Light-front dynamics of gravity and supergravity theories

विद्या वाचस्पति की
उपाधि की अपेक्षाओं की आंशिक पूर्ति में प्रस्तुत शोध प्रबंध

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Date: April 7, 2026

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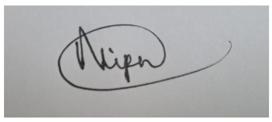
I declare that this written submission represents my idea in my own words and where others' ideas have been included; I have adequately cited and referenced the original sources. I declare that I have acknowledged collaborative work and discussions wherever such work has been included. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

The work reported in this thesis is the original work done by me under the guidance of Prof. Sudarshan Ananth.



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Nipun Bhave

I dedicate this thesis to my parents for all their love and constant support.

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Abstract

This thesis presents investigations regarding symmetries and the constraints they impose on the perturbation theory of pure gravity in four dimensions and maximal supergravity theories in various space-time dimensions. A limitation of existing studies in gauge and gravity theories has been the focus on the Maximally Helicity Violating (MHV) structures. Using a closed-form action for pure gravity in the light-cone gauge in four dimensions, we extract and present all six-point interaction vertices, extending this program of study to non-MHV interactions. We invoke symmetry arguments to derive helicity selection rules for arbitrary spin theories to all orders in perturbation theory and discuss their consequences for scattering amplitudes in pure gravity and its higher derivative extensions.

A major part of this thesis is devoted to the study of maximal supergravity theories in light-cone superspace. The $\mathcal{N} = 8$ supergravity in four dimensions exhibits the best ultraviolet properties amongst any gravitational field theory due to maximal supersymmetry and exceptional symmetry. We explain a systematic procedure for obtaining quartic and higher-point interactions using this exceptional symmetry. We also construct maximal supergravity in five dimensions through dimensional oxidation and derive the generators of the corresponding exceptional symmetry. This analysis leads to a universal form for global symmetries in maximal supergravity theories in all space-time dimensions $d \geq 4$, with the notable feature that it applies to both classical and exceptional symmetry groups.

List of Publications

1. * Sudarshan Ananth and **Nipun Bhave**, *An 'exceptional' form for symmetries in maximal supergravity*, Phys.Lett.B 871 (2025) 140031, arXiv:2508.14425[hep-th]

2. * Sudarshan Ananth and **Nipun Bhave**, *Exceptional symmetries in light-cone superspace* , JHEP 02 (2025) 203, arXiv:2410.19463[hep-th]

3. **Nipun Bhave** and Saurabh Pant, *Nicolai maps and uniqueness in the light-cone gauge*, Journal of High Energy Physics (JHEP) 09 (2024) 121 , arXiv: 2406.18238 [hep-th].

4. * Sudarshan Ananth, **Nipun Bhave**, Chetan Pandey and Saurabh Pant, *Deriving interaction vertices in higher derivative theories*, Phys. Lett. B **853** (2024), 138704, arXiv: 2306.05074 [hep-th].

5. * Sudarshan Ananth, **Nipun Bhave** and S.I Aadharsh Raj, *The structure of interaction vertices in pure gravity in the light-cone gauge*, Phys.Lett.B 838 (2023) 137743, arXiv:2207.10416[hep-th]

This thesis is based on the publications that are marked with the *.

Contents

Chapter 1

Introduction

1.1 Quantum field theory : symmetries and scattering amplitudes

Quantum field theory provides the theoretical framework for describing the quantum mechanics of relativistic particles. The Standard Model of particle physics, which successfully accounts for the electromagnetic, weak, and strong interactions among elementary particles, is formulated within this framework. Experimental observations spanning several decades have confirmed its predictions with remarkable precision.

The triumphs of this framework are numerous. The theory of quantum electrodynamics (QED) predicts the anomalous magnetic dipole moment of the electron with an astonishing accuracy that matches experimental measurements to about one part in a trillion. This theory has also succesfully explained the Casimir effect and the Lamb shift in atomic spectra. Further, the observation of the Higgs boson at the Large Hadron Collider in 2012 provided a crucial piece of the Standard Model. Together, these milestones firmly establish quantum field theory as an indispensable mathematical language and one of the most rigorously tested scientific frameworks.

Two main objects of interest in a quantum field theory are *symmetries* and *scattering amplitudes*. Symmetries are operations that leave a physical system invariant. Most often, they form a group. This mathematical property is extremely useful in the study of various dynamical processes in the physical systems. Broadly, three kinds of symmetries govern a quantum field theory:

- Space-time symmetries, which constitute the Poincaré group of transformations.
- Gauge Symmetries - The Yang-Mills theories that appear in the Standard Model Lagrangian are based on the gauge group $SU(3) \times SU(2) \times U(1)$.
- Global internal symmetries.

Elementary particles are classified using the irreducible representations of the Poincaré group. A simple example of how symmetries restrict the kinds of interactions between these elementary particles can be found in the $SU(3)$ Yang-Mills theory. The absence of three point interaction terms involving purely negative (or positive) helicity gluons is a direct consequence of little group invariance and dimension analysis.

The physical observables in a quantum field theory are encoded in scattering amplitudes. These amplitudes describe the probability for a given set of incoming particles to scatter into a specified set of outgoing particles. Traditionally, scattering amplitudes are computed using the Feynman diagram expansion derived from the Lagrangian of the theory. However, this approach is met with severe challenges, as the number of diagrams increases significantly with each order in perturbation theory.

Surprisingly, many times scattering amplitudes involving massless particles have rather simple expressions. Manifest covariance conceals the inherent simplicity of physical observables. A direct consequence of this covariance is that the theory has gauge redundancies and unphysical degrees of freedom, leading to tedious intermediate calculations where vast numbers of terms eventually cancel. *This is the theme of this thesis. We sacrifice manifest Lorentz invariance to study the physics of gauge and gravity theories.*

A well–known example is the Parke–Taylor formula for gluon scattering amplitudes in Yang–Mills theory, which expresses a class of tree-level amplitudes in a surprisingly compact form [1]. Over the past few decades, this observation has led to significant developments in our understanding of scattering amplitudes. Specifically, the modern techniques such as spinor-helicity variables, on-shell recursion relations, and color–kinematics duality have played an important role in finding efficient ways to calculate amplitudes [2–5]. These findings have uncovered intriguing relations connecting gauge theory to gravitation.

1.2 Gravity as a quantum field theory

Despite its immense successes, it is clear that the Standard Model is not the complete story of nature. One important limitation is the absence of gravity. While general relativity provides an extremely successful classical description of gravity at macroscopic scales, incorporating it into the framework of quantum field theory remains a long-standing challenge.

Naive quantization of pure gravity leads to an ultraviolet divergent and non-renormalizable theory. The origin of these divergences can be traced to the dimensionful nature of the gravitational coupling constant. Unlike the gauge couplings appearing in the Standard Model, the gravitational coupling constant carries dimensions of length. In pure gravity, the first divergence appears at the two-loop level, where a non-vanishing counterterm must be introduced in the action [6]. This indicates that Einstein gravity should be interpreted as an effective field theory, valid below a certain energy scale.

There are several indications that the perturbative (and possibly non-perturbative) structure of gravity possesses features that are not immediately apparent in the conventional Lagrangian formulation. For example, gravitational scattering amplitudes exhibit remarkable squaring relations to gauge theory amplitudes, as illustrated by the Kawai-Lewellen-Tye [7] relations and more recent developments such as color-kinematics duality and double copy [4].

One possible avenue for improving the high energy behaviour of gravity is to enlarge the symmetry structure of the theory. This motivates the study of theories in which gravity is combined with new classes of symmetries. A particularly important example of such an extension is *supersymmetry*.

1.3 Supersymmetry

Supersymmetry provides a striking extension of the space-time symmetries of quantum field theory. Unlike ordinary symmetries, which relate particles of the same statistics, supersymmetry connects bosonic and fermionic degrees of freedom. Although such a symmetry has not been observed yet, it serves as an important tool for calculating amplitudes in the standard model.

An important consequence of this symmetry appears in the quantum behaviour of the theory. Contributions from bosonic and fermionic degrees of freedom enter loop calculations with opposite signs, leading to cancellations that significantly soften divergences at high energies [8]. One of the simplest models where such cancellations were studied was the Wess–Zumino theory [9]. In theories with extended supersymmetry, the effect is even more dramatic. For example, four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory [10] is known to be finite to all orders in perturbation theory.

These observations naturally motivate the study of supersymmetric extensions of gravity. Theories of supergravity contain in addition to the gravitational field, a fermionic partner called the gravitino. We describe below various aspects of supergravity.

1.4 Supergravity

Supergravity theories with the largest possible number of supersymmetries have a great theoretical significance. The maximally supersymmetric theory of gravity in four-dimensions was first constructed by Cremmer and Julia [11]. This theory was obtained through the dimensional reduction of the minimal supergravity theory in eleven dimensions (which is the maximum dimension of space-time where a supersymmetric theory of gravity can be formulated) [12]. One of the surprising outcomes of this construction was the appearance of a large global symmetry, namely the exceptional Lie group $E_{7(7)}$ [13].

The ultraviolet properties of maximal supergravity has been the subject of considerable interest. Early expectations based on power-counting arguments suggested that the theory should diverge at three loops. However, explicit calculations of multi-loop scattering amplitudes have revealed surprising cancellations. In particular, a series of computations by Bern and collaborators showed that four-dimensional $\mathcal{N} = 8$ supergravity exhibits significantly improved high-energy behaviour compared to naive expectations [14–16].

An interesting perspective was proposed by Curtright [17], who suggested that the cancellation of divergences in supersymmetric theories may be related to the matching of Dynkin indices between bosonic and fermionic representations of the little group of the parent theory in higher dimensions. In the case of maximally supersymmetric Yang-Mills, this implies that the $SO(8)$ triality property renders the theory finite to all orders in perturbation theory. In the case of maximal supergravity, the relevant little group is $SO(9)$ with the Dynkin indices of the bosonic representations (columns 2 and 3) and fermionic representations (column 1) [18] shown in the table below:

Rep	(1001)	(2000)	(0010)
I_0	128	44	84
I_2	256	88	168
I_4	640	232	408
I_6	1792	712	1080
I_8	5248	2440	3000

We note the mismatch of the eighth-order Dynkin indices from the table. While this approach has been verified at one loop, extension to higher loops is challenging. A careful study of loop amplitudes to explain the emergence of these indices would indeed be interesting, but remains outside the scope of this thesis.

The latest findings in the $\mathcal{N} = 8$ theory remarkably establish the finiteness of the four-point amplitude at five-loops [16]. Supersymmetry solely cannot explain these findings. One therefore studies the exceptional symmetry $E_{7(7)}$ with a focus on the constraints it imposes on amplitudes and possible counterterms [19–22].

1.5 Motivation and central questions

In this thesis, we work with the light-cone formulation of gravity and supergravity theories. In this approach, we work directly with the physical degrees of freedom sacrificing manifest locality and Poincaré invariance. Symmetries are realized as non-linear transformations of the physical fields. This is useful in building interacting theories order by order in perturbation theory. One key advantage is that amplitude structures naturally arise in this language.

We provide below the motivation and two closely related questions that we attempt to answer in this thesis:

- First question is concerned with pure gravity in four-dimensions. One of the limitations of the existing studies in the literature has been the focus on Maximally Helicity Violating (MHV) structures. There are no findings in the literature regarding non-MHV interaction vertices in pure gravity in the light-cone gauge. We extend this program of study by explicitly deriving these vertices. This derivation leads naturally to the question of the allowed helicity configurations of graviton vertices at each order in perturbation theory.

  Parallely, another motivation to revisit pure gravity comes from recent studies which suggest that gravitational amplitudes (although still divergent) may be better behaved in the high energy limit than previously anticipated [23]. It is therefore important to analyse the structure of interaction vertices in the theory. Closely related to this, is the role of higher-derivative interactions, which appear naturally in effective field theory and are often associated with possible counterterms in quantum gravity. This raises the first question:

  What constraints do space-time symmetries impose on the structure of interaction vertices in theories of gravity (with and without higher-derivative terms)?

- The second question is concerned with exceptional symmetries in maximal supergravity theories. These include the large global symmetry groups such as the exceptional Lie groups $E_{7(7)}$ in four dimensions and $E_{6(6)}$ in five dimensions. In light-cone superspace, it was shown that every Feynman supergraph arising in the $\mathcal{N} = 4$ superYang-Mills theory was power counting finite using superspace manipulations [24, 27]. In this formulation, the $\mathcal{N} = 8$ supergravity shares many structural similarities with the $\mathcal{N} = 4$ superYang-Mills theory. It therefore makes sense to apply similar techniques to supergravity. The exceptional symmetries in light-cone superspace, act non-linearly on the entire supermultiplet (unlike the covariant formulation) and therefore serve as an important tool to derive interaction vertices. While there have been several attempts at this, existing expressions are extremely cumbersome (running into hundreds of terms) making them unsuitable for tractable calculations. A compact expression for vertices at higher orders in the coupling constant is a crucial step towards analyzing Feynman diagrams in superspace.

Another important aspect is to understand the dependence of the degree of divergence of these diagrams on the space-time dimensions. This requires a careful study of the relevant symmetries (which constraint counter-terms) of maximal supergravity theories in each space-time dimension. This raises the second question:

What constraints do exceptional symmetries impose on the Hamiltonian of maximal supergravity in different space-time dimensions?

The investigations presented in this thesis address these questions. We present below the outline of the thesis.

1.6 Outline of the thesis

The thesis is organized as follows:

Chapter 2 reviews the light-cone formulation of relativistic field theories. In this framework, the dynamics of a theory are expressed directly in terms of its physical degrees of freedom, with unphysical gauge components eliminated through constraint equations. We discuss the structure of the Poincaré algebra in light-cone coordinates and describe how the closure of the algebra can be used to systematically determine interaction vertices. This formalism provides the main technical framework used throughout the thesis.

Chapter 3 investigates the interaction vertices in pure gravity within the light-cone formula tion. Starting from the closed-form expression for the light-cone gravitational action, we derive the six-point interaction vertices. In particular, these correspond to the first genuinely non-MHV (Maximally Helicity Violating) vertices that arise in the perturbative expansion. Using symmetry arguments, we then derive the helicity structure of graviton vertices to all orders in perturbation theory. We further extend this analysis to higher-derivative theories of gravity.

Chapter 4 turns to supersymmetric theories and introduces the light-cone superspace formulation of maximal supergravity. In this framework, the entire supermultiplet is described by a single chiral superfield. We review the structure of the $\mathcal{N} = 8$ supergravity theory in four-dimensions and the realization of its symmetries in light-cone superspace.

Chapter 5 studies the derivation of interaction vertices using the exceptional symmetry $E_{7(7)}$ in maximal supergravity in four dimensions. In particular, we show that this symmetry relates lower *even* (*odd*) point vertices to higher *even* (*odd*)-point vertices. We demonstrate that the equations satisfied by the higher-point vertices are analogous to inhomogeneous differential equations. The inhomogeneous part of these vertices is fixed by exceptional symmetry and the homogeneous part is fixed by the space-time symmetries.

Chapter 6 extends these ideas to the maximal supergravity theory in five dimensions. We use the method of dimensional-oxidation to construct the five-dimensional theory starting from the four dimensional theory. We examine the construction of the relevant symmetry algebras $USp(8)$ and $E_{6(6)}$ in light-cone superspace.

Chapter 7 demonstrates that the global symmetries in maximally supersymmetric theories of gravity admit a universal form in light-cone superspace for all dimensions $d \geq 4$. We verify this by discussing the construction of the relevant symmetries $E_{7(7)}$, $E_{6(6)}$, $SO(5,5)$ and $SL(5,\mathbb{R})$. We also discuss the procedure to derive an all-order expression for the generators of the exceptional algebra $E_{7(7)}$.

Finally, in Chapter 8, we summarize the main results of the thesis and discuss possible directions for future work.

Chapter 2

Quantum Field theory in light-cone gauge

In this chapter, we discuss the isometry algebra of four-dimensional Minkowski space-time with a particular emphasis on the massless representations of this algebra. We explain various aspects of field theories formulated in light-cone gauge through the examples of Yang-Mills theory and pure gravity.

2.1 Space-time symmetries : the Poincaré group

A fundamental requirement of any physical theory is that the laws of physics be invariant under inertial change of frames of reference. The principle of relativity also states that the speed of light in vacuum must be constant in any inertial frame of reference. These two requirements lead to a set of space-time transformations known as the isometries of four-dimensional Minkowski space-time. These transformations leave the space-time distance

$$ds^2 \;=\; \eta_{\mu\nu}\, dx^\mu dx^\nu \tag{2.1}$$

invariant, where $\mu, \nu = 0, 1, 2, 3$ and $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric. The collection of all such isometries forms a group known as the Poincaré group.

Explicitly, a Poincaré transformation acts on space-time coordinates as

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu + a^\mu, \tag{2.2}$$

where $\Lambda^\mu{}_\nu$ represents Lorentz transformations and a^μ denote constant space-time translations. The Lorentz transformations include spatial rotations and boosts between inertial frames.

The generators of space-time translations P_μ and Lorentz transformations $J_{\mu\nu}$ satisfy the Poincaré algebra

$$
\begin{aligned}
&[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho}) \\
&[J_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\
&[P_\mu, P_\nu] = 0.
\end{aligned} \tag{2.3}
$$

In a quantum theory, space-time symmetries must be realized as unitary transformations acting on the Hilbert space of physical states. These unitary irreducible representations of the Poincaré algebra classify the elementary particles [28, 29]. The Poincaré algebra admits two independent Casimir operators,

$$
P^2 = P^\mu P_\mu, \qquad W^2 = W^\mu W_\mu, \tag{2.4}
$$

where

$$
W_\mu = \frac{1}{2}\,\epsilon_{\mu\nu\rho\sigma}\,J^{\nu\rho}P^\sigma \tag{2.5}
$$

is the Pauli-Lubanski vector. The eigenvalues of these Casimirs label the irreducible representations and are identified with the mass and spin of the particle, respectively.

In this thesis, we will be concerned primarily with massless representations, for which $P^2 = 0$. Since the Pauli-Lubanski vector satisfies $[W_\mu, P_\nu] = 0$, its action preserves momentum eigenstates and therefore generates the little group associated with a fixed four-momentum.

For massless representations, one may choose a standard momentum frame

$$
P^\mu = (E, 0, 0, E). \tag{2.6}
$$

In this frame, the independent non-vanishing components of the Pauli-Lubanski vector reduce to

$$
W_1 = E\,(J_{23}\,+\,J_{20}), \qquad W_2 = E\,(J_{13}\,+\,J_{10}), \qquad W_3 = E\,J_{12}. \tag{2.7}
$$

Defining the operators

$$T_1 \equiv \frac{1}{E} W_1, \qquad T_2 \equiv \frac{1}{E} W_2, \qquad J \equiv \frac{1}{E} W_3, \tag{2.8}$$

one finds that they satisfy the algebra

$$\begin{aligned} [J\,,\,T_1] \,&=\, -\,i\,T_2, \\ [J\,,\,T_2] \,&=\, +i\,T_1, \\ [T_1\,,\,T_2] \,&=\, 0. \end{aligned} \tag{2.9}$$

This is precisely the Lie algebra of the two-dimensional Euclidean group $ISO(2)$, consisting of one rotational generator J and two commuting translation generators T_1 and T_2.

While the full little group for massless particles is $ISO(2)$, physical particles observed in nature do not carry continuous internal quantum numbers associated with the translational generators T_1 and T_2. Consequently, only the compact $SO(2)$ subgroup generated by J is realized non-trivially. The eigenvalue of J defines the helicity of the particle and determines its local physical degrees of freedom.

In the following, we explain the Poincaré algebra in light-cone gauge realized on the helicity states ϕ and $\bar{\phi}$. We explain the connection of light-cone coordinates with spinor-helicity variables which are crucial to writing scattering amplitudes.

2.1.1 Light-cone Poincaré algebra

We use the light-cone coordinates

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^3)\,, \quad x = \frac{1}{\sqrt{2}}\,(\,x_1 \,+\, i\,x_2\,)\,, \quad \bar{x} = x^*\,, \tag{2.10}$$

and their respective derivatives $\partial_{\pm}, \bar{\partial}, \partial$. In these coordinates, x^+ plays the role of time. We routinely use the operation $\frac{1}{\partial_-}$ (or $-\frac{1}{\partial^+}$) which plays the role of 'inverse' derivative and is defined as [24]

$$\frac{1}{\partial_-}F(x^-) \;=\; \frac{1}{2}\int dy^-\,\epsilon(x^- - y^-)\,F(y^-)\,, \tag{2.11}$$

where $\epsilon(x^- - y^-)$ is the sign function[1]. It is straightforward to verify that when ∂_- is acted from the left in (2.11), one recovers the original function back. In the calculations, we continue to use this inverse derivative operator to write unphysical fields in terms of the propagating degrees of freedom. We routinely perform partial integrations wrt ∂_- to simplify calculations using the property $\partial_-\left[\frac{1}{\partial_-}F(x)\right] = F(x)$. Another property we frequently require is

$$\int A\left[\frac{1}{\partial_-}B\right] \;=\; -\int\left[\frac{1}{\partial_-}A\right]\,B. \tag{2.12}$$

where A, B are some composite functions of the fields and their derivatives. While this operator introduces mild non-locality, one can check that this formulation reproduces the known amplitude results obtained from manifestly local and Lorentz covariant theories. In momentum space, we use the Mandelstam-Leibbrandt prescription for dealing with the spurious singularities introduced by $1/p_-$ in loop calculations [24, 25].

At the light-cone time $x^+ = 0$, the generators of the Poincaré algebra are the momenta [30]

$$P^- = -i\frac{\partial\bar{\partial}}{\partial^+} = -P_+\,,\qquad P^+ = -i\partial^+ = -P_-\,,\qquad P = -i\partial\,,\qquad \bar{P} = -i\bar{\partial}\,, \tag{2.13}$$

the rotations and boosts

$$\begin{aligned} &J = (x\bar{\partial} - \bar{x}\partial - \lambda)\,,\qquad J^+ = ix\partial^+\,,\qquad \bar{J}^+ = i\bar{x}\partial^+ \\ J^{+-} = ix^-\partial^+\,,\qquad &J^- = i(x\frac{\partial\bar{\partial}}{\partial^+} - x^-\partial - \lambda\frac{\partial}{\partial^+})\,,\qquad \bar{J}^- = i(\bar{x}\frac{\partial\bar{\partial}}{\partial^+} - x^-\bar{\partial} + \lambda\frac{\bar{\partial}}{\partial^+}). \end{aligned} \tag{2.14}$$

The generators are of two kinds: kinematical that do not involve time derivatives

$$P^+\,,\;\; P\,,\;\; \bar{P}\,,\;\; J\,,\;\; J^+\,,\;\; \bar{J}^+ \;\text{ and }\; J^{+-}\,, \tag{2.15}$$

[1]One could in principle add a function $h(x)$ independent of x^- to the RHS of (2.11). Here we work with boundary conditions such that $h(x) = 0$. However when working with non-trivial boundary conditions, a careful treatment of these zero modes is necessary [26].

and dynamical that do

$$P^- \ , \quad J^- \ , \quad \bar{J}^- \ . \tag{2.16}$$

Dynamical generators pick up corrections when interactions are switched on. We introduce the Hamiltonian variation

$$\delta_{\mathcal{H}}\phi \equiv \partial^-\phi = \{\phi, H\} = \frac{\partial\bar{\partial}}{\partial^+}\,\phi \ , \tag{2.17}$$

where the last equality only holds for the free theory. Hence, the Poincare algebra is realized non-linearly on the physical degrees of freedom. This is used to build interaction vertices by requiring the commutation relations of the algebra to hold, as done in [30].

While this method builds the classical action order by order in the coupling constant, one has to also ensure that in the quantum theory, no anomalies appear specifically from the path integral measure. One can in principle compute the Jacobian of the non-linear boost transformations to show that the interactions do no contribute to the Jacobi determinant.

