partial_z\phi)^2 + (\partial_\mu\phi)^2 + \frac{(m^2+2)}{z^2}\phi^2\big) - \frac{1}{2}\int dzd^3x\big((\partial_z\chi)^2 + (\partial_\mu\chi)^2 + \frac{(m_\chi^2+2)}{z^2}\chi^2\big)$$
$$- g\int \frac{dz}{z}d^3x\,\phi^2\chi. \tag{6.103}$$

The relevant equations of motion read,

$$\begin{aligned}(\partial_z^2 + \Box - \frac{(m^2+2)}{z^2})\phi(z,x) &= \frac{2g}{z}\phi(z,x)\chi(z,x),\\ (\partial_z^2 + \Box - \frac{(m_\chi^2+2)}{z^2})\chi(z,x) &= \frac{g}{z}\phi^2.\end{aligned} \tag{6.104}$$

We solve the above equations perturbatively by expanding the fields as follows:

$$\begin{aligned}\phi(z,x) &= \sum_{n=0}^{\infty} g^n\phi^{(n)}(z,x),\\ \chi(z,x) &= \sum_{n=0}^{\infty} g^n\chi^{(n)}(z,x).\end{aligned} \tag{6.105}$$

Substituting these expansions into the EOMs, we solve it perturbatively. For the correlator of interest to us, we require the expression at $O(g^2)$. The corrections that we shall require are,

$$\begin{aligned}\phi^{(1)}(z_1,p_1) &= 2i\int_0^\infty \frac{dz}{z}\int\frac{d^3l}{(2\pi)^3}\,\mathcal{G}_\Delta(z,z_1,p_1)\phi^{(0)}(z,l)\chi^{(0)}(z,p_1-l),\\ \chi^{(1)}(z_5,p) &= i\int_0^\infty \frac{dz}{z}\int\frac{d^3l}{(2\pi)^3}\,\mathcal{G}_{\Delta'}(z_5,z,p)\phi^{(0)}(z,l)\phi^{(0)}(z,p-l),\end{aligned} \tag{6.106}$$

and,

$$\phi^{(2)}(z_1, p_1) = 2i \int_0^\infty dz \int \frac{d^3l}{(2\pi)^3}\, \mathcal{G}_\Delta(z, z_1, p_1)\phi^{(0)}(z, l)\chi^{(1)}(z, p_1 - l)$$
$$= -2\int_0^\infty \frac{dz}{z} \int \frac{d^3l}{(2\pi)^3} \int_0^\infty \frac{dz'}{z'} \int \frac{d^3l'}{(2\pi)^3} \mathcal{G}_\Delta(z, z_1, p_1)\mathcal{G}_{\Delta'}(z, z', p_1 - l)\phi^{(0)}(z, l)\phi^{(0)}(z', l')\phi^{(0)}(z', p_1 - l - l'). \tag{6.107}$$

The four point function of interest is then given by a sum of ten terms:

$$\langle 0|O_\Delta(p_1)O_\Delta(p_2)O_\Delta(p_3)O_\Delta(p_4)|0\rangle_{O(g^2)}$$
$$= g^2\Big(\prod_{i=1}^{4}\lim_{z_i\to 0} z_i^{1-\Delta}\Big)\Big(\langle 0|\phi^{(0)}(z_1, p_1)\phi^{(1)}(z_2, p_2)\phi^{(1)}(z_3, p_3)\phi^{(0)}(z_4, p_4)\,|0\rangle$$
$$+ \langle 0|\phi^{(1)}(z_1, p_1)\phi^{(0)}(z_2, p_2)\phi^{(1)}(z_3, p_3)\phi^{(0)}(z_4, p_4)\,|0\rangle + \langle 0|\phi^{(1)}(z_1, p_1)\phi^{(0)}(z_2, p_2)\phi^{(0)}(z_3, p_3)\phi^{(1)}(z_4, p_4)\,|0\rangle$$
$$+ \langle 0|\phi^{(0)}(z_1, p_1)\phi^{(1)}(z_2, p_2)\phi^{(1)}(z_3, p_3)\phi^{(0)}(z_4, p_4)\,|0\rangle + \langle 0|\phi^{(0)}(z_1, p_1)\phi^{(1)}(z_2, p_2)\phi^{(0)}(z_3, p_3)\phi^{(1)}(z_4, p_4)\,|0\rangle$$
$$+ \langle 0|\phi^{(0)}(z_1, p_1)\phi^{(0)}(z_2, p_2)\phi^{(1)}(z_3, p_3)\phi^{(1)}(z_4, p_4)\,|0\rangle + \langle 0|\phi^{(2)}(z_1, p_1)\phi^{(0)}(z_2, p_2)\phi^{(0)}(z_3, p_3)\phi^{(0)}(z_4, p_4)\,|0\rangle$$
$$+ \langle 0|\phi^{(0)}(z_1, p_1)\phi^{(2)}(z_2, p_2)\phi^{(0)}(z_3, p_3)\phi^{(0)}(z_4, p_4)\,|0\rangle + \langle 0|\phi^{(0)}(z_1, p_1)\phi^{(0)}(z_2, p_2)\phi^{(2)}(z_3, p_3)\phi^{(0)}(z_4, p_4)\,|0\rangle$$
$$+ \langle 0|\phi^{(0)}(z_1, p_1)\phi^{(0)}(z_2, p_2)\phi^{(0)}(z_3, p_3)\phi^{(2)}(z_4, p_4)\,|0\rangle\Big). \tag{6.108}$$

Evaluating these correlators using the field corrections results in the following long expression for the boundary correlator:

$$-4g^2\theta(-p_1^2)\theta(-p_1^0)\theta(-p_4^2)\theta(p_4^0)\theta(-s^2)\theta(-s^0)\Big(\int_0^\infty \frac{dz}{z}\int_0^\infty \frac{dz'}{z'}$$
$$\Big[\ \mathcal{G}_\Delta(z, p_1)\mathcal{G}_\Delta(z', p_2)\Big(W_{\Delta',+}(z, z', u)W_{\Delta,+}(z, -p_3)W_{\Delta,+}(z', -p_4) + W_{\Delta',+}(z, z', t)W_{\Delta,+}(z', -p_3)W_{\Delta,+}(z, -p_4)\Big)$$
$$+ \mathcal{G}_\Delta(z, p_1)\mathcal{G}_\Delta(z', p_3)\Big(W_{\Delta',+}(z, z', s)W_{\Delta,+}(z, -p_2)W_{\Delta,+}(z', -p_4) + W_{\Delta',+}(z, z', t)W_{\Delta,-}(z', -p_2)W_{\Delta,+}(z, -p_4)\Big)$$
$$+ \mathcal{G}_\Delta(z, p_1)\mathcal{G}_\Delta(z', p_4)\Big(W_{\Delta',+}(z, z', s)W_{\Delta,+}(z, -p_2)W_{\Delta,-}(z', -p_3) + W_{\Delta',+}(z, z', u)W_{\Delta,+}(z, -p_3)W_{\Delta,-}(z', -p_2)\Big)$$
$$+ \mathcal{G}_\Delta(z, p_2)\mathcal{G}_\Delta(z', p_3)\Big(W_{\Delta',+}(z, z', s)W_{\Delta,-}(z, -p_1)W_{\Delta,+}(z', -p_4) + W_{\Delta',+}(z, z', -u)W_{\Delta,+}(z, -p_4)W_{\Delta,-}(z', -p_1)\Big)$$
$$+ \mathcal{G}_\Delta(z, p_2)\mathcal{G}_\Delta(z', p_4)\Big(W_{\Delta',+}(z, z', s)W_{\Delta,-}(z, -p_1)W_{\Delta,-}(z', -p_3) + W_{\Delta'+}(z, z', -t)W_{\Delta,+}(z, -p_3)W_{\Delta,-}(z', -p_1)\Big)$$
$$+ \mathcal{G}_\Delta(z, p_3)\mathcal{G}_\Delta(z', p_4)\Big(W_{\Delta',+}(z, z', u)W_{\Delta,-}(z, -p_1)W_{\Delta,-}(z', -p_2) + W_{\Delta',+}(z, z', -t)W_{\Delta,-}(z, -p_2)W_{\Delta,-}(z', -p_1)\Big)$$
$$+ \mathcal{G}_\Delta(z, p_1)\Big(\mathcal{G}_{\Delta'}(z, z', s)W_{\Delta,+}(z, -p_2)W_{\Delta,+}(z', -p_3)W_{\Delta,+}(z', -p_4) + \mathcal{G}_{\Delta'}(z, z', u)W_{\Delta,+}(z, -p_3)W_{\Delta,-}(z', -p_2)W_{\Delta,-}(z', -p_4)$$
$$+ \mathcal{G}_{\Delta'}(z, z', t))W_{\Delta,+}(z, -p_4)W_{\Delta,+}(z', -p_2)W_{\Delta,+}(z', -p_3)\Big) + \mathcal{G}_\Delta(z, p_2)\Big(\mathcal{G}_{\Delta'}(z, z', s)W_{\Delta,-}(z, -p_1)W_{\Delta,+}(z', -p_3)W_{\Delta,+}(z', -p_4)$$
$$+ \mathcal{G}_{\Delta'}(z, z', -t)W_{\Delta,+}(z, -p_3)W_{\Delta,-}(z', -p_1)W_{\Delta,+}(z', -p_4) + \mathcal{G}_{\Delta'}(z, z', -u)W_{\Delta,+}(z, -p_4)W_{\Delta,-}(z', -p_1)W_{\Delta,+}(z', -p_3)\Big)$$
$$+ \mathcal{G}_\Delta(z, p_3)\Big(G_{\Delta'}(z, z', u)W_{\Delta,-}(z, -p_1)W_{\Delta,-}(z', -p_2)W_{\Delta,+}(z', -p_4) + \mathcal{G}_{\Delta'}(z, z', -t)W_{\Delta,-}(z, -p_2)W_{\Delta,-}(z', -p_1)W_{\Delta,+}(z', -p_4)$$
$$+ \mathcal{G}_{\Delta'}(z, z', -s)W_{\Delta,+}(z, -p_4)W_{\Delta,-}(z', -p_1)W_{\Delta,-}(z', -p_2)\Big) + \mathcal{G}_\Delta(z, p_4)\Big(\mathcal{G}_{\Delta'}(z, z', t)W_{\Delta,-}(z, -p_1)W_{\Delta,-}(z', -p_2)W_{\Delta,-}(z', -p_3)$$
$$+ \mathcal{G}_{\Delta'}(z, z', -u)W_{\Delta,-}(z, -p_2)W_{\Delta,-}(z', -p_1)W_{\Delta,-}(z', -p_3) + \mathcal{G}_{\Delta'}(z, z', -s)W_{\Delta,-}(z, -p_3)W_{\Delta,-}(z', -p_1)W_{\Delta,-}(z', -p_2)\Big)\Big]\Big), \tag{6.109}$$

where $s^\mu = p_1^\mu + p_2^\mu, t^\mu = p_1^\mu + p_4^\mu$ and $u^\mu = p_1^\mu + p_3^\mu$. Their magnitudes are denoted $|s|, |t|$ and $|u|$. The first six lines of the above expression (blue colour) represent the contributions due to the first order corrected field occurring at two locations in the correlator. The remaining six lines (violet colour) of the expression are the contribution due to the second order correction of a single field in the correlator.

### Via analytic continuation

One can try to ask whether (6.109) can be obtained via analytic continuation from Euclidean space. However, based on the behemoth expression (6.109), an analytic continuation to obtain it from the corresponding Euclidean correlator is clearly going to be extremely difficult. Generalizing the methods of [34] to four point exchange diagrams is certainly possible but would be quite challenging in practice. This shows the importance of intrinsically computing Wightman functions rather than resorting to analytic continuation from Euclidean space.

## 6.3.4 Four point functions in special kinematics

The general expression for the scalar exchange correlator (6.109) is quite complicated. However, for particular kinematics where we take the middle operators to have space-like momenta viz $p_2^2 > 0, p_3^2 > 0$, the expression dramatically simplifies which causes all but one of the above terms in (6.109) to vanish This is what we shall refer to as special kinematics[12]. One can check that this statement is not just true for the scalar correlator but also for ones involving particles with spin. These kinematics make four point Wightman functions extremely tractable and simplify them greatly. Further, we shall find that performing analytic continuation from the Euclidean correlator to the Wightman functions with these kinematics is a simple extension of the procedure at three points (2.101). We simply take discontinuities with respect to the momenta we want time-like.

We now proceed to discuss a variety of examples including the previously discussed scalar ones in these kinematics.

#### Scalar contact and exchange

First of all, as we see from (6.97), when we work in the special kinematics with $p_2^2 > 0, p_3^2 > 0$, the contact contribution vanishes as we also discussed in footnote 11.

$$\langle\langle 0|O_{\Delta_1}(p_1)O_{\Delta_2}(p_2)O_{\Delta_3}(p_3)O_{\Delta_4}(p_4)|0\rangle\rangle_{O(\lambda)}\theta(p_2^2)\theta(p_3^2) = 0. \tag{6.110}$$

Let us move on to the exchange contribution example (6.109). We obtain in the

[12]The reason for all but one term in (6.109) to drop out in these kinematics is that the only terms that contribute with $p_2^2 > 0, p_3^2 > 0$ are when the $\mathcal{G}$ propagator occurs with momentum $p_2$ and $p_3$ as we see from table 6.1. When they occur with a Wightman propagator, they do not contribute since it has support only for time-like momenta as we one can see from table 6.1 yet again.

special kinematics,

$$\langle\langle 0|O_\Delta(p_1)O_\Delta(p_2)O_\Delta(p_3)O_\Delta(p_4)|0\rangle\rangle_{O(g^2)}\theta(p_2^2)\theta(p_3^2)$$
$$= \prod_{i=2}^{3}\theta(p_i^2)\Big(\prod_{i=1}^{4}\lim_{z_i\to 0} z_i^{1-\Delta_i}\Big)\langle\langle 0|\phi^{(0)}(z_1,p_1)\phi^{(1)}(z_2,p_2)\phi^{(1)}(z_3,p_3)\phi^{(0)}(z_4,p_4) = \theta(-p_1^0)\theta(-p_4^2)\theta(p_4^0)\theta(p_2^2)\theta(p_3^2)\mathcal{A}_{\Delta,\Delta,\Delta,\Delta},$$
$$\mathcal{A}_{\Delta,\Delta,\Delta,\Delta} = -4g^2\Bigg[\int_0^\infty \frac{dz}{z}\int_0^\infty \frac{dz'}{z'}\mathcal{G}_\Delta(z,p_2)\mathcal{G}_\Delta(z',p_3)\Big(W_{\Delta',+}(z,z',s)W_{\Delta,-}(z,-p_1)W_{\Delta,+}(z',-p_4)$$
$$+ W_{\Delta',+}(z,z',-u)W_{\Delta,+}(z,-p_4)W_{\Delta,-}(z',-p_1)\Big)\Bigg] \tag{6.111}$$

Although there are two contributions in the above equation, only the first one is non-zero due to kinematic constraints. For the second term to be non-zero, we require the conditions,

$$p_1^2 < 0, p_3^2 > 0, p_1^0 < 0 \text{ with } u^0 > 0, -u^2 < 0. \tag{6.112}$$

which come from the Wightman propagator with momentum $-u^\mu = -p_1^\mu - p_3^\mu$. This is kinematically inconsistent as the sum of a backward pointing time-like vector and a space-like vector cannot result in a forward pointing time-like vector. Thus our result is only one term viz,

$$\mathcal{A}_{\Delta,\Delta,\Delta,\Delta} = -4g^2\int_0^\infty \frac{dz}{z}\int_0^\infty \frac{dz'}{z'}\mathcal{G}_\Delta(z,p_2)\mathcal{G}_\Delta(z',p_3)W_{\Delta',+}(z,z',s)W_{\Delta,-}(z,-p_1)W_{\Delta,+}(z',-p_4). \tag{6.113}$$

Further, using the expressions provided in table 6.1 we see that the bulk to bulk Wightman propagator factorizes into a product of bulk to boundary propagators resulting in,

$$W_{\Delta,+}(z,z',p) = \frac{4^{\Delta-2}\Gamma(\Delta)\Gamma(\Delta-\frac{1}{2})}{\sqrt{\pi}}W_{\Delta',+}(z,p)\frac{1}{|p|^{2\Delta-3}}W_{\Delta',+}(z',p). \tag{6.114}$$

Thus, the above integral factorizes[13]:

$$\mathcal{A}_{\Delta\Delta\Delta\Delta} = c(\Delta')\int_0^\infty \frac{dz}{z}W_{\Delta,-}(z,-p_1)\mathcal{G}_\Delta(z,p_2)W_{\Delta',+}(z,s)\frac{1}{|s|^{2\Delta'-3}}\int_0^\infty \frac{dz'}{z'}W_{\Delta',-}(z,-s)\mathcal{G}_\Delta(z',p_3)W_{\Delta,+}(z',-p_4), \tag{6.115}$$

with $c(\Delta') = -\frac{4^{\Delta'-1}\Gamma(\Delta')\Gamma(\Delta'-\frac{1}{2})}{\sqrt{\pi}}g^2$. This is a great simplification as there are no nested bulk integrals in contrast to traditional Witten diagrams involving the Feynman bulk to bulk propagator. This is a general feature of these kinematics and holds for spinning correlators too. Evaluating the integrals in terms of Appell $F_4$ functions using (6.55) and replacing $g$ with the OPE coefficient (6.67), we get,

$$\mathcal{A}_{\Delta\Delta\Delta\Delta} = \frac{c_{123}^2 4^{7-2\Delta}\pi^{\frac{19}{2}}\Gamma(\Delta')}{\Gamma(\Delta-\frac{1}{2})^2\Gamma(\Delta-\frac{\Delta'}{2})^2\Gamma(\Delta'-\frac{1}{2})\Gamma(\frac{\Delta'}{2})^2}\theta(-s^2)\theta(-s^0)|p_1|^{2\Delta-3}|p_2|^{-\Delta'}|s|^{2\Delta'-3}|p_3|^{-\Delta'}|p_4|^{2\Delta-3}$$
$$\times F_4\Big(\frac{\Delta'}{2},\Delta+\frac{\Delta'}{2}-\frac{3}{2};\Delta-\frac{1}{2},\Delta'-\frac{1}{2};-\frac{|p_1|^2}{|p_2|^2},-\frac{|s|^2}{|p_2|^2}\Big)\frac{1}{|s|^{2\Delta'-3}}F_4\Big(\frac{\Delta'}{2},\Delta+\frac{\Delta'}{2}-\frac{3}{2};\Delta-\frac{1}{2},\Delta'-\frac{1}{2};-\frac{|p_4|^2}{|p_3|^2},-\frac{|s|^2}{|p_3|^2}\Big). \tag{6.116}$$

[13] We used the property $W_{\Delta,+}(z,p) = W_{\Delta,-}(z,-p)$ as can be ascertained from table 6.1.

For example, consider all the scaling dimensions to be 1. In this case the result is,

$$\mathcal{A}_{1111} = 1024\pi^6 c_{123}^2 \frac{1}{|p_1||p_2||p_3||p_4|} \frac{1}{|s|} \theta(-s^2)\theta(-s^0). \tag{6.117}$$

One important point to note is that the general result (6.116) is perfectly finite for any $\Delta, \Delta'$ that satisfy the unitarity bound. This is in contrast to the Euclidean case where one needs to perform renormalization in many cases [29][14] due to the presence of UV divergences for the CFT correlator which are not present for Wightman functions. This can even be seen at the level of two points, check out appendix $K$ of [237] for instance.

### Via Analytic Continuation

Let us discuss how to obtain this correlator via analytic continuation. Clearly, as we discussed earlier, there is no obvious pattern to obtain the general result (6.109) except of course by guessing the pattern via trial and error and taking a sum of products of discontinuities with respect to the squares of the momenta taking a leaf out of the three point case (2.101)! For the special kinematics, however, the procedure is simple as we discussed at the beginning of this section. Let us see this now. Start with the Euclidean correlator,

$$\begin{aligned}
&\langle\langle O_\Delta(p_1)O_\Delta(p_2)O_\Delta(p_3)O_\Delta(p_4)\rangle\rangle \\
&= (p_1p_2p_3p_4)^\nu \int_0^\infty dz\,\sqrt{z}\int_0^\infty dz'\sqrt{z'}\,K_\nu(p_1z)K_\nu(p_2z)\Big(K_\nu(sz)I_\nu(sz')\theta(z-z') + (z\leftrightarrow z')\Big)K_\nu(p_3z)K_\nu(p_4z) \\
&+ \text{t-channel} + \text{u-channel}
\end{aligned} \tag{6.118}$$

We want to reach the Wightman function in the kinematics where the middle operators have spacelike momenta as well as the exchanged momentum $s$ being time-like. For this, we take a discontinuity with respect to $p_1^2, p_4^2$ and $s^2$.

$$\text{Disc}_{p_1^2}\text{Disc}_{p_4^2}\text{Disc}_{s^2}\langle\langle O_\Delta(p_1)O_\Delta(p_2)O_\Delta(p_3)O_\Delta(p_4)\rangle\rangle. \tag{6.119}$$

We then Wick rotate following (2.99). The t and u channels drop out in this process (or are not kinematically consistent) yielding the correct result (6.115).

### Scalar-Photon Bhabha Scattering

Let us move on to an example of scalars exchanging a photon. The action we work with is the scalar QED action (6.68) with the equations of motion given in (6.69). We require the $O(e^2)$ correction for this correlator.

[14] These occur when $\frac{d}{2} \pm (\Delta_1 - \frac{d}{2}) \pm (\Delta_2 - \frac{d}{2}) \pm (\Delta_3 - \frac{d}{2}) = -2k, k \in \mathbb{Z}_{\geq 0}$.

**Via the Equation of Motion**

We obtain,

$$
\begin{aligned}
&\langle\langle 0|O_\Delta(p_1)O^*_\Delta(p_2)O_\Delta(p_3)O^*_\Delta(p_4)|0\rangle\rangle_{O(e^2)}\theta(p_2^2)\theta(p_3^2)\\
&= -e^2\prod_{i=1}^{4}(\lim_{z_i\to 0} z_i^{1-\Delta_i})\theta(p_2^2)\theta(p_3^2)\langle\langle 0|\phi^{(0)}(z_1,p_1)\phi^{*(1)}(z_2,p_2)\phi^{(1)}(z_3,p_3)\phi^{*(0)}(z_4,p_4)|0\rangle\rangle\\
&= -e^2\theta(p_2^2)\theta(p_3^2)\int_0^\infty dz\int_0^\infty dz'\mathcal{G}_\Delta(z,p_2)\mathcal{G}_\Delta(z,p_3)\\
&\times\left(\int\frac{d^3l}{(2\pi)^3}\int\frac{d^3l'}{(2\pi)^3}l_\mu l'_\nu\, W_{\Delta,-}(z,-p_1)\pi^{\mu\nu}(s)W_{\Delta=2,+}(z,z',s)W_{\Delta,+}(z',-p_4)\delta^3(l+p_1)\delta^3(l'+p_4)\right)\\
&= -e^2\theta(p_2^2)\theta(p_3^2)\theta(-p_1^2)\theta(-p_4^2)\theta(-p_1^0)\theta(p_4^0)\mathcal{A}^{\text{Bhabha}}_{\Delta\Delta\Delta\Delta},
\end{aligned}
\tag{6.120}
$$

with,

$$
\begin{aligned}
\mathcal{A}^{\text{Bhabha}}_{\Delta\Delta\Delta\Delta} &= \pi^{\mu\nu}(s)p_{1\mu}p_{4\nu}\theta(-s^0)\theta(-s^2)\int_0^\infty dz\int_0^\infty dz' W_{\Delta,-}(z,-p_1)\mathcal{G}_\Delta(z,p_2)W_{\Delta=2,+}(z,z',s)\mathcal{G}(z',p_3)W_{\Delta,+}(z',-p_4)\\
&= \frac{\pi^{\mu\nu}(s)p_{1\mu}p_{4\nu}\theta(-s^0)\theta(-s^2)}{2|s|}\int_0^\infty dz W_{\Delta,-}(z,-p_1)\mathcal{G}_\Delta(z,p_2)W_{\Delta=2,+}(z,s)\int_0^\infty dz' W_{\Delta=2,-}(z',-s)\mathcal{G}_\Delta(z',p_3)W_{\Delta,+}(z',-p_4).
\end{aligned}
\tag{6.121}
$$

Therefore the result is simply an $s-$ channel contribution. Evaluating the integrals result in the extremely simple answer viz,

$$
\begin{aligned}
\mathcal{A}^{\text{Bhabha}}_{\Delta\Delta\Delta\Delta} &= \frac{\pi^{\mu\nu}(s)p_{1\mu}p_{4\nu}\theta(-s^0)\theta(-s^2)}{|s|}\left(-\frac{2^{5-4\nu}\pi}{\Gamma(1+\nu)^2\Gamma(\frac{3}{2}+\nu)^2}\right)\frac{|p_1|^{2\nu}|p_4|^{2\nu}|s|^2}{|p_2|^3|p_3|^3}\\
&\times F_4(\frac{3}{2},\frac{3}{2}+\nu;1+\nu,\frac{3}{2};-\frac{|p_1|^2}{|p_2|^2},-\frac{|s|^2}{|p_2|^2})F_4(\frac{3}{2},\frac{3}{2}+\nu;1+\nu,\frac{3}{2};-\frac{|p_4|^2}{|p_3|^2},-\frac{|s|^2}{|p_3|^2}),
\end{aligned}
\tag{6.122}
$$

where $\nu = \Delta - \frac{3}{2}$.

**Via Analytic Continuation**

Let us start with the Euclidean correlator. Performing the standard Witten diagram computation results in,

$$
\begin{aligned}
&\langle\langle O_\Delta(p_1)O^*_\Delta(p_2)O_\Delta(p_3)O^*_\Delta(p_4)\rangle\rangle = -e^2\pi^{\mu\nu}(s)p_{1\mu}p_{4\nu}\\
&\times\int_0^\infty dz\int_0^\infty dz' G_{F,\Delta}(z,p_1)G_{F,\Delta}(z,p_2)G_{F,\Delta=2}(z,z',s)G_{F,\Delta}(z',p_3)G_{F,\Delta}(z',p_4) + \text{t+u channels},
\end{aligned}
\tag{6.123}
$$

where since these are Euclidean momenta, only the spacelike contributions to the Feynman propagators in table 6.1 contribute to this expression. Rather than evaluating this expression and then performing the analytic continuation, we choose to do it at the integrand level. Simply taking a discontinuity with respect to three momenta we want time-like viz $p_1^2, p_4^2$ and $s^2$ shows that the associated Euclidean propagators turn into the corresponding Wightman propagator in table 6.1, which yields the result (6.121).

We now proceed to discuss three more examples: Compton scattering, Yang-Mills gluon four point function and Einstein gravity four point function. The details of the calculations are identical to the ones discussed albeit with more tensor structures involved. Thus, we skip the steps and present the final results when the middle two operators have spacelike momenta.

