Trans-Planckian issues & Analogue Gravity: from BEC to acoustic Black Holes

A thesis

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by:

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Dedicated to my
Maa & Baba

Certificate

Certified that the work incorporated in the thesis entitled **"Trans-Planckian issues** & **Analogue Gravity: from BEC to acoustic Black Holes"** Submitted by ***Supratik Sarkar*** was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other University or Institutions.

Pune, India
Date: July 18, 2018

(Dr. Arijit Bhattacharyay)
Thesis Supervisor

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included as well, but I have adequately cited and referenced the original sources for the corresponding works. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Contents

Abstract

To account for the non-local interactions in a Bose-Einstein Condensate (BEC), an addition of a minimal correction term to the standard Gross-Pitaevskii model effectively can make the healing length or characteristic length (ξ) decrease more rapidly with the increase of s-wave scattering length (a). From emergent gravity perspectives, this shrinking of ξ via tuning a through Feshbach resonance, in principle, does make the short-wavelength (i.e. high energy) regime more accessible experimentally by pushing the Lorentz-breaking dispersion even more towards the UV side [1]. The effects of the Lorentz-breaking quantum potential term in the BEC-dynamics on independent multiple scales can be captured through a UV-IR coupling of the phonon-excitation-modes with the emergence of the massive minimally coupled free Klein-Gordon field. The analysis was argued on a (3+1)D flat spacetime where the presence of the mass term gives a hint to cure the infrared divergences through a nonzero threshold for the large-wavelength phonon excitations (say, $K_z \neq 0$ modes as perceived by an observer sitting on $x - y$ plane) on lower dimensional models of emergent spacetimes [2]. The analysis was extended for a canonical acoustic black hole on a (3+1)D curved spacetime through presenting an emergent gravity model up to $O(\xi^2)$ accuracy. In our formalism, the growth rate of the large-wavelength 'secondary' ω modes is found to hold the clue to extract the lost information regarding the short-wavelength 'primary' ω_1 modes. And hence this can actually reveal the relative abundance of the originally Hawking radiated quanta in a (3+1)D canonically curved background [3]. This way we could address the trans-Planckian issues in Physics via introducing the analogue Planck length (the healing length ξ of the underlying cond.mat system, say BEC in our case, is the analogue of the Planck length $\ell_p \approx 10^{-35}$m. in Gravity) and shrinking this ξ to an arbitrary small value by experimentally (Feshbach resonances) increasing the s-wave scattering length a. We showed that the UV-IR coupling between two different bands of phonon excitations is inevitable as perceived by an observer in the emergent gravity experiments. And finally we tackled the way to retrieve the "information loss" in Hawking radiation within the scope emergent spacetimes.

Chapter 1

Introduction

Reasoning by 'analogies' is a natural human tendency that associates new and unknown situations to a stack of known and previously encountered ones. These analogies have played a very crucial role in Physics and Mathematics by providing new ways of looking at various problems and critical issues which allow cross-fertilization of ideas among different branches of science [4]. It is therefore quite evident that the 'analogue models' are kind of invaluable platforms towards providing the progress of our knowledge and understanding of the surrounding nature. A suitably chosen analogy thus can play a very useful role in solving specific problems in Physics and sometimes suggest unexpected and fruitful routes to possible solutions.

Analogue Gravity gives a general overview of simulating or recreating certain phenomena which are attributed to the effects of gravity but surprisingly can be shown to generically emerge in a variety of laboratory systems ranging from flowing fluid to nonlinear optics [5]. Analogue Gravity basically studies condensed matter systems that share properties and features of gravitational theories[1] in $(3 + 1)$D.

It is a new proposed methodology where an 'analogy' is so chosen to address specific issues in General Relativity (GR) and curved spacetime Quantum Field Theory (QFT) [6] through a mathematical framework based on the emergence of an effective acoustic metric $\mathsf{g}_{\mu\nu}$ (pseudo-Riemannian structure) which is formed from the underlying condensed matter system, but is not a solution of Einstein's field equations in Gravity. Since the underlying condensed matter systems are experimentally realizable, hence through this gravity/cond.mat correspondence, opens up a new experimental window to probe various aspects of GR and curved spacetime QFT [7, 8].

The analogue gravity models usually do not take any attempts towards reproducing the 'dynamics' of a gravitational system, e.g., a black hole in gravity, because the dynamics

[1] "The flowing fluid, under appropriate conditions, is expected to reproduce or mimic the flow of spacetime as generated by a gravitational field... the analogue models are always "analogies" and never, or hardly ever "identities", meaning we should not confuse the two systems under comparison." - D. Faccio, 2012 [5]. In no way, one should claim that the analogue models are completely equivalent to GR, but it should at least capture and accurately reproduce some aspects (kinematical) of curved spacetime QFT.

typically get governed by Einstein's field equations [with or without a gravitational source term ($T_{\mu\nu}$) which is actually the stress-energy tensor]. On the other hand, the 'kinematics' of a gravitational black hole refer to particle (photon) trajectories that are solely determined by the system's spacetime metric. The analogue gravity models provide a window to reproduce these features systematically through the emergence of acoustic metric $\mathsf{g}_{\mu\nu}$ that bears a close analogy with the corresponding gravitational spacetime metric. Whether this $\mathsf{g}_{\mu\nu}$ is a result of a gravitational field or a flowing fluid, it becomes irrelevant since the analysis is restricted to the description of wave propagation and evolution within the fluid medium - thus the kinematics are quite identical and the analogy seems legit.

1.1 'Kinematical' effects addressed by Analogue Gravity: *Trans-Planckian problem*

The most important and obvious member of this list is undoubtedly Hawking radiation which is one of the cornerstone results of curved spacetime QFT. The collapse of a distribution of matter ends up forming a black hole that, according to quantum theory, does evaporate and starts emitting particles (in the form of thermal emission) from its event horizon towards the future null infinity. This is called *Hawking effect* as was predicted by Stephen Hawking through his groundbreaking[2] work(s) in 1974 [10, 11].

Hawking radiation is a spontaneous emission of blackbody radiation due to the distortion of the quantum vacuum in the vicinity of a black hole event horizon. Since its very inception in 1974, the phenomenon of Hawking effect had attracted the efforts of innumerable scientists globally, all looking for a deeper understanding of why and how this radiation is emitted and the significance of the implications of this emission with major concerns like "information loss" and even the final fate of our universe.

Now quantum mechanically the black holes are not really 'black' and, to be more precise, the vacuum is not really empty, but they have their quantum fluctuations on top of the classical mean-fields. These vacuum fluctuations give rise to virtual pairs (i.e., particle-antiparticle pairs with very high $+/-$ energies) which get created and annihilated within a very short span of time ($\Delta t \sim \frac{\hbar}{\Delta E}$). Near the horizon, the virtual pairs experience a larger gravitational pull which is strong enough to tear them apart from getting reconciled with their respective partners. It finally ends up emitting particles with positive mass (or, energy) out of the event horizon while the antiparticle with negative mass (or, energy) falls into the black hole because it is

[2]The first hints did come from Bekenstein in 1973 [9] where he had already showed that the gravitational black holes should have non-zero equilibrium temperatures.

classically forbidden w.r.t the external observer sitting at the future null infinity. If one follows the null-geodesic of a Hawking-radiated wave packet at certain frequency at the future infinity backwards in time, it is found to contain arbitrarily high frequency modes very near to the event horizon w.r.t a local freely-falling observer. Hence the associated wavelength of these high energy Hawking-radiated modes goes well below the Planck length $\ell_P = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35}$m. Thus Hawking's original calculation typically relies on the validity of physical laws to arbitrarily high energies, i.e., at sufficiently short length scales ($< \ell_P$) and nobody knows exactly what this sub-Planckian physics is.

1.2 Aims of this Thesis

Thus it is remarkable that Hawking radiation in realistic gravitational systems, such as stellar and galactic black holes, is sufficiently weak so that there's a very tiny hope (practically impossible) of directly observing it in the gravitational context. In order for a gravitational black hole (with mass, say, M_{BH}) to be observed to emit Hawking radiation, it must have a temperature (say, T_{H}) greater than that of the present-day Cosmic Microwave Background (CMB) radiation (say, T_{CMB}). $T_{\text{CMB}} = 2.725$K, while $T_{\text{H}} = \frac{\hbar c^3}{8\pi G M_{\text{BH}} k_{\text{B}}} \approx 6.169 \times 10^{-8}\,\text{K} \times \frac{M_{\text{sun}}}{M_{\text{BH}}}$ in SI units[3]; and hence the direct detection of the gravitational Hawking radiation for a Schwarzschild black hole with a mass equivalent to at least the solar mass becomes far below beyond the limit of the current observational techniques. So its extremely difficult to verify Hawking radiation in nature and the reason being T_{H} is seven orders of magnitude smaller than T_{CMB}. In the words of D. Faccio, "This is somewhat of a set-back for what is without doubt one of the most fascinating and prolific ideas of modern physics. Analogue gravity will probably not be able to claim that this unfortunate glitch will be overcome, simply because analogue gravity experiments do not deal with gravitational black holes, yet it certainly does give us the opportunity to study Hawking radiation from a fresh and rather unexpected perspective" [5].

In response to the growing evidence of lack of deep understanding of the nature and narrow scope of observational verifications, Analogue Gravity is formulated as a new approach. Within the scope of it, in the context of analogue Hawking radiation, there's a window of experimental evidences and the counterpart of ℓ_P is the characteristic length (say, ξ) of the underlying condensed matter system here and this ξ can be tuned to an arbitrarily small value through Feshbach resonance. Hence one can now tackle the sufficiently high frequency excitations (i.e., supposedly Hawking radiated modes).

[3] This is a straightforward result from linear QFT in time-dependent curved spacetime of the black hole (be it gravitational or even acoustic). It is widely accepted by the community that this would occur even for interacting fields as well [12].

1.2.1 "Would a black hole radiate if there were a Planck scale cutoff in its rest frame ?" [12]

In this context of tackling the trans-Planckian issues, T. Jacobson and his co-workers did a number of very significant jobs since 1991 where the issues related to ultrahigh frequencies or ultra-short distances in the standard derivation of Hawking radiation is discussed and criticized [12]. Avoiding the reference to field modes above some short distance cutoff frequency (given by $\omega_c >> M_{\rm BH}^{-1}$, where $M_{\rm BH}$ being the mass of the black hole) in the free-fall frame of the black hole, the standard derivation of black hole radiation is shown in [13]. In this regard, guidance is imported from Unruh's fluid-flow analog of black hole radiation or Hawking effect in the analogue spacetime. If Bose-Einstein condensate (BEC) is chosen to be the underlying condensed matter platform, then the analogue gravity model is usually built by throwing away the "quantum potential" term drastically. The primary reason is this being the Lorentz-breaking term (containing k^4) that breaks the symmetry (Diffeomorphism) towards building up the model of gravity (analogue). This is a term of immense importance in the context of analogue gravity because this gives rise to the dispersion relation which is used to present an alternative scenario of the analogue Hawking radiation bypassing the trans-Planckian problem [14–18].

In 1996, Corley and Jacobson did consider this Lorentz-breaking k^4 term without throwing it away, but they kept it in the dispersion and studied the spectrum of Hawking radiated particles in $(1 + 1)$D black hole geometries for a linear, Hermitian scalar field satisfying a Lorentz noninvariant field equation (for a fundamental momentum scale k_0) [19]. In this model[4], two different types of particle production were found:

- a thermal Hawking flux generated by "mode conversion" at the black hole horizon. For this thermal spectrum, Hawking temperature came out to be $T_H \simeq 0.0008 k_0$
- and a nonthermal spectrum generated by scattering off the background into -ve free-fall (the preferred frame in the black hole spacetime) frequency modes which don't propagate outside the event horizon with +ve Killing frequency. This nonthermal flux was found to dominate at large ω.

In this thesis, we have looked into the effects of the Lorentz-breaking quantum potential term in a different way. Unlike keeping it in the dispersion relation, we have considered the Lorentz-breaking term (quantum potential) in the underlying BEC-dynamics (nonrelativistic) and studied the corresponding effects in the context of analogue spacetime and acoustic Hawking radiation. But the presence of this quantum potential term in the dynamics is somewhat analogous to that of a diffusion term which should spread the small scale modes into the large scale ones.

[4] This model is a variation of Unruh's acoustic black hole analogy.

Effectively, due to the presence of the quantum potential term, massive scalar excitations came into play and Hawking radiation from acoustic black holes practically were killed because of the rapid dissipation of the basic ϑ_1 field ('primary' modes that are supposedly Hawking radiated) from which the massive φ field ('secondary' modes) gains energy. We have systematically addressed these issues related to the presence of the quantum potential term in the dynamics of a nonlocal BEC in the context of acoustic Hawking radiation where this φ field is actually received by the external observer sitting at a distance very close to Killing horizon. For this observer, the supposedly Hawking radiated modes (i.e., ϑ_1 field) practically don't survive.

1.3 A brief history of the Analogue gravity program

It is well known that from an experimentally realizable condensed matter model, through some rigorous mathematical framework, gravity comes out as an 'analogue' phenomenon as seen by the sonic excitations. William Unruh's seminal work [20] practically opened up this field of research, which has been extensively pursued over the last couple of decades and has a host of theoretical proposals around it [4, 12].

The underlying idea can be briefed as the following:

- Acoustic disturbances in a moving Newtonian fluid with a velocity gradient mimics the dynamics of light waves in a curved spacetime in (3 + 1)D.
- The wave (sonic) propagation takes place against a counter-flowing medium, and it is easy to appreciate that if the medium flow-velocity $[\mathbf{v}(\mathbf{r})_{\text{fluid}}]$ is smaller than the local velocity of the phonons, then the sonic wave will definitely be allowed to propagate upstream.
- Due to the velocity gradient, the background fluid flow accelerates up to supersonic speeds, then the upstream sonic disturbances will inevitably be slowed down until it is completely blocked by the counter-propagating medium.
- The moving Newtonian fluid would drag the sound waves along with it and, in the supersonic region (i.e., $|\mathbf{v}(\mathbf{r})_{\text{fluid}}| \geq c_s$ where c_s being the local speed sound inside the fluid), the phonons would never be able to fight their way back upstream.
- Thus these phonons will obviously be trapped and they would experience a surface of no-return inside the fluid medium and hence this generates a singularity which clearly bears a very close analogy with the gravitational event horizon. "The wave-blocking point lies at the transition from sub to super-sonic flow and, as such, represents the analogue of a gravitational wave-blocking horizon" [5]. This indeed paves a way for the scope of observing Hawking radiation in a laboratory set up.

This is just a schematic description; the underlying mathematics opens up a far more significant and unexpected aspect: the sound wave propagation (phonon trajectory) is fully described by a spacetime metric (called acoustic metric in $(3+1)$D curved spacetime which is not a solution of Einstein equations) that is distorted by the flowing medium close to the velocity gradient. This distortion can be regarded as something identical or something mimicking the similar scenario for the usual spacetime metric close to the event horizon of a gravitational black hole. Now going one step ahead - Hawking radiation, in the form of sound waves, is predicted to be emitted from the "sonic" horizon or acoustic analog of the gravitational event horizon. Most interestingly, the acoustic metric is found to be a function of the background velocity of the moving fluid as well as the local speed of sound across the horizon and therefore can in principle be engineered and optimized in the laboratory.

There is an expanding community that kind of thrives on the search of these new settings of analogue spacetime in which a flowing Newtonian medium of some sort can be generated within the scope of a laboratory set up and controlled in such a way as to recreate curved spacetime metrics (acoustic metric in $(3+1)$D) with various applications [5].

1.4 Theorem: Analogue Gravity

Statement[5]: *Given a barotropic, inviscid fluid with irrotational[6] flow (velocity gradient), the propagation of sound waves through the fluid medium is governed by the e.o.m for the linear perturbation of the velocity potential which is nothing but the d'Alembert e.o.m for a massless minimally coupled free scalar field on a* $(3+1)$*D curved spacetime (Lorentzian geomtery),*

$$\Box\vartheta_1(t,\mathbf{x}) \equiv \frac{1}{\sqrt{|\mathsf{g}|}}\partial_\mu\left(\sqrt{|\mathsf{g}|}\mathsf{g}^{\mu\nu}\partial_\nu\vartheta_1(t,\mathbf{x})\right) = 0, \tag{1.1}$$

where the analogue effective metric or acoustic metric (switching from $(t,\mathbf{x}) \to x$ in the argument) is

$$[\mathsf{g}_{\mu\nu}(x)] = \Omega\begin{pmatrix} -\left(c_s^2 - \mathbf{v}^2(x)\right) & \vdots & -\mathbf{v}^{\mathrm{T}}(x) \\ \cdots\cdots & . & \cdots\cdots\cdots \\ -\mathbf{v}(x) & \vdots & \mathcal{I}_{3\times 3} \end{pmatrix}, \tag{1.2}$$

where, Ω is some conformal factor, mentioned later.

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[5]Refer to p.8 of [4].

[6]The fluid is considered to be vorticity-free, i.e., locally irrotational.

Here $\mathbf{v}(x)$ is the background 3D velocity of the Newtonian fluid and c_s is the local speed of sound within the fluid medium. These quantities would be discussed later in great detail. $\mathcal{I}_{3\times 3}$ is nothing but the identity matrix.

It is important to note that, in general, for a moving non-homogeneous Newtonian fluid, the acoustic Riemann curvature tensor associated with this $\mathsf{g}_{\mu\nu}(x)$ is non-vanishing which pretty much makes the background to be curved as perceived by the sonic excitations.

Proof: The following proof very closely follows the discussions in [21] as well as Section **2.3** in Chapter 2 of [4].

Let's consider a classical fluid of non-relativistic particles with irrotational flow described by local density ρ, velocity $\mathbf{v}$ and pressure p in an infinite space[7] The fundamental equations of fluid dynamics are 'continuity equation' and 'Euler's equation'. The first one suggests the conservation of number of fluid particles, i.e.,

$$\partial_t \rho + \nabla.(\rho \mathbf{v}) = 0. \tag{1.3}$$

Due to the fluid being assumed to be barotropic, p is only a function of ρ (with $\frac{dp}{d\rho} > 0$). Let's consider that all the forces acting on the fluid are conservative and $\mathcal{V}$ be their sum of potentials (scalar), except due to pressure, therefore the total force acting on the fluid is

$$\mathbf{F} = -\nabla p - \nabla \mathcal{V}. \tag{1.4}$$

Thus Newton's second law ($force = mass \times acceleration$) in classical mechanics applied to an infinitesimal volume of fluid basically gives the Euler's equation:

$$-\nabla p - \nabla \mathcal{V} = \rho \frac{d\mathbf{v}}{dt} = \rho \Big[\partial_t \mathbf{v} + (\mathbf{v}.\nabla)\mathbf{v}\Big]. \tag{1.5}$$

With the help of $\mathbf{v} \times (\nabla \times \mathbf{v}) = \nabla \left(\frac{1}{2}v^2\right) - (\mathbf{v}.\nabla)\mathbf{v}$, after a slight manipulation, the above Eq.(1.5) is rewritten as

$$\partial_t \mathbf{v} = \mathbf{v} \times (\nabla \times \mathbf{v}) - \nabla \left(\frac{1}{2}v^2\right) - \frac{1}{\rho}\nabla p - \nabla \mathcal{V}. \tag{1.6}$$

Since the fluid flow is irrotational, the velocity potential is introduced as

$$\mathbf{v} = -\nabla \vartheta, \qquad \text{at least locally.} \tag{1.7}$$

[7] Similar kind of model with boundaries would reduce the effective sound velocity inside the fluid medium [22].

Hence Eq.(1.6) becomes

$$-\partial_t(\boldsymbol{\nabla}\vartheta) + \frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{\nabla}\vartheta)^2 = -\frac{1}{\rho}\boldsymbol{\nabla}p - \boldsymbol{\nabla}\mathcal{V}. \tag{1.8}$$

Now the fluid being barotropic, one can define the specific enthalpy to be

$$h : p \mapsto \int_0^p \frac{d\tilde{p}}{\rho} :$$
$$\therefore \quad \boldsymbol{\nabla}h = \frac{\boldsymbol{\nabla}p}{\rho}. \tag{1.9}$$

Thus Eq.(1.8) can be rewritten as

$$\boldsymbol{\nabla}\left(-\partial_t\vartheta + \frac{1}{2}(\boldsymbol{\nabla}\vartheta)^2 + h(p) + \mathcal{V}\right) = 0, \tag{1.10}$$

$\Rightarrow \quad -\partial_t\vartheta+\frac{1}{2}(\boldsymbol{\nabla}\vartheta)^2+h(p)+\mathcal{V} = c_1(t)$ [where, c_1 is a constant which is function of time only].

For simplicity, one can choose $c_1 = 0$, and obtain

$$-\partial_t\vartheta + \frac{1}{2}(\boldsymbol{\nabla}\vartheta)^2 + h(p) + \mathcal{V} = 0, \quad \text{which is a version of Bernoulli's equation.} \tag{1.11}$$

Let's assume we have a solution for the background (ϑ_0, ρ_0) around which the linearized perturbations are considered as

$$\rho = \rho_0 + \epsilon\,\rho_1 + O(\epsilon^2), \tag{1.12a}$$
$$p = p_0 + \epsilon\,p_1 + O(\epsilon^2), \tag{1.12b}$$
$$\vartheta = \vartheta_0 + \epsilon\,\vartheta_1 + O(\epsilon^2). \tag{1.12c}$$

Linearizing the continuity equation [i.e., Eq.(1.3)], it gives a pair of equations:

$$\partial_t\rho_0 + \boldsymbol{\nabla}.(\rho_0\mathbf{v}_0) = 0, \tag{1.13a}$$
$$\partial_t\rho_1 + \boldsymbol{\nabla}.(\rho_1\mathbf{v}_0 + \rho_0\mathbf{v}_1) = 0, \tag{1.13b}$$

where $\mathbf{v}_0 = -\boldsymbol{\nabla}\vartheta_0$ is the velocity field of the unperturbed solution, while $\mathbf{v}_1 = -\boldsymbol{\nabla}\vartheta_1$ being that of the linearized perturbations. Since the fluid is barotropic, it implies that

$$h(p) = h(\,p_0 + \epsilon p_1 + O(\epsilon^2)\,) = h_0 + \epsilon\frac{p_1}{\rho_0} + O(\epsilon^2). \tag{1.14}$$

Now the Euler equation [i.e., Eq.(1.11)] gets linearized to produce a pair of equations:

$$-\partial_t \vartheta_0 + \tfrac{1}{2}(\boldsymbol{\nabla}\vartheta_0)^2 + h_0 + \mathcal{V} = 0, \tag{1.15a}$$

$$-\partial_t \vartheta_1 - \mathbf{v}_0.\boldsymbol{\nabla}\vartheta_1 + \tfrac{p_1}{\rho_0} = 0, \tag{1.15b}$$

$$\Rightarrow \qquad p_1 = \rho_0(\partial_t\vartheta_1 + \mathbf{v}_0.\boldsymbol{\nabla}\vartheta_1). \tag{1.16}$$

Due to the fluid being barotropic, ρ is a function of p only, and hence the linearized density is given by

$$\rho_1 = \frac{\partial \rho}{\partial p} p_1 = \frac{\partial \rho}{\partial p}\rho_0(\partial_t\vartheta_1 + \mathbf{v}_0.\boldsymbol{\nabla}\vartheta_1). \tag{1.17}$$

On substitution of this above expression of ρ_1 into Eq.(1.13b), the final form (up to an overall sign) of the wave equation is given by

$$-\partial_t\left[\frac{\partial \rho}{\partial p}\rho_0(\partial_t\vartheta_1 + \mathbf{v}_0.\boldsymbol{\nabla}\vartheta_1)\right] + \boldsymbol{\nabla}.\left[\rho_0\boldsymbol{\nabla}\vartheta_1 - \mathbf{v}_0\frac{\partial \rho}{\partial p}\rho_0(\partial_t\vartheta_1 + \mathbf{v}_0.\boldsymbol{\nabla}\vartheta_1)\right] = 0. \tag{1.18}$$

To simplify things even further, one can introduce the local speed of the sound (c_s) in the fluid medium. This c_s can be obtained by setting $\mathbf{v}_0 = 0$ and considering ρ_0 to be effectively constant, i.e., $\boldsymbol{\nabla}\rho_0 = 0 = \partial_t\rho_0$. Hence from Eq.(1.18), one easily obtains the standard form of wave equation for the ϑ_1 field:

$$-\left(\frac{\partial \rho}{\partial p}\right)\partial_{tt}\vartheta_1 + \boldsymbol{\nabla}^2\vartheta_1 = 0,$$
$$\Rightarrow \qquad \frac{1}{c_s^2} = \frac{\partial \rho}{\partial p}. \tag{1.19}$$

Eq.(1.18) describes the propagation of sound waves (or linearized perturbations ϑ_1 in other words) through the fluid medium and, by introducing $(d+1)$D spacetime coordinates[8] in order to bring one-to-one correspondence with gravity, one considers $(t, x^i) = x^\mu \equiv x$, $\forall\, \mu, \nu, \ldots = 0(1)d$ and $\forall\, i, j, \ldots = 1(1)d$, this dynamics can be written in a compact form as

$$\partial_\mu\, f^{\mu\nu}(x)\, \partial_\nu \vartheta_1(x) = 0, \tag{1.20}$$

[8]The picture in $(d+1)$D is the most general one, but usually to keep things physically relevant, the formulation is done on $(3+1)$D spacetime where $f^{\mu\nu}(x)$ in Eq.(1.20) is obviously constructed as a 4×4 symmetric matrix.

where $[f^{\mu\nu}(x)]$ is constructed as a $(d+1)\times(d+1)$ symmetric matrix, given by,

$$[f^{\mu\nu}(x)] = \frac{\rho_0}{c_s^2}\begin{pmatrix} -1 & \vdots & -v_0^j(x) \\ \cdots\cdots & . & \cdots\cdots\cdots \\ -v_0^i(x) & \vdots & \left(c_s^2\delta^{ij} - v_0^i(x)\, v_0^j(x)\right) \end{pmatrix}, \quad \text{where } \delta \text{ denotes Kronecker delta.} \tag{1.21}$$

A straightforward calculation shows

$$f = \det[f^{\mu\nu}(x)] = \left(\frac{\rho_0}{c_s^2}\right)^{d+1} .(c_s^2)^d.(-1)^d = \boxed{(-1)^d\,\frac{\rho_0^{d+1}}{c_s^2}}. \tag{1.22}$$

In order to identify the corresponding covariant structure to make a connection with Gravity (and curved spacetime QFT), it is extremely essential to figure out the effective metric of the spacetime (i.e., analogue spacetime). Say, $\mathsf{g}_{\mu\nu}(x)$ be the effective metric or acoustic metric for which

$$f^{\mu\nu}(x) = \sqrt{|\mathsf{g}|}\,\mathsf{g}^{\mu\nu}(x) \quad \text{holds true,} \tag{1.23}$$

where $\mathsf{g} = \det[\mathsf{g}_{\mu\nu}(x)]$.

By this above construction, one readily finds the determinant of the effective metric as

$$f = \det[f^{\mu\nu}(x)] = \det\left[\sqrt{|\mathsf{g}|}\,\mathsf{g}^{\mu\nu}(x)\right] = (\sqrt{|\mathsf{g}|})^{d+1}\det[\mathsf{g}^{\mu\nu}(x)] = \mathsf{g}^{\frac{d+1}{2}}\mathsf{g}^{-1} = \mathsf{g}^{\frac{d-1}{2}},$$

$$\Rightarrow \quad \mathsf{g} = f^{\frac{2}{d-1}} = \left[(-1)^d\,\frac{\rho_0^{d+1}}{c_s^2}\right]^{\frac{2}{d-1}} \quad \text{[using Eq.(1.22)].} \tag{1.24}$$

Thus it is quite evident from Eq.(1.24) that the whole prescription breaks down for $(1+1)$D spacetime, but holds good for any spacetime with $d \geq 2$.

Thus in any $(d+1)$D Lorentzian (or, pseudo-Riemannian) manifold, the d'Alembertian equation of motion is evidently identified through the above wave equation for ϑ_1 in Eq.(1.20) as

$$\boxed{\Box\vartheta_1(t,\mathbf{x}) \equiv \boldsymbol{\nabla}_\mu\boldsymbol{\nabla}^\mu\vartheta_1(x) = \frac{1}{\sqrt{|\mathsf{g}|}}\partial_\mu\left(\sqrt{|\mathsf{g}|}\,\mathsf{g}^{\mu\nu}(x)\,\partial_\nu\vartheta_1(x)\right) = 0}, \tag{1.25}$$

where $\boldsymbol{\nabla}_\mu$ is the usual covariant derivative for the effective curved spacetime and the corresponding $[\mathsf{g}^{\mu\nu}(x)]_{(d+1)\times(d+1)}$ is given by

$$\mathsf{g}^{\mu\nu}(x) = \frac{1}{\sqrt{|\mathsf{g}|}} f^{\mu\nu}(x) = \left(\frac{c_s^2}{\rho_0^{d+1}}\right)^{\frac{1}{d-1}} f^{\mu\nu}(x)\ ; \quad \text{[using Eq.(1.24)]} \tag{1.26a}$$

and $[f^{\mu\nu}(x)]_{(d+1)\times(d+1)}$ can be picked from Eq.(1.21), thus we have $\forall\, d \geq 1$,

$$\Rightarrow \quad [\mathsf{g}^{\mu\nu}(x)]_{(d+1)\times(d+1)} = \left(\frac{c_s^2}{\rho_0^{d+1}}\right)^{\frac{1}{d-1}} \cdot \frac{\rho_0}{c_s^2} \begin{pmatrix} -1 & \vdots & -v_0^j(x) \\ \cdots\cdots & . & \cdots\cdots\cdots \\ -v_0^i(x) & \vdots & \left(c_s^2\delta^{ij} - v_0^i(x)\, v_0^j(x)\right) \end{pmatrix}. \tag{1.26b}$$

Just taking an inverse of the above Eq.(1.26b) would give us the required effective metric, for simplicity on $(3+1)$D analogue spacetime, as

$$[\mathsf{g}_{\mu\nu}(x)]_{4\times4} = \frac{\rho_0}{c_s} \begin{pmatrix} -\left(c_s^2 - \mathbf{v}_0^2(x)\right) & \vdots & -\mathbf{v}_0^{\mathrm{T}}(x) \\ \cdots\cdots & . & \cdots\cdots\cdots \\ -\mathbf{v}_0(x) & \vdots & \mathcal{I}_{3\times3} \end{pmatrix}. \tag{1.27}$$

It is redundant to find the line-element or 'acoustic interval' for the effective analogue spacetime on $(3+1)$D as

$$ds^2 = \mathsf{g}_{\mu\nu}(x)\, dx^\mu dx^\nu = \frac{\rho_0}{c_s}\Big[- c_s^2 dt^2 + \left(dx^i - v_0^i(x)dt\right)\delta_{ij}\left(dx^j - v_0^j(x)dt\right)\Big]. \tag{1.28}$$

Hence the theorem is proved.

1.5 Bose-Einstein condensate: an ideal *analogue gravity* tool

Bose-Einstein condensate (BEC) is considered to be one of the prominent candidates of *analogue gravity* systems [23] for various reasons[9]. One can probe some aspects of curved spacetime Quantum Field Theory (QFT) via the analogue models due to amenability of accurate experimental control and observational verification - a quantum system characterized by

- very cold temperature (~ 100 nK),

[9] Apart from BEC, some other condensed matter systems, such as superfluid helium [24, 25], superconductors [26, 27], polariton superfluid [28] and degenerate Fermi gas [29], have also been used as the tools to probe Analogue gravity.

- local speed of sound being much smaller and easier to be achieved in the laboratory (i.e., $c_s << 2.997 \times 10^8\ \text{ms}^{-1}$),
- and high degree of quantum coherence

does offer the best test-field. Thus BEC, which is a superfluid quantum phase of matter, happens to be the best-fitted tool and one of the most prominent candidates in setting up an Analogue Gravity model, especially in order to simulate quantum blackhole geometries and Hawking radiation. The small amplitude collective modes (phonons) in a BEC can see a Lorentzian metric depending on a suitable background Newtonian bulk flow of the condensate. Under suitable conditions, such a barotropic, inviscid, irrotational background flow field can generate an "ergo-region" for the phonons which cannot overflow the boundary where the bulk velocity exceeds the local velocity of sound. As has been already mentioned, it was Unruh's 1981 paper [20] which practically started this field of analogue gravity by proposing the possibility of observing Hawking radiation within the scope of a laboratory set up. Following Unruh's seminal paper, a lot of attention had been attracted by the field not only in the connection with the classic trans-Planckian problem [12] but also for a wide spectrum of issues elegantly compiled in the review by Barceló, Liberati and Visser [4]. One of the prime targets was to observe *analogue* Hawking radiation [30–32] as a thermal bath of phonons with the temperature proportional to the *surface gravity* [4, 21]. In this context, Parentani and co-workers have already proposed some novel ideas based on density correlations, studying the hydrodynamics over several length scales and even surface-gravity-independent-temperature etc., [14–17]. In order to experimentally detect analogue Hawking radiation, a stack of recent works are obviously worth mentioning here [33–41].

In a BEC, the small amplitude collective excitations [42] of the uniform density moving phase (to be precise, the first order phase-fluctuation-field) obeys the quantum hydrodynamics which, ignoring the quantum potential term (refer to Section-4.2.1 of [4]), can be cast into the d'Alembertian equation of motion of a massless minimally coupled free scalar field on a $(3 + 1)$D Lorentzian manifold with an effective metric to be regarded as the acoustic metric of the curved background[10].

1.5.1 Numerous published works in the context of nonlocal interactions in a BEC and Analogue Gravity

Varied nonlocal forms of the interaction potential have been considered to capture new solutions not admitted by local Gross-Pitaevskii (GP) theory [43, 44]. There is a symmetry based recent

[10] Refer to Eq.(254) of [4].

analysis of nonlocality in GP equation in [45]. With the help of nonlocal nonlinearities, roton minimum in the dispersion relation has been predicted which is a typical characteristic of superfluids [46–48]. An interesting work [49] considers non-locality in the interactions in the presence of a particular periodic global potential to show that, although the GP free energy and the corresponding solutions show asymptotic correspondence as one moves from the nonlocal to the local limit, but, the asymptotic correspondence in the stability of the solutions is not present [50]. In [51], the effect of quantum fluctuations on the mean field model is treated semi-classically and the work by Pérez-García [52] in 2000 presents an interesting analysis considering nonlocal interactions preventing collapse of the condensate. The work of Andreev *et al.* [53] on degenerate boson-fermion system derives a similar dispersion relation based on a third order correction as is there in this paper but focuses on different issues [54–56]. Rosanov *et al* have done an analysis on internal oscillations of solitons with nonlocal nonlinearity on a similar system that considers a correction term similar to ours with an attractive interaction at large distances [57]. There is a nice review article by Yukalov [58] addressing theoretical challenges of BEC theory covering nonlocal and disordered potentials.

Some worthwhile efforts have been taken in order to regularize the dynamics taking the quantum potential term into account [59]. In 2005, Visser *et al.* had shown the emergence of a massive Klein-Gordon (KG) equation considering a two-component-BEC where a laser induced transition between the two components was exploited [60]. Liberati *et al.* proposed a weak $U(1)$ symmetry breaking of the analogue BEC model by the introduction of an extra quadratic term in the Hamiltonian to make the scalar field massive [61]. Considering the flow in a Laval nozzle, Cuyubamba has shown the emergence of a massive scalar field in the context of analogue gravity arguing for the possibility of observation of quasi-normal ringing of the massive scalar field within the laboratory setup [62]. In the context of *trans-Planckian* backaction issues on low-energy predictions in Analogue Gravity, Fischer *et al.* has recently done a remarkable work [63].

1.6 A brief outline

Before going into the chapters, the content of the thesis may be divided into the following main results as summarized below:

1. we show that the healing length (ξ) of the BEC which serves as the analog Planck scale for the analogue gravity system can be tuned to very small values to work within this linear dispersion regime where the main results of Hawking radiation appears. [Paper: *Supratik Sarkar and A. Bhattacharyay, Journal of Physics A: Mathematical and Theoretical 47, 092002 (2014)* [1]]

2. We use this tunability of analogue Planck scale in order to develop a set of multiple-scale-modes where the physics below ξ is essentially considered and there's an emergence of the massive scalar excitations for the large wavelength phonon modes. The analysis is done on a $(3 + 1)$D flat spacetime. [Paper: *Supratik Sarkar and Arijit Bhattacharyay, Phys. Rev. D 93, 024050 (2016)* [2]]

3. We find that the consequences of this UV-IR coupling can be captured through the generation of large wavelength IR modes and their energetic dependence on the short wavelength UV modes (supposedly Hawking-radiated) when the analysis is extended to a $(3 + 1)$D curved spacetime for a canonical acoustic black hole. [Paper: *Supratik Sarkar and A. Bhattacharyay, Phys. Rev. D 96, 064027 (2017)* [3]]

   - We further show that the 'secondary' IR modes in their growth rate will retain the lost information regarding the 'primary' Hawking-radiated modes and there does exist a possibility to retrieve the "information loss" within the scope of analogue Hawking radiation.
   - Our way of looking at the Lorentz-breaking trans-Planckian physics is completely new as compared to existing standard literature in this specific field of research where the trans-Planckian issues are dealt through the presence of dispersion and its effect on mode conversion.