The *non-vanishing* commutators of the Poincaré algebra are

$$
\begin{aligned}
[P^-, J^{+-}] &= -iP^- , \quad && [P^-, J^+] = -iP , \quad && [P^-, \bar{J}^+] = -i\bar{P} , \\
[P^+, J^{+-}] &= iP^+ , \quad && [P^+, J^-] = -iP , \quad && [P^+, \bar{J}^-] = -i\bar{P} , \\
[P, \bar{J}^-] &= -iP^- , \quad && [P, \bar{J}^+] = -iP^+ , \quad && [P, J] = P , \\
[\bar{P}, J^-] &= -iP^- , \quad && [\bar{P}, J^+] = -iP^+ , \quad && [\bar{P}, J] = -\bar{P} , \\
[J^-, J^{+-}] &= -iJ^- , \quad && [J^-, \bar{J}^+] = iJ^{+-} + J , \quad && [J^-, J] = J^- , \\
[\bar{J}^-, J^{+-}] &= -i\bar{J}^- , \quad && [\bar{J}^-, J^+] = iJ^{+-} - J , \quad && [\bar{J}^-, J] = -\bar{J}^- , \\
[J^{+-}, J^+] &= -iJ^+ , \quad && [J^{+-}, \bar{J}^+] = -i\bar{J}^+ , && \\
[J^+, J] &= J^+ , \quad && [\bar{J}^+, J] = -\bar{J}^+ . &&
\end{aligned}
\tag{2.18}
$$

2.1.2 Spinor-Helicity variables

We demonstrate the close link between the light-cone gauge and the spinor-helicity variables here [31]. A four-vector p_μ may be rewritten as a matrix $p_{a\dot{a}}$ using the Pauli matrices $\sigma_\mu = (-\mathbf{1}, \sigma)$

$$
p_\mu \left(\sigma^\mu\right)_{a\dot{a}} \equiv p_{a\dot{a}} = \begin{pmatrix} -p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_0 - p_3 \end{pmatrix} = \sqrt{2} \begin{pmatrix} -p_- & \bar{p} \\ p & -p_+ \end{pmatrix} . \tag{2.19}
$$

The determinant of this matrix is the length of the four-vector p_μ

$$
\det\left(p_{a\dot{a}}\right) = -2\left(p\bar{p} - p_+p_-\right) = -p^\mu p_\mu . \tag{2.20}
$$

For massless particles, one can write the matrix as product of holomorphic and anti-holomorphic spinors using the on-shell condition $p_+ = \frac{p\bar{p}}{p_-}$

$$\lambda_a = \frac{2^{\frac{1}{4}}}{\sqrt{p_-}} \begin{pmatrix} p_- \\ -p \end{pmatrix}, \quad \tilde{\lambda}_{\dot{a}} = -(\lambda_a)^* = -\frac{2^{\frac{1}{4}}}{\sqrt{p_-}} \begin{pmatrix} p_- \\ -\bar{p} \end{pmatrix}, \tag{2.21}$$

such that $\lambda_a \tilde{\lambda}_{\dot{a}}$ reproduces (2.19) on-shell. We define the off-shell holomorphic and anti-holomorphic spinor products [32],

$$\langle ij \rangle = \sqrt{2}\, \frac{p^i\, p_-^j \, - \, p^j\, p_-^i}{\sqrt{p_-^i\, p_-^j}}, \quad [ij] = \sqrt{2}\, \frac{\bar{p}^{\,i}\, p_-^j \, - \, \bar{p}^{\,j}\, p_-^i}{\sqrt{p_-^i\, p_-^j}}. \tag{2.22}$$

2.2 A review of Yang-Mills theory in the light-cone gauge

Yang–Mills theory describes the dynamics of non-abelian gauge fields associated with a compact Lie group G. The fundamental field is the gauge potential $A^a_\mu(x)$, where $\mu = 0, 1, 2, 3$ is a spacetime index and a labels the generators T^a of the Lie algebra $\mathfrak{g}$ of G. The gauge-invariant field strength is defined as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \tag{2.23}$$

where g is the gauge coupling and f^{abc} are the structure constants of the Lie algebra. The Yang–Mills action is given by

$$S_{\text{YM}} = -\frac{1}{4} \int d^4x\, F^a_{\mu\nu} F^{a\,\mu\nu}. \tag{2.24}$$

The Yang–Mills action is invariant under local gauge transformations. Under an infinitesimal gauge transformation with parameter $\Lambda^a(x)$, the gauge field transforms as

$$A^a_\mu \;\to\; A'^{\,a}_\mu = A^a_\mu + \partial_\mu \Lambda^a + g f^{abc} \Lambda^b A^c_\mu. \tag{2.25}$$

This allows to fix one degree of freedom by imposing the light-cone gauge condition $A^a_- = 0$.

We then examine the equations of motion

$$\partial_\mu F^{\mu\nu\, a} + g f^{abc} A_\mu^b F^{\mu\nu\, c} = 0. \tag{2.26}$$

In light-cone gauge, these equations split into dynamical and constraint equations. In particular, the $\nu = +$ component contains no light-cone time derivatives ∂_+ and therefore acts as a constraint equation.

Using $A_- = 0$, the $\nu = +$ equation becomes

$$\partial_-^2 A_+^a = \partial_- \partial_i A_i^a + g f^{abc} A_i^b \partial_- A_i^c, \tag{2.27}$$

where $i = 1, 2$ denote the transverse directions.

This equation can be solved for the non-dynamical field A_+^a

$$A_+^a = \frac{1}{\partial_-} \partial_i A_i^a + g f^{abc} \frac{1}{\partial_-^2} \left(A_i^b \partial_- A_i^c \right). \tag{2.28}$$

Hence, in light-cone gauge the field A_+ is not an independent degree of freedom but is completely determined by the transverse physical fields A_i. It is convenient to combine the two transverse components of the gauge field into complex helicity states. We define

$$A \;=\; \frac{1}{\sqrt{2}} \left(A_1 + i A_2 \right), \qquad \bar{A} \;=\; \frac{1}{\sqrt{2}} \left(A_1 - i A_2 \right). \tag{2.29}$$

These fields correspond to helicity $+1$ and -1 gluon states, respectively, and constitute the physical degrees of freedom of the Yang–Mills field in light-cone gauge. The Yang–Mills action (2.24) may be written purely in terms of the physical helicity modes A^a and $\bar{A}^a$. The resulting Lagrangian takes the form

$$\begin{aligned} \mathcal{L} \;=\;& \bar{A}^a \, \Box \, A^a - 2 g f^{abc} \left(\frac{\bar{\partial}}{\partial^+} A^a \, \partial^+ \bar{A}^b \, A^c + \frac{\partial}{\partial^+} \bar{A}^a \, \partial^+ A^b \, \bar{A}^c \right) \\ & - 2 g^2 f^{abc} f^{ade} \frac{1}{\partial^+} \left(\partial^+ A^b \, \bar{A}^c \right) \frac{1}{\partial^+} \left(\partial^+ \bar{A}^d \, A^e \right). \end{aligned} \tag{2.30}$$

It is now straightforward to make contact with on-shell scattering amplitudes. In particular, Maximally Helicity Violating (MHV) amplitudes correspond to processes involving exactly two negative-helicity gluons and an arbitrary number of positive-helicity gluons. At the three-point level, we may write the MHV interaction term in (2.30) in momentum space and express it in terms of spinor-helicity variables (2.22). This yields the color-stripped three-point amplitude

$$\mathcal{A}_3(1^-, 2^-, 3^+) = g \, \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}. \tag{2.31}$$

Similarly, the four-point MHV amplitude takes a remarkably simple form [1]

$$\mathcal{A}_4(1^-, 2^-, 3^+, 4^+) = g^2 \, \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}. \tag{2.32}$$

2.3 A review of light-cone gravity

The Einstein-Hilbert action reads

$$S_{EH} = \int d^4x \, \mathcal{L} \; = \; \frac{1}{2\,\kappa^2} \int d^4x \, \sqrt{-g} \, \mathcal{R} \, , \tag{2.33}$$

where $g \, = \, det\,(\,g_{\mu\nu}\,)$, $\mathcal{R}$ is the curvature scalar and $\kappa^2 \, = \, 8\pi\, G$ is the coupling constant. The corresponding field equations are

$$\mathcal{R}_{\mu\nu} \; - \; \frac{1}{2}\, g_{\mu\nu} \mathcal{R} \; = \; 0 \, . \tag{2.34}$$

The Einstein-Hilbert action is invariant under space-time diffeomorphisms

$$x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu(x) \tag{2.35}$$

Under such a transformation, the metric tensor transforms as

$$g_{\mu\nu} \;\; \rightarrow \;\; g'_{\mu\nu} = g_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \xi^\rho \partial_\rho g_{\mu\nu}. \tag{2.36}$$

This allows us to make four gauge choices. At this stage, we impose *three* gauge choices [33, 34]

$$g_{--} \; = \; g_{-i} \; = \; 0 \quad , \; i = 1, 2 \, , \tag{2.37}$$

and parameterize the other components as

$$
\begin{aligned}
g_{+-} &= -\, e^{\phi} \, , \\
g_{i\,j} &= e^{\psi}\, \gamma_{ij} \, .
\end{aligned}
\tag{2.38}
$$

γ^{ij} is real, symmetric and unimodular while ϕ and ψ are real. Field equations in which ∂_+ (time derivatives) do not appear serve as constraint relations, as opposed to 'true' equations of motion. The constraint relation $\mu = \nu = -$ from (2.34) is

$$
2\,\partial_-\phi\,\partial_-\psi \;-\; 2\,\partial_-^2\psi \;-\; (\partial_-\psi)^2 \;+\; \frac{1}{2}\partial_-\gamma^{ij}\,\partial_-\gamma_{ij} \;=\; 0\,. \tag{2.39}
$$

This can be solved by making the specific choice

$$
\phi \;=\; \frac{\psi}{2}\,, \tag{2.40}
$$

which is the *fourth* (and final) gauge choice. With this choice

$$
\psi \;=\; \frac{1}{4}\,\frac{1}{\partial_-^2}\,(\partial_-\gamma^{ij}\,\partial_-\gamma_{ij})\,. \tag{2.41}
$$

Unphysical components of the metric are systematically eliminated by constraint relations. This results in a 'physical' closed-form action [33, 34] written entirely in terms of the γ^{ij} (we still write ϕ and ψ despite (2.40), (2.41) because they make the calculation easy to check).

$$
\begin{aligned}
S \;=\; & \frac{1}{2\kappa^2}\int d^4x\; e^{\psi}\left(2\,\partial_+\partial_-\phi \;+\; \partial_+\partial_-\psi - \frac{1}{2}\,\partial_+\gamma^{ij}\partial_-\gamma_{ij}\right) \\
& - e^{\phi}\gamma^{ij}\left(\partial_i\partial_j\phi + \frac{1}{2}\partial_i\phi\partial_j\phi - \partial_i\phi\partial_j\psi - \frac{1}{4}\partial_i\gamma^{kl}\partial_j\gamma_{kl} + \frac{1}{2}\partial_i\gamma^{kl}\partial_k\gamma_{jl}\right) \\
& - \frac{1}{2}e^{\phi-2\psi}\gamma^{ij}\frac{1}{\partial_-}R_i\frac{1}{\partial_-}R_j\,,
\end{aligned}
\tag{2.42}
$$

where

$$
R_i \;\equiv\; e^{\psi}\left(\frac{1}{2}\partial_-\gamma^{jk}\partial_i\gamma_{jk} - \partial_-\partial_i\phi - \partial_-\partial_i\psi + \partial_i\phi\partial_-\psi\right) + \partial_k(e^{\psi}\,\gamma^{jk}\partial_-\gamma_{ij})\,.
$$

A perturbative expansion of the action in (2.42), in powers of κ, is achieved by choosing [34]

$$\gamma_{ij} = \left(\mathrm{e}^{\sqrt{2}\,\kappa\, h}\right)_{ij} \ , \qquad \gamma_{ij}\,\gamma^{jk} = \delta_i^k \ , \qquad h_{ij} = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & -h_{11} \end{pmatrix} \ . \tag{2.43}$$

These relate to the physical helicity states of the graviton as

$$h_{ij} = \frac{1}{\sqrt{2}} \begin{pmatrix} h+\bar{h} & -i(h-\bar{h}) \\ -i(h-\bar{h}) & -h-\bar{h} \end{pmatrix} \ , \tag{2.44}$$

with h and $\bar{h}$ representing gravitons of helicity $+2$ and -2 respectively.

The Lagrangian may be expanded in orders of κ

$$\mathcal{L} = \mathcal{L}_0 \, + \, \mathcal{L}_{\kappa} \, + \, \mathcal{L}_{\kappa^2} \, + \, \mathcal{L}_{\kappa^3} + \mathcal{L}_{\kappa^4} \tag{2.45}$$

The Lagrangian reads

$$\mathcal{L} \;\; = \;\; \bar{h}\,\Box\,h + 2\kappa\,\bar{h}\,\partial_-^2 \left(\frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h - h\frac{\bar{\partial}^2}{\partial_-^2}h\right) + 2\kappa\,h\,\partial_-^2\left(\frac{\partial}{\partial_-}\bar{h}\frac{\partial}{\partial_-}\bar{h} - \bar{h}\frac{\partial^2}{\partial_-^2}\bar{h}\right) \tag{2.46}$$

The expressions for $\mathcal{L}_{\kappa^2}$ and $\mathcal{L}_{\kappa^3}$ in (2.45) are known [34–36] and are presented below.

$$\begin{aligned}
\mathcal{L}_{\kappa^2} = 2\kappa^2 \Bigg\{ & \frac{1}{\partial_-^2}(\partial_- h \partial_- \bar{h}) \frac{\partial\bar{\partial}}{\partial_-^2}(\partial_- h\partial_-\bar{h}) + \frac{1}{\partial_-^3}(\partial_- h\partial_-\bar{h})\,\left(\partial\bar{\partial}h\,\partial_-\bar{h} + \partial_- h\partial\bar{\partial}\bar{h}\right) \\
& -\frac{1}{\partial_-^2}(\partial_- h\partial_-\bar{h})\,\left(2\,\partial\bar{\partial}h\,\bar{h} + 2\,h\partial\bar{\partial}\bar{h} + 9\,\bar{\partial}h\partial\bar{h} + \partial h\bar{\partial}\bar{h} - \frac{\partial\bar{\partial}}{\partial_-}h\,\partial_-\bar{h} - \partial_- h\frac{\partial\bar{\partial}}{\partial_-}\bar{h}\right) \\
& -2\frac{1}{\partial_-}\left(2\bar{\partial}h\,\partial_-\bar{h} + h\partial_-\bar{\partial}\bar{h} - \partial_-\bar{\partial}h\bar{h}\right)h\,\partial\bar{h} - 2\frac{1}{\partial_-}\left(2\partial_- h\,\partial\bar{h} + \partial_-\partial h\,\bar{h} - h\partial_-\partial\bar{h}\right)\bar{\partial}h\,\bar{h} \\
& -\frac{1}{\partial_-}\left(2\bar{\partial}h\,\partial_-\bar{h} + h\partial_-\bar{\partial}\bar{h} - \partial_-\bar{\partial}h\bar{h}\right)\frac{1}{\partial_-}\left(2\partial_- h\,\partial\bar{h} + \partial_-\partial h\,\bar{h} - h\partial_-\partial\bar{h}\right) \\
& -h\,\bar{h}\,\left(\partial\bar{\partial}h\,\bar{h} + h\partial\bar{\partial}\bar{h} + 2\,\bar{\partial}h\partial\bar{h} + 3\frac{\partial\bar{\partial}}{\partial_-}h\,\partial_-\bar{h} + 3\partial_- h\frac{\partial\bar{\partial}}{\partial_-}\bar{h}\right)\Bigg\} \ .
\end{aligned} \tag{2.47}$$

The terms containing a ∂_+ at order κ^2 were eliminated using a field redefinition [34]

$$h \to h - \kappa^2\,\frac{1}{\partial_-}\left\{2\,\partial_-^2 h\frac{1}{\partial_-^3}(\partial_- h\partial_-\bar{h}) + \partial_- h\frac{1}{\partial_-^2}(\partial_- h\partial_-\bar{h}) + \frac{1}{3}(h\bar{h}\partial_- h - hh\partial_-\bar{h})\right\} \ . \tag{2.48}$$

Moving to five-point interaction vertices, ie. at order κ^3 [36], we find

$$\mathcal{L}_{\kappa^3} = 2\sqrt{2}\kappa^3\, L_5\ , \tag{2.49}$$

where L_5 reads

$$\begin{aligned}
L_5 = \;& -\frac{1}{\sqrt{2}} h\bar{\partial}h\bar{\partial}\bar{h}\frac{1}{\partial_-^2}(\partial_-\bar{h}\partial_-h) + \frac{\sqrt{2}}{3}\bar{h}hh\bar{\partial}h\bar{\partial}\bar{h} + \frac{\sqrt{2}}{3}h\bar{\partial}h\bar{\partial}(\bar{h}\bar{h}h) + \frac{\sqrt{2}}{3}h\bar{\partial}\bar{h}\bar{\partial}(\bar{h}hh) \\
& +\frac{1}{\sqrt{2}} h\bar{\partial}h\bar{\partial}\bar{h}\frac{1}{\partial_-^2}(\partial_-\bar{h}\partial_-h) - \frac{\sqrt{2}}{3}\bar{h}hh\bar{\partial}h\bar{\partial}\bar{h} - \frac{\sqrt{2}}{3}h\bar{\partial}h\bar{\partial}(\bar{h}\bar{h}h) - \frac{\sqrt{2}}{3}h\bar{\partial}\bar{h}\bar{\partial}(\bar{h}hh) \\
& -\frac{3}{4\sqrt{2}}h\frac{\bar{\partial}}{\partial_-^2}(\partial_-\bar{h}\partial_-h)\frac{\bar{\partial}}{\partial_-^2}(\partial_-\bar{h}\partial_-h) + \frac{1}{2\sqrt{2}}h\frac{1}{\partial_-^2}(\partial_-\bar{h}\partial_-h)\frac{\bar{\partial}\bar{\partial}}{\partial_-^2}(\partial_-\bar{h}\partial_-h) \\
& -\frac{1}{3\sqrt{2}}\bar{h}hh\frac{\bar{\partial}\bar{\partial}}{\partial_-^2}(\partial_-\bar{h}\partial_-h) - \frac{1}{3\sqrt{2}}h\frac{\bar{\partial}\bar{\partial}}{\partial_-^2}(\partial_-h\partial_-[\bar{h}\bar{h}h]) - \frac{1}{3\sqrt{2}}h\frac{\bar{\partial}\bar{\partial}}{\partial_-^2}(\partial_-[\bar{h}hh]\partial_-\bar{h}) \\
& -\frac{1}{2\sqrt{2}}h\frac{\bar{\partial}\bar{\partial}}{\partial_-^2}(\partial_-[\bar{h}h]\partial_-[\bar{h}h]) - 2\sqrt{2}\,\bar{h}\bar{\partial}h\Bigg[\frac{\bar{\partial}}{\partial_-}\{h\partial_-(\bar{h}h)\} + \frac{\bar{\partial}}{\partial_-}\{\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})\partial_-h\} \\
& -\frac{\bar{\partial}}{\partial_-}(h\bar{h}\partial_-h) - \frac{1}{3}\bar{\partial}(hh\bar{h}) + \frac{1}{\partial_-}\{\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})\bar{\partial}\partial_-h\}\Bigg] + \frac{1}{\sqrt{2}}\,h\,\frac{1}{\partial_-}\Bigg[\partial_-h\bar{\partial}\bar{h} \\
& +\partial_-\bar{h}\bar{\partial}h - \frac{3}{2}\frac{\bar{\partial}}{\partial_-}(\partial_-\bar{h}\partial_-h) + 2\bar{\partial}(h\partial_-\bar{h}) - \bar{\partial}\partial_-(\bar{h}h)\Bigg] \times \frac{1}{\partial_-}\Bigg[\partial_-h\bar{\partial}\bar{h} + \partial_-\bar{h}\bar{\partial}h \\
& -\frac{3}{2}\frac{\bar{\partial}}{\partial_-}(\partial_-\bar{h}\partial_-h) + 2\bar{\partial}(h\partial_-\bar{h}) - \bar{\partial}\partial_-(\bar{h}h)\Bigg] + \bar{\partial}h\bar{\partial}h\left[\frac{\sqrt{2}}{3}\bar{h}\bar{h}h + \frac{3}{\sqrt{2}}\bar{h}\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})\right] \\
& +\sqrt{2}\bar{\partial}h\,\frac{1}{\partial_-}\Bigg\{-\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})\partial_-h\bar{\partial}\bar{h} + \frac{1}{3}\partial_-h\bar{\partial}(\bar{h}\bar{h}h) + \frac{1}{3}\partial_-(\bar{h}hh)\bar{\partial}\bar{h} + \frac{1}{3}\partial_-\bar{h}\bar{\partial}(\bar{h}hh) \\
& -\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})\partial_-\bar{h}\bar{\partial}h + \frac{1}{3}\partial_-(\bar{h}\bar{h}h)\bar{\partial}h - \partial_-(\bar{h}h)\bar{\partial}(\bar{h}h) - \frac{1}{2}\frac{\bar{\partial}}{\partial_-}[\partial_-h\partial_-(\bar{h}\bar{h}h)] \\
& +\frac{3}{2}\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})\frac{\bar{\partial}}{\partial_-}(\partial_-\bar{h}\partial_-h) - \frac{1}{2}\frac{\bar{\partial}}{\partial_-}[\partial_-(\bar{h}hh)\partial_-\bar{h}] + \frac{3}{4}\frac{\bar{\partial}}{\partial_-}[\partial_-(\bar{h}h)\partial_-(\bar{h}h)] \\
& -\frac{1}{2}\frac{\bar{\partial}}{\partial_-^2}(\partial_-\bar{h}\partial_-h)\frac{1}{\partial_-}(\partial_-\bar{h}\partial_-h) + \frac{2}{3}\bar{\partial}[h\partial_-(\bar{h}\bar{h}h)] + \frac{2}{3}\bar{\partial}[\bar{h}hh\partial_-\bar{h}] - \frac{1}{6}\bar{\partial}\partial_-(\bar{h}\bar{h}hh) \\
& -2\bar{\partial}[\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})h\partial_-\bar{h}] - \bar{\partial}[\bar{h}h\partial_-(\bar{h}h)] + \bar{\partial}[\frac{1}{\partial_-^2}(\partial_-\bar{h}\partial_-h)\partial_-(\bar{h}h)]\Bigg\} \\
& -\sqrt{2}\frac{1}{\partial_-}\Bigg\{\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})\bar{\partial}\partial_-h - \frac{1}{3}\bar{\partial}\partial_-(hh\bar{h}) - \bar{\partial}(h\bar{h}\partial_-h) + \bar{\partial}[\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})\partial_-h] \\
& +\bar{\partial}(h\partial_-(\bar{h}h)\Bigg\} \times \frac{1}{\partial_-}\Bigg\{\partial_-h\bar{\partial}\bar{h} + \partial_-\bar{h}\bar{\partial}h - \frac{3}{2}\frac{\bar{\partial}}{\partial_-}(\partial_-\bar{h}\partial_-h) + 2\bar{\partial}(h\partial_-\bar{h}) - \bar{\partial}\partial_-(\bar{h}h)\Bigg\}
\end{aligned}$$

$$
+\sqrt{2}[\bar{h}h+\frac{3}{2}\frac{1}{\partial_-^2}(\partial_- h\partial_-\bar{h})]\,\frac{1}{\partial_-}\left\{\partial_- h\bar{\partial}\bar{h}+\partial_-\bar{h}\bar{\partial}h-\frac{3}{2}\frac{\bar{\partial}}{\partial_-}(\partial_-\bar{\partial}\partial_- h)+2\bar{\partial}(h\partial_-\bar{h})-\bar{\partial}\partial_-(\bar{h}h)\right\}\bar{\partial}h
$$
$$
+\left\{2\,\partial_-^2\bar{h}\frac{1}{\partial_-^3}(\partial_- h\partial_-\bar{h})+\partial_-\bar{h}\frac{1}{\partial_-^2}(\partial_- h\partial_-\bar{h})+\frac{1}{3}(h\bar{h}\partial_-\bar{h}-\bar{h}\bar{h}\partial_- h)\right\}\partial_-\left(\frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h-h\frac{\bar{\partial}^2}{\partial_-^2}h\right)
$$
$$
+\left\{2\,\partial_-^2 h\frac{1}{\partial_-^3}(\partial_- h\partial_-\bar{h})+\partial_- h\frac{1}{\partial_-^2}(\partial_- h\partial_-\bar{h})+\frac{1}{3}(h\bar{h}\partial_- h-hh\partial_-\bar{h})\right\}\times\left[2\frac{\bar{\partial}}{\partial_-^2}(\partial_-^2\bar{h}\frac{\bar{\partial}}{\partial_-}h)\right.
$$
$$
\left.+\partial_-(\frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h-h\frac{\bar{\partial}^2}{\partial_-^2}h)-\frac{1}{\partial_-}(\partial_-^2\bar{h}\frac{\bar{\partial}^2}{\partial_-^2}h)-\frac{\bar{\partial}^2}{\partial_-^3}(h\partial_-^2\bar{h})\right]\;+\;\text{c.c.} \tag{2.50}
$$

The last three lines of (2.50) represent the quintic interaction vertices produced by applying the field redefinition in (2.48) to the cubic interaction vertices in (2.46). Although L_5 appears to contain non-MHV structures, these are related by conjugation to MHV vertices.