### Scalar-Photon Compton scattering

Next, we consider the classic Compton scattering process involving two photons and two scalars. The relevant action and equations of motion are those in scalar QED which are respectively (6.68) and (6.69). Solving the equation of motion at $O(e^2)$ and computing the Wightman function with $p_2^2 > 0, p_3^2 > 0$ results in,

$$\begin{aligned}
&\epsilon_1^\mu \epsilon_4^\nu \langle\langle 0|J_\mu(p_1)O_\Delta(p_2)O^*_\Delta(p_3)J_\nu(p_4)|0\rangle\rangle\theta(p_2^2)\theta(p_3^2)\\
&= \epsilon_1^\mu\epsilon_4^\nu \lim_{z_i\to 0}\frac{z_2^{1-\Delta}z_3^{1-\Delta}}{z_1z_4}\langle\langle 0|A_\mu^{(0)}(z_1,p_1)\phi^{(1)}(z_2,p_2)\phi^{*(1)}(z_3,p_3)A_\nu^{(0)}(z_4,p_4)|0\rangle\rangle\\
&= e^2\theta(p_2^2)\theta(p_3^2)(\epsilon_1\cdot p_2)(\epsilon_4\cdot p_3)\int_0^\infty dz\int_0^\infty dz'\, W_{\Delta=2,-}(z,-p_1)\mathcal{G}_\Delta(z,p_2)W_{\Delta,+}(z,z',s)\mathcal{G}_\Delta(z',p_3)W_{\Delta=2,+}(z',-p_4)\\
&= e^2\theta(-p_1^2)\theta(p_2^2)\theta(p_3^2)\theta(-p_4^2)\theta(-p_1^0)\theta(p_4^0)(\epsilon_1\cdot p_4)(\epsilon_4\cdot p_1)\mathcal{A}_{\gamma\phi_\Delta\phi^*_\Delta\gamma}. \qquad (6.124)
\end{aligned}$$

The function $\mathcal{A}_{\gamma\phi_\Delta\phi^*_\Delta\gamma}$ can be evaluated in terms of a product of Appell $F_4$ functions resulting in,

$$\begin{aligned}
\mathcal{A}_{\gamma\phi_\Delta\phi^*_\Delta\gamma} &= \frac{-4^{4-\Delta}\sqrt{\pi}|p_1||p_4|}{|p_2|^3|p_3|^3\Gamma(\Delta)\Gamma(\Delta-\frac{1}{2})}F_4(\frac{3}{2},\Delta;\frac{3}{2},\Delta-\frac{1}{2};-\frac{|p_1|^2}{|p_2|^2},-\frac{|s|^2}{|p_2|^2})\\
&\times\frac{1}{|s|^{3-2\Delta}}F_4(\frac{3}{2},\Delta;\frac{3}{2},\Delta-\frac{1}{2};-\frac{|p_4|^2}{|p_3|^2},-\frac{|s|^2}{|p_3|^2}), \qquad (6.125)
\end{aligned}$$

For example, consider conformally coupled scalars that have $\Delta = 2$. In that case, the result is simply,

$$\begin{aligned}
&\mathcal{A}_{\gamma\phi_{\Delta=2}\phi^*_{\Delta=2}\gamma}\\
&= 32i\frac{p_1p_2sp_3p_4}{(p_1+p_2+s)(p_1+p_2-s)(p_1-p_2+s)(-p_1+p_2+s)(p_3+p_4+s)(p_3+p_4-s)(p_3-p_4+s)(-p_3+p_4+s)}. \qquad (6.126)
\end{aligned}$$

### Yang-Mills theory

We now proceed to an example involving gluons. The Yang-Mills action was given in (6.79) with the equation of motion provided in (6.80). Following the usual method, we find the four point function,

$$\begin{aligned}
&\langle\langle 0|J^{A_1}(p_1,\epsilon_1)J^{A_2}(p_2,\epsilon_2)J^{A_3}(p_3,\epsilon_3)J^{A_4}(p_4,\epsilon_4)|0\rangle\rangle\theta(p_2^2)\theta(p_3^2)\\
&= (\prod_{i=1}^4 z_i^{-1})\langle\langle 0|A^{A_1(0)}(z_1,p_1,\epsilon_1)A^{A_2(1)}(z_2,p_2,\epsilon_2)A^{A_3(1)}(z_3,p_3,\epsilon_3)A^{A_4(0)}(z_4,p_4,\epsilon_4)|0\rangle\rangle\theta(p_2^2)\theta(p_3^2)\\
&= \theta(-p_1^2)\theta(p_2^2)\theta(p_3^2)\theta(-p_4^2)\theta(-p_1^0)\theta(p_4^0)\theta(-s^2)\theta(-s^0)\mathcal{A}_{4,YM}, \qquad (6.127)
\end{aligned}$$

with,

$$\mathcal{A}_{4,YM} = \frac{g^2}{|s|} f^{A_1A_2A} f_A^{A_3A_4} V^{\mu}_{3,YM}(\epsilon_1,\epsilon_2)\pi_{\mu\nu}(s)V^{\nu}_{3,YM}(\epsilon_3,\epsilon_4)\int_0^\infty dz \sin(|p_1|z)e^{-|p_2|z}\sin(|s|z)\int_0^\infty dz' \sin(|s|z')e^{-|p_3|z}\sin(|p_4|z')$$
$$= \frac{4g^2 f^{A_1A_2A} f_A^{A_3A_4} V^{\mu}_{3,YM}(\epsilon_1,\epsilon_2)\pi_{\mu\nu}(s)V^{\nu}_{3,YM}(\epsilon_3,\epsilon_4)|p_1||p_2||s||p_3||p_4|}{((|p_1|^2+|p_2|^2)^2+|s|^4+2|s|^2(|p_2|^2-|p_1|^2))((|p_3|^2+|p_4|^2)^2+|s|^4+2|s|^2(|p_4|^2-|p_3|^2))} \tag{6.128}$$

Note that the Yang-Mills contact diagram drops out in these kinematics for the same reasons discussed in footnote 11.

### Einstein Gravity

The computation in Einstein gravity for the graviton four point function yields the result,

$$\begin{aligned}&\langle\langle 0|T(p_1,\epsilon_1)T(p_2,\epsilon_2)T(p_3,\epsilon_3)T(p_4,\epsilon_4)|0\rangle\rangle\theta(p_2^2)\theta(p_3^2)\\&=\theta(-p_1^2)\theta(p_2^2)\theta(p_3^2)\theta(-p_4^2)\theta(-p_1^0)\theta(p_4^0)\theta(-s^2)\theta(-s^0)\mathcal{A}_{4,GR},\end{aligned} \tag{6.129}$$

where,

$$\mathcal{A}_{4,GR} = \frac{4\kappa^2 V^{\mu\nu}_{3,GR}(\epsilon_1,\epsilon_2)\Pi_{\mu\nu\rho\sigma}(s)V^{\rho\sigma}_{3,GR}(\epsilon_3,\epsilon_4)|p_1|^3|p_2|^3|s|^3|p_3|^3|p_4|^3}{\Big(((|p_1|^2+|p_2|^2)^2+|s|^4+2|s|^2(|p_2|^2-|p_1|^2))((|p_3|^2+|p_4|^2)^2+|s|^4+2|s|^2(|p_4|^2-|p_3|^2))\Big)^2} \tag{6.130}$$

One can easily check that the discontinuity with respect to $p_1^2, p_4^2, s^2$ of the Euclidean AdS graviton correlator results in this expression. Note that yet again, the quartic contact does not contribute in these kinematics. Comparing (6.130) to its Yang-Mills counterpart (6.128) suggests a double copy relation. We will have much more to say on this in section 6.6.

## 6.3.5 Factorization and Wightman Conformal Partial Waves

In the previous subsection, we have observed that every four point function we have considered factorizes into a product of three point functions when we take the middle two operators to have spacelike momenta. For example, the Yang-Mills theory four point function (6.128) can be written as,

$$\begin{aligned}&\langle\langle 0|J^{A_1}(p_1,\epsilon_1)J^{A_2}(p_2,\epsilon_2)J^{A_3}(p_3,\epsilon_3)J^{A_4}(p_4,\epsilon_4)|0\rangle\rangle_{YM}\theta(p_2^2)\theta(p_3^2)\\&=\langle\langle 0|J^{A_1}(p_1,\epsilon_1)J^{A_2}(p_2,\epsilon_2)J^{\mu A}(-s)|0\rangle\rangle\frac{\pi_{\mu\nu}(s)}{|s|}\langle\langle 0|J^{\nu}_A(s)J^{A_3}(p_3,\epsilon_3)J^{A_4}(p_4,\epsilon_4)|0\rangle\rangle\\&=\mathcal{W}^{(s)}_{(JJ|J|JJ)}(p_1,p_2|s|p_3,p_4),\end{aligned} \tag{6.131}$$

where $\mathcal{W}^{(s)}_{(JJ|J|JJ)}$ is a conformal partial wave (CPW). Indeed, all our factorized expressions can be re-interpreted in terms of these quantities. In momentum space, CPWs are simply given by a product of three point functions [52, 85, 280],

$$\mathcal{W}^{(s)}_{(O_1O_2|J_s|O_3O_4)}(p_1,p_2|s|p_3,p_4)$$
$$=\frac{\Pi_{\mu_1\cdots\mu_s,\nu_1\cdots\nu_2}(s)}{|s|^{2s-1}}\langle\langle 0|O_1(p_1)O_2(p_2)J^{\mu_1\cdots\mu_s}(-p_1-p_2)|0\rangle\rangle\langle\langle 0|J^{\nu_1\cdots\nu_s}(-p_3-p_4)O_3(p_3)O_4(p_4)|0\rangle\rangle,$$
$$\mathcal{W}^{(s)}_{(O_1O_2|O_\Delta|O_3O_4)}(p_1,p_2|s|p_3,p_4)$$
$$=\frac{1}{|s|^{2\Delta-3}}\langle\langle 0|O_1(p_1)O_2(p_2)O_\Delta(-p_1-p_2)|0\rangle\rangle\langle\langle 0|O_\Delta(-p_3-p_4)O_3(p_3)O_4(p_4)|0\rangle\rangle. \tag{6.132}$$

The superscript $(s)$ denotes that this is an s-channel exchange. $\Pi_{\mu_1\cdots\mu_s\nu_1\cdots\nu_s}$ is the transverse traceless projector which is the transverse projector for spin-1 and the transverse traceless projector for spin-2. Our factorized Wightman functions written are just s-channel conformal partial waves.

Let us now compare in more detail to the construction of [275]. The author shows for a time-ordered correlator with all external momentum space-like,

$$\mathrm{Disc}_{s^2}\langle 0|T\{O_1(p_1)O_1(p_2)O_3(p_3)O_4(p_4)\}|0\rangle_s\theta(p_1^2)\theta(p_2^2)\theta(p_3^2)\theta(p_4^2)$$
$$=\langle T\{O_1(p_1)O_2(p_2)\}\bar{T}\{O_3(p_3)O_4(p_4)\}\rangle+(1,2)\leftrightarrow(3,4)=g^{(s)}_{(O_1O_2|O_{\mathrm{exchange}}|O_3O_4)}(p_1,p_2|s|p_3,p_4), \tag{6.133}$$

where the kinematics enforcing that all momenta are space-like is also implicit in the right hand side of this equation. Of course, $s$ is assumed to be time-like in this construction as well and we further make the choice that $s^0<0$[15]. The object $g^{(s)}_{(O_1O_2|O_{\mathrm{exchange}}|O_3O_4)}(p_1,p_2|s|p_3,p_4)$ is the conformal partial wave associated to the causal double commutator.

To obtain the Wightman conformal partial wave (6.132), we need to take two more discontinuities with respect to $p_1^2,p_4^2$ in addition to (6.133). This is because the spectral conditions (2.91) demand in a Wightman function (with the ordering $1>2>3>4$) that $p_1^2<0,p_4^2<0$ as well as the fact that $p_1^0<0,p_4^0>0$. We also make no assumptions about $s^\mu$ as since the spectral conditions already render it time-like with $s^0<0$. Taking a discontinuity of (6.133) with respect to $p_1^2$ and $p_4^2$ assuming they have negative and positive energy respectively results in,

$$\mathrm{Disc}_{p_1^2}\mathrm{Disc}_{p_4^2}\mathrm{Disc}_{s^2}\langle 0|T\{O_\Delta(p_1)O_\Delta(p_2)O_\Delta(p_3)O_\Delta(p_4)\}|0\rangle_s\left(\prod_{i=2}^{3}\theta(p_i^2)\right)\theta(-s^2)\theta(-p_1^2)\theta(-p_4^2)\theta(-p_1^0)\theta(p_4^0)$$
$$=\mathrm{Disc}_{p_1^2}\mathrm{Disc}_{p_4^2}g^{(s)}_{(O_1O_2|O_{\mathrm{exchange}}|O_3O_4)}(p_1,p_2|s|p_3,p_4)$$
$$=\mathcal{W}^{(s)}_{(O_\Delta O_\Delta|O_\Delta|O_\Delta O_\Delta)}(p_1,p_2|s|p_3,p_4). \tag{6.134}$$

which is precisely our factorized expression equal to the Wightman conformal partial wave[16]. To sum up the discussion, Wightman functions in the special kinematics

[15]See appendix B of [275] for the details. The author considers the $t-$ channel which can easily be generalized to the $s-$ channel case.

[16]Note that the Feynman propagators and EOM inverter propagators are equal for space-like momenta, see table 6.1.

naturally arise by taking three cuts of the associated time-ordered correlator: One with respect to the exchanged momenta square and the remaining two with respect to the momenta squares of the first and fourth operators. This result is naturally the Wightman conformal partial wave[17].

### 6.3.6 Higher point correlators

The formalism outline so far can easily be extended to the computation of higher point Wightman functions. Here, we show this for the five point function of identical scalars with interacting with a cubic potential. The action and equations of motion are the same as in (6.103) and (6.104). We shall also focus on special kinematics with the middle three operators having space-like momenta for concreteness. This receives contribution iff there is a field correction for all three middle operators. This is because we need a EOM inverter propagator with their momenta since only it has support for space-like momenta as we see from table 6.1. This is exactly the same reasoning we gave for four point functions earlier in subsection 6.3.3. Thus we have,

$$\begin{aligned}&\langle\langle 0|O_\Delta(p_1)O_\Delta(p_2)O_\Delta(p_3)O_\Delta(p_4)O_\Delta(p_5)|0\rangle\rangle_{O(g^3)}\theta(p_2^2)\theta(p_3^2)\theta(p_4^2)\\&=\prod_{i=1}^{5}\big(\lim_{z_i\to 0} z_i^{1-\Delta_i}\big)\langle\langle 0|\phi^{(0)}(z_1,p_1)\phi^{(1)}(z_2,p_2)\phi^{(1)}(z_3,p_3)\phi^{(1)}(z_4,p_4)\phi^{(0)}(z_5,p_5)|0\rangle\rangle\end{aligned} \tag{6.135}$$

Plugging in the field corrections (6.106) and simplifying the resulting expression yields,

$$\begin{aligned}&\langle\langle 0|O_\Delta(p_1)O_\Delta(p_2)O_\Delta(p_3)O_\Delta(p_4)O_\Delta(p_5)|0\rangle\rangle_{O\left(g^3\right)}\theta(p_2^2)\theta(p_3^2)\theta(p_4^2)\\&=(2ig)^3\theta(-p_1^2)\theta(-p_1^0)\theta(p_2^2)\theta(p_3^2)\theta(p_4^2)\theta(-p_5^2)\theta(p_5^0)\mathcal{A}_{5\Delta\Delta\Delta\Delta\Delta},\\&\mathcal{A}_{5\Delta\Delta\Delta\Delta\Delta}=\Big(\prod_{i=2}^{4}\frac{dz_i}{z_i}\Big)\mathcal{G}_\Delta(z_2,p_2)\mathcal{G}_\Delta(z_3,p_3)\mathcal{G}_\Delta(z_4,p_4)W_{\Delta,-}(z_2,-p_1)W_{\Delta,+}(z_2,z_3,s_{12})W_{\Delta,+}(z_3,z_4,s_{123})W_+(z_4,-p_5)\\&=\left(\frac{i2^{\frac{29}{2}-5\Delta}\Gamma(\Delta-\frac{1}{2})^2\sqrt{\pi}}{\Gamma(\Delta)^7}\Big(\prod_{i=1}^{5}|p_i|^\nu\Big)\int_0^\infty dz_2\sqrt{z_2}J_\nu(|p_1|z_2)K_\nu(|p_2|z_2)J_\nu(|s_{12}|z_2)\int_0^\infty dz_3\sqrt{z_3}J_\nu(|s_{12}|z_3)K_\nu(|p_3|z_3)J_\nu(|s_{123}|z_3)\right.\\&\left.\times\int_0^\infty dz_4\sqrt{z_4}J_\nu(|s_{123}|z_4)K_\nu(|p_4|z_4)J_\nu(|p_5|z_4)\right)\theta(-s_{12}^2)\theta(-s_{12}^0)\theta(-s_{123}^2)\theta(-s_{123}^0).\end{aligned} \tag{6.136}$$

We have defined $s_{12}^\mu = p_1^\mu + p_2^\mu$ and $s_{123}^\mu = p_1^\mu + p_2^\mu + p_3^\mu$ as well as $\nu = \Delta - \frac{3}{2}$ in the above equation. Note that in these special kinematics, the five point functions also factorize.

As an example, let us evaluate the above integrals for $\Delta = 1$ scalars. The result is,

$$\mathcal{A}_{5,11111} \propto \frac{1}{|p_1||p_2||p_3||p_4||p_5|}\frac{1}{|s_{12}||s_{123}|}. \tag{6.137}$$

We leave a more detailed analysis of higher point functions to the future. The purpose of this section was to illustrate that our formalism provides a simple and systematic way to compute them.

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[17] It would be interesting to understand if there is a generalization of the geodesic Witten diagram construction of conformal partial waves [281] for these real-time Wightman conformal partial wave. Perhaps a construction using time-like geodesics such as in [282] would be a potential starting point. We thank Allic Sivaramakrishnan for this comment.

## 6.4 From Wightman functions to Euclidean correlators

In this section, we demonstrate how one can recover a Euclidean correlator from the four point Wightman functions in the special kinematics. The reader might be worried since it seems like we lose a lot of information restricting to these special kinematics. However, as we shall in this section, we can actually recover the Euclidean correlator up to contact diagram contributions. We shall show on the other hand is[18],

$$\langle 0|T\{O_1(p_1)O_2(p_2)O_3(p_3)O_4(p_4)\}|0\rangle_s = F(p_1,p_2,p_3,p_4,s)$$
$$\sim \int_0^\infty \frac{\omega_4 d\omega_4}{\omega_4^2-p_4^2}\int_0^\infty \frac{\omega_1 d\omega_1}{\omega_1^2-p_1^2}\int_0^\infty \frac{\omega_s d\omega_s}{\omega^2-s^2}(\langle 0|O_1(p_1)O(p_2)O(p_3)O(p_4)|0\rangle'/.\{p_1\to\omega_1, p_4\to\omega_4, s\to\omega_s\}), \tag{6.138}$$

where the prime indicates that our Wightman function is in the special kinematics $p_2^2>0, p_3^2>0$. The contour choice is also decided by the Wightman $i\epsilon$ prescription (2.99). Similarly, one can add the contributions of the other channels by permutations[19].

We consider Yang-Mills theory as a illustrative, yet non trivial example to show this procedure. Also, note that after performing the dispersive integrals, we implicitly Wick rotate following the Wightman to obtain the Euclidean correlator from its time-ordered counterpart. First, to set the stage, let us do so at the level of three point functions.

### 6.4.1 Three point functions

We have seen earlier (2.101) that in order to obtain a three point Wightman function with the middle operator having space-like momenta, from a Euclidean correlator, we need to take discontinuities with respect to the time-like momenta and Wick rotate following the Wightman $i\epsilon$ prescription (2.99). In this subsection, we want to essentially reverse this process. The Yang-Mills three point function (6.82) repeated here for convenience is,

$$\frac{ig}{2}\theta(p_2^2)\theta(-p_1^2)\theta(-p_3^2)\theta(-p_1^0)\theta(p_3^0)f^{ABC}V_3^{YM}\left(\frac{1}{E}-\frac{1}{E-2p_1}-\frac{1}{E-2p_2}-\frac{1}{E-2p_3}\right), \tag{6.140}$$

where $V_3^{YM}$ is the Yang-Mills three point vertex (6.83).

Stripping off the theta functions, we get,

$$\langle\langle 0|J^A(p_1,\epsilon_1)J^B(p_2,\epsilon_2)J^C(p_3,\epsilon_3)|0\rangle\rangle' = i\frac{g}{2}f^{ABC}V_3^{YM}f_W(p_1,p_2,p_3), \tag{6.141}$$

[18] The subscript $s$ denotes that this is an s-channel contribution.

[19] This is similar to the statement in [275] for recovering time-ordered correlators from their discontinuities up to contact diagrams.

$$\langle 0|T\{O_1(p_1)O_2(p_2)O_3(p_3)O_4(p_4)\}|0\rangle_s = F(p_1,p_2,p_3,p_4,s)\sim\int_0^\infty\frac{\omega_s d\omega_s}{\omega^2-s^2}\text{Disc}_{\omega_s^2}F(p_1,p_2,p_4,p_4,\omega_s). \tag{6.139}$$

This RHS of this equation also has the nice property that it factorizes just like the Wightman functions in the special kinematics. However, for going to twistor space, we require the Wightman function for reasons argued in the introduction such as the fact that they satisfy current conservation Ward-Takahashi identities without contact terms and hence enjoy a simple representation in twistor space.

with,

$$f_W(p_1,p_2,p_3) = \left(\frac{1}{E} - \frac{1}{E-2p_1} - \frac{1}{E-2p_2} - \frac{1}{E-2p_3}\right). \tag{6.142}$$

To obtain the Euclidean form-factor we need to "invert" the discontinuity procedure. This can be achieved by performing dispersive integrals over the two time-like momenta magnitudes as follows[20]:

$$f_E(p_1,p_2,p_3) = \int_0^\infty \frac{d\omega_1^2}{\omega_1^2 - p_1^2}\int_0^\infty \frac{d\omega_3^2}{\omega_3^2-p_3^2} f_W(\omega_1,p_2,\omega_3). \tag{6.143}$$

As for the contour choice to avoid the poles at $\omega_i = \pm p_i, i = 1,3$ there is no ambiguity thanks to the Wightman $i\epsilon$ prescription (2.99) using the fact that $p_1^0 < 0, p_3^0 > 0$ for this Wightman function. Evaluating the integrals results in,

$$f_E(p_1,p_2,p_3) = \frac{1}{p_1+p_2+p_3}, \tag{6.144}$$

which is the correct Euclidean form-factor. The same method holds for all the other spinning three point functions which one can readily check with the results of subsection 6.3.2 and comparing with the known Euclidean results. For three point correlators involving generic scalar operators such as (6.65), (6.73), this procedure is much more complicated due to the fact that they are not simple rational functions of the $p_i^2$ but rather have a more intricate branch cut structure (that of the Appell $F_4$ functions in the referenced expressions). Thus, the dispersive integrals have to be performed taking this into account and will be more involved .

### 6.4.2 Four point functions

Consider the the Yang-Mills four point function in the special kinematics (6.128). Stripping off the theta functions and pre-factors we obtain from (6.128),

$$\begin{aligned}
&\langle\langle 0|J^{A_1}(p_1,\epsilon_1)J^{A_2}(p_2,\epsilon_2)J^{A_3}(p_3,\epsilon_3)J^{A_4}(p_4,\epsilon_4)|0\rangle\rangle' \\
&= f^{A_1A_2A}f_A^{A_3A_4}\frac{p_1p_2s}{(p_1^4+(p_2^2-s^2)^2-2p_1^2(p_2^2+s^2))}V_3^{YM,\mu}(\epsilon_1,\epsilon_2)\pi_{\mu\nu}(s)V_3^{YM,\nu}(\epsilon_3,\epsilon_4)\frac{p_3p_4s}{(p_3^4+(p_4^2-s^2)^2-2p_3^2(p_4^2+s^2))} \\
&= f^{A_1A_2A}f_A^{A_3A_4}V_3^{YM,\mu}(\epsilon_1,\epsilon_2)V_3^{YM,\nu}f^W(p_1,p_2,s,p_3,p_4)\left(\eta_{\mu\nu}-\frac{s_\mu s_\nu}{s^2}\right).
\end{aligned} \tag{6.145}$$

To obtain the Euclidean correlator, let us take a leaf out of the three point case and perform dispersive integrals with respect to the squares of the three time-like momenta viz $p_1$, $s$ and $p_4$. We consider,

$$f^{A_1A_2A}f_A^{A_3A_4}V_3^{YM,\mu}(\epsilon_1,\epsilon_2)V_3^{YM,\nu}\int_0^\infty\frac{d\omega_1^2}{\omega_1^2-p_1^2}\int_0^\infty\frac{d\omega_4^2}{\omega_4^2-p_4^2}\int_0^\infty\frac{d\omega_s^2}{\omega_s^2-s^2}f^W(\omega_1,p_2,\omega_s,p_3,\omega_4)\left(\eta_{\mu\nu}-\frac{p_{12\mu}p_{12\nu}}{\omega_s^2}\right). \tag{6.146}$$

[20]The three point dispersive integrals have previously been performed in [236, 237]. The difference from our analysis here is that they performed dispersive integrals with respect to all three momenta. What we see here is that it is sufficient to do it with respect to just any two of them.

To perform the $\omega_1$ integral, we enclose the poles at $\omega_1 = p_1, \omega_1 = p_2 - \omega_s$ and $\omega_1 = p_2 + \omega_s$ which is the generalization of the prescription used for three point functions in [236]. Then, to perform the $\omega_4$ integral we enclose the poles at $\omega_4 = p_4, \omega_4 = p_3 - \omega_s$ and $\omega_4 = p_3 + \omega_s$. Finally, the $\omega_s$ integral is performed by enclosing the poles at $\omega_s = p_1 + p_2, \omega_s = p_3 + p_4$ and $\omega_s = s$. The result of this computation is,

$$V_3^{YM,\mu}(\epsilon_1,\epsilon_2)V_3^{YM,\nu}(\epsilon_3,\epsilon_4)\frac{1}{(p_1+p_2+p_3+p_4)(p_1+p_2+s)(p_3+p_4+s)}\left(\eta_{\mu\nu}+\frac{s_\mu s_\nu}{(p_1+p_2)(p_3+p_4)s}(p_1+p_2+p_3+p_4+s)\right), \tag{6.147}$$

which is precisely the correct answer for the s-channel contribution to the Euclidean correlator, see for instance [65].

To obtain the $t$ and $u$ channels, we simply re-instate colour factors, permute labels and add them up to obtain the sum of all the exchange diagrams. This gives us,

$$\begin{aligned}
&\langle\langle J^{A_1}(p_1,\epsilon_1)J^{A_2}(p_2,\epsilon_2)J^{A_3}(p_3,\epsilon_3)J^{A_4}(p_4,\epsilon_4)\rangle\rangle_{\text{exchange}}\\
&= \text{Disp}_{p_1^2}\text{Disp}_{p_4^2}\text{Disp}_{s^2}\langle\langle 0|J^{A_1}(p_1,\epsilon_1)J^{A_2}(p_2,\epsilon_2)J^{A_3}(p_3,\epsilon_3)J^{A_4}(p_4,\epsilon_4)|0\rangle\rangle'\\
&+ \text{Disp}_{p_1^2}\text{Disp}_{p_4^2}\text{Disp}_{u^2}\langle\langle 0|J^{A_1}(p_1,\epsilon_1)J^{A_3}(p_3,\epsilon_3)J^{A_2}(p_2,\epsilon_2)J^{A_4}(p_4,\epsilon_4)|0\rangle\rangle'\\
&+ \text{Disp}_{p_1^2}\text{Disp}_{p_3^2}\text{Disp}_{t^2}\langle\langle 0|J^{A_1}(p_1,\epsilon_1)J^{A_4}(p_4,\epsilon_4)J^{A_2}(p_2,\epsilon_2)J^{A_3}(p_3,\epsilon_3)|0\rangle\rangle'.
\end{aligned} \tag{6.148}$$

Disp is shorthand for the dispersive integrals we performed above. Every term on the RHS is a stripped Wightman function in special kinematics where the middle two operators have space-like momenta in each of the terms. This procedure yields the correct exchange contribution to the four point function as can be checked using our expressions and comparing with the known results.

This method however, misses the contact diagram contribution. However, this is not too bad since contact diagrams are in general very simple and can be added to the final result by hand by demanding that the current conservation Ward-Takahashi identtiy for the Euclidean correlator has to be obeyed. Thus we get,

$$\begin{aligned}
&\langle\langle J^{A_1}(p_1,\epsilon_1)J^{A_2}(p_2,\epsilon_2)J^{A_3}(p_3,\epsilon_3)J^{A_4}(p_4,\epsilon_4)\rangle\rangle\\
&= \langle\langle J^{A_1}(p_1,\epsilon_1)J^{A_2}(p_2,\epsilon_2)J^{A_3}(p_3,\epsilon_3)J^{A_4}(p_4,\epsilon_4)\rangle\rangle_{\text{exchange}} + \langle\langle J^{A_1}(p_1,\epsilon_1)J^{A_2}(p_2,\epsilon_2)J^{A_3}(p_3,\epsilon_3)J^{A_4}(p_4,\epsilon_4)\rangle\rangle_{\text{contact}},
\end{aligned} \tag{6.149}$$

where the contact diagram expression can be found in the literature.

We summarize this procedure in figure 6.1.