Chapter 2

Elementary theory of dilute Bose-Einstein condensates (BEC)

The basic idea of Bose-Einstein condensation (BEC) takes us back to the year of 1925 when Albert Einstein [64], on the basis of a scientific paper (1924) by an Indian physicist S. N. Bose, predicted the occurrence of a phase transition in a gas of non-interacting particles (bosons) with integral spin.

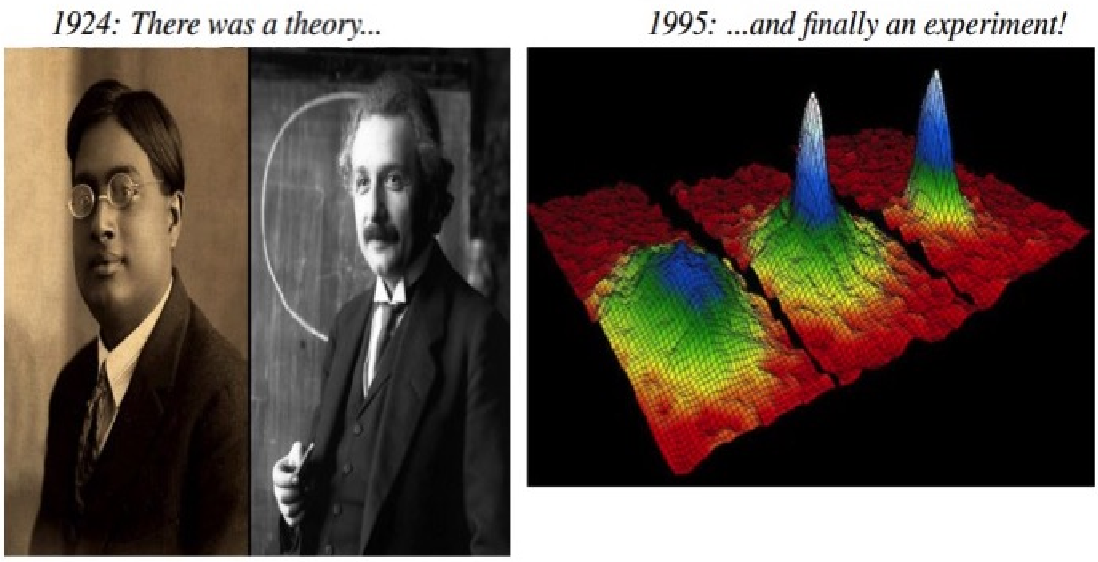

Figure 2.1: Source: A presentation by R. Roth, GSI Theory group, Germany, (2001). Click here to find the presentation on web.

This phase transition is associated with the condensation of atoms with integral spin (i.e., bosons) in the state of lowest energy (ground state) and is the direct upshot of the quantum statistical effects. The experimental realization of BEC was first achieved in 1995 in dilute atomic gases [65] and since then, the study of BEC as the quantum gas with high degeneracy has kept attracting the interest of scientists from different areas on a massive scale around the globe [42]. The year 1995 definitely marked the beginning of a very rapid development in the study of condensed matter physics, specially theory of *ultracold* atoms.

2.1 The Ideal Bose gas

The statistics of non-interacting bosons is described in most of the textbooks and this happens to be the simplest demonstration of the possibility of realization of BEC and predicts a few important properties of the actual system quite correctly. The equilibrium properties of the systems are usually calculated in this treatment in a semi-classical approximation.

2.1.1 Grand-canonical partition function

We will be discussing the quantum statistical description of ideal Bose gas in the context of a grand canonical ensemble, say, there exists a configuration of N' particles (bosons) in a state k of energy E_k. In this section, I will just follow the formalism of the text book by Pitaevskii and Stringari [42].

Thermodynamics of this given system (at, say, some temperature T) is often derived from its partition function,

$$Z(\beta, \mu) = \sum_{N'=0}^{\infty} \sum_{k} e^{\beta(\mu N' - E_k)}, \qquad \text{where, } \beta = \frac{1}{k_{\mathrm{B}} T}. \tag{2.1}$$

Evidently k_{B} is the Boltzmann constant and μ is chemical potential of the reservoir with which the system under consideration stays in a thermal and chemical equilibrium, defined as,

$$\mu = E(N') - E(N' - 1) \sim \frac{\partial E}{\partial N'} \qquad \text{(at constant entropy)}. \tag{2.2}$$

It is to be noted that the natural variables in the grand canonical ensemble are T (or, β) and μ. Now probability of realizing such a configuration (N', k, E_k) is given by

$$P_{N'}(E_k) = \frac{e^{\beta(\mu N' - E_k)}}{Z(\beta, \mu)}. \tag{2.3}$$

The grand canonical potential Ω is defined as

$$-k_{\mathrm{B}} T \ln Z(\beta, \mu) \;=\; \Omega(T, \mu) \;=\; E - TS - \mu N, \tag{2.4}$$

where,

$$\text{entropy of the system: } S = -\left(\frac{\partial\Omega}{\partial T}\right)_\mu, \tag{2.5a}$$

$$\text{total number of particles: } \sum_{N'=0}^{\infty}\sum_k N' P_{N'}(E_k) = N = -\left(\frac{\partial\Omega}{\partial\mu}\right)_T, \tag{2.5b}$$

$$\text{total energy: } \sum_{N'=0}^{\infty}\sum_k E_k P_{N'}(E_k) = E = \Omega(T,\mu) - T\left(\frac{\partial\Omega}{\partial T}\right)_\mu - \mu\left(\frac{\partial\Omega}{\partial\mu}\right)_T. \tag{2.5c}$$

Considering the system to be described by independent particle Hamiltonian, the eigenstates k are defined through the single-particle states by specifying the set $\{n_i\}$ where n_i is the microscopic occupancy of the i^{th} single-particle state with eigen-energy ε_i (determined by the solution of the single-particle Schrödinger equation, see Eq.(3.9) of [42]). Thus in this case of total N' number bosons in an eigenstate k with energy E_k,

$$N' = \sum_i n_i \tag{2.6a}$$

$$E_k = \sum_i n_i \varepsilon_i. \tag{2.6b}$$

Now in order to work out the grand canonical partition function $Z(\beta,\mu)$ in details, from Eq.(2.1), we identify $Z_{N',k}$ as the following

$$Z(\beta,\mu) = \sum_{N'=0}^{\infty}\sum_k Z_{N',k} \tag{2.7}$$

and using Eq.(2.6), it is evaluated as

$$\begin{aligned} Z_{N',k} &= e^{\beta\left(\mu\sum_i n_i - \sum_i n_i\varepsilon_i\right)} \\ &= e^{\beta\left(\mu n_0 - n_0\varepsilon_0\right)} \times e^{\beta\left(\mu n_1 - n_1\varepsilon_1\right)} \times \ldots \\ &= \left[e^{\beta\left(\mu-\varepsilon_0\right)}\right]^{n_0} \times \left[e^{\beta\left(\mu-\varepsilon_1\right)}\right]^{n_1} \times \ldots \quad . \end{aligned} \tag{2.8}$$

Therefore from Eq.(2.7), in order to find $Z(\beta,\mu)$, one can safely replace the summations over particle number N' and state k by the set of all single-particle states $\{n_i\}$,

$$
\begin{aligned}
Z(\beta,\mu) &= \sum_{\{n_0,n_1,\dots\}} \left[e^{\beta(\mu-\varepsilon_0)}\right]^{n_0} \times \left[e^{\beta(\mu-\varepsilon_1)}\right]^{n_1} \times \dots \\
&= \sum_{n_0} \left[e^{\beta(\mu-\varepsilon_0)}\right]^{n_0} \times \sum_{n_1} \left[e^{\beta(\mu-\varepsilon_1)}\right]^{n_1} \times \dots \\
&= \prod_i \underbrace{\sum_{n_i} \left[e^{\beta(\mu-\varepsilon_i)}\right]^{n_i}}. \qquad (2.9)
\end{aligned}
$$

Now for Bose-Einstein statistics of an ideal gas sample, for particles with integral spin, the microscopic occupation number has no restriction and can accommodate as many bosons (indistinguishable particles) as it may for each single-particle eigenstate, i.e., $n_i = 0, 1, 2, \dots, \infty$ $\forall i$ and hence the underbracketted part in the above Eq.(2.9) constitutes an infinite GP series with $e^{\beta(\mu-\varepsilon_i)} < 1$ $\forall i$ on physical ground.

Thus from Eq.(2.9),

$$
\begin{aligned}
Z(\beta,\mu) &= \prod_i \sum_{n_i=0,1,\dots}^{\infty} \left[e^{\beta(\mu-\varepsilon_i)}\right]^{n_i} \\
&= \prod_i \frac{1}{1-e^{\beta(\mu-\varepsilon_i)}} \qquad (2.10)
\end{aligned}
$$

2.1.2 Bose-distribution function

Combining this above Eq.(2.10) with Eq.(2.4),

$$
\begin{aligned}
\Omega &= -k_{\mathrm{B}}T \ln \prod_i \frac{1}{1-e^{\beta(\mu-\varepsilon_i)}} \\
&= k_{\mathrm{B}}T \sum_i \ln\left(1-e^{\beta(\mu-\varepsilon_i)}\right) \qquad (2.11)
\end{aligned}
$$

Using the above Eq.(2.5b), the total number of particles at some temperature T is given by,

$$
\begin{aligned}
N &= \sum_i \frac{1}{e^{\left(\beta(\varepsilon_i-\mu)\right)}-1} = \sum_i \bar{n}_i, \\
\text{i.e.,}\quad & \boxed{\bar{n}_i(T,\mu) = \frac{1}{e^{\left(\beta(\varepsilon_i-\mu)\right)}-1}}, \qquad (2.12)
\end{aligned}
$$

where $\bar{n}_i$ is obviously the average occupation number for each single-particle state.

Similarly, from Eqs.(1.5a) and (1.5c), the entropy and total energy of the system are given by, respectively,

$$S = k_B \sum_i \left[\frac{\beta(\varepsilon_i - \mu)}{e^{\left(\beta(\varepsilon_i - \mu)\right)} - 1} - \ln\left(1 - e^{\beta(\mu - \varepsilon_i)}\right) \right], \tag{2.13}$$

$$E = \sum_i \frac{\varepsilon_i}{e^{\left(\beta(\varepsilon_i - \mu)\right)} - 1}. \tag{2.14}$$

Now Eq.(2.12) does provide the important physical constraint on the chemical potential of the ideal Bose gas, as

$$\mu < \varepsilon_0, \tag{2.15}$$

where ε_0 is the lowest ($i = 0$ for BEC ground state) energy eigenvalue of the single-particle Hamiltonian. The above inequality has to hold good otherwise it will result in a negative value of microscopic occupation number of states with corresponding energy being smaller than the chemical potential μ. For $i = 0$ state, the occupation number of the lowest energy eigenstate is

$$N_0(T, \mu) \equiv \bar{n}_0(T, \mu) = \frac{1}{e^{\left(\beta(\varepsilon_0 - \mu)\right)} - 1} \tag{2.16}$$

which becomes increasingly large in the limit $\mu \to \varepsilon_0$ and is in fact divergent exactly at $\mu = \varepsilon_0$. This is actually at the heart of the mechanism of BEC where one can write the total number of particles by segregating the ground state occupancy from the rest of the excited states in the following manner

$$N = N_0 + N_T, \quad \text{where, } N_T(T, \mu) = \sum_{i \neq 0} \bar{n}_i(T, \mu). \tag{2.17}$$

Here $N_T(T, \mu)$ is nothing but the number of bosons that are out of the condensate, i.e., residing at higher order excited eigenstates. It is also referred to as the thermal component of the Bose gas.

2.1.3 Transition temperature T_c and condensate fraction N_0

Let's notice the following features of an ideal Bose gas.

- For a fixed value of T, the function $N_T(T, \mu)$ has a smooth behavior as a function of the chemical potential μ and reaches its maximum value when μ attains its maxima which is

ε_0, i.e.,

$$\lim_{\mu \to \varepsilon_0} N_T(T, \mu) = \left(\text{maximum value at } T\right) \equiv N_c(T) = \sum_{i \neq 0} \frac{1}{e^{\left(\beta(\varepsilon_i - \varepsilon_0)\right)} - 1}. \tag{2.18}$$

$N_c(T)$ is an increasing function of T. In this prescription, T_c is introduced and is defined to be the highest temperature over which the condensate of bosons doesn't form. Thus at $T = T_c$, N_c reaches its maxima which is evidently N as far as the formation of the condensate is restored, i.e.,

$$N_T(T = T_c, \mu = \varepsilon_0) \equiv N_c(T = T_c) = N. \tag{2.19}$$

At this limiting condition, the contribution of the condensate is negligible, i.e., $N_0 \to 0$ (according to Eq.(2.17)).

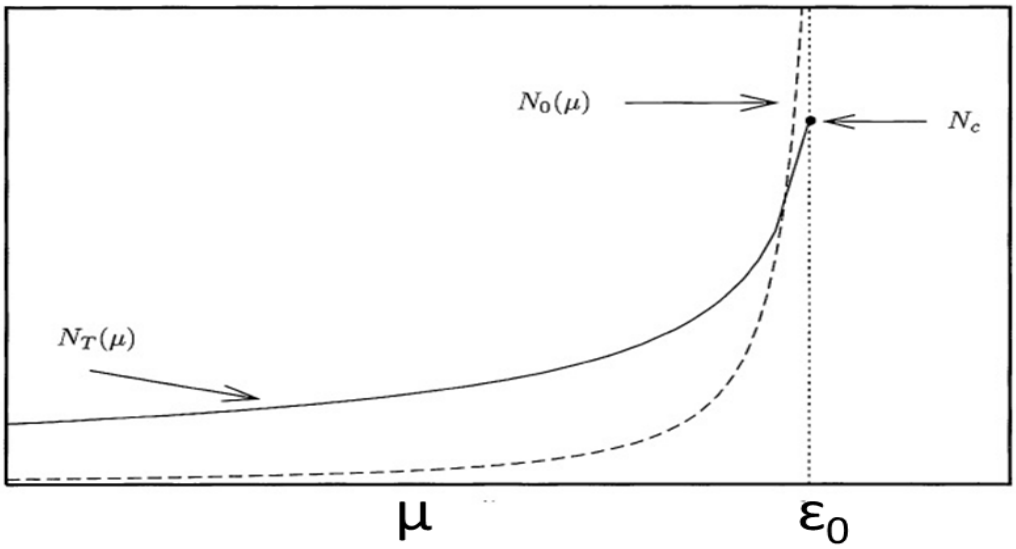

Figure 2.2: Ideal gas model. The number of bosons which are out of the condensate (N_T, solid line) and in the condensate (N_0, dashed line) as a function of chemical potential μ and temperature T. The actual value of μ is fixed by the normalization condition given by Eq.(2.17). If $N > N_c$, the solution is given by $\mu \sim \varepsilon_0$ and the system forms a BEC ($N_0/N \neq 0$) in the thermodynamic limit. [Source: Fig.3.1 of [42]]

- The behavior of N_0 is rather different. In the $\lim_{\mu \to \varepsilon_0}$, we have a zero in the denominator and N_0 diverges according to Eq.(2.16). In this case, the contribution from the non-condensed segment of the system is negligibly small.

- **Semi-classical distribution function**: Quantum mechanically, the density profile of a non-interacting ideal BEC is given by Eq.(2.42) of [66]. From the semi-classical distribution function, the density profile and the velocity distribution of the particles are obtained in order to determine the shape of an anisotropic cloud followed by free expansion. Various thermodynamic properties of Bose gases can be then calculated as functions of the temperature (T).

- Ideal BEC has infinite compressibility in the absence of interaction between particles.

The BEC is formed experimentally within an ansiotropic harmonic trap[11] which is specified by its respective frequencies, say, in 3D ($\omega_x, \omega_y, \omega_z$), then due to zero-point energies, there may be some corrections to the measure of T_c. But that can again be neglected if the number of particles in the system is sufficiently large (i.e., $N \to \infty$).

One may obtain the expression[12] for the transition temperature for an ideal Bose gas of N bosons in a 3D harmonic trap ($\omega_x, \omega_y, \omega_z$) as

$$T_c \approx 0.94 \frac{\hbar}{k_{\rm B}} \bar{\omega} N^{1/3}, \qquad \text{where,} \quad \bar{\omega} = \left(\omega_x \omega_y \omega_z\right)^{1/3}. \tag{2.20}$$

The condensate fraction is defined to be the number of bosons in the condensate (or, the occupancy of the condensate state). As a function of temperature T, it is given by

$$N_0(T) = N \left[1 - \left(\frac{T}{T_c}\right)^{\varkappa}\right], \tag{2.21}$$

where $\varkappa$ is a parameter. For an ideal Bose gas inside a 3D box, $\varkappa = 3/2$, and in case of an isotropic/anisotropic 3D harmonic trapping potential, $\varkappa = 3$.

The value of T_c predicted by the ideal BEC has provided a window to correctly measure some important properties of the system within the scope of laboratory experiments and reaching the BEC regime for the dilute atomic gases confined within harmonic trapping potentials.

[11] Refer to Eq.(2.6) of [66].
[12] Refer to Eq.(2.20) of [66].

2.2 Many-body formalism

To study interacting non-uniform Bose gases at very low temperature, one uses the Bogoliubov prescription for the field operator $\hat{\psi}(t, \mathbf{r})$ (in coordinate space) that obeys the well known equal-time commutation relations

$$\left[\hat{\psi}(t, \mathbf{r}), \hat{\psi}^\dagger(t, \mathbf{r}')\right] = \delta(\mathbf{r} - \mathbf{r}') \; ; \tag{2.22a}$$

$$\left[\hat{\psi}(t, \mathbf{r}), \hat{\psi}(t, \mathbf{r}')\right] = 0 = \left[\hat{\psi}^\dagger(t, \mathbf{r}), \hat{\psi}^\dagger(t, \mathbf{r}')\right] . \tag{2.22b}$$

Here $\mathbf{r}$ and $\mathbf{r}'$ are two different positions for two different particles (i.e., nearest bosons) wrt some specified origin. We start by introducing the definition (see Eq.(2.1) of the text book by Pitaevskii and Stringari [42]) of one-body density matrix as

$$n^{(1)}(\mathbf{r}, \mathbf{r}') = \langle \hat{\psi}^\dagger(t, \mathbf{r}) \, \hat{\psi}(t, \mathbf{r}') \rangle \tag{2.23}$$

for which $n^{(1)}$ naturally becomes Hermitian. This definition is general and irrespective of any statistics, be it in equilibrium or out of equilibrium (obviously in case of inequilibrium, $n^{(1)}$ would depend on time as well). By setting $\mathbf{r}' = \mathbf{r}$ in Eq.(2.23), what we come up with is called the diagonal density of the system, given by,

$$n^{(1)}(\mathbf{r}, \mathbf{r}) = \langle \hat{\psi}^\dagger(t, \mathbf{r}) \, \hat{\psi}(t, \mathbf{r}) \rangle \equiv n(\mathbf{r}). \tag{2.24}$$

The normalization of the one-body density matrix is defined through the notion of total number of particles present in the system,

$$N = \int d\mathbf{r} \; n(\mathbf{r}). \tag{2.25}$$

Similarly like Eq.(2.23), the one-body density matrix in the momentum distribution is given by

$$n^{(1)}(\mathbf{p}, \mathbf{p}') = \langle \hat{\psi}^\dagger(t, \mathbf{p}) \, \hat{\psi}(t, \mathbf{p}') \rangle, \tag{2.26a}$$

where,

$$\hat{\psi}(t, \mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int d\mathbf{r} \; e^{i\mathbf{p}.\mathbf{r}/\hbar} \, \hat{\psi}(t, \mathbf{r}) \tag{2.26b}$$

is defined to be the field-operator in the momentum space.

If one considers the uniform and isotropic system of N particles occupying volume $\mathcal{V}$ in the absence of external potentials, then in the thermodynamic limit, i.e.,

$$\lim_{N, \mathcal{V} \to \infty} \frac{N}{\mathcal{V}} = n \; \text{ is kept finite,} \tag{2.27}$$

the one-body density matrix is given by

$$n^{(1)}(\mathbf{r},\mathbf{r}') \equiv n^{(1)}(\mathbf{r}-\mathbf{r}') = \frac{1}{\mathcal{V}}\int d\mathbf{p}\, n(\mathbf{p})\, e^{-i\mathbf{p}.(\mathbf{r}-\mathbf{r}')/\hbar} \tag{2.28}$$

where the momentum distribution exhibits a singular behavior given by,

$$n(\mathbf{p}) = N_0\,\delta(\mathbf{p}) + \tilde{n}(\mathbf{p}). \tag{2.29}$$

Here N_0 is the statistical weight which is $\propto N$. The physical interpretation of the above Eqs.(2.28) and (2.29) are - there is indeed a macroscopic occupation of the single particle state (BEC ground state) with momentum $\mathbf{p} = 0$ and this provides a general prescription of BEC on a very fundamental level with the quantity $N_0/N \lesssim 1$ being called as the 'condensate fraction'.

2.2.1 Long-range order

Plugging Eq.(2.29) into Eq.(2.28), one finds that the one-body density matrix doesn't really vanish at large distances and rather approaches a finite constant value,

$$\lim_{|\mathbf{r}-\mathbf{r}'|\to\infty} n^{(1)}(\mathbf{r},\mathbf{r}') \longrightarrow n_0 = \frac{N_0}{\mathcal{V}}. \tag{2.30}$$

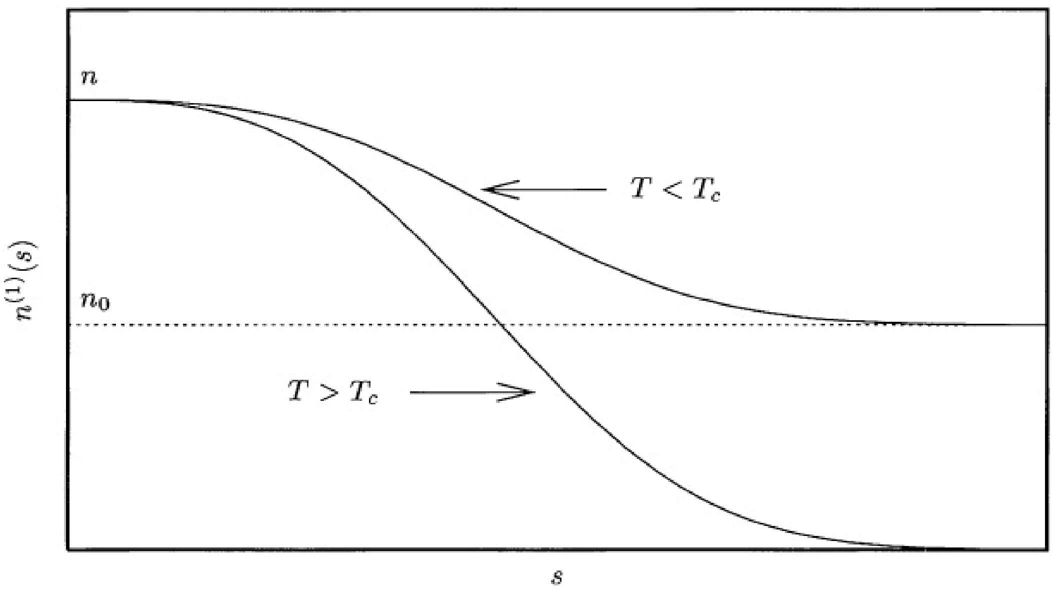

Figure 2.3: Off-diagonal one-body density $n^{(1)}(s)$ as a function of the relative distance s, where $\delta\mathbf{r} \equiv s$ be the parameter. For $T < T_c$, $n^{(1)}(s)_{s\to\infty} = n_0 = N_0/V$ and $n^{(1)}(s)_{s\to 0} \approx n = N/V$. [Source: Fig.2.1 of [42]]

It is often referred to as 'off-diagonal' long-range order since it involves the non-diagonal terms ($\mathbf{r} \neq \mathbf{r}'$) in the one-body density matrix. BEC is formed only when one of the single-particle states (BEC ground state) is occupied in a macroscopic way while the single-particle states have a microscopic occupation of $O(1)$. For completeness, one can find the expression of the one-body density matrix $n^{(1)}(\mathbf{r}, \mathbf{r}')$ in the close range where $|\mathbf{r} - \mathbf{r}'| \to 0$ limit is carried out. The result[13] is given by

$$n^{(1)}(\mathbf{r}, \mathbf{r}')|_{\text{close range}} \equiv \lim_{\delta\mathbf{r}\to 0} n^{(1)}(\delta\mathbf{r}) = n\left(1 - \frac{1}{2} < p_z^2 > \frac{(\delta\mathbf{r})^2}{\hbar^2} + \ldots\right). \tag{2.31}$$

One may refer to Fig.2.3 for a clearer justification. The physics is shown schematically there.

2.2.2 Many-body Hamiltonian - emergence of 'interaction'

The many-body Hamiltonian $\hat{\mathcal{H}}$ of the full interacting Bose-system, say N interacting bosons confined by an external potential, in terms of the field operators $\hat{\psi}(t, \mathbf{r})$ is given by

$$\begin{aligned}\hat{\mathcal{H}} = &\int \left(\frac{\hbar^2}{2m}\nabla\hat{\psi}^\dagger(t, \mathbf{r})\nabla\hat{\psi}(t, \mathbf{r})\right) d\mathbf{r} + \int \hat{\psi}^\dagger(t, \mathbf{r}) V_{\text{ext}}(t, \mathbf{r})\hat{\psi}(t, \mathbf{r})\, d\mathbf{r} \\ &+ \frac{1}{2}\int \hat{\psi}^\dagger(t, \mathbf{r})\hat{\psi}^\dagger(t, \mathbf{r}') V(\mathbf{r}' - \mathbf{r})\hat{\psi}(t, \mathbf{r})\hat{\psi}(t, \mathbf{r}')\, d\mathbf{r}' d\mathbf{r},\end{aligned} \tag{2.32}$$

with $\hbar = h/2\pi$, h being the Planck constant, m being the mass of a single boson. Obviously, $V_{\text{ext}}(t, \mathbf{r})$ is an external (trapping) potential and $V(\mathbf{r}' - \mathbf{r})$ is the two-body interaction potential.

2.3 The Weakly-interacting Bose gas: Bogoliubov theory

It is not surprising that even for a dilute sample, the picture of interaction (whatsoever weak it may be) does affects the various properties of the system in quite an extent. In the absence of boson-boson interaction, the ground state energy was zero and this is obviously not trivial and traditional perturbation techniques can't be applied.

This problem was rescued by Bogoliubov in 1947 when he proposed a new methodology of perturbation techniques in order to give the structure of the modern approaches to BEC in dilute gases [67]. As obvious, in this case the compressibility comes out to be finite.

[13] Refer to Eq.(2.11) of [42].

A very significant feature of ultracold atomic vapours is that the inter-particle separations (which are typically $\sim 10^{-7}$m.) are usually an order of magnitude larger than the length scales associated with the atom-atom interaction. As a consequence, the two-body interaction between atoms in a BEC dominates, and three- or even higher-body interactions kind of lose the importance.

2.3.1 The effective soft potential $V_{\text{eff}}(\mathbf{r}' - \mathbf{r})$

In the experimentally realizable systems, $V(\mathbf{r}'-\mathbf{r})$ always contains a short-range interaction term by presence of which the dynamical equation becomes difficult to be solved at the microscopic level. In order to calculate the low-energy properties of the system (BEC), the actual form of the two-body interaction potential $V(\mathbf{r}' - \mathbf{r})$ is not important for describing the macroscopic properties of the Bose gas. Therefore as per convenience, it may be replaced by an effective, soft proportional $V_{\text{eff}}(\mathbf{r}' - \mathbf{r})$ to which the standard perturbation theory can easily be applied.

Since in the theory of dilute BEC, at low energies, one can safely consider the inter-particle scattering to be solely spherically symmetric, the whole formulation is sort of parametrized through a single parameter called the s-wave scattering length a. Physical properties of the system would practically depend on a. So, as far as both $V(\mathbf{r}' - \mathbf{r})$ and $V_{\text{eff}}(\mathbf{r}' - \mathbf{r})$ yield the same a, they provide the correct results in the many-body problems in Physics on a macroscopic level. In the following Fig.1.1, this is schematically depicted.

The actual interaction potential gets replaced [in Eq.(2.32)] by some effective soft repulsive potential for sake of convenience and without harming the level of accuracy of the physical problem and macroscopic properties,

$$V(\mathbf{r}' - \mathbf{r}) \longrightarrow V_{\text{eff}}(\mathbf{r}' - \mathbf{r}). \tag{2.33}$$

2.3.2 Bogoliubov theorem: the second-quantized prescription & quantum fluctuations

Under the Bogoliubov approximation (N. Bogoliubov, 1947 [67]), one gets the liberty to treat the macroscopic component of the BEC field operator as a classical mean field (i.e., the condensate wave function), given by,

$$\hat{\psi}(t, \mathbf{r}) = \psi(t, \mathbf{r}) + \Delta\hat{\psi}(t, \mathbf{r}). \tag{2.34}$$

In case of dilute Bose gases at very low temperatures, the non-condensate fraction $\Delta\hat{\psi}(t, \mathbf{r}) \to 0$ and hence the quantum field operator completely coincides with the classical mean field because of the macroscopic occupation ($N_0 >> 1$) in the BEC ground state. Because of this, there's

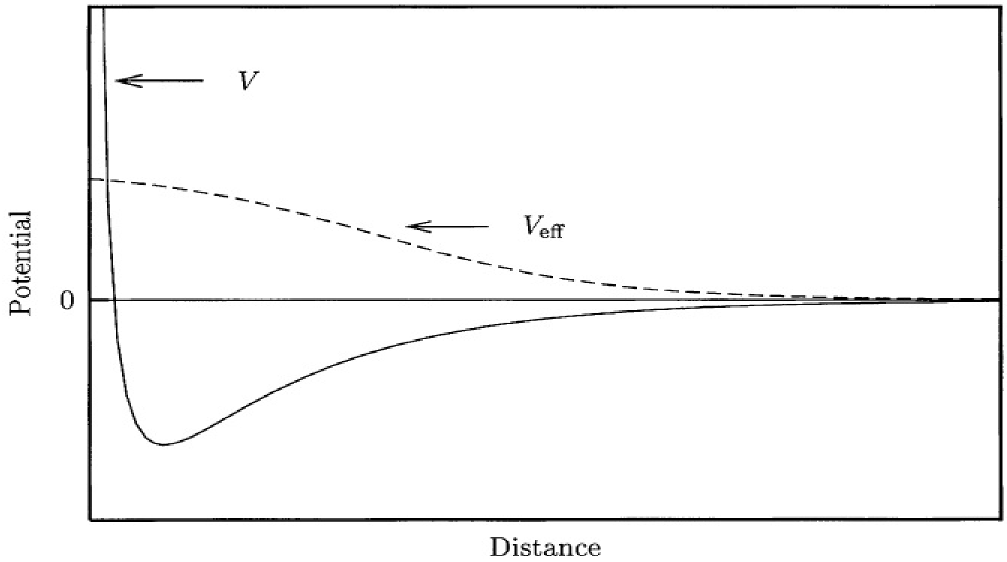

Figure 2.4: Schematic representation of the actual interaction potential $V \equiv V(\mathbf{r}' - \mathbf{r})$ (solid line) and the effective soft potential $V_{\rm eff}$ (dashed line). V and $V_{\rm eff}$ produce the same s-wave scattering length a. In order to apply Bogoliubov theory, the average range of both potentials should be $<< n^{-1/3}$. [Source: Fig.4.1 on p.28 of [42]]

something called Bogoliubov prescription by virtue of which one can safely replace the quantum operator(s) (creation/annihilation) with just a c-number in the Hamiltonian by neglecting the non-commutativity in the theory of BEC.

In the second quantized picture, for a gas occupying a volume $\mathcal{V}$, the field operator(s) $\hat{\psi}(t, \mathbf{r})$ can be expressed as

$$\hat{\psi}(t, \mathbf{r}) = \frac{1}{\sqrt{\mathcal{V}}} \sum_{\mathbf{p}} \hat{a}_{\mathbf{p}}(t)\ e^{i\mathbf{p}.\mathbf{r}/\hbar}, \tag{2.35}$$

where $\hat{a}_{\mathbf{p}}(t)$ is the operator annihilating[14] a particle in the single-particle state with momentum $\mathbf{p}$ at time t (obviously the values of $\mathbf{p}$ satisfy the usual cyclic boundary conditions). In the absence of external potential (i.e., $V_{\rm ext} = 0$) for simplicity, plugging the above Eq.(2.35) into Eq.(2.32), one comes up with

$$\hat{\mathcal{H}} = \sum_{\mathbf{p}} \frac{p^2}{2m} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \frac{1}{2\mathcal{V}} \sum_{\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}} V_{\mathbf{q}}\ \hat{a}^\dagger_{\mathbf{p}_1+\mathbf{q}} \hat{a}^\dagger_{\mathbf{p}_2-\mathbf{q}} \hat{a}_{\mathbf{p}_1} \hat{a}_{\mathbf{p}_2}, \tag{2.36}$$

[14] For sake of clarity, the annihilation/creation operator(s), at some momentum state $\mathbf{p}$, are defined as
annihilation operator: $\hat{a}_p|n_0, n_1, ..., n_p, ...\rangle = \sqrt{n_p}|n_0, n_1, ..., n_p - 1, ...\rangle$ and
creation operator: $\hat{a}^\dagger_p|n_0, n_1, ..., n_p, ...\rangle = \sqrt{n_p + 1}|n_0, n_1, ..., n_p + 1, ...\rangle$

where the interaction potential is redefined through

$$V_{\mathbf{q}} = \int d\mathbf{r}\ \underbrace{V(\mathbf{r})}\ e^{-i\mathbf{q}.\mathbf{r}/\hbar} \equiv \int d\mathbf{r}\ \underbrace{V_{\text{eff}}(\mathbf{r})}\ e^{-i\mathbf{q}.\mathbf{r}/\hbar}. \tag{2.37}$$

As of now, the whole formulation is based on assumptions made on physical grounds in the low energy regime. Therefore only the lower momenta are involved in the solution of the dynamical equation in this many-body problem. We are allowed to consider $\mathbf{p} = 0$ value, i.e., the BEC ground state. The Bogoliubov prescription prescribes that

$$\hat{a}_0 = \sqrt{N_0} = \hat{a}_0^\dagger \qquad [\because\ N_0 \pm 1 \simeq N_0 \text{ in a BEC}] \tag{2.38}$$

to be inserted in Eq.(2.36). This approximation of neglecting the non-commutativity of the quantum operators are pretty accurate only when one employs the effective soft potential rather than the actual interaction potential. That is because had it been $V(\mathbf{r}' - \mathbf{r})$, at short interatomic distances, the potential being strong enough (see Fig.1.1), the quantum correlations would have been necessary and important.