Analogous to the Yang-Mills theory, the cubic MHV vertex in (2.46), in momentum space can be written in terms of the spinor-helicity variables to obtain the three point amplitude

$$
M_3(1^-,2^-,3^+)=\frac{\langle 1\,2\rangle^6}{\langle 2\,3\rangle^2\langle 3\,1\rangle^2}\;, \tag{2.51}
$$

Similarly, from (2.47), we find at four-point

$$
M_4(1^-,2^-,3^+,4^+)=\frac{\langle 1\,2\rangle^8[1\,2]}{\langle 1\,2\rangle\langle 1\,3\rangle\langle 1\,4\rangle\langle 2\,3\rangle\langle 2\,4\rangle\langle 3\,4\rangle^2}\;, \tag{2.52}
$$

These MHV amplitudes (vertices) factorise, as expected from the KLT relations [7], into products of off-shell MHV vertices in Yang-Mills theory [37].

We demonstrated in this chapter, that amplitude structures naturally emerge in the light-cone formulations of Yang-Mills theory and pure gravity. We focussed particularly on the MHV structures [37–40]. This provides evidence for the fact that scattering amplitudes in gravitational theories are the square of those in gauge theories describing spin-1 fields :

$$
\text{Gravitation}\sim(\text{Yang-Mills})^2 \tag{2.53}
$$

In the next chapter, we present the perturbative expansion of pure gravity in the light-cone gauge to order κ^4 where the first truly non-MHV interaction vertices appear. We will also demonstrate relations of the type (2.53) in higher derivative gravity and gauge theories.

Chapter 3

Gravitation : Perturbation theory and amplitudes

(*The material presented in this chapter is primarily based on the works done by the author in [41, 42]*)

In this chapter, we present the 6-point interaction vertices in pure gravity in four-dimensions. We also derive the helicity selection rules from symmetry principles to explain the structure of interaction vertices to all orders in perturbation theory. We argue the origin of MHV vertices in this formulation. Finally, we discuss interaction vertices in higher derivative theories of gravity. The issue of uniqueness of quartic and higher order interaction vertices obtained using closure of Poincare algebra for field theories of spin > 1 is also discussed.

3.1 The 6-point result

We start with the closed form action derived in (2.42)

$$\begin{aligned} S \;=\; & \frac{1}{2\kappa^2}\int d^4x\; e^{\psi}\left(2\,\partial_+\partial_-\phi \,+\, \partial_+\partial_-\psi - \frac{1}{2}\,\partial_+\gamma^{ij}\partial_-\gamma_{ij}\right) \\ & -e^{\phi}\gamma^{ij}\left(\partial_i\partial_j\phi + \frac{1}{2}\partial_i\phi\partial_j\phi - \partial_i\phi\partial_j\psi - \frac{1}{4}\partial_i\gamma^{kl}\partial_j\gamma_{kl} + \frac{1}{2}\partial_i\gamma^{kl}\partial_k\gamma_{jl}\right) \\ & -\frac{1}{2}e^{\phi-2\psi}\gamma^{ij}\frac{1}{\partial_-}R_i\frac{1}{\partial_-}R_j\;, \end{aligned} \tag{3.1}$$

where

$$R_i \;\equiv\; e^{\psi}\left(\frac{1}{2}\partial_-\gamma^{jk}\partial_i\gamma_{jk} - \partial_-\partial_i\phi - \partial_-\partial_i\psi + \partial_i\phi\partial_-\psi\right) + \partial_k(e^{\psi}\,\gamma^{jk}\partial_-\gamma_{ij})\;,$$

and $\phi = \psi/2$ with

$$\psi \;=\; \frac{1}{4}\,\frac{1}{\partial_-^2}\,(\partial_-\gamma^{ij}\,\partial_-\gamma_{ij})\;. \tag{3.2}$$

Transverse variables and the metric

We work with the helicity states directly, by using transverse variables a and b (so $x^a\,,x^b$ represent $x\,,\bar{x}$) [36]. The relevant metric is

$$\eta_{ab}=\frac{\partial x^i}{\partial x^a}\frac{\partial x^j}{\partial x^b}\,\eta_{ij}\;. \tag{3.3}$$

So

$$\eta_{xx}=\eta_{\bar{x}\bar{x}}=0\;,\qquad \eta_{x\bar{x}}=\eta_{\bar{x}x}=1\;. \tag{3.4}$$

The inverse metric ($\eta^{ab}\eta_{bc}=\delta^a{}_c$) has components

$$\eta^{xx}=\eta^{\bar{x}\bar{x}}=0\;,\qquad \eta^{x\bar{x}}=\eta^{\bar{x}x}=1\;. \tag{3.5}$$

In this new basis, the expressions for γ from (7.13) read

$$\gamma_{ab}\;=\;\sum_{n=0}^{\infty}\begin{pmatrix} 2\kappa^{2n+1}\,\dfrac{(4h\bar{h})^n}{(2n+1)!}\,\bar{h} & \kappa^{2n}\,\dfrac{(4h\bar{h})^n}{(2n)!} \\[3ex] \kappa^{2n}\,\dfrac{(4h\bar{h})^n}{(2n)!} & 2\kappa^{2n+1}\,\dfrac{(4h\bar{h})^n}{(2n+1)!}\,h \end{pmatrix}\;, \tag{3.6}$$

$$\gamma^{ab}\;=\;\sum_{n=0}^{\infty}\begin{pmatrix} -2\kappa^{2n+1}\,\dfrac{(4h\bar{h})^n}{(2n+1)!}\,h & \kappa^{2n}\,\dfrac{(4h\bar{h})^n}{(2n)!} \\[3ex] \kappa^{2n}\,\dfrac{(4h\bar{h})^n}{(2n)!} & -2\kappa^{2n+1}\,\dfrac{(4h\bar{h})^n}{(2n+1)!}\,\bar{h} \end{pmatrix}\;. \tag{3.7}$$

We only need terms up to $n=2$ in the series (3.6) and (3.7) for the discussion here. It is then easy to also calculate ψ at orders 2 and 4

$$\begin{aligned}\psi\;&=\;2\kappa^2\psi_2\;+\;4\kappa^4\psi_4 \\ &=2\kappa^2\left\{-\frac{1}{\partial_-^2}(\partial_-\bar{h}\partial_-h)\right\} \\ &+4\kappa^4\left\{-\frac{1}{3\partial_-^2}(\partial_-h\partial_-[\bar{h}\bar{h}h])-\frac{1}{3\partial_-^2}(\partial_-[\bar{h}hh]\partial_-\bar{h})+\frac{1}{2\partial_-^2}(\partial_-[\bar{h}h]\partial_-[\bar{h}h])\right\}\;.\end{aligned} \tag{3.8}$$

We now present here, a new result: the perturbative expansion to order κ^4 of the closed form action (2.42) in terms of the helicity states h and $\bar{h}$. This is the lowest order at which truly non-MHV structures first appear.

$$\mathcal{L}_{\kappa^4} = 4\kappa^4\, L_6 \;, \tag{3.9}$$

where

$$\begin{aligned}
L_6 &= 4\psi_4\partial\bar{\partial}\psi_2 + \psi_2^2\partial\bar{\partial}\psi_2 - \frac{1}{15}\left(h\partial\bar{\partial}[(h\bar{h})^2\bar{h}] + \bar{h}\partial\bar{\partial}[(h\bar{h})^2 h]\right) + \frac{1}{3}h\bar{h}\partial\bar{\partial}[(h\bar{h})^2] \\
&-\frac{2}{9}h\bar{h}h\partial\bar{\partial}[h\bar{h}\bar{h}] + \frac{\psi_2}{3}\left(\frac{\partial\bar{\partial}}{\partial_-}h\partial_-[h\bar{h}\bar{h}] + \partial_-\bar{h}\frac{\partial\bar{\partial}}{\partial_-}[h\bar{h}h] + \partial_- h\frac{\partial\bar{\partial}}{\partial_-}[h\bar{h}\bar{h}] + \frac{\partial\bar{\partial}}{\partial_-}\bar{h}\partial_-[h\bar{h}h]\right) \\
&-\psi_2\frac{\partial\bar{\partial}}{\partial_-}[h\bar{h}]\partial_-[h\bar{h}] + \left(\psi_4 + \frac{\psi_2^2}{2}\right)\left(\frac{\partial\bar{\partial}}{\partial_-}h\partial_-\bar{h} + \partial_- h\frac{\partial\bar{\partial}}{\partial_-}\bar{h}\right) - \frac{1}{6}(h\bar{h}h\bar{h})\partial\bar{\partial}\psi_2 - h\bar{h}\partial\bar{\partial}\psi_4 \\
&-\frac{\psi_2}{2}h\bar{h}\partial\bar{\partial}\psi_2 + \frac{3}{4}h\bar{h}\partial\psi_2\bar{\partial}\psi_2 - \frac{5}{2}\psi_4\partial\bar{\partial}\psi_2 - \frac{5}{16}\psi_2^2\partial\bar{\partial}\psi_2 \\
&+(\partial h\bar{\partial}\bar{h} - \partial\bar{h}\bar{\partial}h)\left(\frac{1}{6}h\bar{h}h\bar{h} + \frac{1}{2}\psi_2 h\bar{h} + \frac{1}{2}\psi_4 + \frac{1}{8}\psi_2^2\right) \\
&+\left(\frac{1}{3}h\bar{h} + \frac{1}{6}\psi_2\right)(\partial h\bar{\partial}[h\bar{h}\bar{h}] + \bar{\partial}\bar{h}\partial[h\bar{h}h] - \bar{\partial}h\partial[h\bar{h}\bar{h}] - \partial\bar{h}\bar{\partial}[h\bar{h}h]) \\
&+\left(\frac{h\bar{h}}{3} + \frac{\psi_2}{2}\right)(h\partial\bar{h} - \bar{h}\partial h)\bar{\partial}[h\bar{h}] + \left(\frac{h\bar{h}}{3} + \frac{\psi_2}{2}\right)(\bar{h}\bar{\partial}h - h\bar{\partial}\bar{h})\partial[h\bar{h}] + \frac{1}{6}(h\partial\bar{h} - \bar{h}\partial h)\bar{\partial}[h\bar{h}h\bar{h}] \\
&+\frac{1}{6}(\bar{h}\bar{\partial}h - h\bar{\partial}\bar{h})\partial[h\bar{h}h\bar{h}] + \frac{h}{3}(\bar{\partial}[h\bar{h}]\partial[h\bar{h}\bar{h}] - \bar{\partial}[h\bar{h}\bar{h}]\partial[h\bar{h}]) + \frac{\bar{h}}{3}(\partial[h\bar{h}]\bar{\partial}[h\bar{h}h] - \partial[h\bar{h}h]\bar{\partial}[h\bar{h}]) \\
&+\Bigg\{2h\partial\bar{h}\frac{\bar{\partial}}{\partial_-}\left[\frac{2}{3}h\bar{h}\bar{h}\partial_- h - \frac{2}{3}h\bar{h}h\partial_-\bar{h} - 2\psi_2 h\partial_-\bar{h} + \psi_2\partial_-[h\bar{h}]\right] - 2h\partial\bar{h}\frac{1}{\partial_-}\Bigg[\frac{1}{3}(\partial_- h\bar{\partial}[h\bar{h}\bar{h}] \\
&+\bar{\partial}\bar{h}\partial_-[h\bar{h}h] + \bar{\partial}h\partial_-[h\bar{h}\bar{h}] + \partial_-\bar{h}\bar{\partial}[h\bar{h}h]) - \partial_-[h\bar{h}]\bar{\partial}[h\bar{h}] + \psi_2\left(\partial_- h\bar{\partial}\bar{h} + \bar{\partial}h\partial_-\bar{h}\right) \\
&+\frac{3}{2}\psi_2\partial_-\bar{\partial}\psi_2 - \frac{1}{2}\bar{\partial}\psi_2\partial_-\psi_2 + \frac{3}{2}\partial_-\bar{\partial}\psi_4\Bigg] + \left(\bar{\partial}\left[h\bar{h} - 2\frac{1}{\partial_-}\left[h\partial_-\bar{h}\right]\right] - \frac{1}{\partial_-}\left[\partial_- h\bar{\partial}\bar{h} + \bar{\partial}h\partial_-\bar{h}\right] - \frac{3}{2}\bar{\partial}\psi_2\right) \\
&\times\left(2h\partial\left[\frac{1}{3}h\bar{h}\bar{h} + \frac{1}{\partial_-}\left[\psi_2\partial_-\bar{h} - \bar{h}\bar{h}\partial_- h\right]\right] + 2\partial\bar{h}\left[\frac{1}{3}h\bar{h}h - \frac{3}{2}\psi_2 h\right]\right) + c.c.\Bigg\} \\
&-\left(\frac{1}{3}(h\bar{h})^2 - 3\psi_4 - 3\psi_2 h\bar{h} + \frac{9}{4}\psi_2^2\right)\bar{\partial}h\partial\bar{h} \\
&-\left(h\bar{h} - \frac{3}{2}\psi_2\right)\left(2\bar{\partial}h\partial\left[\frac{1}{3}h\bar{h}\bar{h} + \frac{1}{\partial_-}\left[\psi_2\partial_-\bar{h} - \bar{h}\bar{h}\partial_- h\right]\right] + \text{ c.c. }\right) \\
&-\left(h\bar{h} - \frac{3}{2}\psi_2\right)\left\{\bar{\partial}\left[h\bar{h} - 2\frac{1}{\partial_-}\left[h\partial_-\bar{h}\right]\right] - \frac{1}{\partial_-}\left[\partial_- h\bar{\partial}\bar{h} + \bar{\partial}h\partial_-\bar{h}\right] - \frac{3}{2}\bar{\partial}\psi_2\right\}\Bigg\{\partial\left[h\bar{h} - 2\frac{1}{\partial_-}\left[\bar{h}\partial_- h\right]\right] \\
&-\frac{1}{\partial_-}\left[\partial_-\bar{h}\partial h + \partial\bar{h}\partial_- h\right] - \frac{3}{2}\partial\psi_2\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&-\left\{2\bar{\partial}h\frac{\partial}{\partial_-}\left(-\frac{2}{15}\bar{h}\partial_-[h\bar{h}]^2-\frac{2}{3}\psi_2\bar{h}\partial_-[h\bar{h}]+\left(\frac{8}{15}(h\bar{h})^2+\frac{4}{3}\psi_2h\bar{h}+\psi_4+\frac{1}{2}\psi_2^2\right)\partial_-\bar{h}\right)+\text{ c.c. }\right\}\\
&-\left\{\left(\partial\left[h\bar{h}-2\frac{1}{\partial_-}[\bar{h}\partial_-h]\right]-\frac{1}{\partial_-}[\partial_-\bar{h}\partial h+\partial\bar{h}\partial_-h]-\frac{3}{2}\partial\psi_2\right)\right.\\
&\times\left(\frac{\bar{\partial}}{\partial_-}\left[\frac{2}{3}h\bar{h}\bar{h}\partial_-h-\frac{2}{3}h\bar{h}h\partial_-\bar{h}-2\psi_2h\partial_-\bar{h}+\psi_2\partial_-[h\bar{h}]\right]\right.\\
&-\frac{1}{\partial_-}\left[\frac{1}{3}\left(\partial_-h\bar{\partial}[h\bar{h}\bar{h}]+\bar{\partial}\bar{h}\partial_-[h\bar{h}h]+\bar{\partial}h\partial_-[h\bar{h}\bar{h}]+\partial_-\bar{h}\bar{\partial}[h\bar{h}h]\right)-\partial_-[h\bar{h}]\bar{\partial}[h\bar{h}]+\psi_2\left(\partial_-h\bar{\partial}\bar{h}+\bar{\partial}h\partial_-\bar{h}\right)\right.\\
&\left.\left.\left.+\frac{3}{2}\psi_2\partial_-\bar{\partial}\psi_2-\frac{1}{2}\bar{\partial}\psi_2\partial_-\psi_2+\frac{3}{2}\partial_-\bar{\partial}\psi_4\right]\right)+\text{c.c.}\right\}\\
&-2\partial\left[\frac{1}{3}h\bar{h}\bar{h}+\frac{1}{\partial_-}\left[\psi_2\partial_-\bar{h}-\bar{h}\bar{h}\partial_-h\right]\right]\bar{\partial}\left[\frac{1}{3}hh\bar{h}+\frac{1}{\partial_-}\left[\psi_2\partial_-h-hh\partial_-\bar{h}\right]\right]\\
&+\left\{\frac{\partial\bar{\partial}}{\partial_-}\bar{M}\left[\partial_-^2h\frac{1}{\partial_-^3}\left(\partial_-h\partial_-\bar{h}\right)+\frac{1}{3}\left(h\bar{h}\partial_-h-hh\partial_-\bar{h}\right)\right]-\frac{1}{\partial_-^2}\left(\partial_-M\partial_-\bar{h}\right)\left[-\frac{1}{2}\frac{\partial\bar{\partial}}{\partial_-^2}\left(\partial_-h\partial_-\bar{h}\right)\right.\right.\\
&\left.+\frac{1}{\partial_-}\left(\partial\bar{\partial}h\partial_-\bar{h}\right)\right]-M\bar{h}\left[\bar{\partial}h\partial\bar{h}+\frac{4}{3}h\partial\bar{\partial}\bar{h}+4\partial_-h\frac{\partial\bar{\partial}}{\partial_-}\bar{h}\right]-2\frac{1}{\partial_-^2}\left(\partial_-M\partial_-\bar{h}\right)\left[\frac{1}{2}\bar{\partial}h\partial\bar{h}+h\partial\bar{\partial}\bar{h}-\partial_-h\frac{\partial\bar{\partial}}{\partial_-}\bar{h}\right.\\
&\left.\left.+\frac{1}{\partial_-}\left(\partial\bar{\partial}h\partial_-\bar{h}\right)-\frac{1}{4}\frac{\partial\bar{\partial}}{\partial_-^2}\left(\partial_-h\partial_-\bar{h}\right)\right]+2M\partial\bar{h}\frac{1}{\partial_-}\left(2\bar{\partial}h\partial_-\bar{h}+h\partial_-\bar{\partial}\bar{h}-\partial_-\bar{\partial}h\bar{h}\right)+c.c\right\}\\
&-\frac{1}{\partial_-}\left(2\bar{\partial}M\partial_-\bar{h}+M\partial_-\bar{\partial}\bar{h}-\partial_-\bar{\partial}M\bar{h}\right)\frac{1}{\partial_-}\left(2\partial_-h\partial\bar{h}+\partial_-\partial h\bar{h}-h\partial_-\partial\bar{h}\right)\\
&+\left\{\frac{\partial\bar{\partial}}{\partial_-}\bar{h}\left[\partial_-^2M\frac{1}{\partial_-^3}\left(\partial_-h\partial_-\bar{h}\right)+\frac{1}{3}\left(M\bar{h}\partial_-h-Mh\partial_-\bar{h}\right)\right]-\frac{1}{\partial_-^2}\left(\partial_-h\partial_-\bar{M}\right)\left[-\frac{1}{2}\frac{\partial\bar{\partial}}{\partial_-^2}\left(\partial_-h\partial_-\bar{h}\right)\right.\right.\\
&\left.+\frac{1}{\partial_-}\left(\partial\bar{\partial}h\partial_-\bar{h}\right)\right]-h\bar{M}\left[\bar{\partial}h\partial\bar{h}+\frac{4}{3}h\partial\bar{\partial}\bar{h}+4\partial_-h\frac{\partial\bar{\partial}}{\partial_-}\bar{h}\right]-2\frac{1}{\partial_-^2}\left(\partial_-h\partial_-\bar{M}\right)\left[\frac{1}{2}\bar{\partial}h\partial\bar{h}+h\partial\bar{\partial}\bar{h}-\partial_-h\frac{\partial\bar{\partial}}{\partial_-}\bar{h}\right.\\
&\left.\left.+\frac{1}{\partial_-}\left(\partial\bar{\partial}h\partial_-\bar{h}\right)-\frac{1}{4}\frac{\partial\bar{\partial}}{\partial_-^2}\left(\partial_-h\partial_-\bar{h}\right)\right]+2h\partial\bar{M}\frac{1}{\partial_-}\left(2\bar{\partial}h\partial_-\bar{h}+h\partial_-\bar{\partial}\bar{h}-\partial_-\bar{\partial}h\bar{h}\right)+c.c\right\}\\
&-\frac{1}{\partial_-}\left(2\bar{\partial}h\partial_-\bar{M}+h\partial_-\bar{\partial}\bar{M}-\partial_-\bar{\partial}h\bar{M}\right)\frac{1}{\partial_-}\left(2\partial_-h\partial\bar{h}+\partial_-\partial h\bar{h}-h\partial_-\partial\bar{h}\right)\\
&+\left\{\frac{\partial\bar{\partial}}{\partial_-}\bar{h}\left[\partial_-^2h\frac{1}{\partial_-^3}\left(\partial_-M\partial_-\bar{h}\right)+\frac{1}{3}\left(h\bar{M}\partial_-h-hM\partial_-\bar{h}\right)\right]-\frac{1}{\partial_-^2}\left(\partial_-h\partial_-\bar{h}\right)\left[-\frac{1}{2}\frac{\partial\bar{\partial}}{\partial_-^2}\left(\partial_-M\partial_-\bar{h}\right)\right.\right.\\
&\left.+\frac{1}{\partial_-}\left(\partial\bar{\partial}M\partial_-\bar{h}\right)\right]-h\bar{h}\left[\bar{\partial}M\partial\bar{h}+\frac{4}{3}M\partial\bar{\partial}\bar{h}+4\partial_-M\frac{\partial\bar{\partial}}{\partial_-}\bar{h}\right]-2\frac{1}{\partial_-^2}\left(\partial_-h\partial_-\bar{h}\right)\left[\frac{1}{2}\bar{\partial}M\partial\bar{h}+M\partial\bar{\partial}\bar{h}\right.\\
&\left.-\partial_-M\frac{\partial\bar{\partial}}{\partial_-}\bar{h}+\frac{1}{\partial_-}\left(\partial\bar{\partial}M\partial_-\bar{h}\right)-\frac{1}{4}\frac{\partial\bar{\partial}}{\partial_-^2}\left(\partial_-M\partial_-\bar{h}\right)\right]\\
&\qquad\left.+2h\partial\bar{h}\frac{1}{\partial_-}\left(2\bar{\partial}M\partial_-\bar{h}+M\partial_-\bar{\partial}\bar{h}-\partial_-\bar{\partial}M\bar{h}\right)+c.c\right\}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\partial_-}\left(2\bar\partial h\partial_-\bar h+h\partial_-\bar\partial\bar h-\partial_-\bar\partial h\bar h\right)\frac{1}{\partial_-}\left(2\partial_-M\partial\bar h+\partial_-\partial M\bar h-M\partial_-\partial\bar h\right)\\
&+\Bigg\{\frac{\partial\bar\partial}{\partial_-}\bar h\left[\partial_-^2h\frac{1}{\partial_-^3}\left(\partial_-h\partial_-\bar M\right)+\frac{1}{3}\left(h\bar h\partial_-M-hh\partial_-\bar M\right)\right]-\frac{1}{\partial_-^2}\left(\partial_-h\partial_-\bar h\right)\left[-\frac{1}{2}\frac{\partial\bar\partial}{\partial_-^2}\left(\partial_-h\partial_-\bar M\right)\right.\\
&\left.+\frac{1}{\partial_-}\left(\partial\bar\partial h\partial_-\bar M\right)\right]-h\bar h\left[\bar\partial h\partial\bar M+\frac{4}{3}h\partial\bar\partial\bar M+4\partial_-h\frac{\partial\bar\partial}{\partial_-}\bar M\right]-2\frac{1}{\partial_-^2}\left(\partial_-h\partial_-\bar h\right)\left[\frac{1}{2}\bar\partial h\partial\bar M+h\partial\bar\partial\bar M\right.\\
&\left.-\partial_-h\frac{\partial\bar\partial}{\partial_-}\bar M+\frac{1}{\partial_-}\left(\partial\bar\partial h\partial_-\bar M\right)-\frac{1}{4}\frac{\partial\bar\partial}{\partial_-^2}\left(\partial_-h\partial_-\bar M\right)\right]\\
&\qquad+2h\partial\bar h\frac{1}{\partial_-}\left(2\bar\partial h\partial_-\bar M+h\partial_-\bar\partial\bar M-\partial_-\bar\partial h\bar M\right)+c.c\Bigg\}\\
&-\frac{1}{\partial_-}\left(2\bar\partial h\partial_-\bar h+h\partial_-\bar\partial\bar h-\partial_-\bar\partial h\bar h\right)\frac{1}{\partial_-}\left(2\partial_-h\partial\bar M+\partial_-\partial h\bar M-h\partial_-\partial\bar M\right)\ ,
\end{aligned}
\tag{3.10}
$$

with

$$
M=-\frac{1}{2}\frac{1}{\partial_-}\left\{2\,\partial_-^2h\frac{1}{\partial_-^3}(\partial_-h\partial_-\bar h)+\partial_-h\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar h)+\frac{1}{3}(h\bar h\partial_-h-hh\partial_-\bar h)\right\}\,, \tag{3.11}
$$