Let us note that we have obtained the result for the Yang-Mills Euclidean four point function without ever having to perform a nested bulk integral that one would do in traditional Witten diagram approaches. Just starting from the Wightman function in these special kinematics allows us to construct the Euclidean correlator, which has knowledge of all kinematics[21].

## 6.5 Wightman functions in Twistor space

As indicated in the introduction, the motivation for us to study holographic Wightman functions is due to the fact that they enjoy a simple representation in the twistor space

[21] Of course, it is a different matter of performing the analytic continuation from the Euclidean correlator to a Wightman function in arbitrary kinematics in practice. Look at (6.109) for example. The point is, the Euclidean correlator in principle knows about all kinematic configurations of Wightman functions. It would be very interesting to carefully analyze why this is the case and find out how general it is. At the very least, it works for all the spinning four point examples we have considered in this chapter (including those with $\Delta = 1, 2$ scalars).

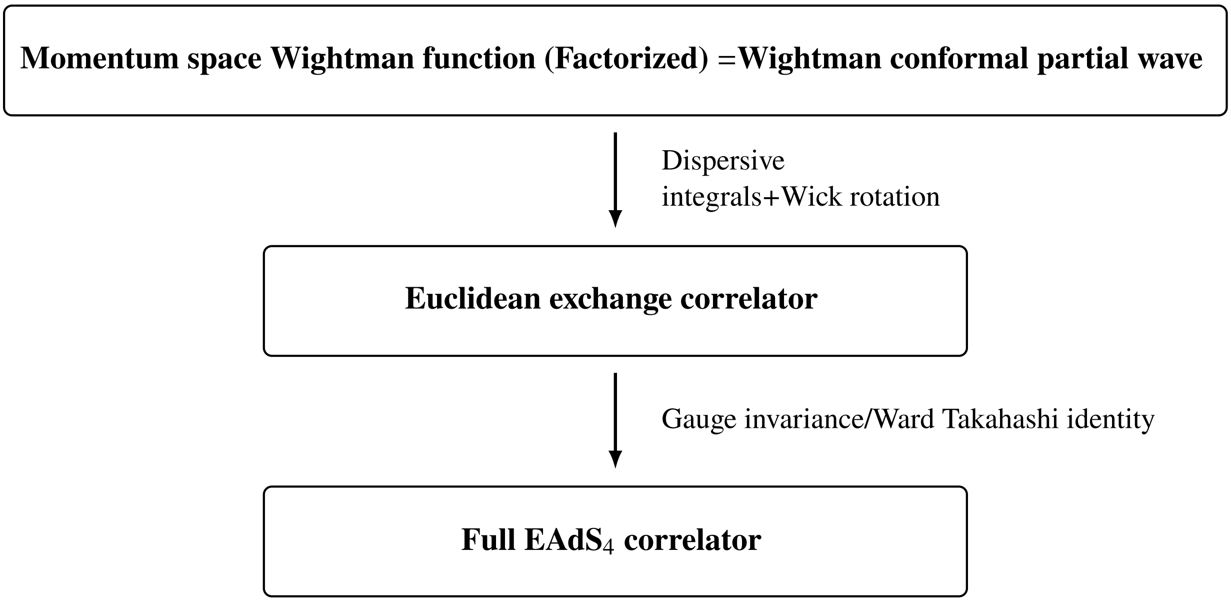


Figure 6.1: Recovering the full correlator from just the special kinematics Wightman function. Note that there is not a single nested bulk integral in sight in this procedure! Once we have the $\mathrm{EAdS}_4$ correlator, one can in principle obtain any Lorentzian correlator in any desired kinematics.

construction for 3d CFT correlators. In this section, after briefly reviewing the construction of twistor variables from momentum space, we first show that the two and three point Wightman functions we have explicitly calculated in the earlier section are precisely the results obtained from twistor space via a half-Fourier transform and a suitable "analytic" continuation. We then take first steps in extending the 3d CFT twistor framework to four point functions.

### 6.5.1 From momentum space to twistors

Let us first review the construction of spinor helicity variables from momentum space, see [57] for a more detailed discussion. If $p_\mu$ is a space-like vector, we can trade it for a pair of real $SL(2,\mathbb{R})$ spinors $\lambda$ and $\bar{\lambda}$ as follows:

$$p_\mu \to p^{ab} = p_\mu(\sigma^\mu)^{ab} = \frac{(\lambda^a\bar{\lambda}^b + \lambda^b\bar{\lambda}^a)}{2}, \tag{6.150}$$

where $(\sigma^\mu)$ are the Lorentzian Pauli matrices. This description has the little group redundancy, $\lambda \to r\lambda, \bar{\lambda} \to \frac{\bar{\lambda}}{r}, p_{ab} \to p_{ab},\ r \in \mathbb{R}/\{0\}$. Lorentz invariant quantities in this description are obtained by contracting spinors using the $SL(2,\mathbb{R})$ Levi-Civita symbol:

$$\langle ij\rangle = \lambda_{ia}\lambda_j^b, \langle\bar{i}\bar{j}\rangle = \bar{\lambda}_{ia}\bar{\lambda}_j^b, \langle i\bar{j}\rangle = \lambda_{ia}\bar{\lambda}_j^a, \tag{6.151}$$

where $i, j$ label spinors associated to momenta $p_{i\mu}, p_{j\mu}$.

Twistor space is related to spinor helicity variables by a *half-Fourier transform* first introduced by Witten [20]. We go from the projective coordinates $(\lambda, \bar{\lambda})$ to projective coordinates $(\lambda, \bar{\mu})$. This is a twistor, which is a fundamental representation of the double cover of the 3d conformal group Sp(4; ℝ).

$$Z^A = (\lambda^a, \bar{\mu}_{a'}), \tag{6.152}$$

Given a conserved current $J_s^{\mu_1\cdots\mu_s}(p_\mu)$ dual to a massless boson, we can trade it for two quantities viz positive and negative helicity components as follows[22]:

$$J_s^{\pm}(\lambda, \bar{\lambda}) = \epsilon^{\pm}_{\mu_1}\cdots\epsilon^{\pm}_{\mu_s} J_s^{\mu_1\cdots\mu_s}(p_\mu), \tag{6.153}$$

with,

$$\epsilon^-_\mu = \frac{-1}{\lambda\cdot\bar{\lambda}}(\sigma_\mu)^b_a\lambda^a\lambda_b\,,\epsilon^+_\mu = \frac{-1}{\lambda\cdot\bar{\lambda}}(\sigma_\mu)^b_a\bar{\lambda}^a\bar{\lambda}_b. \tag{6.154}$$

It is also not too difficult to show that [57],

$$J_{s\mu_1\cdots\mu_s} = \left(-\frac{1}{2}\right)^s\left(\epsilon^-_{\mu_1}\cdots\epsilon^-_{\mu_s}J_s^+ + \epsilon^+_{\mu_1}\cdots\epsilon^+_{\mu_s}J_s^-\right). \tag{6.155}$$

The twistor space counterparts of these currents are obtained via a *half Fourier transform*:

$$\hat{J}_s^{\pm}(Z) = \hat{J}_s^{\pm}(\lambda,\bar{\mu}) = \int \frac{d^2\bar{\lambda}}{(2\pi)^2}e^{i\bar{\lambda}\cdot\bar{\mu}}\frac{J_s^{\pm}(\lambda,\bar{\lambda})}{p^{s-1}}, \tag{6.156}$$

The conformal generators in these variables combine to form a Sp(4; ℝ) covariant quantity,

$$T^{AB} = Z^{(A}\frac{\partial}{\partial Z_{B)}}. \tag{6.157}$$

Note that we have lowered one of the indices in the above expression which we can naturally do using the Sp(4; ℝ) invariant symplectic form Ω as follows:

$$Z^A = \Omega^{AB}Z_B, Z_A = \Omega_{BA}Z^B, \Omega = \begin{pmatrix} 0 & 1_{2\times2} \\ -1_{2\times2} & 0 \end{pmatrix}. \tag{6.158}$$

Thus, one can expect simple results for current correlators due to the simplicity of (6.157). In particular the, twistor space currents are dimensionless[23]. The twistor space correlators obey the conformal Ward identities,

$$\sum_{i=1}^{N}\langle 0|\cdots[T^{AB},\hat{J}^{\pm}_{s_i}(Z_i)]\cdots 0|\rangle = 0. \tag{6.159}$$

[22]The remaining components of the current constructed using (6.154) like in (5.25) are all proportional to its divergence which is zero for a conserved current.

[23]The position space operator has $\Delta = s+1$ which the Fourier transform and rescaling by $\frac{1}{p^{s-1}}$ bring down to $-1$ with the $d^2\bar{\lambda}$ in the half-Fourier transform (6.156) contributing scaling dimension $+1$ thus yielding a dimensionless twistor space current. Note in particular that this is also true for $\Delta = 1$ scalars which can be obtained by setting $s = 0$. On the other hand, the twistor space description of other operators are complicated and involve non-local terms [257].

The solutions to this equation come in two flavors. Symplectic dot products of twistors and the $\delta^4$ distribution

$$\text{Sp}(4;\mathbb{R})\text{ Invariants}:\left\{Z_i\cdot Z_j=-\Omega_{AB}Z_i^A Z_j^B=\lambda_i\cdot\bar{\mu}_j-\lambda_j\cdot\bar{\mu}_i\right\},\left\{\delta^4(\sum_{k=1}^{N}c_kZ_k)=\delta^2(\sum_{k=1}^{N}c_k\lambda_k)\delta^2(\sum_{k=1}^{N}c_k\bar{\mu}_k),c_k\in\mathbb{R}\right\}. \tag{6.160}$$

Our results below shall include both classes of solutions. We will also work with scalar operators which in twistor space are given by,

$$\hat{O}_\Delta(Z)=\int\frac{d^2\bar{\lambda}}{(2\pi)^2}e^{i\bar{\lambda}\cdot\bar{\mu}}p\,O_\Delta(\lambda,\bar{\lambda}). \tag{6.161}$$

For correlators involving $\Delta=1$ scalars, the invariants will indeed be of the form (6.160). Before we proceed to the examples, let us write down the inverse of the half-Fourier transforms (6.156), (6.161) which will be useful later.

$$\hat{J}_s^{\pm}(\lambda,\bar{\lambda})=\frac{J_s^{\pm}(\lambda,\bar{\lambda})}{p^{s-1}}=\int d^2\bar{\mu}e^{-i\bar{\lambda}\cdot\bar{\mu}}\hat{J}_s^{\pm}(Z),$$
$$\hat{O}_\Delta(p)=pO_\Delta(p)=\int d^2\bar{\mu}e^{-i\bar{\lambda}\cdot\bar{\mu}}\hat{O}_\Delta(Z). \tag{6.162}$$

### 6.5.2 What are we computing?

The objects of interest to us are twistor space "Wightman functions" of these operators. Strictly speaking, we have defined twistor space via a half-Fourier transform with momenta that are space-like (5.24). However, even Wightman two point functions have support only when the momenta are time-like which is required by the spectral conditions. As we shall see, what we compute in twistor space are the Wightman functions with the spectral theta functions stripped off and the momenta analytically continued to be spacelike, thus clarifying the relation between twistor correlators and Wightman functions: The former are close spacelike "cousins" of the latter. Let us also note that we re-instate the momentum conserving delta functions in this section as it is required when transforming to twistor space.

### 6.5.3 Two point functions

The general spin-s current two point function in twistor space takes the form[24],

$$\langle 0|\hat{J}_s^{h_1}(Z_1)\hat{J}_s^{h_2}(Z_2)|0\rangle=\delta_{h_1,h_2}\frac{i^{2h_1+2}}{(Z_1\cdot Z_2)^{2h_1+2}}. \tag{6.164}$$

[24]One can also work in an ambidextrous basis, choosing to represent positive helicity currents in twistor space with coordinates $Z^A$ and negative helicity currents in the dual twistor space with coordinates $W_A$. One can go form one description to the other using

$$\hat{J}_s^h(W)=\int\frac{d^4Z}{(2\pi)^2}e^{iW\cdot Z}\hat{J}_s^h(Z). \tag{6.163}$$

Its half-Fourier transform to spinor helicity variables using (6.156) yields,

$$\langle 0|\hat{J}_s^+(p_1)\hat{J}_s^+(p_2)|0\rangle' \propto \frac{\langle\bar{1}\bar{2}\rangle^{2s}}{\langle 1\bar{1}\rangle^{2s-1}}\text{Sign}(\langle 1\bar{1}\rangle)\delta^3(p_1+p_2), \langle 0|\hat{J}_s^-(p_1)\hat{J}_s^-(p_2)|0\rangle'$$
$$\propto \frac{\langle 12\rangle^{2s}}{\langle 1\bar{1}\rangle^{2s-1}}\text{Sign}(\langle 1\bar{1}\rangle)\delta^3(p_1+p_2). \tag{6.165}$$

Let us relate this result to our two point Wightman functions starting with the spin-1 example (6.53). We first contract it with general transverse null polarization vectors $\epsilon_i$.

$$\begin{aligned}\langle 0|J(\epsilon_1,p_1)J(\epsilon_2,p_2)|0\rangle &= 2|p_1|\epsilon_{1\mu}\epsilon_{2\nu}(\eta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2})\theta(-p_1^2)\theta(-p_1^0)(2\pi)^3\delta^3(p_1+p_2)\\ &= 2|p_1|(\epsilon_1\cdot\epsilon_2)\theta(-p_1^2)\theta(-p_1^0)(2\pi)^3\delta^3(p_1+p_2).\end{aligned} \tag{6.166}$$

We then define a stripped Wightman function denoted with a prime:

$$\begin{aligned}&\langle 0|J(\epsilon_1,p_1)J(\epsilon_2,p_2)|0\rangle = \langle 0|J(\epsilon_1,p_1)J(\epsilon_2,p_2)|0\rangle'\theta(-p_1^2)\theta(-p_1^0),\\ &\langle 0|J(\epsilon_1,p_1)J(\epsilon_2,p_2)|0\rangle' = 2|p_1|(\epsilon_1\cdot\epsilon_2)(2\pi)^3\delta^3(p_1+p_2).\end{aligned} \tag{6.167}$$

Although the full Wightman function has support only for time-like momenta, we can analytically continue the stripped Wightman function (6.167) to spacelike momenta. Making use of the spinor helicity construction (5.24) and choosing the polarization vectors to have positive and negative helicity (6.154) respectively yields,

$$\begin{aligned}\langle 0|J^+(p_1)J^+(p_2)|0\rangle' &= 2\text{Sign}(\langle 1\bar{1}\rangle)\frac{\langle\bar{1}\bar{2}\rangle^2}{\langle 1\bar{1}\rangle}(2\pi)^3\delta^3(p_1+p_2),\\ \langle 0|J^-(p_1)J^-(p_2)|0\rangle' &= 2\text{Sign}(\langle 1\bar{1}\rangle)\frac{\langle 12\rangle^2}{\langle 1\bar{1}\rangle}(2\pi)^3\delta^3(p_1+p_2)\end{aligned} \tag{6.168}$$

Note that the stripped correlators have $(1\leftrightarrow 2)$ permutation symmetry since the spectral theta functions that distinguish the operator ordering have been stripped off. Thus, we see that the twistor space two point function found in earlier works is actually the counterpart of the stripped Wightman function (6.167). This also explains the origin of the sign factor in the above equations first found in [236]. Similarly, for the stress tensor two point function, one can follow the same procedure showing that its stripped Wightman function corresponds to (6.164) with $h_i=\pm 2$.

For the general scalar case (6.51), the twistor space answer is more complicated as it involves what is known as the *infinity twistor* of $\mathbb{R}^{2,1}$ [257]. The half-Fourier transform for generic scalar operators from momentum space to twistor space is given in (6.161). The twistor space result corresponding to the scalar two point function was found in [257],

$$\langle 0|\hat{O}_\Delta(Z_1)\hat{O}_\Delta(Z_2)|0\rangle = \frac{|\langle Z_1 I Z_2\rangle|^{2(\Delta-1)}}{(Z_1\cdot Z_2)^{2\Delta}}, \tag{6.169}$$

where,

$$\langle Z_1 I Z_2\rangle = Z_1^A I_{AB} Z_2^B = \langle 12\rangle \text{ with } I_{AB} = \begin{pmatrix}\epsilon_{ab} & 0\\ 0 & 0.\end{pmatrix} \tag{6.170}$$

Going to momentum space using (6.162) yields,

$$\langle 0|\hat{O}_\Delta(p_1)\hat{O}_\Delta(p_2)|0\rangle' \propto (2\pi)^3\delta^3(p_1+p_2)p_1^{2\Delta-3}\text{Sign}(p_1), \tag{6.171}$$

which is precisely the scalar two point Wightman function (6.51) with the spectral theta functions stripped and the result analytically continued to spacelike momenta.

### 6.5.4 Three point functions

We now move on to the case of three point functions. We focus on Wightman functions with $p_2^2 > 0$. Analogous to the two point case, we define the stripped three point Wightman functions,

$$\langle 0|O_1(p_1)O_2(p_2)O_3(p_3)|0\rangle\theta(p_2^2) = \theta(-p_1^2)\theta(-p_1^0)\theta(p_2^2)\theta(-p_3^2)\theta(p_3^0)\langle 0|O_1(p_1)O_2(p_2)O_3(p_3)|0\rangle', \tag{6.172}$$

where $O_i$ are arbitrary (spinning) operators. When converting to twistor space, we analytically continue the momenta in the stripped correlator to be spacelike. When some of these operators are identical, we shall see that the stripped correlator also possesses permutation symmetry.

Below, we discuss the twistor counterparts of the various three point Wightman functions we have computed earlier. Let us also mention that the scalars we shall focus on henceforth are conformally coupled ones with Neumann boundary conditions imposed. This implies that the dual conformal operator has $\Delta = 1$ which as we discussed in footnote 23, enjoys a simpler description in twistor space compared to its generic $\Delta$ counterparts. Also, we ignore overall numerical constants as well as the bulk couplings in the expressions for notational simplicity but it is a simple matter to reinstate them if desired.

For general spins $s_1, s_2$ and $s_3$ (including the $\Delta = 1$ scalar by setting spin to zero), the general solution at the conformal Ward identities is [257][25],

$$\begin{aligned}
&\langle 0|\hat{J}_{s_1}^{h_1}(Z_1)\hat{J}_{s_2}^{h_2}(Z_2)\hat{J}_{s_3}^{h_3}(Z_3)|0\rangle \\
&= i^{\alpha+\beta+\gamma}\Big(c_{h_1h_2h_3}\delta^{[\alpha]}(Z_1\cdot Z_2)\delta^{[\beta]}(Z_2\cdot Z_3)\delta^{[\gamma]}(Z_3\cdot Z_1) \\
&+ k_{h_1h_2h_3}\delta^{[\alpha+\beta+\gamma]}(Z_1\cdot Z_2)\int dc_1dc_2c_1^{-\beta}c_2^{-\gamma}\delta^4(c_1Z_1+c_2Z_2-Z_3)\Big),
\end{aligned} \tag{6.173}$$

where the coefficients $c_{h_1h_2h_3}, k_{h_1h_2h_3}$ and $\alpha, \beta, \gamma$ depend on the helicity. Using the fact that the conserved currents have helicity $h_i = \pm s_i$, we have,

$$(h_1h_2h_3)\ \text{helicity}:\ \alpha = h_1+h_2-h_3, \beta = h_2+h_3-h_1, \gamma = h_3+h_1-h_2. \tag{6.174}$$

Our notation for the delta function derivatives is as follows:

$$\delta^{[n]}(x) = \int_{-\infty}^{\infty} dc(-ic)^n e^{-icx}. \tag{6.175}$$

[25]One can also choose to work in a mixture of twistor and dual twistor variables like in [236, 237]. These are related to the solutions in (6.173) by a twistor Fourier transform (6.163).

We also make sense of this formula by suitably regularizing the cases where $n$ is a negative integer following earlier works. For example, the anti-derivative of $\delta^{[0]}(x) = \delta(x)$ is $\delta^{[-1]}(x) = \frac{\text{Sgn}(x)}{2}$,$\delta^{[-2]}(x) = \frac{|x|}{2}$ and so on.

The inverse half-Fourier transform to spinor helicity variables (6.162) of the general solution (6.173) results in,

$$\begin{aligned}&\langle 0|\hat{J}_{s_1}^{h_1}(p_1)\hat{J}_{s_2}^{h_2}(p_2)\hat{J}_{s_3}^{h_3}(p_3)|0\rangle' \\ &= \frac{(-1)^{\alpha+\beta+\gamma}}{4}(2\pi)^3\delta^3(p_1+p_2+p_3)\langle\bar{1}\bar{2}\rangle^{\alpha}\langle\bar{2}\bar{3}\rangle^{\beta}\langle\bar{3}\bar{1}\rangle^{\gamma}\left(\frac{c_{h_1h_2h_3}}{E^{\alpha+\beta+\gamma}} + \frac{k_{h_1h_2h_3}}{(E-2p_1)^{\alpha}(E-2p_2)^{\gamma}(E-2p_3)^{\beta}}\right),\end{aligned} \tag{6.176}$$

where $E = p_1 + p_2 + p_3$ as usual. We will now see how (6.176) reproduces all the three point results we have computed so far.

### Conformally coupled scalars

Let us begin with the simplest three point function: That of a $\Delta = 1$ scalar operator. The twistor space expression corresponding to this correlator is found by setting $k_{s_1s_2s_3} = 1$ and $c_{s_1s_2s_3} = 0$ in our general solution (6.173), see [257].

$$\langle 0|\hat{O}_1(Z_1)\hat{O}_1(Z_2)\hat{O}_1(Z_3)|0\rangle = \delta(Z_1\cdot Z_2)\int_{-\infty}^{\infty} dc_1\int_{-\infty}^{\infty} dc_2\, \delta^4(c_1Z_1+c_2Z_2-Z_3). \tag{6.177}$$

The inverse half Fourier transform found using (6.176) yields,

$$\begin{aligned}&\langle 0|\hat{O}_1(p_1)\hat{O}_1(p_2)\hat{O}_1(p_3)|0\rangle' = \frac{1}{4}(2\pi)^3\delta^3(p_1+p_2+p_3)\\ &\implies \langle 0|O_1(p_1)O_1(p_2)O_1(p_3)|0\rangle' = \frac{(2\pi)^3\delta^3(p_1+p_2+p_3)}{p_1p_2p_3}\end{aligned} \tag{6.178}$$

This is indeed the correct stripped Wightman function which can be obtained by evaluating the integral with $\theta(p_2^2)$ in (6.65), setting all the scaling dimensions to 1 and stripping the theta functions as in (6.172).

### Photon-Scalar-Scalar

Similar to the scalar example, one can choose the values of the coefficients in the general solution (6.176) to match with the stripped Wightman functions corresponding to this scalar QED example.

$$\begin{aligned}&\langle 0|\hat{J}^+(Z_1)\hat{O}_1(Z_2)\hat{O}_1(Z_3)|0\rangle = ic_{100}\left(\delta^{[1]}(Z_1\cdot Z_2)\text{Sgn}(Z_2\cdot Z_3)\delta^{[1]}(Z_3\cdot Z_1) + \delta^{[1]}(Z_1\cdot Z_2)\int \frac{dc_1}{c_1}\frac{dc_2}{c_2}\,\delta^4(c_1Z_1+c_2Z_2-Z_3)\right),\\ &\langle 0|\hat{J}^-(Z_1)\hat{O}_1(Z_2)\hat{O}_1(Z_3)|0\rangle = ic_{-100}\left(\text{Sgn}(Z_1\cdot Z_2)\delta^{[1]}(Z_2\cdot Z_3)\text{Sgn}(Z_3\cdot Z_1) + \text{Sgn}(Z_1\cdot Z_2)\int dc_1dc_2c_1c_2\,\delta^4(c_1Z_1+c_2Z_2-Z_3)\right),\end{aligned} \tag{6.179}$$

with their spinor helicity counterparts found using (6.176). Contracting our Wightman function result (6.73),(6.75) with the null transverse polarization vector $\epsilon_{1\mu}$, stripping off the spectral functions as in (6.172)and converting to spinor helicity variables shows that it matches with (6.179).

### Yang-Mills theory

Let us move on to a case where all three operators have spin. In this case, we have eight helicity configurations to deal with. In appendix D, we have provided all these expressions both in spinor helicity and twistor space.

For example, consider the $(+++)$ helicity configutation. As the reader can easily verify using (6.162), the twistor space expression corresponding to our Wightman function (6.82) is none other than (6.173) with $s_1 = s_2 = s_3 = 1, c_{111} = 0, k_{111} = 1$ viz[26].

$$\langle 0|\hat{J}^{+A}(Z_1)\hat{J}^{+B}(Z_2)\hat{J}^{+C}(Z_3)|0\rangle = f^{ABC}\text{Sgn}(Z_1\cdot Z_2)\int \frac{dc_1 dc_2}{c_1 c_2}\delta^4(c_1 Z_1 + c_2 Z_2 - Z_3). \tag{6.181}$$

### More examples also including gravity

Similar analysis can be carried out for the remaining examples. Here, we quote the final results in twistor space which give rise to the stripped momentum space correlators.

### Minimal coupling between scalars and gravitons

For $\langle 0|T^+ O_1 O_1|0\rangle$, we obtain the positive helicity stripped Wightman function using our result (6.87) and the definition (6.172). The twistor space counterpart to it is obtained by setting $\alpha = 2, \beta = -2, \gamma = 2, k_{200} = 1, c_{200} = 1$, which results in,

$$\begin{aligned}\langle 0|\hat{T}^+(Z_1)\hat{O}_1(Z_2)\hat{O}_1(Z_3)|0\rangle' &= -\delta^{[2]}(Z_1\cdot Z_2)\delta^{[-2]}(Z_2\cdot Z_3)\delta^{[2]}(Z_3\cdot Z_1)\\ &\quad + \delta^{[2]}(Z_1\cdot Z_2)\int dc_1 dc_2 c_1^2 c_2^{-2}\delta^4(c_1 Z_1 + c_2 Z_2 - Z_3),\end{aligned} \tag{6.182}$$

as can be verified using (6.162) and the result (6.176). It is easy to obtain the negative helicity case similarly.

### Einstein Gravity

Finally, we move on to the example of Einstein gravity in the $(+++)$ helicity which can be found using (6.90), writing down the stripped correlator (6.172) and choosing all helicities to be positive using (6.153). The spinor helicity and twistor results are

[26] It is easy to show using (6.163) that this in dual twistor space equals,

$$\langle 0|\hat{J}^{+A}(W_1)\hat{J}^{+B}(W_2)\hat{J}^{+C}(W_3)|0\rangle = f^{ABC}\text{Sgn}(W_1\cdot W_2)\text{Sgn}(W_2\cdot W_3)\text{Sgn}(W_3\cdot W_1), \tag{6.180}$$

matching with the result of [236].

provided in appendix D.1. For example we find[27],

$$\langle 0|\hat{T}^+(Z_1)\hat{T}^+(Z_2)\hat{T}^+(Z_3)|0\rangle = \int \frac{dc_{12}dc_{23}dc_{31}}{c_{12}^2 c_{23}^2 c_{31}^2}\delta^4(c_{12}Z_3 + c_{23}Z_1 + c_{31}Z_2)e^{\frac{iZ_1\cdot Z_2}{c_{12}}}. \tag{6.184}$$

This concludes our discussion of three point functions in twistor space. What we have established is the fact that the two and three point twistor space correlators considered in literature so far actually correspond to stripped Wightman functions.

### 6.5.5 Four point functions

Given the fact that we have obtained factorized formulae for Wightman functions when the middle two operators have spacelike momenta, it is natural to leverage our knowledge of the three point functions in twistor space to obtain the results at four points in this kinematic regime. In this section, we focus exactly on this problem. To begin with, we define the stripped four point Wightman function,

$$\begin{aligned}&\langle 0|O_1(p_1)O_2(p_2)O_3(p_3)O_4(p_4)|0\rangle\theta(p_2^2)\theta(p_3^2)\\ &= \theta(-p_1^2)\theta(-p_1^0)\theta(p_2^2)\theta(-s^2)\theta(-s^0)\theta(p_3^2)\theta(-p_4^2)\theta(p_4^0)\langle 0|O_1(p_1)O_2(p_2)O_3(p_3)O_4(p_4)|0\rangle'.\end{aligned} \tag{6.185}$$

As we showed in subsection 6.3.5, this coincides with a Wightman conformal partial wave which through figure 6.1 has enough information to recover the full correlator. Thus, it is an extremely important object to construct in twistor space.