Since only the small momenta are involved in the solution of many-body problem, for $\mathbf{q} = 0$, replacing $V_{\mathbf{q}}$ by V_0 in Eq.(2.36),

$$\hat{\mathcal{H}} = \sum_{\mathbf{p}} \frac{p^2}{2m}\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{V_0}{2\mathcal{V}}\hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 + \frac{V_0}{2\mathcal{V}}\sum_{\mathbf{p}\neq 0}\left(4\hat{a}_0^\dagger \hat{a}_{\mathbf{p}}^\dagger \hat{a}_0 \hat{a}_{\mathbf{p}} + \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{-\mathbf{p}}^\dagger \hat{a}_0 \hat{a}_0 + \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_{\mathbf{p}} \hat{a}_{-\mathbf{p}}\right). \tag{2.39}$$

The normalization relation, with higher degree of accuracy, is given by

$$\hat{a}_0^\dagger \hat{a}_0 + \sum_{\mathbf{p}\neq 0} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} = N \tag{2.40a}$$

$$\Rightarrow \qquad \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \simeq N^2 - 2N \sum_{\mathbf{p}\neq 0} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}}. \tag{2.40b}$$

Using Eqs.(2.38) and (2.40), one can rewrite $\mathcal{H}$ from Eq.(2.39) as

$$\hat{\mathcal{H}} = \sum_{\mathbf{p}} \frac{p^2}{2m}\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{1}{2}V_0 n\left(N - 2\sum_{\mathbf{p}\neq 0} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}}\right) + \frac{1}{2}V_0 n \sum_{\mathbf{p}\neq 0}\left(4\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{-\mathbf{p}}^\dagger + \hat{a}_{\mathbf{p}} \hat{a}_{-\mathbf{p}}\right), \tag{2.41}$$

where $n = N/\mathcal{V}$ is the density of the Bose gas. Now the diagonalization of these particle operators $\hat{a}_{\mathbf{p}}$ and $\hat{a}_{\mathbf{p}}^\dagger$ can be done through *Bogoliubov transformation*. It is basically a linear transformation through introducing a new set of operators $\hat{b}_{\mathbf{p}}$ and $\hat{b}^{\dagger}_{\mathbf{p}}$ constructed as

$$\hat{a}_{\mathbf{p}} = u_{\mathbf{p}}\hat{b}_{\mathbf{p}} + v^{*}_{-\mathbf{p}}\hat{b}^{\dagger}_{-\mathbf{p}}, \qquad \hat{a}^{\dagger}_{\mathbf{p}} = u^{*}_{\mathbf{p}}\hat{b}^{\dagger}_{\mathbf{p}} + v_{-\mathbf{p}}\hat{b}_{-\mathbf{p}}. \tag{2.42}$$

If one imposes the Bose commutation relations (as obeyed by the original particle operators $\hat{a}_{\mathbf{p}}$ and $\hat{a}^{\dagger}_{\mathbf{p}}$) on these new set of operators $\hat{b}_{\mathbf{p}}$ and $\hat{b}^{\dagger}_{\mathbf{p}}$, then we have

$$\left[\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{p}'}\right] = \delta_{\mathbf{pp}'} = \left[\hat{b}_{\mathbf{p}}, \hat{b}^{\dagger}_{\mathbf{p}'}\right], \tag{2.43a}$$

$$\therefore \quad \left|u_{\mathbf{p}}\right|^2 - \left|v_{-\mathbf{p}}\right|^2 = 1. \tag{2.43b}$$

Our target is to diagonalize the Hamiltonian $\mathcal{H}$ in Eq.(2.39) and by virtue of the Bogoliubov transformation [Eq.(2.42)], the complex coefficients $u_{\mathbf{p}}$ and $v_{-\mathbf{p}}$ are so chosen that the non-diagonal terms of $\mathcal{H}$ do vanish. Demanding this constraint on the coefficients, one can explicitly obtain them[15]. Finally, the Hamiltonian can be expressed in its diagonal form as

$$\mathcal{H} = E_0 + \sum_{\mathbf{p}} \varepsilon(p)\, \hat{b}^{\dagger}_{\mathbf{p}}\hat{b}_{\mathbf{p}}, \tag{2.44}$$

where E_0 is the ground state energy of the system calculated with higher degree of accuracy.

In language of standard quantum mechanics, the scattering of a pair of particles is dominated by only the s-wave contribution[16] to the particle wave function (by neglecting the orbital and spin contributions) in the low energy limit.

In case of an interacting Bose gas, the lowest order Born approximation yields the s-wave interaction coupling constant to be directly $\propto$ (the s-wave scattering length a), given by,

$$\mathrm{g} = \frac{4\pi\hbar^2 a}{m}. \tag{2.45}$$

In this picture of quantum scattering, using higher-order perturbation theory[17], one finds

$$V_0 = \mathrm{g}\left(1 + \frac{\mathrm{g}}{\mathcal{V}} \sum_{\mathbf{p}\neq 0} \frac{m}{p^2}\right). \tag{2.46}$$

[15] Refer to Eq.(4.28) of [42].

[16] That is the scattering is effectively spherically symmetric between a pair of first nearest neighbours and parametrized by the s-wave scattering length a.

[17] Refer to a text book, e.g. Landau & Lifshitz, 1987b.

Using Eq.(2.46), the expression of E_0 in Eq.(2.44) can be explicitly given by

$$E_0 = \frac{1}{2}Ngn + \frac{1}{2}\sum_{\mathbf{p}\neq 0}\left[\varepsilon(p) - gn - \frac{p^2}{2m} + \frac{mg^2n^2}{p^2}\right], \tag{2.47}$$

where

$$\boxed{\varepsilon(p) = \left[\frac{gn}{m}p^2 + \left(\frac{p^2}{2m}\right)^2\right]^{1/2}} \tag{2.48}$$

is the celebrated Bogoliubov dispersion law for elementary excitations of the system

In the above prescription, we have always talked about interactions between a pair of bosons, i.e. first nearest neighbors. However a three-body (or even higher) interaction could have also been considered but the condensate would have lost its stability then and in the quantum degenerate regime, such system(s) in fact represents a metastable configuration where thermalization is ensured by two-body collisions. This metastable phase of the system is well explained through the Bogoliubov theory described above. For a three-body interaction, it effectively drives the system to form a solid configuration from the gaseous / fluid state and hence its experimentally difficult to achieve due to the rapid instability of the system.

2.3.3 Local speed of sound in the fluid medium, c_s

From the Bogoliubov dispersion [Eq.(2.48)], in the low momentum regime, the contribution of p^4 is negligibly small wrt that of p^2 and hence for small values of $|\mathbf{p}|$, the Bogoliubov dispersion becomes linear (the usual phonon distribution). Therefore,

$$\varepsilon(p) \approx \sqrt{\frac{gn}{m}}p, \qquad \therefore \qquad \boxed{c_s = \frac{\varepsilon(p)}{p} \approx \sqrt{\frac{gn}{m}}}. \tag{2.49}$$

2.3.4 Healing (or, coherence) length of the condensate, ξ_0

If one considers a BEC confined within a box of hard walls, then the condensate wave function is assumed to be well-behaved and hence must vanish at the hard wall while the condensate density reaches its bulk value in the interior of the box. The spatial distance over which the wave function rises from zero at the wall to its bulk value in the interior may be estimated by balance between the kinetic energy contributions ($p^2/2m$) and the interaction[18] energy term (gn). Say the spatial distance be some $\xi_0 = \hbar/p$ for which the balance between the above two

[18] As far as the interactions are considered to be *local*, the corresponding contribution is given by gn. Refer to the local Gross-Pitaevskii equation later.

contributions takes place, i.e.,

$$\underbrace{\frac{p^2}{2m}}_{\text{free particle regime}} \sim \underbrace{gn}_{\text{phonon regime}}, \quad \text{for } p = \frac{\hbar}{\xi_0},$$
$$\Rightarrow \frac{(\hbar/\xi_0)^2}{2m} = gn \quad \Rightarrow \boxed{\xi_0 = \frac{\hbar}{\sqrt{2mgn}} = \frac{1}{\sqrt{8\pi an}}}, \tag{2.50}$$

where $g = 4\pi\hbar^2 a/m$ is inserted from Eq.(2.45). This ξ_0 is called the 'healing' length or 'coherence' length of the condensate. In some literature, it is also referred to be the characteristic interaction length. This is basically the length scale that demarcates the transition between the free-particle regime and the phonon regime.

I will resume talking about the healing length of the condensate in case of nonlocal interactions and how it gets modified from this usual ξ_0 and what the main significances are in the context of BEC being used as a useful tool in Analogue Gravity to probe some aspects of high-energy (short-wavelength due to smaller healing length) phenomena.

2.4 Nonuniform BEC at $T = 0$: *local* Gross-Pitaevskii model

To study nonuniform BEC is significant and important for mainly two reasons - gases are naturally nonuniform and experimentally it was first achieved in traps and secondly, the consideration of nonuniformity happens to produce the famous Bogoliubov dispersion and also gives rise to various quantum phenomena of the system.

2.4.1 The diluteness condition

In the dilute systems, where

$$\text{the range of interatomic forces} << \text{average interparticle separation } (n^{-1/3}), \tag{2.51}$$

where $n = \frac{N}{V}$ is the density of the gas sample, one has to consider interactions between a pair of first nearest neighbours. Due to the diluteness of the gas sample, configurations involving interactions between three or more particles can easily be cast out.

Moreover, since the atoms have low velocities, many properties of these systems may be calculated in terms of a single parameter, the scattering length. To describe scattering in BEC, one usually writes the condensate wave function as a superposition of an incoming plane wave and a scattered wave[19]. By neglecting the effects due to the coupling between orbital and spin degrees of freedom, we assume that the interaction between particles in the theory of weakly-interacting BEC is spherically symmetric. This assumption is legit at very low energies and characterized through a parameter called a which is nothing but the s-wave scattering length[20].

Having the notion of a defined, the condition of diluteness [i.e., Eq.(2.51)] is given by

$$|a| << n^{-1/3}. \tag{2.52}$$

A different scenario might take place near a *Feshbach resonance*[21] where the above inequality might not hold good and hence the condition of diluteness in a BEC is in general not satisfied.

2.4.2 Mean-field approximation from Heisenberg picture

In the Heisenberg picture, the dynamics of this Bose-field-operator $\hat{\psi}(t, \mathbf{r})$ is given by the exact equation[22]

$$i\hbar\partial_t\hat{\psi}(t,\mathbf{r}) = \left[\hat{\psi}(t,\mathbf{r}), \hat{\mathcal{H}}\right] = \left(-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V_{\text{ext}}(t,\mathbf{r}) + \int \hat{\psi}^\dagger(t,\mathbf{r}')\,V(\mathbf{r}'-\mathbf{r})\,\hat{\psi}(t,\mathbf{r}')\,d\mathbf{r}'\right)\hat{\psi}(t,\mathbf{r})\,, \tag{2.53}$$

where $\hat{\mathcal{H}}$ is plugged in from Eq.(2.32).

Now one gets the license to replace the quantum field operator $\hat{\psi}(t, \mathbf{r})$ by the classical wave function $\psi(t, \mathbf{r})$ of the condensate due to the macroscopic occupation of a large number of atoms in a single quantum state (BEC ground state). In this regard, the Bogoliubov prescription was already mentioned through Eq.(2.34) and specially Eq.(2.38) which was the direct upshot of the macroscopic occupation in the condensate. The mean field approximation is

$$\hat{\psi}(t,\mathbf{r}) \;\rightarrow\; <\hat{\psi}> = \psi(t,\mathbf{r}) \quad , \tag{2.54}$$

by which one sort of neglects the noncommutativity of the field operators $\hat{\psi}(t, \mathbf{r})$ as defined above. This mean-field approximation has its implication from a physical point of view as well - since $\hat{\psi}(t, \mathbf{r})$ or $\hat{\psi}^\dagger(t, \mathbf{r})$ act as 'annihilation' or 'creation' operator(s) respectively to annihilate

[19]Refer to Eq.(5.7) of [66].

[20]Refer to Eq.(5.10) of [66].

[21]Feshbach resonance is basically a phenomenon which was first investigated by Herman Feshbach in 1958 in the context of nuclear physics [68, 69]. Then it was studied by Fano in 1961 in the context of atomic physics [70] and followed by a more general treatment in relation with their use in cold atomic gases as proposed by Stwalley in 1976 [71]. These phenomena of resonances have become an important tool in order to study the basic atomic physics of ultracold atoms, because they provide a possibility to vary the effective interaction between particles just by adjusting an external parameter such as the magnetic field.

[22]Refer to Eq.(5.1) of [42].

or create a particle at $(t, \mathbf{r})$, now if a particle is subtracted from or added to the condensate, it does not really change the physical properties of the whole system which is actually governed by the order parameter $\psi(t, \mathbf{r})$.

In terms of the order parameter of the system, the dynamical equation [Eq.(2.53)] becomes

$$i\hbar\partial_t \psi(t, \mathbf{r}) = \left(-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V_{\text{ext}}(t, \mathbf{r}) + \underbrace{\int \psi^*(t, \mathbf{r}') V(\mathbf{r}' - \mathbf{r}) \psi(t, \mathbf{r}') \, d\mathbf{r}'}_{\text{say, } \mathbb{T}_{\text{int}}} \right) \psi(t, \mathbf{r}) \,. \tag{2.55}$$

► **Locality** - In the usual *local* picture, one makes a 'drastic' δ-function approximation to the range of $V(\mathbf{r}' - \mathbf{r})$ by considering

$$V(\mathbf{r}' - \mathbf{r}) \equiv V_{\text{eff}}(\mathbf{r}' - \mathbf{r}) = \mathrm{g}\delta(\mathbf{r}' - \mathbf{r}), \tag{2.56a}$$

$$\therefore \mathbb{T}_{\text{int}} = \int d\mathbf{r}' \psi^*(t, \mathbf{r}') V(\mathbf{r}' - \mathbf{r}) \psi(t, \mathbf{r}') \equiv \int d\mathbf{r}' |\psi(t, \mathbf{r}')|^2 V_{\text{eff}}(\mathbf{r}' - \mathbf{r}) = \int d\mathbf{r}' |\psi(t, \mathbf{r}')|^2 \mathrm{g}\delta(\mathbf{r}' - \mathbf{r})$$
$$= \mathrm{g}|\psi(t, \mathbf{r})|^2. \tag{2.56b}$$

Now, from Eq.(2.55), one writes the local Gross-Pitaevskii (GP) equation in the following form

$$\boxed{i\hbar\partial_t \psi(t, \mathbf{r}) = \left(-\frac{\hbar^2 \boldsymbol{\nabla}^2}{2m} + V_{\text{ext}}(t, \mathbf{r}) + \mathrm{g}|\psi(t, \mathbf{r})|^2 \right) \psi(t, \mathbf{r})}\,. \tag{2.57}$$

This above equation for the complex order parameter of the inhomogenous condensate was independently derived by Gross in 1961 [72] and Pitaevskii in 1961 [73]. The GP equation not only correctly reproduces the quantum mechanical Bogoliubov spectrum[23], i.e., Eq.(2.48), it is extremely useful in getting explicit density variations of the inhomogeneous condensate in closed form expressions. It also admits the vortex solutions which are particle like (dispersion relation is not linear like that of massless phonons) and are higher energy states stabilized by rotating the the BEC [42].

The local GP equation [Eq.(2.57)] can be shown to be derived variationally by minimizing a free energy functional of the system, i.e., for an energy functional

$$E = \int d\mathbf{r} \left[\frac{\hbar^2}{2m} |\boldsymbol{\nabla}\psi(t, \mathbf{r})|^2 + V_{\text{ext}}(t, \mathbf{r}) \, |\psi(t, \mathbf{r})|^2 + \frac{\mathrm{g}}{2} \, |\psi(t, \mathbf{r})|^4 \right], \tag{2.58}$$

[23] Refer to Eq.(5.70) of [42].

the variational equation (which can be derived from minimizing the corresponding action[24])

$$i\hbar\frac{\partial\psi(t,\mathbf{r})}{\partial t} = \frac{\delta E}{\delta\psi^*(t,\mathbf{r})} \tag{2.59}$$

yields the dynamical equation for a nonuniform local BEC.

The asymptotic exactness of the GP energy functional at the dilute limit with two-body interactions was rigorously shown by Lieb *et al* [74, 75]. In this proof, at the thermodynamic limit $N \to \infty$, $a \to 0$ to keep Na finite and this limit corresponds well with $V(\mathbf{r}' - \mathbf{r})$ being replaced by a δ-function.

[24] The action is given by $\mathcal{S} = -i\hbar \int dt \int d\mathbf{r}\, \psi^*(t,\mathbf{r})\partial_t\psi(t,\mathbf{r}) + \int dt E$, refer to Eq.(5.6) of [42].

Chapter 3

Analogue gravity from BEC: *Nonlocal* interactions and modified Bogoliubov spectrum

3.1 Analogue model of gravity from GP theory of BEC with contact interaction

In the present section, I will show the standard derivation of the acoustic metric that emerges out of a nonrelativistic BEC and demonstrate that the dynamics of the phonons (acoustic disturbances) of the condensate closely mimic the dynamical equation for a minimally coupled scalar field in a $(3+1)$D curved spacetime (i.e., the d'Alembertian equation of motion). According to the standard literature, this scalar field is a massless Klein-Gordon field.

On considering $|\psi(t,\mathbf{r})|^2 = n(t,\mathbf{r})$ to be the condensate density, a general single particle state of the BEC, by adopting the Madelung ansatz, is

$$\psi(t,\mathbf{r}) = \sqrt{n(t,\mathbf{r})}\, e^{i\vartheta(t,\mathbf{r})/\hbar}. \tag{3.1}$$

The standard GP equation for a nonuniform BEC [i.e., Eq.(2.57) in Chapter-2] with contact interactions is

$$i\hbar\partial_t\, \psi(t,\mathbf{r}) = \left(-\frac{\hbar^2\boldsymbol{\nabla}^2}{2m} + V_{\text{ext}}(t,\mathbf{r}) + g|\psi(t,\mathbf{r})|^2\right)\psi(t,\mathbf{r}),$$

where $\psi(t, \mathbf{r})$ is substituted according to the above ansatz in Eq.(3.1) and it gives rise to a set of coupled equations[25];

Continuity equation:
$$\partial_t n(t, \mathbf{r}) + \frac{1}{m}\boldsymbol{\nabla}.\Big(n(t, \mathbf{r})\boldsymbol{\nabla}\vartheta(t, \mathbf{r})\Big) = 0, \tag{3.2}$$

Euler equation:
$$\partial_t \vartheta(t, \mathbf{r}) + \left(\frac{[\boldsymbol{\nabla}\vartheta(t, \mathbf{r})]^2}{2m} + V_{\text{ext}}(t, \mathbf{r}) + g n(t, \mathbf{r}) \underbrace{- \frac{\hbar^2}{2m}\frac{\boldsymbol{\nabla}^2\sqrt{n(t, \mathbf{r})}}{\sqrt{n(t, \mathbf{r})}}}\right) = 0. \tag{3.3}$$

The above underbracketted term is referred to as the "quantum potential" which is the central concept of de Broglie-Bohm formulation of quantum mechanics as introduced by David Bohm in 1952 [76, 77]. Later it was elaborated in the light of Thermodynamics and Statistical Mechanics by Bohm and Hiley in 1996 [78]. It is given by

$$\mathsf{V}_{\text{quantum}} = -\frac{\hbar^2}{2m}\frac{\boldsymbol{\nabla}^2\sqrt{n(t, \mathbf{r})}}{\sqrt{n(t, \mathbf{r})}}. \tag{3.4}$$

The qualitative features of this term $\mathsf{V}_{\text{quantum}}$ represent a drastic departure from the classical physics [79]. If the density of the gas changes slowly in space, which is pretty much the case for a dilute BEC, then this quantum potential term can safely be neglected (because double derivatives of $\sqrt{n}$ would leave even small contributions then) and this issue is discussed in the context of Thomas-Fermi limit in [42]-p.43.

Now one can consider the fluctuations[26] to the density [i.e. $n(t, \mathbf{r})$] and phase [i.e. $\vartheta(t, \mathbf{r})$] of the single-particle BEC state in the following manner,

$$n(t, \mathbf{r}) \to n_0(t, \mathbf{r}) + n_1(t, \mathbf{r}) \quad , \quad \vartheta(t, \mathbf{r}) \to \vartheta_0(t, \mathbf{r}) + \tilde{\vartheta}_1(t, \mathbf{r}); \tag{3.5}$$

i.e., $n_0(t, \mathbf{r})$ and $\vartheta_0(t, \mathbf{r})$ are basically the classical mean-field density and phase respectively, such that $< n >= n_0(t, \mathbf{r})$ and $< \vartheta >= \vartheta_0(t, \mathbf{r})$. These are obviously the macroscopic descriptions of the condensate within the classical scenario. On the other hand, $n_1(t, \mathbf{r})$ is the first order density-fluctuation and $\tilde{\vartheta}_1(t, \mathbf{r})$ is the first order phase-fluctuation[27]. In other words, these fluctuations

[25] Here, Euler equation [Eq.(3.3)] is written in Hamilton-Jacobi form for convenience. One may refer to Eq.(5.15) of [42].

[26] These linearised fluctuations in the dynamical quantities are more generally referred to as *acoustic disturbances*. To be precise and according to the convention, the low-frequency large-wavelength disturbances are called *wind gusts*, while the high-frequency short-wavelength disturbances being described as acoustic disturbances. Refer to [21]-p.1770.

[27] There's a reason why I use the 'tilde' over ϑ_1, its because we will later decompose these linearized fluctuations into two further modes to investigate amplitude modulations; i.e., $\tilde{\vartheta}_1 \to \vartheta_2(X)\vartheta_1(x)$. This will be discussed in details in the following chapter(s).

are the quantum fields in nature and can be described as "quantum acoustic representation" [28].

Inserting Eq.(3.5) back into Eq.(3.2) and (3.3), one comes up with the linearized dynamics for these quantum fluctuations, given by Eq.(3.6) (obtained from the continuity equation) coupled with Eq.(3.7) (obtained from the Euler equation) as the following,

$$\partial_t n_1(t,\mathbf{r}) + \frac{1}{m}\boldsymbol{\nabla}.\Big(n_1(t,\mathbf{r})\boldsymbol{\nabla}\vartheta_0(t,\mathbf{r}) + n_0(t,\mathbf{r})\boldsymbol{\nabla}\tilde{\vartheta}_1(t,\mathbf{r})\Big) = 0, \tag{3.6}$$

and

$$\partial_t \tilde{\vartheta}_1(t,\mathbf{r}) + \frac{1}{m}\boldsymbol{\nabla}\vartheta_0(t,\mathbf{r}).\boldsymbol{\nabla}\tilde{\vartheta}_1(t,\mathbf{r}) + \mathrm{g}n_1(t,\mathbf{r}) - \frac{\hbar^2}{2m}\hat{D}_2\, n_1(t,\mathbf{r}) = 0, \tag{3.7}$$

where,

$$\hat{D}_2\, n_1(t,\mathbf{r}) = -\frac{n_1(t,\mathbf{r})}{2} n_0^{-3/2}(t,\mathbf{r})\,\boldsymbol{\nabla}^2 n_0^{1/2}(t,\mathbf{r}) + \frac{n_0^{-1/2}(t,\mathbf{r})}{2}\,\boldsymbol{\nabla}^2\Big(n_0^{-1/2}(t,\mathbf{r})\, n_1(t,\mathbf{r})\Big), \tag{3.8}$$

represents a second-order differential operator obtained by linearizing $\mathsf{V}_{\text{quantum}}$ from Eq.(3.4). From Eq.(3.7), $n_1(t,\mathbf{r})$ can be obtained as the following,

$$n_1(t,\mathbf{r}) = -\hat{\mathcal{A}}\Big(\partial_t \tilde{\vartheta}_1(t,\mathbf{r}) + \tfrac{1}{m}\boldsymbol{\nabla}\vartheta_0(t,\mathbf{r}).\boldsymbol{\nabla}\tilde{\vartheta}_1(t,\mathbf{r})\Big), \tag{3.9}$$
$$\text{where,} \quad \hat{\mathcal{A}} = \Big(\mathrm{g} - \tfrac{\hbar^2}{2m}\hat{D}_2\Big)^{-1}.$$

This expression of $n_1(t,\mathbf{r})$ is again substituted back into the Eq.(3.6) to get a second order partial differential equation in terms of the phase fluctuations, i.e.,

$$\begin{aligned}\partial_t \Big[-\hat{\mathcal{A}}\,\Big(\partial_t \tilde{\vartheta}_1(t,\mathbf{r}) + \tfrac{1}{m}\boldsymbol{\nabla}\vartheta_0(x).\boldsymbol{\nabla}\tilde{\vartheta}_1(t,\mathbf{r})\Big)\Big] \\ +\tfrac{1}{m}\boldsymbol{\nabla}.\Big(\Big[-\hat{\mathcal{A}}\,\Big(\partial_t \tilde{\vartheta}_1(t,\mathbf{r}) + \tfrac{1}{m}\boldsymbol{\nabla}\vartheta_0(t,\mathbf{r}).\boldsymbol{\nabla}\tilde{\vartheta}_1(t,\mathbf{r})\Big)\Big]\boldsymbol{\nabla}\vartheta_0(t,\mathbf{r}) + n_0(t,\mathbf{r})\,\boldsymbol{\nabla}\tilde{\vartheta}_1(t,\mathbf{r})\Big) = 0.\end{aligned} \tag{3.10}$$

As far as setting up the analogue model of gravity is concerned, one usually considers the context of passing a sonic disturbance through a barotropic inviscid fluid[29], the background fluid flow $\mathbf{v}(t,\mathbf{r})$ is considered to be vorticity-free, or in other words, locally irrotational; i.e.,

$$\mathbf{v}(t,\mathbf{r}) = \frac{1}{m}\boldsymbol{\nabla}\vartheta_0(t,\mathbf{r}). \tag{3.11}$$

Clearly, the classical mean-field phase $\vartheta_0(t,\mathbf{r})$ of the BEC state now gets to act as the velocity

[28] Refer to eqn.(242) of [4].

[29] For a detailed justification, refer to [4]-p.9.

potential here. This background velocity being irrotational plays a crucial role in order to determine the metric (which eventually turns out to be *Lorentzian* as seen by the phonons inside the fluid medium, i.e., BEC here) of this particular (3 + 1)D curved spacetime. In the present context, the BEC is completely non-relativistic, i.e. $|\mathbf{v}| << c$ where $c \approx 2.997 \times 10^8$ ms^{-1} is the speed of light in vacuum.

In order to identify the local speed of sound inside the fluid medium, as already mentioned before in Eq.(2.49) in Chapter-2, is given by

$$c_s(t, \mathbf{r}) = \sqrt{\frac{n_0(t, \mathbf{r})\, \mathrm{g}}{m}} \quad \text{(so obviously, } 0 < c_s << c \text{ in magnitude).} \tag{3.12}$$

N.B. If one starts by assuming the density n_0 to be position independent (which is pretty much the argument for considering a canonical acoustic black hole[30] out of an incompressible and spherically symmetric fluid flow), due to the barotropic assumption, the pressure also becomes position independent. Thus for a barotropic fluid, c_s becomes a position independent constant, refer to Eq.(25) of [4]. In our analysis, for the sake of simplicity, we keep c_s as a constant all through. This is an approximation on our part in the present context. We make this approximation here because we don't need to consider the actual structure of the sonic horizon in that details. Obviously, when c_s becomes position dependent, the sonic horizon might have a spread over space. But what we will be basically concerned with later is analysing "secondary" waves (low frequency sonic modes) being generated outside the acoustic horizon by gaining energy from the "primary" waves (high frequency Hawking radiated sonic modes) due to quantum potential induced UV-IR coupling in a (3 + 1)D curved spacetime. Hence, for that reason, we regard c_s to be a position independent constant and this approximation is legit. I'll demonstrate this issue in details in the next chapter(s).

Introducing (3 + 1)D spacetime coordinates, in Cartesian for sake of simplicity,

$$(t, \mathbf{r}) \to (t, x^i) \equiv x^\mu \tag{3.13}$$

(or, just x instead of x^μ when it is used in the argument of any function),

where Roman indices run over only the spatial dimensions 1 − 3 and the Greek indices run from 0 − 3. The 3D velocity is rewritten in terms of its components $v^i(x) = \boldsymbol{\nabla}^i \vartheta_0(x)/m$ and the dynamical equation for the phase fluctuations $\tilde{\vartheta}_1(x)$ [i.e., Eq.(3.10)] can be easily rewritten as

$$\boxed{\partial_\mu f^{\mu\nu} \partial_\nu \tilde{\vartheta}_1(x) = 0}\,, \tag{3.14}$$

[30] I'll discuss this problem later.

where $[f^{\mu\nu}]$ is constructed as a 4×4 matrix given by,

$$[f^{\mu\nu}] = \begin{pmatrix} f^{00} & \vdots & f^{0j} \\ \cdots\cdots & . & \cdots\cdots\cdots\cdots \\ f^{i0} & \vdots & f^{ij} \end{pmatrix}, \tag{3.15}$$

$$\text{where,} \quad f^{00} = -\left(\mathfrak{g} - \frac{\hbar^2}{2m}\hat{D}_2\right)^{-1}, \; f^{0j} = -\left(\mathfrak{g} - \frac{\hbar^2}{2m}\hat{D}_2\right)^{-1} v^j(x),$$
$$f^{i0} = -v^i(x)\left(\mathfrak{g} - \frac{\hbar^2}{2m}\hat{D}_2\right)^{-1}, \; f^{ij} = \frac{n_0(x)\,\delta^{ij}}{m} - v^i(x)\left(\mathfrak{g} - \frac{\hbar^2}{2m}\hat{D}_2\right)^{-1} v^j(x)\,. \tag{3.16}$$

3.1.1 The acoustic approximation

At this stage, this $f^{\mu\nu}$ is not the standard rank $(2, 0)$ contravariant tensor since it doesn't respect Diffeomorphism due to the matrix-elements being operators, but not functions/numbers. In the usual literature, this $\hat{D}_2$ term is neglected on physical ground due to the fact that, in the spectral decomposition of the phase fluctuations $\tilde{\vartheta}_1(x)$, the contributions from the linearization of $\mathsf{V}_{\text{quantum}}$ become negligibly small at wavelengths larger than the healing length (ξ_0) of the system. From Eq.(2.50) in Chapter-2,

$$\xi_0 = \frac{\hbar}{\sqrt{2m\mathfrak{g}n_0}} \sim \frac{\hbar}{mc_s} \Rightarrow \text{analogue Compton wavelength, } \lambda_{c,\,\text{acoustic}}\,. \tag{3.17}$$

With the omission of $\hat{D}_2$ term from the matrix-elements, one comes up with the momentum independent representation of $[f^{\mu\nu}]$. From Eq.(3.16), in $(3+1)$D Cartesian, it is given by

$$[f^{\mu\nu}]|_{\hat{D}_2 \to 0} \equiv [f^{\mu\nu}_{\text{New}}(x)] = \frac{1}{\mathfrak{g}} \begin{pmatrix} -1 & -v_1(x) & -v_2(x) & -v_3(x) \\ -v_1(x) & \left(c_s^2 - v_1^2(x)\right) & -v_1(x)\,v_2(x) & -v_1(x)\,v_3(x) \\ -v_2(x) & -v_2(x)\,v_1(x) & \left(c_s^2 - v_2^2(x)\right) & -v_2(x)\,v_3(x) \\ -v_3(x) & -v_3(x)\,v_1(x) & -v_3(x)\,v_2(x) & \left(c_s^2 - v_3^2(x)\right) \end{pmatrix}. \tag{3.18}$$

3.1.2 Covariant picture: massless minimally coupled Klein-Gordon field

It is required to identify the corresponding covariant structure of the dynamics in order to uphold the Lorentz invariance that by default comes out to be a striking feature from an underlying non-relativistic condensed matter tool and hence the introduction of the effective metric (or in other words, acoustic metric) in place of the respective f-matrix is essential.

Say, $[\mathsf{g}_{\mu\nu}(x)]$ be the general acoustic metric that actually defines the (3 + 1)D curved/flat spacetime under consideration of its determinant, given by, $\mathsf{g} = \det[\mathsf{g}_{\mu\nu}(x)]$. Considering Eq.(3.18), one identifies

$$f^{\mu\nu}_{\text{New}} = \sqrt{|\mathsf{g}|}\mathsf{g}^{\mu\nu}, \tag{3.19}$$

$$\begin{aligned} \Rightarrow \quad \det[f^{\mu\nu}_{\text{New}}] = \det[\sqrt{|\mathsf{g}|}\mathsf{g}^{\mu\nu}] = (\sqrt{|\mathsf{g}|})^4 \det[\mathsf{g}^{\mu\nu}] = (\sqrt{|\mathsf{g}|})^4 \mathsf{g}^{-1} = \mathsf{g}. \\ \because \det[f^{\mu\nu}_{\text{New}}] = -\frac{c_s^6}{g^4}, \quad \therefore\ \mathsf{g} = -\frac{c_s^6}{g^4}, \\ \text{and obviously,} \quad \mathsf{g}^{\mu\nu} = \frac{1}{\sqrt{|\mathsf{g}|}} f^{\mu\nu}_{\text{New}} = \frac{g^2}{c_s^3} f^{\mu\nu}_{\text{New}}. \end{aligned} \tag{3.20}$$

Now, from Eq.(3.20), it is just trivial to find the acoustic metric, which is of the following form,

$$[\mathsf{g}_{\mu\nu}(x)] = \frac{c_s}{g} \begin{pmatrix} -\left(c_s^2 - \mathbf{v}^2(x)\right) & -v_1(x) & -v_2(x) & -v_3(x) \\ -v_1(x) & 1 & 0 & 0 \\ -v_2(x) & 0 & 1 & 0 \\ -v_3(x) & 0 & 0 & 1 \end{pmatrix}. \tag{3.21}$$

It should be noted that, in general relativity, the spacetime metric (which does bear the feature of the background geometry) is related to the distribution of matter (i.e. the stress-energy tensor) through the Einstein's-Field-Equations; whereas, the acoustic metric $[\mathsf{g}_{\mu\nu}(x)]$ here happens to be related to the background velocity field $[\mathbf{v(r)}]$ as well as the local speed of sound (c_s) in a way more simpler algebraic fashion. Some striking features of this $[\mathsf{g}_{\mu\nu}(x)]$ from topological aspect and regarding 'stable causality' have been discussed by Visser in [21]-pp.1773-1774.

Finally, Eq.(3.14) is rewritten in the standard covariant form, given by,

$$\frac{1}{\sqrt{|\mathsf{g}|}}\partial_\mu\left(\sqrt{|\mathsf{g}|}\mathsf{g}^{\mu\nu}\partial_\nu\right)\tilde{\vartheta}_1(x)=0,$$
$$\text{i.e.}\quad \boxed{\nabla_\mu\nabla^\mu\tilde{\vartheta}_1(x)=0}, \tag{3.22}$$

where ∇_μ is obviously the covariant derivative. Here $\tilde{\vartheta}_1(x)$ has emerged as the massless minimally coupled Klein-Gordon field in $(3+1)$D curved/flat background. With the acoustic metric $[\mathsf{g}_{\mu\nu}(x)]$ now in hand, the analogy is fully established and the platform to investigate newer physics within the scope of Analogue Gravity from BEC is set up.