The last 18 lines of (3.10) appear as a consequence of the field redefinition applied to remove terms with light-cone time derivative ∂_+ at the quartic level. As done at order κ^2, interaction vertices containing a ∂_+ at this order, need to be eliminated using an appropriate field redefinition. This is achieved by using

$$
\begin{aligned}
h\to h+2\kappa^4\Bigg\{&\frac{1}{\partial_-}\Bigg[\frac{2}{3}h\partial_-\bar hM-\frac{5}{6}\partial_-h\bar hM-\frac{1}{2}h\bar h\partial_-M+\frac{1}{2}\partial_-\bar Mhh+\frac{1}{6}\bar Mh\partial_-h\\
&-\partial_-^2h\frac{1}{\partial_-^3}\left(\partial_-M\partial_-\bar h\right)-\partial_-^2M\frac{1}{\partial_-^3}\left(\partial_-h\partial_-\bar h\right)-\partial_-^2h\frac{1}{\partial_-^3}\left(\partial_-\bar M\partial_-h\right)-\partial_-h\left(\psi_4+\frac{\psi_2^2}{2}\right)\\
&+\frac{1}{3}h\partial_-[h\bar h]^2+h\psi_2\partial_-[h\bar h]-\frac{h^2}{3}\left(\frac{1}{3}\partial_-[h\bar h\bar h]+2\psi_2\partial_-\bar h\right)-\frac{2h\bar h}{3}\left(\frac{1}{3}\partial_-[h\bar hh]+2\psi_2\partial_-h\right)\Bigg]\\
&-2\partial_-h\frac{1}{\partial_-^3}(\partial_-M\partial_-\bar h)-2\partial_-h\frac{1}{\partial_-^3}(\partial_-h\partial_-\bar M)+\partial_-h\frac{1}{\partial_-}(4\psi_4+\psi_2^2)-\frac{1}{15}(h\bar h)^2h\\
&-\frac{1}{2}M\psi_2+\frac{1}{2}\partial_-h\frac{1}{\partial_-^2}(M\partial_-\bar h)+\frac{1}{2}\partial_-h\frac{1}{\partial_-^2}(\bar M\partial_-h)\Bigg\}\,.
\end{aligned}
\tag{3.12}
$$

Since the ∂_+ that occurs in (2.42) appears linearly, it can always be shifted to a higher order.

3.2 The helicity selection rules and structure of interaction vertices

We note and emphasize a key interesting finding from (3.10) : *all* six-point interaction vertices at order κ^4, in light-cone gravity, are of the form $h^3\,\bar{h}^3$. For example, there is no term of the form $h^2\,\bar{h}^4$. In the following, we provide a detailed derivation of helicity structure of interaction vertices for an arbitrary spin theory and then discuss the case of pure gravity. Specifically, we explain the findings reported in the previous section, solely using symmetries.

A massless field of integer spin λ has two physical degrees of freedom ϕ and $\bar{\phi}$ corresponding to the $+$ and $-$ helicity states respectively. In the light-cone gauge, Poincaré algebra is realised on these physical fields as explained in section (2.2). In particular, the algebra is non-linearly realized in presence of interaction terms. Therefore, we start with an ansatz for the Hamiltonian variation of the field, work through the Poincaré commutators to deduce the structure of the interaction vertices in the case of pure gravity [30].

We start with the $n-$point interaction vertex containing massless fields of spin λ. This would be at order κ^{n-2} and has the schematic form $\kappa^{n-2}\,\phi^c\,\bar{\phi}^d$ ($c, d \in \{0, 1, .., n\}$ and $c + d = n$). We begin with the ansatz

$$\delta_{\mathcal{H}}^{\kappa^{n-2}}\phi = \kappa^{n-2}\,\partial^{+\mu_0}\Big\{[\bar{\partial}^{\alpha_1}\,\partial^{\beta_1}\,\partial^{+\mu_1}\phi]\,...\,[\bar{\partial}^{\alpha_c}\,\partial^{\beta_c}\,\partial^{+\mu_c}\phi]\;\;[\bar{\partial}^{\alpha_{c+1}}\,\partial^{\beta_{c+1}}\,\partial^{+\mu_{c+1}}\bar{\phi}]\,...\,[\bar{\partial}^{\alpha_{n-1}}\,\partial^{\beta_{n-1}}\,\partial^{+\mu_{n-1}}\bar{\phi}]\Big\}$$

where α_i, β_i, μ_i are integers to be fixed by the algebra. The commutator $[\delta_J, \delta_H]$ yields

$$\sum_{i=1}^{n-1}(\alpha_i - \beta_i) + (d - c)\lambda = 0\,. \tag{3.13}$$

Another commutator from appendix C, $[\delta_{J^{+-}}, \delta_H]$ produces the condition

$$\sum_{i=0}^{n-1}\mu_i = -1\,. \tag{3.14}$$

Using (3.14) and dimensional analysis we find (noting that $[\kappa] = [L]^{\lambda-1}$)

$$\sum_{i=1}^{n-1}(\alpha_i + \beta_i) = 2 + (\lambda - 2)(c + d - 2) \, . \tag{3.15}$$

The non-negativity of the powers of the transverse derivatives in the Hamiltonian together with equations (3.13) and (3.15) imply

$$\sum_{i=1}^{n-1} \alpha_i = (\lambda - 1)c - d - \lambda + 3 \geq 0 \, , \tag{3.16}$$

$$\sum_{i=1}^{n-1} \beta_i = (\lambda - 1)d - c - \lambda + 3 \geq 0 \, . \tag{3.17}$$

Now consider the consequences of these inequalities in the simplest case: for spin-1 fields. $\lambda = 1$ implies $c\,, d \leq 2$ from which it follows that $n \leq 4$. Thus verifying the known result, that the Yang-Mills theory action terminates with four-point interaction vertices.

Now we consider the case of gravity with $\lambda = 2$. Expressions (3.16) and (3.17) imply that

$$\sum_{i=1}^{n-1} \alpha_i = c - d + 1 \geq 0 \, , \tag{3.18}$$

$$\sum_{i=1}^{n-1} \beta_i = d - c + 1 \geq 0 \, . \tag{3.19}$$

For a MHV vertex, $d = 2$. From (3.18) and (3.19), we see that $1 \leq c \leq 3$. As $c + d = n$, it follows that $3 \leq n \leq 5$ for a MHV vertex. Exactly as demonstrated in section 2, MHV vertices arise from the cubic, quartic and quintic interaction vertices. Also, as verified explicitly by our calculations, the first entirely non-MHV vertices appear in the 6-point result. Using $c + d = n$ in (3.18) and (3.19) yields

$$\frac{n-1}{2} \leq c\,, d \leq \frac{n+1}{2} \, . \tag{3.20}$$

This relation allows us to extract information about the general form of even- and odd-point vertices. Consider an even point vertex, where $n = 2m$ for some positive integer m. We find

$$m - \frac{1}{2} \leq c\,,d \leq m + \frac{1}{2}\,, \tag{3.21}$$

which implies $c = d = m$. Thus, the only allowed structure at order $2m$ is $h^m\ \bar{h}^m$. Now consider an odd-point vertex, where $n = 2m + 1$ for some positive integer m. We find

$$m \leq c\,,d \leq m+1\,. \tag{3.22}$$

If $c = m$, then $d = m+1$ and consequently, odd-point vertices will have the structure $h^m\ \bar{h}^{m+1}$. The other case ($c = m + 1$) produces the structure $h^{m+1}\ \bar{h}^m$. These are the only two allowed possibilities at odd order. Further, due to the reality of the Lagrangian, both structures appear together (as complex conjugates of one another).

This clearly demonstrates the fact that MHV terms do not occur beyond the quintic interaction vertex in light cone gravity. This brings us naturally to the question of how MHV vertices can occur beyond quintic interaction vertices. In the following section, we demonstrate that this happens through the idea of MHV Lagrangians [39, 40, 43].

3.3 MHV Lagrangians

In this section, we first motivate the need for a field redefinition to remove the cubic anti-MHV vertices in both Yang-Mills theories and pure gravity. The resulting Lagrangian after applying the field redefinition produces MHV interaction vertices. In the light-cone gauge, the Lagrangian for Yang-Mills theory (2.30) has the following helicity structure

$$L_{\mathrm{YM}} = L_{+-}\ +\ L_{++-}\ +\ L_{+--}\ +\ L_{++--}\,. \tag{3.23}$$

Feynman diagrams primarily constructed using the first cubic vertex vanish since these correspond to amplitudes involving all 'plus' or all but one 'plus' helicities [32, 44, 45]. It therefore makes sense to look for a canonical field redefinition that will map the kinetic term and the L_{++-} (non-MHV) cubic vertex to a purely kinetic term, ie.

$$L_{+-}\ +\ L_{++-}\ \rightarrow\ L'_{+-}\,. \tag{3.24}$$

This is achieved through field redefinitions of the form [39, 40]

$$
\begin{aligned}
A &\sim u_1\, B + u_2\, BB + u_3\, BBB + ... \,, \\
\bar{A} &\sim v_1\, \widetilde{B} + v_2\, \widetilde{B}B + v_3\, \widetilde{B}BB + ... \,,
\end{aligned}
\tag{3.25}
$$

where $A, \bar{A}$ are $+$ and $-$ helicity Yang-Mills fields and $B, \widetilde{B}$ are the new fields. The relations (3.25) are schematic. The details may be found references [39, 40]. The coefficients u_n, v_n are functions of momenta and can be worked out, order by order, based on the requirement (3.24). In the process of eliminating the cubic anti-MHV vertex from (3.23), the terms L_{+--} and L_{++--} generate an infinite series of MHV vertices. The resulting 'MHV Lagrangian' has the following structure

$$
L_{YM} = L'_{+-} + L'_{+--} + L'_{++--} + L'_{+++--} + L'_{++++--} + \ldots \quad . \tag{3.26}
$$

The key advantage being that all MHV tree-amplitudes are trivial to read off (by simply taking the relevant momenta on-shell).

Gravity in light-cone gauge is similar in helicity structure to Yang-Mills theory and therefore the same idea may be applied there, eliminating the anti-MHV cubic vertex from the Lagrangian in (2.46). The canonical field redefinitions for $h, \bar{h}$ in momentum space take the form [37]

$$
\begin{aligned}
h &\sim y_1\, C + y_2\, CC + y_3\, CCC + ... \,, \\
\bar{h} &\sim z_1\, \widetilde{C} + z_2\, \widetilde{C}C + z_3\, \widetilde{C}CC + ... \,,
\end{aligned}
\tag{3.27}
$$

where $C, \widetilde{C}$ are the new fields and the coefficients y_n, z_n are functions of momenta.

Once these field redefinitions are applied to the gravity Lagrangian, exactly like with Yang-Mills theory, new structures appear. In particular, the redefinitions in (3.27) produce MHV vertices in the theory. A concrete example, relevant to this discussion is the $h \to C^3$ field redefinition applied to the quintic vertex $h^3\,\bar{h}^2$. this produces manifestly MHV vertices, $C^4\,\bar{C}^2$ and its complex conjugate, at the six-point interaction level. The MHV canonical field redefinition is therefore the origin of MHV vertices in the theory at higher orders.

Having explained various perturbative aspects of pure gravity, we now generalize the construction of interaction vertices to higher derivative theories. These play an important role in effective field theories and analysis of counter-terms. In [46], one-loop counter terms were studied in the light-cone gauge. It was shown that terms of type R^2 essentially produce vertices proportional to the free equations of motion and may be removed by a suitable field redefinition (thus all n-point graviton amplitudes produced by the R^2 term vanish as expected [47]). We therefore study the simplest non-trivial case of a higher derivative theory of gravity corresponding to addition of terms of the type R^3 in the next section.

3.4 Higher derivative theories

The usual approach of gauge-fixing the covariant action turns out to be extremely tedious for the case of higher derivative theories. We instead build the interaction terms in the Hamiltonian by requiring closure of Poincaré algebra. Specifically, we construct cubic interaction vertices for three arbitrary integer spin fields with a coupling constant $[\alpha] = L^{\lambda_1+\lambda_2+\lambda_3}$. We then specialize the discussion to the case when $\lambda_1 = \lambda_2 = \lambda_3 = 2$.

The Hamiltonian for the free field theory is

$$H = -\int d^3x\, \bar{\phi}_i\, \partial\bar{\partial}\, \phi_i \;=\; \int d^3x\, \partial^+\bar{\phi}_i\, \delta_{P^-}\phi_i\;, \tag{3.28}$$

with ϕ_i referring to a field of helicity λ_i with $i \in \mathbb{Z}^+$. As discussed in section (2.2), the δ_{P^-} operator picks up non-linear corrections, order by order, in the coupling constant α, when interactions are switched on.

In reference [30] cubic interaction vertices between fields, all having helicity λ were constructed with a coupling constant α having dimensions of $L^{\lambda-1}$.

In this discussion, we consider the two most general - ansatze for cubic interactions (based on dimensional analysis and helicity counting)

$$\text{Type-1}: \qquad \delta^{\prime\alpha}_{P^-}\,\phi_1 = \alpha\, A\, \partial^{+\,\mu}\left[\bar{\partial}^a\partial^c\partial^{+\,\rho}\bar{\phi}_2\;\bar{\partial}^b\partial^d\partial^{+\,\sigma}\bar{\phi}_3\right]\;, \tag{3.29}$$

$$\text{Type-2}: \qquad \delta^{\prime\prime\alpha}_{P^-}\,\phi_1 = \alpha\, C\, \partial^{+\,\mu}\left[\bar{\partial}^a\partial^c\partial^{+\,\rho}\phi_2\;\bar{\partial}^b\partial^d\partial^{+\,\sigma}\phi_3\right]\;, \tag{3.30}$$

where μ, ρ, σ, a, b are integers and A and C are numerical factors.

The key departure from [30] for both types of ansatze being that the dimension of the coupling constant is $[\alpha] = L^{\lambda_2+\lambda_3+\lambda_1-1}$. It can be shown that the type-2 vertices at the cubic order, are proportional to the free equations of motion and can be removed using appropriate field-redefinitions. We therefore focus on the derivation of type-1 vertices.

We first start with $\delta'^{\alpha}_{P^-}\phi_1$ and use the commutation relations and dimensional analysis to find the unknown parameters. The commutators

$$
\begin{aligned}
&[\,\delta_J\,,\delta'^{\alpha}_{P^-}\,]\phi_1 = 0\,, \\
&[\,\delta_{J^{+-}}\,,\delta'^{\alpha}_{P^-}\,]\phi_1 = -\delta'^{\alpha}_{P^-}\phi_1\,,
\end{aligned}
\tag{3.31}
$$

impose the following conditions on our ansatz

$$
\begin{aligned}
&(c+d)-(a+b) = \lambda_2+\lambda_3+\lambda_1\,, \\
&\mu+\rho+\sigma = -1\,.
\end{aligned}
\tag{3.32}
$$

Let $\lambda = \lambda_2+\lambda_3+\lambda_1$ so the first equation of (3.32) reads $(c+d)-(a+b) = \lambda$. The dimensional analysis of (3.29) gives us the following relation

$$
(a+b)+(c+d) = \lambda\,. \tag{3.33}
$$

The addition of first equation in (3.32) and (3.33) results in

$$
c+d = \lambda\,, \tag{3.34}
$$

$$
a+b = 0\,. \tag{3.35}
$$

As $a, b, c, d > 0$, this implies that $a = b = 0$. Therefore, mixed derivative terms are not allowed for type-1 vertices (3.29). As $c + d = \lambda$, there are $\lambda + 1$ possible values for a pair (c, d). We rewrite the ansatz (3.29) as a sum of these $\lambda + 1$ terms

$$
\delta'^{\alpha}_{P^-}\phi_1 = \alpha\sum_{n=0}^{\lambda} A_n\partial^{+\,\mu_n}\left[\bar{\partial}^{(\lambda-n)}\partial^{+\,\rho_n}\bar{\phi}_2\,\partial^n\partial^{+\,\sigma_n}\bar{\phi}_3\right]\,. \tag{3.36}
$$

The next commutator $[\,\delta_{\bar{J}^+}\,,\delta'^{\alpha}_{P^-}\,]\phi_1 = 0$ gives the following condition

$$\sum_{n=0}^{\lambda} A_n \Big\{ (\lambda-n)\,\partial^{+\,\mu_n}\,\left[\partial^{(\lambda-n-1)}\partial^{+\,(\rho_n+1)}\bar{\phi}_2\,\partial^n\partial^{+\,\sigma_n}\bar{\phi}_3\right]$$
$$+\,n\,\partial^{+\,\mu_n}\,\left[\partial^{(\lambda-n)}\partial^{+\,\rho_n}\bar{\phi}_2\,\partial^{(n-1)}\partial^{+\,(\sigma_n+1)}\bar{\phi}_3\right]\Big\} = 0\,. \tag{3.37}$$

The above condition is satisfied if the coefficients obey the following recursion relations

$$A_{n+1} = -\frac{(\lambda-n)}{(n+1)}A_n = (-1)^{n+1}\binom{\lambda}{n+1}A_0\,, \tag{3.38}$$
$$\rho_{n+1} = \rho_n + 1\,,\quad \sigma_{n+1} = \sigma_n - 1\,,\quad \mu_{n+1} = \mu_n\,. \tag{3.39}$$

To determine the exact values of ρ, μ, and σ, we need the dynamical commutators $[\,\delta_{J^-}\,,\delta'_{P^-}\,]^{\alpha}\,\phi_1 = 0$ and $[\,\delta_{\bar{J}^-}\,,\delta'_{P^-}\,]^{\alpha}\,\phi_1 = 0$. The boost generators J^- and $\bar{J}^-$ also get corrected when interactions are turned on and are of the form

$$\delta_{J^-}\phi = -x\,\delta'^{\alpha}_{P^-}\phi + \delta^{\alpha}_{s}\phi\,, \qquad\qquad \delta_{\bar{J}^-}\phi = -\bar{x}\,\delta'^{\alpha}_{P^-}\phi + \delta^{\alpha}_{\bar{s}}\phi\,. \tag{3.40}$$

The boost generators are determined if we know the spin parts $\delta^{\alpha}_{S}\phi\,,\,\delta^{\alpha}_{\bar{S}}\phi$. For type-1 vertices they are structurally of the form

$$\delta^{\alpha}_{\bar{S}}\phi \sim \partial^{\lambda-1}\bar{\phi}\,\bar{\phi}\,, \qquad\qquad \delta^{\alpha}_{S}\bar{\phi} \sim \bar{\partial}^{\lambda-1}\phi\phi\,. \tag{3.41}$$

Due to helicity, the transformations $\delta^{\alpha}_{S}\phi$ and $\delta^{\alpha}_{\bar{S}}\bar{\phi}$ do not exist. We now compute $[\,\delta_{J^-}\,,\delta'_{P^-}\,]^{\alpha}\,\phi_1 = 0$ to obtain

$$\sum_{n=0}^{\lambda} A_n \Big\{ (\mu_n+\lambda_1+1)\,\frac{\partial}{\partial^+}\,\partial^{+\,\mu_n}\,\left[\partial^{(\lambda-n)}\partial^{+\,\rho_n}\bar{\phi}_2\,\partial^n\partial^{+\,\sigma_n}\bar{\phi}_3\right]$$
$$+\,(\rho_n+\lambda_2)\,\partial^{+\,\mu_n}\,\left[\partial^{(\lambda-n+1)}\partial^{+\,(\rho_n-1)}\bar{\phi}_2\,\partial^n\partial^{+\,\sigma_n}\bar{\phi}_3\right]$$
$$+\,(\sigma_n+\lambda_3)\,\partial^{+\,\mu_n}\,\left[\partial^{(\lambda-n)}\partial^{+\,\rho_n}\bar{\phi}_2\,\partial^{(n+1)}\partial^{+\,(\sigma_n-1)}\bar{\phi}_3\right]\Big\} = 0\,. \tag{3.42}$$

The solution of the above recursion relation for ρ, σ and μ subject to the boundary conditions $\sigma_{n=\lambda} = -\lambda_3\,,\ \rho_{n=0} = -\lambda_2$ is

$$\rho_n = n - \lambda_2\,,\qquad \sigma_n = \lambda - \lambda_3 - n\,,\quad \mu_n = -1-\lambda_1\,. \tag{3.43}$$

Plugging the values of ρ, σ and μ in our ansatz, we find

$$\delta'^{\alpha}_{p^-}\phi_1 = \sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\,\partial^{+\,(-1-\lambda_1)}\,\left[\bar\partial^{(\lambda-n)}\,\partial^{+\,(n-\lambda_2)}\,\bar\phi_2\,\partial^n\,\partial^{+\,(\lambda-\lambda_3-n)}\,\bar\phi_3\right]\,. \tag{3.44}$$

Since

$$H = \int d^3x\;\partial^+\bar\phi_1\,\delta'^{\alpha}_{p^-}\phi_1\,, \tag{3.45}$$

the interaction Hamiltonian is

$$H^{\alpha} = \alpha\int d^3x\;\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\,\frac{1}{\partial^{+\lambda_1}}\bar\phi_1\,\left[\bar\partial^{(\lambda-n)}\,\partial^{+\,(n-\lambda_2)}\,\bar\phi_2\,\partial^n\,\partial^{+\,(\lambda-\lambda_3-n)}\,\bar\phi_3\right] + c.c.\,. \tag{3.46}$$

As is well known, for odd λ, non-trivial cubic vertices require the introduction of an antisymmetric structure constant f^{abc}. For obtaining interaction vertices corresponding to terms of type R^3, we simply substitute $\lambda_i = 2$, $\lambda = 6$ and $\phi_i = h, \bar\phi_i = \bar h$ in the above expression

$$H_G = \alpha\int d^3x\;\sum_{n=0}^{6}(-1)^n\binom{\lambda}{n}\,\frac{1}{\partial^{+\,2}}\bar h\,\left[\bar\partial^{(6-n)}\,\partial^{+\,(n-2)}\,\bar h\,\partial^n\,\partial^{+\,(4-n)}\,\bar h\right] + c.c.\,. \tag{3.47}$$

Amplitude structures

In momentum space, the cubic vertices (3.46) (without the complex conjugate part) have the following structure (with measure and constants suppressed)

$$\begin{aligned}
H \;&\sim\; \sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\,\frac{1}{p^{+\lambda_1}}\,\left[\bar k^{(\lambda-n)}\,k^{+\,(n-\lambda_2)}\,l^n\,l^{+\,(\lambda-\lambda_3-n)}\right]\bar\phi_1(p)\,\bar\phi_2(k)\,\bar\phi_3(l)\,,\\
&=\; \frac{k^{+\,(\lambda_1+\lambda_3)}\,l^{+\,(\lambda_1+\lambda_2)}}{p^{+\lambda_1}}\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\,\left[\left(\frac{\bar k}{k^+}\right)^{(\lambda-n)}\left(\frac{\bar l}{l^+}\right)^n\right]\bar\phi_1(p)\,\bar\phi_2(k)\,\bar\phi_3(l)\,,\\
&=\; \frac{(\bar k l^+ - \bar l k^+)^{\lambda}}{p^{+\lambda_1}\,k^{+\lambda_2}\,l^{+\lambda_3}}\;\bar\phi_1(p)\,\bar\phi_2(k)\,\bar\phi_3(l)\,.
\end{aligned} \tag{3.48}$$

In terms of spinor helicity variables [31], the vertex reads

$$V^{\alpha}(p,k,l) = \frac{1}{\sqrt{2^{\lambda}}} \langle pk\rangle^{\lambda_1+\lambda_2-\lambda_3} \langle kl\rangle^{\lambda_2+\lambda_3-\lambda_1} \langle lp\rangle^{\lambda_3+\lambda_1-\lambda_2} + c.c.\,. \tag{3.49}$$

Consider $\lambda_i = 1$ which correspond to the vertices from terms of type F^3. We get

$$V_{YM}(p,k,l) = \frac{1}{\sqrt{2^3}} \langle pk\rangle \langle kl\rangle \langle lp\rangle + c.c.\,. \tag{3.50}$$

With $\lambda_i = 2$, the vertices (3.49) read

$$V_{G}(p,k,l) = \frac{1}{\sqrt{2^6}} \langle pk\rangle^2 \langle kl\rangle^2 \langle lp\rangle^2 + c.c.\,, \tag{3.51}$$

confirming the KLT relations in higher derivative theories. This is consistent with the general result for three-point amplitudes derived in [44, 49] using S-matrix arguments, and little group scaling and in [50, 51] using a Fock-space approach. We explain in appendix A how a class of higher-point tree level amplitudes can be constructed starting from the seed amplitudes (3.50) and (3.51).