#### Conformally coupled scalars

We begin with the simplest example of $\Delta = 1$ scalars exchanging $\Delta = 1$ scalars. In this case we have the extremely simple expression for the rescaled correlator,

$$\langle 0|\hat{O}_1(p_1)\hat{O}_1(p_2)\hat{O}_1(p_3)\hat{O}_1(p_4)|0\rangle' = \frac{1}{|s|}(2\pi)^3\delta^3(p_1+p_2+p_3+p_4). \tag{6.186}$$

Introducing an auxiliary integral we can write it as,

$$\int \frac{d^3p}{(2\pi)^3|p|}\langle 0|\hat{O}_1(p_1)\hat{O}_1(p_2)\hat{O}_1(p)|0\rangle'\langle 0|\hat{O}_1(-p)\hat{O}_1(p_3)\hat{O}_1(p_4)|0\rangle'. \tag{6.187}$$

In this form, conversion to twistor space is straightforward!. For the integrated over momentum $p$, we re-express the measure in spinor helicity using the formula [257],

$$\int d^3p = \frac{1}{2\mathrm{Vol}(GL(1,\mathbb{R}))}\int d^2\lambda d^2\bar{\lambda}|p|. \tag{6.188}$$

[27] Using the twistor Fourier transform (6.163) we see that the dual twistor space expression equals,

$$\langle 0|\hat{T}^{+A}(W_1)\hat{T}^{+B}(W_2)\hat{T}^{+C}(W_3)|0\rangle = |(W_1\cdot W_2)||W_2\cdot W_3||W_3\cdot W_1|, \tag{6.183}$$

which matches with the result of [236].

As for the operators with this momenta we use,

$$\hat{O}_1(p) = \hat{O}_1(\lambda, \bar{\lambda})$$
$$\hat{O}_1(-p) = \hat{O}_1(\lambda, -\bar{\lambda}). \tag{6.189}$$

Essentially we describe a momenta using $(\lambda, \bar{\lambda})$ and minus of the momenta by flipping the sign of $\bar{\lambda}$. For the external operators we directly perform the half-Fourier transform (6.161). For the exchanged one, we simply write it in terms of its twistor space counterpart using (6.162). Putting all these steps together we have,

$$\begin{aligned}
&\langle 0|\hat{O}_1(Z_1)\hat{O}_1(Z_2)\hat{O}_1(Z_3)\hat{O}_1(Z_4)|0\rangle \\
&= \frac{1}{2\text{Vol}(GL(1,\mathbb{R}))}\int \frac{d^2\lambda d^2\bar{\lambda} d^2\bar{\mu} d^2\bar{\nu}}{(2\pi)^3}\langle 0|\hat{O}_1(Z_1)\hat{O}_2(Z_2)\hat{O}_1(\lambda,\bar{\mu})|0\rangle\langle 0|\hat{O}_1(\lambda,\bar{\nu})\hat{O}_1(Z_3)\hat{O}_1(Z_4)|0\rangle e^{i\bar{\lambda}\cdot(\bar{\nu}-\bar{\mu})} \\
&= \frac{1}{2\text{Vol}(GL(1,\mathbb{R}))}\int \frac{d^2\lambda d^2\bar{\mu} d^2\bar{\nu}}{(2\pi)}\delta^2(\bar{\nu}^a - \bar{\mu}^a)\langle 0|\hat{O}_1(Z_1)\hat{O}_2(Z_2)\hat{O}_1(\lambda,\bar{\mu})|0\rangle\langle\langle 0|\hat{O}_1(\lambda,\bar{\nu})\hat{O}_1(Z_3)\hat{O}_1(Z_4)|0\rangle \\
&= \frac{1}{4\pi\text{Vol}(GL(1,\mathbb{R}))}\int d^2\lambda d^2\bar{\mu}\ \langle 0|\hat{O}_1(Z_1)\hat{O}_2(Z_2)\hat{O}_1(\lambda,\bar{\mu})|0\rangle\langle 0|\hat{O}_1(\lambda,\bar{\mu})\hat{O}_1(Z_3)\hat{O}_1(Z_4)|0\rangle \\
&= \frac{1}{4\pi\text{Vol}(GL(1,\mathbb{R}))}\int d^4Z\ \langle 0|\hat{O}_1(Z_1)\hat{O}_2(Z_2)\hat{O}_1(Z)|0\rangle\langle 0|\hat{O}_1(Z)\hat{O}_1(Z_3)\hat{O}_1(Z_4)|0\rangle,
\end{aligned} \tag{6.190}$$

where to go from the first to the second line, we performed the $d^2\bar{\lambda}$ integral and in going to the third from the second, we used the resulting delta function for the $d^2\bar{\nu}$ integral setting $\bar{\nu}^a = \bar{\mu}^a$. Finally, we defined the twistor space $\mathbb{RP}^3$ measure,

$$\frac{d^4Z}{\text{Vol}(GL(1,\mathbb{R}))} = \frac{d^2\lambda d^2\bar{\lambda}}{\text{Vol}(GL(1,\mathbb{R}))}, Z^A = (\lambda^a, \bar{\mu}_{a'}). \tag{6.191}$$

Note that this four point function is simply a product of three point functions with the common twistor integrated over all of $\mathbb{RP}^3$. Let us now use the expressions for these three point functions (6.177) in the above expression. We obtain the beautiful expression[28],

$$\begin{aligned}
&\langle 0|\hat{O}_1(Z_1)\hat{O}_1(Z_2)\hat{O}_1(Z_3)\hat{O}_1(Z_4)|0\rangle \\
&= \frac{c_{123}^2}{8\pi\text{Vol}(GL(1,\mathbb{R}))}\delta(Z_1\cdot Z_2)\delta(Z_3\cdot Z_4)\int dc_1dc_2dc_3dc_4\int d^4Z\delta^4(c_1Z_1+c_2Z_2-Z)\delta^4(c_3Z_3+c_4Z_4+Z) \\
&= \frac{c_{123}^2}{8\pi\text{Vol}(GL(1,\mathbb{R}))}\delta(Z_1\cdot Z_2)\delta(Z_3\cdot Z_4)\int dc_1dc_2dc_3dc_4\delta^4(c_1Z_1+c_2Z_2+c_3Z_3+c_4Z_4) \\
&= \frac{c_{123}^2}{8\pi|Z_1\cdot Z_3||Z_2\cdot Z_4|}\delta(Z_1\cdot Z_2)\delta(Z_3\cdot Z_4)\delta(v-1),
\end{aligned} \tag{6.192}$$

where the conformal cross ratio $v$ is defined as,

$$v = \frac{Z_1\cdot Z_4 Z_2\cdot Z_3}{Z_1\cdot Z_3 Z_2\cdot Z_4}. \tag{6.193}$$

[28] As we discussed, this correlator is the twistor space counterpart of the Wightman function in the special kinematics. It would be interesting to explore the twistor correlators corresponding to more general kinematic configurations. It would also be interesting to explore if there exist twistor space expressions corresponding to time-ordered or Euclidean correlators. One possible way would be to come up with a prescription for analytic continuation in twistor space or in Schwinger parameter space to obtain these other correlators. We thank Guilherme Pimentel for this comment.

Following our discussion in subsection 6.3.5, we can identify the final expression as a twistor space Wightman conformal partial wave corresponding to the $\Delta = 1$ scalar exchange:

$$\mathcal{W}^{(s)}_{(O_1O_1|O_1|O_1O_1)}(Z_1, Z_2, Z_3, Z_4) = \frac{1}{8\pi|Z_1 \cdot Z_3||Z_2 \cdot Z_4|}\delta(Z_1 \cdot Z_2)\delta(Z_3 \cdot Z_4)\delta(v-1). \tag{6.194}$$

Note the remarkable simplicity of this result. The expression is localized at the value of the cross ratio $v = 1$.

### Yang Mills theory

Let us start with the stripped momentum space correlator viz,

$$\begin{aligned}
&\langle 0|J^{A_1}(p_1,\epsilon_1)J^{A_2}(p_2,\epsilon_2)J^{A_3}(p_3,\epsilon_3)J^{A_4}(p_4,\epsilon_4)|0\rangle' = (2\pi)^3\delta^3(p_1+p_2+p_3+p_4)\\
&\times \langle\langle 0|J^{A_1}(p_1,\epsilon_1)J^{A_2}(p_2,\epsilon_2)J^{A\mu}(-s)|0\rangle\rangle'\frac{\pi_{\mu\nu}(s)}{|s|}\langle\langle 0|J^{\nu}_{A}(-p_3-p_4)J^{A_3}(p_3,\epsilon_3)J^{A_4}(p_4,\epsilon_4)|0\rangle'\\
&= \int\frac{d^3p}{(2\pi)^3|p|}\int\langle 0|J^{A_1}(p_1,\epsilon_1)J^{A_2}(p_2,\epsilon_2)J^{A\mu}(p)|0\rangle'\pi_{\mu\nu}(p)\langle 0|J^{\nu}_{A}(-p)J^{A_3}(p_3,\epsilon_3)J^{A_4}(p_4,\epsilon_4)|0\rangle'.
\end{aligned} \tag{6.195}$$

The most convinient way to deal with this expression is the helicity basis. For the exchanged current, we use the formula,

$$J^{A\mu}(p)|0\rangle\pi_{\mu\nu}(p)\langle 0|J^{\nu}_{A}(-p) = \Big(J^{A+}(p)|0\rangle\langle 0|J^{-}_{A}(-p) + J^{A-}(p)|0\rangle\langle 0|J^{+}_{A}(-p)\Big). \tag{6.196}$$

For the external operators we consider the MHV configuration with $\epsilon_1, \epsilon_3$ being positive helicity polarizations and $\epsilon_2, \epsilon_4$ being negative helicity ones, using their form in (6.154). This converts the above expression into,

$$\begin{aligned}
&\langle 0|J^{A_1+}(p_1)J^{A_2-}(p_2)J^{A_3+}(p_3)J^{A_4-}(p_4)|0\rangle'\\
&= \frac{1}{2(2\pi)^3}\int d^2\lambda d^2\bar{\lambda}\langle 0|J^{A_1+}(p_1)J^{A_2-}(p_2)J^{A+}(p)|0\rangle'\langle 0|J^{-}_{A}(-p)J^{A_3+}(p_3)J^{A_4-}(p_4)|0\rangle' + (1\leftrightarrow 3, 2\leftrightarrow 4).
\end{aligned} \tag{6.197}$$

where we used (6.188) for writing the measure in spinor helicity variables. Also note that we are using the fact that these stripped Wightman functions have permutation symmetry so we can perform a $(1 \leftrightarrow 3, 2 \leftrightarrow 4)$ exchange to obtain the contribution due to the second term in (6.196). Performing a half-Fourier transform (6.162) for the external operators and expressing the exchanged operator following the scalar case viz (6.189), (6.190), we obtain,

$$\begin{aligned}
&\langle 0|J^{A_1+}(Z_1)J^{A_2-}(Z_2)J^{A_3+}(Z_3)J^{A_4-}(Z_4)|0\rangle\\
&= \frac{1}{4\pi\mathrm{Vol}(GL(1,\mathbb{R}))}\int d^4Z\langle 0|J^{A_1+}(Z_1)J^{A_2-}(Z_2)J^{A+}(Z)|0\rangle\langle J^{-}_{A}(Z)J^{A_3+}(Z_3)J^{A_4-}(Z_4)|0\rangle + (1\leftrightarrow 3, 2\leftrightarrow 4).
\end{aligned} \tag{6.198}$$

Plugging in the explicit forms of these three point functions results in[29],

$$\frac{\langle 0|J^{A_1+}(Z_1)J^{A_2-}(Z_2)J^{A_3+}(Z_3)J^{A_4-}(Z_4)|0\rangle}{f^{A_1A_2A}f_A^{A_3A_4}}$$
$$= \frac{8i^2\mathrm{Sgn}(Z_1\cdot Z_2)\mathrm{Sgn}(Z_3\cdot Z_4)}{4\pi\mathrm{Vol}(GL(1,\mathbb{R}))}\int dc_3\int dc_4\frac{c_3^3}{c_4}\int d^4Z\mathrm{Sgn}(Z_2\cdot Z)\delta^{[3]}(Z\cdot Z_1)\delta^4(Z-c_3Z_3-c_4Z_4)+(1\leftrightarrow 3,2\leftrightarrow 4)$$
$$= \frac{2i^2\mathrm{Sgn}(Z_1\cdot Z_2)\mathrm{Sgn}(Z_3\cdot Z_4)}{\pi\mathrm{Vol}(GL(1,\mathbb{R}))}\int dc_3\int dc_4\frac{c_3^3}{c_4}\mathrm{Sgn}(c_3Z_2\cdot Z_3+c_4Z_2\cdot Z_4)\delta^{[3]}(c_3Z_3\cdot Z_1+c_4Z_4\cdot Z_1)+(1\leftrightarrow 3,2\leftrightarrow 4)$$
$$= \frac{4i^4\mathrm{Sgn}(Z_1\cdot Z_2)\mathrm{Sgn}(Z_3\cdot Z_4)}{\pi\mathrm{Vol}(GL(1,\mathbb{R}))}\int dc_1dc_2dc_3dc_4\frac{c_1^3c_3^3}{c_2c_4}e^{-ic_2c_3Z_2\cdot Z_3-ic_2c_4Z_2\cdot Z_4}e^{ic_1c_3Z_1\cdot Z_3+ic_1c_4Z_1\cdot Z_4}+(1\leftrightarrow 3,2\leftrightarrow 4)$$
$$= \frac{4i^4\mathrm{Sgn}(Z_1\cdot Z_2)\mathrm{Sgn}(Z_3\cdot Z_4)}{\pi\mathrm{Vol}(GL(1,\mathbb{R}))}\int dc_1dc_2dc_3dc_4\frac{c_1^3c_3^3}{c_2c_4}e^{ic_1c_4Z_1\cdot Z_4-ic_2c_3Z_2\cdot Z_3}\cos\left(c_1c_3Z_1\cdot Z_3-c_2c_4Z_2\cdot Z_4\right)$$
$$= \frac{16}{\pi}\mathrm{Sgn}(Z_1\cdot Z_2)\mathrm{Sgn}(Z_2\cdot Z_3)\mathrm{Sgn}(Z_3\cdot Z_4)\mathrm{Sgn}(Z_4\cdot Z_1)\frac{1}{|Z_1\cdot Z_3|^4}\frac{d^3}{dv^3}(v^3\mathrm{Sign}(1-\frac{1}{v})), \tag{6.199}$$

which is a simple and pleasing result. $v$ is the conformal cross ratio (6.193). Although it is not obvious from the above, we note that the $(1\leftrightarrow 3, 2\leftrightarrow 4)$ exchange term simply contributed an extra factor of 2. The fact that the two terms are equal is not obvious in spinor helicity variables due to the presence of degeneracies and Schouten identities[30]. Given the fact that the two terms are identical in twistor space, we can now go back to spinor helicity variables to verify it. We have checked numerically that,

$$\langle 0|J^{A_1+}(p_1)J^{A_2-}(p_2)J^{A+}(-p_1-p_2)|0\rangle'\frac{1}{|s|}\langle 0|J_A^-(p_1+p_2)J^{A_3+}(p_3)J^{A_4-}(p_4)|0\rangle'$$
$$= \langle 0|J^{A_1+}(p_1)J^{A_2-}(p_2)J_A^-(-p_1-p_2)|0\rangle'\frac{1}{|s|}\langle 0|J^{A+}(p_1+p_2)J^{A_3+}(p_3)J^{A_4-}(p_4)|0\rangle'. \tag{6.200}$$

Therefore, the answer in spinor helicity variables (6.197) is simply,

$$\langle 0|J^{A_1+}(p_1)J^{A_2-}(p_2)J^{A_3+}(p_3)J^{A_4-}(p_4)|0\rangle$$
$$= \frac{1}{(2\pi)^3}\int d^2\lambda d^2\bar{\lambda}\langle 0|J^{A_1+}(p_1)J^{A_2-}(p_2)J^{A+}(p)|0\rangle\langle 0|J_A^-(-p)J^{A_3+}(p_3)J^{A_4-}(p_4)|0\rangle$$
$$= \frac{4}{|s|}\langle 0|J^{A_1+}(p_1)J^{A_2-}(p_2)J^{A_3+}(-p_1-p_2)|0\rangle'\langle 0|J_A^-(-p_3-p_4)J^{A_3+}(p_3)J^{A_4-}(p_4)|0\rangle' \tag{6.201}$$

Twistor space expressions on the other hand, are much more easier to manipulate and handle at four points.

Finally, we note that the pre-factor in (6.199) is

$$\mathrm{Sgn}(Z_1\cdot Z_2)\mathrm{Sgn}(Z_2\cdot Z_3)\mathrm{Sgn}(Z_3\cdot Z_4)\mathrm{Sgn}(Z_4\cdot Z_1), \tag{6.202}$$

[29] One can obtain the dual twistor space expressions using (6.163) if desired.

[30] These arise due to the fact that spinor helicity variables deal with two component spinors out of which only two can be linearly independent. The same could be said for twistor space where the variables are $\lambda_a$ and $\bar{\mu}_a$ for each operator out of which only two of them are linearly independent. However, the results for conserved currents and $\Delta = 1$ scalars reorganize themselves in terms of $Z^A$ as can be seen by the structure of the conformal generators (6.157). In these variables, there are no degeneracies at four points since $Z^A$ is itself a four component object.

which is identical in form to the flat space gluon MHV amplitude [252]! Similar to the scalar case, we identify this twistor space expression (6.199) with the Wightman CPW corresponding to the exchange of a spin-1 current:

$$\mathcal{W}^{(s)}_{(JJ|J|JJ)}(Z_1, Z_2, Z_3, Z_4) = \frac{16}{\pi}\text{Sgn}(Z_1 \cdot Z_2)\text{Sgn}(Z_2 \cdot Z_3)\text{Sgn}(Z_3 \cdot Z_4)\text{Sgn}(Z_4 \cdot Z_1)\frac{1}{|Z_1 \cdot Z_3|^4}\frac{d^3}{dv^3}(v^3\text{Sign}(1-\frac{1}{v})). \tag{6.203}$$

We proceed to the case of Einstein gravity next.

### Einstein Gravity

Moving on to Einstein gravity, the stripped four point function in the MHV configuration is given by,

$$\begin{aligned}&\langle 0|T^+(p_1)T^-(p_2)T^+(p_3)T^-(p_4)|0\rangle' \\ &= \int \frac{d^3p}{(2\pi)^3|p|}\langle 0|T^+(p_1)T^-(p_2)T^{\mu\nu}(p)|0\rangle'\Pi_{\mu\nu\rho\sigma}(p)\langle 0|T^{\rho\sigma}(-p)T^+(p_3)T^-(p_4)|0\rangle'.\end{aligned} \tag{6.204}$$

We then analytically continuing all momenta to be space-like and use the helicity basis for the exchanged operator (6.155). Performing a half-Fourier transform for the external operators (6.156) and expressing the exchanged ones using the inverse half-Fourier transform (6.162) and performing the same sequence of steps as we did for the previous examples results in,

$$\begin{aligned}&\langle 0|T^+(Z_1)T^-(Z_2)T^+(Z_3)T^-(Z_4)|0\rangle \\ &= \frac{1}{4\pi\text{Vol}(GL(1,\mathbb{R}))}\int d^4Z\langle 0|T^+(Z_1)T^-(Z_2)T^+(Z)|0\rangle\langle T^-(Z)T^+(Z_3)T^-(Z_4)|0\rangle + (1\leftrightarrow 3, 2\leftrightarrow 4).\end{aligned} \tag{6.205}$$

Using the expressions for the three point functions and performing the $d^4Z$ integral results in,

$$\begin{aligned}&\langle 0|T^+(Z_1)T^-(Z_2)T^+(Z_3)T^-(Z_4)|0\rangle \\ &= \frac{4}{\pi\text{Vol}(GL(1,\mathbb{R}))}|Z_1\cdot Z_2||Z_3\cdot Z_4|\int dc_1dc_2dc_3dc_4\frac{c_1^6c_3^6}{c_2^2c_4^2}e^{ic_1c_4Z_1\cdot Z_4 - ic_2c_3Z_2\cdot Z_3}\cos\left(c_1c_3Z_1\cdot Z_3 - c_2c_4Z_2\cdot Z_4\right) \\ &= 32|Z_1\cdot Z_2||Z_2\cdot Z_3||Z_3\cdot Z_4||Z_4\cdot Z_1|\frac{1}{|Z_1\cdot Z_3|^8}\frac{1}{v}\frac{d^6}{dv^6}\left(v^7|1-\frac{1}{v}|\right).\end{aligned} \tag{6.206}$$

Very interestingly, the pre-factor $|Z_1\cdot Z_2||Z_2\cdot Z_3||Z_3\cdot Z_4||Z_4\cdot Z_1|$ is exactly of the same form as the four graviton scattering amplitude in flat space [252] exactly like the gluon case. Similar to the previous examples, we can interpret this result as the Wightman CPW corresponding to the exchange of the stress tensor,

$$\mathcal{W}^{(s)}_{(TT|T|TT)}(Z_1, Z_2, Z_3, Z_4) = 32|Z_1\cdot Z_2||Z_2\cdot Z_3||Z_3\cdot Z_4||Z_4\cdot Z_1|\frac{1}{|Z_1\cdot Z_3|^8}\frac{1}{v}\frac{d^6}{dv^6}\left(v^7|1-\frac{1}{v}|\right). \tag{6.207}$$

Also note the similarity of (6.206) to the gluon MHV result (6.199). In fact, that $(1 \leftrightarrow 3)$, $(2 \leftrightarrow 4)$ term just contributes an extra factor of 2 just like the gluon case. This implies that even in spinor helicity variables where there appear to be two independent terms (6.204) as we see using (6.155), they are actually equal! This is not obvious to show directly in spinor helicity due to the degeneracies but the twistor space analysis makes it clear! Thus we have in spinor helicity,

$$\langle\langle 0|T^{+}(p_1)T^{-}(p_2)T^{+}(p_3)T^{-}(p_4)|0\rangle\rangle' = \frac{4}{|s|^3}\langle\langle 0|T^{+}(p_1)T^{-}(p_2)T^{+}(-p_1-p_2)|0\rangle\rangle'\langle\langle 0|T^{-}(-p_3-p_4)T^{+}(p_3)T^{-}(p_4)|0\rangle\rangle', \tag{6.208}$$

which one can verify numerically. It is however more satisfying that twistor space on the other hand has given us simple analytic proof that they are equal. This indicates a double copy with the gluon Wightman function which we will make this much more concrete in section 6.6.

This concludes our discussion of correlators in twistor space. One can similarly convert our other results such as for Bhabha and Compton scattering to twistor space if desired. In short, what we have seen in this section is that twistor space yields extremely simple and elegant expressions for correlators, whose conformal invariance is manifest. We leave a more comprehensive analysis to the future.

### 6.5.6 Higher point functions

In this brief subsection, we convert our result for the $\Delta = 1$ scalar five point function to twistor space. We have the factorized expression (6.137). After rescaling it for suitablitiy to twistor space (6.162), we can re-write the stripped five point Wightman function as,

$$\begin{aligned}
&\langle 0|\hat{O}_1(p_1)\hat{O}_1(p_2)\hat{O}_1(p_3)\hat{O}_1(p_4)\hat{O}_1(p_5)|0\rangle' = \frac{1}{|s_{12|}|s_{123}|}(2\pi)^3\delta^3(p_1+p_2+p_3+p_4+p_5)\\
&= \int \frac{d^3p}{(2\pi)^3|p|}\frac{d^3q}{(2\pi)^3|q|}\langle 0|\hat{O}_1(p_1)\hat{O}_1(p_2)\hat{O}_1(p)|0\rangle'\langle 0|\hat{O}_1(-p)\hat{O}_1(p_3)\hat{O}_1(q)|0\rangle'\langle 0|\hat{O}_1(-q)\hat{O}_1(p_4)\hat{O}_1(p_5)|0\rangle'.
\end{aligned} \tag{6.209}$$

Performing steps analogous to the above four point examples to convert this into twistor space, we obtain the simple result,

$$\begin{aligned}
&\langle 0|\hat{O}_1(Z_1)\hat{O}_1(Z_2)\hat{O}_1(Z_3)\hat{O}_1(Z_4)\hat{O}_1(Z_5)|0\rangle\\
&= \frac{1}{(4\pi)^2}\delta(Z_1\cdot Z_2)\delta(Z_4\cdot Z_5)\int\frac{dc_1dc_2dc_3dc_4dc_5}{\mathrm{Vol}(GL(1,\mathbb{R}))}\delta(c_1Z_1\cdot Z_3+c_2Z_2\cdot Z_3)\delta^4(c_1Z_1+c_2Z_2+c_3Z_3+c_4Z_4+c_5Z_5)\\
&= \frac{1}{4\pi^2}\frac{\delta(Z_1\cdot Z_2)\delta(Z_4\cdot Z_5)}{|Z_1\cdot Z_3||Z_2\cdot Z_4||Z_3\cdot Z_5|}\delta(u_{14,23}-1-u_{34,25}(u_{15,23}-1)),
\end{aligned} \tag{6.210}$$

where the five-point cross ratios are given by,

$$u_{ij,kl} = \frac{(Z_i\cdot Z_j)(Z_k\cdot Z_l)}{(Z_i\cdot Z_l)(Z_k\cdot Z_j)}. \tag{6.211}$$

We leave a systematic and comprehensive analysis of higher point functions in twistor space for a future work.

## 6.6 Double Copy

Broadly speaking, the double copy in the context of scattering amplitudes states that graviton amplitudes can (roughly) be obtained by squaring gauge theory amplitudes. Since the initial works, a lot of progress has been made including double copy at loop level, double copy for classical solutions in gauge theory and gravity, double copy involving many other theories, see [283] and citations thereof for a review of the developments. Given the fact that gravity is generally much harder to study than gauge theory, it is desirable to want to further understand and utilize this phenomena.

In this section, we study the double copy in the context of Yang-Mills theory and Einstein gravity. First, we discuss three point functions and show a double copy between stripped gluon and graviton Wightman functions in momentum space, spinor helicity and twistor variables. For four point functions, we stick to twistor variables since momentum space and spinor helicity approaches become more complicated due to the presence of too many degenerate tensor structures. We derive an extremely simple double copy relation between the correlators in Yang-Mills theory and Einstein gravity obtainaing the latter as a simple square of the former. This correlator corresponds in momentum space to the stripped factorized Wightman function.

### 6.6.1 Three point functions

The Einstein gravity three point scattering amplitude is the square of its colour stripped Yang-Mills counterpart.

$$A_{3,GR} = A_{YM}^2 = \Big((\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot p_1) + (\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot p_3) + (\epsilon_3 \cdot \epsilon_1)(\epsilon_2 \cdot p_3)\Big)^2. \quad (6.212)$$

For three point AdS/CFT correlators, double copy structures have been found first in the Euclidean context [32,41] and more recently in the Lorentzian setting [236]. Let us check this property for our Wightman functions. We focus on the case where the middle operator has spacelike momenta. We begin by removing the colour factor and spectral theta functions and squaring our result for $\langle JJJ\rangle$ (6.82) yielding,

$$(\langle\langle 0|JJJ|0\rangle\rangle'_{YM})^2 \sim V_{3,YM}^2 \frac{p_1^2 p_2^2 p_3^2}{E^2(E-2p_1)^2(E-2p_2)^2(E-2p_3)^2}. \quad (6.213)$$

Comparing with the expression for the graviton three point Wightman function (6.90) shows that,

$$\langle\langle 0|TTT|0\rangle\rangle'_{GR} \sim p_1 p_2 p_3 (\langle\langle 0|JJJ|0\rangle\rangle'_{YM})^2, \quad (6.214)$$

As a consequence of this relation, one can also check in spinor helicity variables that the double copy holds in every helicity configuration. The expressions for these eight helicity configurations are provided in appendix D.1. Similarly, one can see using the twistor space expressions in appendix D.1, the double copy in twistor space corresponds to squaring the Schwinger parameters when the correlator is written in said parametrization. For concreteness, consider the $(+++)$ helicity configuration. We have,

$$\langle 0|J^{+A}(Z_1)J^{+B}(Z_2)J^{+C}(Z_3)|0\rangle' = -4f^{ABC}\delta^{[3]}(Z_1\cdot Z_2)\int\frac{dc_1}{c_1}\int\frac{dc_2}{c_2}\delta^4(c_1Z_1+c_2Z_2-Z_3)$$
$$= -4if^{ABC}\int dc_1dc_2dc_{12}\frac{c_{12}^3}{c_1c_2}e^{-ic_{12}Z_1\cdot Z_2}\delta^4(c_1Z_1+c_2Z_2-Z_3). \qquad (6.215)$$

Similarly, for gravity we have,

$$\langle 0|T^{+}(Z_1)T^{+}(Z_2)T^{+}(Z_3)|0\rangle = -4\delta^{[6]}(Z_1\cdot Z_2)\int\frac{dc_1}{c_1^2}\int\frac{dc_2}{c_2^2}\delta^4(c_1Z_1+c_2Z_2-Z_3)$$
$$= 4\int dc_1dc_2dc_{12}\left(\frac{c_{12}^3}{c_1c_2}\right)^2 e^{-ic_{12}Z_1\cdot Z_2}\delta^4(c_1Z_1+c_2Z_2-Z_3), \qquad (6.216)$$

thus showing that the squaring the Schwinger parameter function $\frac{c_{12}^3}{c_1c_2}$ in the gluon correlator (6.215) and stripping off the colour factor results in (6.216). Similarly, one can check in every helicity configuration using the expressions that this is a general feature.