3.2 GP theory of BEC with *nonlocal* interactions

Due to the possibility of increasing the s-wave scattering length practically from $-\infty$ to ∞ near a Feshbach resonance, which has already been experimentally achieved [80], one can take nonlocal s-wave scattering into account by considering an effective interaction potential of the form[31]

$$V_{\text{eff}}(\mathbf{r}'-\mathbf{r})=\frac{\mathfrak{g}}{(\sqrt{2\pi}a)^{\aleph}}e^{-\frac{|\mathbf{r}'-\mathbf{r}|^2}{2a^2}}. \tag{3.23}$$

Like Eq.(2.56), the interaction term ($\mathbb{T}_{\text{int}}$) in Eq.(2.55) can be calculated in details in the following manner in order to incorporate nonlocality at its minimal level. We have

$$\mathbb{T}_{\text{int}}=\int_{\mathbf{r}'=\mathbf{r}}^{\infty}d\mathbf{r}'\;|\psi(t,\mathbf{r}')|^2\;\underline{V(\mathbf{r}'-\mathbf{r})}\equiv\int_{\mathbf{r}'=\mathbf{r}}^{\infty}d\mathbf{r}'\;|\psi(t,\mathbf{r}')|^2\;\underline{V_{\text{eff}}(\mathbf{r}'-\mathbf{r})}. \tag{3.24}$$

Now one can Taylor expand $|\psi(t,\mathbf{r}')|^2$ in 3D about the point $\mathbf{r}'=\mathbf{r}$ given by

$$|\psi(t,\mathbf{r}')|^2=|\psi(t,\mathbf{r})|^2+(\mathbf{r}'-\mathbf{r}).\boldsymbol{\nabla}'|\psi(t,\mathbf{r}')|^2\Big|_{\mathbf{r}'=\mathbf{r}}+\frac{1}{2}\Big((\mathbf{r}'-\mathbf{r}).\boldsymbol{\nabla}'\Big)^2|\psi(t,\mathbf{r}')|^2\Big|_{\mathbf{r}'=\mathbf{r}}+\ldots \tag{3.25}$$

Inserting Eq.(3.25) into Eq.(3.24), one obtains

[31] In Eq.(3.23), $\aleph$ is the spatial dimension in which the interactions are considered. Evidently, throughout our analysis, $\aleph=3$.

$$\mathbb{T}_{\rm int} = \int_{\mathbf{r}'=\mathbf{r}}^{\infty} d\mathbf{r}' \left(|\psi(t,\mathbf{r})|^2 + (\mathbf{r}'-\mathbf{r}).\boldsymbol{\nabla}'|\psi(t,\mathbf{r}')|^2\Big|_{\mathbf{r}'=\mathbf{r}} + \frac{1}{2}\Big((\mathbf{r}'-\mathbf{r}).\boldsymbol{\nabla}'\Big)^2|\psi(t,\mathbf{r}')|^2\Big|_{\mathbf{r}'=\mathbf{r}} + \ldots \right) \times V_{\rm eff}(\mathbf{r}'-\mathbf{r})$$

$$= \int_{\mathbf{r}'=\mathbf{r}}^{\infty} d\mathbf{r}' \left(|\psi(t,\mathbf{r})|^2 + (\mathbf{r}'-\mathbf{r}).\boldsymbol{\nabla}'|\psi(t,\mathbf{r}')|^2\Big|_{\mathbf{r}'=\mathbf{r}} + \frac{1}{2}\Big((\mathbf{r}'-\mathbf{r}).\boldsymbol{\nabla}'\Big)^2|\psi(t,\mathbf{r}')|^2\Big|_{\mathbf{r}'=\mathbf{r}} + \ldots \right) \times \frac{\mathfrak{g}}{(\sqrt{2\pi}a)^3} e^{-\frac{|\mathbf{r}'-\mathbf{r}|^2}{2a^2}}$$

[using Eq.(3.23)]

$$= \mathbb{T}_{\rm int}^{(0)} + \mathbb{T}_{\rm int}^{(1)} + \mathbb{T}_{\rm int}^{(2)} + \cdots \quad \text{(say.)} \tag{3.26}$$

For convenience and simplicity, we choose 3D Cartesian coordinates to work out further and evaluate the three terms identified just above [Eq.(3.26)].

$$\begin{aligned}
\mathbb{T}_{\rm int}^{(0)} &= \int_{\mathbf{r}'=\mathbf{r}}^{\infty} d\mathbf{r}'\, |\psi(t,\mathbf{r})|^2 \frac{\mathfrak{g}}{(\sqrt{2\pi}a)^3} e^{-\frac{|\mathbf{r}'-\mathbf{r}|^2}{2a^2}} = |\psi(t,\mathbf{r})|^2 \frac{\mathfrak{g}}{(\sqrt{2\pi}a)^3} \int_{\mathbf{r}'=\mathbf{r}}^{\infty} d\mathbf{r}'\, e^{-\frac{|\mathbf{r}'-\mathbf{r}|^2}{2a^2}} \\
&= |\psi(t,\mathbf{r})|^2 \frac{\mathfrak{g}}{(\sqrt{2\pi}a)^3} \int_{-\infty}^{\infty} dx' e^{-\frac{|x'-x|^2}{2a^2}} \int_{-\infty}^{\infty} dy' e^{-\frac{|y'-y|^2}{2a^2}} \int_{-\infty}^{\infty} dz' e^{-\frac{|z'-z|^2}{2a^2}} \\
&= |\psi(t,\mathbf{r})|^2 \frac{\mathfrak{g}}{(\sqrt{2\pi}a)^3} \left(\int_{-\infty}^{\infty} dx' e^{-\frac{|x'-x|^2}{2a^2}} \right)^3 = |\psi(t,\mathbf{r})|^2 \frac{\mathfrak{g}}{(\sqrt{2\pi}a)^3} \left(\sqrt{2\pi}a\right)^3 \\
&= \mathfrak{g}|\psi(t,\mathbf{r})|^2 \ .
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
\mathbb{T}_{\rm int}^{(1)} &= \int_{\mathbf{r}'=\mathbf{r}}^{\infty} d\mathbf{r}' \left((\mathbf{r}'-\mathbf{r}).\boldsymbol{\nabla}'|\psi(t,\mathbf{r}')|^2\Big|_{\mathbf{r}'=\mathbf{r}} \right) \frac{\mathfrak{g}}{(\sqrt{2\pi}a)^3} e^{-\frac{|\mathbf{r}'-\mathbf{r}|^2}{2a^2}} \\
&= \frac{\mathfrak{g}}{(\sqrt{2\pi}a)^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'\, dy'\, dz' \Big((x'-x)\,\partial_{x'} + (y'-y)\,\partial_{y'} + (z'-z)\,\partial_{z'} \Big) |\psi(t,\mathbf{r}')|^2\Big|_{\mathbf{r}'=\mathbf{r}} \times e^{-\frac{|\mathbf{r}'-\mathbf{r}|^2}{2a^2}} \\
&= \frac{\mathfrak{g}}{(\sqrt{2\pi}a)^3} \times 3\Bigg(\partial_x |\psi(t,\mathbf{r})|^2 \underbrace{\int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dx' \ (x'-x)\, e^{-\frac{|\mathbf{r}'-\mathbf{r}|^2}{2a^2}}}_{=\int_{-\infty}^{\infty} d\tilde{z} \int_{-\infty} d\tilde{y} \int_{-\infty}^{\infty} d\tilde{x}\ \ \tilde{x}\, e^{-\frac{\tilde{x}^2+\tilde{y}^2+\tilde{z}^2}{2a^2}}} \Bigg),
\end{aligned}$$

[assuming, $\tilde{x} = x' - x,\ \tilde{y} = y' - y,\ \tilde{z} = z' - z$]

where the above under-braced part vanishes since the integrand is an odd function with the integration limits being from $-\infty$ to $+\infty$ and hence

$$\mathbb{T}^{(1)}_{\text{int}} = 0. \tag{3.28}$$

It is evident that all the odd-higher-order terms, viz. $\mathbb{T}^{(3)}_{\text{int}}$, $\mathbb{T}^{(5)}_{\text{int}}$, ... etc. would also vanish likewise. It is a characteristic feature of the symmetry of the potential, or in other words, the s-wave scattering.

Now,

$$\begin{aligned}
\mathbb{T}^{(2)}_{\text{int}} &= \int_{\mathbf{r}'=\mathbf{r}}^{\infty} d\mathbf{r}' \left(\frac{1}{2}\Big((\mathbf{r}'-\mathbf{r}).\boldsymbol{\nabla}'\Big)^2 |\psi(t,\mathbf{r}')|^2\Big|_{\mathbf{r}'=\mathbf{r}}\right) \frac{\mathfrak{g}}{(\sqrt{2\pi}a)^3} e^{-\frac{|\mathbf{r}'-\mathbf{r}|^2}{2a^2}} \\
&= \frac{\mathfrak{g}}{(\sqrt{2\pi}a)^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'\, dy'\, dz'\ \frac{1}{2} \underbrace{\Big((x'-x)\,\partial_{x'} + (y'-y)\,\partial_{y'} + (z'-z)\,\partial_{z'}\Big)^2} |\psi(t,\mathbf{r}')|^2\Big|_{\mathbf{r}'=\mathbf{r}} \\
&\qquad \times\ e^{-\frac{|\mathbf{r}'-\mathbf{r}|^2}{2a^2}} \\
&= \frac{\mathfrak{g}}{(\sqrt{2\pi}a)^3}\frac{1}{2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'\, dy'\, dz'\ \overbrace{\Big((x'-x)^2\,\partial_{x'x'} + (y'-y)^2\,\partial_{y'y'} + (z'-z)^2\,\partial_{z'z'}\Big)} \\
&\qquad \times\ |\psi(t,\mathbf{r}')|^2\Big|_{\mathbf{r}'=\mathbf{r}} e^{-\frac{|\mathbf{r}'-\mathbf{r}|^2}{2a^2}}
\end{aligned}$$

(Expanding the under-bracketted part, the cross-terms vanish individually and hence the over-bracketted part is left.)

$$\begin{aligned}
&= \frac{\mathfrak{g}}{(\sqrt{2\pi}a)^3}\frac{1}{2}\Big(\partial_{xx}+\partial_{yy}+\partial_{zz}\Big)|\psi(t,\mathbf{r})|^2 \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'\, dy'\, dz'\ (x'-x)^2\ e^{-\frac{|\mathbf{r}'-\mathbf{r}|^2}{2a^2}}}_{=\int_{-\infty}^{\infty} d\tilde{z}\ e^{-\frac{\tilde{z}^2}{2a^2}} \int_{-\infty}^{\infty} d\tilde{y}\ e^{-\frac{\tilde{y}^2}{2a^2}} \int_{-\infty}^{\infty} d\tilde{x}\ \ \tilde{x}^2\ e^{-\frac{\tilde{x}^2}{2a^2}}}
\end{aligned}$$

[assuming, $\tilde{x} = x'-x,\ \tilde{y} = y'-y,\ \tilde{z} = z'-z$]

$$\begin{aligned}
&= \frac{\mathfrak{g}}{(\sqrt{2\pi}a)^3}\frac{1}{2}\boldsymbol{\nabla}^2|\psi(t,\mathbf{r})|^2 \times \Big((a\sqrt{2\pi})^2\ (a^3\sqrt{2\pi})\Big) \\
&= \frac{1}{2}a^2\mathfrak{g}\boldsymbol{\nabla}^2|\psi(t,\mathbf{r})|^2\ .
\end{aligned} \tag{3.29}$$

Using Eqs.(3.27), (3.28) and (3.29), the full interaction term in Eq.(3.26) is given by

$$\mathbb{T}_{\text{int}} = \mathfrak{g}|\psi(t,\mathbf{r})|^2 + 0 + \frac{1}{2}a^2\mathfrak{g}\boldsymbol{\nabla}^2|\psi(t,\mathbf{r})|^2 + \cdots \tag{3.30}$$

3.2.1 Our proposed model

In order to incorporate the non-locality, here we have considered the first minimal correction (through a 3D Gaussian interaction) on top of the standard local picture within the scope of GP theory. Thus from the general GP equation [i.e., Eq.(2.55)], given by,

$$i\hbar\partial_t\psi(t,\mathbf{r}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(t,\mathbf{r}) + \underbrace{\int \psi^*(t,\mathbf{r}')\,V(\mathbf{r}'-\mathbf{r})\,\psi(t,\mathbf{r}')\,d\mathbf{r}'}_{\text{say, } \mathbb{T}_{\text{int}}}\right)\psi(t,\mathbf{r})\,,$$

the minimal GP model for a BEC with a correction due to nonlocality in 3D Cartesian system, in presence of the s-wave scattering, is characterized by

$$\boxed{i\hbar\partial_t\,\psi(t,\mathbf{r}) = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{\text{ext}}(t,\mathbf{r}) + \mathrm{g}|\psi(t,\mathbf{r})|^2\right)\psi(t,\mathbf{r}) + \underbrace{\frac{1}{2}a^2\mathrm{g}\,\psi(t,\mathbf{r})\,\nabla^2|\psi(t,\mathbf{r})|^2}_{\text{minimal correction for nonlocality}}}\,, \qquad (3.31)$$

where $\mathbb{T}_{\text{int}} \simeq \mathrm{g}|\psi(t,\mathbf{r})|^2 + \frac{1}{2}a^2\mathrm{g}\nabla^2|\psi(t,\mathbf{r})|^2$ up to minimal correction due to nonlocal interactions from Eq.(3.30).

A change of symmetry might change the small numerical pre-factor of the last term on r.h.s. in the above Eq.(3.31) without affecting the dependence on s-wave scattering length a whose critical value would be determined by the condensate density n which is in fact a very large number (usually $\sim 10^{21}$ m.$^{-3}$). Therefore, this numerical pre-factor doesn't have any major impact on any of the physical results and outcomes. In the following section(s)/chapter(s), we would replace this $1/2$-factor by some κ of $O(1)$ to keep things more general and independent of any specific coordinate systems.

It should be noted that even when Feshbach resonance is not there to increase a, still to look at the dynamics at smaller length scales (with greater accuracy), one must consider the first minimal correction term to the standard δ-interaction configuration which is obviously localized. In this regard, our present treatment stands as more legit as given the fact that there already exists a Laplacian in the kinetic term of the dynamics and thus the minimal nonlocal correction term which is also of the same order should not be ignored and omitted.

3.2.2 A few features of this nonlocal GP model for a nonrelativistic BEC

Let us identify a few properties of the Eq.(3.31).

- Like in case of local GP equation, it (see Eq.(2.58) in Chapter-2) can also be derived from a free energy functional, given by,

$$E = \int d\mathbf{r} \left[\frac{\hbar^2}{2m} |\nabla\psi(t,\mathbf{r})|^2 + V_{\text{ext}}(t,\mathbf{r})\, n(t,\mathbf{r}) + \frac{g}{2}\Big(|\psi(t,\mathbf{r})| - n(t,\mathbf{r})\Big)^2 + \kappa\, a^2 g\, n(t,\mathbf{r})\, \nabla n(t,\mathbf{r}) \right], \tag{3.32}$$

where, as already mentioned, κ is just a numeric constant of $O(1)$ and is replaced by a factor of $\frac{1}{2}$ in 3D by Pendse in his recent paper [81]. Plugging this above E in Eq.(2.59), given by,

$$i\hbar\frac{\partial\psi(t,\mathbf{r})}{\partial t} = \frac{\delta E}{\delta\psi^*(t,\mathbf{r})},$$

one lands up with our proposed model, i.e., Eq.(3.31) for $\kappa = 1/2$. This result is non-trivial in the sense that this corrections exist only for s-wave scattering for corresponding spherically symmetric effective potential.

- Secondly and very importantly, the spatially uniform but oscillating ground state (ψ_0) solution[32] of the local GP Eq.(2.57) is still a solution of the system. Let's discuss this issue briefly here. If $\psi_0(t,\mathbf{r})$ has to be the ground state solution of the nonlocal GP Eq.(3.31), then

$$i\hbar\partial_t\, \psi_0(t,\mathbf{r}) = \Big(-\tfrac{\hbar^2\nabla^2}{2m} + V_{\text{ext}}(t,\mathbf{r}) + g\, n_0(t,\mathbf{r})\Big)\, \psi_0(t,\mathbf{r}) + \kappa\, a^2 g\, \psi_0(t,\mathbf{r})\, \nabla^2 n_0(t,\mathbf{r}), \tag{3.33}$$

where, $\quad n_0(t,\mathbf{r}) = |\psi_0(t,\mathbf{r})|^2$ is the density of the BEC ground state.

$$\Rightarrow \Big(\psi_0^*(t,\mathbf{r}) \times \text{Eq.(3.33)} - c.c.\Big) = 0,$$

$$\Rightarrow \partial_t n_0(t,\mathbf{r}) + \nabla.\mathbf{j}(t,\mathbf{r}) = 0, \tag{3.34}$$

which is nothing but the continuity equation (see Eq.(5.9) of [42]) by identifying the current density to be of the following form

$$\mathbf{j}(t,\mathbf{r}) = -\frac{i\hbar}{2m}\Big(\psi_0^*(t,\mathbf{r})\, \nabla\psi_0(t,\mathbf{r}) - \text{c.c.}\Big) \equiv \frac{n_0(t,\mathbf{r})}{m}\nabla\vartheta_0(t,\mathbf{r}). \tag{3.35}$$

Here $\vartheta_0(t,\mathbf{r})$ is the phase of the BEC ground state

$$\psi_0(t,\mathbf{r}) = \sqrt{n_0(t,\mathbf{r})}\, e^{i\vartheta_0(t,\mathbf{r})/\hbar} \tag{3.36}$$

and, in comparison with Eq.(5.10) of [42], the current density $\mathbf{j}(t,\mathbf{r})$ remains unaltered wrt that in case of local GP theory.

[32] Refer to Eq.(5.11) of [42].

In case of stationary solutions, the BEC ground state evolves in time as

$$\psi_0(t, \mathbf{r}) = \psi_0'(\mathbf{r}) e^{-i\mu t/\hbar}, \quad \text{where, } \mu = \frac{\partial E}{\partial N} \tag{3.37}$$

is the chemical potential of the system and ψ_0' is some time-independent complex function. Plugging Eq.(3.37) into Eq.(3.33), one gets

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r}) - \mu + \mathrm{g}|\psi_0'(\mathbf{r})|^2 + \kappa a^2 \mathrm{g}\, \nabla^2 |\psi_0'(\mathbf{r})|^2\right) \psi_0'(\mathbf{r}) = 0, \tag{3.38}$$

by considering the external potential to be independent of time. A mathematically rigorous proof that the above Eq.(3.38) (without the nonlocal correction term) adopts a solution that is the lowest energy state of the dilute BEC where bosons interacting with repulsive forces is provided by Lieb in 2000 [74].

- For a uniform Bose gas, $\psi_0'(\mathbf{r})$ is merely a constant and hence the above Eq.(3.38) trivially gives

$$\mu = \mathrm{g}|\psi_0'|^2 = \mathrm{g}\, n_0. \tag{3.39}$$

Therefore the conservation of mass is an essential ingredient which is kept preserved despite the addition of the term representing the non-locality to the standard local GP model.

3.3 1D small-amplitude excitations for nonlocal BEC

From the evaluation of $\mathbb{T}^{(2)}_{\text{int}}$ in the Eq.(3.30), it is trivial to obtain the corresponding expression in 1D Cartesian coordinates. Hence one writes the GP equation for 1D inhomogeneities with a minimal nonlocal correction term as,

$$i\hbar\partial_t\, \psi(t, \mathbf{r}) = \left(-\frac{\hbar^2}{2m}\partial_x^2 + V_{\text{ext}}(t, \mathbf{r}) + \mathrm{g}|\psi(t, \mathbf{r})|^2\right) \psi(t, \mathbf{r}) + \underbrace{\boxed{\frac{1}{6}}}_{\kappa = 1/6} a^2 \mathrm{g}\, \psi(t, \mathbf{r})\, \partial_x^2 |\psi(t, \mathbf{r})|^2, \tag{3.40}$$

which effectively considers that, the $\psi(t, \mathbf{r})$ is uniform on the $y - z$ plane (for sake of simplicity) and also that, there are no small amplitude global modes with spherical or circular symmetry in the system for the unavoidable spatial dependences of the amplitude of such modes.

We are not considering the interaction potential to be a δ-function. We are considering a to be not small (but finite) and that legitimizes the inclusion of the correction (nonlinear) term with a second order derivative in the above Eq.(3.40) [or, Eq.(3.31)]. Since, there already exists a term with a second order spatial derivative in the local GP equation, the slowness of density variation over space cannot prevent the additional term from appearing in it. The slowness of the spatial variation of the density is considered here to help drop the higher order terms of the expansion. We would like to emphasize the fact that, this is the first minimal correction term that could be added, keeping the GP dynamics local, in order to incorporate the effect of non-locality of the interactions on top of the δ-correlated interaction. At the limit $a \to 0$ (corresponding to the limit taken by Lieb *et al* [74, 75]), this additional term will obviously disappear and we get back the traditional local GP equation (i.e. Eq.(5.2) of [42]). Apart from relaxing the width of the δ-potential, in all other respect, we are following the standard assumptions of the local GP theory. Moreover, our model is also a local one where the extra term brings in the effect of non-locality to some extent.

Small-amplitude oscillations are indeed one important and significant class of time-dependent solutions of the GP (local/nonlocal) equation. The small-amplitude oscillations are the elementary excitations of the system and they admit a natural quantum description [42].

The BEC order parameter is perturbed around the ground state (equilibrium) in terms of the small-amplitude fluctuations as

$$\psi(t,\mathbf{r}) \equiv \psi(t,x) = \left(\psi_0(x) + \underbrace{\sum_j \left[u_j(x) e^{-i\omega_j t} + v_j^*(x) e^{i\omega_j t} \right]}_{\text{say,}\ \ \wp(t,x)} \right) e^{-\frac{i\mu t}{\hbar}}, \tag{3.41}$$

where μ is the chemical potential of the interacting Bose gas[33], ω_j is the oscillation frequency of the j^{th} excited state. The BEC ground state is a metastable state and is dynamically stable as far as

$$\int d\mathbf{r}\, |u_j|^2 \neq \int d\mathbf{r}\, |v_j|^2 \tag{3.42}$$

holds good[34]. From Eq.(3.41), the condensate density is given by the ground state occupancy,

$$|\psi_0(x)|^2 = n(x), \qquad \therefore\ |\psi(t,x)|^2 \simeq \Big(n(x) + \psi_0(x)\, \wp^*(t,x) + \psi_0^*(x)\, \wp(t,x) \Big). \tag{3.43}$$

[33] Refer to Eq.(4.16) of [42].

[34] Refer to Eq.(5.7167) of [42].

3.3.1 Modified Bogoliubov spectrum in 1D

Plugging Eq.(3.41) in Eq.(3.40), and collecting all the terms with their respective factors $e^{-i\omega_j t/\hbar}$ and $e^{i\omega_j t/\hbar}$, one comes up with the following pair of coupled differential equations,

$$\begin{aligned}\hbar\omega_j u_j(x) &= \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_{\text{ext}} - \mu + 2\text{g}n(x)\right)u_j(x) + \text{g}\left(\psi_0(x)\right)^2 v_j(x) \\ &\quad + \frac{1}{6}a^2\text{g}\left(\psi_0(x) + u_j(x)\right)\frac{\partial^2}{\partial x^2}\left(n(x) + \psi_0^*(x)u_j(x) + \psi_0(x)v_j(x)\right), \qquad (3.44a)\\ -\hbar\omega_j v_j(x) &= \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_{\text{ext}} - \mu + 2\text{g}n(x)\right)v_j(x) + \text{g}\left(\psi_0^*(x)\right)^2 u_j(x) \\ &\quad + \frac{1}{6}a^2\text{g}\left(\psi_0^*(x) + v_j(x)\right)\frac{\partial^2}{\partial x^2}\left(n(x) + \psi_0^*(x)u_j(x) + \psi_0(x)v_j(x)\right). \qquad (3.44b)\end{aligned}$$

The above Eq.(3.44a) is obtained by setting the total coefficient of $e^{-i\omega_j t/\hbar}$ to zero while Eq.(3.44b) is obtained by setting the coefficient of $e^{i\omega_j t/\hbar}$ to zero followed by a c.c. on both sides. This formalism of linearized small-amplitude fluctuations was first developed by Pitaevskii in 1961 [73] in order to demonstrate the oscillatory motions of a quantum vortex line.

These solutions in fact give the characteristic nature of the collective phenomena exhibited by the interacting Bose gases. For simplicity, we restrict ourselves within the regime of uniformity and in the absence of any trapping potential; i.e.,

$$\begin{aligned}V_{\text{ext}} &= 0,\\ \psi_0 = \sqrt{n} = \psi_0^* \quad &(\because n(x) = \text{const.} = n),\\ \text{and} \quad \mu &= \text{g}n. \end{aligned} \qquad (3.45)$$

Thus from Eq.(3.44), after linearizing the dynamics, one can get the following,

$$\begin{aligned}\hbar\omega_j u_j(x) &= \text{g}n u_j(x) + \left(\frac{a^2\text{g}n}{6} - \frac{\hbar^2}{2m}\right)u_j''(x) + \text{g}n v_j(x) + \frac{a^2\text{g}n}{6}v_j''(x), \qquad (3.46a)\\ -\hbar\omega_j v_j(x) &= \text{g}n v_j(x) + \left(\frac{a^2\text{g}n}{6} - \frac{\hbar^2}{2m}\right)v_j''(x) + \text{g}n u_j(x) + \frac{a^2\text{g}n}{6}u_j''(x)\ , \qquad (3.46b)\end{aligned}$$

where our notation is $u_j''(x) = \frac{\partial^2}{\partial x^2}u_j(x)$. Considering the ansatz

$$u_j(x) = u_j e^{ikx}, \qquad v_j(x) = v_j e^{ikx}, \qquad (3.47)$$

one simplifies the above Eqs.(3.46) and obtains the following condition for which there exists a nontrivial unique set of solutions for u_j & v_j. For any general set of u & v, it is given by,

$$\underbrace{\begin{pmatrix} \left(\mathfrak{g}n - \frac{1}{6}a^2\mathfrak{g}nk^2 + \frac{\hbar^2k^2}{2m} - \hbar\omega\right) & \left(\mathfrak{g}n - \frac{1}{6}a^2\mathfrak{g}nk^2\right) \\ \left(\mathfrak{g}n - \frac{1}{6}a^2\mathfrak{g}nk^2\right) & \left(\mathfrak{g}n - \frac{1}{6}a^2\mathfrak{g}nk^2 + \frac{\hbar^2k^2}{2m} + \hbar\omega\right) \end{pmatrix}}_{\text{the determinant of this matrix has to vanish for a nontrivial \& unique set of solutions for } u \text{ and } v} \begin{pmatrix} u \\ v \end{pmatrix} = 0,$$

$$\text{Therefore,} \quad \boxed{\hbar^2\omega^2 = \frac{\hbar^2k^2\mathfrak{g}n}{m} + \left(\frac{\hbar^4}{4m^2} - \frac{\hbar^2a^2\mathfrak{g}n}{6m}\right)k^4.} \tag{3.48}$$

The standard "Bogoliubov dispersion" (**BD**) law is given by Eq.(4.31)/(5.70) of [42] and the above dispersion is indeed modified wrt that due to the presence of the nonlocal correction term (in 1D) in Eq.(3.40). This "modified Bogoliubov dispersion" (**MBD**) immediately tells us that, upto the limit

$$\frac{4\pi}{3}a^3 = \frac{1}{2n}, \quad \Rightarrow \text{(limiting value of } a\text{:)} \;\; a \to \tilde{a} = \left(\frac{8}{3}\pi n\right)^{-1/3}, \tag{3.49}$$

the qualitative nature of the dispersion relation remains the same as that of the BD specturm. The excitation energy of MBD, however, reduces at larger k compared to the BD as a approaches its limiting value $\tilde{a} = \left(\frac{8}{3}\pi n\right)^{-1/3}$. The possibility of keeping the MBD linear by tuning the s-wave scattering length a is the main point of interest from the perspective of BEC as an analogue gravity system.

For an $a > \tilde{a}$, the sign of the coefficient of the quartic term in k in Eq.(3.48) would be negative and more than two-body interactions might be important at this stage if superposition of two-body interactions practically fails to accommodate the physics. The bending of the dispersion curve at long-range interactions shows some tendency to the roton minimum, but, in the present context we are confined to 1D and do not take into account the structures of the interaction potential which stabilizes excitations and brings the dispersion curve back upwards again. However, the indication that a long range interaction is a generic reason for the creation of another minimum for small wavelength excitations is present here.

3.4 Healing length for the nonlocal BEC, ξ

Let us have a look at the change in the healing length (ξ_0 of the local GP model) which is instrumental in demarcating the phonon length scales from the particle like excitations. The

BD spectrum is given by

$$\hbar^2\omega^2 = \frac{\hbar^2 k^2 \mathrm{g} n}{m} + \frac{\hbar^4 k^4}{4m^2}. \tag{3.50}$$

At small k, this dispersion relation takes the form of a phonon dispersion relation $\hbar\omega = |\mathbf{p}|c_s$, where $|\mathbf{p}| = \hbar k$. The healing length indicates the point of transition from the phonon spectrum to the particle spectrum where one considers the kinetic and the potential energy balance as $\frac{p^2}{2m} = \frac{\hbar^2}{2m\xi_0^2} \simeq \mathrm{g}n$. This relation gives a healing length

$$\xi_0 = \frac{\hbar}{\sqrt{2m\mathrm{g}n}} = \frac{\hbar}{\sqrt{2}mc_s} = \frac{1}{\sqrt{8\pi a n}}. \tag{3.51}$$

The relationship from which the healing length is derived, being a balance between the kinetic and the potential energy of excitations, indicates that excitations with a smaller length scale than the healing length are treated as particles (wave packets with mass). Phonon-like excitations typically have wavelengths larger than the healing length. In the modified model, to find the healing length (say, some ξ which is different from ξ_0), we have to consider a balance between the quadratic and the quartic terms in the wave number on the right hand side of Eq.(3.48) considering

$$p^2 c_s^2 = \left(\frac{1}{4m^2} - \frac{a^2 \mathrm{g} n}{6m\hbar^2}\right) p^4, \tag{3.52}$$

we get

$$\xi = \xi_0 \left(\tfrac{1}{2} - \tfrac{4}{3}\pi a^3 n\right)^{1/2} \qquad \text{where, } \xi_0 = \tfrac{1}{\sqrt{8\pi a n}} \quad \text{[from Eq.(2.50)],} \tag{3.53a}$$

$$\text{thus more generally,} \quad \boxed{\xi = \xi_0 \left(\frac{1}{2} - 8\kappa\pi a^3 n\right)^{1/2} = \xi_0\,\epsilon}. \tag{3.53b}$$

Here $\epsilon = \left(\frac{1}{2} - 8\kappa\pi a^3 n\right)^{1/2}$ is set to be a very small quantity that is used as a parameter. The s-wave scattering length a can practically be increased from $-\infty$ to ∞ near a *Feshbach resonance* as experimentally verified by Cornish *et al.* in 2000 [80]. Evidently, the tuning of a will keep increasing or decreasing the value of the parameter ϵ as per requirement [1, 82]. In fact, by increasing a through Feshbach resonance, the value of $8\pi\kappa a^3 n_0$ can be made as close to $\frac{1}{2}$ as possible and hence, naturally, ϵ can be experimentally set to be a very small quantity via Eq.(2.11). So, the healing length (ξ) of the MBD decreases with the increase in a which is a much rapid decrease for a constant n than the $a^{-1/2}$ scale of decrease obtained from the conventional local GP model. In fact, where the healing length (ξ_0) scaling as $a^{-1/2}$ at a constant n becomes zero at $a \to \infty$, but here it becomes zero at an $a \sim n^{-1/3}$; i.e., at a finite s-wave scattering length, $\tilde{a} = (8\pi n/3)^{-1/3}$,

$$\lim_{a \to \tilde{a}} \xi = 0. \tag{3.54}$$

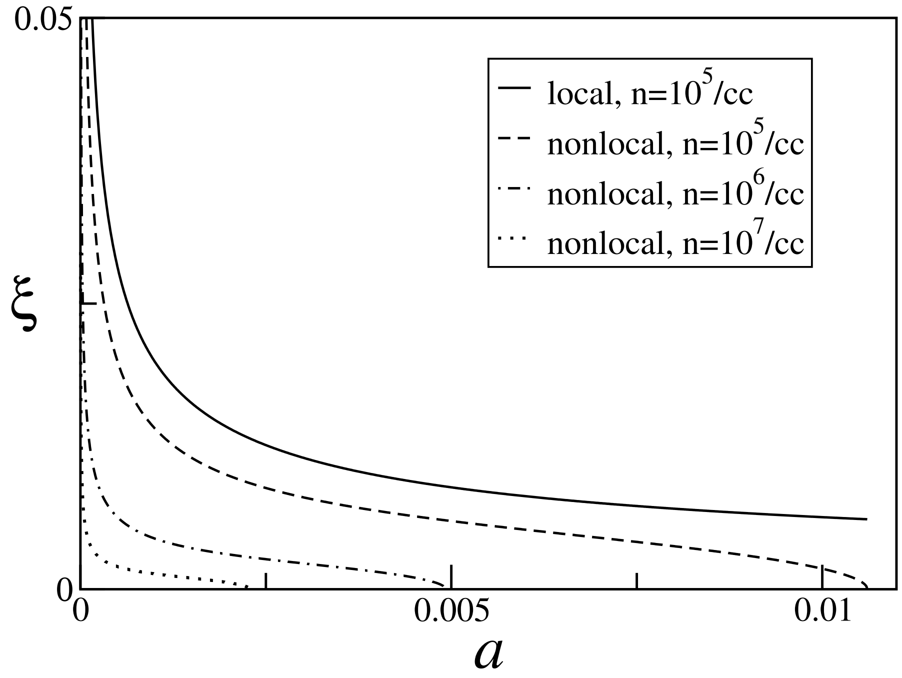

Figure 3.1: Comparison of the changes of healing length for local (ξ_0) versus nonlocal (ξ) BEC (for $\kappa = 1/6$) with the change of s-wave scattering length a at various concentrations (n).

In Fig.2.1, we compare the variation of healing length with a for both local (ξ_0) and nonlocal (ξ) cases at various densities of the condensate. The equivalent way of looking at the present scenario is an increase of the effective mass of the particles with an enhancement of the scattering length. It is the Laplacian term appearing as the minimal correction with an opposite sign to the kinetic energy term that renormalizes the effective mass. Interesting to note that, this increase of the effective mass of the particles is not affecting the expression of the velocity of the sound wave in the BEC which comes from the interactions itself. It is the δ-function (local) interaction that fixes the velocity of sound just as in the local GP equation. Thus, the velocity of sound scales as $\sqrt{a}$ at a constant n and this finite velocity of sound at a vanishing of the healing length ($\xi \to 0$) is an interesting situation for an analogue system.

3.5 Analogue gravity from *nonlocal* BEC: the modified $\hat{D}_2$

As we have already discussed that the continuity equation is preserved even after the advent of nonlocal correction term on top of the conventional GP model for a dilute Bose gas. Thus Eq.(3.3) is unaltered for a nonlocal BEC while the Euler equation [Eq.(3.3)] is modified as

$$\partial_t \vartheta(t,\mathbf{r}) + \left(\frac{[\nabla\vartheta(t,\mathbf{r})]^2}{2m} + V_{\text{ext}}(t,\mathbf{r}) + \mathrm{g}n(t,\mathbf{r}) \underbrace{- \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n(t,\mathbf{r})}}{\sqrt{n(t,\mathbf{r})}} + \kappa a^2 \mathrm{g}\,\nabla^2 n(t,\mathbf{r})} \right) = 0, \quad (3.55)$$

where the underbracketted part is the modified quantum potential due to nonlocal interactions and is given by

$$\mathsf{V}_{\text{quantum}} = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n(t,\mathbf{r})}}{\sqrt{n(t,\mathbf{r})}} + \kappa a^2 \mathrm{g}\,\nabla^2 n(t,\mathbf{r}). \quad (3.56)$$

In comparison with Eq.(3.4), it is evident that the last term in the above equation is the nonlocal correction creeping in $\mathsf{V}_{\text{quantum}}$. Thus, linearizing the dynamics as we have already shown previously, Eqs.(3.6) and (3.7) are retained, but the second-order differential operator ($\hat{D}_2$) gets modified due to this $\mathsf{V}_{\text{quantum}}$ in Eq.(3.56). Keeping Eq.(3.8) in mind, the modified expression of $\hat{D}_2$ for a nonlocal BEC is given by

$$\hat{D}_2\, n_1(t,\mathbf{r}) = -\frac{n_1(t,\mathbf{r})}{2} n_0^{-3/2}(t,\mathbf{r})\,\nabla^2 n_0^{1/2}(t,\mathbf{r}) + \frac{n_0^{-1/2}(t,\mathbf{r})}{2}\nabla^2\left(n_0^{-1/2}(t,\mathbf{r})\, n_1(t,\mathbf{r})\right) \underbrace{-\kappa\frac{2m}{\hbar^2} a^2 \mathrm{g}\,\nabla^2 n_1(t,\mathbf{r})}, \quad (3.57)$$

where the underbracketted term indicates the nonlocal correction. This would modify the corresponding entries of the f-matrix in Eq.(3.16) as a result of which, the dynamical equation in $(3+1)$D, given by,

$$\partial_\mu \tilde{f}^{\mu\nu}\partial_\nu \tilde{\vartheta}_1(x) = 0 \quad (3.58)$$

would give rise to a whole new physics and a stack of interesting phenomena. Here the old $f_{\mu\nu}$ is simply replaced by $\tilde{f}_{\mu\nu}$ just for notational ease because later we would decompose this basic $\tilde{f}$-matrix into f, f_1 and f_2 at different orders of ϵ [see Eq.(3.53b)].

As has been mentioned already, the most important result of this minimal generalization of the local GP equation, which can be achieved by tuning the s-wave scattering length a, is that the speed of sound $c_s = \sqrt{\mathrm{g}n/m}$ is not affected but one can get rid of the under-root quartic term in the dispersion relation. So, in principle, one can go to high momentum scales keeping the dispersion relation linear and at the same time keeping c_s small. The possibility of tuning the kinetic energy term out helps not only avoid complications at the sonic horizon $c_s \sim |\mathbf{v}|$, but also gives control over moving from massless (linear dispersion relation) to massive (wave packets) excitations and vice versa. Small wave length excitations with linear dispersion are very important when finite size geometrical constraints apply to the sonic horizon. Obviously, keeping up to the third order term in the expansion of the order parameter would impose restrictions on going to small length scales at some point beyond which one should consider other higher order terms. However, the next higher order term in the Taylor expansion of the order parameter would again be canceled by symmetry of interactions and one has to consider an even higher order term, the effect of which should not be felt if one does not go to a much smaller length scale.

In the high frequency regime of analogue systems, one in general writes the dispersion relation as

$$E^2 = m^2 c_s^4 + \hbar^2 k^2 c_s^2 + \hbar^2 c_s^2 \triangle (k, K), \tag{3.59}$$

where $K \sim \xi^{-1}$ (e.g. in the present case) is the inverse length scale set by the system [4]. The ξ is the healing length, as already mentioned, can be considered the analogue Planck scale of the system. One considers various forms for the last term in the above equation doing an expansion about $k = 0$ which may or may not converge. Some examples being $\pm k^3/K$ and $\pm k^4/k^2$ where the positive and the negative sign correspond to so-called super and subluminal scenarios [4]. In the superluminal case the group velocity becomes larger than the velocity of sound for small k modes whereas in the subluminal case it can go below the velocity of sound. The mechanism of thermal radiation emitted by the horizon is very different in these two cases. The MBD [Eq.(3.48)] actually shows that the coefficient of the large k part of the spectrum can go through zero providing a means to smoothly move between the sub and superluminal regimes.