Comments on n-point interaction vertices in higher derivative theories

In this section, we first deduce the structure of interaction vertices at higher orders purely from dimensional and kinematical constraints and then prove that only those n-point vertices can be uniquely fixed by the Poincaré algebra which contain purely one type of transverse derivatives .

We work here with a special class of higher derivative theories where $\lambda_i = \lambda$, and work out the structure of interaction vertices at higher orders purely from symmetry constraints. In a perturbative expansion, the dimension of the coupling for a n-point interaction vertex is

$$[\alpha^{n-2}] = 3\lambda - 1 + (n-3)(\lambda-1)\,. \tag{3.52}$$

where α is the 3-point coupling. The n-point Hamiltonian is of the form $\phi^p\,\bar{\phi}^q$. We start with the ansatz

$$\delta_{P^-}^{\alpha^{n-2}} \phi \;=\; \alpha^{n-2}\,\partial^{+\mu_0}\Big\{[\bar\partial^{a_1}\,\partial^{c_1}\,\partial^{+\mu_1}\phi]\;[\bar\partial^{a_2}\,\partial^{c_2}\,\partial^{+\mu_2}\phi]\,........\,[\bar\partial^{a_p}\,\partial^{c_p}\,\partial^{+\mu_p}\phi]$$
$$[\bar\partial^{a_{p+1}}\,\partial^{c_{p+1}}\,\partial^{+\mu_{p+1}}\bar\phi]\,.......\,[\bar\partial^{a_{n-1}}\,\partial^{c_{n-1}}\,\partial^{+\mu_{n-1}}\bar\phi]\Big\}\,,$$

where $p+q=n$ and the a_i, c_i are non-negative integers and the μ_i are integers. The commutator $[\delta_J, \delta_{P^-}]$ yields

$$\sum_{i=1}^{n-1}(a_i - c_i) = (p-q)\lambda\,. \tag{3.53}$$

Using the commutator $[\delta_{J^{+-}}, \delta_{P^-}]$ gives $\sum_{i=0}^{n-1}\mu_i = -1$. Dimensional analysis gives the following constraint

$$\sum_{i=1}^{n-1}(a_i + c_i) = 6 + (\lambda-2)(p+q)\,. \tag{3.54}$$

Using (3.53), (3.54) and the non-negativity of powers of transverse derivatives we obtain

$$\sum_{i=1}^{n-1} a_i = (\lambda-1)p - q + 3 \geq 0\,, \tag{3.55}$$
$$\sum_{i=1}^{n-1} c_i = (\lambda-1)q - p + 3 \geq 0\,. \tag{3.56}$$

For $\lambda = 1$, we get $p, q \leq 3$. At cubic order, note that the $(p=3, q=0)$ $A\,A\,A$ structure and $(p=0, q=3)$ $\bar A\,\bar A\,\bar A$ structure follow from this. At the next order, two new structures $A^3\,\bar A$ and $\bar A^3\,A$ are allowed as compared to the usual Yang-Mills quartic vertices.

We then consider $\lambda = 2$, and use $q = n-p$ in (3.55) and (3.56)

$$\frac{n-3}{2} \;\leq\; p \;\leq\; \frac{n+3}{2}\,. \tag{3.57}$$

The interaction vertex may be odd or even. For an odd point vertex, $n = 2m+1$ where m is a positive integer. This gives

$$m-1 \,\leq\, p \,\leq\, m+2\,. \tag{3.58}$$

Thus, for a cubic vertex where $m=1$, (3.58) allows helicity structures $h\,h\,h$ and $\bar{h}\,\bar{h}\,\bar{h}$.

For an even point vertex $n=2m$ we get

$$m-\frac{3}{2} \,\leq\, p \,\leq\, m+\frac{3}{2}\,. \tag{3.59}$$

Since p is an integer, the condition (3.59) is

$$m-1 \,\leq\, p \,\leq\, m+1\,. \tag{3.60}$$

For a quartic vertex where $m=2$, (3.60) allows terms of helicity structure $h\,\bar{h}^3$ and $\bar{h}\,h^3$. Thus, higher derivative operators produce new helicity configurations at each order. For $n=6$, in addition to terms of type $h^3\,\bar{h}^3$, vertices of type $h^2\,\bar{h}^4$ and $\bar{h}^2\,h^4$ appear.

We now prove that: *all n- point vertices containing purely one type of transverse derivatives can be uniquely fixed by the Poincaré algebra.*

We start with an ansatz for the n-point interaction vertex (3.53) and plug $a_i=0$. It reads

$$\begin{aligned}\delta^{\alpha^{n-2}}_{P^-}\phi \;\;=\;\; \alpha^{n-2}\,\partial^{+\mu_0}\Big\{&[\,\partial^{c_1}\,\partial^{+\mu_1}\phi]\;\;[\,\partial^{c_2}\,\partial^{+\mu_2}\phi]\,........\,[\,\partial^{c_p}\,\partial^{+\mu_p}\phi] \\ &[\,\partial^{c_{p+1}}\,\partial^{+\mu_{p+1}}\bar{\phi}]\,.......\,[\,\partial^{c_{n-1}}\,\partial^{+\mu_{n-1}}\bar{\phi}]\Big\}\,,\end{aligned}$$

Consistency with the helicity generator j requires

$$\sum_{i=1}^{n-1} c_i=(q-p)\lambda \tag{3.61}$$

The commutator with the generators J^{+-} , $\bar{J}^{+}$ determines the vertex upto the powers of ∂^+s. The exact powers of ∂^+s are fixed by the dynamical generators. We argue that the spin transformation appearing in the dynamical generator J^- at this order is trivial and hence can be used to uniquely determine the vertex.

The number of transverse derivatives in the spin transformation $\delta_S^{\alpha^{n-2}}\phi \;\sim\; \alpha^{n-2}\,\phi_1......\bar{\phi}_{n-1}$ (derivatives suppressed) must be one less than that in $\delta_{P^-}^{\alpha^{n-2}}\phi$ due to its dimensionality. The dynamical generator J^- has helicity $+1$. In order for the spin transformation $\delta_S^{\alpha^{n-2}}\phi$ to have helicity $+1$, the number of transverse derivatives in it must be one greater than that in $\delta_{P^-}^{\alpha^{n-2}}\phi$. This proves that the spin transformation $\delta_S^{\alpha^{n-2}}\phi$ cannot be consistent with the helicity and dimensionality simultaneously and hence must vanish. This allows us to uniquely determine the vertex. Therefore, all n- point vertices containing purely one type of transverse derivatives can be uniquely fixed by the Poincaré algebra. In the case of vertices containing both kinds of transverse derivatives, the spin transformations appearing in both J^- and $\bar{J}^-$ are non-trivial. Therefore, these cannot be fixed uniquely by Poincaré algebra.

We emphasise here that, in our approach, one only needs to feed the dimension of the coupling constant and the helicity of the interacting fields to construct the vertices by closure of Poincaré algebra. However, this approach when applied to quartic and higher-point vertices is met with many limitations. In the subsequent sections, we study maximally supersymmetric theories of gravity which display large symmetries in the form of exceptional groups in four and five-dimensions. We explain how these exceptional symmetries are useful in constraining the perturbation series of these theories.

Chapter 4

Aspects of supergravity in light-cone gauge

In this chapter, we begin by introducing the space-time symmetries that include fermionic generators, forming the superPoincaré algebras. We discuss and review various aspects of the maximally supersymmetric theory of gravity in four-dimensions. Specifically, we explain the formulation of this theory and its symmetries in light-cone superspace.

4.1 Graded Lie algebras

The Coleman-Mandula theorem provides a powerful constraint on the possible continuous symmetries of the S–matrix in an interacting relativistic quantum field theory. Under certain physical assumptions, the theorem shows that all symmetries generated by ordinary bosonic operators must factorize into the direct product of the Poincaré group and an internal symmetry group generated by Lorentz scalars [52]. Any attempt to introduce additional bosonic symmetry generators carrying non-trivial Lorentz quantum numbers leads to excessively strong constraints on scattering amplitudes and forces the S–matrix to become trivial.

Supersymmetry evades this restriction by extending the concept of symmetry beyond ordinary Lie algebras to graded algebras containing fermionic generators. These graded Lie algebras contain both commutation and anti-commutation relations based on the nature of the generators. In this way, supersymmetry provides a consistent and nontrivial extension of space-time symmetries compatible with an interacting S-matrix.

We introduce the matrices $\sigma^{\mu}_{\alpha\dot{\alpha}}$ and $\bar{\sigma}^{\mu\,\dot{\alpha}\alpha}$, which provide the map between vector and spinor indices in four-dimensional Minkowski space. The spinor indices are raised and lowered using the anti-symmetric Levi-Civita symbols $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$. They are defined by

$$\sigma^{\mu}_{\alpha\dot{\alpha}} = (\mathbf{1},\, \vec{\sigma}), \qquad \bar{\sigma}^{\mu\,\dot{\alpha}\alpha} = (\mathbf{1},\, -\vec{\sigma}), \tag{4.1}$$

where $\vec{\sigma}$ are the Pauli matrices. Using these, one defines the antisymmetric matrices $\sigma_{\mu\nu}$, which generate Lorentz transformations in the $(\frac{1}{2}, 0)$ spinor representation, as

$$(\sigma_{\mu\nu})_\alpha{}^\beta = \frac{1}{4}\left(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu\right)_\alpha{}^\beta. \tag{4.2}$$

Similarly, the matrices $\bar{\sigma}_{\mu\nu}$ generate Lorentz transformations in the $(0, \frac{1}{2})$ representation and are given by

$$(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}\left(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu\right)^{\dot{\alpha}}{}_{\dot{\beta}}. \tag{4.3}$$

We introduce the generators Q_α and $\bar{Q}_{\dot{\alpha}}$ which transform in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lorentz group

$$[\, J_{\mu\nu} \,, Q_\alpha \,] \;=\; i\, (\sigma_{\mu\nu})_\alpha{}^\beta \, Q_\beta, \tag{4.4}$$

$$[\, J_{\mu\nu} \,, \bar{Q}^{\dot{\alpha}} \,] \;=\; i\, (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \, \bar{Q}^{\dot{\beta}}. \tag{4.5}$$

These anti-commute to produce the translation generators P_μ

$$\{\, Q_\alpha \,,\, \bar{Q}_{\dot{\alpha}} \,\} \;=\; 2\, \sigma^\mu_{\alpha\dot{\alpha}} \, P_\mu \,. \tag{4.6}$$

The supercharges Q_α and $\bar{Q}_{\dot{\alpha}}$ commute with the translations P_μ

$$\begin{aligned} [\, Q_\alpha \,,\, P_\mu \,] &= \; 0 \,, \\ [\, \bar{Q}_{\dot{\alpha}} \,,\, P_\mu \,] &= \; 0 \,. \end{aligned} \tag{4.7}$$

The above (anti-) commutation relations together with the Poincaré algebra form the super-Poincaré algebra. It is straightforward to write this algebra in the case of extended super-symmetry. In such a case, the supercharges have an extra index and are denoted as Q^m_α with $m = 1, 2, ..\mathcal{N}$. These contain an additional symmetry corresponding to the rotation of super-charges termed as R-symmetry. Schematically we have

$$[R, Q_\alpha] = i\, Q_\alpha, \qquad [R, \bar{Q}_{\dot{\alpha}}] = -i\, \bar{Q}_{\dot{\alpha}}. \tag{4.8}$$

$\mathcal{N} = 4$ superYang-Mills theory

In a supersymmetric theory, the number of bosonic states is equal to the fermionic states and they are related by the action of the supercharges. Since a supercharge squares to zero, one can then count the total number of bosonic and fermionic degrees of freedom. Once, the highest spin is fixed, the maximal possible number of supersymmetry charges also get fixed. In the case of spin-1, this corresponds to the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with 16 real supercharges. These 16 supercharges transform under the $SU(4)$ R-symmetry of the theory. The supermultiplet is summarized below :

Helicity	$+1$	$+\frac{1}{2}$	0	$-\frac{1}{2}$	-1
Field content	A	χ^m	ϕ^{mn}	$\bar{\chi}_m$	$\bar{A}$
Multiplicity	1	4	6	4	1
$SU(4)_R$ rep.	$\mathbf{1}$	$\mathbf{4}$	$\mathbf{6}$	$\bar{\mathbf{4}}$	$\mathbf{1}$

Table 4.1: On-shell supermultiplet of $\mathcal{N} = 4$ super Yang–Mills theory, organized by helicity.

This is an interesting theory and serves as a theoretical laboratory for the study of symmetries and scattering amplitudes. The first *all*-orders proof of ultraviolet finiteness was provided using light-cone superspace formulation of this theory. Given the interesting relations shared by Yang-Mills theory and gravity, one would like to understand what this formulation may teach us about the ultraviolet properties of maximally supersymmetric theory of gravity in four-dimensions. Below, we first give a brief introduction of this theory and then discuss in detail, its formulation in light-cone superspace, particularly the action and its symmetries.

4.2 Maximal supergravity

The $\mathcal{N} = 1$ supergravity in eleven-dimensions produces maximally supersymmetric theories of gravity in lower dimensions [11–13]. This theory is also the low energy limit of M-theory [55, 56]. The degrees of freedom in this theory are governed by the little group $SO(9)$. They include the 128 bosonic degrees of freedom spanned by the metric G_{MN} which transforms as the **44** (the symmetric traceless tensor) and the three-form field A_{MNP} which transforms as the **84** of $SO(9)$. The 128 fermionic degrees of freedom are captured by the Rarita-Schwinger field ψ^M_α where the index M is the $SO(9)$ index and α is the spinor-index.

A striking and unique feature of maximal supergravity theories is the emergence of enhanced global symmetries in the form of exceptional groups in three, four and five-dimensions. While a part of these symmetries can be traced to the original eleven-dimensional theory, much of it is relegated to the bucket of hidden symmetries. We summarize the global symmetries in supergravity in the chart below:

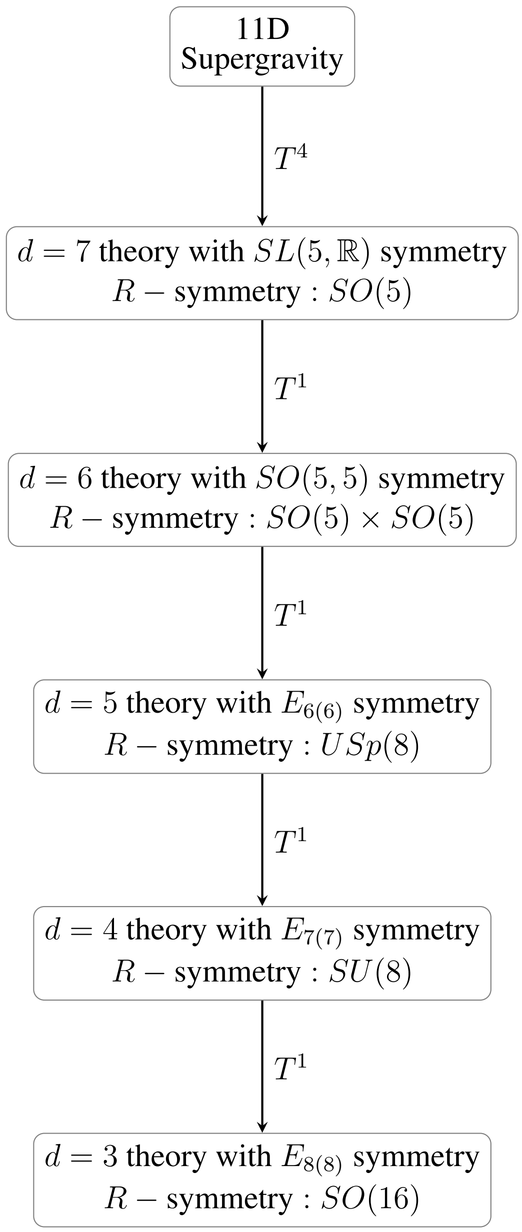

Figure 4.1: Chart of Global symmetries

4.3 Exceptional groups

The finite-dimensional simple Lie algebras are completely classified by their Dynkin diagrams into four infinite classical families, A_n, B_n, C_n, and D_n, together with exactly five exceptional algebras, namely G_2, F_4, E_6, E_7, and E_8. While the classical series exist for arbitrary rank n, the exceptional algebras occur only at fixed ranks. In this section, we briefly introduce some aspects of the exceptional E-type algebras, which play an important role in maximal supergravity. The model in $(11-n)$ dimensions exhibits a global $E_{n(n)}$ symmetry. The associated Dynkin diagrams starting from E_8 on the left are shown below (for $n \leq 5$, the exceptional groups degenerate into the classical ones)

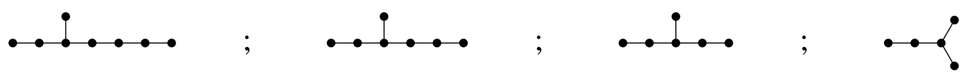

Figure 4.2: Dynkin diagrams of E_8,E_7, E_6 and D_5

$E_{6(6)}$ algebra

The exceptional group E_6 is a simple Lie group of rank six with 78 generators. In the context of maximal supergravity in five-dimensions, the non-compact version of E_6 denoted as $E_{6(6)}$ appears as a symmetry. This group has 36 compact generators corresponding to the unitary symplectic group $USp(8)$ and 42 non-compact generators [57]. The difference between the non-compact and compact generators is indicated in the bracket. The $USp(8)$ group is a set of unitary matrices that also preserve an anti-symmetric symplectic form which we denote as C_{mn}. The symplectic invariant satisfies $C^{mn}\, C_{np} = -\delta^m_p$.

The fundamental representation of E_6 is $\mathbf{27}$ dimensional spanned by z^{mn} satisfying $z^{mn} = -z^{nm} = (z_{mn})^*$ and $C_{mn}\, z^{mn} = 0$. Under an infinitesimal E_6 transformation we get

$$\delta z_{mn} \;=\; \Lambda^p{}_{[m}\, z_{n]p} \,+\, \Sigma_{mnpq}\, z^{pq} \tag{4.9}$$

where $\Lambda^p{}_m$ correspond to 36 $USp(8)$ parameters and Σ_{mnpq} correspond to the 42 remaining parameters. The Σ_{mnpq} is a completely anti-symmetric traceless tensor satisfying $\Sigma_{mnpq} = (\Sigma^{mnpq})^*$. Corresponding relations are true for the $\overline{\mathbf{27}}$ representation.

$E_{7(7)}$ algebra

The exceptional group E_7 is a simple Lie group of rank seven with 133 generators. In the context of maximal supergravity in four dimensions, the relevant non-compact form is $E_{7(7)}$, whose maximal compact subgroup is $SU(8)$ [58]. Accordingly, the algebra of $E_{7(7)}$ decomposes into 63 compact generators corresponding to $SU(8)$ and 70 non-compact generators. The difference between compact and non-compact generators is again indicated by the signature in the brackets.

The fundamental representation of E_7 is **56**-dimensional and can be described in terms of anti-symmetric tensors x^{AB} and $\bar{x}_{AB}$, where $A, B = 1, \ldots, 8$ are $SU(8)$ indices. Under an infinitesimal E_7 transformation, we find

$$\delta x_{AB} = \Lambda_A{}^C\, x_{CB} + \Lambda_B{}^C\, x_{AC} + \Sigma_{ABCD}\, \bar{x}^{CD}, \tag{4.10}$$

$$\delta \bar{x}^{AB} = \bar{\Lambda}^A{}_C\, \bar{x}^{CB} + \bar{\Lambda}^B{}_C\, \bar{x}^{AC} + \overline{\Sigma}^{ABCD}\, x_{CD}. \tag{4.11}$$

Here $\Lambda_A{}^B$ are the 63 parameters of $SU(8)$, while Σ_{ABCD} is totally antisymmetric tensor satisfying

$$\Sigma_{ABCD} = \frac{1}{2}\, \varepsilon_{ABCDEFGH}\, \overline{\Sigma}^{EFGH},$$

capturing the remaining 70 parameters

Other exceptional groups

E_8 is the largest exceptional group having a rank 8 with its Lie algebra containing 248 generators. In the context of maximal supergravity in three-dimensions, the relevant group is $E_{8(8)}$. The algebra comprises of 120 compact generators corresponding to its maximal compact subalgebra $SO(16)$ and 128 non-compact generators which transform in the spinor-representation of $SO(16)$. We shall describe the Lie algebra of $E_{8(8)}$ and discuss some features in chapter 7.
Other exceptional groups include G_2 and F_4. Although, very interesting, these are outside the scope of this thesis.

4.4 $(\mathcal{N}=8, d=4)$ supergravity in light-cone superspace

When the eleven-dimensional supergravity is reduced to four-dimensions, it produces the $\mathcal{N}=8$ supergravity. This theory has a R-symmetry given by $SU(8)$. The 256 degrees of freedom are comprised of two gravitons of helicity ± 2, 28 gauge fields and their conjugates with helicity ± 1, 70 scalars, 8 gravitinos each with helicity $\pm\frac{3}{2}$ and 56 gauginos each with helicity $\pm\frac{1}{2}$.

It is useful to work with the superspace formalism to describe the dynamics of supersymmetric field theories. In superspace, one extends the usual Minkowski space-time with Grassmann coordinates which anti-commute and square to zero. The fields then depend on these extended coordinates and are called superfields. A superfield elegantly capture all the components of a supermultiplet. We shall work here with the light-cone superspace formalism.

In light-cone superspace, in addition to the usual space-time coordinates, we have the anti-commuting coordinates (Grassmann variables) θ^m and $\bar{\theta}_m$ $(m = 1, 2, ...8)$ which transform as the $\mathbf{8}$ and $\bar{\mathbf{8}}$ of $SU(8)$.