### 6.6.2 Four point functions

Consider the expressions for our MHV four point functions in twistor space viz (6.199) and (6.206). Let us re-write these expressions also expressing the pre-factors in Schwinger parametrization. For the gluon case we have,

$$\frac{\langle 0|J^{A_1+}(Z_1)J^{A_2-}(Z_2)J^{A_3+}(Z_3)J^{A_4-}(Z_4)|0\rangle}{f^{A_1A_2A}f_A^{\ A_3A_4}}$$
$$= \int\frac{-4dc_1dc_2dc_3dc_4dc_{12}dc_{34}}{\pi\mathrm{Vol}(GL(1,\mathbb{R}))}\frac{c_1^3c_3^3}{c_{12}c_{34}c_2c_4}e^{ic_1c_4Z_1\cdot Z_4-ic_2c_3Z_2\cdot Z_3}e^{-ic_{12}Z_1\cdot Z_2-ic_{34}Z_3\cdot Z_4}\cos\left(c_1c_3Z_1\cdot Z_3-c_2c_4Z_2\cdot Z_4\right), \qquad (6.217)$$

We define the Schwinger parameter Yang-Mills correlator viz,

$$\mathcal{M}_{4,YM}(c_1,c_2,c_3,c_4,c_{12},c_{34}) = \frac{c_1^3c_3^3}{c_{12}c_{34}c_2c_4}. \qquad (6.218)$$

Let us now do the same for the graviton case. We have,

$$\langle 0|T^{+}(Z_1)T^{-}(Z_2)T^{+}(Z_3)T^{-}(Z_4)|0\rangle$$
$$= \int\frac{4dc_1dc_2dc_3dc_4dc_{12}dc_{34}}{\pi\mathrm{Vol}(GL(1,\mathbb{R}))}\left(\frac{c_1^3c_3^3}{c_{12}c_{34}c_2c_4}\right)^2 e^{ic_1c_4Z_1\cdot Z_4-ic_2c_3Z_2\cdot Z_3}e^{-ic_{12}Z_1\cdot Z_2-ic_{34}Z_3\cdot Z_4}\cos\left(c_1c_3Z_1\cdot Z_3-c_2c_4Z_2\cdot Z_4\right). \qquad (6.219)$$

The graviton Schwinger parameter correlator is thus,

$$\mathcal{M}_{4,GR}(c_1,c_2,c_3,c_4,c_{12},c_{34}) = \left(\frac{c_1^3c_3^3}{c_{12}c_{34}c_2c_4}\right)^2 = \mathcal{M}_{4,YM}^2, \qquad (6.220)$$

which is simply the square of its Yang-Mills counterpart! Using the helicity flipping operator of [236], one can induce the double copy in every other helicity configuration. For example, consider the $(+++-)$ helicity. Our answers for the $(+-+-)$ helicity are expressed in the variables $(Z_1, Z_2, Z_3, Z_4)$. To $(+++-)$, we simply replace $Z_2$ by $W_2$ which converts $J^{A_2-}(Z_2)$ to $J^{A_2+}(W_2)$. The expressions for gluons (6.217) and gravitons (6.219) becomes,

$$
\begin{aligned}
&\frac{\langle 0|J^{A_1+}(Z_1)J^{A_2+}(W_2)J^{A_3+}(Z_3)J^{A_4-}(Z_4)|0\rangle}{f^{A_1A_2A}f_A^{A_3A_4}}\\
&=\int \frac{-4dc_1dc_2dc_3dc_4dc_{12}dc_{34}}{\pi \mathrm{Vol}(GL(1,\mathbb{R}))}\frac{c_1^3c_3^3}{c_{12}c_{34}c_2c_4}e^{ic_1c_4Z_1\cdot Z_4-ic_2c_3W_2\cdot Z_3}e^{-ic_{12}Z_1\cdot W_2-ic_{34}Z_3\cdot Z_4}\cos\left(c_1c_3Z_1\cdot Z_3-c_2c_4W_2\cdot Z_4\right),\\
&\langle 0|T^+(Z_1)T^+(W_2)T^+(Z_3)T^-(Z_4)|0\rangle\\
&=\int \frac{4dc_1dc_2dc_3dc_4dc_{12}dc_{34}}{\pi \mathrm{Vol}(GL(1,\mathbb{R}))}\left(\frac{c_1^3c_3^3}{c_{12}c_{34}c_2c_4}\right)^2 e^{ic_1c_4Z_1\cdot Z_4-ic_2c_3W_2\cdot Z_3}e^{-ic_{12}Z_1\cdot W_2-ic_{34}Z_3\cdot Z_4}\cos\left(c_1c_3Z_1\cdot Z_3-c_2c_4W_2\cdot Z_4\right),
\end{aligned}
\tag{6.221}
$$

and similarly for other helicity configurations. Thus, the double copy is present in general. This analysis really shows us how twistor space can shed light on structures that are obscured in momentum space and spinor helicity variables. Given the fact that our twistor analysis unveiled that the spinor helicity expressions correspond to (6.201) and (6.208), we obtain the following induced double copy in spinor helicity variables:

$$
\langle 0|T^+(p_1)T^-(p_2)T^+(p_3)T^-(p_4)|0\rangle' = |s|p_1p_2p_3p_4(\langle 0|J^{A_1+}(p_1)J^{A_2-}(p_2)J^{A_3+}(p_3)J^{A_4-}(p_4)|0\rangle')^2, \tag{6.222}
$$

highlighting the utility of twistor space to derive such formulae[31].

### 6.6.3 From the twistor double copy to the full graviton correlator

We have see that twistor space provides a pleasing and simple double copy relation. Given this fact, we can easily obtain a graviton four point Wightman function from its gluon counterpart using (6.220). These are of course, expressions that correspond to the factorized results in momentum space (D.12) and (D.13) where the middle two operators have space-like momenta. However, we have shown that there is a straightforward and simple way to obtain the Euclidean AdS correlators from this factorized expression in section 6.4. We emphasize yet again that there is not a single nested bulk integral in sight in this process and the process is straightforward.

## 6.7 Summary of this Chapter

In this chapter, we have set the stage for the analysis of higher point Wightman functions in twistor space. We began our discussion by setting up the computation of holographic Wightman functions in the context of AdS$_4$. We systematically computed many examples of two, three and four point Wightman functions involving general scalars, photons,

[31] It would be interesting to compare our results with other double copy constructions, such as for on-shell correlators [284], or using a differential representation for AdS boundary correlators such as in [285].

gluons and gravitons and also five point scalar correlators. Interestingly, we found that the four point functions factorize into a product of three point functions in special kinematics when the momenta of the middle operators are taken to be space-like. This expression is a conformal partial wave associated to the operator dual to the particle exchanged in the bulk. Converting the resulting expressions into spinor helicity variables and performing a half-Fourier transform to twistor space yielded results for two and three point functions consistent with previous works. We then proceeded to the case of four point functions in twistor space associated to the factorized expressions in momentum space. Utilizing the helicity basis, we found extremely simple and compact results. In particular, we discovered a simple double copy relation between gluon and graviton correlators with the latter simply being a square of the former in Schwinger parameter space. Along the way, we discussed the analytic continuation from our factorized Lorentzian Wightman functions to Euclidean AdS showing that even the simple factorized expression, contains enough information to recover the Euclidean correlator up to contact diagram contributions.

There are a number of interesting future works that this chapter opens the door towards:

## Twistor conformal bootstrap

One of our goals is to set up the conformal bootstrap program in twistor space. Given our holographic results, it seems promising to pursue this program. General four point functions receive contributions from intermediate states involving scalars and non-conserved spinning operators and thus it is important to compute three point functions involving at least one insertion of these operators. The twistor framework was extended in [257] to accommodate generic operators. However, it remains an open problem to bootstrap their three point functions. Once that is done, the stage will be set for the bootstrap.

## Generalization to higher dimensions

The machinery to compute Wightman functions in this chapter cam easily extended to any spacetime dimension. The twistor space framework on the other hand, is quite dimension dependent [238]. An $AdS_5$ twistor formulation was presented in [286]. It would be interesting to develop the twistor space machinery for 4d CFT and connect it to the formalism of [286].

## Black hole backgrounds

Given the simplicity of our algebraic approach using the equation of motion to compute real-time Wightman functions, a natural question is to extend the formalism of this chapter to AdS black hole and more general backgrounds. From the CFT perspective, these correspond to real time correlators and as such are an important and interesting direction to pursue.

### General kinematics and other correlators

In this chapter, we have mostly focused on four point spinning correlators in the special kinematics where the middle two operators have space-like momenta. Although these kinematics are quite powerful and even allow us to construct the Euclidean correlator which has in principle information about all possible Wightman functions and their kinematics, it is still a different matter to explicitly be able to reach these kinematics via analytic continuation. That is also of course an important problem to pursue but so is the computation of spinning Wightman functions in general kinematics much like what we did for the scalar case (6.115).

### Higher point functions

The machinery developed in this chapter can be generalized to compute higher point correlators as we also exemplified with the computation of a scalar five point function. Given the simplicity of the twistor answers, a possible approach is to intrinsically setup our formalism in twistor space from the get go which might lead to an even more systematic approach.

### Recursion relations

Recursion relations such as BCFW [287], and BG [288] for scattering amplitudes paved the way for many developments in field theory. BCFW recursion has also been developed in twistor space where they take their most natural and simplified form [22, 23]. In AdS, recursion relations have also been developed for time-ordered or Euclidean boundary correlators [63, 64, 76, 289]. However, they are yet to reach the simplicity of their counterparts in scattering amplitudes, where at tree-level, one can simply stitch together three point amplitudes to obtain higher point ones. What we would like to emphasize yet again is that Wightman functions are simpler quantities than other correlators as we have seen in this chapter. They satisfy factorization formulae and also do not possess contact terms in current conservation Ward-Takahashi identities. Thus, developing recursion relations for Wightman functions might be an interesting direction to pursue.

### Extension to loops

It would be interesting to extend our formalism for computing Wightman functions to one-loop and beyond, including understanding the analytic continuation at loop level that relates Wightman functions to Euclidean counterparts.

# Chapter 7

# Summary and Future Directions

The main subject of my PhD research in general has been the development and applications of amplitudes inspired methods for conformal field theories. Some of the key successes covered here include,

- A systematic analysis of conformal current correlators in momentum space spinor helicity variables especially at three points which allowed us to uncover the holographic dual of $AdS_4$ chiral higher spin theory (Chapter 2 and Chapter 4).
- Mapping interacting theory higher spin equations to their free theory counterparts thus obtaining a possible solution to four and higher point current correlators in Chern-Simons matter theories (Chapter 3).
- Discovering that $n-$point momentum space correlators in conformal quantum mechanics take the form of Lauricella $E_A^{(n-2)}$ functions with $n-3$ undetermined parameters that are the analogs of conformal cross ratios (Chapter 2).
- Developing real time techniques to compute four point holographic Wightman functions in $AdS_4$ in theories with scalars, photons, gluons and gravitons, finding novel factorization kinematics and a connection between Witten diagrams for Wightman functions and conformal partial waves. Systematized the principle of analytic continuation to recover the full correlator from just the conformal partial waves (Chapter 6).
- Deriving the Penrose and Witten transforms in the twistor space framework for $CFT_3$. Extending the formalism to include arbitrary operators of any spin and scaling dimension by using the infinity twistor of $\mathbb{R}^{2,1}$ which from the bulk $AdS_4$ perspective implies the ability to describe particles of any mass and any spin. Deriving a parity odd version of the Penrose transform. Classifying all the conformal invariants in twistor space (Chapter 5).
- Deriving the supersymmetric version of the Penrose transform. Classifying the super-conformal invariants in twistor space. Extending the formalism to super-scalars and parity odd super-correlators (Chapter 5).
- Recasting holographic current correlators in twistor space. Deriving a simple squaring double copy relation between four point functions in Einstein gravity and Yang-Mills theory (Chapter 6).

For reasons of length and focus, not all of my related publications are included in this thesis. However, we shall summarize their key results briefly to provide a broader context for the work presented above.

- Understanding the details of how OPE consistency constrains the structure of three point current correlators that do not obey the spin triangle inequality. Deriving the non-conservation identities in slightly broken higher spin theories using exactly conserved Ward-Takahashi identities using the epsilon transform [45]. The results of these analysis played a key role in chapters 2 and 3.

- Solving the twist conformal block equation in (slightly broken) higher spin theories for spinning current correlators and using these results to learn about the locality of their bulk duals. Finding the possibility of a local sub-sector that was later realized [159] which is also the subject of chapter 4.

- Developing a super spinor-helicity and Grassmann twistor formalism for $\mathcal{N} = 1, 2$ $SCFT_3$ that exponentially simplifies the analysis compared to position and momentum space methods. Discovered super double copy relations between $\mathcal{N} = 1$ and $\mathcal{N} = 2$ super-correlators [48]. This extends the analysis of chapter 2 to the superspace setting.

- Systematically solving the Ward identities for two and three point conformal correlators in twistor space with a focus on how the distributional solutions arise [237]. This analysis complements the first part of chapter 5.

- Developing a super-twistor formalism for $\mathcal{N} = 1, 2, 3, 4$ $SCFT_3$ by solving the $OSp(\mathcal{N}|4)$ super-conformal Ward identities [241]. This analysis complements the second part of chapter 5 on super-twistors.

## Future Directions

Looking forward, the spinor helicity and twistor perspectives offer a natural path toward recursion relations for (A)dS/CFT correlators analogous to BCFW recursion relations in flat space [22,23,287], and even toward a geometric formulation of conformal correlators perhaps akin to the amplituhedron [252]. Twistors in particular, being sensitive only to the null structure of space-time present an inviting approach to describe physics in flat space, AdS and dS in a unified framework. The recent Grassmannian integral formalism developed for $CFT_3$ [290] also presents an exciting avenue for further research. Below, we discuss some concrete future directions.

## Recursion relations and a Parke-Taylor formula for $AdS_4$ boundary correlators

The Parke-Taylor formula [19] provides a remarkably simple expression for MHV n-point gluon scattering amplitudes in flat space. The simplicity of this formula inspired and paved the way for many discoveries such on-shell recursion relations. The analogous result for gluons (or any other particles) in curved backgrounds such as AdS remains unknown. Even at four points, the time-ordered boundary correlator of gluons is quite complicated in spinor helicity variables. A promising potential way forward is to focus on Wightman functions as they satisfy current conservation Ward-Takahashi identities without any contact terms and admit a simple twistor space description at three points. We found evidence for this in chapter 6, uncovering an elegant structure for gluon four point Wightman functions in special kinematics, including a simple squaring procedure to obtain the analogous graviton result. Computing higher point functions and also the results for other kinematic configurations would be interesting and help shed light on the utility of Wightman functions in this quest.

## A CFT Amplituhedron

In four dimensional flat space, certain scattering amplitudes can be interpreted as the volume of a geometric object: The amplituhedron [252]. This makes manifest among other things the infinite dimensional Yangian symmetry of $\mathcal{N} = 4$ SYM amplitudes. A natural question to ask is: what is the geometric problem to which AdS/CFT correlators are the answer? Momentum twistor techniques [291] have been crucial in flat space and thus developing their AdS/CFT counterparts could be a useful tool to have especially to tackle theories that possess dual conformal invariance. Further, for the conformally invariant $\mathcal{N} = 4$ SYM theory, it would be very interesting to explore how these structures change when we go from flat space to the conformally flat Poincare patch of $AdS_4$.

## Connections between holography in Minkowski spacetime, de-Sitter and AdS/CFT

An outstanding open problem is the holographic description of physics in flat and de-Sitter spacetimes. Formally, AdS/CFT contains information about both these cases via a flat space limit or a suitable Wick rotation. However, in practice these procedures are subtle. This is where Twistor variables might possibly pave a path forward. Since twistor space is insensitive to the conformal factor of a metric, it may provide a unified description of holography in flat space and the Poincare patches of AdS and dS which are conformally flat. The only distinction among these spaces arises in the form of the infinity twistor and for conformally invariant theories like Yang-Mills at tree level, the dependence on this object is just through simple sign factors [22] indicating a promising point for further exploration.

## The conformal bootstrap

The conformal bootstrap programme in $d > 2$ has seen immense progress [15, 16] since the pioneering work [14]. However, most progress has been limited to cases where the external operators are scalars due to the difficulty in handling operators with spin such as conserved currents. This is potentially a place where twistor space can provide a pivoting point given the fact that it makes both conformal symmetry and current conservation manifest. Therefore, it would be interesting to explore whether the spinning conformal bootstrap can be benefited by working in twistor space. As a first step, one could formulate the operator product expansion and construct conformal blocks in Twistor space, followed by crossing symmetry and imposing constraints on the spectrum and OPE coefficients starting with interesting theories like the 3d Ising model at criticality. It would also be interesting to develop similar constructions in other dimensions especially in 4d CFTs where ambi-twistors or bi-twistors seem to be a potential way forward.

# Chapter A

# Dimensional Reduction from four to three dimensions

## A.1 From four dimensional to three dimensional spinor helicity variables

In this appendix, we show how one can obtain a general three dimensional momenta starting from a null momentum in four dimensions.

### A.1.1 $\mathbb{R}^{3,1} \to \mathbb{R}^3$

A four dimensional massless momenta can be written in an unconstrained form as a outer product of two spinors as follows: $p_a^{\dot{a}} = \lambda_a \tilde{\lambda}^{\dot{a}}$ with a little group redundancy $\lambda \to e^{-\frac{i\theta}{2}}\lambda, \tilde{\lambda} \to e^{+\frac{i\theta}{2}}\tilde{\lambda}, \theta \in \mathbb{R}$. In matrix form we have,

$$p_a^{\dot{a}} \sim \begin{pmatrix} p_t + p_z & p_x - ip_y \\ p_x + ip_y & p_t - p_z \end{pmatrix}. \tag{A.1}$$

The reality condition for the four momenta components is that $p$ is Hermitian and thus $(\lambda_a)^\dagger = \tilde{\lambda}^{\dot{a}}$ and $(\tilde{\lambda}^{\dot{a}})^\dagger = \lambda_a$. A three dimensional Euclidean momenta can be obtained from this null 4d Minkowskian momenta as follows: Introduce a special vector,

$$\epsilon_{a\dot{a}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.2}$$

The three dimensional momentum is then given by,

$$p_{ab} = -\frac{1}{2}\left(p^{\dot{a}}_{(a}\epsilon_{b)\dot{a}}\right). \tag{A.3}$$

In matrix form this procedure is as follows: First form,

$$p_a^{\dot{a}}\epsilon_{b\dot{a}} \sim \begin{pmatrix} p_t + p_z & p_x - ip_y \\ p_x + ip_y & p_t - p_z \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -p_x + ip_y & p_t + p_z \\ -p_t + p_z & p_x + ip_y \end{pmatrix}. \tag{A.4}$$

Now symmetrize this matrix and multiply by a factor of $-\frac{1}{2}$. The result is,

$$p_{ab} = -\frac{1}{2}\begin{pmatrix} -p_x + ip_y & p_t + p_z \\ -p_t + p_z & p_x + ip_y \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -p_x + ip_y & -p_t + p_z \\ p_t + p_z & p_x + ip_y \end{pmatrix} = \begin{pmatrix} -p_x + ip_y & p_z \\ p_z & p_x + ip_y \end{pmatrix}. \tag{A.5}$$

Thus, we see that $p_t$ has completely dropped out in this procedure and left behind a three dimensional general momenta. In the langauge of spinors we have,

$$p_{ab} = -\frac{1}{2}\left(\lambda_{(a}\epsilon_{b)\dot{a}}\tilde{\lambda}^{\dot{a}}\right) = \frac{1}{2}\left(\lambda_{(a}\bar{\lambda}_{b)}\right), \tag{A.6}$$

where we have defined,

$$\bar{\lambda}_b = \epsilon_{\dot{a}b}\tilde{\lambda}^{\dot{a}}. \tag{A.7}$$

The induced reality condition on the three dimensional spinors is,

$$\lambda_a^{\dagger} = \bar{\lambda}^a, (\bar{\lambda}^a)^{\dagger} = \lambda_a. \tag{A.8}$$

### A.1.2 $\mathbb{R}^{2,2} \longrightarrow \mathbb{R}^{2,1}$

The procedure is very similar to the above case. A null momenta in the four dimensional Klein space can be written as an outerproduct of two spinors, $p_a^{\dot{a}} = \lambda_a\tilde{\lambda}^{\dot{a}}$ with a little group redundancy $\lambda \to \frac{1}{r}\lambda, \tilde{\lambda} \to r\tilde{\lambda}, r \in \mathbb{R}$. In matrix form we have,

$$p_a^{\dot{a}} \sim \begin{pmatrix} p_t + p_z & p_x - p_w \\ p_x + p_w & p_t - p_z \end{pmatrix}. \tag{A.9}$$

The $p_w$ component that appears here is related to the corresponding Euclidean $p_y$ via $ip_y = p_w$. The reality condition here is simply that the above matrix is real and hence gives $\lambda_a^* = \lambda_a$ and $(\tilde{\lambda}^{\dot{a}})^* = \tilde{\lambda}^{\dot{a}}$, that is, the spinors are real and independent. One can now follow the same procedure as in the previous subsection, i.e., defining a special vector $\epsilon_{a\dot{a}}$ (A.2) and defining a 3d momentum via (A.3). The difference now is that $p_w$ is a timelike coordinate unlike $p_y$ and thus what we end up with is a three dimensional Lorentzian momentum. Again, in the language of the spinors we have,

$$p_{ab} = -\frac{1}{2}\left(\lambda_{(a}\epsilon_{b)\dot{a}}\tilde{\lambda}^{\dot{a}}\right) = \frac{1}{2}\left(\lambda_{(a}\bar{\lambda}_{b)}\right), \tag{A.10}$$

where we have defined,

$$\bar{\lambda}_b = \epsilon_{\dot{a}b}\tilde{\lambda}^{\dot{a}}. \tag{A.11}$$

The reality condition induced on the three dimensional momenta is now,

$$\lambda_a^* = \lambda_a, (\bar{\lambda}^a)^* = \bar{\lambda}^a. \tag{A.12}$$

### A.1.3 Understanding discrete symmetries in Klein space

We have seen above, the natural dimensional reduction of Klein space leads to three dimensional Minkowski space. It is thus interesting to ask how the discrete symmetries in Klein space reduce to those in $\mathbb{R}^{2,1}$. The starting point is the position bi-spinor:

$$x_a^{\dot{a}} = \begin{pmatrix} t + z & x - w \\ x + w & t - z \end{pmatrix}. \tag{A.13}$$

Parity is the transformation $z \rightarrow -z$ whereas time-reversal is $t \rightarrow -t$. The alternate transformations $x \rightarrow -x$ and $w \rightarrow -w$ are connected to the above choices via $SO(2)$ spatial and temporal rotations respectively. Our aim is to translate these transformations to spinor helicity variables. A null Kleinian momenta is given by,

$$p_a^{\dot{a}} = \begin{pmatrix} p_t + p_z & p_x - p_w \\ p_x + p_w & p_t - p_z \end{pmatrix} = \begin{pmatrix} \lambda_1\tilde{\lambda}_2 & -\lambda_1\tilde{\lambda}_1 \\ \lambda_2\tilde{\lambda}_2 & -\lambda_2\tilde{\lambda}_1 \end{pmatrix}. \tag{A.14}$$

Under parity, $p_z$ flips to $-p_z$. This leads to the following transformation for the spinors:

$$P : \begin{pmatrix} \lambda_1, & \lambda_2 \end{pmatrix} \rightarrow \begin{pmatrix} -\tilde{\lambda}_1, & \tilde{\lambda}_2 \end{pmatrix}, \tag{A.15}$$

which is identical to the parity transformation on the spinors in $\mathbb{R}^{2,1}$ (2.75) after a dimensional reduction. What about under time-reversal? Lets say we flip $t \rightarrow -t$. Then should one flip $p_x, p_z$ and $p_w$ to $-p_x, -p_z$ and $-p_w$? In Minkowski space $\mathbb{R}^{3,1}$, time-reversal is anti-linear and thus under $t \rightarrow -t$, $(p_t, p_x, p_z, p_y) \rightarrow (p_t, -p_x, -p_z, -p_y)$. This also ensures that energy, that is $p_t$ is invariant under a time-reversal. Klein space is connected to Minkowski space by a Wick rotation $p_y = ip_w$. Thus, $p_y$ flipping under time-reversal implies that $p_w$ must not flip since $i \rightarrow -i$ takes care of the sign flip. Therefore, the Klein space time-reversal should not change $p_w \rightarrow -p_w$. Thus we have time-reveral taking $(p_t, p_w, p_x, p_z) \rightarrow (p_t, p_w, -p_x, -p_z)$. On the spinors it amounts to,

$$T : \begin{pmatrix} \lambda_1, & \lambda_2 \end{pmatrix} \rightarrow \begin{pmatrix} -\lambda_2, & \lambda_1, \end{pmatrix} \tag{A.16}$$

and similarly for $\tilde{\lambda}$. This leads to the same time-reversal operation we defined in $\mathbb{R}^{2,1}$ in (2.70) after a dimensional reduction.

In short, what we have shown is that the discrete symmetries of four dimensional Klein space naturally give rise to the discrete symmetries of three dimensional Minkowski space.

## A.2 Dimensional reduction from the Twistor space perspective: $SL(4, \mathbb{R}) \rightarrow$ Sp$(4, \mathbb{R})$

In this appendix, we shall start with the generators of $SL(4, \mathbb{R})$, which is the 4d conformal group and dimensionally reduce to three dimensions and obtain the Sp(4) generators (5.43). For $U \in SL(4, \mathbb{R})$ we have,

$$\text{Det}[U] = 1. \tag{A.17}$$

Exponentiating $U = e^{i\alpha G}$ with $G \in \mathfrak{sl}(4, \mathbb{R})$ we obtain,

$$\text{Det}[e^{i\alpha G}] = e^{i\alpha \text{Tr}(G)} = 1 \implies \text{Tr}(G) = 0. \tag{A.18}$$

Let us now construct a projection of $\mathfrak{sl}(4, \mathbb{R})$ onto the subalgebra $\mathfrak{sp}(4, \mathbb{R})$ using the symplectic form $\Omega$. In index notation we have,

$$T_B^A = \Omega_{EB}(\Omega^{C(A}G_C^{E)}) = \Omega_{EB}T^{AE}. \tag{A.19}$$

Let us exponentiate this quantity and check whether it satisfies the Sp(4) group property viz,

$$M \in Sp(4) \implies M^T \Omega M = \Omega \implies T^T \Omega + \Omega T = 0 \text{ for } T \in \mathfrak{sp}(4, \mathbb{R}). \tag{A.20}$$

Putting $M = e^{i\alpha T}$ and using the relation (A.19), it is easy to see (A.20) is indeed satisfied. Let us now see how to derive the particular representation of Sp(4, $\mathbb{R}$) that acts on twistor space from the twistor space representation of the generators of $SL(4, \mathbb{R})$. The $SL(4, \mathbb{R})$ generators that act on twistor space associated to $\mathbb{R}^{2,2}$ [1]. are given by,

$$G^A_B = Z^A \frac{\partial}{\partial Z^B} - \frac{1}{4} \delta^A_B Z^C \frac{\partial}{\partial Z^C}. \tag{A.21}$$

They obey the $\mathfrak{sl}(4, \mathbb{R})$ algebra viz,

$$[G^A_B, G^C_D] = \delta^C_B G^A_D - \delta^A_D G^C_B. \tag{A.22}$$

Using the definition (A.19) with,

$$T^{AB} = \Omega^{C(A} T^{B)}_C = Z^{(A} \frac{\partial}{\partial Z_{B)}}, \tag{A.23}$$

it is easy to show that,

$$[T^{AB}, T^{CD}] \propto \Omega^{AC} T^{BD} + \Omega^{AD} T^{BC} + \Omega^{BC} T^{AD} + \Omega^{BD} T^{AC}, \tag{A.24}$$

which is exactly the Lie-algebra of $\mathfrak{sp}(4, \mathbb{R})$.