3.6 Discussions

To conclude, our simple analysis on the basis of a minimal correction to the local GP equation to incorporate the non-locality might be giving a similar dispersion relation people have observed previously in other contexts, but, here we identify that, this minimal correction indicates a tunability of the healing length leaving almost all the other results of the local GP theory qualitatively the same. Although we are looking at the present result from an analogue gravity perspective for obvious reasons, it can prove to be extremely important from a condensed matter perspective as well. The healing length is a measure of the size of a vortex and being able to change this healing length to a good extent at finite a can generate a whole lot of possibilities to deal with vortex dynamics. We hope to discuss similar things in our later communications. Nevertheless, in the context of analogue gravity, the present result indicates a range of tunability of the dispersion. This important fact was unnoticed in the context of the BEC so far. Thus, the generic large momentum divergence of the group velocity of wave packets, where undesirable, can be controlled to a good extent and excitations of small wave number can be accessed.

Chapter 4

Effects of quantum potential: Massive scalar modes - essence of UV-IR coupling

The healing length (ξ) of a dilute BEC, in the standard condensed matter context, is found to be $\frac{\lambda_{C,\text{ acoustic}}}{\sqrt{2}}$ where $\lambda_{C,\text{ acoustic}}$ (see Eq.(3.17) in the previous chapter) is the effective Compton wavelength (see Eq.(3.17) in the previous chapter) of a particle where the speed of light (c) is replaced by the speed of sound (c_s). Below this length scale, the quantum potential term of BEC becomes non-negligible and there happens a Lorentz breakdown in the analogue picture. Dispersion becomes important at small length scales. One takes advantage of this fact of knowing the dispersion relation and tries to address the analogue trans-planckian problem in this regime [19, 83–86]. The basic idea of most of such works is to understand the robustness of the Hawking radiation (Planckian spectrum) in the presence of Lorentz-breaking dispersions.

In the present chapter, we are addressing the effects of the presence of the Lorentz-breaking term (i.e., the 'Quantum Potential' term[35]) using a completely different approach through *multiple scale perturbations*. Unlike looking at the effects of $\mathsf{V}_{\text{quantum}}$ in the dispersion spectrum as described in the published literature, we keep the $\mathsf{V}_{\text{quantum}}$ in the BEC-dynamics.

Our main result in this work is to show that as a consequence of quantum potential induced Lorentz symmetry breaking of the massless scalar field at smallest length scales, there emerges a massive scalar field at larger length scales on a spacetime of 'lower' dimensions. This is a general result within the scope of analogue systems, because it does not take into account anything special about the condensate, namely *multicomponent, special geometry* or *forced symmetry breaking*.

[35] Refer to Eq.(3.56) in the previous chapter; i.e., $\mathsf{V}_{\text{quantum}} = -\frac{\hbar^2}{2m}\frac{\boldsymbol{\nabla}^2\sqrt{n(x)}}{\sqrt{n(x)}} + \kappa a^2 \mathrm{g}\,\boldsymbol{\nabla}^2 n(x)$.

4.1 The Setting of Analogue Gravity model

In the following prescription, we'll be deriving our proposed model in detail where we show that the underlying non-relativistic BEC-system under consideration is very much capable of simulating the massive KG equation for some scalar field even in a flat background. Since many elementary particles in nature do have non-zero mass, this happens to be a very significant and essential step, as already pointed out by Visser *et al.* in 2005 [60], towards building some realistic analogue models rather than just providing some tentative mathematical methodology.

4.1.1 The control over $\hat{D}_2$ because of tuning of ξ

In section-2.5 in the previous chapter, the modified expression of $\hat{D}_2$ for a nonlocal BEC was given by Eq.(3.57) which is

$$\hat{D}_2\, n_1(x) = -\frac{n_1(x)}{2} n_0^{-3/2}(x)\, \nabla^2 n_0^{1/2}(x) + \frac{n_0^{-1/2}(x)}{2} \nabla^2\Big(n_0^{-1/2}(x)\, n_1(x)\Big) \underbrace{-\kappa\frac{2m}{\hbar^2} a^2 \mathrm{g}\, \nabla^2 n_1(x)}_{\text{correction due to nonlocality}} . \tag{4.1}$$

We restrict our formalism for a uniform condensate, i.e., when $V_{\text{ext}} = 0$ and naturally the classical mean-field density becomes just a constant[36]: $n_0(x) \equiv n_0$ where usually $n_0 \sim 10^{21}$ m.$^{-3}$ in the BEC experiments. This obviously simplifies the above expression of $\hat{D}_2$ from Eq.(4.1) and we write it in terms of the healing length ξ as

$$\hat{D}_2 \equiv \frac{2m\mathrm{g}}{\hbar^2}\left(\frac{\hbar^2}{4m\mathrm{g}n_0} - \kappa a^2\right)\nabla^2 = \frac{2m\mathrm{g}}{\hbar^2}\xi^2\nabla^2, \tag{4.2}$$

where ξ is imported from the combination of Eq.(3.53b) and (3.51) in the previous chapter as $\xi^2 = \frac{\hbar^2}{2m\mathrm{g}n_0}\left(1/2 - 8\kappa\pi a^3 n_0\right)$ with the s-wave coupling constant being $\mathrm{g} = 4\pi\hbar^2 a/m$. When non-local interactions are *not* taken into account even at the lowest order as has been considered by us in our proposed model in the previous chapter, the second term inside the bracket in the above expression of $\hat{D}_2$ in Eq.(4.2) would have been missing.

Now, as we have already expressed $n_1(x)$ in terms of $\tilde{\vartheta}_1(x)$ in the previous chapter, given by Eq.(3.9), i.e.,

$$n_1(x) = -\hat{\mathcal{A}}\Big(\partial_t\tilde{\vartheta}_1(x) + \frac{1}{m}\nabla\vartheta_0(x).\nabla\tilde{\vartheta}_1(x)\Big),$$

[36] Refer to Eq.(3.5) in previous chapter where linear fluctuations were considered on top of the classical mean-fields: $n(x) \to n_0(x) + n_1(x)$, $\vartheta(x) \to \vartheta_0(x) + \tilde{\vartheta}_1(x)$.

where one can consider a controlled expansion through a binomial approximation of $\hat{\mathscr{A}}$ since ξ can now be tuned to a very small value ($\xi << 1$) by increasing a via Feshbach resonance and hence, from Eq.(4.2), one comes up with

$$\hat{\mathscr{A}} = \left(\mathfrak{g} - \frac{\hbar^2}{2m}\hat{D}_2\right)^{-1} = \left(\mathfrak{g} - \mathfrak{g}\,\xi^2\,\boldsymbol{\nabla}^2\right)^{-1} \simeq \mathfrak{g}^{-1}\left(1 + \xi^2\,\boldsymbol{\nabla}^2\right). \tag{4.3}$$

This is exactly what is the advantage of considering the nonlocal interactions in a BEC because the healing length ξ becomes tunable and one can make it shrink to a very small value. This is how one can access the trans-Planckian physics in a controlled manner in this context of analogue gravity (analogue Hawking radiation from BEC) by going below a certain length scale. This advantage in turn plays a crucial role to tackle the Lorentz-breaking $\hat{D}_2$-term which comes from the linearization of $\mathsf{V}_{\text{quantum}}$ in the BEC-dynamics and saves it from getting thrown away drastically.

By substituting $n_1(x)$, we have already obtained the second-order p.d.e [i.e., Eq.(2.13)] for the linearized phase fluctuations $\tilde{\vartheta}_1(x)$ before. In terms of the velocity components, as imported from the previous chapter[37], Eq.(2.13) can now be rewritten as

$$\partial_t\left[-\hat{\mathscr{A}}\left(\partial_t\tilde{\vartheta}_1(x) + \mathbf{v}(x).\boldsymbol{\nabla}\tilde{\vartheta}_1(x)\right)\right] + \boldsymbol{\nabla}.\left(\left[-\hat{\mathscr{A}}\left(\partial_t\tilde{\vartheta}_1(x) + \mathbf{v}(x).\boldsymbol{\nabla}\tilde{\vartheta}_1(x)\right)\right]\mathbf{v}(x) + \frac{c_s^2}{\mathfrak{g}}\boldsymbol{\nabla}\tilde{\vartheta}_1(x)\right) = 0,$$

$$\text{where, } \hat{\mathscr{A}} \equiv \mathfrak{g}^{-1}\left(1 + \xi^2\,\boldsymbol{\nabla}^2\right), \text{ from Eq.(4.3).} \tag{4.4}$$

Thus the dynamical equation[38] for $\tilde{\vartheta}_1(x)$ is given by

$$\partial_\mu \tilde{f}^{\mu\nu}\partial_\nu\tilde{\vartheta}_1(x) = 0, \qquad \text{where we had} \qquad [\tilde{f}^{\mu\nu}] = \left[\begin{array}{c:c} \tilde{f}^{00} & \tilde{f}^{0j} \\ \hdashline \tilde{f}^{i0} & \tilde{f}^{ij} \end{array}\right]. \tag{4.5}$$

From Eq.(3.16), the respective elements, after the controlled expansion in Eq.(4.3), are explicitly approximated by

$$\text{where,} \quad \tilde{f}^{00} \simeq -\mathfrak{g}^{-1}\left(1 + \xi^2\,\boldsymbol{\nabla}^2\right) \quad , \quad \tilde{f}^{0j} \simeq -\mathfrak{g}^{-1}\left(1 + \xi^2\,\boldsymbol{\nabla}^2\right)v^j(x),$$
$$\tilde{f}^{i0} \simeq -v^i(x)\,\mathfrak{g}^{-1}\left(1 + \xi^2\,\boldsymbol{\nabla}^2\right) \quad , \quad \tilde{f}^{ij} \simeq \frac{c_s^2\,\delta^{ij}}{\mathfrak{g}} - v^i(x)\,\mathfrak{g}^{-1}\left(1 + \xi^2\,\boldsymbol{\nabla}^2\right)v^j(x)\,. \tag{4.6}$$

[37]From Eq.(3.11), the background flow of fluid is $\mathbf{v}(x) = \boldsymbol{\nabla}\vartheta_0(x)/m$ and, from Eq.(3.12), the local speed of sound is $c_s = \sqrt{n_0\mathfrak{g}/m}$.

[38]One may refer to Eq.(3.58) in the previous chapter.

At this stage, we clearly realize that the elements of $\tilde{f}$-matrix contains a 3D Laplacian which obviously breaks the Lorentz symmetry and a standard covariant structure (in order to look into some aspects of gravity or flat/curved spacetime features) can not be identified from $\partial_\mu \tilde{f}^{\mu\nu} \partial_\nu \tilde{\vartheta}_1(x) = 0$ in the above Eq.(4.5). In the absence of quantum potential (or in other words $\xi \approx 0$), this Lorentz-breaking terms are not residing inside the elements of $\tilde{f}$-matrix and thats why in the standard practice, through acoustic approximation[39], one can easily identify the covariant picture from $\tilde{f}^{\mu\nu}$.

Thus one gets an insight that, in order to realize the effects of the quantum potential, one can't throw away the Lorentz-breaking 3D Laplacians (each comes with a pre-factor of $\xi^2 = \xi_0^2 \epsilon^2$ where ξ itself being very small[40]) from the $\tilde{f}$-matrix-elements and one can capture the upshot only at length scales [$\sim \mathcal{O}\left(\frac{1}{\xi}\right)$] much larger than the usual small length scales at which the standard prescription of massless Klein-Gordon field[41] is emerged from Eq.(4.5).

4.1.2 Multiple-scale-perturbations

Let's note the following feature of Eq.(4.4)/(4.6) :

- Each operation with $\hat{\mathcal{A}}$ naturally contains a 3D Laplacian which is considered to be at small scales (i.e. $\boldsymbol{\nabla}^2 \equiv \partial^{\mathrm{j}}\partial_{\mathrm{j}}$ where j is the dummy index that runs only over spatial dimensions, but not temporal). As already mentioned above, this comes with a pre-factor of ξ^2 at each operation. This fact is being stressed upon at this stage because an analogue gravity model will be set up on different length scales in order to systematically address the effects of keeping quantum potential in the BEC-dynamics.

Now we explicitly mention the decomposition of the spacetime derivatives over independent multiple scales (viz: one is the usual small scale while the other being the larger scale) in order to separate out the dynamics upto $\mathcal{O}(\epsilon^2)$ accuracy. One has to understand that before this decomposition, until now, these two independent scales or the spacetime derivatives are kind of mixed together and hidden inside the ∂_μ-s in Eq.(4.5). To avoid notational discrepancies and for sake of easy understanding, we denote the parent scale (from which two independent scales are born) to be $\tilde{\partial}_\mu$ and Eq.(4.5) becomes

$$\tilde{\partial}_\mu \tilde{f}^{\mu\nu}(x) \tilde{\partial}_\nu \tilde{\vartheta}_1(x) = 0. \tag{4.7}$$

[39] Refer to section-2.1.1 .

[40] Refer to Eq.(3.53b) where $\xi = \xi_0 \left(\frac{1}{2} - 8\kappa\pi a^3 n\right)^{1/2} = \xi_0\, \epsilon$.

[41] Refer to Eq.(3.22) in the previous chapter; i.e., $\nabla_\mu \nabla^\mu \tilde{\vartheta}_1(x) = 0$.

The multiple-scale-perturbation as considered here is defined by

$$\tilde{\partial}_\mu \to \partial_\mu + \epsilon\,\partial_{\bar{\mu}} \;\;, \;\; \text{where small scales: } x \equiv x^\mu \text{ and large scales: } X \equiv X^{\bar{\mu}}, \tag{4.8}$$

the functional dependence in Eq.(4.7) changes: $\tilde{f}^{\mu\nu}(x) \to \tilde{f}^{\mu\nu}(x, X)$, $\tilde{\vartheta}_1(x) \to \tilde{\vartheta}_1(x, X)$.

Here μ, ν, ... etc. along with $\bar{\mu}$, $\bar{\nu}$, ... etc. are the two different sets of free/dummy indices subject to the restriction which is they have to separately run over the small and large scales respectively[42].

From now on, throughout the thesis,

- by small-scale spacetime derivative, we would refer to this $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ on r.h.s of Eq.(4.8),
- and by large-scale spacetime derivative, we would refer to the above mentioned $\partial_{\bar{\mu}} \equiv \frac{\partial}{\partial X^{\bar{\mu}}}$.
- In the following prescription, we restrict the formalism where $\hat{\mathcal{A}}$ in Eq.(4.3) is to remain unperturbed and to be involving only the small scale derivatives because, even the small scale 3D-Laplacian in $\hat{\mathcal{A}}$ already bears a pre-factor of ϵ^2.
- The background flow field $\mathbf{v}(x) = \boldsymbol{\nabla}\vartheta_0(x)/m$ (see Eq.(3.11) in the previous chapter) is also unperturbed and kept at the usual small scales.

Here, within the scope of analogue spacetime, we should be able to exploit the UV-IR coupling between different bands of phonon excitations in order to look at the small-scale-effects (UV correspondence at $\sim \mathcal{O}(\xi)$, where ξ is tuned to a small value) captured at larger length-scales (IR correspondence at $\sim \mathcal{O}(1/\xi)$ and above).

4.1.3 Amplitude modulations in $\tilde{\vartheta}_1(x, X)$

Let us go by the choice of scales as suggested by the dynamics itself with the linearized phase fluctuation field being expressed in a product form as

$$\tilde{\vartheta}_1(x, X) \to \vartheta_2(X)\,\vartheta_1(x). \tag{4.9}$$

In the absence of quantum potential term, the amplitude ϑ_2 becomes just a spacetime-independent constant; whereas the effect of quantum potential is likely to rule away the possibility of the flatness of this amplitude of the basic $\vartheta_1(x)$-field. Our motivation is to capture the emergence of the massive scalar excitations, as the direct upshot of the presence of quantum potential in the underlying BEC-dynamics, in the dynamics of this amplitude $\vartheta_2(X)$.

[42] It is quite redundant from Eq.(4.8) that in Cartesian: $\tilde{\partial}_t \to \partial_t + \epsilon\,\partial_{\mathcal{T}}$, $\tilde{\partial}_x \to \partial_x + \epsilon\,\partial_X$, $\tilde{\partial}_y \to \partial_y + \epsilon\,\partial_Y$, $\tilde{\partial}_z \to \partial_z + \epsilon\,\partial_Z$ and similarly in spherical polar: $\tilde{\partial}_t \to \partial_t + \epsilon\,\partial_{\mathcal{T}}$, $\tilde{\partial}_r \to \partial_r + \epsilon\,\partial_R$, $\tilde{\partial}_\theta \to \partial_\theta + \epsilon\,\partial_\Theta$, $\tilde{\partial}_\phi \to \partial_\phi + \epsilon\,\partial_\Phi$.

4.1.4 The Model: dynamics at different orders of ϵ

In this section, we will set up the complete model of analogue gravity out of a nonlocal nonrelativistic BEC and deduce the explicit structure of the general acoustic metric in $(3+1)$D. Before proceeding further, it is important to mention, for sake of transparency, that I'll stick to the signature[43] of the metric tensor as $(-,+,+,+)$.

Now Eq.(4.8) is applied on Eq.(4.5) subject to Eq.(4.9) and gives rise to a set of equations at different orders of ϵ. The full model, considered till $O(\epsilon^2)$ accuracy, is given by,

$$\mathcal{O}(1) \quad \Rightarrow \quad \partial_\mu f^{\mu\nu}\partial_\nu\vartheta_1(x) = 0, \tag{4.10}$$

$$\mathcal{O}(\epsilon) \quad \Rightarrow \quad \left\{\partial_\mu f_1^{\mu\bar{\mu}}\partial_{\bar{\mu}} + \partial_{\bar{\nu}} f_1^{\bar{\nu}\nu}\partial_\nu\right\}\Big(\vartheta_2(X)\vartheta_1(x)\Big) = 0, \tag{4.11}$$

$$\text{and,}\ \mathcal{O}(\epsilon^2) \quad \Rightarrow \quad \partial_{\bar{\mu}} f_2^{\bar{\mu}\bar{\nu}}\partial_{\bar{\nu}}\vartheta_2(X) - \mathfrak{m}^2\,\vartheta_2(X) = 0; \tag{4.12}$$

where the original $\tilde{f}^{\mu\nu}$ gets decomposed in to three different matrices, viz. $f^{\mu\nu}(x)$, $f_1^{\mu\bar{\mu}}(x,X)$ or $f_1^{\bar{\nu}\nu}(X,x)$ and $f_2^{\bar{\mu}\bar{\nu}}(X)$ which are all constructed as symmetric 4×4 matrices explicitly written later, at three different orders $\mathcal{O}(1)$, $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ of dynamics respectively. Here $\mathfrak{m}$ is the mass of the large length scale phonon modes, while it is strikingly found to be a finite function of the $\vartheta_1(x)$ field and thus Eq.(4.12) is the 'massive' free Klein-Gordon (KG) equation for the amplitude-field $\vartheta_2(X)$.

It is important to acknowledge the fact that, in the standard literature, one usually identifies this Eq.(4.10) as the massless minimally coupled KG equation for a scalar field $\vartheta_1(x)$, see Eqs.(248) & (254) of [4]. But in our present framework, on top this usual massless picture at $\mathcal{O}(1)$, we come up with a massive KG field in larger length scales at $\mathcal{O}(\epsilon^2)$ subject to some constraint, given by Eq.(4.11), obtained at the intermediate $\mathcal{O}(\epsilon)$.

For the time being, we have restrained ourselves from giving the full expression of the mass term $\mathfrak{m}$ which is extremely cluttered, but under some physical and relevant approximations, it comes out to be relatively tidier. In this regard, we have worked out each and every step using *Mathematica 9.0* package in order to obtain the respective final expressions. However, later in the Appendix, the full expression of $\mathfrak{m}$ is presented without any restriction.

In the following section, we choose a specific coordinate system according to our convenience in order to provide the explicit structures the f-matrices as introduced in Eqs.(4.10)-(4.12).

[43] In fact this convention of $(-,+,+,+)$ is clearly the reason that leads to generating a "$-$" sign in front of $\mathfrak{m}^2$ in Eq.(4.12).

4.2 Effective acoustic metric in $(3+1)$D Cartesian coordinates

Let's consider a general background velocity in 3D Cartesian system,

$$\frac{1}{m}\nabla\vartheta_0(x) = \boxed{\mathbf{v}(x)} = v_1(x)\,\hat{x}_1 + v_2(x)\,\hat{x}_2 + v_3(x)\,\hat{x}_3. \tag{4.13}$$

At $\mathcal{O}(1)$ in Eq.(4.10), the $[f^{\mu\nu}(x)]$ is of the following form

$$[f^{\mu\nu}(x)] = \frac{1}{\mathfrak{g}}\begin{pmatrix} -1 & -v_1(x) & -v_2(x) & -v_3(x) \\ -v_1(x) & \left(c_s^2 - v_1^2(x)\right) & -v_1(x)\,v_2(x) & -v_1(x)\,v_3(x) \\ -v_2(x) & -v_2(x)\,v_1(x) & \left(c_s^2 - v_2^2(x)\right) & -v_2(x)\,v_3(x) \\ -v_3(x) & -v_3(x)\,v_1(x) & -v_3(x)\,v_2(x) & \left(c_s^2 - v_3^2(x)\right) \end{pmatrix}. \tag{4.14}$$

Since the background velocity fields are kept unperturbed, the coordinate signature doesn't creep into the above form of $[f^{\mu\nu}(x)]$ and hence, for the dynamics at all orders of ϵ, the respective forms of $[f_1^{\mu\bar{\mu}}(x,X)]$ or $[f_1^{\bar{\nu}\nu}(X,x)]$ and $[f_2^{\bar{\mu}\bar{\nu}}(X)]$ are exactly the same as $[f^{\mu\nu}(x)]$, i.e.,

$$\text{structure wise}\quad f^{\mu\nu}(x) \text{ at } \mathcal{O}(1) = \begin{cases} f_1^{\mu\bar{\mu}}(x,X) \equiv f_1^{\bar{\nu}\nu}(X,x) & \text{at } \mathcal{O}(\epsilon) : \text{Eq.(4.11)}, \\ f_2^{\bar{\mu}\bar{\nu}}(X) & \text{at } \mathcal{O}(\epsilon^2) : \text{Eq.(4.12)}. \end{cases} \tag{4.15}$$

In any curvilinear coordinates, say spherical polar, these forms do differ structure wise due to the coordinate dependence of the matrix-elements, but this issues are dealt in great detail in the following chapter.

4.2.1 The covariant formulation for the massive KG field $\vartheta_2(X)$

$[f_2^{\bar{\mu}\bar{\nu}}]$ in Eq.(4.12) can be replaced by the structure of $[f^{\mu\nu}]$ and in order to cast this Eq.(4.12) into a Lorentz invariant form, it is required to identify the corresponding covariant structure and hence the introduction of the effective metric (or in other words, acoustic metric) in place of $[f^{\mu\nu}]$ is very essential.

Say, $[\mathsf{g}_{\mu\nu}(x)]$ be the general acoustic metric that actually defines the $(3+1)$D Cartesian spacetime under consideration with its determinant, given by, $\mathsf{g} = \det[\mathsf{g}_{\mu\nu}(x)]$. Considering Eq.(4.12), one identifies

$$f^{\mu\nu}(x) = \sqrt{|\mathsf{g}|}\,\mathsf{g}^{\mu\nu}(x). \tag{4.16}$$

$$\Rightarrow \quad \det[f^{\mu\nu}(x)] \;=\; \det[\sqrt{|\mathsf{g}|}\,\mathsf{g}^{\mu\nu}(x)] \;=\; (\sqrt{|\mathsf{g}|})^4 \det[\mathsf{g}^{\mu\nu}(x)] \;=\; (\sqrt{|\mathsf{g}|})^4\, \mathsf{g}^{-1} \;=\; \mathsf{g}.$$

$$\because \text{ from Eq.(4.14)}, \quad \det[f^{\mu\nu}(x)] = -\frac{c_s^6}{\mathsf{g}^4}\,, \qquad \therefore\; \mathsf{g} = -\frac{c_s^6}{\mathsf{g}^4},$$

$$\text{and obviously,} \quad \mathsf{g}^{\mu\nu}(x) = \frac{1}{\sqrt{|\mathsf{g}|}} f^{\mu\nu}(x) = \frac{\mathsf{g}^2}{c_s^3} f^{\mu\nu}(x). \quad (4.17)$$

Thus we write the inverse metric as

$$[\mathsf{g}^{\mu\nu}(x)] = \frac{\mathsf{g}}{c_s^3} \begin{pmatrix} -1 & -v_1(x) & -v_2(x) & -v_3(x) \\ -v_1(x) & \left(c_s^2 - v_1^2(x)\right) & -v_1(x)\,v_2(x) & -v_1(x)\,v_3(x) \\ -v_2(x) & -v_2(x)\,v_1(x) & \left(c_s^2 - v_2^2(x)\right) & -v_2(x)\,v_3(x) \\ -v_3(x) & -v_3(x)\,v_1(x) & -v_3(x)\,v_2(x) & \left(c_s^2 - v_3^2(x)\right) \end{pmatrix}, \quad (4.18)$$

and the required effective acoustic metric as

$$[\mathsf{g}_{\mu\nu}(x)] = \frac{c_s}{\mathsf{g}} \begin{pmatrix} -\left(c_s^2 - \mathbf{v}^2(x)\right) & -v_1(x) & -v_2(x) & -v_3(x) \\ -v_1(x) & 1 & 0 & 0 \\ -v_2(x) & 0 & 1 & 0 \\ -v_3(x) & 0 & 0 & 1 \end{pmatrix}, \qquad \therefore \text{ easy to verify that } \boxed{\mathsf{g} = -\frac{c_s^6}{\mathsf{g}^4}}. \quad (4.19)$$

It should be noted that, in general relativity, the spacetime metric (which does bear the feature of the background geometry) is related to the distribution of matter (i.e. the stress-energy tensor) through the Einstein's-Field-Equations; whereas, the acoustic metric $[\mathsf{g}_{\mu\nu}(x)]$ here happens to be related to the background velocity field $[\mathbf{v}(x)]$ as well as the local speed of sound (c_s) in a way more simpler algebraic fashion. Some striking features of this $[\mathsf{g}_{\mu\nu}(x)]$ from topological aspect and regarding 'stable causality' have been discussed by Visser in [21]-pp.1773-1774.

4.2.2 An example of a special case: analysis on flat spacetime

Since the background flow of fluid defines the effective metric of the analogue spacetime, if we choose a special case where the background flow is uniform over spacetime, then what one comes up with is a flat spacetime for all the metric entries [i.e., $\mathsf{g}_{\mu\nu}(x)$] becoming constants. A uniform background flow of the fluid means $\mathbf{v}$ is constant, i.e.,

$$v_1(x) \equiv v_1 \;,\; v_2(x) \equiv v_2 \;,\; v_3(x) \equiv v_3 \qquad \text{where, } v_1, v_2, v_3 \text{ all being constants.} \tag{4.20}$$

We consider the phase-fluctuations being Fourier-expanded as the following ansatz:

$$\vartheta_1(x) \to A_1\, e^{ik_\mu x^\mu} \;,\; \vartheta_2(X) \to A_2\, e^{iK_{\bar{\mu}} X^{\bar{\mu}}} \text{ where, } k^\mu = (\omega, \mathbf{k}) \text{ and } K^{\bar{\mu}} = (W, \mathbf{K}). \tag{4.21}$$

Obviously ω is the frequency of the basic field $\vartheta_1(x)$ w.r.t a laboratory frame while W being the same of the amplitude field $\vartheta_2(X)$. We have A_1 and A_2 as some constants and for sake of clarity, the 3D wave-vectors be expressed in terms of the respective set of components as

$$\mathbf{k} = (k_1, k_2, k_3) \quad , \quad \mathbf{K} = (K_{\bar{1}}, K_{\bar{2}}, K_{\bar{3}}) \quad . \tag{4.22}$$

Using the fact as mentioned in Eq.(4.15), we rewrite Eqs.(4.11) and (4.12) with a little bit manipulations as

$$\begin{aligned}
\text{Eq.(4.11)} : \qquad & \left\{\partial_\mu f^{\mu\bar{\mu}}\, \vartheta_1(x)\right\} \partial_{\bar{\mu}} \vartheta_2(X) \;+\; \partial_{\bar{\nu}} \vartheta_2(X) \left\{ f^{\bar{\nu}\nu}\, \partial_\nu \vartheta_1(x) \right\} = 0, \\
\Rightarrow \; & \left\{\partial_\mu f^{\mu\bar{\mu}}\, \vartheta_1(x)\right\} \partial_{\bar{\mu}} \vartheta_2(X) \;+\; \partial_{\bar{\mu}} \vartheta_2(X) \left\{ f^{\bar{\mu}\nu}\, \partial_\nu \vartheta_1(x) \right\} = 0, \\
\Rightarrow \; & \partial_{\bar{\mu}} \vartheta_2(X) \left\{ \partial_\mu \Big(f^{\mu\bar{\mu}} \vartheta_1(x) \Big) \;+\; f^{\bar{\mu}\nu} \Big(\partial_\nu \vartheta_1(x) \Big) \right\} = 0,
\end{aligned} \tag{4.23}$$

$$\text{and Eq.(4.12)} : \qquad f_2^{\bar{\mu}\bar{\nu}} \Big(\partial_{\bar{\mu}}\, \partial_{\bar{\nu}}\, \vartheta_2(X) \Big) - \mathfrak{m}^2\, \vartheta_2(X) = 0. \tag{4.24}$$

In (3 + 1)D Cartesian, the mass term $\mathfrak{m}^2$ in the above Eq.(4.24) takes a simpler expression that can be compacted in terms of matrix multiplications and its functional form is given by

$$\mathfrak{m}^2 = \frac{1}{\vartheta_1(x)} \mathcal{D} \mathcal{F} \mathcal{D}^{\mathrm{T}}\, \vartheta_1(x), \tag{4.25}$$

where,

$$\mathcal{D}_{1\times4} \text{ is a row matrix } \equiv \Big(\partial_0, \partial_1, \partial_2, \partial_3\Big) \quad \text{and } \mathcal{F}_{4\times4} \equiv \frac{\xi_0^2}{\mathfrak{g}} \begin{pmatrix} \boldsymbol{\nabla}^2 & \boldsymbol{\nabla}^2 v_1 & \boldsymbol{\nabla}^2 v_2 & \boldsymbol{\nabla}^2 v_3 \\ v_1 \boldsymbol{\nabla}^2 & v_1 \boldsymbol{\nabla}^2 v_1 & v_1 \boldsymbol{\nabla}^2 v_2 & v_1 \boldsymbol{\nabla}^2 v_3 \\ v_2 \boldsymbol{\nabla}^2 & v_2 \boldsymbol{\nabla}^2 v_1 & v_2 \boldsymbol{\nabla}^2 v_2 & v_2 \boldsymbol{\nabla}^2 v_3 \\ v_3 \boldsymbol{\nabla}^2 & v_3 \boldsymbol{\nabla}^2 v_1 & v_3 \boldsymbol{\nabla}^2 v_2 & v_3 \boldsymbol{\nabla}^2 v_3 \end{pmatrix}.$$

4.3 Massive scalar excitations on flat spacetime - UV-IR coupling

Plugging the ansatz of the amplitude-modulated phase fluctuations as mentioned in Eq.(4.21) in Eq.(4.23), the $O(\epsilon)$-dynamics basically gives rise to

$$\frac{2}{\mathfrak{g}}\Big[W(\omega-\mathbf{k}.\mathbf{v})-K_{\bar{1}}\Big(\omega v_1+K_{\bar{1}}(c_s^2-v_1^2)-K_{\bar{2}}v_1v_2-K_{\bar{3}}v_3v_1\Big)$$
$$-K_{\bar{2}}\Big(\omega v_2-K_{\bar{1}}v_1v_2+K_{\bar{2}}(c_s^2-v_2^2)-K_{\bar{3}}v_2v_3\Big)-K_{\bar{3}}\Big(\omega v_3-K_{\bar{1}}v_3v_1-K_{\bar{2}}v_2v_3)+K_{\bar{3}}(c_s^2-v_3^2)\Big)\Big]$$
$$\times\,\vartheta_2\,\vartheta_1=0,$$

$$\Rightarrow\ \boxed{W=\mathbf{K}.\mathbf{v}+\frac{c_s^2\,\mathbf{K}.\mathbf{k}}{\omega-\mathbf{k}.\mathbf{v}}}. \tag{4.26}$$

Similarly, with the help of Eq.(4.25), the $O(\epsilon^2)$-dynamics from Eq.(4.24) gives rise to

$$\frac{1}{\mathfrak{g}}\Big[-c_s^2\mathbf{K}^2+(W-\mathbf{K}.\mathbf{v})^2\Big]\vartheta_2-\left(\frac{\xi_0^2}{\mathfrak{g}\,\vartheta_1}\right)\mathbf{k}^2\,(\omega-\mathbf{k}.\mathbf{v})^2\,\vartheta_2\,\vartheta_1=0,$$
$$\Rightarrow\ \boxed{W=\mathbf{K}.\mathbf{v}\pm\sqrt{\xi_0^2\,\mathbf{k}^2(\omega-\mathbf{k}.\mathbf{v})^2+c_s^2\,\mathbf{K}^2}}. \tag{4.27}$$

In the above dispersion for the massive scalar excitations, the contribution of the mass term $\mathfrak{m}$ is separately read off as

$$\mathcal{M}_0=\xi_0|\mathbf{k}|\underbrace{(\omega-\mathbf{k}.\mathbf{v})}_{\tilde{\omega}}\ \text{where, } \tilde{\omega} \text{ is obviously the co-moving frequency of } \vartheta_1(x)\text{-field.} \tag{4.28}$$

The appearance of ξ_0 in the above expression of mass ($\mathcal{M}_0$) makes the role of the quantum potential quite prominent and explicit. One can note that Eq.(4.26) is the dispersion relation which always exists over and above the massive KG equation [i.e., Eq.(4.12) or equivalently Eq.(4.24)] which itself in turn gives rise to another dispersion relation given by Eq.(4.27)

Now having both Eq.(4.26) and Eq.(4.27) held simultaneously, it is clearly visible that if the magnitude of wave-vector ($|\mathbf{K}|$) has to vanish, then $W=0$ which means either $|\mathbf{k}|=0$ or $\tilde{\omega}=0$. The simultaneous validity of Eq.(4.26) and Eq.(4.27) also implies that the $\mathbf{K}$-modes are always excited for a nonzero co-moving frequency ($\tilde{\omega}\neq 0$).

A selection of **K**-modes, given a fixed **k** band of excitations, should come into play via the combination of both the equation of constraint and the massive KG equation, i.e.,

$$\frac{c_s^2\,\mathbf{K}.\mathbf{k}}{\omega - \mathbf{k}.\mathbf{v}} = \pm\sqrt{\mathcal{M}_0^2 + c_s^2\,\mathbf{K}^2} \quad , \Rightarrow \boxed{\xi_0^2\,c_s^4\,|\mathbf{k}|^2\,(\mathbf{K}.\mathbf{k})^2 = \mathcal{M}_0^4 + \mathcal{M}_0^2\,c_s^2\,\mathbf{K}^2}. \tag{4.29}$$

Given an $\mathcal{M}_0 \neq 0$, for a co-moving observer, the **K**-modes must be excited above a minimum $|\mathbf{K}|$ threshold. So, the presence of mass here would observationally rule out a flat amplitude of the basic ϑ_1-modes. One would also see here an anisotropic velocity of **K** excitations as the existence of **k**-modes has broken the symmetry.

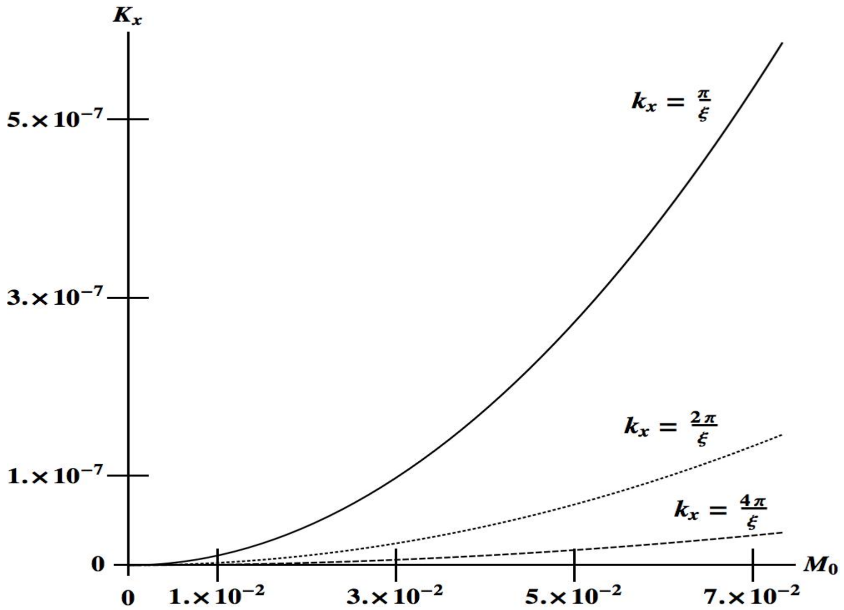

Figure 4.1: The plot of K_X vs $\mathcal{M}_0$ in S.I. units. Here we consider three cases separately for $k_x = \frac{\pi}{\xi}, \frac{2\pi}{\xi}, \frac{4\pi}{\xi}$ and they are denoted by continuous, dotted and dashed lines respectively. (Here, k_x and K_X are used instead of k_1 and $K_{\bar{1}}$ respectively to keep the notations clear and vivid.)