All physical degrees of freedom of the $\mathcal{N}=8$ theory are written in terms of a single complex superfield [59]

$$
\begin{aligned}
\phi\,(\,y\,) = & \frac{1}{\partial^{+2}}\,h\,(y) \,+\, i\,\theta^m\,\frac{1}{\partial^{+2}}\,\bar{\psi}_m\,(y) \,+\, \frac{i}{2}\,\theta^m\,\theta^n\,\frac{1}{\partial^+}\,\bar{A}_{mn}\,(y)\;, \\
& -\,\frac{1}{3!}\,\theta^m\,\theta^n\,\theta^p\,\frac{1}{\partial^+}\,\bar{\chi}_{mnp}\,(y) \,-\, \frac{1}{4!}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\bar{D}_{mnpq}\,(y)\;, \\
& +\,\frac{i}{5!}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\theta^r\,\epsilon_{mnpqrstu}\,\chi^{stu}\,(y)\;, \\
& +\,\frac{i}{6!}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\theta^r\,\theta^s\,\epsilon_{mnpqrstu}\,\partial^+\,A^{tu}\,(y)\;, \\
& +\,\frac{1}{7!}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\theta^r\,\theta^s\,\theta^t\,\epsilon_{mnpqrstu}\,\partial^+\,\psi^u\,(y)\;, \\
& +\,\frac{4}{8!}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\theta^r\,\theta^s\,\theta^t\,\theta^u\,\epsilon_{mnpqrstu}\,\partial^{+2}\,\bar{h}\,(y)\;.
\end{aligned}
\tag{4.12}
$$

with $y = (x^+, x^- - \frac{i}{\sqrt{2}}\theta^m\bar{\theta}_m, x, \bar{x})$. We note that the superfield described above depends explictly on θ^m and implicitly on $\bar{\theta}_m$. This ensures that the superfield is an irreducible representation of the superPoincaré algebra and contains the correct degrees of freedom. This condition can be described in terms of the superspace covariant derivatives d^m and $\bar{d}_m$ which are defined as

$$d^m = -\frac{\partial}{\partial \bar\theta_m} - \frac{i}{\sqrt{2}}\theta^m \partial^+ \;\; ; \qquad \bar d_m = \frac{\partial}{\partial \theta^m} + \frac{i}{\sqrt{2}}\bar\theta_m \partial^+ . \tag{4.13}$$

The superfield ϕ and its complex conjugate $\bar\phi$ then satisfy the chirality constraints

$$d^m \,\phi \;=\; 0\,; \qquad \bar d_m \,\bar\phi \;=\; 0\;, \tag{4.14}$$

We note another key feature that $\phi(y)$ contains all the 256 degrees of freedom of the theory. Therefore $\bar\phi$ must be related to ϕ. This is a consequence of 'maximal' supersymmetry. This is captured by the "inside-out" constraints

$$\bar d_m \,\bar d_n \,\bar d_p \,\bar d_q \,\phi \;=\; \frac{1}{2}\,\epsilon_{mnpqrstu} \,d^r \,d^s \,d^t \,d^u \,\bar\phi\;. \tag{4.15}$$

The light-cone superPoincaré algebra

On the light-cone, the SuperPoincaré algebra splits up into generators that are independent of interactions/dynamics and those which deform in presence of interactions. Like the Poincaré algebra generators, the supercharges Q_α and $\bar Q_{\dot\alpha}$ can be written in terms of kinematical and dynamical parts. The kinematical supersymmetries are given by

$$q^m = -\frac{\partial}{\partial \bar\theta_m} \,+\, \frac{i}{\sqrt{2}}\,\theta^m \,\partial^+ \,; \qquad \bar q_m = \;\; \frac{\partial}{\partial \theta^m} \,-\, \frac{i}{\sqrt{2}}\,\bar\theta_m \,\partial^+ \;, \tag{4.16}$$

which satisfy

$$\{\,q^{\,m}\,,\,\bar q_n\,\} \;=\; i\,\sqrt{2}\,\delta^m{}_n\,\partial^+ \;. \tag{4.17}$$

These anti-commute with the chiral derivatives

$$\{\,q^{\,m}\,,\,\bar d_n\,\} \;=\; \{\,d^m\,,\,\bar q_n\,\} \;=\; 0\;, \tag{4.18}$$

At the light-cone time $x^+ = 0$, we have the three momenta

$$p^+ \;=\; -i\,\partial^+\,; \qquad p \;=\; -i\,\partial\,; \qquad \bar p \;=\; -i\,\bar\partial\;, \tag{4.19}$$

the transverse space rotations

$$j \;=\; x\,\bar\partial - \bar x\,\partial + S^{12}\,, \tag{4.20}$$

where

$$S^{12} \;=\; \frac{1}{4i\sqrt{2}\partial^+}(q^j\bar q_j - \bar q_j q^j). \tag{4.21}$$

and the “plus-rotations"

$$j^+ \;=\; i\,x\,\partial^+\,; \qquad \bar j^+ \;=\; i\,\bar x\,\partial^+\,, \tag{4.22}$$

$$j^{+-} \;=\; i\,x^-\,\partial^+ - \frac{i}{2}\,(\,\theta^m\frac{\partial}{\partial\bar\theta_m} + \bar\theta_m\,\frac{\partial}{\partial\theta^m}\,)\,. \tag{4.23}$$

The dynamical generators include the light-cone Hamiltonian

$$p^- \;=\; -i\frac{\partial\bar\partial}{\partial^+} \tag{4.24}$$

and the boosts

$$\begin{aligned} j^- \;&=\; i\,x\,\frac{\partial\bar\partial}{\partial^+} \;-\; i\,x^-\,\partial \;+\; i\,\Big(\theta^m\frac{\partial}{\partial\theta^m} \;+\; \frac{i}{4\sqrt{2}\,\partial^+}\,(\,d^m\,\bar d_m - \bar d_m\,d^m\,)\Big)\frac{\partial}{\partial^+}\;\;, \\ \bar j^- \;&=\; i\,\bar x\,\frac{\partial\bar\partial}{\partial^+} \;-\; i\,x^-\,\bar\partial \;+\; i\,\Big(\bar\theta_m\frac{\partial}{\partial\bar\theta_m} + \frac{i}{4\sqrt{2}\,\partial^+}\,(\,d^m\,\bar d_m - \bar d_m\,d^m\,)\Big)\frac{\bar\partial}{\partial^+}\;\;. \end{aligned} \tag{4.25}$$

while the commutator of the dynamical supersymmetries $q_-^{\,m}$ and $\bar q_{-\,m}$ close on p^- whose expressions are given by

$$\begin{aligned} Q^m \;&\equiv i\,[\,\bar j^-\,,\,q^m\,] \;=\; \frac{\bar\partial}{\partial^+}\,q^m\,, \\ \bar Q_n \;&\equiv i\,[\,j^-\,,\,\bar q_n\,] \;=\; \frac{\partial}{\partial^+}\,\bar q_n\,. \end{aligned} \tag{4.26}$$

The Lagrangian

We now describe the action for the $\mathcal{N} = 8$ supergravity theory to order κ (the gravitational coupling constant)

$$\int d^4x \int d^8\theta \, d^8\bar{\theta} \, \mathcal{L} \; \equiv \; \int \mathcal{L} \, , \tag{4.27}$$

where

$$\mathcal{L} = -\bar{\phi} \, \frac{\Box}{\partial^{+4}} \, \phi - \; 2 \, \kappa \, \frac{1}{\partial^{+2}} \, \bar{\phi} \;\; \bar{\partial} \, \phi \, \bar{\partial} \, \phi + c.c.,$$

with Grassmann integration normalized such that $\int d^8\theta \, (\theta)^8 = 1$.

The action was first derived in the ref [59] using closure of superPoincar e algebra and then simplified using numerous partial integrations and use of inside-out relations in ref. [35]. We expect to be able to find such simple forms at higher orders. In fact, one can prove interesting magic identities in superspace, which we explain in appendix.

If one puts all fields except the gravitons to zero and performs the Grassmann integration, it is straightforward to show that one recovers the cubic vertices (2.46) of pure gravity. Apart from the space-time symmetry, one finds that it is possible to extend the R-symmetry $SU(8)$ to a larger symmetry $E_{7(7)}$. We describe below the light-cone form of the generators of this symmetry.

4.4.1 Exceptional symmetry in $d = 4$

The action (4.27) displays a global $E_{7(7)}$ symmetry. The Lie algebra of this symmetry can be decomposed in to: those generators which correspond to its maximal compact subgroup $SU(8)$

$$\delta_{SU(8)}\phi \;=\; \omega^{j}_{\;\;i} \, \frac{1}{i\sqrt{2}\,\partial^+}(q^i \, \bar{q}_j \, - \, \frac{1}{8} \, \delta^i_j \, q^k \bar{q}_k)\phi \, , \tag{4.28}$$

where the $\omega^j_{\ i}$ $(i, j = 1, \ldots, 8)$ capture the 63 parameters of $SU(8)$ and those that correspond to the non-linear coset space $E_{7(7)}/SU(8)$ given by

$$\delta_{E_{7(7)}/SU(8)}\phi = -\frac{2}{\kappa}\theta^{klmn}\overline{\Xi}_{klmn} \tag{4.29}$$
$$+\frac{\kappa}{4!}\Xi^{klmn}\frac{1}{\partial^{+2}}\left(\bar{d}_{klmn}\frac{1}{\partial^{+}}\phi\,\partial^{+3}\phi - 4\,\bar{d}_{klm}\,\phi\,\bar{d}_n\,\partial^{+2}\phi + 3\,\bar{d}_{kl}\,\partial^{+}\phi\,\bar{d}_{mn}\,\partial^{+}\phi\right),$$

where the Ξ^{klmn} is the rank four antisymmetric tensor of $SU(8)$) which corresponds to the 70 parameters and satisfies $\Xi^{klmn} = \frac{1}{2}\epsilon^{klmnpqrs}\,\overline{\Xi}_{pqrs}$. These generators satisfy the algebra

$$\left[\,\delta_{su(8)}\,,\,\delta_{su(8)}\,\right]\phi \;=\; \delta_{su(8)}\,\phi, \qquad \left[\,\delta_{su(8)}\,,\,\delta_{E_{7(7)}/SU(8)}\,\right] \;=\; \delta_{E_{7(7)}/SU(8)}\,\phi,$$
$$\left[\,\delta_{E_{7(7)}/SU(8)}\,,\,\delta_{E_{7(7)}/SU(8)}\,\right] \;=\; \delta_{su(8)}\,\phi. \tag{4.30}$$

Invariance of the four-dimensional theory under both the superPoincaré symmetry and this exceptional symmetry was shown, in this language, in [60].

While, several efforts have been made to build the quartic interaction vertices in light-cone superspace [35, 61, 62], the expressions derived are very lengthy and cumbersome (some of them include about hundreds of terms) rendering them unsuitable for use in explicit calculations. However, we explain in the next chapter how a compact expression for these quartic vertices may be arrived at.

Chapter 5

An ‘exceptional’ bootstrap in $d = 4$

(*The material presented in this chapter is primarily based on the work done by the author in [63]*)

In this chapter, we explain how one may exploit the non-linear form of the $E_{7(7)}$ symmetry in light-cone superspace to derive quartic and higher-point interaction vertices in the theory. We demonstrate that the equations satisfied by these vertices are analogous to inhomogeneous differential equations. Invariance of these vertices under various symmetries is checked and specific forms are derived.

5.1 Leveraging exceptional symmetries

Symmetry transformations are realized non-linearly in terms of the physical variables in the light-cone approach. This non-linearity is used as a tool to build interacting theories. A simple example is that of using Poincare invariance which is not manifest in this formalism to derive cubic interaction vertices. As demonstrated in section 3, this approach suffers from many challenges when applied to quartic and higher-point vertices, one of them being the non-uniqueness. We describe in this section, a new way of building interacting theories making use of the exceptional symmetry.

We saw in eqn (4.29) that the $E_{7(7)}$ symmetry is represented as series in the coupling constant κ with only the odd powers. An immediate and interesting consequence of this is that, demanding invariance of the Hamiltonian under this symmetry relates higher *odd* (*even*) point vertices to lower *odd* (*even*) point vertices as described in the figure (5.1) below. This suggests that the role of the dynamical generators in the usual light-cone procedure is played instead by the exceptional symmetry. Schematically we have

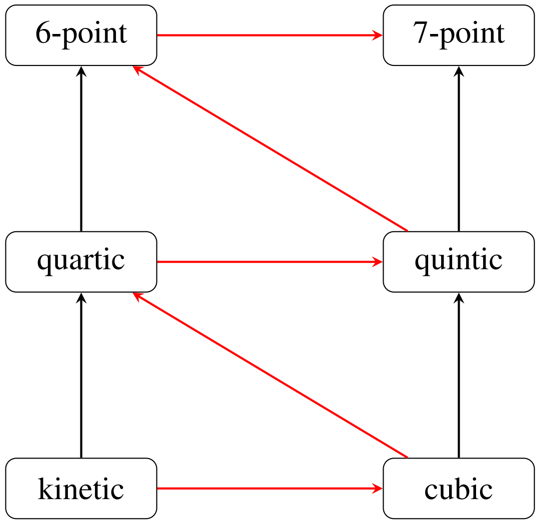

Figure 5.1: The red-arrows indicate the usual light-cone approach to deriving interaction vertices using dynamical generators while the black arrows indicate the alternate path, using the non-linearly realized exceptional symmetries

It is interesting to note that in the case of maximal supergravity in three-dimensions, only even-point vertices exist and there are no odd-point vertices. This is because the dynamical fields in the theory under the R-symmetry, transform in the spinor-representation of $SO(16)$ and it is not possible to construct a scalar out of three such fields. Thus, the two approaches are equivalent in that case.

We now describe the method for a maximal supergravity theoy in d-dimensions. We start by demanding invariance of the supergravity Hamiltonian under the exceptional symmetry group in d-dimensions. This produces the following equation

$$\delta_{1/\kappa}(\mathcal{V}_4) \;=\; -\delta_\kappa(\mathcal{V}_2), \tag{5.1}$$

where $\mathcal{V}_4$ is the unknown quartic vertex while $\mathcal{V}_2$ is the kinetic term in the Hamiltonian. Since the order $\frac{1}{\kappa}$ transformations map the superfield to a constant, we can decompose $\mathcal{V}_4$ as the sum of a homogeneous piece and an inhomogeneous piece

$$\mathcal{V}_4 = \mathcal{V}_4' + \mathcal{V}_4'' \,, \tag{5.2}$$

such that

$$\delta_{1/\kappa}\mathcal{V}_4' = 0 \quad ; \quad \delta_{1/\kappa}(\mathcal{V}_4'') = -\delta_\kappa(\mathcal{V}_2) \,, \tag{5.3}$$

with the boundary condition

$$\mathcal{V}_4\big|_{\{\text{all fields except graviton set to zero}\}} \;=\; \mathcal{M}_4\ , \tag{5.4}$$

where $\mathcal{M}_4$ is the quartic interaction vertex for pure gravity in the light-cone gauge [34, 35].

Note that any quartic vertex with a spacetime derivative on each superfield will satisfy the relation (5.3) (since the order $\frac{1}{\kappa}$ maps a superfield to a constant). Therefore, we find that the homogeneous piece $\mathcal{V}'_4$ cannot be fixed by the exceptional symmetry.

Instead, we first find the inhomogeneous piece $\mathcal{V}''_4$ using the exceptional symmetry. The next step is to then then require invariance of the full quartic interaction vertex $\mathcal{V}_4$ under the kinematical generators of the Lorentz algebra. This is used to derive the $\mathcal{V}'_4$. It then automatically guarantees that the resulting vertex $\mathcal{V}_4$ satisfies (5.1) since $\mathcal{V}'_4$ and $\mathcal{V}''_4$ satisfy (5.3). We apply this method to the four-dimensional theory below.

5.1.1 Four dimensions

In four-dimensions, we first find the inhomogeneous quartic pieces $\mathcal{V}''_4$ using $E_{7(7)}$. The Hamiltonian of the $d=4$ theory is

$$\mathcal{H} \;=\; \int d^3x\, d^8\theta\, d^8\bar\theta \left[\, 2\,\bar\phi\,\frac{\partial\bar\partial}{\partial^{+\,4}}\,\phi \;+\; 2\,\kappa\,\left(\frac{1}{\partial^{+2}}\,\bar\phi\;\;\bar\partial\,\phi\,\bar\partial\,\phi + c.c.\right)\right]\,. \tag{5.5}$$

We use the generators of $E_{7(7)}$ presented in equation (4.29).

Varying ϕ in the kinetic term of (5.5) with respect to the non-linear $E_{7(7)}$ symmetry yields a contribution at order κ (since the order $\frac{1}{\kappa}$ variation vanishes)

$$2\,\frac{\kappa}{4!}\,\Xi^{klmn}\,\frac{\partial\bar\partial}{\partial^{+\,6}}\,\bar\phi\,\left(\frac{1}{\partial^{+}}\phi\,\bar d_{klmn}\partial^{+\,3}\phi + 4\,\phi\,\bar d_{klmn}\,\partial^{+\,2}\phi + 3\,\partial^{+}\phi\,\bar d_{klmn}\,\partial^{+}\phi\right)\,, \tag{5.6}$$

which is obtained after performing some partial integrations (the measures and integrals are suppressed). This can be simplified to

$$2\,\frac{\kappa}{4!}\Xi^{klmn}\left(2\,\frac{\partial\bar\partial}{\partial^{+\,4}}\,\bar\phi\,\frac{1}{\partial^+}\phi\,\bar d_{klmn}\partial^+\phi \right. \qquad (5.7)$$
$$\left. -\,\frac{\partial\bar\partial}{\partial^{+\,6}}\,\bar\phi\,\frac{1}{\partial^+}\phi\,\bar d_{klmn}\partial^{+\,3}\phi\,+\,\frac{\partial\bar\partial}{\partial^{+\,6}}\,\bar\phi\,\partial^+\phi\,\bar d_{klmn}\,\partial^+\phi\right).$$

When we vary the superfield $\bar\phi$ in the kinetic term of (5.5), we obtain the complex conjugate of the above contribution. The inhomogeneous part $\mathcal{V}_4''$ is then given by

$$\begin{aligned}\mathcal{V}_4'' \;=\;& +\frac{1}{4!}\,\kappa^2\bigg[2\,\frac{\partial\bar\partial}{\partial^{+\,4}}\,\bar\phi\,\frac{1}{\partial^+}\phi\,\bar d_{klmn}\partial^+\phi \\ & -\,\frac{\partial\bar\partial}{\partial^{+\,6}}\,\bar\phi\left(\frac{1}{\partial^+}\phi\,\bar d_{klmn}\partial^{+\,3}\phi\,-\,\partial^+\phi\,\bar d_{klmn}\,\partial^+\phi\right)\bigg]\,d^{klmn}\,\bar\phi\,+\,c.c.\,.\end{aligned} \qquad (5.8)$$

It is quite straightforward to verify that this result is consistent with the kinematical generators j, j^{+-}. It is equally easy to verify that the expression (5.8) is not invariant under the remaining kinematical generators, $j^+, \bar j^+$. We ask for invariance of (5.8) under the remaining kinematic generators to find the homogeneous part of $\mathcal{V}_4$. This produces V_4' and and results in a quartic interaction vertex

$$\begin{aligned}\mathcal{V}_4 \;=\;& +\frac{1}{4!}\,\kappa^2\bigg[2\,\frac{\partial\bar\partial}{\partial^{+\,4}}\,\bar\phi\,\frac{1}{\partial^+}\phi\,\bar d_{klmn}\partial^+\phi \\ & -\,\frac{\partial\bar\partial}{\partial^{+\,6}}\,\bar\phi\left(\frac{1}{\partial^+}\phi\,\bar d_{klmn}\partial^{+\,3}\phi\,-\,\partial^+\phi\,\bar d_{klmn}\,\partial^+\phi\right)\bigg]\,d^{klmn}\,\bar\phi \\ & -\frac{1}{4!}\,\kappa^2\bigg[2\,\frac{\partial}{\partial^{+\,3}}\,\bar\phi\,\frac{1}{\partial^+}\phi\,\bar d_{klmn}\partial^+\phi \\ & -\,\frac{\partial}{\partial^{+\,5}}\,\bar\phi\left(\frac{1}{\partial^+}\phi\,\bar d_{klmn}\partial^{+\,3}\phi\,-\,\partial^+\phi\,\bar d_{klmn}\,\partial^+\phi\right)\bigg]\,\frac{\bar\partial}{\partial^+}d^{klmn}\,\bar\phi \\ & -\frac{1}{4!}\,\kappa^2\bigg[2\,\frac{\bar\partial}{\partial^{+\,3}}\,\bar\phi\,\frac{1}{\partial^+}\phi\,\bar d_{klmn}\partial^+\phi \\ & -\,\frac{\bar\partial}{\partial^{+\,5}}\,\bar\phi\left(\frac{1}{\partial^+}\phi\,\bar d_{klmn}\partial^{+\,3}\phi\,-\,\partial^+\phi\,\bar d_{klmn}\,\partial^+\phi\right)\bigg]\,\frac{\partial}{\partial^+}d^{klmn}\,\bar\phi \\ & +\frac{1}{4!}\,\kappa^2\bigg[2\,\frac{1}{\partial^{+\,2}}\,\bar\phi\,\frac{1}{\partial^+}\phi\,\bar d_{klmn}\partial^+\phi \\ & -\,\frac{1}{\partial^{+\,4}}\,\bar\phi\left(\frac{1}{\partial^+}\phi\,\bar d_{klmn}\partial^{+\,3}\phi\,-\,\partial^+\phi\,\bar d_{klmn}\,\partial^+\phi\right)\bigg]\,\frac{\partial\bar\partial}{\partial^{+\,2}}d^{klmn}\,\bar\phi\,+c.c.\,,\end{aligned} \qquad (5.9)$$

which is consistent with the exceptional symmetry $E_{7(7)}$ and all the kinematical generators of the Poincaré group. The vertex is a sum of homogeneous and inhomogeneous pieces as explained earlier. While deriving interaction vertices in the light-cone gauge, one first uses kinematical generators to fixes the overall form of the vertices. The dynamical generators are then used to fix the exact ∂^+ structure. But as we saw, here the ∂^+ structure is inherited and fixed from the kinetic term due to the exceptional symmetry thus suggesting that the vertex (5.9) is consistent with the dynamical generators. Finally, to the point of whether there could be additional homogeneous terms, which do not mix with the above terms, the boundary condition (5.4) would serve as a check.

One can now in principle find the dynamical boost generators to order κ^2 using the quartic Hamiltonian. To ensure that no Lorentz anomalies appear in the quantum theory, one can explictly check that the interaction pieces of the boost transformations do not contribute to any Jacobian, so that the path integral measure remains Lorentz invariant.

We can also determine the inhomogeneous part of the quintic vertices [36], using this procedure starting from the cubic vertices in (5.5). The order $\frac{1}{\kappa}$ variation of the cubic vertices vanishes so the order κ variation must be cancelled by the order $\frac{1}{\kappa}$ variation of the inhomogeneous quintic vertex. Accordingly, we find

$$
\begin{aligned}
\mathcal{V}_5'' \;=\; & -2\,\frac{\kappa^3}{4!}\left\{\sum_{n=0}^{4}\binom{4}{n}\,\frac{\bar\partial}{\partial^{+\,2}}\left[\frac{1}{\partial^{+\,2}}\bar\phi\,\bar\partial\,\bar d^{\,n}\phi\right]\frac{1}{\partial^+}\phi\,\partial^{+\,3}\,\bar d^{\,4-n}\phi\right. \\
& + \left(2\sum_{n=0}^{3}\binom{3}{n}\,\bar\partial\left[\frac{1}{\partial^{+\,2}}\bar\phi\,\bar\partial\,\bar d^{\,n}\phi\right]\frac{1}{\partial^+}\phi\,\partial^+\,\bar d^{\,4-n}\phi\right. \\
& \left. -\,2\,\frac{\bar\partial}{\partial^{+\,2}}\left[\frac{1}{\partial^{+\,2}}\bar\phi\,\bar\partial\,\bar d^{\,n}\phi\right]\left[\partial^+\phi\,\partial^+\,\bar d^{\,4-n}\phi\,+\,\frac{1}{\partial^+}\phi\,\partial^{+\,3}\,\bar d^{\,4-n}\phi\right]\right) \\
& \left. +\,3\,\frac{\bar\partial}{\partial^{+\,2}}\left[\frac{1}{\partial^{+\,2}}\bar\phi\,\bar\partial\,\phi\right]\bar d_{ij}\,\partial^+\,\phi\,\bar d_{kl}\,\partial^+\,\phi\right\}d^{ijkl}\,\bar\phi \\
& +\,\frac{\kappa^3}{4!}\left[2\,\frac{1}{\partial^{+\,2}}(\,\partial\,\bar\phi\,\partial\,\bar\phi\,)\,\frac{1}{\partial^+}\phi\;\bar d_{klmn}\partial^+\phi\right. \\
& \left. -\frac{1}{\partial^{+\,4}}(\partial\,\bar\phi\,\partial\,\bar\phi)\left(\frac{1}{\partial^+}\phi\,\bar d_{klmn}\partial^{+\,3}\phi\,-\,\partial^+\phi\,\bar d_{klmn}\,\partial^+\phi\right)\right]d^{klmn}\,\bar\phi\,+\,c.c.\;.
\end{aligned}
\tag{5.10}
$$

In principle, repeat the same procedure as described for the quartic vertex to determine the homogeneous pieces of quintic vertices using the kinematical symmetries. We stress that deriving six-point and higher vertices [41, 42] would need expressions for the coset transformations to order κ^3).