What we see via the above group theory exercise is that the twistor space associated to $\mathbb{R}^{2,1}$ is a subspace of the twistor space of the twistor space associated to $\mathbb{R}^{2,2}$. Both of these are subsets of $\mathbb{RP}^3$ where the latter allows for general volume preserving transformations whereas the former restricts to those transformations that also preserve the symplectic form.

[1] For instance, these can be found by starting with the $GL(4, \mathbb{R})$ generators in [238] and subtracting out the trace part.

# Chapter B

# Projective Integral identities

In this appendix, we derive two useful formulae that relate projective integrals to ordinary nonprojective integrals by quotienting by $\text{Vol}(GL(1,\mathbb{R}))$. This volume factor is given by the following divergent integral,

$$\text{Vol}(GL(1,\mathbb{R})) = \int_{-\infty}^{\infty} \frac{dc}{|c|}. \tag{B.1}$$

## B.1 From $\mathbb{RP}^1$ to $\mathbb{R}^2$

Consider the following quantity

$$\int \langle \lambda d\lambda \rangle f(\lambda), \tag{B.2}$$

such that,

$$f(r\lambda) = \frac{1}{r^2} f(\lambda). \tag{B.3}$$

The Penrose transform (5.16) is an example of such an integral with $f(\lambda) \sim \lambda^{a_1} \cdots \lambda^{a_{2s}} J_s^+(\lambda, \bar{\mu})|_X$. We can for instance, locally choose a chart where $\lambda_a = (\lambda_1, \lambda_2) = (1, \xi)$ such that (B.2) becomes,

$$\int \langle \lambda d\lambda \rangle f(\lambda_1, \lambda_2) = \int_{-\infty}^{\infty} d\xi f(1, \xi). \tag{B.4}$$

We now want to show that,

$$\int \langle \lambda d\lambda \rangle f(\lambda) = \frac{1}{\text{Vol}(GL(1,\mathbb{R}))} \int d^2\lambda f(\lambda). \tag{B.5}$$

Lets write out the right hand-side of (B.5) explicitly:

$$\frac{1}{\text{Vol}(GL(1,\mathbb{R}))} \int d^2\lambda f(\lambda) = \frac{1}{\text{Vol}(GL(1,\mathbb{R}))} \int d\lambda_1 \wedge d\lambda_2 f(\lambda_1, \lambda_2). \tag{B.6}$$

Now perform a variable change,

$$\lambda_1 = \rho_1, \lambda_2 = \rho_1 \rho_2 \implies d\lambda_1 \wedge d\lambda_2 = |\rho_1| d\rho_1 \wedge d\rho_2. \tag{B.7}$$

We thus have,

$$\frac{1}{\text{Vol}(GL(1,\mathbb{R}))}\int d\lambda_1\wedge d\lambda_2 f(\lambda_1,\lambda_2) = \frac{1}{\text{Vol}(GL(1,\mathbb{R}))}\int d\rho_1\wedge d\rho_2|\rho_1|f(\rho_1,\rho_1\rho_2)$$
$$= \frac{1}{\text{Vol}(GL(1,\mathbb{R}))}\int d\rho_1\wedge d\rho_2|\rho_1|\frac{1}{\rho_1^2}f(1,\rho_2) \tag{B.8}$$

where we used the projective property of the function $f$ (B.3). The integral now factorizes and the one above $\rho_1$ completely cancels out the volume factor (B.1). Relabeling $\rho_2 = \xi$ then yields (B.4) thus proving (B.5).

## B.2 From $\mathbb{R}^3$ to $\mathbb{R}^2\times\mathbb{R}^2$

We now consider the following integral,

$$\int d^3x f(x). \tag{B.9}$$

We now write $x$ as a bispinor by contracting with the Pauli matrices,

$$x^{ab} = \frac{\lambda^a\rho^b + \lambda^b\rho^a}{2}. \tag{B.10}$$

By definition we have a redundancy,

$$\lambda\to\frac{1}{r}\lambda, \rho\to r\rho \implies x\to x. \tag{B.11}$$

We want to show that,

$$\int d^3x f(x) = \frac{1}{4\text{Vol}(GL(1,\mathbb{R}))}\int d^2\lambda d^2\rho|\lambda\cdot\rho|f(\lambda,\rho), \tag{B.12}$$

where the redundancy (B.11) implies that,

$$f(\frac{\lambda}{r}, r\rho) = f(\lambda,\rho). \tag{B.13}$$

To prove this statement, let us start with the RHS of (B.12). Writing it out explicitly we have,

$$\frac{1}{2\text{Vol}(GL(1,\mathbb{R}))}\int d^2\lambda d^2\rho|\lambda\cdot\rho|f(\lambda,\rho) = \frac{1}{\text{Vol}(GL(1,\mathbb{R}))}\int d\lambda_1\wedge d\lambda_2\wedge d\rho_1\wedge d\rho_2|\lambda_1\rho_2-\lambda_2\rho_1|f(\lambda_1,\lambda_2,\rho_1,\rho_2). \tag{B.14}$$

We now make a variable change $\lambda_1 = \frac{\chi_1}{\nu_2}, \lambda_2 = \frac{\chi_2}{\nu_2}, \rho_1 = \nu_1\nu_2, \rho_2 = \nu_2$. (B.14) along with the rescaling property (B.13) becomes,

$$\frac{1}{\text{Vol}(GL(1,\mathbb{R}))}\int\frac{1}{|\nu_2|}d\chi_1\wedge d\chi_2\wedge d\nu_1\wedge d\nu_2|\chi_1-\chi_2\nu_1|f(\chi_1,\chi_2,\nu_1,1)$$
$$= \int d\chi_1\wedge d\chi_2\wedge d\nu_1|\chi_1-\chi_2\nu_1|f(\chi_1,\chi_2,\nu_1,1). \tag{B.15}$$

Let us now deal with the LHS of (B.12) and prove that it is also equal to (B.15). We have,

$$
\begin{aligned}
&\int d^3x f(x) = \frac{1}{2}\int dx^{11}\wedge dx^{12}\wedge dx^{22} f(x_{11}, x_{12}, x_{22})\\
&\frac{1}{4}\int d\lambda_1\wedge d\lambda_2\wedge d\rho_1 |\lambda_1 - \lambda_2\rho_1| f(\lambda_1, \lambda_2, \rho_1, 1).
\end{aligned}
\tag{B.16}
$$

To go from the first line to the second we used the relation between $x$ and $\lambda, \rho$ viz (B.11) and chose to set $\rho_2 = 1$. Relabeling $\lambda_1 \to \chi_1, \lambda_2 \to \chi_2$ and $\rho_1 \to \nu_1$ and multiplying by a factor of 4 yields (B.15), thus proving the formula (B.12).

# Chapter C

# More on Wightman functions

## C.1 Derivation of Ward-Takahashi identity for a time ordered correlator

In this appendix, we show how the current conservation Ward-Takahashi identity for a time ordered correlator arises in the canonical approach. A time ordered three point function of a $U(1)$ current $J^\mu$ and two charged scalars $\phi, \chi$ is given by,

$$\begin{aligned}
&\langle 0|T\{J^\mu(x_1)\phi(x_2)\chi(x_3)\}|0\rangle \\
&= \theta(t_{12})\theta(t_{23})\langle 0|J^\mu(x_1)\phi(x_2)\chi(x_3)|0\rangle + \theta(t_{13})\theta(t_{32})\langle 0|J^\mu(x_1)\chi(x_3)\phi(x_2)|0\rangle \\
&+ \theta(t_{21})\theta(t_{13})\langle 0|\phi(x_2)J^\mu(x_1)\chi(x_3)|0\rangle + \theta(t_{23})\theta(t_{31})\langle 0|\phi(x_2)\chi(x_3)J^\mu(x_1)|0\rangle \\
&+ \theta(t_{31})\theta(t_{12})\langle 0|\chi(x_3)J^\mu(x_1)\phi(x_2)|0\rangle + \theta(t_{32})\theta(t_{21})\langle 0|\chi(x_3)\phi(x_2)J^\mu(x_1)|0\rangle, \qquad \text{(C.1)}
\end{aligned}$$

where we used the shorthand $t_{ij} = t_i - t_j$. Let us take the derivative of (C.1) with respect to $\partial_{1\mu}$. The only contributions are when it acts on the Heaviside theta functions as the six Wightman functions appearing on the RHS are identically conserved. The result is,

$$\begin{aligned}
&\partial_{1\mu}\langle 0|T\{J^\mu(x_1)\phi(x_2)\chi(x_3)\}|0\rangle \\
&= \delta(t_{12})\theta(t_{23})\langle 0|J^0(x_1)\phi(x_2)\chi(x_3)|0\rangle + \delta(t_{13})\theta(t_{32})\langle 0|J^0(x_1)\chi(x_3)\phi(x_2)|0\rangle \\
&- \delta(t_{12})\theta(t_{13})\langle 0|\phi(x_2)J^0(x_1)\chi(x_3)|0\rangle + \theta(t_{21})\delta(t_{13})\langle 0|\phi(x_2)J^0(x_1)\chi(x_3)|0\rangle \\
&- \theta(t_{23})\delta(t_{31})\langle 0|\phi(x_2)\chi(x_3)J^0(x_1)|0\rangle - \delta(t_{31})\theta(t_{12})\langle 0|\chi(x_3)J^0(x_1)\phi(x_2)|0\rangle \\
&+ \theta(t_{31})\delta(t_{12})\langle 0|\chi(x_3)J^0(x_1)\phi(x_2)|0\rangle - \theta(t_{32})\delta(t_{21})\langle 0|\chi(x_3)\phi(x_2)J^0(x_1)|0\rangle. \qquad \text{(C.2)}
\end{aligned}$$

Grouping like terms together, this quantity can be written as,

$$\begin{aligned}
\partial_{1\mu}\langle 0|T\{J^\mu(x_1)\phi(x_2)\chi(x_3)\}|0\rangle &= \delta(t_{12})\langle 0|T\{[J^0(x_1),\phi(x_2)]\chi(x_3)\}|0\rangle \\
&+ \delta(t_{13})\langle 0|T\{\phi(x_2)[J^0(x_1),\chi(x_3)]\}|0\rangle. \qquad \text{(C.3)}
\end{aligned}$$

Recall that,

$$\begin{aligned}
[Q,\phi(t_2,\vec{x}_2)] &= \int_{\Sigma_{t_2}} d^{d-1}\vec{x}\,[J^0(t_2,\vec{x}),\phi(t_2,\vec{x}_2)] = q_\phi\,\phi(t_2,\vec{x}_2), \\
[Q,\chi(t_3,\vec{x}_3)] &= \int_{\Sigma_{t_3}} d^{d-1}\vec{x}\,[J^0(t_3,\vec{x}),\chi(t_3,\vec{x}_3)] = q_\chi\,\chi(t_3,\vec{x}_3). \qquad \text{(C.4)}
\end{aligned}$$

This implies,

$$[J^0(x_1), \phi(x_2)] = q_\phi\, \delta^{d-1}(\vec{x}_{12})\phi(x_2)\ ,\ [J^0(x_1), \chi(x_3)] = q_\chi\, \delta^{d-1}(\vec{x}_{13})\chi(x_3). \tag{C.5}$$

Substituting these commutators back in (C.3) we obtain the familiar charge conservation Ward-Takahashi identity:

$$\partial_{1\mu}\langle 0|T\{J^\mu(x_1)\phi(x_2)\chi(x_3)\}|0\rangle = \big(q_\phi\delta^d(x_{12}) + q_\chi\delta^d(x_{13})\big)\langle 0|T\{\phi(x_2)\chi(x_3)\}|0\rangle. \tag{C.6}$$

Integrating (C.6) with respect to $x_1$ over the entire manifold $\mathbb{R}^{1,d-1}$, we obtain (after using Gauss' theorem on the LHS),

$$0 = (q_\phi + q_\chi)\langle 0|T\{\phi(x_2)\chi(x_3)\}|0\rangle \implies q_\chi = -q_\phi, \tag{C.7}$$

thus imposing conservation of charge.

## C.2 Properties of conformal Wightman functions

In this Appendix we discuss several properties of Wightman functions which are useful for our discussion in the main text.

### C.2.1 Vanishing of Wightman function when any of the momenta vanish

We give an argument following [34] on why three point Wightman functions vanish whenever any of the momenta vanish. The position space form factor of generic three point (scalar, spinor, spinning) Wightman functions takes the form (with $\epsilon > 0$),

$$\begin{aligned}&\mathbf{ff}(x_1, x_2, x_3) = \\ &\frac{1}{\big(-(t_1-t_2)^2 + (\vec{x}_1-\vec{x}_2)^2 + i\epsilon(t_1-t_2)\big)^\alpha \big(-(t_2-t_3)^2 + (\vec{x}_2-\vec{x}_3)^2 + i\epsilon(t_2-t_3)\big)^\beta \big(-(t_1-t_3)^2 + (\vec{x}_1-\vec{x}_3)^2 + i\epsilon(t_1-t_3)\big)^\gamma},\end{aligned} \tag{C.8}$$

where $\alpha, \beta, \gamma$ are functions of the spins and scaling dimensions of the external operators. Its Fourier transform is given by,

$$\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} d^dx_1 d^dx_2 d^dx_3 e^{ip_1\cdot x_1 + ip_2\cdot x_2 + p_3\cdot x_3}\, \mathbf{ff}(x_1, x_2, x_3). \tag{C.9}$$

Let us now make the following change of variables:

$$x_{12}^\mu = x_1^\mu - x_2^\mu, x_{23}^\mu = x_2^\mu - x_3^\mu, X^\mu = \frac{x_1^\mu + x_2^\mu + x_3^\mu}{3}. \tag{C.10}$$

The Fourier transform is then given by,

$$\int \frac{d^dX d^dx_{12} d^dx_{23} e^{i(p_1+p_2+p_3)\cdot X + \frac{i}{3}(2p_1-p_2-p_3)\cdot x_{12} + \frac{i}{3}(p_1+p_2-2p_3)\cdot x_{23}}}{\big(-t_{12}^2 + \vec{x}_{12}^2 + i\epsilon t_{12}\big)^\alpha \big(-t_{23}^2 + \vec{x}_{23}^2 + i\epsilon t_{23}\big)^\beta \big(-(t_{12}+t_{23})^2 + (\vec{x}_{12}+\vec{x}_{23})^2 + i\epsilon(t_{12}+t_{23})\big)^\gamma}. \tag{C.11}$$

The integral over $X$ results in $(2\pi)^d\delta(p_1+p_2+p_3)$, that is, the momentum conserving delta function. Upon imposing it, we are left with the following integral:

$$\int \frac{d^dx_{12}d^dx_{23}e^{ip_1\cdot x_{12}+i(p_1+p_2)\cdot x_{23}}}{\big(-t_{12}^2+\vec{x}_{12}^2+i\epsilon t_{12}\big)^{\alpha}\big(-t_{23}^2+\vec{x}_{23}^2+i\epsilon t_{23}\big)^{\beta}\big(-(t_{12}+t_{23})^2+(\vec{x}_{12}+\vec{x}_{23})^2+i\epsilon(t_{12}+t_{23})\big)^{\gamma}}. \tag{C.12}$$

We now wish to show that when say, $p_1=0$, the integral vanishes. To that end consider the $x_{12}$ integral decomposed into its time and space components:

$$\int_{\mathbb{R}\times\mathbb{R}^{d-1}} dt_{12}d^{d-1}\vec{x}_{12}\frac{e^{-ip_1^0t_{12}+i\vec{p}_1\cdot\vec{x}_{12}}}{\big(-t_{12}^2+\vec{x}_{12}^2+i\epsilon t_{12}\big)^{\alpha}\big(-(t_{12}+t_{23})^2+(\vec{x}_{12}+\vec{x}_{23})^2+i\epsilon(t_{12}+t_{23})\big)^{\gamma}}. \tag{C.13}$$

Our focus is now on the integral over $t_{12}$. Consider the case when $p_1=0 \implies p_1^0=\vec{p}_1=0$. The integral becomes,

$$\int_{\mathbb{R}\times\mathbb{R}^{d-1}} dt_{12}d^{d-1}\vec{x}_{12}\frac{1}{\big(-t_{12}^2+\vec{x}_{12}^2+i\epsilon t_{12}\big)^{\alpha}\big(-(t_{12}+t_{23})^2+(\vec{x}_{12}+\vec{x}_{23})^2+i\epsilon(t_{12}+t_{23})\big)^{\gamma}}. \tag{C.14}$$

Consider the case when $\alpha$ and $\gamma$ are integers. The integrand, viewed then has poles at the following locations:

$$\{\pm|\vec{x}_{12}|+i\epsilon, -t_{23}\pm\sqrt{\vec{x}_{12}^2+2\vec{x}_{12}\cdot\vec{x}_{23}+\vec{x}_{23}^2}+i\epsilon\}. \tag{C.15}$$

In particular, they are only in the upper half plane. If $\alpha,\gamma$ are fractions, then the situation is that the branch points are only in the upper half plane. Moreover, note that the spectral condition implies that $-p_1^0>|\vec{p}_1| \implies p_1^0<0$. Either way, we can choose a semi-circular contour in the lower half plane to evaluate this integral. Since there are no poles/branch-points, the result is identically zero.

A similar argument shows that the correlator vanishes when the momentum $p_2$ or $p_3$ vanishes.

### C.2.2 Wightman function conjugation property

### Two point correlators

In position space, the two point Wightman functions of a scalar $O_\Delta$ are given by

$$\begin{aligned}\langle 0|O_\Delta(t_1,\vec{x}_1)O_\Delta(t_2,\vec{x}_2)|0\rangle &= \frac{1}{(-t_{12}^2+\vec{x}_{12}^2+i\epsilon t_{12})^{\Delta}},\\ \langle 0|O_\Delta(t_2,\vec{x}_2)O_\Delta(t_1,\vec{x}_1)|0\rangle &= \frac{1}{(-t_{21}^2+\vec{x}_{21}^2+i\epsilon t_{21})^{\Delta}},\end{aligned} \tag{C.16}$$

where $\epsilon>0$. It is clear that the complex conjugation of the first Wightman function results in the second as $(i\epsilon t_{12})^*=-i\epsilon t_{12}=i\epsilon t_{21}$.

$$\langle 0|O_\Delta(t_1,\vec{x}_1)O_\Delta(t_2,\vec{x}_2)|0\rangle^* = \langle 0|O_\Delta(t_2,\vec{x}_2)O_\Delta(t_1,\vec{x}_1)|0\rangle. \tag{C.17}$$

Thus we see that complex conjugation reverses the operator ordering. Let us derive the analogous statement in Fourier space. We have by definition,

$$\langle 0|O_\Delta(p_1)O_\Delta(p_2)|0\rangle = \int d^3x_1\, d^3x_2\, e^{ip_1\cdot x_1+ip_2\cdot x_2}\langle 0|O_\Delta(t_1,\vec{x}_1)O_\Delta(t_2,\vec{x}_2)|0\rangle. \tag{C.18}$$

Taking complex conjugate on both sides we get,

$$\begin{aligned}\langle 0|O_\Delta(p_1)O_\Delta(p_2)|0\rangle^* &= \int d^3x_1\, d^3x_2\, e^{-ip_1\cdot x_1-ip_2\cdot x_2}\langle 0|O_\Delta(t_1,\vec{x}_1)O_\Delta(t_2,\vec{x}_2)|0\rangle^* \\ &= \int d^3x_1\, d^3x_2\, e^{-ip_1\cdot x_1-ip_2\cdot \vec{x}_2}\langle 0|O_\Delta(t_2,\vec{x}_2)O_\Delta(t_1,\vec{x}_1)|0\rangle \\ &= \langle 0|O_\Delta(-p_2)O_\Delta(-p_1)|0\rangle. \end{aligned} \tag{C.19}$$

Thus, in Fourier space, we see that complex conjugation reverses the order of the operators and flips the sign of the momenta. Let us illustrate this for the two point function in momentum space. We have ,

$$\begin{aligned}\langle 0|O_\Delta(p_1)O_\Delta(p_2)|0\rangle &\propto (-p_1^2)^{\Delta-\frac{d}{2}}\theta(-p_1^0-|\vec{p}_1|)(2\pi)^d\delta^d(p_1+p_2), \\ \langle 0|O_\Delta(p_2)O_\Delta(p_1)|0\rangle &\propto (-p_1^2)^{\Delta-\frac{d}{2}}\theta(p_1^0-|\vec{p}_1|)(2\pi)^d\delta^d(p_1+p_2). \end{aligned} \tag{C.20}$$

It is easy to see that the above reality condition is indeed satisifed.

## Three point correlators

Similar to the reality condition for two points, we see at three points the following reality condition:

$$\langle 0|O_\Delta(t_1,\vec{x}_1)O_\Delta(t_2,\vec{x}_2)O_\Delta(t_3,\vec{x}_3)|0\rangle^* = \langle 0|O_\Delta(t_3,\vec{x}_3)O_\Delta(t_2,\vec{x}_2)O_\Delta(t_1,\vec{x}_1)|0\rangle. \tag{C.21}$$

Tracing through the same steps as for the two point function we obtain the analogous momentum space reality condition:

$$\langle 0|O_\Delta(p_1)O_\Delta(p_2)O_\Delta(p_3)|0\rangle^* = \langle 0|O_\Delta(-p_3)O_\Delta(-p_2)O_\Delta(-p_1)|0\rangle. \tag{C.22}$$

Please note that above, it is the vector that flip and not their magnitudes. It is obvious that this property extends to arbitrary three point (even and odd) Wightman functions involving spinning or spinorial operators. Since the conjugation in Lorentzian signature imposes a reality condition (A.12) on spinors $\lambda$, $\bar{\lambda}$; the imaginary parts of the Wightman functions are purely due to the $i\epsilon$ prescription which gets flipped by complex conjugation.

# Chapter D

# More on Holographic Wightman functions

In this appendix, we discuss in more detail, the expressions for three point Wightman functions in Yang-Mills and Einstein gravity in spinor helicity and twistor space. We also discuss the relationship between four point Wightman functions in special kinematics and Wightman conformal partial waves in many examples.

## D.1 Three point functions in all helicities

In this appendix, we present the expressions for the stripped three point Wightman functions in Yang-Mills theory and Einstein gravity in all eight helicities: First in spinor helicity variables and then in twistor space. We express all results with net negative helicity in terms of $\langle ij \rangle$ and $p_i$ and net positive helicity in terms of $\langle \bar{i}\bar{j} \rangle$ and $p_i$. However, there are many spinor helicity identities derived from momentum conservation and the Schouten identities[1] that we use in the main-text to bring these results to a more convenient form depending on the application.

### D.1.1 Yang-Mills theory

Consider the Yang-Mills stripped three point function in the eight helicity configurations (also suppressing the momentum conserving delta function). The chiral sector (net positive helicity) expressions are,

$$\begin{aligned}
\langle\langle 0|J^{+A}(p_1)J^{+B}(p_2)J^{+C}(p_3)|0\rangle\rangle' &= i f^{ABC} \frac{\langle \bar{1}\bar{2}\rangle\langle \bar{2}\bar{3}\rangle\langle \bar{3}\bar{1}\rangle}{(E-2p_1)(E-2p_2)(E-2p_3)},\\
\langle\langle 0|J^{+A}(p_1)J^{+B}(p_2)J^{-C}(p_3)|0\rangle\rangle' &= -i f^{ABC} \frac{\langle \bar{1}\bar{2}\rangle^3}{\langle \bar{2}\bar{3}\rangle\langle \bar{3}\bar{1}\rangle E},\\
\langle\langle 0|J^{+A}(p_1)J^{-B}(p_2)J^{+C}(p_3)|0\rangle\rangle' &= -i f^{ABC} \frac{\langle \bar{3}\bar{1}\rangle^3}{\langle \bar{1}\bar{2}\rangle\langle \bar{2}\bar{3}\rangle E},\\
\langle\langle 0|J^{-A}(p_1)J^{+B}(p_2)J^{+C}(p_3)|0\rangle\rangle' &= -i f^{ABC} \frac{\langle \bar{2}\bar{3}\rangle^3}{\langle \bar{1}\bar{2}\rangle\langle \bar{3}\bar{1}\rangle E}.
\end{aligned} \tag{D.1}$$

[1]The Schouten identity is simply the statement that there are at most $d$ linearly independent vectors in a $d$ dimensional space.