Fig. **4.1** shows a plot of $K_{\bar{1}}$ vs $\mathcal{M}_0$ (in 1D for simplicity) for $_{87}$**Rb** with $m \approx 1.45 \times 10^{-25}$ kg., the s-wave scattering length $a \sim 109a_0$ where $a_0 \sim 0.529 \times 10^{-10}$ m. is the Bohr radius; density of the condensate is $n_0 \sim 10^{21}$ m.$^{-3}$. This gives the speed of sound in the condensate to be $c_s \sim 61.909 \times 10^{-4}$ ms.$^{-1}$ and the healing length $\xi \sim 5.86 \times 10^{-8}$ m. . The frequency of ϑ_1 band of excitations w.r.t the lab-frame is $\omega = c_s|\mathbf{k}|$. So if a $k_1 \equiv |\mathbf{k}|$ is specified, ω gets fixed automatically. Thus $K_{\bar{1}}$ is evaluated as a function of $\mathcal{M}_0$ only while $\mathcal{M}_0$ being run independently from 0 to ~ 0.07 in S.I. units, $K_{\bar{1}}$ is plotted accordingly for each fixed k_1.

There's a different interpretation for an observer on *lower dimensional space*, $x - y$ plane (say). Then the system must be excited at a specified non-zero $K_{\bar{3}}$ mode at $K_{\bar{1}} = 0 = K_{\bar{2}}$ such that[44]

$$K_{\bar{3}} = \mathcal{M}_0^2 \left(c_s^4 \, \xi_0^2 \, |\mathbf{k}|^2 \, k_3^2 - c_s^2 \, \mathcal{M}_0^2 \right)^{-\frac{1}{2}}. \tag{4.30}$$

This gives the selection of $K_{\bar{3}}$ (or, K_Z) which must be excited due to $\mathcal{M}_0 > 0$, or in other words, because of the presence of quantum potential in the underlying BEC-dynamics. This situation would be seen by the observer living on $x-y$ plane as the presence of massive phonon-excitations which is supported by the necessary presence of the transverse K_Z-modes. The existence of transverse excitations here are not giving rise to the mass; the mass is entirely determined by the **k**-modes in the presence of the quantum potential. However, the existence of the transverse modes plays a necessary role in supporting the mass to explicitly show up in lower dimensions. These transverse excitations are unavoidable experimentally and are very important in the sense that they can act as an effective mass for the phonons generated by analogue Hawking emission from an acoustic black hole in a BEC and can cure the infrared divergence which appears in a $(1+1)$ dimensional case [87, 88].

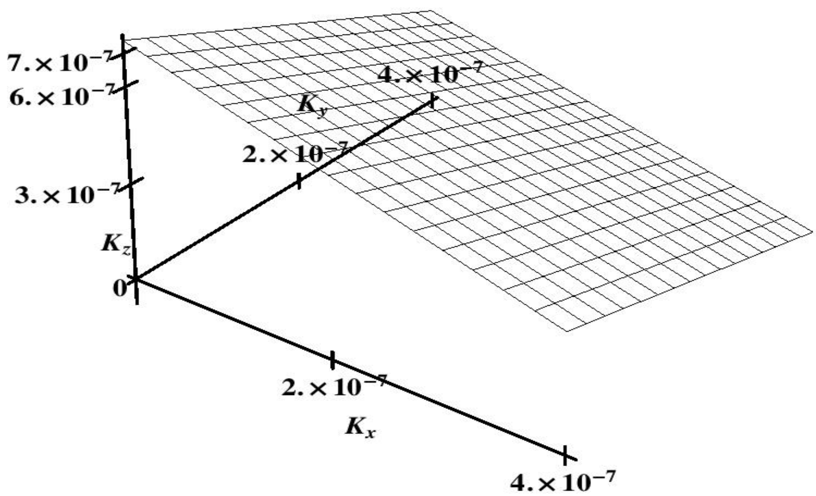

Figure 4.2: The 3D plot of K_Z vs $K_X - K_Y$ for $_{87}$**Rb** with $\mathcal{M}_0 = 0.25$ in S.I. units, where we have considered $k_x = \frac{1}{\sqrt{3}} \cdot \frac{4\pi}{\xi} = k_y = k_z$ and $\mathrm{Cos}(\mathbf{k}, \mathbf{v}) = 1$. The planar surface intersects the K_Z-axis at $K_Z \simeq 7.39 \times 10^{-7}$ in S.I. units which agrees with Eq.(4.30).

Now for the appearance of K_X, K_Y modes, these K_Z modes can be adjusted such that Eq.(4.26) and Eq.(4.27) satisfy simultaneously. We have plotted Fig. **4.2** where the regular interdependence between the observed K_X, K_Y -modes on $x-y$ plane and the unobserved transverse K_Z -mode for a given nonzero $\mathcal{M}_0$ is shown. This figure basically shows a wide range of validity of the dispersion relation [i.e. Eq.(4.29)] to create massive scalar modes.

[44]Eq.(4.30) is nothing but the 1D version of Eq.(4.29).

4.4 Discussions

By a controlled consideration of the quantum potential term in the hydrodynamics of BEC, which could be experimentally achievable, we see the possible emergence of mass based on how close the small scale dynamics approaches to the analogue Compton wavelength of the system. The standard massless covariant KG equation (see Eq.(3.22) in the previous chapter) has negligible correction if the length scales of the free scalar field is large enough compared to ξ. The correction becomes large at small length scales, but shows up in the large length scale dynamics through an intrinsic coupling between the small and large scales. In general, if Hawking radiation is to be seen in laboratories, it has to be in $(2+1)$D or even lower dimensional spacetime. This is because one has to keep at least one dimension free to handle the source/sink of the flow.

Under such considerations, any excitations in the left out dimensions would support massive excitations to appearing through the quantum potential coupling and these excitations could be seen at larger length scales. Proper correction has to be taken into account to filter out the 'expected' Hawking spectrum in such a scenario and the framework presented here can be used for that purpose.

Inferences from this flat spacetime analysis can be jotted down as the following

1. There's indeed a nontrivial distribution of $\vartheta_2(X)$-modes and the presence of the mass term ($\mathcal{M}_0 \neq 0$) observationally rules out the flat amplitude of the basic $\vartheta_1(x)$-modes.

2. For an observer on lower dimensional space, say $x - y$ plane, given a fixed $\mathbf{k}$, even for $K_X = 0 = K_Y$, there has to be nonzero threshold for the transverse K_Z-modes.

3. This threshold for the $K_Z \neq 0$ modes in presence of the massive scalar excitations is unavoidable in the analogue gravity experiments.

4. They can act as effective mass for the Hawking radiated phonons from analogue black holes and might cure the IR divergences on lower dimensional analogue gravity models.

Chapter 5

Analysis on curved spacetime for a canonical acoustic black hole from a BEC

In the previous chapter, we have looked into the effect of the Lorentz-breaking quantum potential term in a different way. This is a term of immense importance in the context of analogue gravity because this gives rise to the dispersion relation which is used to present an alternative scenario of the analogue Hawking radiation bypassing the trans-Planckian problem [14–19]. But the presence of this quantum potential term in the dynamics is somewhat analogous to that of a diffusion term which should spread the small scale modes into the large scale ones. In our previous work, we had guessed the existence of this coupling between the small and large scale dynamics and had captured this picture through the massive large wave length excitations as the amplitude modes over the usual small scale excitations. This whole work was presented on flat spacetime for the sake of simplicity and in order to introduce the idea.

In the present chapter, we analyze in details, the effect and consequence of the presence of the quantum potential term in the context of spreading out of the small scale excitations into the large scale ones on the curved spacetime of a canonical (a model first proposed by Visser [21] in 1998) acoustic black hole. This is an important analysis in its own right, because, the ultraviolet (UV) to infrared (IR) coupling is inevitable in these types of systems. This, in turn, results in the presence of instabilities to the short wavelength (UV correspondence) modes which are predominant in the Hawking spectrum as seen by a free-falling observer in a local Minkowski spacetime. This is because of the fact that, at very large curvatures, the local Minkowski flat space can only account for the short wavelength (UV correspondence) modes. The large wavelength (IR correspondence) modes which would be subsequently generated out of the UV-modes are characterized by a mass term solely dependent on the small scales. This fact does manifest the energetic dependence of these IR-modes on the UV ones and, at some sufficiently large time, there would be a transfer of energy from the UV-modes to the IR ones and this can completely mask the Hawking signal of the Hawking radiated modes. A detailed analysis of the UV-IR coupled dynamics is quite essential in that respect. In this context, it is important to note a work by Vieira *et al.* who discussed the analogue Hawking radiation of the massless scalar particles and the features of the Hawking spectrum associated in the sapcetime of rotating and canonical acoustic black holes [89].

5.1 The Model in $(3 + 1)$D Spherical polar coordinates

Here we have obtained the series solution to the free minimally coupled "massive" KG equation on a $(3 + 1)$D canonical curved background[45]. And we have followed the method shown by Elizalde [90] in 1988. The acoustic situation although being similar to the Schwarzschild geometry, however, the metric (i.e., acoustic metric) itself is quite different from the standard Schwarzschild metric. As a result, our present study here is quite different in details from the one done by Elizalde on the Schwarzschild background.

Using the series solution truncated to the desired accuracy, we show that the IR-modes, having grown from the UV-modes (supposedly Hawking radiated), would have a dominant "power law growth over space" which is characterized by the quartic power of the UV-frequency of the original Hawking radiated modes. This has the clear indication that there remains the information of relative abundance of the Hawking radiated quanta within the growth of the IR-modes generated by the UV Hawking radiated modes. In view of the inevitability of the appearance of the UV-IR coupling through the presence of the quantum potential term in these analogue gravity models, the present exercise of ours does provide a better look at the prevailing situation in such systems.

Let's import the model as set up in the previous chapter till $\mathcal{O}(\epsilon^2)$ accuracy on two independent scales (i.e., usual small scales: $x \equiv x^\mu$ and the other being large scales: $X \equiv X^{\bar{\mu}}$) as given by Eqs.(4.10), (4.11) and (4.12). For convenience, lets recall these informations below:

$$\text{analogue gravity model}\begin{cases} \mathcal{O}(1): & \partial_\mu f^{\mu\nu}\partial_\nu \vartheta_1(x) = 0, \\ \mathcal{O}(\epsilon): & \Big\{\partial_\mu f_1^{\mu\bar{\mu}}\partial_{\bar{\mu}} + \partial_{\bar{\nu}} f_1^{\bar{\nu}\nu}\partial_\nu\Big\}\Big(\vartheta_2(X)\vartheta_1(x)\Big) = 0, \\ \mathcal{O}(\epsilon^2): & \partial_{\bar{\mu}} f_2^{\bar{\mu}\bar{\nu}}\partial_{\bar{\nu}}\vartheta_2(X) - \mathfrak{m}^2\,\vartheta_2(X) = 0. \end{cases} \tag{5.1}$$

Here $f^{\mu\nu}(x)$, $f_1^{\mu\bar{\mu}}(x, X)$ or $f_1^{\bar{\nu}\nu}(X, x)$ and $f_2^{\bar{\mu}\bar{\nu}}(X)$ which are all constructed as symmetric 4×4 matrices explicitly written later, at three different orders $\mathcal{O}(1)$, $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ of dynamics respectively. Here $\mathfrak{m}$ is the mass of the large length scale phonon modes, while it is strikingly

[45] Refer to the line element given by Eq.(55) of ref.[21]

found to be a finite function of the $\vartheta_1(x)$ field (and thus $O(\epsilon^2)$-dynamics gives rise to the 'massive' free Klein-Gordon (KG) field $\vartheta_2(X)$ in large length scales).

Similarly like Eq.(4.13) in the previous chapter, the general background velocity[46] of the fluid is

$$\frac{1}{m}\nabla\vartheta_0(x) \;=\; \mathbf{v}(x) \;=\; v_r(\mathbf{r})\,\hat{r} + v_\theta(\mathbf{r})\,\hat{\theta} + v_\phi(\mathbf{r})\,\hat{\phi}. \tag{5.2a}$$

For sake of clarity,

$$v_r(\mathbf{r}) = \frac{1}{m}\,\partial_r\vartheta_0(x)\,,\quad v_\theta(\mathbf{r}) = \frac{1}{m}\frac{1}{r}\,\partial_\theta\,\vartheta_0(x)\,,\quad v_\phi(\mathbf{r}) = \frac{1}{m}\frac{1}{r\sin\theta}\,\partial_\phi\,\vartheta_0(x). \tag{5.2b}$$

From Eq.(3.12), the local speed of sound is $c_s = \sqrt{\frac{n_0\,\mathfrak{g}}{m}}$. (5.2c)

Now we are going to show the structures of the f-matrices to present our model. At $O(1)$ [in Eq.(4.10)], the $[f^{\mu\nu}(x)]$ matrix is of the following form,

$$[f^{\mu\nu}] \;=\; \frac{r^2\sin\theta}{\mathfrak{g}}\begin{pmatrix} -1 & -v_r & -\frac{v_\theta}{r} & -\frac{v_\phi}{r\sin\theta} \\ -v_r & \left(c_s^2 - v_r^2\right) & -\frac{v_r v_\theta}{r} & -\frac{v_r v_\phi}{r\sin\theta} \\ -\frac{v_\theta}{r} & -\frac{v_\theta v_r}{r} & \frac{\left(c_s^2-v_\theta^2\right)}{r^2} & -\frac{v_\theta v_\phi}{r^2\sin\theta} \\ -\frac{v_\phi}{r\sin\theta} & -\frac{v_\phi v_r}{r\sin\theta} & -\frac{v_\phi v_\theta}{r^2\sin\theta} & \frac{\left(c_s^2-v_\phi^2\right)}{r^2\sin^2\theta} \end{pmatrix}. \tag{5.3}$$

[46] As imported from the previous chapter(s), say from Eq.(3.11).

In Eq.(4.11), $[f_1^{\mu\bar{\mu}}(x,X)]$ and $[f_1^{\bar{\nu}\nu}(X,x)]$ are the two matrices with their each and every corresponding entry being exactly the same, i.e.,

$$[f_1^{\mu\bar{\mu}}] \equiv \frac{R^2\sin\Theta\, r^2\sin\theta}{\mathfrak{g}}\begin{pmatrix} -1 & -v_r & -\frac{v_\theta}{R} & -\frac{v_\phi}{R\sin\Theta} \\ -v_r & \left(c_s^2 - v_r^2\right) & -\frac{v_r v_\theta}{R} & -\frac{v_r v_\phi}{R\sin\Theta} \\ -\frac{v_\theta}{r} & -\frac{v_\theta v_r}{r} & \frac{\left(c_s^2-v_\theta^2\right)}{Rr} & -\frac{v_\theta v_\phi}{Rr\sin\Theta} \\ -\frac{v_\phi}{r\sin\theta} & -\frac{v_\phi v_r}{r\sin\theta} & -\frac{v_\phi v_\theta}{Rr\sin\theta} & \frac{\left(c_s^2-v_\phi^2\right)}{R\sin\Theta\, r\sin\theta} \end{pmatrix} \equiv [f_1^{\bar{\nu}\nu}]. \quad (5.4)$$

And finally the $[f_2^{\bar{\mu}\bar{\nu}}(X)]$ matrix, appearing in $O(\epsilon^2)$-dynamics [i.e., Eq.(4.12)], is given by,

$$[f_2^{\bar{\mu}\bar{\nu}}] = \frac{R^2\sin\Theta}{\mathfrak{g}}\begin{pmatrix} -1 & -v_r & -\frac{v_\theta}{R} & -\frac{v_\phi}{R\sin\Theta} \\ -v_r & \left(c_s^2 - v_r^2\right) & -\frac{v_r v_\theta}{R} & -\frac{v_r v_\phi}{R\sin\Theta} \\ -\frac{v_\theta}{R} & -\frac{v_\theta v_r}{R} & \frac{\left(c_s^2-v_\theta^2\right)}{R^2} & -\frac{v_\theta v_\phi}{R^2\sin\Theta} \\ -\frac{v_\phi}{R\sin\Theta} & -\frac{v_\phi v_r}{R\sin\Theta} & -\frac{v_\phi v_\theta}{R^2\sin\Theta} & \frac{\left(c_s^2-v_\phi^2\right)}{R^2\sin^2\Theta} \end{pmatrix}. \quad (5.5)$$

Up to this point, we have pretty much sketched the basic introduction of our proposed model describing the dynamics of the phonon modes at different length scales and at different orders of the parameter ϵ. Our main motive is to try investigating the massive KG equation found at $O(\epsilon^2)$ in detail through a simple mathematical framework.

5.2 Scale reversion of the dynamics ($\partial_{\bar{\mu}} \to \frac{1}{\epsilon}\partial_\mu$)

In the previous chapter, we analyzed the situation on a flat background. But, for the large scale dynamics to see the curvature of spacetime, we must revert back systematically to the small length scales and this is exactly where our present analysis stands out to be very different from our previous analysis on flat spacetime.

On reversion back to the small scale dynamics from the large scales, each large scale spacetime derivative does generate a factor of $1/\epsilon$ (i.e., $\partial_{\bar{\mu}} \to \frac{1}{\epsilon}\partial_\mu$) while undergoing the switching of scales.

Till now, we are yet to talk about the constraint equation we found at $O(\epsilon)$, see Eq.(4.11). First of all, we would start by identifying the construction of two matrices, viz. $[f^{\mu\bar{\mu}}_{1,\mathbb{I}}(x,X)]$ and $[f^{\bar{\nu}\nu}_{1,\mathbb{II}}](X,x)$, given by

$$\left(R^2\sin\Theta\times[f^{\mu\bar{\mu}}_{1,\mathbb{I}}]\right)=[f^{\mu\bar{\mu}}_{1}]\equiv[f^{\bar{\nu}\nu}_{1}]=\left(r^2\sin\theta\times[f^{\bar{\nu}\nu}_{1,\mathbb{II}}]\right)\ , \tag{5.6}$$

with reference to Eq.(5.4). This readily gives the permit to rewrite the $O(\epsilon)$-dynamics in the following manner,

$$\left\{\left(R^2\sin\Theta\right)\times\partial_\mu f^{\mu\bar{\mu}}_{1,\mathbb{I}}\,\partial_{\bar{\mu}}+\left(r^2\sin\theta\right)\times\partial_{\bar{\nu}}f^{\bar{\nu}\nu}_{1,\mathbb{II}}\,\partial_\nu\right\}\left(\vartheta_2(X)\vartheta_1(x)\right)=0 \tag{5.7}$$

Through scale reversion, $R^2\sin\Theta$ written above naturally becomes $r^2\sin\theta\neq 0$, $\partial_{\bar{\mu}}\to\frac{1}{\epsilon}\partial_\mu$ as just mentioned above and we find that both $[f^{\mu\bar{\mu}}_{1,\mathbb{I}}(x,X)]$ and $[f^{\bar{\nu}\nu}_{1,\mathbb{II}}(X,x)]$ strikingly take the form of $[f^{\mu\nu}(x)]$ as given by Eq.(5.3), the field-amplitude $\vartheta_2(X)$ gets scale transformed to give rise to a new complex scalar function, hence $\left(\vartheta_2(X)\,\theta_1(x)\right)\to\Xi(x)$. Thus the above Eq.(5.7) is rewritten in the scale-reversed form as,

$$\partial_\mu f^{\mu\nu}\partial_\nu\ \Xi(x)=0. \tag{5.8}$$

Hence we come up with exactly the same dynamics for $\Xi(x)$ at $O(\epsilon)$ as we had obtained for only $\vartheta_1(x)$ at $O(1)$, see Eq.(4.10) in the previous chapter. This clearly indicates that we are not going to get anything new at this stage because $O(\epsilon)$-dynamics practically captures the same field over small length scales. This happens particularly because there is no source term at $O(\epsilon)$.

So we have to move on to analyzing the next order, i.e. $O(\epsilon^2)$- dynamics, to see the new structure in the field after having it reverted back to the small scales. In the beginning, the full expression of the mass term in $O(\epsilon^2)$-dynamics contains a factor[47] of $R^2\sin\Theta$. But through the process of scale reversion, this factor of $R^2\sin\Theta$ obviously becomes $r^2\sin\theta$ as discussed above giving rise to a re-scaled mass $\mathfrak{m}$ which gets inserted in the scale-reversed massive KG equation [i.e., Eq.(5.9)] below. From now on, throughout the paper, whenever we speak about $\mathfrak{m}$, we would only refer to this re-scaled $\mathfrak{m}$ appearing upon scale reversion. The general expression of $\mathfrak{m}$, which was obtained using *Mathematica 9.0* package, is extremely lengthy and too cluttered and hence is called off from being explicitly shown here.

[47] This can be regarded as a coordinate-artifact due to the spherical polar coordinates. In cartesian, the formation of $\partial_{\bar{\mu}}f^{\bar{\mu}\bar{\nu}}_2\partial_{\bar{\nu}}\vartheta_2(X)$ while constructing Eq.(4.12) would not have required a multiplication by any factor from left and hence, consequently, the large scale signature would have been completely absent in the expression of $\mathfrak{m}$. See footnote 17 of our published work [3].

By inspection, its quite evident that $[f_2^{\bar{\mu}\bar{\nu}}(X)]$ in Eq.(4.12) would again take the form exactly as $[f^{\mu\nu}(x)]$ on switching back to the small scales and $\vartheta_2(X) \rightarrow \varphi(x)$ as mentioned already. Therefore Eq.(4.12) gets scale-transformed as the following,

$$\partial_\mu f^{\mu\nu} \partial_\nu \varphi(x) - \epsilon^2 \mathfrak{m}^2 \varphi(x) = 0, \tag{5.9}$$

where $[f^{\mu\nu}]$ is already given by Eq.(5.3).

Eq.(5.9) is purposely multiplied by a real scalar constant $\mathfrak{g}/c_s$ and gives rise to,

$$\partial_\mu f^{\mu\nu}_{\text{New}} \partial_\nu \varphi(x) - \frac{\mathfrak{g}}{c_s} \epsilon^2 \mathfrak{m}^2 \varphi(x) = 0, \tag{5.10}$$

where the contravariant f-matrix now being scaled up as

$$f^{\mu\nu}_{\text{New}} = \frac{\mathfrak{g}}{c_s}[f^{\mu\nu}] = \frac{r^2 \sin\theta}{c_s} \begin{pmatrix} -1 & -v_r & -\frac{v_\theta}{r} & -\frac{v_\phi}{r\sin\theta} \\ -v_r & \left(c_s^2 - v_r^2\right) & -\frac{v_r v_\theta}{r} & -\frac{v_r v_\phi}{r\sin\theta} \\ -\frac{v_\theta}{r} & -\frac{v_\theta v_r}{r} & \frac{\left(c_s^2 - v_\theta^2\right)}{r^2} & -\frac{v_\theta v_\phi}{r^2\sin\theta} \\ -\frac{v_\phi}{r\sin\theta} & -\frac{v_\phi v_r}{r\sin\theta} & -\frac{v_\phi v_\theta}{r^2\sin\theta} & \frac{\left(c_s^2 - v_\phi^2\right)}{r^2\sin^2\theta} \end{pmatrix}. \tag{5.11}$$

This process of scale reversion is actually a very important step here. Had we not switched back to the small scales, the spacetime metric arising out of $[f_2^{\bar{\mu}\bar{\nu}}(X)]$ corresponding to the large scale dynamics would have defined the background geometry to be effectively flat because its entries are basically the velocity field components all of which were retained at small scales (see Eq.(5.5)) and thus act as just constants w.r.t the large scale spacetime derivatives.

5.3 The covariant massive KG field $\varphi(x)$ on a $(3+1)$D curved spacetime

In order to cast the above Eq.(5.10) into a Lorentz invariant form (strictly speaking - invariance under General Coordinate Transformation or *Diffeomorphism*), it is required to identify the corresponding covariant structure and hence the introduction of the effective metric (or in other words, acoustic metric) in place of the respective f-matrix is essential. From this point onward, we talk about the 'covariant' massive minimally coupled free KG equation.

Let, $[\mathsf{g}_{\mu\nu}(x)]$ be the general acoustic metric that actually defines the $(3+1)$D curved spacetime under consideration with its determinant, given by, $\mathsf{g} = \det[\mathsf{g}_{\mu\nu}(x)]$. Considering Eq.(5.10), one identifies

$$f^{\mu\nu}_{\text{New}} = \sqrt{|\mathsf{g}|}\mathsf{g}^{\mu\nu}. \tag{5.12}$$

Thus, similarly like Eq.(4.17) in the previous chapter, we have

$$\mathsf{g}^{\mu\nu} = \frac{1}{\sqrt{|\mathsf{g}|}} f^{\mu\nu}_{\text{New}} = \frac{1}{c_s r^2 \sin\theta} f^{\mu\nu}_{\text{New}}. \tag{5.13}$$

Now, from the above Eq.(5.13), it is just trivial to find the acoustic metric which is of the following form,

$$[\mathsf{g}_{\mu\nu}(x)] = \begin{pmatrix} -\left(c_s^2 - \mathbf{v}^2\right) & -v_r & -v_\theta r & -v_\phi r \sin\theta \\ -v_r & 1 & 0 & 0 \\ -r v_\theta & 0 & r^2 & 0 \\ -r \sin\theta v_\phi & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}. \tag{5.14}$$

It should be noted that, in general relativity, the spacetime metric (which does bear the feature of the background geometry) is related to the distribution of matter (i.e., the stress-energy tensor) through the Einstein's-Field-Equations; whereas, the acoustic metric $[\mathsf{g}_{\mu\nu}(x)]$ in the context of analogue spacetime happens to be related to the background velocity field $\mathbf{v}(x)$ as well as the local speed of sound (c_s) in a way more simpler algebraic fashion. Some striking features of this $[\mathsf{g}_{\mu\nu}(x)]$ from topological aspects and regarding 'stable causality' have been discussed by Visser in [21]-pp.1773-1774.

Finally, Eq.(5.10) is re-written in the standard covariant form, given by,

$$\frac{1}{\sqrt{|\mathsf{g}|}}\partial_\mu \left(\sqrt{|\mathsf{g}|}\mathsf{g}^{\mu\nu}\partial_\nu\right) \varphi(x) - \frac{1}{\sqrt{|\mathsf{g}|}} \frac{\mathfrak{g}}{c_s} \epsilon^2 \mathfrak{m}^2 \varphi(x) = 0,$$

$$\text{i.e.,} \quad \boxed{\left(\nabla_\mu \nabla^\mu - \mathcal{M}^2\right) \varphi(x) = 0}\ , \quad [\text{covariant massive free KG field } \varphi(x)] \tag{5.15}$$

where ∇_μ is obviously the covariant derivative and the final mass-term $\mathcal{M}$ is determined through

$$\boxed{\mathcal{M}^2 = \frac{1}{\sqrt{|\mathsf{g}|}}\frac{\mathfrak{g}}{c_s}\epsilon^2\mathfrak{m}^2}\ , \qquad \text{where, } \mathfrak{m}^2 = \frac{\xi_0^2}{\mathfrak{g}}\frac{\sin\theta}{r^2}\frac{1}{\vartheta_1(x)}Q_1(x), \tag{5.16}$$

where $\mathsf{g} = -c_s^2 r^4 \sin^2\theta$ in our present model [refer to Eq.(5.13)] and $Q_1(x)$ is complicated function of $\vartheta_1(x)$ and the components of the background velocity field $\mathbf{v}(x)$. For a canonical acoustic black hole, which we will study in the following section where $\mathbf{v}(x) \equiv v_r(\mathbf{r})\hat{r}$ with $v_\theta = 0 = v_\phi$, the expression of $Q_1(x)$ takes a relatively simpler form and that is explicitly found later in the Appendix section towards the end of this thesis.

5.4 The canonical acoustic Black Hole (from a nonlocal BEC)

With a strong motive to examine how closely the acoustic metric $[\mathsf{g}_{\mu\nu}(x)]$ in Eq.(5.14) can get to mimic the standard Schwarzschild geometry in gravity, one usually considers some specific symmetry in the analogue spacetime to move ahead. If one starts by considering an analogue gravity scenario in a spherically symmetric flow (we'll consider the flow to be non-relativistic here) of a barotropic incompressible inviscid fluid, one comes up with a solution called *canonical acoustic black hole* found by Visser [21] in 1998.

In principle, we would restrict ourselves only to the stationary[48], non-rotating, asymptotically flat canonical acoustic black holes. Thus, in our following prescription, the notions of *apparent* and *event horizons* (acoustic)[49] coincide and the distinction becomes immaterial. In the language of the standard general relativity, an event horizon is a null hypersurface that separates those spacetime points connected to infinity through timelike path from those that are not [91].

Since a canonical acoustic BH, as considered here, is indeed stationary and asymptotically flat, every event horizon is a Killing horizon for some Killing vector field, say σ^μ. Due to the time-translational and axial symmetry of the metric [refer to Eq.(5.21) later], obviously there are two Killing vector fields, viz. $\sigma_{(t)} \equiv \partial_t$ and $\sigma_{(\phi)} \equiv \partial_\phi$ which go from timelike to spacelike and vice-versa at the event horizon. *Killing horizon* is formally defined to be the null hypersurface on which the Killing vector field becomes null.

The acoustic (Killing) horizon is formed once the radial component of the background fluid-velocity (v_r) exceeds the local speed of sound (c_s), refer to Eq.(45) of [21].

[48] Stationary solutions are of special interest and significance because they are regarded as the "end states" of a gravitational collapse.

[49] The event horizon is a global feature, it could be difficult to actually locate such a boundary while being handed with a metric in an arbitrary set of coordinates. Usually it is defined to be the boundary of the region from which even the null geodesics can not escape - strictly speaking, this is *future* event horizon.

5.4.1 Covariant massive KG equation in canonically curved analogue spacetime [$\mathbf{v}(x) \equiv v_r(r)\hat{r}$]

Since, the classical mean-field density n_0 and phase $\vartheta_0(x)$ have got to satisfy the continuity equation (see Eq.(3.2) in Chapter-2), so clearly,

$$\because\ 0 = \partial_t n_0 + \frac{1}{m}\boldsymbol{\nabla}.\Big(n_0\,\boldsymbol{\nabla}\vartheta_0(x)\Big) = n_0\,\boldsymbol{\nabla}.\mathbf{v}(x), \ \therefore\ |\mathbf{v}(x)| \propto \frac{1}{r^2}. \tag{5.17}$$

And thus, through a normalization constant finite $r_0 > 0$, the background velocity field [50] is set to be

$$|\mathbf{v}(x)| \equiv v_r(r) = c_s\frac{r_0^2}{r^2} \quad (\forall\, 0 < r < \infty). \tag{5.18}$$

Considering $v_\theta = 0 = v_\phi$, Eq.(5.14) gives rise to the exact acoustic metric that describes the present scenario. The corresponding line element is given by,

$$ds^2 = -c_s^2 dt^2 + \Big(dr \pm v_r\,dt\Big)^2 + r^2\Big(d\theta^2 + \sin^2\theta\,d\phi^2\Big). \tag{5.19}$$

It is to be noted that when $v_r > 0$, which is also the usual convention, then it would be a "−" sign in front of $v_r dt$ in the second term on r.h.s of the above Eq.(5.19). Otherwise, it would be a "+" sign over there when $v_r < 0$, i.e. when the fluid flow is considered to be in the opposite direction.

Instead of the laboratory time t, one can now introduce the analogue-Schwarzschild-time coordinate τ via the simple coordinate transformation as

$$t \longrightarrow \tau = t \mp \left(\frac{r_0}{2c_s}\tan^{-1}(r/r_0) + \frac{r_0}{4c_s}\ln\left|\frac{1-\frac{r}{r_0}}{1+\frac{r}{r_0}}\right|\right), \tag{5.20}$$

and, using Eq.(5.18) and Eq.(5.19), it readily gives rise to a somewhat "Schwarzschild-like" line element[51] describing a canonical acoustic black hole, given by,

$$ds^2 = -\frac{c_s^2}{r^4}\Delta(r)\,d\tau^2 \ + \ \frac{r^4}{\Delta(r)}dr^2 \ + \ r^2\Big(d\theta^2 \ + \ \sin^2\theta\,d\phi^2\Big), \quad \text{where, } \Delta(r) = r^4 - r_0^4\,. \tag{5.21}$$

It is redundant to read off the acoustic metric from the above line element [Eq.(5.21)] as the following,

$$[\mathrm{g}_{\mu\nu}]_{\text{Canonical BH}} \equiv \left(-\frac{c_s^2}{r^4}\Delta(r)\,,\ \frac{r^4}{\Delta(r)}\,,\ r^2\,,\ r^2\sin^2\theta\right) \tag{5.22}$$

[50] Refer to Eq.(54) of [21].

[51] Refer to Eq.(56) of [89].

and we see that the spacetime of a canonical acoustic BH is asymptotically flat and naturally has a "physical singularity" at $r = 0$, which is again quite obvious from Eq.(5.18) - the background fluid velocity diverges at the center of the canonical acoustic BH. Evidently, r_0 is the Killing horizon (or, *sonic horizon* to be more precise) of the canonical acoustic black hole. As far as the physical picture is concerned, beyond this point $r = r_0$; the fluid essentially becomes supersonic w.r.t an observer sitting at some large $r \to \infty$; i.e. $|\mathbf{v}(x)| \geqslant c_s$ holds true $\forall\, r \leqslant r_0$ which is again quite obvious from Eq.(5.18) as well.

Through the acoustic metric $[\mathsf{g}_{\mu\nu}(x)]_{\text{Canonical BH}}$ in Eq.(5.22), the covariant massive KG equation [Eq.(5.15), in the spacetime of a canonical acoustic black hole, boils down to the following form

$$-\frac{r^4}{c_s^2\Delta(r)}\partial_{\tau\tau}\,\varphi(\tau,\mathbf{r}) + \frac{1}{r^2}\partial_r\left(\frac{\Delta(r)}{r^2}\partial_r\,\varphi(\tau,\mathbf{r})\right) + \frac{1}{r^2\sin\theta}\partial_\theta\Big(\sin\theta\,\partial_\theta\,\varphi(\tau,\mathbf{r})\Big) + \frac{1}{r^2\sin^2\theta}\partial_{\phi\phi}\,\varphi(\tau,\mathbf{r}) - \mathcal{M}^2\,\varphi(\tau,\mathbf{r}) = 0. \tag{5.23}$$

The above Eq.(5.23) has some important features:

1. The spacetime given by Eq.(5.21) is clearly *static* and has a time-translational symmetry. Thus the temporal part of $\varphi(\tau,\mathbf{r})$ that solves the above differential equation [Eq.(5.23)] can easily be separated out as $e^{-i\omega\tau}$ $(\forall\, 0 < \omega < \infty)$, where ω is the frequency (or, equivalently *energy* in $\hbar = 1$ unit) of the particles associated to the $\varphi(\tau,\mathbf{r})$ field.

2. $[\mathsf{g}_{\mu\nu}(x)]_{\text{Canonical BH}}$ describes a spacetime that also has a rotational invariance with respect to ϕ and similarly the azimuthal part of the solution to Eq.(5.23) is obviously $e^{i\mathsf{m}\phi}$, where $\mathsf{m} = \pm 1, \pm 2, \pm 3, \ldots$ is the azimuthal quantum number.

3. From Eq.(5.23), its evident that the general angular solution can be given in terms of the standard spherical harmonics,

$$\mathcal{Y}^l_{\mathsf{m}}(\theta,\phi) = \mathcal{P}^l_{\mathsf{m}}(\cos\theta)e^{i\mathsf{m}\phi}, \tag{5.24}$$

   where $\mathcal{P}^l_{\mathsf{m}}$'s are obviously the Legendre polynomials with l being an integer such that $|\mathsf{m}| \leqslant l$.

5.4.2 The Radial solution $\mathcal{R}(r)$

Therefore, to solve Eq.(5.23), we can consider the following ansatz as

$$\varphi(\tau, \mathbf{r}) = \frac{1}{r}\mathcal{R}(r)\mathcal{Y}^l_{\mathrm{m}}(\theta, \phi)e^{-i\omega\tau}, \tag{5.25}$$

where $\mathcal{R}(r)$ is just the radial function to be determined.

Substituting Eq.(5.25) back into Eq.(5.23), we find that

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{\Delta(r)}{r^2}\frac{d}{dr}\left(\frac{\mathcal{R}(r)}{r}\right)\right) + \left[\frac{\omega^2 r^4}{c_s^2\Delta(r)} - \left(\mathcal{M}^2 + \frac{l(l+1)}{r^2}\right)\right]\frac{\mathcal{R}(r)}{r} = 0. \tag{5.26}$$

Now this has become a linear second-order ordinary differential equation in r for an undetermined function $\mathcal{R}(r)$. We go on reducing Eq.(5.26) further to check the singularity (if any) at various points, because in order to solve the radial differential equation, we are about to pick the Frobenius ansatz for $\mathcal{R}(r)$ and adopt the method of series solution.