While working with non-trivial boundary conditions, one needs a careful treatment of the longitudinal zero modes, as discussed in chapter 2. It would be interesting to perform this analysis of supergravity similar to the one performed for light-cone electromagnetism in ref [26]. In such a case, one can expect to have additional contributions in the Hamiltonian describing the supergravity theory. To study non-perturbative aspects of supergravity, it would be interesting to derive a non-perturbative form for these exceptional symmetries. A first step in this direction, is to understand if one could derive an all order result for these symmetries. We describe how this may be achieved in chapter 7.

Chapter 6

Supergravity in $d = 5$

(*The material presented here is based on the work of the author in [63]*)

In this chapter, we discuss the construction of maximal supergravity theory in five-dimensions by the method of dimensional-oxidation starting from the four-dimensional theory. This theory is also interesting, as it is expected to be finite till four loops in four graviton scattering. We derive the form of the generators of the relevant symmetry algebras in light-cone superspace.

6.1 From $D = 11$ to $d = 5$

We explain briefly the degrees of freedom in the five-dimensional theory as representation of the little group $SO(3)$ and particularly how they arise from the eleven-dimensional theory. We split the little group up as $SO(9) \supset SO(3) \times SO(6)$ with $SO(3)$ being the little group in five-dimensions and $SO(6)$, an internal symmetry group. Under this division, we have

$$\mathbf{44} \;=\; \mathbf{21}_0 + \mathbf{18}_1 + \mathbf{5}_2 \;, \tag{6.1}$$

and

$$\mathbf{84} \;=\; \mathbf{20}_0 + \mathbf{1}_0 + \mathbf{45}_1 + \mathbf{18}_1, \tag{6.2}$$

where the subscript denotes the associated $SO(3)$ representation. The internal $SO(6)$ symmetry can be enhanced to an $USp(8)$ symmetry [57]. This $USp(8)$ group is the relevant R-symmetry and also the maximal compact subgroup of the exceptional group $E_{6(6)}$. This is then the key global symmetry associated with maximal supergravity in five-dimensions. In this chapter, we also construct the generators for both $USp(8)$ and $E_{6(6)}$ in light-cone superspace and show explicitly the invariance of the supergravity action, to first order in the coupling constant, under these symmetries.

The generators of the exceptional symmetries E_7 and E_8 were derived in ref [60–62] in the light-cone superspace. Here we explain how this program of study may be extended to E_6 as well.

We begin with the well understood and studied four-dimensional theory and use dimensional 'oxidation'[64] to arrive at the action in $d = 5$ instead of attempting to formulate directly or via dimensional reduction of the eleven-dimensional theory. The reason we do this is because it is easier to 'lift' and adapt the $d = 4$ formalism. In this process, we will also show that some of the exceptional structures encountered in $d = 4$ survive when moving to $d = 5$.

6.2 From $d = 4$ to $d = 5$

We emphasize here that we will use the **same** $(\mathcal{N} = 8, d = 4)$ superfield from (4.12) to describe the dynamics of five-dimensional supergravity theory. The reason is that in the five-dimensional supergravity theory, the four-dimensional degrees of freedom are re-organized. The little group is upgraded when moving to five-dimensions from $SO(2)$ to $SO(3)$. Similarly, the role of $SU(8)$ is played instead by $USp(8)$, ie. $SU(8)$ representations in the superfield need to be 'viewed' as representations of $USp(8)$. Specifically, the $\mathbf{70}$ of $SU(8)$ splits as

$$\mathbf{70} = \mathbf{42} + \mathbf{27} + \mathbf{1}. \tag{6.3}$$

We therefore have, in five dimensions, $\mathbf{42}$ scalar fields, $\mathbf{27}$ vector bosons and $\mathbf{1}$ graviton degree of freedom (each with Cartan eigenvalue zero). The $\mathbf{28}$ of $SU(8)$ splits into

$$\mathbf{28} \;=\; \mathbf{27} + \mathbf{1}\,, \tag{6.4}$$

producing $\mathbf{27}$ vector bosons and $\mathbf{1}$ graviton each with Cartan eigenvalue $+1$ (and -1 for the $\overline{\mathbf{28}}$ of $SU(8)$). The $\mathbf{56}$ of $SU(8)$ is now

$$\mathbf{56} \;=\; \mathbf{48} + \mathbf{8}\,, \tag{6.5}$$

representing $\mathbf{48}$ spin-$1/2$ and $\mathbf{8}$ spin-$3/2$ fields with the Cartan eigenvalue $+1/2$ (and $-1/2$ for the $\overline{\mathbf{56}}$). The physical degrees of freedom in four and five dimensions are summarized below.

Dimension	R-symmetry	Gravitons	Gravitinos	Vectors	Gauginos	Scalars
4	SU(8)	2	16	56	112	70

5	USp(8)	5	32	81	96	42

6.2.1 Grassmann variables and spinors in five dimensions

Theories of supergravity with extended supersymmetry in five-dimensions have the symplectic group $USp(2n)$ as their R-symmetry [57] (a light-cone related reference this group is [65]). The $USp(8)$ which is the relevant R-symmetry group here, has a 36-dimensional Lie algebra. The little group in five-dimensions is $SO(3) \sim SU(2)$. We therefore introduce the symplectic Grassmann variables $\theta^{i\alpha}$ (where $i = 1, 2, ..8$ is now an $USp(8)$ index and $\alpha = 1, 2$ is the $SU(2)$ index) satisfying

$$\bar{\theta}_{i\alpha} \;=\; \theta^{j\beta}\, C_{ji}\, \epsilon_{\beta\alpha}\, , \tag{6.6}$$

where $\epsilon_{\alpha\beta}$ (antisymmetric in $\alpha\,,\beta$) is the invariant tensor of $SU(2)$. The C_{ij} is the anti-symmetric symplectic invariant (or the charge-conjugation matrix) and satisfies $C^{ij}C_{jk} = -\delta^i_{\ k}$.

These Grassmann variables satisfy the algebra

$$\{\theta^{i\alpha}, \theta^{j\beta}\} \;=\; 0 \quad ; \quad \{\theta^{i\alpha}, \frac{\partial}{\partial\theta^{j\beta}}\} \;=\; \delta^i_{\ j}\, \delta^\alpha_{\ \beta}\, . \tag{6.7}$$

The kinematical supersymmetries are defined as

$$\begin{aligned} q^{i\alpha} &= -\frac{\partial}{\partial\bar{\theta}_{i\alpha}} + \frac{i}{\sqrt{2}}\theta^{i\alpha}\partial^+\, , \\ \bar{q}_{j\beta} &= \frac{\partial}{\partial\theta^{j\beta}} - \frac{i}{\sqrt{2}}\bar{\theta}_{j\beta}\partial^+\, , \end{aligned} \tag{6.8}$$

and obey

$$\{q^{i\alpha}, \bar{q}_{j\beta}\} \;=\; i\sqrt{2}\delta^i_{\ j}\, \delta^\alpha_{\ \beta}\, \partial^+ \quad ; \quad \{q^{i\alpha}, q^{j\beta}\} \;=\; -i\sqrt{2}\, C^{ij}\, \epsilon^{\alpha\beta}\, \partial^+ . \tag{6.9}$$

It is easy to check that the second component of $\theta^{j\alpha}$ ($\alpha = 2$) is related to the first component of $\bar{\theta}_{i\beta}$ ($\beta = 1$) ($\bar{\theta}_{i1} = C_{ij}\theta^{j2}$ from (6.6)), we can simply work with the first component ($\theta^{i1} \equiv \theta^i$) [65]. We find that $q^{i1}\,, \bar{q}_{j1}$ satisfy an algebra similar to that of the $SU(8)$ kinematical supersymmetries defined in equation (4.16) of section 3. This can be checked by substituting $\alpha, \beta = 1$ in (6.9). In the following, we will always work with the first component of the spinors and their conjugates.

6.2.2 Five dimensions

In this section, we explain how starting with (4.27), we can 'lift' the theory up one dimension so we have a five-dimensional theory.

We first introduce an additional coordinate x^5 and the derivative ∂_5. We then add to the four-dimensional little group $SO(2)$, the coset generators $SO(3)/SO(2)$ [65] to obtain all the generators of $SO(3)$

$$
\begin{aligned}
j^5 &= i(x^5\partial - x\partial^5) - \frac{1}{4i\partial^+} q^k\, C_{kl} q^l \ , \\
\bar{j}^5 &= i(x^5\bar{\partial} - \bar{x}\partial^5) + \frac{1}{4i\partial^+} \bar{q}_k\, C^{kl} \bar{q}_l .
\end{aligned}
\tag{6.10}
$$

We use three conditions to fix the spin part of the generators. We demand that the spin transformations be onsistent with helicity. Additionally, we require that the spin variation of the superfield commute with the chiral derivatives. We find that this fixes the spin part to be a quadratic function of the kinematical supersymmetries (q's in the case of j^5 and $\bar{q}$'s in the case of $\bar{j}^5$). Finally, the dimension of the generator fixes the exact ∂^+ structure.

The $SO(3)$ Lie algebra is generated by j , j^5 , $\bar{j}^5$ which obey

$$
[j^5, \bar{j}^5] \;=\; j \; ; \qquad [j, j^5] \;=\; +j^5 \; ; \qquad [j, \bar{j}^5] \;=\; -\bar{j}^5 . \tag{6.11}
$$

In addition to these, we also have the kinematical generator $j^{+\,5} = i x^5 \partial^+$ (at $x^+ = 0$) in the five-dimensional Lorentz algebra. For the purposes of this paper, we do not require the explicit form of the dynamical generators but we provide their form here for completeness

$$
\begin{aligned}
j^5 &= i\,(\,x^5\,\partial \,-\, x\,\partial^5\,) \,-\, \frac{1}{4\,i\,\partial^+}\, q^k\, C_{kl}\, q^l \ , \\
\bar{j}^5 &= i\,(\,x^5\,\bar{\partial} \,-\, \bar{x}\,\partial^5\,) \,+\, \frac{1}{4\,i\,\partial^+}\, \bar{q}_k\, C^{kl}\, \bar{q}_l \ .
\end{aligned}
\tag{6.12}
$$

The $SO(3)$ algebra is spanned by j , j^5 , $\bar{j}^5$ which obey

$$[\,j^5\,,\,\bar{j}^5\,]\;=\;j\;;\qquad [\,j\,,\,j^5\,]\;=+j^5\;;\qquad [\,j\,,\,\bar{j}^5\,]\;=-\bar{j}^5. \tag{6.13}$$

Additional generators include the rotation $j^{+5}\;=\;i\,x^5\,\partial^+$ (at $x^+\;=\;0$) and the momenta $p^5\;=\;-i\,\partial^5$. The expression for p^- now reads

$$p^-\;=\;-i\frac{\partial\bar{\partial}+\frac{1}{2}\partial^5\partial^5}{\partial^+}. \tag{6.14}$$

The first term in the expressions for j^- and $\bar{j}^-$ (4.25) is similarly modified . The dynamical generator j^{-5} is obtained from the commutator $[\,j^-\,,\,\bar{j}^5\,]\;=\;j^{-5}$.

As a first step, we allow the superfield to depend on the extra coordinate x^5. As shown in [64], covariance under the space-time symmetries in higher-dimensions can be achieved by replacing the $SO(2)$ derivatives by the generalized derivative $\overline{\nabla}$ and its conjugate. For five dimensions, we conjecture the following form for the generalized derivative

$$\overline{\nabla}=\bar{\partial}+\frac{\sigma}{16}\,\bar{d}_m\,C^{mn}\,\bar{d}_n\frac{\partial^5}{\partial^+}. \tag{6.15}$$

We prove that this form is indeed correct by using two rotations as described below (with the coset transformations j^5 and $\bar{j}^5$)

$$[\,\overline{\nabla},j^5]=\nabla^5=-i\partial^5+i\frac{\sigma}{16}\bar{d}_m\,C^{mn}\,\bar{d}_n\frac{\partial}{\partial^+}\;, \tag{6.16}$$

and

$$[\nabla^5,\bar{j}^5]=\overline{\nabla}\;, \tag{6.17}$$

This proves that $(\overline{\nabla},\nabla^5)$ has the correct transformation, as that of a vector in five dimensions. We fix the value of the constant σ by demanding Lorentz invariance of the cubic vertex (obtained by replacing $\bar{\partial}$ by $\overline{\nabla}$) in $d=5$. We discuss this below in detail.

6.2.3 Invariance of the action

The kinetic term, in (4.27) is rendered $SO(3)$-invariant by simply including the extra transverse derivative in the d'Alembertian. Moving to the cubic vertex in (4.27), the proposal is that it remains the same for the five-dimensional theory except for the replacement $\partial, \bar{\partial} \;\rightarrow\; \nabla, \overline{\nabla}$ (and the added dependence of the superfields on the fifth coordinate), ie.

$$\mathcal{V} \;=-\; 2\,\kappa \int d^5x \int d^8\theta\, d^8\bar{\theta}\; \frac{1}{\partial^{+2}}\; \bar{\phi}\;\; \overline{\nabla}\, \phi\, \overline{\nabla}\, \phi \,, \tag{6.18}$$

together with its complex conjugate. To show $SO(3)$ invariance, we consider the variations

$$\delta_{j^5}\, \phi \;\;=\;\; \frac{i}{4\partial^+}\, \bar{\omega}_5\;\; q^m\; C_{mn}\, q^n\, \phi \;, \tag{6.19}$$

$$\delta_{j^5}\, \bar{\phi} \;\;=\;\; \frac{i}{4\partial^+}\, \bar{\omega}_5\;\; q^m\; C_{mn}\, q^n\, \bar{\phi} \;, \tag{6.20}$$

$$\delta_{j^5}\, \overline{\nabla} \;\;=\;\; -\,\bar{\omega}_5\, \nabla^5 \;, \tag{6.21}$$

where $\bar{\omega}_5$ is the parameter corresponding to the coset transformation j^5. We show the invariance under the coset $SO(3)/SO(2)$ transformations (since invariance under the $SO(2)$ is clear), which proves the invariance under the full $SO(3)$.

The contributions from the variation of the superfields and the generalised derivative are given below (terms that involve two $SO(2)$ derivatives or two ∂^5s, cancel trivially). The non-trivial terms all involve a single $SO(2)$ derivative and a single ∂^5.

from $\int\, \frac{1}{\partial^{+2}}\, \left(\, \delta_{j^5}\, \bar{\phi}\,\right)\, \overline{\nabla}\, \phi\, \overline{\nabla}\, \phi$:

$$\begin{aligned}\int\; \frac{i\,\sigma}{4}\bigg\{ & \frac{8}{\partial^{+3}}\, \bar{\phi}\; \bar{\partial}\, \phi\, \partial^5\, \partial^+\, \phi \;-\; \frac{4}{\partial^{+3}}\, \bar{\phi}\; \bar{\partial}\, \phi\, \partial^5\, \partial^+\, \phi, \\ & +\; \frac{1}{\partial^{+3}}\, \bar{\phi}\; \bar{\partial}\, \phi\, \theta^m\, \bar{d}_m\, \partial^5\, \partial^+\, \phi, \\ & +\; \frac{1}{\partial^{+3}}\, \bar{\phi}\; \theta^m\; \bar{\partial}\, \partial^+\, \phi\; \bar{d}_m\, \partial^5\, \phi\; \bigg\}. \end{aligned} \tag{6.22}$$

from $\int \frac{1}{\partial^{+2}}\,\bar{\phi}\,\left(\delta_{j^5}\,\overline{\nabla}\right)\phi\,\overline{\nabla}\,\phi$:

$$\int\,\frac{2\,i}{\partial^{+2}}\,\bar{\phi}\,\bar{\partial}\,\phi\,\partial^5\,\phi. \tag{6.23}$$

from $\int \frac{1}{\partial^{+2}}\,\bar{\phi}\,\overline{\nabla}\left(\delta_{j^5}\,\phi\right)\overline{\nabla}\,\phi$:

$$\int\,\frac{i\,\sigma}{4}\left\{\,\frac{4}{\partial^{+2}}\,\bar{\phi}\,\bar{\partial}\,\phi\,\partial^5\,\phi\;+\;\frac{1}{\partial^{+2}}\,\bar{\phi}\,\theta^m\,\bar{\partial}\,\phi\,\bar{d}_m\,\partial^5\,\phi\,\right\}. \tag{6.24}$$

A useful identity when simplifying this is

$$\int\frac{1}{\partial^{+3}}\,\bar{\phi}\,\bar{\partial}\,\phi\,\partial^5\,\partial^+\,\phi\;=\;\int\frac{1}{\partial^{+3}}\,\bar{\phi}\,\partial^5\,\phi\,\bar{\partial}\,\partial^+\,\phi\,, \tag{6.25}$$

which is a consequence of the inside-out constraints (plus partial integrations). We give a proof of some magic identities in superspace in the appendix. We find

$$\delta_{j^5}\,\mathcal{V}\;\propto\;\int\left(\frac{i\,\sigma}{2}+\,2i\right)\frac{1}{\partial^{+2}}\,\bar{\phi}\,\bar{\partial}\,\phi\,\partial^5\,\phi\,. \tag{6.26}$$

As stated already, the invariance requires this variation to vanish so that $\sigma = -4$ and

$$\bar{\nabla}\;=\;\bar{\partial}\;-\;\frac{1}{4}\,\bar{d}_m\,C^{mn}\,\bar{d}_n\,\frac{\partial^5}{\partial^+}\,, \tag{6.27}$$

completing the proof of $SO(3)$ invariance of our ansatz for the cubic vertex in (6.18). In this light-cone form, the Lorentz invariance in five-dimensions is automatic, once the invariance under the little group has been proven.

6.3 $E_{6(6)}$

We now focus on the relevant global symmetry group, the exceptional group $E_{6(6)}$ which has 78 generators. The Lie algebra splits into that of the $USp(8)$ and the coset $E_{6(6)}/USp(8)$ so we have

$$\mathbf{78} \;=\; \mathbf{36}+\mathbf{42}. \tag{6.28}$$

The $\mathbf{36}$ generators of $USp(8)$ are constructed from the kinematic supercharges and are given by

$$T^{ij} = \frac{1}{i\sqrt{2}\partial^+}(q^i\, C^{jk}\, \bar{q}_k \;+\; q^j\, C^{ik}\, \bar{q}_k)\;, \tag{6.29}$$

where, the generators are symmetric in the indices i, j and satisfy the algebra

$$[T^{ij}, T^{mn}] \;=\; C^{jm}\, T^{in} + C^{jn}\, T^{im} + C^{im}\, T^{jn} + C^{in}\, T^{jm}\;. \tag{6.30}$$

The action of these 36 $USp(8)$ generators on the superfield is given by

$$\delta_{USp(8)}\phi \;=\; \omega_{ij}\, T^{ij}\phi\,, \tag{6.31}$$

where ω_{ij} correspond to the $\mathbf{36}$ parameters.

The $\mathbf{42}$ coset transformations act non-linearly on the superfield. Specifically, they shift the scalars by a constant at the lowest order [57]. These are therefore of the form $a\kappa^{-1}+b\kappa+\mathcal{O}(\kappa^3)$. In terms of the $USp(8)$ representations, the $\mathbf{42}$ corresponds to the completely anti-symmetric traceless rank four tensor. We thus conjecture that the lowest order (κ^{-1}) transformations are

$$\delta_{E_{6(6)}/USp(8)}\phi \;=\; -\frac{2}{\kappa}\theta^{klmn}\overline{\Sigma}_{klmn}, \tag{6.32}$$

where $\theta^{klmn} = \theta^k\theta^l\theta^m\theta^n$ and $\overline{\Sigma}_{klmn}$ are the $\mathbf{42}$ parameters which like the scalar fields satisfy $\Sigma^{klmn} = \Sigma^*_{klmn} \equiv \overline{\Sigma}_{klmn}$. At this order, it is only the scalar fields which are affected by these transformations- they act like a shift symmetry. This variation (6.32) is fixed due to the constraints imposed by helicity and dimension.

The generators at order κ are fixed by the requirement that the commutator of two coset generators close on an $USp(8)$ generator. We find

$$\delta_{E_{6(6)}/USp(8)}\phi = -\frac{2}{\kappa}\theta^{klmn}\overline{\Sigma}_{klmn} \tag{6.33}$$
$$+\frac{\kappa}{4!}\Sigma^{klmn}\frac{1}{\partial^{+2}}\left(\bar{d}_{klmn}\frac{1}{\partial^+}\phi\,\partial^{+3}\phi - 4\,\bar{d}_{klm}\,\phi\,\bar{d}_n\,\partial^{+2}\phi + 3\,\bar{d}_{kl}\,\partial^+\phi\,\bar{d}_{mn}\,\partial^+\phi\right),$$

where $\bar{d}_{i_1 i_2..i_n} = \bar{d}_{i_1}\bar{d}_{i_2}....\bar{d}_{i_n}$. It can be checked that the above variation is chiral.

With $\boldsymbol{\delta}\phi \;\equiv\; \delta_{E_{6(6)}/USp(8)}$ and $\delta\phi \;\equiv\; \delta_{USp(8)}\phi$ we have, schematically, for the $E_{6(6)}$ algebra

$$[\boldsymbol{\delta}_1, \boldsymbol{\delta}_2]\phi \;=\; \delta\phi \quad ; \quad [\boldsymbol{\delta}, \delta]\phi \;=\; \boldsymbol{\delta}\phi \quad ; \quad [\delta_1, \delta_2]\phi \;=\; \delta\phi\,, \tag{6.34}$$

closely resembling the $E_{7(7)}$ results in four dimensions.

The five-dimensional action to the cubic order is trivially invariant under the linear $USp(8)$ generators. Since the order $\frac{1}{\kappa}$ variation maps a superfield to a constant, it is straightforward to verify that the action is invariant to this order.

In order to verify invariance at the next order, one needs the quartic interaction vertex. We can apply a similar approach as in the case of the four-dimensional theory to derive quartic and higher-point vertices in the five-dimensional theory.

In this chapter, we constructed the maximal supergravity in five-dimensions and its relevant symmetries in light-cone superspace. These symmetries can then in principle be used to find the interacting Hamiltonian using the bootstrap approach discussed in chapter 5.

In the next chapter, we generalize the construction of symmetries in light-cone superspace to all dimensions $d \geq 4$ and arrive at a universal form.

Chapter 7

Exceptional symmetries in light-cone superspace

(*The material presented in this chapter is primarily based on the work done by the author in [66]*)

In this chapter, we derive a universal form for global symmetries in maximally supersymmetric theories of gravity in all dimensions $d \geq 4$. An interesting feature of this form is that it describes both classical and exceptional groups. Finally we discuss how one may arrive at an all order expression for each of these groups with an example of $E_{7(7)}$ symmetry. We also comment on why $E_{8(8)}$ does not comply with this universal form.