Their twistor space counterparts are given by,

$$
\begin{aligned}
\langle 0|J^{+A}(Z_1)J^{+B}(Z_2)J^{+C}(Z_3)|0\rangle' &= -4f^{ABC}\delta^{[3]}(Z_1\cdot Z_2)\int\frac{dc_1}{c_1}\int\frac{dc_2}{c_2}\delta^4(c_1Z_1+c_2Z_2-Z_3),\\
\langle 0|J^{+A}(Z_1)J^{+B}(Z_2)J^{-C}(Z_3)|0\rangle' &= -f^{ABC}\text{Sgn}(Z_3\cdot Z_1)\text{Sgn}(Z_2\cdot Z_3)\delta^{[3]}(Z_1\cdot Z_2),\\
\langle 0|J^{+A}(Z_1)J^{-B}(Z_2)J^{+C}(Z_3)|0\rangle' &= -f^{ABC}\text{Sgn}(Z_1\cdot Z_2)\text{Sgn}(Z_2\cdot Z_3)\delta^{[3]}(Z_3\cdot Z_1),\\
\langle 0|J^{-A}(Z_1)J^{+B}(Z_2)J^{+C}(Z_3)|0\rangle' &= -f^{ABC}\text{Sgn}(Z_1\cdot Z_2)\text{Sgn}(Z_3\cdot Z_1)\delta^{[3]}(Z_2\cdot Z_3).
\end{aligned}
\tag{D.2}
$$

The anti-chiral sector (net negative helicity) on the other hand is given by,

$$
\begin{aligned}
\langle\langle 0|J^{-A}(p_1)J^{-B}(p_2)J^{-C}(p_3|0\rangle\rangle' &= -if^{ABC}\frac{\langle 12\rangle\langle 23\rangle\langle 31\rangle}{(E-2p_1)(E-2p_2)(E-2p_3)},\\
\langle\langle 0|J^{-A}(p_1)J^{-B}(p_2)J^{+C}(p_3|0\rangle\rangle' &= if^{ABC}\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle E},\\
\langle\langle 0|J^{-A}(p_1)J^{+B}(p_2)J^{-C}(p_3|0\rangle\rangle' &= if^{ABC}\frac{\langle 31\rangle^3}{\langle 12\rangle\langle 23\rangle E},\\
\langle\langle 0|J^{+A}(p_1)J^{-B}(p_2)J^{-C}(p_3|0\rangle\rangle' &= if^{ABC}\frac{\langle 23\rangle^3}{\langle 12\rangle\langle 31\rangle E},
\end{aligned}
\tag{D.3}
$$

with their twistor versions,

$$
\begin{aligned}
\langle 0|J^{-A}(Z_1)J^{-B}(Z_2)J^{-C}(Z_3)|0\rangle &= -\frac{1}{2}f^{ABC}\text{Sgn}(Z_1\cdot Z_2)\text{Sgn}(Z_2\cdot Z_3)\text{Sgn}(Z_3\cdot Z_1),\\
\langle 0|J^{-A}(Z_1)J^{-B}(Z_2)J^{+C}(Z_3)|0\rangle &= -2f^{ABC}\text{Sgn}(Z_1\cdot Z_2)\int\frac{dc_1}{c_1}\int\frac{dc_2}{c_2}\delta^4(c_1Z_1+c_2Z_2-Z_3),\\
\langle 0|J^{-A}(Z_1)J^{+B}(Z_2)J^{-C}(Z_3)|0\rangle &= -2f^{ABC}\text{Sgn}(Z_1\cdot Z_2)\int\frac{dc_1}{c_1}\int dc_2\, c_2^3\,\delta^4(c_1Z_1+c_2Z_2-Z_3),\\
\langle 0|J^{+A}(Z_1)J^{-B}(Z_2)J^{-C}(Z_3)|0\rangle &= -2f^{ABC}\text{Sgn}(Z_1\cdot Z_2)\int dc_1c_1^3\int\frac{dc_2}{c_2}\delta^4(c_1Z_1+c_2Z_2-Z_3),
\end{aligned}
\tag{D.4}
$$

### D.1.2 Einstein Gravity

A similar analysis can be performed for Einstein gravity. The chiral sector is given by,

$$
\begin{aligned}
\langle\langle 0|T^+(p_1)T^+(p_2)T^+(p_3)|0\rangle\rangle' &= \frac{\langle\bar{1}\bar{2}\rangle^2\langle\bar{2}\bar{3}\rangle^2\langle\bar{3}\bar{1}\rangle^2 p_1p_2p_3}{(E-2p_1)^2(E-2p_2)^2(E-2p_3)^2},\\
\langle\langle 0|T^+(p_1)T^+(p_2)T^-(p_3)|0\rangle\rangle' &= \frac{\langle\bar{1}\bar{2}\rangle^6 p_1p_2p_3}{\langle\bar{2}\bar{3}\rangle^2\langle\bar{3}\bar{1}\rangle^2E^2},\\
\langle\langle 0|T^+(p_1)T^-(p_2)T^+(p_3)|0\rangle\rangle' &= \frac{\langle\bar{3}\bar{1}\rangle^6 p_1p_2p_3}{\langle\bar{1}\bar{2}\rangle^2\langle\bar{2}\bar{3}\rangle^2E^2},\\
\langle\langle 0|T^-(p_1)T^+(p_2)T^+(p_3)|0\rangle\rangle' &= \frac{\langle\bar{2}\bar{3}\rangle^6 p_1p_2p_3}{\langle\bar{1}\bar{2}\rangle^2\langle\bar{3}\bar{1}\rangle^2E^2},
\end{aligned}
\tag{D.5}
$$

with the corresponding twistor space correlators,

$$
\begin{aligned}
\langle 0|T^{+}(Z_1)T^{+}(Z_2)T^{+}(Z_3)|0\rangle &= -4\delta^{[6]}(Z_1\cdot Z_2)\int\frac{dc_1}{c_1^2}\int\frac{dc_2}{c_2^2}\delta^4(c_1Z_1+c_2Z_2-Z_3),\\
\langle 0|T^{+}(Z_1)T^{+}(Z_2)T^{-}(Z_3)|0\rangle &= -|Z_3\cdot Z_1||Z_2\cdot Z_3|\delta^{[6]}(Z_1\cdot Z_2),\\
\langle 0|T^{+}(Z_1)T^{-}(Z_2)T^{+}(Z_3)|0\rangle &= -|Z_1\cdot Z_2||Z_2\cdot Z_3|\delta^{[6]}(Z_3\cdot Z_1),\\
\langle 0|T^{-}(Z_1)T^{+}(Z_2)T^{+}(Z_3)|0\rangle &= -|Z_1\cdot Z_2||Z_3\cdot Z_1|\delta^{[6]}(Z_2\cdot Z_3).
\end{aligned}
\tag{D.6}
$$

Their counterparts in the anti-chiral sector are given by,

$$
\begin{aligned}
\langle\langle 0|T^{-}(p_1)T^{-}(p_2)T^{-}(p_3)|0\rangle\rangle' &= \frac{\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2 p_1p_2p_3}{(E-2p_1)^2(E-2p_2)^2(E-2p_3)^2},\\
\langle\langle 0|T^{-}(p_1)T^{-}(p_2)T^{+}(p_3)|0\rangle\rangle' &= \frac{\langle 12\rangle^6 p_1p_2p_3}{\langle 23\rangle^2\langle 31\rangle^2E^2},\\
\langle\langle 0|T^{-}(p_1)T^{+}(p_2)T^{-}(p_3)|0\rangle\rangle' &= \frac{\langle 31\rangle^6 p_1p_2p_3}{\langle 12\rangle^2\langle 23\rangle^2E^2},\\
\langle\langle 0|T^{+}(p_1)T^{-}(p_2)T^{-}(p_3)|0\rangle\rangle' &= \frac{\langle 23\rangle^6 p_1p_2p_3}{\langle 12\rangle^2\langle 31\rangle^2E^2},
\end{aligned}
\tag{D.7}
$$

and,

$$
\begin{aligned}
\langle 0|T^{-}(Z_1)T^{-}(Z_2)T^{-}(Z_3)|0\rangle &= -\frac{1}{2}|Z_1\cdot Z_2||Z_2\cdot Z_3||Z_3\cdot Z_1|,\\
\langle 0|T^{+}(Z_1)T^{-}(Z_2)T^{-}(Z_3)|0\rangle &= -2|Z_1\cdot Z_2|\int\frac{dc_1}{c_1^2}\int\frac{dc_2}{c_2^2}\delta^4(c_1Z_1+c_2Z_2-Z_3),\\
\langle 0|T^{-}(Z_1)T^{+}(Z_2)T^{-}(Z_3)|0\rangle &= -2|Z_1\cdot Z_2|\int\frac{dc_1}{c_1^2}\int dc_2c_2^6\delta^4(c_1Z_1+c_2Z_2-Z_3),\\
\langle 0|T^{+}(Z_1)T^{-}(Z_2)T^{-}(Z_3)|0\rangle &= -2|Z_1\cdot Z_2|\int dc_1c_1^6\int\frac{dc_2}{c_2^2}\delta^4(c_1Z_1+c_2Z_2-Z_3).
\end{aligned}
\tag{D.8}
$$

## D.2 Four point correlators in special kinematics as CPWs

In this appendix, we present the interpretation of our special kinematics four point functions in terms of conformal partial waves as discusssed in subsection 6.3.5.

We begin with the scaling dimension $\Delta$ scalar tree level four point function (6.115) arising due to the exchange of a $\Delta'$ scalar. The result can simply be written as,

$$
\begin{aligned}
&\langle\langle 0|O_\Delta(p_1)O_\Delta(p_2)O_\Delta(p_3)O_\Delta(p_4)|0\rangle\rangle_{\text{phi-cube}}\theta(p_2^2)\theta(p_3^2)\\
&\propto \langle\langle 0|O_\Delta(p_1)O_\Delta(p_2)O_{\Delta'}(-s)|0\rangle\rangle\frac{1}{|s|^{2\Delta'-3}}\langle\langle 0|O_{\Delta'}(s)O_\Delta(p_3)O_\Delta(p_4)|0\rangle\rangle\\
&= \mathcal{W}^{(s)}_{(O_\Delta O_\Delta|O_{\Delta'}|O_\Delta O_\Delta)}(p_1,p_2|s|p_3,p_4),
\end{aligned}
\tag{D.9}
$$

that is, the Wightman function in these kinematics is simply the s-channel Wightman scalar exchange CPW!

Similarly, for scalar Bhabha scattering we obtain,

$$\begin{aligned}
&\langle\langle 0|O_\Delta(p_1)O^*_\Delta(p_2)O_\Delta(p_3)O^*_\Delta(p_4)|0\rangle\rangle_{\text{Bhabha}}\theta(p_2^2)\theta(p_3^2)\\
&\propto \langle\langle 0|O_\Delta(p_1)O_\Delta(p_2)J^\mu(-s)|0\rangle\rangle\frac{\pi_{\mu\nu}(s)}{|s|}\langle\langle 0|J^\nu(s)O_\Delta(p_3)O_\Delta(p_4)|0\rangle\rangle\\
&= \mathcal{W}^{(s)}_{(O_\Delta O_\Delta|J|O_\Delta O_\Delta)}(p_1,p_2|s|p_3,p_4),
\end{aligned} \tag{D.10}$$

which is yet again the s-channel conformal partial wave due to the exchange of a spin-1 current dual to the bulk photon.

For Compton scattering we obtain,

$$\begin{aligned}
&\langle\langle 0|J(p_1,\epsilon_1)O_\Delta(p_2)O^*_\Delta(p_3)J(p_4,\epsilon_4)|0\rangle\rangle_{\text{Compton}}\theta(p_2^2)\theta(p_3^2)\\
&= \langle\langle 0|J(p_1,\epsilon_1)O_\Delta(p_2)O^*_\Delta(-s)|0\rangle\frac{1}{|s|^{2\Delta-3}}\langle\langle 0|O_\Delta(s)O^*_\Delta(p_3)J(p_4,\epsilon_4)|0\rangle\rangle\\
&= \mathcal{W}^{(s)}_{(JO_\Delta|O_\Delta|O^*_\Delta J)}(p_1,p_2|s|p_3,p_4).
\end{aligned} \tag{D.11}$$

Finally, for Yang-Mills and Einstein-Hilbert gravity we find,

$$\begin{aligned}
&\langle\langle 0|J^{A_1}(p_1,\epsilon_1)J^{A_2}(p_2,\epsilon_2)J^{A_3}(p_3,\epsilon_3)J^{A_4}(p_4,\epsilon_4)|0\rangle\rangle_{YM}\theta(p_2^2)\theta(p_3^2)\\
&= \langle\langle 0|J^{A_1}(p_1,\epsilon_1)J^{A_2}(p_2,\epsilon_2)J^{\mu A}(-s)|0\rangle\rangle\frac{\pi_{\mu\nu}(s)}{|s|}\langle\langle 0|J^\nu_A(s)J^{A_3}(p_3,\epsilon_3)J^{A_4}(p_4,\epsilon_4)|0\rangle\rangle\\
&= \mathcal{W}^{(s)}_{(JJ|J|JJ)}(p_1,p_2|s|p_3,p_4),
\end{aligned} \tag{D.12}$$

and,

$$\begin{aligned}
&\langle\langle 0|T(p_1,\epsilon_1)T(p_2,\epsilon_2)T(p_3,\epsilon_3)T(p_4,\epsilon_4)|0\rangle\rangle_{EH}\theta(p_2^2)\theta(p_3^2)\\
&= \langle\langle 0|T(p_1,\epsilon_1)T(p_2,\epsilon_2)T^{\mu\nu}(-s)|0\rangle\frac{\Pi_{\mu\nu\rho\sigma}(s)}{|s|^3}\langle\langle 0|T^{\rho\sigma}(s)T(p_3,\epsilon_3)T(p_4,\epsilon_4)|0\rangle\rangle\\
&= \mathcal{W}^{(s)}_{(TT|T|TT)}(p_1,p_2|s|p_3,p_4).
\end{aligned} \tag{D.13}$$

# Chapter E

# Super-Conformal Quantum Mechanics

In this appendix, we constrain the form of $\mathcal{N} = 1, 2$ super-conformal quantum mechanical correlators.

## E.1 The Super Conformal Algebra and Ward Identities

### E.1.1 $\mathcal{N} = 1$ Superconformal Algebra and Ward Identities

We begin with the description of $\mathcal{N} = 1$ superspace. A point in this superspace is described by the pair $(t, \theta)$ where $t$ is the usual time coordinate and $\theta$ is Grassmann valued. The generators of the $\mathcal{N} = 1$ superconformal algebra consists of the usual conformal generators $H, K$ and $D$, the supersymmetry generator $Q$ and the special superconformal generator $S$. They obey the following $\mathfrak{osl}(1|2, \mathbb{R})$. graded Lie superalgebra:

$$\begin{gathered}
[D, H] = -iH, \quad [D, K] = iK, \quad [K, H] = -2iD, \\
\{Q, Q\} = H, \quad \{S, S\} = -K, \quad \{Q, S\} = iD, \\
[D, Q] = -\frac{i}{2}Q, \quad [D, S] = \frac{i}{2}S, \;\; [K, Q] = -S, \quad [H, S] = -Q.
\end{gathered} \tag{E.1}$$

Their action on primary operators is as follows:

$$\begin{aligned}
[H, \mathbf{O}_\Delta] &= \omega \mathbf{O}_\Delta, \\
[D, \mathbf{O}_\Delta] &= -i\left(\omega\frac{\partial}{\partial\omega} + (1-\Delta) - \frac{1}{2}\theta\frac{\partial}{\partial\theta}\right)\mathbf{O}_\Delta, \\
[K, \mathbf{O}_\Delta] &= -\left(\omega\frac{\partial^2}{\partial\omega^2} + 2(1-\Delta)\frac{\partial}{\partial\omega} - \theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\omega}\right)\mathbf{O}_\Delta, \\
[Q, \mathbf{O}_\Delta] &= \left(\frac{\partial}{\partial\theta} + \frac{\theta}{2}\omega\right)\mathbf{O}_\Delta, \\
[S, \mathbf{O}_\Delta] &= \left(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\omega} + \left(\frac{1}{2} - \Delta\right)\theta + \frac{\theta\omega}{2}\frac{\partial}{\partial\omega}\right)\mathbf{O}_\Delta.
\end{aligned} \tag{E.2}$$

### E.1.2 $\mathcal{N}=2$ Superconformal Algebra and Ward Identities

We now move on to the arena of $\mathcal{N}=2$ superspace. A point in this superspace is described by the triplet $(t,\theta,\bar{\theta})$ where $t$ is the usual time coordinate and $\theta$ is a complex Grassmann variable. The momentum (or rather, frequency) superspace is spanned by the triplet $(\omega,\theta,\bar{\theta})$. The generators of the $\mathcal{N}=2$ superconformal algebra consists of the usual conformal generators $H,K$ and $D$, the supersymmetry generators $Q,\bar{Q}$, the special superconformal generators $S,\bar{S}$ and the $U(1)$, $R$-symmetry generator $R$. Their algebra can be found for instance in [96]. The action of the $\mathcal{N}=2$ superconformal algebra generators on primary operators is as follows:

$$[H,\mathbf{O}_\Delta] = \omega\,\mathbf{O}_\Delta, \tag{E.3}$$

$$[D,\mathbf{O}_\Delta] = -i\left(\omega\frac{\partial}{\partial\omega} + (1-\Delta) - \frac{1}{2}\theta\frac{\partial}{\partial\theta} - \frac{1}{2}\bar{\theta}\frac{\partial}{\partial\bar{\theta}}\right)\mathbf{O}_\Delta, \tag{E.4}$$

$$[K,\mathbf{O}_\Delta] = -\left(\omega\frac{\partial^2}{\partial\omega^2} + 2(1-\Delta)\frac{\partial}{\partial\omega} - \theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\omega} - \bar{\theta}\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\omega}\right)\mathbf{O}_\Delta, \tag{E.5}$$

$$[Q,\mathbf{O}_\Delta] = \left(\frac{\partial}{\partial\theta} + \frac{\bar{\theta}}{2}\omega\right)\mathbf{O}_\Delta, \quad [\bar{Q},\mathbf{O}_\Delta] = \left(\frac{\partial}{\partial\bar{\theta}} + \frac{\theta}{2}\omega\right)\mathbf{O}_\Delta, \tag{E.6}$$

$$[S,\mathbf{O}_\Delta] = \left(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\omega} + \left(\frac{1}{2}-\Delta\right)\bar{\theta} + \frac{\bar{\theta}\omega}{2}\frac{\partial}{\partial\omega} - \frac{\bar{\theta}\theta}{2}\frac{\partial}{\partial\theta}\right)\mathbf{O}_\Delta, \tag{E.7}$$

$$[\bar{S},\mathbf{O}_\Delta] = \left(\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\omega} + \left(\frac{1}{2}-\Delta\right)\theta + \frac{\theta\omega}{2}\frac{\partial}{\partial\omega} - \frac{\theta\bar{\theta}}{2}\frac{\partial}{\partial\bar{\theta}}\right)\mathbf{O}_\Delta, \tag{E.8}$$

$$[R,\mathbf{O}_\Delta] = \left(\theta\frac{\partial}{\partial\theta} - \bar{\theta}\frac{\partial}{\partial\bar{\theta}}\right)\mathbf{O}_\Delta. \tag{E.9}$$

These imply the following Ward identities for the correlation functions:

$$\begin{aligned}
\sum_{i=1}^{n} \omega_i\, f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n) &= 0,\\
\sum_{i=1}^{n}\left(\omega_i\frac{\partial}{\partial\omega_i} + (1-\Delta_i) - \frac{1}{2}\theta_i\frac{\partial}{\partial\theta_i} - \frac{1}{2}\bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i}\right) f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n) &= 0,\\
\sum_{i=1}^{n}\left(\omega_i\frac{\partial^2}{\partial\omega_i^2} + 2(1-\Delta_i)\frac{\partial}{\partial\omega_i} - \theta_i\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\omega_i} - \bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i}\frac{\partial}{\partial\omega_i}\right) f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n) &= 0,\\
\sum_{i=1}^{n}\left(\frac{\partial}{\partial\theta_i} + \frac{\bar{\theta}_i}{2}\omega_i\right) f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n) &= 0,\\
\sum_{i=1}^{n}\left(\frac{\partial}{\partial\bar{\theta}_i} + \frac{\theta_i}{2}\omega_i\right) f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n) &= 0,\\
\sum_{i=1}^{n}\left(\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\omega_i} + \left(\frac{1}{2}-\Delta_i\right)\bar{\theta}_i + \frac{\bar{\theta}_i\omega_i}{2}\frac{\partial}{\partial\omega_i} - \frac{\bar{\theta}_i\theta_i}{2}\frac{\partial}{\partial\theta_i}\right) f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n) &= 0,\\
\sum_{i=1}^{n}\left(\frac{\partial}{\partial\bar{\theta}_i}\frac{\partial}{\partial\omega_i} + \left(\frac{1}{2}-\Delta_i\right)\theta_i + \frac{\theta_i\omega_i}{2}\frac{\partial}{\partial\omega_i} - \frac{\theta_i\bar{\theta}_i}{2}\frac{\partial}{\partial\bar{\theta}_i}\right) f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n) &= 0,\\
\sum_{i=1}^{n}\left(\theta_i\frac{\partial}{\partial\theta_i} - \bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i}\right) f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n) &= 0.
\end{aligned} \tag{E.10}$$

where, $f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n)$ is a general $\mathcal{N}=2$, $n$-point function.

## E.2 Correlators in $\mathcal{N} = 1$ Super Conformal Quantum Mechanics

In this section, we shall extend our analysis of section 2.4 to theories that have in addition to the $\mathfrak{sl}(2,\mathbb{R})$ conformal symmetry, $\mathcal{N}=1$ super symmetry. We first discuss the superspace formalism that we employ, which we then follow by solving the super conformal Ward identities for two, three and four point functions.

In terms of component fields, a $\mathcal{N}=1$ superfield can be expanded as follows:

$$\mathbf{O}_\Delta(\omega,\theta) = \Phi_\Delta(\omega) + \theta\,\Psi_{\Delta+\frac{1}{2}}(\omega). \tag{E.11}$$

Our aim is to constrain the correlation functions of these superfields by solving the superconformal ward identities. These identities read,

$$\langle[\mathcal{L},\mathbf{O}_{\Delta_1}(\omega_1,\theta_1)]\cdots\mathbf{O}_{\Delta_n}(\omega_n,\theta_n)\rangle + \cdots\langle\mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\cdots[\mathcal{L},\mathbf{O}_{\Delta_n}(\omega_n,\theta_n)]\rangle = 0,\ \ \mathcal{L}\in\{H,K,D,Q,S\}. \tag{E.12}$$

Using the action of the generators on primary operators provided in (E.2) and (E.12) yields the following equations:

$$\sum_{i=1}^{n} \omega_i f_n(\omega_1, \theta_1; \ldots; \omega_n, \theta_n) = 0, \tag{E.13}$$

$$\sum_{i=1}^{n} \left( \omega_i \frac{\partial}{\partial \omega_i} + (1 - \Delta_i) - \frac{1}{2}\theta_i \frac{\partial}{\partial \theta_i} \right) f_n(\omega_1, \theta_1; \ldots; \omega_n, \theta_n) = 0, \tag{E.14}$$

$$\sum_{i=1}^{n} \left( \omega_i \frac{\partial^2}{\partial \omega_i^2} + 2(1 - \Delta_i) \frac{\partial}{\partial \omega_i} - \theta_i \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial \omega_i} \right) f_n(\omega_1, \theta_1; \ldots; \omega_n, \theta_n) = 0, \tag{E.15}$$

$$\sum_{i=1}^{n} \left( \frac{\partial}{\partial \theta_i} + \frac{\theta_i}{2} \omega_i \right) f_n(\omega_1, \theta_1; \ldots; \omega_n, \theta_n) = 0, \tag{E.16}$$

$$\sum_{i=1}^{n} \left( \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial \omega_i} + \left( \frac{1}{2} - \Delta_i \right) \theta_i + \frac{\theta_i \omega_i}{2} \frac{\partial}{\partial \omega_i} \right) f_n(\omega_1, \theta_1; \ldots; \omega_n, \theta_n) = 0, \tag{E.17}$$

where, $f_n(\omega_1, \theta_1; \ldots; \omega_n, \theta_n)$ is a general $\mathcal{N} = 1$, $n$-point function. Armed with the above equations, we now proceed to investigate its implications for correlation functions. We begin with the two-point case.

### Two Point Functions

Consider an arbitrary two-point function:

$$\langle \mathbf{O}_{\Delta_1}(\omega_1, \theta_1) \mathbf{O}_{\Delta_2}(\omega_2, \theta_2) \rangle = \langle \Phi_{\Delta_1}(\omega_1) \Phi_{\Delta_2}(\omega_2) \rangle - \theta_1 \theta_2 \langle \Psi_{\Delta_1}(\omega_1) \Psi_{\Delta_2}(\omega_2) \rangle, \tag{E.18}$$

where we used the superfield expansion [1] (E.11).
Translation, dilatation and special conformal invariance (equations (E.13), (E.14) and (E.15)) constrain the correlator to take the following form:

$$\langle \mathbf{O}_{\Delta_1}(\omega_1, \theta_1) \mathbf{O}_{\Delta_2}(\omega_2, \theta_2) \rangle = \delta(\omega_1 + \omega_2) \delta_{\Delta_1, \Delta_2} \omega_1^{2\Delta_1 - 1} \big( c_0 - c_1 \omega_1 \theta_1 \theta_2 \big). \tag{E.19}$$

The Q supersymmetric Ward identity (E.16) then fixes $c_1 = -\frac{c_0}{2}$. Therefore, the final result for the two point correlator reads,

$$\langle \mathbf{O}_{\Delta_1}(\omega_1, \theta_1) \mathbf{O}_{\Delta_2}(\omega_2, \theta_2) \rangle = c_0 \delta(\omega_1 + \omega_2) \delta_{\Delta_1, \Delta_2} \omega_1^{2\Delta_1 - 1} \left( 1 + \frac{\omega_1}{2} \theta_1 \theta_2 \right). \tag{E.20}$$

Note that the result is quite reminiscent of what was obtained in three dimensions in [48].

### Three Point Functions

We now move to the three point case. A generic correlator reads:

$$\begin{aligned}\langle \mathbf{O}_{\Delta_1}(\omega_1, \theta_1) \mathbf{O}_{\Delta_2}(\omega_2, \theta_2) \mathbf{O}_{\Delta_3}(\omega_3, \theta_3) \rangle = \delta(\omega_1 + \omega_2 + \omega_3) \big( & c_0(\omega_1, \omega_2) + c_4(\omega_1, \omega_2) \theta_1 \theta_2 \\ & + c_5(\omega_1, \omega_2) \theta_2 \theta_3 + c_6(\omega_1, \omega_2) \theta_1 \theta_3 \big).\end{aligned} \tag{E.21}$$

[1] Note that we did not retain any terms with an odd number of the $\theta_i$ as they multiply component correlators which are grassmann odd and hence zero.

This correlator is constrained by the $Q$ Ward identity (E.16) which yields the following constraints:

$$c_5(\omega_1,\omega_2) = \left(c_4(\omega_1,\omega_2) + \frac{1}{2}c_0(\omega_1,\omega_2)\omega_2\right), \quad c_6(\omega_1,\omega_2) = \left(-c_4(\omega_1,\omega_2) + \frac{1}{2}c_0(\omega_1,\omega_2)\omega_1\right). \tag{E.22}$$

Our correlator thus takes the form:

$$\begin{aligned}\langle \mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2)\mathbf{O}_{\Delta_3}(\omega_3,\theta_3)\rangle &= \delta(\omega_1+\omega_2+\omega_3)\Bigg(c_0(\omega_1,\omega_2) + c_4(\omega_1,\omega_2)\theta_1\theta_2 \\ &+ \left(-c_4(\omega_1,\omega_2) + \frac{1}{2}c_0(\omega_1,\omega_2)\omega_1\right)\theta_1\theta_3 + \left(c_4(\omega_1,\omega_2) + \frac{1}{2}c_0(\omega_1,\omega_2)\omega_2\right)\theta_2\theta_3\Bigg).\end{aligned} \tag{E.23}$$

Dilatation invariance (E.14) then fixes the overall scaling of the correlator thus leading to,

$$\begin{aligned}\langle \mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2)\mathbf{O}_{\Delta_3}(\omega_3,\theta_3)\rangle &= \delta(\omega_1+\omega_2+\omega_3)\omega_1^{\Delta_t-2}\Big(c_0(x)\,(2+\omega_1\,\theta_1\theta_3+\omega_1\,x\,\theta_2\theta_3) \\ &+ 2\omega_1\,c_4(x)(\theta_1\theta_2-\theta_1\theta_3+\theta_2\theta_3)\Big),\end{aligned} \tag{E.24}$$

where $\Delta_t = \Delta_1+\Delta_2+\Delta_3, \;\; x = \dfrac{\omega_2}{\omega_1}$.