By inspection, we find the nature of singularities (for detailed explanations, see Appendix) of the above differential equation [Eq.(5.26)] as the following

1. $r = 0$ is an apparent 'regular' singular point (it is to be understood clearly that $r = 0$ is indeed a point of physical singularity for the metric [Eq.(5.22)] of the ODE, but not for the above ordinary differential equation [Eq.(5.26)]. For Eq.(5.26), $r = 0$ is a regular or removable singular point. Refer to Appendix for a detailed argument).

2. $r = r_0$ is also a 'regular' singular point.

3. And the point $r \to \infty$ is an 'irregular' singular point.

We introduce a new coordinate χ in order to simplify the structure of the above Eq.(5.26). The coordinate transformation, actually known as the *Eddington-Finkelstein* tortoise coordinates (also known as *Regge-Wheeler* coordinates), basically allows one to use the new coordinate χ even in the interior region of the acoustic black hole (i.e., when $r < r_0$).

In the present context[52], the coordinate transformation is given by,

$$r \longrightarrow \chi \equiv \chi(r) = \pm\frac{r}{c_s} \mp \frac{r_0}{2c_s}\tan^{-1}\left(\frac{r}{r_0}\right) \pm \frac{r_0}{4c_s}\ln\left|\frac{1-\frac{r}{r_0}}{1+\frac{r}{r_0}}\right|. \tag{5.27}$$

Conventionally, $\chi = +\frac{r}{c_s} - \frac{r_0}{2c_s}\tan^{-1}\left(\frac{r}{r_0}\right) + \frac{r_0}{4c_s}\ln\left|\frac{1-\frac{r}{r_0}}{1+\frac{r}{r_0}}\right|$ and it is evident that

[52] In case of the usual Schwarzschild metric in the standard theory of gravity, this tortoise coordinate becomes $\chi = r + r_s \ln\left|\frac{r}{r_s} - 1\right|$ where r_s is the Schwarzschild radius. See Eq.(5.108) of [91].

1. χ approaches 0 as $r \to 0$,

2. χ approaches $-\infty$ as $r \to r_0$ from the either sides of the acoustic Killing horizon,

3. and, as $r \to +\infty$, χ approaches $+\infty$.

Hence in the exterior region of the acoustic black hole, i.e., $\forall\, r_0 < r < +\infty$, χ is found to be continuous: $-\infty < \chi < +\infty$.

The tortoise coordinate is intended to grow infinite at the appropriate rate such as to cancel out the singular behavior of the spacetime at $r = r_0$ (the coordinate-singularity is quite vivid from the Eq.(5.22)) which is essentially nothing but the artifact of the choice of coordinates.

Via the above transformation described in Eq.(5.27), one can easily reduce Eq.(5.26) to the following form,

$$\frac{d^2\mathcal{R}(r)}{d\chi^2} + \left[\omega^2 - \left(\mathcal{M}^2 + \frac{\bar{l}}{r^2} + \frac{4r_0^4}{r^6}\right) c_s^2 \left(1 - \frac{r_0^4}{r^4}\right)\right] \mathcal{R}(r) = 0, \quad \text{where, } \bar{l} \equiv l(l+1). \tag{5.28}$$

Our aim is to find a series solution of the above Eq.(5.28) valid in the exterior region of the space-time; i.e., $\forall\, r > r_0$. We pick an ansatz, as chosen by Elizalde [90], of the following form,

$$\mathcal{R}(r) = \alpha\, e^{\pm i\,[\Bbbk\,\chi\, +\, h(\rho)]}\ ; \tag{5.29a}$$

$$\text{where, } \rho = 1 - \frac{r_0}{r}, \quad h(\rho) = \beta\, \ln(1-\rho) + \sum_{n=1}^{\infty} \left(a_n\, \rho^n\right), \quad \text{necessarily, } a_1 \neq 0. \tag{5.29b}$$

Here α is any arbitrary constant while $\Bbbk$ in Eq.(5.29a) and β in Eq.(5.29b) are the constants to be determined. One should note that all the a_n-s above are nothing but the Frobenius-coefficients.

In order to avoid any conflict of notations, we would like to clearly mention here that this n in Eq.(5.29b) is simply the dummy index of the infinite sum and has nothing to do with the n which was first introduced previously as density in Eq.(3.1) in Chapter-2. From now on, we would only consider the + sign in front of i on r.h.s of Eq.(5.29a); but one should notice that an ansatz with just $-i$ over there would also do equally.

Clearly in the exterior region, i.e., $\forall\, r \geqslant r_0$, we always have $0 \leqslant \rho \leqslant 1$ (see Eq.(5.29b)) because

$$\lim_{r\to r_0} \rho = 0 \quad ; \quad \lim_{r\to\infty} \rho = 1. \tag{5.30}$$

This is exactly what justifies the form of $h(\rho)$ as considered in Eq.(5.29b) to be legit in the exterior region.

Most of the tedious algebraic expressions are explicitly shown in the Appendix, and we will be sketching only the important steps here. Inserting Eqs.(5.29a) and (5.29b) back into Eq.(5.28), followed by further simplifications (see Appendix, for details), one can rewrite Eq.(5.28) in terms of the variable ρ as the following,

$$\left(-\frac{(1-\rho)^{10}}{r_0^2}\right)\times\Bigg[\frac{\Bbbk^2 r_0^2}{(1-\rho)^{10}}+\frac{2\beta\Bbbk\left(\rho^3-4\rho^2+6\rho-4\right)\rho r_0 c_s}{(1-\rho)^9}+\beta c_s^2\left\{\frac{1}{(1-\rho)^4}-1\right\}\left\{\frac{\beta-i}{(1-\rho)^4}-\beta+5i\right\}$$
$$+\sum_{n=1}^{\infty}\Bigg\{\frac{n^2\left(\rho^3-4\rho^2+6\rho-4\right)^2 a_n^2 c_s^2\boxed{\rho^{2n}}}{(1-\rho)^6}-\frac{1}{(1-\rho)^8}\Big(ic_s n a_n\boxed{\rho^n}\left(\rho^3-4\rho^2+6\rho-4\right)\Big(-c_s(1-\rho)$$
$$\times\left[n(\rho^4-5\rho^3+10\rho^2-10\rho+4)+\rho\Big(2i\beta\left(\rho^3-4\rho^2+6\rho-4\right)+5\rho^3-19\rho^2+26\rho-14\Big)\right]-2i\Bbbk r_0\Big)\Big)\Bigg\}\Bigg]$$
$$+\left[\omega^2+\frac{1}{r_0^2}\left\{c_s^2\rho(\rho-1)^2\left(\rho^3-4\rho^2+6\rho-4\right)\left(\bar{l}+4(\rho-1)^4\right)+\mathcal{M}^2\rho\left(\rho^3-4\rho^2+6\rho-4\right)r_0^2c_s^2\right\}\right]=0. \tag{5.31}$$

By clubbing the corresponding coefficients of the various powers of ρ from the first square-bracket of the above equation, one keeps all of them on the left-hand-side; while the second square-bracket of the above equation is giving rise to the n-independent terms, all of which are moved to the right-hand-side. Thus the above Eq.(5.31) can be neatly re-written as Eq.(C.3), see Appendix.

In order to find the recursion relation, this Eq.(C.3) can now be compacted as,

$$\text{l.h.s of Eq.(C.3)} = \underbrace{\sum_{n=1}^{\infty}\sum_{k=0}^{10}\rho^{n+k}\,\mathfrak{f}_k^{\mathbb{I}}(n)}_{\mathcal{S}_1\ \ \text{(say)}}+\underbrace{\sum_{n=1}^{\infty}\sum_{p=0}^{10}\rho^{2n+p}\,\mathfrak{f}_p^{\mathbb{II}}(n)}_{\mathcal{S}_2\ \ \text{(say)}}$$
$$\Updownarrow$$
$$\text{r.h.s of Eq.(C.3)} = \mathcal{F}(\rho)\ \equiv F_0\rho^0+F_1\rho^1+\ldots+F_{11}\rho^{11}\ , \tag{5.32}$$

having identified the respective coefficient(s) of each power of ρ on both sides by some specified functions, given by, $\mathfrak{f}_k^{\mathbb{I}}(n)$ $(\forall\, k)$ and $\mathfrak{f}_p^{\mathbb{II}}(n)$ $(\forall\, p)$ are obviously defined in consistence with their corresponding explicit forms written in the l.h.s of Eq.(C.3). On the other hand, the F_0, F_1, $\cdots$, F_{11}, as described above in Eq.(5.32), are all independent of n. These are the respective coefficients of ρ^0, ρ^1,..., ρ^{11} in the full source term $\mathcal{F}(\rho)$.

Our motive is to exhaust each and every term of $\mathcal{F}(\rho)$ by the corresponding term(s) picked from $\mathcal{S}_1$ and $\mathcal{S}_2$ via the power-matching of ρ on both sides of Eq.(C.3) and then try investigating the *recursion relation*. For the sake of clarity and lucidness, a vast part of the calculation[53] is shown in details and step by step in the Appendix section towards the end of the thesis.

At this point, we need to refer to the Eqs.(C.6), (C.9), (C.11) and (C.13) from Appendix and having these equations clubbed together, one can rewrite Eq.(5.32) in the following manner,

$$\because \quad (\mathcal{S}_1 + \mathcal{S}_2) = \mathcal{F}(\rho)$$

$$\Rightarrow \sum_{n=1}^{12-k-1} \sum_{k=0}^{10} \rho^{n+k}\, \mathfrak{f}_k^{\mathrm{I}}(n) + \sum_{n=1}^{\frac{12-p_1}{2}-1} \sum_{p_1=0,2,..}^{10} \rho^{2n+p_1}\, \mathfrak{f}_{p_1}^{\mathrm{III}}(n) + \sum_{n=1}^{\frac{13-p_2}{2}-1} \sum_{p_2=1,3,..}^{9} \rho^{2n+p_2}\, \mathfrak{f}_{p_2}^{\mathrm{III}}(n)$$
$$+ \sum_{j=12,13,..}^{\infty} \left[\sum_{k=0}^{10} \mathfrak{f}_k^{\mathrm{I}}(j-k) + \underbrace{\sum_{p=0,1,..}^{10} \mathfrak{f}_p^{\mathrm{III}}\left(\frac{j-p}{2}\right)}_{\forall\,(j-p)=0,2,4,...} \right] \rho^j$$

$$= F_0\rho^0 + F_1\rho^1 + F_2\rho^2 + \cdots + F_{11}\rho^{11} \tag{5.33}$$
$$\Big(\text{obviously, } F_0 = -\omega^2 r_0^2,\;\; F_{11} = 0 \text{ from r.h.s of Eq.(C.3)}\Big)\,.$$

From the above Eq.(5.33), now one can evaluate the undetermined constants (*viz.* $\Bbbk$, β, a_n-s) one by one which were introduced previously in Eqs.(5.29a) and (5.29b).

1. By equating the coefficients of ρ^0 on both sides of Eq.(5.33), we get

$$-\Bbbk^2 r_0^2 = -\omega^2 r_0^2 \;\;\Rightarrow\;\; \Bbbk = \pm\omega. \tag{5.34}$$

   Its to be noted that we take $\Bbbk = +\omega$ from now on in order to consider only the outgoing modes from the sonic horizon towards the external observer.

2. By equating the coefficients of ρ^1 on both sides of Eq.(5.33), we get

$$8a_1 c_s\left(-r_0\omega + 2ic_s\right) - 4c_s\Big(c_s(4i\beta + \bar{l} + 4) + r_0^2\mathcal{M}^2 c_s - 2\beta r_0\omega\Big) = 0,$$

$$\Rightarrow \beta = \frac{2a_1\left(4c_s^2+r_0^2\omega^2\right)+r_0\omega c_s\left(\bar{l}+\mathcal{M}^2r_0^2+4\right)}{8c_s^2+2r_0^2\omega^2} + i\left(\frac{c_s^2\left(\bar{l}+\mathcal{M}^2r_0^2+4\right)}{4c_s^2+r_0^2\omega^2}\right). \tag{5.35}$$

[53]The calculation being extremely tedious, a part of it has been worked out via Mathematica-9.0 package, however the key steps are mentioned systematically.

3. By equating the coefficients of ρ^2 on both sides of Eq.(5.33), we get a_2 given by Eq.(**??**), see Appendix.

4. And so on and so forth, by equating the coefficients of ρ^{11} on both sides of Eq.(5.33) in the same manner, one can determine a_{11} explicitly (in terms of a_1 which is kept nonzero arbitrary since the beginning).

5. Finally, for any general j, one can find the coefficient a_j ($\forall j = 12, 13, 14, \ldots$) from the recursion relation which is deduced later in Appendix, see Eq.(C.14).

It is to be noted that the recursion relation arises out of the square bracket on l.h.s of Eq.(5.33) after having all the source terms fully exhausted. Though the explicit form of the mass term $\mathcal{M}$ is yet to be shown, the coefficients $\Bbbk$, β, a_n-s ($\forall n \neq 1$) are all determined at this stage. Hence through the Eqs.(5.29a) and (5.29b), one basically gets the full structure of the radial solution $\mathcal{R}(r)$.

In order to find $\mathcal{M}$ explicitly, one requires the exact form of $\vartheta_1(x)$ which is nothing but the solution of the $O(1)$-dynamics, i.e., Eq.(5.1).

5.4.3 Obtaining the usual mass-less scalar field $\vartheta_1(x)$

Like Eq.(5.25), one can consider an ansatz for the mass-less scalar field of the following form,

$$\vartheta_1(x) = \frac{1}{r}\mathcal{R}_1(r)\mathcal{Y}^l_{\mathrm{m}}(\theta,\phi)e^{-i\omega_1\tau}, \qquad \text{typically,} \quad \omega_1 >> \omega. \tag{5.36}$$

Inserting this in Eq.(5.23) with $\mathcal{M} = 0$ gives rise to a radial equation [similarly like Eq.(5.28)] of the following form,

$$\frac{d^2\mathcal{R}_1(r)}{d\chi^2} + \left[\omega_1^2 - \left(\frac{\bar{l}}{r^2} + \frac{4r_0^4}{r^6}\right)c_s^2\left(1 - \frac{r_0^4}{r^4}\right)\right]\mathcal{R}_1(r) = 0. \tag{5.37}$$

Keeping the Eqs.(5.29a) and (5.29b) in mind, we pick $\mathcal{R}_1(r)$ to be of the following form,

$$\mathcal{R}_1(r) = \alpha_1 e^{\pm i\,[\Bbbk_1\,\chi\, +\, h_1(\rho)]}\,; \tag{5.38a}$$

$$\text{where, } \rho = 1 - \frac{r_0}{r}, \quad h_1(\rho) = \beta_1 \ln(1-\rho) + \sum_{n=1}^{\infty}\left(b_n\,\rho^n\right), \qquad \text{necessarily, } b_1 \neq 0. \tag{5.38b}$$

By inspection, we figure out the following:

1. Exactly like Eq.(5.34), we conclude that

$$\Bbbk_1 = \pm\omega_1 \tag{5.39}$$

and we again take the '+' sign for the outgoing modes.

2. If we just drop the mass term in Eq.(5.35), we simply come up with β_1. Therefore,

$$\beta_1 \;\equiv\; \beta|_{\mathcal{M}=0} \;=\; \left(b_1 + \frac{(\bar{l}+4)r_0\omega_1 c_s}{8c_s^2 + 2r_0^2\omega_1^2}\right) + i\left(\frac{(\bar{l}+4)c_s^2}{4c_s^2 + r_0^2\omega_1^2}\right). \tag{5.40}$$

3. Similarly, having the mass term dropped from the expression of a_2 in Eq.(**??**), we come up with b_2 given by Eq.(**??**) in the Appendix.

4. And so on till b_{11} in the same manner, followed by the recursion relation for some general b_j ($\forall\, j = 12, 13, 14, \ldots$) helps determine the rest of the coefficients explicitly and all in terms of b_1.

Outside at a finite distance from the sonic horizon ($r \gtrsim r_0$): If one considers the massless solution [see Eq.(5.35)] to be residing just outside the sonic horizon w.r.t some external observer, then the radial coordinate of $\vartheta_1(x)$ is obviously almost of the same order of r_0, i.e., $r \gtrsim r_0$.

In this regime, the measure of the variable $\rho = 1 - \frac{r_0}{r}$ gives a very small number and thus one can fairly restrict oneself to the first order of ρ, while neglecting its higher powers throughout the calculations. And therefore, Eq.(5.38b) is approximated to

$$h_1(\rho) \approx \beta_1 \,\ln(1-\rho) + b_1\,\rho. \tag{5.41}$$

Now inserting the Eqs.(5.39), (5.40) and (5.41) back into Eq.(5.38a), we get

$$\mathcal{R}_1(r) \approx \alpha_1 \exp\Big[\pm\, i\omega_1\,\chi \;\pm\; ib_1\left(1 - \frac{r_0}{r}\right) \;\pm\; \left\{\frac{(\bar{l}+4)c_s^2}{4c_s^2 + r_0^2\omega_1^2} + i\left(-b_1 - \frac{r_0\omega_1 c_s(\bar{l}+4)}{8c_s^2 + 2r_0^2\omega_1^2}\right)\right\}\ln\frac{r}{r_0}\Big]. \tag{5.42}$$

And hence from Eq.(5.35),

$$\vartheta_1(x) \approx \frac{1}{r}\,\alpha_1\left(\frac{r}{r_0}\right)^{\pm\frac{(\bar{l}+4)c_s^2}{4c_s^2+r_0^2\omega_1^2}} \exp\left[\pm\, i\left\{\omega_1\,\chi \;+\; b_1\left(1 - \frac{r_0}{r}\right) \;+\; \left(-b_1 - \frac{r_0\,\omega_1\,c_s(\bar{l}+4)}{8c_s^2 + 2r_0^2\omega_1^2}\right)\ln\frac{r}{r_0}\right\}\right] \times \mathcal{Y}_{\mathrm{m}}^{l}(\theta,\phi)\,e^{-i\omega_1\tau}. \tag{5.43}$$

This above Eq.(5.43) gives the final expression of the mass-less scalar field approximated upto linear ρ. Between the $\pm$ signs inside the exponent above, one should consider the "$-$" sign for the in-going modes from the sonic horizon towards the center of the acoustic black hole, and the "+" sign for the out-going modes. An important thing to be noted here is that the spatial growth of these short wavelength modes goes as $\sim r^{(\omega_1^{-2})}$. This would be later compared with the growth of the large wavelength amplitude modes $\varphi(x)$.

Now we move ahead to find the expression of the mass term.

5.4.4 Deriving the full mass term $\mathcal{M}$

We can identify the following steps we had taken towards arriving at the complete expression of $\mathfrak{m}$, for a canonical acoustic BH, given by Eq.(A.1) (see Appendix-I):

1. Out of that huge lot, we collected a pack of terms at $O(\epsilon^2)$-dynamics [see Eq.(4.12) in the previous chapter or Eq.(5.1)] and equated their sum total to zero in order to form an equation in larger scales where $\vartheta_1(x)$ was treated effectively as a constant.

2. Among the terms written on the l.h.s of this equation, a number of terms got compacted as $\partial_{\bar{\mu}} f_2^{\bar{\mu}\bar{\nu}} \partial_{\bar{\nu}} \vartheta_2(X)$ while the rest were being identified as something proportional to the amplitude field, i.e. $-\mathfrak{m}^2 \vartheta_2(X)$. The expression of $\mathfrak{m}$, until this step, would naturally contain a factor of $R^2 \sin\Theta$ while appearing in the third equation of (5.1).

3. After the scale-reversion (i.e., $\partial_{\bar{\mu}} \to \frac{1}{\epsilon}\partial_\mu$), this factor of $R^2 \sin\Theta$ simply became $r^2 \sin\theta$ and gave rise to the expression of a rescaled mass $\mathfrak{m}$ inserted in Eq.(5.9).

4. In order to keep things from getting too messy and unnecessarily cluttered, we consider the background velocity to be $|\mathbf{v}(x)| = v_r(r)$; which is exactly nothing but the case for a canonical acoustic black hole, see Eq.(5.18). Thus we get a bit tidier expression for $\mathfrak{m}$, given by Eq.(A.1).

With $\vartheta_1(x)$ field in hand, as shown[54] in Eq.(5.43), one can just readily evaluate $\mathfrak{m}|_{\text{for Canonical BH}}$ through Eq.(A.1). Now finding the final mass term $\mathcal{M}$ for a canonical acoustic black hole is simply redundant and a one-step-process. Using Eq.(5.16),

$$\mathcal{M}^2|_{\text{for Canonical BH}} = \frac{1}{c_s r^2 \sin\theta} \frac{\mathfrak{g}}{c_s} \epsilon^2 \, \mathfrak{m}^2|_{\text{for Canonical BH}}. \tag{5.44}$$

[54] Out of two possible signs, we are interested only in the outgoing modes and hence consider the $+i$ in the exponent of the r.h.s of Eq.(5.43) throughout the paper.

Since our formalism is restricted only within the domain of a canonical acoustic black hole, from now on, we would call off the subscript for the mass term(s) and write just $\mathcal{M}$ to refer to the mass term as expressed by the above Eq.(5.44).

While deriving $\mathcal{M}$, we would again restrict ourselves to considering only the most dominant term(s).

After some trivial and tedious algebra, the expression of the mass term is finally given by,

$$\mathcal{M} = \xi\left[\left(\frac{-176c_s^2\omega_1^2 + r_0^2\omega_1^4}{256c_s^4r_0^2} + i\,\frac{-48c_s^2\omega_1 + 3r_0^2\omega_1^3}{32c_s^3r_0^3}\right)\rho^{-4} + O(\rho^{-3}) + \ldots\right]^{1/2} , \tag{5.45a}$$

$$\approx \mathcal{M}_{O(\rho^{-4})} , \tag{5.45b}$$

where,

$$\mathcal{M}^2_{O(\rho^{-4})} = \boxed{\frac{\xi^2}{\left(1-\frac{r_0}{r}\right)^4}\frac{1}{32c_s^4r_0^3}\left(\frac{r_0}{8}\left(-176c_s^2\omega_1^2 + r_0^2\omega_1^4\right) + i\,3c_s\left(-16c_s^2\omega_1 + r_0^2\omega_1^3\right)\right)}. \tag{5.45c}$$

It is interesting to note that, as far as the most dominant terms are concerned, $\mathcal{M}_{O(\rho^{-4})}$ happens to be independent of the choice of a particular spherical harmonic while picking $\vartheta_1(x)$ from Eq.(5.43). Hence this is the most general expression of the mass term of the phonon modes associated to $\varphi(x)$ field in a canonical spacetime within the regime not too far from the sonic horizon.

One may be interested in a more accurate measure of $\mathcal{M}$, and hence the sub-leading contributions could be relevant in that scenario. See Eq.(A.2) in Appendix, where we have given the next two sub-leading contributions in the expression of the mass term.

As it is quite evident from the above Eq.(5.45c) that $\mathcal{M}_{O(\rho^{-4})}$ does depend on the position r and this kind of coordinate dependence of the mass term appears in many contexts of physics - e.g.[60] where Visser *et al.* encountered a position-dependent-mass. But if $\mathcal{M}$ is picked from Eq.(5.45a), then for an observer sitting at a very large r, the mass term becomes a real constant,

$$\text{i.e., } \lim_{r\to\infty}\mathcal{M} = \frac{\xi\omega_1^2}{c_s^2} \tag{5.46}$$

in the asymptotic limit for any arbitrary $\omega_1 \in \Re$.

To keep the notations simpler, from now on, we would refer to $\mathcal{M}_{O(\rho^{-4})}$ in Eq.(5.45c) by calling it just as $\mathcal{M}$ since the following calculations are solely based only on the leading order contributions in the mass term.

5.4.5 Obtaining the 'massive' scalar field $\varphi(x)$

Finally we are on the verge of deriving the expression of massive scalar field. Like Eq.(5.42), one can now obtain the radial contribution to the massive field upto linear ρ in order to consider only the leading order contributions.

With the mass term $\mathcal{M}$ in hand, one obtains β from Eq.(5.35) and then using Eqs.(5.29a) and (5.29b), we come up with

$$\mathcal{R}(r) \approx \alpha \exp\left[\pm i\omega\,\chi \pm i a_1 \left(1 - \frac{r_0}{r}\right) \pm \left\{\left(\frac{c_s^2\left(\bar{l} + \mathcal{M}^2 r_0^2 + 4\right)}{4c_s^2 + r_0^2\omega^2}\right) + i\left(-a_1 - \frac{r_0\omega c_s\left(\bar{l} + \mathcal{M}^2 r_0^2 + 4\right)}{8c_s^2 + 2r_0^2\omega^2}\right)\right\} \ln\frac{r}{r_0}\right]. \tag{5.47}$$

Thus the massive scalar field (see Eq.(5.25)) is finally given by,

$$\varphi(x) \approx \frac{\alpha}{r} \exp\left[\pm i\omega\chi \pm i a_1 \left(1 - \frac{r_0}{r}\right) \pm \left\{\left(\frac{c_s^2\left(\bar{l} + \mathcal{M}^2 r_0^2 + 4\right)}{4c_s^2 + r_0^2\omega^2}\right) + i\left(-a_1 - \frac{r_0\omega c_s\left(\bar{l} + \mathcal{M}^2 r_0^2 + 4\right)}{8c_s^2 + 2r_0^2\omega^2}\right)\right\} \ln\frac{r}{r_0}\right] \times \mathcal{Y}^l_{\mathrm{m}}(\theta,\phi)\, e^{-i\omega\tau},$$

$$\boxed{\begin{aligned} &= \frac{1}{r}\alpha\left(\frac{r}{r_0}\right)^{\pm\left(\frac{2c_s^2(\bar{l}+4)+2c_s^2 r_0^2\mathfrak{R}(\mathcal{M}^2)+r_0^3\omega\mathfrak{I}(\mathcal{M}^2)}{8c_s^2+2r_0^2\omega^2}\right)} \\ &\quad\times \exp\left[\pm i\left\{\omega\chi + a_1\left(1 - \frac{r_0}{r}\right) - \left(a_1 + \frac{c_s r_0\Big(\omega\left(\bar{l}+4\right) + r_0^2\omega\mathfrak{R}(\mathcal{M}^2) - 2c_s r_0\mathfrak{I}(\mathcal{M}^2)\Big)}{8c_s^2 + 2r_0^2\omega^2}\right)\ln\frac{r}{r_0}\right\}\right] \\ &\quad\times \mathcal{Y}^l_{\mathrm{m}}(\theta,\phi)\, e^{-i\omega\tau}, \end{aligned}}$$

$$\tag{5.48}$$

where $\mathcal{M}$ is picked from Eq.(5.45c) with $\mathfrak{R}\big(\mathcal{M}^2\big)$ and $\mathfrak{I}\big(\mathcal{M}^2\big)$ being its real and imaginary parts respectively.

The above expression clearly indicates that at a fixed r, the growth rate over space is $\sim r^{\frac{\|\mathcal{M}^2\|}{\omega^2}}$. So, from Eq.(5.45c), the growth rate of these large wavelength modes, for a specific mode of frequency ω, turns out to be actually $\sim r^{(\omega_1^4)}$ which encodes the information of the supposedly Hawking radiated modes (i.e. $\vartheta_1(x)$ field). This gives rise to the low frequency (or larger wavelength) band of ω, i.e. $\varphi(x)$ field which, in the absence of the mass term (or in other words when $\xi^2 \approx 0$), is not distinguishable from the primary ω_1-modes. Obviously the smaller

ω-modes would grow faster and effectively extract more energy from the ω_1-modes which are supposedly Hakwing radiated.

In the present chapter, we systematically analyzed the consequences of the presence of the quantum potential term in the dynamics of a condensate on the perspectives of analogue Hawking radiation. Here we have worked out this formulation for a canonical acoustic BH configuration in (3 + 1)D spacetime. The quantum potential term causes a UV-IR coupling which can be separated as an independent dynamics at larger length scales without disturbing the Lorentz invariance of the basic KG equation (massless) is something that we have already shown [2] in the previous chapter and this current chapter simply extends the same method to a curved spacetime.

5.5 Outlook

The presence of the UV-IR coupling resulting from the quantum potential would make short wavelength modes to lose energy to large wavelength ones which show up as massive amplitude excitations of the high frequency Hawking radiated modes. In the actual experimental evaluation of analogue Hawking radiation, one can not neglect these large wavelength modes which will grow from primary Hawking radiated quanta and would cause an “information loss" of the actually Hawking radiated modes.

Our present analysis shows that the growth rate of these large wavelength (ω) modes, in a canonical spacetime, holds the clue to keep the underlying physics consistent. In general, a massless scalar field would grow over space near the analogue acoustic Killing / sonic horizon (the region which is accessible in the experiments) as something $\sim r^{(\omega_1^{-2})}$. On the contrary, the massive secondary excitations generated by these primary modes would grow over space as $\sim r^{(\omega_1^4 \omega^{-2})}$ for large ω_1, but ω can be obtained easily from the temporal profile, i.e., $e^{-i\omega\tau}$ in Eq.(5.48), of the large wavelength signal as received by the external observer.

So a careful observation of the ω_1 dependance of the growth rates of these secondary modes can actually reveal the relative abundance of the originally Hawking radiated quanta in the (3 + 1)D canonical spacetime. These massive amplitude modes arise from the quantum connection which is $O(\epsilon^2)$ small. But, at the same time, one should be insured that ω_1 is typically large and that makes this mechanism of secondary excitation generation absolutely relevant in the quantum fluids, like BEC.

This work of ours presents a detailed derivation and analysis of these excitations generated by quantum potential which, in every likelihood, would be a dominant contributor to the loss of correlations which are instrumented in probing the analogue Hawking effect in such systems.

At a glance -

1. the ‘secondary’ ω-modes would grow faster and effectively extract more energy from the ‘primary’ ω_1-modes which are supposedly Hawking radiated.

2. The quantum potential term (i.e., $\hat{D}_2$) causes a UV-IR coupling between ω_1 and ω-bands of phonon excitations in the context of analogue Hawking radiation from a canonical acoustic BH and this is inevitable as observed by an observer.

3. The low frequency band of ω, i.e., $\varphi(x)$-field, in the absence of the mass term (or in other words when $\xi^2 \approx 0$), is not distinguishable from the primary ω_1-modes.

4. The growth rate of the massive $\varphi(x)$-field over space is $\sim r^{\frac{\|\mathcal{M}^2\|}{\omega^2}}$ which is actually $\approx r^{(\omega_1^4 \omega^{-2})}$. For specific ω, this encodes the information of the supposedly Hawking radiated modes (i.e., $\vartheta_1(x)$-field).

5. **Rescue to “Information Loss” in analogue Hawking radiation:** A careful observation of the ω_1 dependance of the growth rates of these ‘secondary’ massive $\varphi(x)$-modes can actually reveal the relative abundance of the originally Hawking radiated quanta in the $(3 + 1)$D canonical curved spacetime.

Chapter 6

Summary and Outlook

In this thesis, we choose Bose-Einstein Condensates (BEC) to be the underlying condensed matter system to set up an analogue gravity model up to some desired accuracy in order to investigate some important and inevitable features of analogue Hawking radiation. Before going into the chapter-wise short discussions, the content of the thesis may be divided into the following main results as summarized below:

1. we show that the healing length (ξ) of the BEC which serves as the analog Planck scale for the analogue gravity system can be tuned to very small values to work within this linear dispersion regime where the main results of Hawking radiation appears. (Paper: [1])

2. We use this tunability of analogue Planck scale in order to develop a set of multiple-scale-modes where the physics below ξ is essentially considered and there's an emergence of the massive scalar excitations for the large wavelength phonon modes. The analysis is done on a $(3 + 1)$D flat spacetime. (Paper: [2])

3. We find that the consequences of this UV-IR coupling can be captured through the generation of large wavelength IR modes and their energetic dependence on the short wavelength UV modes (supposedly Hawking-radiated) when the analysis is extended to a $(3 + 1)$D curved spacetime for a canonical acoustic black hole. (Paper: [3])

   - We further show that the 'secondary' IR modes in their growth rate will retain the lost information regarding the 'primary' Hawking-radiated modes and there does exist a possibility to retrieve the "information loss" within the scope of analogue Hawking radiation.

   - Our way of looking at the Lorentz-breaking trans-Planckian physics is completely new as compared to existing standard literature in this specific field of research where the trans-Planckian issues are dealt through the presence of dispersion and its effect on mode conversion.

- It is evident to observe that even if one doesn't take into account nonlocality which is another of saying just to consider local GP equation, ξ would clearly be replaced by the usual healing length ξ_0 in the limit $a \to 0$ (contact interaction picture). Hence those $\xi^2\nabla^2$ terms would just become $\xi_0^2\nabla^2$ and the effects of the quantum potential would still become extremely significant as these laplacians (for $\xi_0 \neq 0$) would indeed break the Lorentz symmetry and finally give rise to massive scalar excitations in our prescribed model of analogue gravity. As far as the analogue gravity experiments are concerned, things are usually done by throwing away the quantum potential term from the underlying BEC dynamics and thus, even if for $a \to 0$ (for contact interaction with moderately large condensate lifetime) where ξ would just be replaced by ξ_0, all the possible deviations from the standard result of analogue Hawking radiation can be described by the coupling between two set of energy modes (UV-IR) as one of the possible reasons.

In Chapter 3, we show the derivation of the effective acoustic metric $\mathsf{g}_{\mu\nu}(x)$ that emerges out of the GP theory of nonrelativistic BEC with contact-interaction. According to the standard literature, we demonstrate that the dynamics of the phonons (acoustic disturbances) of the condensate closely mimic the dynamical equation for a free minimally coupled massless scalar field in a $(3+1)$D curved spacetime (i.e., the d'Alembertian equation of motion). Then we talk about the usual acoustic approximation where the $\hat{D}_2$-term (that comes from the linearization of quantum potential in the BEC-dynamics) is drastically neglected on physical ground for wavelengths $> \xi_0$ (where $\xi_0 \sim \lambda_{\text{acoustic, Compton}}$), basically in order to identify the covariant structure towards setting up an analogue gravity model. Now we introduce a 3D-gaussian interaction potential of width ($\sim a$, say) instead of the usual δ-function contact-interaction with a strong motive to capture the corrections and modifications due to nonlocal interactions between particles in a Bose gas. We formulate the GP model with the minimal correction due to nonlocal interactions where the additional correction term $\sim O(a^2)$ and has a 3D-Laplacian (the presence of which is quite justified and legit since there is already a ∇^2 in the usual kinetic term of the standard GP equation). It is followed by deriving the modified version of the standard Bogoliubov quartic-spectrum where, from analogue gravity perspectives, the Lorentz-breaking dispersion is found to get pushed more towards the UV-side. This immediately gives rise to a modified healing length ξ instead of the usual ξ_0 in case of the standard contact-interaction. We compare the functional dependance of ξ and ξ_0 over the s-wave scattering length a where ξ is found to decrease more rapidly than $\xi_0 \sim \frac{1}{\sqrt{a}}$ with increasing a via Feshbach resonance. From here on, we derive the analogue gravity model from nonlocal BEC (that we have proposed) by keeping the quantum potential term in the BEC-dynamics. It essentially gives rise to a crucial modification in the expression of the $\hat{D}_2$-term that saves it from being drastically thrown away due to the possibility of tuning ξ with increasing a. The analogy is fully built as soon as one arrives at the dynamical equation for the linearized fluctuations $\tilde{\vartheta}_1(x)$ of the BEC single particle phase in (3 + 1)D flat/curved spacetime

$$\partial_\mu \tilde{f}^{\mu\nu} \partial_\nu \tilde{\vartheta}_1(x) = 0.$$

Until this point, the above $\tilde{f}$-matrix is a function of $|\mathbf{v}|_{\text{fluid}}$ and c_s, but contains the Lorentz-breaking $\hat{D}_2$-term(s) being inserted into its elements and hence, unless we systematically tackle these symmetry breaking contributions, a standard covariant (or, Diffeomorphic rank-2 tensor) 'metric' can not be derived out of $\tilde{f}^{\mu\nu}$. We propose to address these issues which are treated step by step in the following chapters.

In the next chapter, we mainly show that the underlying nonrelativistic BEC with minimal corrections due to nonlocal interactions is very much capable of simulating the free massive Klein-Gordon (KG) field in (3 + 1)D. We start from the stepping stone, i.e., $\partial_\mu \tilde{f}^{\mu\nu} \partial_\nu \tilde{\vartheta}_1(x) = 0$, where there was no sign of massive excitations until now. Considering a uniform condensate, we obtain the simplified expression of $\hat{D}_2$ which $\sim \mathcal{O}(\xi^2)$. This readily gives the license to take a controlled expansion (i.e., binomial approximation because $\xi << 1$ can be made arbitrarily small by increasing a experimentally) of the symmetry-breaking terms residing inside $[\tilde{f}^{\mu\nu}]$. Thus the tuning of ξ helps one tackle the presence of the Lorentz-breaking quantum potential term(s) in restoring the usual covariant picture. Then we introduce independent multiple scales (small scale: $x \equiv x^\mu$ and large scale: $X \equiv X^{\bar{\mu}}$) and capture the physics of the amplitude modulations to the linearized phase fluctuations by identifying

$$\tilde{\vartheta}_1(x) \to \vartheta_2(X)\vartheta_1(x) \quad \text{through multiple scale perturbations} \quad \partial_\mu \longrightarrow \partial_\mu + \epsilon \partial_{\bar{\mu}}$$
$$(\epsilon = \xi/\xi_0 \text{ being a small parameter}).$$

Then we set up the full model up to $\mathcal{O}(\epsilon^2)$ accuracy by considering the dynamics at different orders of ϵ where $\mathcal{O}(1)$ recovers the standard prescription of massless excitations and $\mathcal{O}(\epsilon^2)$ gives the desired dynamics for the minimally coupled large wavelength free phonon modes $\vartheta_2(X)$ with a mass term $\mathfrak{m}$ that strikingly comes out to be a finite function of $\vartheta_1(x)$, given by,

$$\partial_{\bar{\mu}} f_2^{\bar{\mu}\bar{\nu}} \partial_{\bar{\nu}} \vartheta_2(X) - \mathfrak{m}^2 \vartheta_2(X) = 0.$$

The dynamics at intermediate $\mathcal{O}(\epsilon)$ is found to be basically a constraint on top of the massive KG equation. Here we confine ourselves within a constant background flow of fluid which immediately makes the entires of $\tilde{f}$-matrix to be all constants and the analysis was done on flat spacetime. The physical implications of this work can be described as - due to the presence of the nonzero mass term, the flat amplitude of $\vartheta_1(x)$-modes are observationally ruled out. There has to be a nonvanishing threshold for the transverse $K_Z(\neq 0)$ modes even when $K_X = 0 = K_Y$ for a fixed $\mathbf{k}$ as perceived by an observer sitting on lower dimensional space, say $X - Y$ plane. This is unavoidable in the analogue gravity experiments and can act as the effective mass for the Hawking radiated phonons.

In Chapter 5, we start by the scale reversion in order to go back to the old small scales by performing

$$\partial_{\bar{\mu}} \rightarrow \frac{1}{\epsilon}\partial_{\mu}, \quad f_2^{\bar{\mu}\bar{\nu}} \rightarrow f^{\mu\nu}, \quad \text{and,} \quad \vartheta_2(X) \rightarrow \varphi(x).$$

We obtain the effective acoustic metric $[\mathsf{g}_{\mu\nu}(x)]$, through identifying $f^{\mu\nu}(x) = \sqrt{|\mathsf{g}|}\mathsf{g}^{\mu\nu}(x)$, for a canonical acoustic black hole where the background flow is considered to be spherically symmetric with $|\mathbf{v}_{\text{fluid}}| \equiv v_r(\mathbf{r}) \sim \frac{1}{r^2}$. We write the standard covariant dynamics for a massive minimally coupled free KG field as

$$\left(\nabla_{\mu}\nabla^{\mu} - \mathcal{M}^2\right)\varphi(x) = 0, \quad \text{where,} \quad \mathcal{M}^2 = \frac{1}{\sqrt{|\mathsf{g}|}}\frac{\mathfrak{g}}{c_s}\epsilon^2\mathfrak{m}^2 \quad \left(\mathfrak{g} = \frac{4\pi\hbar^2 a}{m},\ s\text{-wave coupling const.}\right).$$

Via a linear transformation on time t, we introduce Schwarzschild time coordinate, say τ, and read off the Schwarzschild-like line element for the analogue spacetime where the series solution of $\varphi(x)$ is derived in full details. Then we systematically address the issues of the presence of the quantum potential term being the root cause of a UV-IR coupling between short wavelength 'primary' phonon modes (i.e., ω_1 band) which are supposedly Hawking-radiated through the Killing horizon of the canonical acoustic black hole and the large wavelength 'secondary' phonon modes (i.e., ω band; $\omega << \omega_1$). Our analysis reveals that these ω-modes, which are indistinguishable from the 'primary' modes in the absence of the mass term (i.e., for $\xi \approx 0$), would eventually grow faster and effectively extract more energy from the supposedly Hawking-radiated ω_1 modes. The growth rate of the massive $\varphi(x)$-field over space is $r^{\frac{\|\mathcal{M}^2\|}{\omega^2}} \approx r^{(\omega_1^4\omega^{-2})}$. For a specific ω, this encodes the information regarding the supposedly Hawking radiated bands of frequencies. A careful observation of the ω_1 dependance of the growth rates of these 'secondary' massive $\varphi(x)$ excitations can actually reveal the relative abundance of the originally Hawking radiated quanta in the $(3+1)$D canonically curved spacetime.

In future, we hope that, our present framework and analysis in deriving the correction to the Hawking-radiated quanta in analogue spacetime will find wider application in addressing trans-Planckian physics from analogue gravity models. In our framework, we have been able to separate the dynamics of the models above and below analogue Planck scale with appropriate coupling between those. This particular way of investigating the system doesn't only keep the large scale dynamics as it is when looked at by neglecting the trans-Planckian physics, but also presents a clearer picture as to how the trans-Planckian physics brings in correction to the large wavelength picture from the instability at shorter wavelengths. We hope that, this model of analysis can be extended in the context of analogue Hawking radiation and could also find applications to other analogue gravity pictures such as analogue particle production processes (cosmological particle production in analogue spacetime) when the trans-Planckian issues are being addressed.

Appendix A

The Mass term in details

In order to tackle some untidy and too cluttered expressions in a proper presentable manner, we will be introducing a few new symbols here, viz. $\mathcal{Q}_1$, $\mathcal{Q}_2$, etc., whenever required.

The following expression of $\mathfrak{m}$ is written in case of a canonical acoustic black hole, where the background velocity is picked by Eq.(5.18).

$$\mathfrak{m}^2|_{\text{for Canonical BH}} = \frac{\xi_0^2}{\mathfrak{g}}\frac{\sin\theta}{r^2}\frac{1}{\vartheta_1(x)}\mathcal{Q}_1(x)|_{\text{for Canonical BH}} \tag{A.1a}$$

where,

$$\begin{aligned}
\mathcal{Q}_1(x)|_{\text{for Canonical BH}} \quad &\equiv \quad \mathcal{Q}_1(x)|_{v_\theta=0=v_\phi} \\
= \Bigg[& r^4 v_r'' \frac{\partial^2\vartheta_1}{\partial t\,\partial r} + 2r^3\frac{\partial^3\vartheta_1}{\partial t^2\,\partial r} + 4r^3 v_r' \frac{\partial^2\vartheta_1}{\partial t\,\partial r} + r^2\frac{\partial^4\vartheta_1}{\partial t^2\,\partial\theta^2} + \frac{r^2}{\sin^2\theta}\frac{\partial^4\vartheta_1}{\partial t^2\,\partial\phi^2} + r^2\cot\theta\frac{\partial^3\vartheta_1}{\partial t^2\,\partial\theta} + r^2 v_r'\frac{\partial^3\vartheta_1}{\partial t\,\partial\theta^2} \\
& + \frac{r^2}{\sin^2\theta}v_r'\frac{\partial^3\vartheta_1}{\partial t\,\partial\phi^2} + r^2\cot\theta\, v_r'\frac{\partial^2\vartheta_1}{\partial t\,\partial\theta} + r^4\frac{\partial^4\vartheta_1}{\partial t^2\,\partial r^2} + 3r^4 v_r'\frac{\partial^3\vartheta_1}{\partial t\,\partial r^2} + v_r\Bigg\{2r^2\frac{\partial^4\vartheta_1}{\partial t\,\partial r\,\partial\theta^2} + \frac{2r^2}{\sin^2\theta}\frac{\partial^4\vartheta_1}{\partial t\,\partial r\,\partial\phi^2} \\
& + 2r^2\cot\theta\frac{\partial^3\vartheta_1}{\partial t\,\partial r\,\partial\theta} + 2r^2\frac{\partial^2\vartheta_1}{\partial t\,\partial r} + 2r^2 v_r'\frac{\partial^3\vartheta_1}{\partial r\,\partial\theta^2} + \frac{2r^2}{\sin^2\theta}v_r'\frac{\partial^3\vartheta_1}{\partial r\,\partial\phi^2} + 2r^2\cot\theta\, v_r'\frac{\partial^2\vartheta_1}{\partial r\,\partial\theta} + 2r^4\frac{\partial^4\vartheta_1}{\partial t\,\partial r^3} \\
& + 6r^3\frac{\partial^3\vartheta_1}{\partial t\,\partial r^2} + r^4 v_r^{(3)}\frac{\partial\vartheta_1}{\partial r} + 4r^3 v_r''\frac{\partial\vartheta_1}{\partial r} + 2r^2 v_r'\frac{\partial\vartheta_1}{\partial r} + 4r^4 v_r'\frac{\partial^3\vartheta_1}{\partial r^3} + 3r^4 v_r''\frac{\partial^2\vartheta_1}{\partial r^2} + 10r^3 v_r'\frac{\partial^2\vartheta_1}{\partial r^2}\Bigg\} \\
& + r^2 v_r^2\left(\frac{\partial^4\vartheta_1}{\partial r^2\,\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^4\vartheta_1}{\partial r^2\,\partial\phi^2} + \cot\theta\frac{\partial^3\vartheta_1}{\partial r^2\,\partial\theta} + 4r\frac{\partial^3\vartheta_1}{\partial r^3} + 2\frac{\partial^2\vartheta_1}{\partial r^2} + r^2\frac{\partial^4\vartheta_1}{\partial r^4}\right) \\
& + r^4 v_r' v_r''\frac{\partial\vartheta_1}{\partial r} + 2r^3\,(v_r')^2\,\frac{\partial\vartheta_1}{\partial r} + 2r^4\,(v_r')^2\,\frac{\partial^2\vartheta_1}{\partial r^2}\Bigg].
\end{aligned} \tag{A.1b}$$

In the above expression, $v_r' \equiv \frac{\partial v_r}{\partial r}$, $v_r'' \equiv \frac{\partial^2 v_r}{\partial r^2}$, and $v_r^{(3)} \equiv \frac{\partial^3 v_r}{\partial r^3}$.

Only upto leading order contributions are considered in the final expression of the mass term in Eq.(5.45c). Here we have given a more accurate measure by taking into account the next two sub-leading contributions, given by,

$$\mathcal{M} \approx \Big[Q_3(r) \ + \ iQ_4(r)\Big]^{1/2}, \tag{A.2}$$

where,

$$Q_3(r) = \frac{\xi^2\omega_1^2\left(-176c_s^2 + r_0^2\omega_1^2\right)}{256c_s^4r_0^2\left(1-\frac{r_0}{r}\right)^4} + \frac{\xi^2\omega_1^2(64c_s^4(88+7\bar{l}) - 8c_s^2(-113+\bar{l})r_0^2\omega_1^2 - 6r_0^4\omega_1^4)}{256c_s^4r_0^2\left(4c_s^2 + r_0^2\omega_1^2\right)\left(1-\frac{r_0}{r}\right)^3} + \frac{\xi^2\omega_1^2\,\bar{l}}{16c_s^2r_0^2\left(1-\frac{r_0}{r}\right)^2} \quad , \tag{A.3a}$$

$$Q_4(r) = \frac{\xi^2\omega_1\left(-48c_s^2 + 3r_0^2\omega_1^2\right)}{32c_s^3r_0^3\left(1-\frac{r_0}{r}\right)^4} + \frac{\xi^2\omega_1(64c_s^4(20+\bar{l}) + 2c_s^2(66-7\bar{l})r_0^2\omega_1^2 - 17r_0^4\omega_1^4)}{32c_s^3r_0^3\left(4c_s^2 + r_0^2\omega_1^2\right)\left(1-\frac{r_0}{r}\right)^3} + \frac{\xi^2\omega_1\,\bar{l}}{4c_sr_0^3\left(1-\frac{r_0}{r}\right)^2} \quad . \tag{A.3b}$$

Appendix B

Nature of singularities: radial differential equation for the canonical acoustic BH

In this section, we would investigate the nature of singularities of Eq.(5.26) at various points. We first rewrite it in the following manner

$$\frac{d^2}{dr^2}\mathcal{R}(r)+\underbrace{\frac{4r_0^4}{r\left(r^4-r_0^4\right)}}_{\mathsf{p}(r)\ \ \text{(say)}}\frac{d}{dr}\mathcal{R}(r)+\underbrace{\frac{\omega^2 r^{10}-c_s^2\mathcal{M}^2\left(r^4-r_0^4\right)r^6-c_s^2\left(r^4-r_0^4\right)\left(l(l+1)r^4+4r_0^4\right)}{c_s^2 r^2\left(r^4-r_0^4\right)^2}}_{\mathsf{q}(r)\ \ \text{(say)}}\mathcal{R}(r)=0. \tag{B.1}$$

1. At $r=0$: $\lim_{r\to 0}\mathsf{p}(r)\to-\infty$, $\lim_{r\to 0}\mathsf{q}(r)\to\infty$, i.e. There's a singularity at $r=0$ point. But we find that

$$\lim_{r\to 0}\left[r\mathsf{p}(r)\right]=-4, \qquad \lim_{r\to 0}\left[r^2\mathsf{q}(r)\right]=4, \tag{B.2}$$

   which refer to $r=0$ to be a *regular* singular point.

2. At $r=r_0$: $\lim_{r\to r_0}\mathsf{p}(r)\to\infty$, $\lim_{r\to r_0}\mathsf{q}(r)\to\infty$, i.e. There's again a singularity at $r=r_0$ point. But

$$\lim_{r\to r_0}\left[(r-r_0)\mathsf{p}(r)\right]=1, \qquad \lim_{r\to r_0}\left[(r-r_0)^2\mathsf{q}(r)\right]=\frac{\omega^2 r_0^2}{16c_s^2}, \tag{B.3}$$

   hold true which guarantee that $r=r_0$ is a *regular* singular point.

3. At $r \to \infty$: We take $r = \frac{1}{r_*}$, thus as $r \to \infty$, $r_* \to 0$. Now obviously $\mathsf{p}(r = \frac{1}{r_*})$ becomes some $\mathsf{p}_1(r_*)$ and $\mathsf{q}(r = \frac{1}{r_*})$ becomes some $\mathsf{q}_1(r_*)$. We have,

$$\lim_{r_* \to 0} \left[\frac{2}{r_*} - \frac{\mathsf{p}_1(r_*)}{r_*^2} \right] \to \infty, \qquad \lim_{r_* \to 0} \left[\frac{\mathsf{q}_1(r_*)}{r_*^4} \right] \to \infty, \tag{B.4}$$

which basically mean $r_* \to 0$ or $r \to \infty$ is a singular point; while

$$\lim_{r_* \to 0} \left[r_* \left(\frac{2}{r_*} - \frac{\mathsf{p}_1(r_*)}{r_*^2} \right) \right] = 2, \qquad \underbrace{\lim_{r_* \to 0} \left[r_*^2 \left(\frac{\mathsf{q}_1(r_*)}{r_*^4} \right) \right] \to \infty}_{\text{i.e. the limit doesn't exist}}. \tag{B.5}$$

Appendix C

The nitty-gritty details of the Frobenius series solution for the canonical acoustic BH

Inserting Eqs.(5.29a) and (5.29b) into the Eq.(5.28), we calculate the l.h.s of Eq.(5.28) part by part.

The first part of l.h.s of Eq.(5.28)

$$\Rightarrow \frac{d^2\mathcal{R}(r)}{d\chi^2} = Q_2(r)\times\left(-\frac{1}{r^{10}}\right)\Big[\mathbb{k}^2 r^{10} + \beta c_s^2\left(r^4 - r_0^4\right)\left(r^4(-i+\beta) - (-5i+\beta)r_0^4\right) - 2\mathbb{k}r^5\beta c_s\left(r^4 - r_0^4\right)$$
$$+\sum_{n=1}^{\infty}\Big\{ -ina_n c_s r_0 \underbrace{\left(1-\frac{r_0}{r}\right)^n}\left(r^3 + r_0 r^2 + r_0^2 r + r_0^3\right)\Big(c_s r_0 r^3(n-1) + c_s r_0^2 r^2(n-1) + c_s r_0^4(2i\beta + n + 5)$$
$$+ c_s r_0^3 r(n-1) + c_s(-2-2i\beta)r^4 + 2i\mathbb{k}r^5\Big) + n^2 a_n^2 c_s^2 r_0^2 \underbrace{\left(1-\frac{r_0}{r}\right)^{2n}}\left(r^3 + r_0 r^2 + r_0^2 r + r_0^3\right)^2\Big\}\Big], \tag{C.1}$$

where the above two terms are under-braced for a reason because, while switching from $r \to \rho$, these two terms obviously become ρ^n and ρ^{2n} respectively which are boxed in Eq.(5.31).

And the second part of l.h.s of Eq.(5.28)

$$\Rightarrow \left[\omega^2 - \left(\mathcal{M}^2 + \frac{\bar{l}}{r^2} + \frac{4r_0^4}{r^6}\right)c_s^2\left(1 - \frac{r_0^4}{r^4}\right)\right]\mathcal{R}(r) \;=\; Q_2(r) \times \left[\omega^2 - \frac{c_s^2\left(r^4 - r_0^4\right)\left(\bar{l}r^4 + \mathcal{M}^2 r^6 + 4r_0^4\right)}{r^{10}}\right], \tag{C.2}$$

where,

$$Q_2(r) = \alpha\, \exp\left[i\left\{\frac{\Bbbk}{4c_s}\left(4r - 2r_0 \tan^{-1}\left(\frac{r}{r_0}\right) + r_0 \ln\left|\frac{1 - \frac{r}{r_0}}{1 + \frac{r}{r_0}}\right|\right) + \sum_n \left(a_n\left(1 - \frac{r_0}{r}\right)^n\right)\right\}\right]\left(\frac{r_0}{r}\right)^{i\beta}.$$

Clearly, $Q_2(r) \neq 0$ because $\alpha \neq 0$. Having the above two Eqs.(C.1) and (C.2) attached together, followed by a trivial algebraic manipulation, we actually come up with Eq.(5.31). Now we present the corresponding coefficients of ρ^n, ρ^{n+1},... $\forall\, n = 1(1)\infty$ in a more jotted down way by rewriting the parent equation (i.e. Eq.(5.28)) in the following manner,

$$\begin{aligned}
&\underbrace{-\Bbbk^2 r_0^2} + \sum_{n=1}^{\infty}\Big[\, 8ina_nc_s\,(2nc_s + i\Bbbk r_0)\underbrace{\rho^n} + 4na_nc_s\,(7\Bbbk r_0 + 2c_s(4\beta - 14in - 7i))\underbrace{\rho^{n+1}}\\
&+ 4ina_nc_s\,(c_s(48i\beta + 89n + 89) + 10i\Bbbk r_0)\underbrace{\rho^{n+2}} + 10na_nc_s\,(3\Bbbk r_0 + 2c_s(26\beta - 34in - 51i))\underbrace{\rho^{n+3}}\\
&+ 4ina_nc_s\,(7c_s(30i\beta + 31n + 62) + 3i\Bbbk r_0)\underbrace{\rho^{n+4}} + 2na_nc_s\,(\Bbbk r_0 + c_s(448\beta - 388in - 970i))\underbrace{\rho^{n+5}}\\
&+ ina_nc_s^2(656i\beta + 493n + 1479)\underbrace{\rho^{n+6}} + 110na_nc_s^2(3\beta - 2in - 7i)\underbrace{\rho^{n+7}} + 22ina_nc_s^2(5i\beta + 3n + 12)\underbrace{\rho^{n+8}}\\
&+ 2na_nc_s^2(11\beta - 6in - 27i)\underbrace{\rho^{n+9}} + ina_nc_s^2(2i\beta + n + 5)\underbrace{\rho^{n+10}} - 16n^2a_n^2c_s^2\underbrace{\rho^{2n}} + 112n^2a_n^2c_s^2\underbrace{\rho^{2n+1}}\\
&- 356n^2a_n^2c_s^2\underbrace{\rho^{2n+2}} + 680n^2a_n^2c_s^2\underbrace{\rho^{2n+3}} - 868n^2a_n^2c_s^2\underbrace{\rho^{2n+4}} + 776n^2a_n^2c_s^2\underbrace{\rho^{2n+5}} - 493n^2a_n^2c_s^2\underbrace{\rho^{2n+6}}\\
&+ 220n^2a_n^2c_s^2\underbrace{\rho^{2n+7}} - 66n^2a_n^2c_s^2\underbrace{\rho^{2n+8}} + 12n^2a_n^2c_s^2\underbrace{\rho^{2n+9}} - n^2a_n^2c_s^2\underbrace{\rho^{2n+10}}\ \Big]\\
=\;& \underbrace{-\omega^2 r_0^2} + 4c_s\left(c_s(4i\beta + \bar{l} + 4) + \mathcal{M}^2 r_0^2 c_s - 2\beta\Bbbk r_0\right)\underbrace{\rho}\\
&- 2c_s\left(c_s\left(-8\beta^2 + 68i\beta + 7\bar{l} + 60\right) + 3\mathcal{M}^2 r_0^2 c_s - 10\beta\Bbbk r_0\right)\underbrace{\rho^2}\\
&+ 4c_s\left(\mathcal{M}^2 r_0^2 c_s + 5\left(-\beta\Bbbk r_0 + c_s\left(-4\beta^2 + 24i\beta + \bar{l} + 20\right)\right)\right)\underbrace{\rho^3}\\
&- c_s\left(\mathcal{M}^2 r_0^2 c_s + 5\left(-2\beta\Bbbk r_0 + 3c_s\left(-12\beta^2 + 64i\beta + \bar{l} + 52\right)\right)\right)\underbrace{\rho^4}\\
&+ 2c_s\left(-\beta\Bbbk r_0 + 3c_s\left(-40\beta^2 + 204i\beta + \bar{l} + 164\right)\right)\underbrace{\rho^5} + c_s^2\left(208\beta^2 - 1044i\beta - \bar{l} - 836\right)\underbrace{\rho^6}\\
&- \left(\beta^2 - 5i\beta - 4\right)\left\{120c_s^2\underbrace{\rho^7} + 45c_s^2\underbrace{\rho^8} - 10c_s^2\underbrace{\rho^9} + c_s^2\underbrace{\rho^{10}}\right\}.
\end{aligned} \tag{C.3}$$

Each power of ρ, in the above equation, is again purposely under-braced to depict the whole picture as vividly as possible in front of the reader.

In the following calculation, we would go on reducing Eq.(5.32) and evaluate the summations systematically. The intermediate steps are shown here to arrive at Eq.(5.33). From Eq.(5.32),

$$\mathcal{S}_1 = \sum_{n=1}^{\infty}\sum_{k=0}^{10}\rho^{n+k}\,\mathfrak{f}^{\mathbb{I}}_k(n) = \left[\sum_{n=1}^{\infty}\rho^{n+0}\,\mathfrak{f}^{\mathbb{I}}_0(n) + \sum_{n=1}^{\infty}\rho^{n+1}\,\mathfrak{f}^{\mathbb{I}}_1(n) + \sum_{n=1}^{\infty}\rho^{n+2}\,\mathfrak{f}^{\mathbb{I}}_2(n) + \ldots + \sum_{n=1}^{\infty}\rho^{n+10}\,\mathfrak{f}^{\mathbb{I}}_{10}(n)\right]. \tag{C.4}$$

For any particular $0 \leqslant k \leqslant 10$, the general term being singled out from $\mathcal{S}_1$ is

$$\begin{aligned}
&\sum_{n=1}^{\infty}\rho^{n+k}\,\mathfrak{f}^{\mathbb{I}}_k(n) \\
&= \sum_{n=1}^{12-k-1}\rho^{n+k}\,\mathfrak{f}^{\mathbb{I}}_k(n) + \underbrace{\sum_{n=12-k}^{\infty}\rho^{n+k}\,\mathfrak{f}^{\mathbb{I}}_k(n)}_{\boxed{n+k=\lambda+12,\ \text{say}}} \\
&= \underbrace{\sum_{n=1}^{12-k-1}\rho^{n+k}\,\mathfrak{f}^{\mathbb{I}}_k(n)}_{\text{finite sum }\forall\, k} + \underbrace{\sum_{\lambda=0,1,..}^{\infty}\rho^{\lambda+12}\,\mathfrak{f}^{\mathbb{I}}_k(\lambda+12-k)}_{\boxed{\lambda+12=j,\ \text{say}}}\,.
\end{aligned} \tag{C.5}$$

$$\therefore\ \mathcal{S}_1 = \sum_{n=1}^{12-k-1}\sum_{k=0}^{10}\rho^{n+k}\,\mathfrak{f}^{\mathbb{I}}_k(n) + \underbrace{\sum_{j=12,13,..}^{\infty}\left(\sum_{k=0}^{10}\mathfrak{f}^{\mathbb{I}}_k(j-k)\right)\rho^{j}}_{\mathfrak{s}_1\ \ (\text{say})}\,. \tag{C.6}$$

Similarly like Eq.(C.4),

$$\mathcal{S}_2 = \sum_{n=1}^{\infty} \sum_{p=0}^{10} \rho^{2n+p}\, \mathfrak{f}_p^{\mathbb{III}}(n)$$

$$= \Bigg[\underbrace{\sum_{n=1}^{\infty} \rho^{2n+0}\, \mathfrak{f}_1^{\mathbb{III}}(n) + \sum_{n=1}^{\infty} \rho^{2n+2}\, \mathfrak{f}_2^{\mathbb{III}}(n) + \ldots + \sum_{n=1}^{\infty} \rho^{2n+10}\, \mathfrak{f}_{10}^{\mathbb{III}}(n)}_{\text{say, } \mathcal{S}_2^{\mathbb{I}} \text{ where } \forall\, p \equiv p_1 = 0,2,\ldots,10}$$

$$+ \underbrace{\sum_{n=1}^{\infty} \rho^{2n+1}\, \mathfrak{f}_1^{\mathbb{III}}(n) + \sum_{n=1}^{\infty} \rho^{2n+3}\, \mathfrak{f}_3^{\mathbb{III}}(n) + \ldots + \sum_{n=1}^{\infty} \rho^{2n+9}\, \mathfrak{f}_9^{\mathbb{III}}(n)}_{\text{say, } \mathcal{S}_2^{\mathbb{II}} \text{ where } \forall\, p \equiv p_2 = 1,3,\ldots,9} \Bigg]. \tag{C.7}$$

Now the general term from the above $\mathcal{S}_2^{\mathbb{I}}$ is singled out as the following,

$$\sum_{n=1}^{\infty} \rho^{2n+p_1}\, \mathfrak{f}_{p_1}^{\mathbb{III}}(n)$$

$$= \sum_{n=1}^{\frac{12-p_1}{2}-1} \rho^{2n+p_1}\, \mathfrak{f}_{p_1}^{\mathbb{III}}(n) \quad + \quad \underbrace{\sum_{n=\frac{12-p_1}{2}}^{\infty} \rho^{2n+p_1}\, \mathfrak{f}_{p_1}^{\mathbb{III}}(n)}_{\boxed{2n + p_1 = \lambda_1 + 12, \text{ say}}}$$

$$= \underbrace{\sum_{n=1}^{\frac{12-p_1}{2}-1} \rho^{2n+p_1}\, \mathfrak{f}_{p_1}^{\mathbb{III}}(n)}_{\text{finite sum } \forall\, p_1} + \underbrace{\sum_{\lambda_1=0,2,..}^{\infty} \rho^{\lambda_1+12}\, \mathfrak{f}_{p_1}^{\mathbb{III}}\left(\frac{\lambda_1 + 12 - p_1}{2}\right)}_{\boxed{\lambda_1 + 12 = j, \text{ say}}}. \tag{C.8}$$

$$\therefore\ \mathcal{S}_2^{\mathbb{I}} = \sum_{n=1}^{\frac{12-p_1}{2}-1} \sum_{p_1=0,2,..}^{10} \rho^{2n+p_1}\, \mathfrak{f}_{p_1}^{\mathbb{III}}(n)$$

$$+ \underbrace{\sum_{j=12,14,..}^{\infty} \left(\sum_{p_1=0,2,..}^{10} \mathfrak{f}_{p_1}^{\mathbb{III}}\left(\frac{j-p_1}{2}\right) \right) \rho^j}_{\mathfrak{s}_2 \text{ (say)}}. \tag{C.9}$$

Again, one can single out the general term from $\mathcal{S}_2^{\mathbb{III}}$ as well,

$$\sum_{n=1}^{\infty} \rho^{2n+p_2}\, \mathfrak{f}_{p_2}^{\mathbb{III}}(n)$$

$$= \sum_{n=1}^{\frac{13-p_2}{2}-1} \rho^{2n+p_2}\, \mathfrak{f}_{p_2}^{\mathbb{III}}(n) \quad + \quad \underbrace{\sum_{n=\frac{13-p_2}{2}}^{\infty} \rho^{2n+p_2}\, \mathfrak{f}_{p_2}^{\mathbb{III}}(n)}_{\boxed{2n+p_2 = \lambda_2 + 12, \text{ say}}}$$

$$= \underbrace{\sum_{n=1}^{\frac{13-p_2}{2}-1} \rho^{2n+p_2}\, \mathfrak{f}_{p_2}^{\mathbb{III}}(n)}_{\text{finite sum } \forall\, p_2} + \underbrace{\sum_{\lambda_2=1,3,..}^{\infty} \rho^{\lambda_2+12}\quad \mathfrak{f}_{p_2}^{\mathbb{III}}\left(\frac{\lambda_2+12-p_2}{2}\right)}_{\boxed{\lambda_2+12 = j, \text{ say}}}. \tag{C.10}$$

$$\therefore\ \mathcal{S}_2^{\mathbb{III}} = \sum_{n=1}^{\frac{13-p_2}{2}-1} \sum_{p_2=1,3,..}^{9} \rho^{2n+p_2}\, \mathfrak{f}_{p_2}^{\mathbb{III}}(n)$$

$$+ \underbrace{\sum_{j=13,15,..}^{\infty} \left(\sum_{p_2=1,3,..}^{9} \mathfrak{f}_{p_2}^{\mathbb{III}}\left(\frac{j-p_2}{2}\right) \right) \rho^j}_{\mathfrak{s}_3 \quad (\text{say})}. \tag{C.11}$$

One can notice that,

- for $\mathfrak{s}_2$ in Eq.(C.9), the necessary condition is: $(j - p_1)$ has to always be even with $\forall\, p_1 = 0, 2, ..., 10$ and $\forall\, j = 12, 14, ..., \infty$ and
- for $\mathfrak{s}_3$ in Eq.(C.11), the necessary condition is: $(j - p_2)$ has to always be even with $\forall\, p_2 = 1, 3, ..., 9$ and $\forall\, j = 13, 15, ..., \infty$.

Clearly the j-s are different in $\mathfrak{s}_2$ and $\mathfrak{s}_3$, but since the corresponding ρ^j-s in Eqs.(C.9) and (C.11) are all linearly independent, we can practically club these two separate infinite summations into a single one. Thus

$$\mathcal{S}_2 = \left(\mathcal{S}_2^{\mathbb{I}} + \mathcal{S}_2^{\mathbb{III}}\right) \xRightarrow[\text{summations}]{\text{infinite}} (\mathfrak{s}_2 + \mathfrak{s}_3)$$

$$= \sum_{j=12,14,..}^{\infty} \Bigg(\sum_{p_1=0,2,..}^{10} \mathfrak{f}_{p_1}^{\mathbb{III}} \left(\frac{j-p_1}{2}\right) \Bigg) \rho^j + \sum_{j=13,15,..}^{\infty} \Bigg(\sum_{p_2=1,3,..}^{9} \mathfrak{f}_{p_2}^{\mathbb{III}} \left(\frac{j-p_2}{2}\right) \Bigg) \rho^j$$

$$= \underbrace{\sum_{j=12,13,..}^{\infty} \Bigg(\sum_{p=0,1,..}^{10} \mathfrak{f}_{p}^{\mathbb{III}} \left(\frac{j-p}{2}\right) \Bigg) \rho^j}_{\mathfrak{s}_{2,3}\ \ (\text{say})} \qquad (\text{provided } \forall\, (j-p) = 0, 2, 4, \ldots). \tag{C.12}$$

And through Eqs.(C.6) and (C.12)

$$(\mathcal{S}_1 + \mathcal{S}_2) \xRightarrow[\text{summations}]{\text{infinite}} (\mathfrak{s}_1 + \mathfrak{s}_{2,3})$$

$$= \sum_{j=12,13,..}^{\infty} \Bigg(\sum_{k=0}^{10} \mathfrak{f}_{k}^{\mathbb{I}} (j-k) \Bigg) \rho^j + \sum_{j=12,13,..}^{\infty} \Bigg(\sum_{p=0,1,..}^{10} \mathfrak{f}_{p}^{\mathbb{III}} \left(\frac{j-p}{2}\right) \Bigg) \rho^j$$

$$= \sum_{j=12,13,\ldots}^{\infty} \left[\sum_{k=0}^{10} \mathfrak{f}_{k}^{\mathbb{I}} (j-k) \;+\; \underbrace{\sum_{p=0,1,..}^{10} \mathfrak{f}_{p}^{\mathbb{III}} \left(\frac{j-p}{2}\right)}_{\forall\,(j-p)=0,2,4,\ldots,\infty} \right] \rho^j. \tag{C.13}$$

After having these Eqs.(C.6), (C.9), (C.11) and (C.13) clubbed together, we had written the l.h.s of Eq.(5.33) where the last square bracket now generates a recursion relation quite naturally. This helps find out the Frobenius coefficient(s) for any arbitrary $j = 12, 13, 14, \ldots$ provided

$(j-p)=0,2,4,\ldots$ holds true $\forall p=0,1,\ldots,10$. The recursion relation is explicitly given by (please turn over this page),

$$
\begin{aligned}
a_j = -\tfrac{1}{32j(2c_s j+i\omega r_0)}\Big[& a_{j-1}\Big(-16(j-1)\{28jc_s-14c_s+i\,(8\beta c_s+7r_0\omega)\}\Big)\\
&+a_{j-2}\Big(16(j-2)\{89jc_s-89c_s+i\,(48\beta c_s+10r_0\omega)\}\Big)+a_{j-3}\Big(-40(j-3)\{68jc_s-102c_s+i\,(52\beta c_s+3r_0\omega)\}\Big)\\
&+a_{j-4}\Big(16(j-4)\{217jc_s-434c_s+i\,(210\beta c_s+3r_0\omega)\}\Big)+a_{j-5}\Big(-8(j-5)\{388jc_s-970c_s+i\,(448\beta c_s+r_0\omega)\}\Big)\\
&+a_{j-6}\Big(4(j-6)\,c_s\,\{493j-1479+656i\beta\}\Big)+a_{j-7}\Big(-440(j-7)\,c_s\,\{2j-7+3i\beta\}\Big)+a_{j-8}\Big(88(j-8)\,c_s\,\{3j-12+5i\beta\}\Big)\\
&+a_{j-9}\Big(-8(j-9)\,c_s\,\{6j-27+11i\beta\}\Big)+a_{j-10}\Big(4(j-10)\,c_s\,\{j-5+2i\beta\}\Big)\\
&+\boxed{a^2_{\frac{j}{2}}}\Big(16ic_s j^2\Big)+\boxed{a^2_{\frac{j-1}{2}}}\Big(-112ic_s(j-1)^2\Big)+\boxed{a^2_{\frac{j-2}{2}}}\Big(356ic_s(j-2)^2\Big)+\boxed{a^2_{\frac{j-3}{2}}}\Big(-680ic_s(j-3)^2\Big)\\
&+\boxed{a^2_{\frac{j-4}{2}}}\Big(868ic_s(j-4)^2\Big)+\boxed{a^2_{\frac{j-5}{2}}}\Big(-776ic_s(j-5)^2\Big)+\boxed{a^2_{\frac{j-6}{2}}}\Big(493ic_s(j-6)^2\Big)+\boxed{a^2_{\frac{j-7}{2}}}\Big(-220ic_s(j-7)^2\Big)\\
&+\boxed{a^2_{\frac{j-8}{2}}}\Big(66ic_s(j-8)^2\Big)+\boxed{a^2_{\frac{j-9}{2}}}\Big(-12ic_s(j-9)^2\Big)+\boxed{a^2_{\frac{j-10}{2}}}\Big(ic_s(j-10)^2\Big)\Big]. \qquad \text{(C.14)}
\end{aligned}
$$

Its quite evident that the boxed coefficients written above do not contribute anything to a_j only when $(j-p)$ is found to be an odd number $\forall p=0,1,\ldots,10$. Here, in Eq.(C.14), a_j is expressed in terms of the coefficients all of which are pre-determined and thus the recursion relation is consistent.

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