The final step is to obtain constraints from the special conformal Ward identity (E.15) (The $S$ Ward identity will then trivially follows as $[K,Q] = -S$). This results in four differential equations out of which only three are independent, viz,

$$x(1+x)\frac{d^2c_0(x)}{dx^2} + 2\Big(1-\Delta_2-x(-2+\Delta_2+\Delta_3)\Big)\frac{dc_0(x)}{dx} - (1+2\Delta_1-\Delta_t)(-2+\Delta_t)c_0(x) = 0,$$

$$x(1+x)\frac{d^2c_4(x)}{dx^2} + \Big(1-2\Delta_2+x(3-2\Delta_2-2\Delta_3)\Big)\frac{dc_4(x)}{dx} - (1+2\Delta_1-\Delta_t)(-1+\Delta_t)c_4(x) = 0,$$

$$x^2(1+x)\frac{d^2c_0(x)}{dx^2} + x\Big(3-2\Delta_2-2x(-2+\Delta_2+\Delta_3)\Big)\frac{dc_0(x)}{dx} + \Big(1-2\Delta_2-x(1+2\Delta_1-\Delta_t)(-2+\Delta_t)\Big)c_0(x),$$
$$+2x(1+x)\frac{d^2c_4(x)}{dx^2} + \Big(2-4\Delta_2-4x(-1+\Delta_2+\Delta_3)\Big)\frac{dc_4(x)}{dx} - 2(2\Delta_1-\Delta_t)(-1+\Delta_t)c_4(x) = 0.$$

Solving the first equation gives the solution for $c_0(x)$

$$\begin{aligned}c_0(x) &= c_{01}\,{}_2F_1\Big(2-\Delta_t, 1+2\Delta_1-\Delta_t, 2-2\Delta_2; -x\Big) \\ &+ c_{02}\,x^{2\Delta_2-1}\,{}_2F_1\Big(1+2\Delta_2-\Delta_t, \Delta_t-2\Delta_3, 2\Delta_2; -x\Big),\end{aligned} \tag{E.25}$$

which is exactly the non supersymmetric correlator (2.130) as expected. Similarly, solving the second equation gives,

$$\begin{aligned}c_4(x) &= c_{41}\,{}_2F_1\Big(1-\Delta_t, 1+2\Delta_1-\Delta_t, 1-2\Delta_2; -x\Big) \\ &+ c_{42}\,x^{2\Delta_2}\,{}_2F_1\Big(1+2\Delta_2-\Delta_t, 1+\Delta_t-2\Delta_3, 1+2\Delta_2; -x\Big).\end{aligned} \tag{E.26}$$

The third equation mixes the coefficients giving the following constraints,

$$c_{41} = c_{01}\frac{2\Delta_2 - 1}{2(-1+\Delta_t)}, \qquad c_{42} = -c_{02}\frac{\Delta_t - 2\Delta_3}{4\Delta_2}. \tag{E.27}$$

Using the constraints obtained by the action of $Q$ (E.22), and Hypergeometric function Identities, we obtain the following form for $c_5(x)$:

$$\begin{aligned} c_5(x) = c_{51}\,{}_2F_1\Big(1-\Delta_t, 2\Delta_1-\Delta_t, 1-2\Delta_2; -x\Big) \\ + c_{52}\,x^{2\Delta_2}\,{}_2F_1\Big(1+2\Delta_2-\Delta_t, 2\Delta_1+2\Delta_2-\Delta_t, 1+2\Delta_2; -x\Big), \end{aligned} \tag{E.28}$$

where the coefficients are given by,

$$c_{51} = c_{01}\frac{2\Delta_2 - 1}{2(-1+\Delta_t)}, \quad c_{52} = c_{02}\frac{\Delta_t - 2\Delta_1}{4\Delta_2}. \tag{E.29}$$

Similarly, we can repeat the same procedure to obtain $c_6(x)$:

$$\begin{aligned} c_6(x) = c_{61}\,{}_2F_1\Big(1-\Delta_t, 1+2\Delta_1-\Delta_t, 2-2\Delta_2; -x\Big) \\ + c_{62}\,x^{2\Delta_2-1}\,{}_2F_1\Big(2\Delta_2-\Delta_t, 2\Delta_1+2\Delta_2-\Delta_t, 2\Delta_2; -x\Big), \end{aligned} \tag{E.30}$$

with coefficients $c_{61}$ and $c_{62}$ given by,

$$c_{61} = c_{01}\frac{\Delta_t - 2\Delta_2}{2(-1+\Delta_t)}, \quad c_{62} = \frac{c_{02}}{2}. \tag{E.31}$$

Therefore, our final expression for the three-point function in $\mathcal{N}=1$ SCQM reads,

$$\begin{aligned} \langle \mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2)\mathbf{O}_{\Delta_3}(\omega_3,\theta_3)\rangle = \delta(\omega_1+\omega_2+\omega_3)\omega_1^{\Delta_t-2} \\ \Bigg(c_{01}\Big({}_2F_1\,(2-\Delta_t, 1+2\Delta_1-\Delta_t, 2-2\Delta_2; -x)\,(2+\omega_1\,\theta_1\theta_3+\omega_1\,x\,\theta_2\theta_3) \\ + \omega_1\frac{2\Delta_2-1}{\Delta_t-1}\,{}_2F_1\,(1-\Delta_t, 1+2\Delta_1-\Delta_t, 1-2\Delta_2; -x)\,(\theta_1\theta_2-\theta_1\theta_3+\theta_2\theta_3)\Big) \\ + c_{02}\,x^{2\Delta_2-1}\Big({}_2F_1\,(1+2\Delta_2-\Delta_t, \Delta_t-2\Delta_3, 2\Delta_2; -x)\,(2+\omega_1\,\theta_1\theta_3+\omega_1\,x\,\theta_2\theta_3) \\ - \omega_1\,x\frac{\Delta_t-2\Delta_3}{2\Delta_2}\,{}_2F_1\,(1+2\Delta_2-\Delta_t, 1+\Delta_t-2\Delta_3, 1+2\Delta_2; -x)\,(\theta_1\theta_2-\theta_1\theta_3+\theta_2\theta_3)\Big)\Bigg). \end{aligned} \tag{E.32}$$

where $x = \dfrac{\omega_2}{\omega_1}$.

### Four Point Functions

We now move to the four-point case. A generic correlator reads:

$$\langle \mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2)\mathbf{O}_{\Delta_3}(\omega_3,\theta_3)\mathbf{O}_{\Delta_4}(\omega_4,\theta_4)\rangle = \delta(\omega_1+\omega_2+\omega_3+\omega_4) \tag{E.33}$$

$$\begin{aligned}&\big(c_0(\omega_1,\omega_2,\omega_3) + c_{12}(\omega_1,\omega_2,\omega_3)\theta_1\theta_2\\ &+ c_{13}(\omega_1,\omega_2,\omega_3)\theta_1\theta_3 + c_{14}(\omega_1,\omega_2,\omega_3)\theta_1\theta_4\\ &+ c_{23}(\omega_1,\omega_2,\omega_3)\theta_2\theta_3 + c_{24}(\omega_1,\omega_2,\omega_3)\theta_2\theta_4\\ &+ c_{34}(\omega_1,\omega_2,\omega_3)\theta_3\theta_4 + c_{1234}(\omega_1,\omega_2,\omega_3)\theta_1\theta_2\theta_3\theta_4\big).\end{aligned} \tag{E.34}$$

Constraining the correlator using the $Q$ Ward Identity (E.16), yields the following constraints:

$$\begin{aligned} c_{1234} &= \frac{1}{2}\left(c_{23}\omega_1 - c_{13}\omega_2 + c_{12}\omega_3\right), \quad & c_{24} &= c_{12} - c_{23} + \frac{c_0\omega_2}{2},\\ c_{14} &= -c_{12} - c_{13} + \frac{c_0\omega_1}{2}, & c_{34} &= c_{13} + c_{23} + \frac{c_0\omega_3}{2}. \end{aligned} \tag{E.35}$$

Dilatation invariance (E.14) then fixes the overall scaling of the correlator, thus leading to,

$$\langle \mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2)\mathbf{O}_{\Delta_3}(\omega_3,\theta_3)\mathbf{O}_{\Delta_4}(\omega_4,\theta_4)\rangle = \delta(\omega_1+\omega_2+\omega_3+\omega_4)\omega_1^{\Delta_t-3}$$

$$\begin{aligned}&\Big(c_0(x,y)\left(2+\omega_1\,\theta_1\theta_4 + \omega_2\;\theta_2\theta_4 + \omega_3\,\theta_3\theta_4\right)\\ &+ \omega_1\,c_{23}(x,y)\big(2(\theta_2\theta_3 - \theta_2\theta_4 + \theta_3\theta_4) + \omega_1\theta_1\theta_2\theta_3\theta_4\big)\\ &+ \omega_1\,c_{13}(x,y)\big(2(\theta_1\theta_3 - \theta_1\theta_4 + \theta_3\theta_4) - \omega_2\theta_1\theta_2\theta_3\theta_4\big)\\ &+ \omega_1\,c_{12}(x,y)\big(2(\theta_1\theta_2 - \theta_1\theta_4 + \theta_2\theta_4) + \omega_3\theta_1\theta_2\theta_3\theta_4\big)\Big),\end{aligned} \tag{E.36}$$

where $\Delta_t = \Delta_1+\Delta_2+\Delta_3+\Delta_4, x = \dfrac{\omega_2}{\omega_1}, y = \dfrac{\omega_3}{\omega_1}$.

From the component expansion, it can be seen that the components $c_{12},\ldots$ have the same functional form as $c_0(x,y)$, i.e., Appell functions (2.139), but differ only by the scaling dimensions. However, we can follow the same routine as we did for the three point function and apply $K$ to get the constraints on coefficients. We choose not to do that and instead apply $S$ as it has a first-order action and leads to simpler equations. If we fix the component correlators by their Appell function representation (2.139) and leave the coefficients undetermined, then apply $S$, we obtain constraints connecting these coefficients. For instance at $O(\theta_1)$ and $O(\theta_2)$ we find,

$$2\frac{dc_{12}}{dx} + 2\frac{dc_{13}}{dy} - x\frac{dc_0}{dx} - y\frac{dc_0}{dy} - (2+\Delta_1-\Delta_2-\Delta_3-\Delta_4)c_0 = 0, \tag{E.37}$$

$$-2x\frac{dc_{12}}{dx} - 2y\frac{dc_{12}}{dy} + 2\frac{dc_{23}}{dy} + x\frac{dc_0}{dx} - 2(2-\Delta_t)c_{12} + (1-2\Delta_2)c_0 = 0. \tag{E.38}$$

Proceeding to higher orders in the Grassmann expansion, we obtain similar constraints. To solve these equations (as well as the analysis for higher point functions) is an interesting problem which we defer to the future. We now proceed to the $\mathcal{N}=2$ case.

## E.3 Correlators in $\mathcal{N} = 2$ Super Conformal Quantum Mechanics

In this section, we extend our results to conformal theories that also possess $\mathcal{N} = 2$ supersymmetry.

We bring the reader's attention to the $R$ symmetry generator (last equation of (E.10)). The $\theta_i$ have $R$ charge $+1$ while the $\bar{\theta}_i$ have $R$ charge $-1$. Therefore, every single term in a correlation function which is a singlet of the $R$ symmetry group should contain an equal number of $\theta_i$ and $\bar{\theta}_i$.

In the $\mathcal{N} = 2$ superspace formalism, (bosonic) superfields can be expanded as follows:

$$\mathbf{O}_\Delta(\omega, \theta) = \Phi_\Delta(\omega) + \theta\, \Psi_{\Delta+\frac{1}{2}}(\omega) + \bar{\theta}\, \bar{\Psi}_{\Delta+\frac{1}{2}}(\omega) + \theta\bar{\theta} F_{\Delta+1}(\omega). \tag{E.39}$$

Our aim in what is to follow is to constrain the correlation functions of these superfields by solving the superconformal ward identities. These identities read,

$$\langle [\mathcal{L}, \mathbf{O}_{\Delta_1}(\omega_1, \theta_1)] \cdots \mathbf{O}_{\Delta_n}(\omega_n, \theta_n)\rangle + \cdots \langle \mathbf{O}_{\Delta_1}(\omega_1, \theta_1) \cdots [\mathcal{L}, \mathbf{O}_{\Delta_n}(\omega_n, \theta_n)]\rangle = 0,\ \mathcal{L} \in \{H, K, D, Q, S, \bar{Q}, \bar{S}, R\}. \tag{E.40}$$

The explicit form of these identities can be found in sub-appendix E.1.2. Armed with the above equations, we now proceed to investigate its implications for correlation functions. We now proceed to obtain two and three-point functions that were obtained by solving the $\mathcal{N} = 2$ superconformal Ward identities. Since the procedure of obtaining the solutions is identical to that of the $\mathcal{N} = 1$ case, we shall desist from providing details and instead provide the final results.

### Two Point Functions

The two point function of a generic $\mathcal{N} = 2$ primary superfield takes the following form:

$$\langle \mathbf{O}_{\Delta_1}(\omega_1, \theta_1, \bar{\theta}_1) \mathbf{O}_{\Delta_2}(\omega_2, \theta_2, \bar{\theta}_2)\rangle = \delta_{\Delta_1, \Delta_2} \delta(\omega_1 + \omega_2) \omega_1^{2\Delta_1 - 1} \Big( 4 + 2\omega_1(\theta_1\bar{\theta}_2 - \theta_2\bar{\theta}_1) - \omega_1^2 \theta_1\theta_2\bar{\theta}_1\bar{\theta}_2 \Big). \tag{E.41}$$

It can easily be verified that this expression satisfies the superconformal Ward identities. We also note an interesting relation between (E.41) and it's $\mathcal{N} = 1$ counterpart (E.20):

$$\langle \mathbf{O}_{2\Delta_1}(\omega_1, \theta_1, \bar{\theta}_1) \mathbf{O}_{2\Delta_2}(\omega_2, \theta_2, \bar{\theta}_2)\rangle_{\mathcal{N}=2} = \omega_1 \langle \mathbf{O}_{\Delta_1}(\omega_1, \theta_1) \mathbf{O}_{\Delta_2}(\omega_2, \bar{\theta}_2)\rangle_{\mathcal{N}=1} \langle \mathbf{O}_{\Delta_1}(\omega_1, \bar{\theta}_1) \mathbf{O}_{\Delta_2}(\omega_2, \theta_2)\rangle_{\mathcal{N}=1}. \tag{E.42}$$

The above form is the unique product of $\mathcal{N} = 1$ two point functions that possess the required $R$ symmetry. Indeed, this is reminiscent of the super double copy obtained in three-dimensional conformal field theories in [48]. The authors of [48] also obtained a super double copy at the three point level. Let us also thus move on to the three point case.

**Three Point Functions**

Let us first define,

$$C_i(\Delta_1,\Delta_2,\Delta_3;-x) = c_{i1}\ {}_2F_1(2-\Delta_1-\Delta_2-\Delta_3,1+\Delta_1-\Delta_2-\Delta_3,2(1-\Delta_2);-x) + x^{2\Delta_2-1}\ c_{i2}\ {}_2F_1(1-\Delta_1+\Delta_2-\Delta_3,\Delta_1+\Delta_2-\Delta_3,2\Delta_2,-x), \tag{E.43}$$

and,

$$\begin{aligned} c_0(x) &= C_0\left(\Delta_1,\Delta_2,\Delta_3;-x\right), \quad c_6(x) = C_6\left(\Delta_1+\frac{1}{2},\Delta_2+\frac{1}{2},\Delta_3;-x\right),\\ c_8(x) &= C_8\left(\Delta_1+\frac{1}{2},\Delta_2,\Delta_3+\frac{1}{2};-x\right), \quad c_9(x) = C_9\left(\Delta_1,\Delta_2+\frac{1}{2},\Delta_3+\frac{1}{2};-x\right),\\ c_{11}(x) &= C_{11}\left(\Delta_1,\Delta_2+1,\Delta_3;-x\right), \quad c_{12}(x) = C_{12}\left(\Delta_1+1,\Delta_2+1,\Delta_3;-x\right). \end{aligned} \tag{E.44}$$

Our result for the $\mathcal{N}=2$ SCQM three-point correlator after solving the superconformal Ward identities (E.10) is the following expression:

$$\begin{aligned} &\langle \mathbf{O}_{\Delta_1}(\omega_1,\theta_1,\bar\theta_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2,\bar\theta_2)\mathbf{O}_{\Delta_3}(\omega_3,\theta_3,\bar\theta_3)\rangle = \delta(\omega_1+\omega_2+\omega_3)\omega_1^{\Delta_t-2}\\ &\Bigg[c_0(x)\Bigg(1+\frac{\omega_1}{4}\Big(-2\theta_1\bar\theta_1-2x\theta_1\bar\theta_2+2(x+2)\theta_1\bar\theta_3+2x\theta_2\bar\theta_3-2(1+x)\theta_3\bar\theta_3\\ &\quad+x\,\omega_1\theta_1\theta_2\bar\theta_1\bar\theta_3+x^2\omega_1\theta_1\theta_2\bar\theta_2\bar\theta_3-\omega_1(1+x)\theta_1\theta_3\bar\theta_1\bar\theta_3-x(1+x)\omega_1\theta_1\theta_3\bar\theta_2\bar\theta_3\Big)\Bigg)\\ &+c_6(x)\Bigg(\frac{\omega_1}{2}\Big(-2\theta_1\bar\theta_1+2\theta_1\bar\theta_3+2\theta_2\bar\theta_1-2\theta_2\bar\theta_3+\omega_1\theta_1\theta_2\bar\theta_1\bar\theta_3+x\omega_1\theta_1\theta_2\bar\theta_1\bar\theta_3\\ &\quad-x\omega_1\theta_1\theta_3\bar\theta_1\bar\theta_2-\omega_1\theta_1\theta_3\bar\theta_1\bar\theta_3-x\omega_1\theta_1\theta_3\bar\theta_2\bar\theta_3+x\omega_1\theta_2\theta_3\bar\theta_1\bar\theta_2\\ &\quad+\omega_1\theta_2\theta_3\bar\theta_1\bar\theta_3+x\omega_1\theta_2\theta_3\bar\theta_2\bar\theta_3\Big)+\frac{x(x+1)}{4}\omega_1^3\theta_1\theta_2\theta_3\bar\theta_1\bar\theta_2\bar\theta_3\Bigg)\\ &+c_8(x)\Bigg(\frac{\omega_1}{2}\Big(-2\theta_1\bar\theta_1+2\theta_1\bar\theta_3+2\theta_3\bar\theta_1-2\theta_3\bar\theta_3+x\omega_1\theta_1\theta_2\bar\theta_1\bar\theta_3-x\omega_1\theta_1\theta_3\bar\theta_1\bar\theta_2-x\omega_1\theta_1\theta_3\bar\theta_2\bar\theta_3\\ &\quad+x\omega_1\theta_2\theta_3\bar\theta_1\bar\theta_3\Big)+\frac{x^2\omega_1^3}{4}\theta_1\theta_2\theta_3\bar\theta_1\bar\theta_2\bar\theta_3\Bigg)\\ &+c_9(x)\Bigg(\frac{\omega_1}{2}\Big(-2\theta_1\bar\theta_2+2\theta_1\bar\theta_3+2\theta_3\bar\theta_2-2\theta_3\bar\theta_3+x\omega_1\theta_1\theta_2\bar\theta_2\bar\theta_3+\omega_1\theta_1\theta_3\bar\theta_1\bar\theta_2-\omega_1\theta_1\theta_3\bar\theta_1\bar\theta_3\\ &\quad+\omega_1\theta_1\theta_3\bar\theta_2\bar\theta_3-x\omega_1\theta_1\theta_3\bar\theta_2\bar\theta_3+x\omega_1\theta_2\theta_3\bar\theta_2\bar\theta_3\Big)-\frac{x}{4}\omega_1^3\theta_1\theta_2\theta_3\bar\theta_1\bar\theta_2\bar\theta_3\Bigg)\\ &+c_{11}(x)\Bigg(\frac{\omega_1}{2}\Big(-2\theta_1\bar\theta_2+2\theta_1\bar\theta_3+2\theta_2\bar\theta_2-2\theta_2\bar\theta_3+\omega_1\theta_1\theta_2\bar\theta_2\bar\theta_3+x\omega_1\theta_1\theta_2\bar\theta_2\bar\theta_3\\ &\quad+\omega_1\theta_1\theta_3\bar\theta_1\bar\theta_2-\omega_1\theta_1\theta_3\bar\theta_1\bar\theta_3-x\omega_1\theta_1\theta_3\bar\theta_2\bar\theta_3-\omega_1\theta_2\theta_3\bar\theta_1\bar\theta_2\\ &\quad+\omega_1\theta_2\theta_3\bar\theta_1\bar\theta_3+x\omega_1\theta_2\theta_3\bar\theta_2\bar\theta_3\Big)-\frac{(x+1)}{4}\omega_1^3\theta_1\theta_2\theta_3\bar\theta_1\bar\theta_2\bar\theta_3\Bigg)\\ &+c_{12}(x)\omega_1^2\Bigg((\theta_1\theta_2\bar\theta_1\bar\theta_2-\theta_1\theta_2\bar\theta_1\bar\theta_3+\theta_1\theta_2\bar\theta_2\bar\theta_3-\theta_1\theta_3\bar\theta_1\bar\theta_2+\theta_1\theta_3\bar\theta_1\bar\theta_3-\theta_1\theta_3\bar\theta_2\bar\theta_3+\theta_2\theta_3\bar\theta_1\bar\theta_2\\ &\quad-\theta_2\theta_3\bar\theta_1\bar\theta_3)+\omega_1^2\theta_2\theta_3\bar\theta_2\bar\theta_3\Bigg)\Bigg]. \end{aligned} \tag{E.45}$$

where $x = \dfrac{\omega_2}{\omega_1}$ and the coefficients of the $c_i$ are related as follows:

$$
\begin{aligned}
c_{81} &= -c_{01}\frac{\Delta_1}{\Delta_t - 1} - c_{61}\frac{\Delta_1 + \Delta_2 - \Delta_3}{2\Delta_2 - 1}, & c_{82} &= -\frac{c_{02}\Delta_1 + 2c_{62}\Delta_2}{\Delta_1 - \Delta_2 + \Delta_3},\\
c_{91} &= -\frac{c_{01}(2\Delta_2 - 1)(\Delta_1 - \Delta_3)}{(2\Delta_1 - \Delta_t)(\Delta_t - 1)} + c_{61}\frac{(2\Delta_3 - \Delta_t)}{(2\Delta_1 - \Delta_t)}, & c_{92} &= \frac{c_{02}(\Delta_1 - \Delta_3) - 2c_{62}\Delta_2}{2\Delta_2},\\
c_{121} &= c_{01}\frac{\Delta_2(1 - 2\Delta_2)}{2(\Delta_3 - 1)\Delta_t}, & c_{122} &= -c_{02}\frac{(\Delta_1 + \Delta_2 - \Delta_3)(1 + \Delta_1 + \Delta_2 - \Delta_3)}{8\Delta_2(1 + 2\Delta_2)},\\
c_{111} &= \Delta_2\frac{c_{01}(2\Delta_2 - 1) + 2c_{61}(\Delta_t - 1)}{(2\Delta_1 - \Delta_t)(\Delta_t - 1)}, & c_{112} &= (\Delta_t - 2\Delta_2 - 1)\frac{c_{02}(\Delta_t - 2\Delta_3) - 4c_{62}\Delta_2}{4\Delta_2(1 + 2\Delta_2)}.
\end{aligned}
\tag{E.46}
$$

Notice that in contrast with the $\mathcal{N} = 1$ three point function (E.32) which had two free parameters, its $\mathcal{N} = 2$ counterpart has four $(c_{01}, c_{02}, c_{61}, c_{62})$. This motivates us to check if there exists a double copy like we found for the two point case (E.42). However, when we explicitly checked, we found no such double copy relation. Perhaps we need to employ variables similar to the Grassmann twistor variables of [48] for a double copy relation to be manifest. We leave such an exercise as well as the analysis of higher point functions and higher supersymmetry to the future.

# Chapter F

# Special Functions and their Series Expansions

In this appendix, we discuss the different special functions encountered in this thesis. These include Bessel and modified Bessel functions of various kinds, the ${}_2F_1$ hypergeometric function, Appell $F_2$ and $F_4$ functions and the $E_A^{(m)}$ Lauricella functions. We discuss useful identities and series expansions for these quantities.

## F.1 Bessel and Hankel functions

Bessel's differential equation is,

$$z^2 \frac{d^2 w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2) w = 0. \tag{F.1}$$

The two linearly independent solutions to this ODE are,

$$\begin{aligned} J_\nu(z) &= \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{z^2}{4}\right)^k}{k!\Gamma(\nu + k + 1)}, \\ Y_\nu(z) &= \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}. \end{aligned} \tag{F.2}$$

A different basis of solutions to this ODE are the Hankel functions which are obtained via the connection formulae,

$$\begin{aligned} H_\nu^{(1)}(z) &= J_\nu(z) + iY_\nu(z), \\ H_\nu^{(2)}(z) &= J_\nu(z) - iY_\nu(z). \end{aligned} \tag{F.3}$$

This is essentially the generalization of working with $\sin(z), \cos(z)$ or $e^{iz}$ and $e^{-iz}$. We also deal with modified Bessel functions which obey,

$$z^2 \frac{d^2 w}{dz^2} + z\frac{dw}{dz} - (z^2 + \nu^2) w = 0. \tag{F.4}$$

The solutions are,

$$\begin{aligned} I_\nu(z) &= \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k!\Gamma(\nu+k+1)}, \\ K_\nu(z) &= \frac{\pi}{2}\frac{I_{-\nu}(z)-I_\nu(z)}{\sin(\nu\pi)}. \end{aligned} \tag{F.5}$$

Note that the only difference between the series expansion for $I_\nu(z)$ and $J_\nu(z)$ is that the latter has an extra $(-1)^k$ in the sum.

## F.2 Hypergeometric functions

The hypergeometric function ${}_2F_1(a,b;c;z)$ is defined via the following Gauss series:

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k. \tag{F.6}$$

$(x)_k$ denotes the rising Pochhamer symbol $(x)_k = x(x+1)\cdots(x+k-1)$. This function is the solution to the following differential equation,

$$z(1-z)\frac{d^2w}{dz^2} + (c-(a+b+1)z)\frac{dw}{dz} - abw = 0. \tag{F.7}$$

Some useful identities involving the hypergeometric functions that we used in the main text are:

$$\begin{aligned} &\frac{2\Delta_2-1}{\Delta_t-1}\,{}_2F_1(1-\Delta_t, 1-\Delta_t+2\Delta_1, 1-2\Delta_2; -x) + x\,{}_2F_1(2-\Delta_t, 1-\Delta_t+2\Delta_1, 2-2\Delta_2; -x) \\ &\qquad = \frac{2\Delta_2-1}{\Delta_t-1}\,{}_2F_1(1-\Delta_t, 2\Delta_1-\Delta_t, 1-2\Delta_2; -x). \end{aligned} \tag{F.8}$$

$$\begin{aligned} &\frac{2\Delta_2}{2\Delta_1+2\Delta_2-\Delta_t}\,{}_2F_1(1+2\Delta_2-\Delta_t, 2\Delta_1+2\Delta_2-\Delta_t, 2\Delta_2; -x) \\ &\qquad - {}_2F_1(1+2\Delta_2-\Delta_t, 1+2\Delta_1+2\Delta_2-\Delta_t, 1+2\Delta_2; -x) \\ &\qquad = \frac{\Delta_t-2\Delta_1}{2\Delta_1+2\Delta_2-\Delta_t}\,{}_2F_1(1+2\Delta_2-\Delta_t, 2\Delta_1+\Delta_2-\Delta_t, 1+2\Delta_2; -x). \end{aligned} \tag{F.9}$$

## F.3 Appell Functions

Appell functions are two variable generalizations of the Gauss hypergeometric functions. There are four of these, two of which we have encountered in this thesis. These are

Appell $F_2$ and Appell $F_4$. They have the following series expansions,

$$F_2(a; b, b'; c, c; ; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n m!n!} x^m y^n$$
$$F_4(a, b; c, c', x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n m!n!} x^m y^n. \tag{F.10}$$

They obey the following differential equations,

$$\left(x(1-x)\frac{\partial^2}{\partial x^2} - xy\frac{\partial^2}{\partial x \partial y} + (c-(a+b+1)x)\frac{\partial}{\partial x} - by\frac{\partial}{\partial y} - ab\right)F_2(a; b, b'; c, c'; x, y) = 0,$$
$$\left(y(1-y)\frac{\partial^2}{\partial y^2} - xy\frac{\partial^2}{\partial x \partial y} + (c'-(a+b'+1)x)\frac{\partial}{\partial y} - b'x\frac{\partial}{\partial x} - ab'\right)F_2(a; b, b'; c, c'; x, y) = 0, \tag{F.11}$$

and,

$$\left(x(1-x)\frac{\partial^2}{\partial x^2} - 2xy\frac{\partial^2}{\partial x \partial y} - y^2\frac{\partial^2}{\partial y^2} + (c-(a+b+1)x)\frac{\partial}{\partial x} - (a+b+1)y\frac{\partial}{\partial y} - ab\right)F_4(a; b; c, c'; x, y) = 0,$$
$$\left(y(1-y)\frac{\partial^2}{\partial y^2} - 2xy\frac{\partial^2}{\partial x \partial y} - x^2\frac{\partial^2}{\partial x^2} + (c'-(a+b+1)y)\frac{\partial}{\partial y} - (a+b+1)x\frac{\partial}{\partial x} - ab\right)F_4(a; b; c, c'; x, y) = 0. \tag{F.12}$$

## F.4 Lauricella functions

Lauricella functions are $n-$variable generalizations of the Gauss hypergeometric function. The one of interest to us is Lauricella $E_A^{(m)}$ which we have encountered in chapter 2. A series expansion for the type-A Lauricella function of $m$ variables is the following [112]:

$$E_A^{(m)}(a, b_1, \cdots, b_m, c_1, \cdots, c_m; -x_1, \cdots, -x_m) = \sum_{m_i \in \mathbb{N}_0} \frac{(a)_{m_1+\cdots+m_n} \prod_{i=1}^m (b_i)_{m_i}}{\prod_{i=1}^m (c_i)_{m_i} \prod_{i=1}^m m_i!} \prod_{i=1}^m x_i^{m_i}. \tag{F.13}$$

$\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$ is the rising Pochammer symbol. The above series converges when $\sum_{i=1}^m |x_i| < 1$. As a special case, we obtain ${}_2F_1$, when $m = 1$; Appell $F_2$, when $m = 2$ and so on.

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