7.1 A 'universal' form for global symmetries

When eleven-dimensional supergravity is reduced to d space-time dimensions, its Lorentz symmetry $SO(10,1)$ splits into two factors : the Lorentz group $SO(d-1,1)$ which corresponds to space-symmetries and $SO(11-d)$ which appears as an internal symmetry. In many situations, this internal symmetry is further enlarged by duality transformations to the group $E_{n(n)}$. In a covariant description, the action of this symmetry is limited to the vector and scalar sectors of the theory. However, in the light-cone approach the removal of unphysical degrees of freedom leads to a mixing of all fields, so that the symmetry must act on the full supermultiplet. [61] .

The global symmetry group $G = E_{n(n)}$ naturally decomposes into two parts: the maximal compact subgroup H, which plays the role of the R-symmetry group, and the coset space $K = G/H$. At the level of the Lie algebra, this structure is reflected schematically in the commutation relations

$$[H, H] = H \quad , \quad [H, K] = K \quad , \quad [K, K] = H \, . \tag{7.1}$$

To construct the generators associated with this global symmetry, we introduce Grassmann variables $\theta^{m\alpha}$, where the index m labels the R-symmetry representation and α denotes the spinor representation of the little group. The corresponding kinematic supercharges and covariant derivatives are written as $q^{m\alpha}$ and $d^{m\alpha}$, respectively. Their conjugate counterparts $\bar{q}_{m\alpha}$ and $\bar{d}_{m\alpha}$ are obtained through contraction with the invariant tensors of the R-symmetry group and the little group[1]. As a result, one may restrict attention to a suitable set of independent components of the supercharges and covariant derivatives. As will be demonstrated through explicit examples, these components obey the relations given in (4.17).

In higher dimensions, we can use the same superfield expression as in (4.12) as has been already demonstrated in chapter 6. This superfield is allowed to have an added dependence on the extra coordinates, as the degrees of freedom simply reshuffle when moving to higher-dimensions. We will elaborate on this in the case of six-dimensions. We demand this superfield to vanish under the action of the independent components of $d^{m\alpha}$ in all dimensions $d > 4$. There are two crucial requirements that these symmetry transformations must satisfy:

- Their action on the superfield must preserve chirality so that the transformed superfield also is an irreducible representation of the superPoincaré algebra.

- They must preserve the helicity of the superfield as G is an internal symmetry that commutes with the little group.

The symmetry H is manifest by construction and therefore realized linearly on the superfield [**?**]

$$\delta_H \phi \; = \; \omega^m{}_n \, \frac{q^{n\alpha} \, \bar{q}_{m\alpha}}{i\sqrt{2}\,\partial^+} \, \phi \, . \tag{7.2}$$

The $\omega^m{}_n$ are constant parameters with appropriate symmetry properties such that there independent components match with the dimension of the adjoint representation of H.

At the lowest order, the coset transformations K shift the scalars by a constant [57, 61]. The scalars in the superfield always appear with four powers of θ (indices suppressed). By dimension

[1] This holds for $d \; > \; 4$.

and helicity analysis, an ansatz for K would take the form

$$\delta_K \phi = \frac{1}{\kappa}\, a_1 \;+\; a_2 \;+\; \kappa\, a_3 + \;\kappa^2\, a_4 + \ldots\;, \tag{7.3}$$

with $a_1 \sim \theta^{k\alpha}\,\theta^{l\beta}\,\theta^{m\gamma}\,\theta^{n\delta}\,\Sigma_{klmn}$, where Σ_{klmn} are constant parameters with appropriate symmetry properties. The commutators (7.1) and the fact that H is linear, imply that a_2, a_3 and a_4 are linear, quadratic and cubic functions of the superfield respectively. Since a_2 is linear in the parameters Σ^{klmn} and the superfield, one cannot write any structure compatible with the helicity. Therefore, a_2 must be identically zero and the commutation relations (7.1) then imply $a_4 = 0$. In a similar manner, one can argue that all terms with even powers of the coupling constant vanish.

We now arrive at a 'universal form' for δ_K from the third commutator in (7.1). We find

$$\delta_K\,\phi \;\;=\;\; -\frac{2}{\kappa}\theta^{klmn}_{\alpha\beta\gamma\delta}\;\Sigma_{klmn} \;+\; \xi(d)\,\frac{\kappa}{4!}\Sigma^{klmn}\frac{1}{\partial^{+\,2}}\left(\mathcal{C}\left[\bar{d}^{\,\alpha\beta\gamma\delta}_{klmn}\,\frac{1}{\partial^{+}}\phi\,\partial^{+\,3}\phi\right]\right) + \mathcal{O}(\kappa^3)\;,$$

$$\theta^{klmn}_{\alpha\beta\gamma\delta} \;\equiv \theta^{k\alpha}\,\theta^{l\beta}\,\theta^{m\gamma}\,\theta^{n\delta}\quad,\quad \bar{d}^{\,\alpha\beta\gamma\delta}_{klmn} \;\equiv\; \bar{d}_{k\alpha}\,\bar{d}_{l\beta}\,\bar{d}_{m\gamma}\,\bar{d}_{n\delta}\;, \tag{7.4}$$

where $\xi(d)$ is a constant. The term $\mathcal{C}$ refers to the process of chiralization: the addition of terms (often polynomial in the superfields) to the existing superspace expression such that $d^{m\alpha}\,\delta_K\phi = 0$ [64]. As explained earlier, this ensures that the operation preserves chirality.

The table below summarizes the global symmetry, its compact subgroup (R-symmetry) and the little group for dimensions four through seven.

Dimension	Little group	R-symmetry	Global symmetry
4	SO(2)	SU(8)	$E_{7(7)}$
5	SO(3)	USp(8)	$E_{6(6)}$
6	SO(4)	SO(5) $\times$ SO(5)	SO(5,5)
7	SO(5)	SO(5)	SL(5,R)

We have already examined the construction of the generators of $E_{7(7)}$ and $E_{6(6)}$ and it is easy to verify that they conform with the universal form (7.4). An interesting thing to notice is that the chiralization functions that appear in the non-linear parts of both $E_{6(6)}$ and $E_{7(7)}$ are structurally similar. We demonstrate here for $E_{7(7)}$ that the action of d^m (the $SU(8)$ covariant derivative) indeed vanishes.

The linear-part of E_7 is

$$\delta\phi \equiv \delta_{SU(8)}\phi \;=\; \omega^j{}_k\,\frac{1}{i\sqrt{2}\,\partial^+}(q^k\,\bar{q}_j - \frac{1}{8}\,\delta^j_k\,q^m\bar{q}_m)\phi\,, \tag{7.5}$$

and the non-linear part of E_7 is

$$\begin{aligned}\boldsymbol{\delta}\phi \equiv \delta_{E_{7(7)}/SU(8)}\phi = -\frac{2}{\kappa}\theta^{klmn}\Xi_{klmn}\\ + \frac{\kappa}{4!}\Xi^{klmn}\frac{1}{\partial^{+2}}\bigg(\bar{d}_{klmn}\frac{1}{\partial^+}\phi\,\partial^{+3}\phi - 4\,\bar{d}_{klm}\,\phi\,\bar{d}_n\,\partial^{+2}\phi + 3\,\bar{d}_{kl}\,\partial^+\phi\,\bar{d}_{mn}\,\partial^+\phi\bigg)\,.\end{aligned} \tag{7.6}$$

The $\delta\phi$ and the order κ^{-1} piece in $\boldsymbol{\delta}\phi$ are chiral, trivially. To confirm 'chirality' at the next order in (7.6), we act with the chiral derivative d^p to obtain

$$\begin{aligned} d^p\boldsymbol{\delta}\phi \;&=\; \frac{\kappa}{4!}\,\Xi^{klmn}\,\frac{1}{\partial^{+2}}\bigg(d^p\,\bar{d}_{klmn}\frac{1}{\partial^+}\phi\,\partial^{+3}\phi \;-\; 4\,d^p\,\bar{d}_{klm}\,\phi\,\bar{d}_n\,\partial^{+2}\phi \\ &\quad + 4\,\bar{d}_{klm}\,\phi\,d^p\,\bar{d}_n\,\partial^{+2}\phi \;+\; 6\,d^p\,\bar{d}_{kl}\,\partial^+\phi\,\bar{d}_{mn}\,\partial^+\phi\bigg)\,, \end{aligned} \tag{7.7}$$

and use $\{d^m, \bar{d}_n\} \;=\; -i\sqrt{2}\,\delta^m_n\,\partial^+$ to simplify this to

$$\begin{aligned} d^p\boldsymbol{\delta}\phi \;&=\; -i\sqrt{2}\,\frac{\kappa}{4!}\,\Xi^{klmp}\,\frac{1}{\partial^{+2}}\bigg(4\,\bar{d}_{klm}\phi\,\partial^{+3}\phi \;-\; 12\,\bar{d}_{kl}\partial^+\,\phi\,\bar{d}_m\,\partial^{+2}\phi \\ &\quad - 4\,\bar{d}_{klm}\,\phi\,\partial^{+3}\phi \;+\; 12\,\bar{d}_{kl}\,\partial^+\phi\,\bar{d}_m\,\partial^{+2}\phi\bigg) \;=\; 0\,, \end{aligned} \tag{7.8}$$

Similarly this will hold for the exceptional symmetry $E_{6(6)}$. In the following, we discuss the generators of the global symmetries in six and seven-dimensions.

7.1.1 $E_{5(5)}$ in six dimensions

While climbing up the ladder of dimensions, six-maximal supergravity theory is the first where the symmetry is a classical group: $SO(5,5)$. We list some important features here:

- Maximal compact subgroup/ R-symmetry : $SO(5) \times SO(5)$

- The little group : $SO(4) \sim SU(2) \times SU(2)$.

We use the Lie algebra isomorphism $SO(5) \sim USp(4)$ to define the Grassmann variables $(\theta^{m\alpha}, \theta^{a\dot{\alpha}})$ where $m, a = 1, 2, ..4$ are the internal symmetry indices and $\alpha, \dot{\alpha} = 1, 2$ represent little group indices. The $USp(4)$ symplectic invariants are denoted by C_{mn} and C_{ab}. The supercharges $(q^{m\alpha}, q^{a\dot{\alpha}})$ and the covariant derivatives $(d^{m\alpha}, d^{a\dot{\alpha}})$ within each $SU(2)$ satisfy the algebra

$$\begin{aligned} \{q^{i\alpha}, \bar{q}_{j\beta}\} &= i\sqrt{2}\delta^i_j\, \delta^\alpha_\beta\, \partial^+ \quad ; \quad \{q^{i\alpha}, q^{j\beta}\} = -i\sqrt{2}\, C^{ij}\, \epsilon^{\alpha\beta}\, \partial^+ . \\ \{d^{i\alpha}, \bar{d}_{j\beta}\} &= -i\sqrt{2}\delta^i_j\, \delta^\alpha_\beta\, \partial^+ \quad ; \quad \{d^{i\alpha}, d^{j\beta}\} = +i\sqrt{2}\, C^{ij}\, \epsilon^{\alpha\beta}\, \partial^+ . \end{aligned} \tag{7.9}$$

while they anti-commute with those from the other $SU(2)$. Supercharges anti-commute with the covariant derivatives as usual. Since the second component of the spinors are related to the first component of the conjugate spinors, we shall work with $\alpha = 1, \dot{\alpha} = 1$.

As alluded to earlier, we work with the same superfield with added dependence in higher-dimensions. We explain how the various $SU(8)$ representations in the four-dimensiona superfield (4.12) split in six-dimensions.

How the degrees of freedom re-shuffle in higher-dimensions

In six dimensions, the physical fields are representations of the little group $SO(4) \sim SU(2) \times SU(2)$, labeled by two half-integers (j_1, j_2) (corresponding to the two Cartan generators) with dimension $(2\,j_1 + 1) \times (2\,j_2 + 1)$. The R-symmetry is reduced to the $USp(4) \times USp(4)$ group. The superfield may be expanded in terms of the Grassmann coordinates $(\theta^{m1}, \theta^{a1})$ where $m, a = 1, 2, ..4$ correspond to the two $USp(4)$ groups. The **70** of $SU(8)$ is then

$$\mathbf{70} = \mathbf{5} \times \mathbf{5} + \mathbf{4} \times \mathbf{4} + \mathbf{4} \times \mathbf{4} + \mathbf{5} \times \mathbf{1} + \mathbf{1} \times \mathbf{5} + \mathbf{1} + \mathbf{1} + \mathbf{1}\,, \tag{7.10}$$

representing the 25 scalars, 32 vectors (with Cartan eigenvalues $(\pm\frac{1}{2}\,, \mp\frac{1}{2})$), 10 anti-symmetric bosons and 1 graviton each with Cartan eigenvalue (0,0). We also get two gravitons with Cartan eigenvalues $(\pm 1\,, \mp 1)$. The **28** splits as

$$\mathbf{28} = \mathbf{4} \times \mathbf{4} + \mathbf{5} \times \mathbf{1} + \mathbf{1} \times \mathbf{5} + \mathbf{1} + \mathbf{1}\,, \tag{7.11}$$

producing 16 vectors with Cartan eigenvalue $(\frac{1}{2}, \frac{1}{2})$, 5 anti-symmetric bosons and a graviton with Cartan eigenvalue (1,0) (and similarly with (0,1)). The $\overline{\mathbf{28}}$ yields the conjugate fields with the opposite helicity. The $\mathbf{56}$ is now

$$\mathbf{56} \;=\; \mathbf{5}\times\mathbf{4} \,+\, \mathbf{4}\times\mathbf{5} \,+\, \mathbf{1}\times\mathbf{4} \,+\, \mathbf{4}\times\mathbf{1} \,+\, \mathbf{1}\times\mathbf{4} \,+\, \mathbf{4}\times\mathbf{1}\,, \tag{7.12}$$

producing the 20 gauginos and 4 gravitinos with Cartan eigenvalues $(\,\frac{1}{2}\,,\,0\,)$ (and similarly with $(\,0, \frac{1}{2}\,)$). We also get 4 gravitinos each with Cartan eigenvalue $(\,-\frac{1}{2}\,,\,1\,)$ (and similarly with $(\,1, -\frac{1}{2}\,)$). The $\overline{\mathbf{56}}$ splits in a similar manner yielding the conjugate fields (of opposite helicity). The table below summarizes the degrees of freedom in six dimensions [67].

Degrees of freedom	$SU(2)\times SU(2)$ A	$USp(4)\times USp(4)$ B	Total $A\times B$
Gravitons	9	1	9
Gravitinos	12	4	48
Vectors	4	16	64
Anti-symmetric bosons	6	5	30
Gauginos	4	20	80
Scalars	1	25	25

This analysis can be extended to seven and higher dimensions similarly.

The $SO(5,5)$ transformations

The $SO(5)$ gamma matrices γ^I $(I = 1, 2..5)$ satisfy

$$\{\gamma^I\,,\gamma^J\,\} \;=\; 2\,\delta^{IJ}\,\mathbb{I}_{4\times 4}\;, \tag{7.13}$$

where the $(\gamma^I)^{kl} = C^{lm}(\gamma^I)^k{}_m$ are anti-symmetric in k, l and traceless with respect to C_{kl} [65]. A vector of $SO(5)$ can then be written as a bi-spinor

$$p_{mn} \;=\; p_I\,(\gamma^I)_{mn}\,. \tag{7.14}$$

The commutators of the gamma matrices $(\gamma^{IJ})^{kl} = C^{lm}(\gamma^{IJ})^k{}_m$ are symmetric in k, l [65].

The adjoint of $SO(5,5)$ splits up in terms of $SO(5)\times SO(5)$ representations as $\mathbf{45}=\mathbf{20}+\mathbf{25}$. The $\mathbf{20}$ linearly realized generators of $SO(5)\times SO(5)$ are spanned by T^{IJ} and T^{AB} (I,J and A,B correspond to the two independent $SO(5)$ groups)

$$\begin{aligned}
T^{IJ} &= (\gamma^{IJ})_{mn}\,\frac{1}{i\sqrt{2}\partial^+}(q^m\,C^{nk}\,\bar{q}_k\;+\;q^n\,C^{mk}\,\bar{q}_k)\,, \\
T^{AB} &= (\gamma^{AB})_{ab}\,\frac{1}{i\sqrt{2}\partial^+}(q^a\,C^{bc}\,\bar{q}_c\;+\;q^b\,C^{ac}\,\bar{q}_c)\,,
\end{aligned} \tag{7.15}$$

where $q^m\equiv q^{m1}$, $\bar{q}_m\equiv\bar{q}_{m1}$ and $q^a\equiv q^{a1}$, $\bar{q}_a\equiv\bar{q}_{a1}$.

The $\mathbf{25}$ non-linear transformations are given by

$$\delta_{E_{5(5)}/H}\phi \;=\; -\frac{2}{\kappa}\theta^{klab}\Sigma_{klab}\;+\;\kappa\,\Sigma^{klab}\frac{1}{\partial^{+2}}\left(\mathcal{C}\left[\bar{d}_{klab}\frac{1}{\partial^+}\phi\,\partial^{+3}\phi\right]\right)\;,$$

where $\Sigma_{klab}=\Sigma_{IA}\,(\gamma^I)_{kl}\,(\gamma^A)_{ab}$, $\theta^{klab}\equiv\theta^{k1}\theta^{l1}\theta^{a1}\theta^{b1}$ and $\bar{d}_{klmn}\equiv\bar{d}_{k1}\bar{d}_{l1}\bar{d}_{a1}\bar{d}_{b1}$. The Σ_{IA} are the 25 constant parameters. The chiral function is structurally different (from the one in four and five dimensions) and reads

$$\begin{aligned}
\mathcal{C}\left[\bar{d}_{klab}\frac{1}{\partial^+}\phi\,\partial^{+3}\phi\right] \;=\;& \bar{d}_{klab}\frac{1}{\partial^+}\phi\,\partial^{+3}\phi\;-2\,\bar{d}_{kla}\,\phi\,\bar{d}_b\,\partial^{+2}\phi \\
&-2\,\bar{d}_{kab}\,\phi\,\bar{d}_l\,\partial^{+2}\phi\;+\;\bar{d}_{kl}\,\partial^+\phi\,\bar{d}_{ab}\,\partial^+\phi\;+2\,\bar{d}_{ka}\,\partial^+\phi\,\bar{d}_{bl}\,\partial^+\phi .
\end{aligned}\tag{7.16}$$

We confirm that $\delta_{E_{5,5}/H}\phi\equiv\delta\phi$ is chiral by acting with $d^m\equiv d^{m1}$

$$\begin{aligned}
d^m\delta\phi \;=\;& \kappa\,\Sigma^{klab}\Big(d^m\,\bar{d}_{klab}\frac{1}{\partial^+}\phi\,\partial^{+3}\phi\;-2\,d^m\,\bar{d}_{kla}\,\phi\,\bar{d}_b\,\partial^{+2}\phi\;-\;2\,d^m\,\bar{d}_{kab}\,\phi\,\bar{d}_l\,\partial^{+2}\phi \\
&+2\,\bar{d}_{kab}\,\phi\,d^m\,\bar{d}_l\,\partial^{+2}\phi\;+\;d^m\,\bar{d}_{kl}\,\partial^+\phi\,\bar{d}_{ab}\,\partial^+\phi\;+\;4\,d^m\,\bar{d}_{ka}\,\partial^+\phi\,\bar{d}_{bl}\,\partial^+\phi\Big)\,,
\end{aligned}\tag{7.17}$$

which simplifies to

$$\begin{aligned}
d^m\delta\phi \;=\;& -i\sqrt{2}\,\kappa\,\Sigma^{mlab}\Big(2\,\bar{d}_{lab}\,\phi\,\partial^{+3}\phi\;-4\,\bar{d}_{la}\,\partial^+\,\phi\,\bar{d}_b\,\partial^{+2}\phi\;-2\,\bar{d}_{ab}\,\partial^+\,\phi\,\bar{d}_l\,\partial^{+2}\phi \\
&-2\,\bar{d}_{lab}\,\phi\,\partial^{+3}\phi\;+2\,\bar{d}_{ab}\,\partial^+\,\phi\,\bar{d}_l\,\partial^{+2}\phi+4\,\bar{d}_{la}\,\partial^+\,\phi\,\bar{d}_b\,\partial^{+2}\phi\Big)\;=\;0\,.
\end{aligned}\tag{7.18}$$

It can be similarly shown by acting with $d^a\equiv d^{a1}$ that $d^a\delta\phi=0$. Hence, the transformations (7.16) are of the form in (7.4).

7.1.2 $E_{4(4)}$ in seven dimensions

We list some key features of the seven-dimensional supergravity :

- Global symmetry : $SL(5, R)$

- Maximal compact subgroup : $SO(5) \sim USp(4)$

- Little group : $SO(5) \sim USp(4)$

We define the Grassmann variables $\theta^{m\alpha}$ satisfying $\theta^{m\alpha} = C^{mn} C^{\alpha\beta} \bar{\theta}_{n\beta}$, where both m, α are $USp(4)$ indices, one corresponding to the little group and the other corresponding to the internal symmetry. The corresponding supercharges and covariant derivatives satisfy the algebra

$$\begin{aligned} \{q^{m\alpha}, \bar{q}_{n\beta}\} &= i\sqrt{2}\, \delta^m_n\, \delta^\alpha_\beta\, \partial^+ \quad ; \quad \{q^{m\alpha}, q^{n\beta}\} = -i\sqrt{2}\, C^{mn}\, C^{\alpha\beta}\, \partial^+ \,, \\ \{d^{m\alpha}, \bar{d}_{n\beta}\} &= -i\sqrt{2}\, \delta^m_n\, \delta^\alpha_\beta\, \partial^+ \quad ; \quad \{d^{m\alpha}, d^{n\beta}\} = +i\sqrt{2}\, C^{mn}\, C^{\alpha\beta}\, \partial^+ . \end{aligned} \tag{7.19}$$

We choose a form for the symplectic invariant $C_{\alpha\beta}$ [65]

$$C_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{7.20}$$

Here, we shall work with the $\alpha, \beta = 1, 2$ components of the spinors to write the $SL(5, R)$ transformations.

The $SL(5, R)$ transformations

The adjoint of $SL(5, R)$ splits up in terms of $SO(5)$ representations as $\mathbf{24} = \mathbf{10} + \mathbf{14}$. We use the $SO(5)$ gamma matrices defined in (7.13) to write the $\mathbf{10}$ linear transformations T^{IJ} $(I, J = 1, 2, ..5)$

$$T^{IJ} = (\gamma^{IJ})_{mn} \frac{1}{i\sqrt{2}\,\partial^+} (q^{m1}\, C^{nk}\, \bar{q}_{k1} + q^{n1}\, C^{mk}\, \bar{q}_{k1} + q^{m2}\, C^{nk}\, \bar{q}_{k2} + q^{n2}\, C^{mk}\, \bar{q}_{k2}) \,. \tag{7.21}$$

The $\mathbf{14}$ non-linear transformations are given by

$$\delta_{SL(5,R)/SO(5)}\phi \;=\; -\frac{2}{\kappa}\theta^{kl\underline{mn}}\Sigma_{klmn} \;+\; \kappa\,\Sigma^{klmn}\frac{1}{\partial^{+\,2}}\left(\mathcal{C}\left[\bar{d}_{kl\underline{mn}}\frac{1}{\partial^{+}}\phi\,\partial^{+\,3}\phi\right]\right)\,, \quad (7.22)$$

where $\theta^{kl\underline{mn}} \equiv \theta^{k1}\,\theta^{l1}\,\theta^{m2}\,\theta^{n2}$ and $\bar{d}_{kl\underline{mn}} \equiv \bar{d}_{k1}\,\bar{d}_{l1}\,\bar{d}_{m2}\,\bar{d}_{n2}$. The chiral function in (7.22) is the same as that in (7.16). $\Sigma_{klmn} \;=\; \Sigma_{IJ}\,(\gamma^{I})_{kl}\,(\gamma^{J})_{mn}$ and the Σ_{IJ} is a symmetric traceless tensor representing the 14 constant parameters.

Thus the global symmetries in dimensions four through seven share the universal form. This approach may be useful to understand how the eleven-dimensional model may be formulated in a manner covariant under these global symmetries [68, 69]. In principle, the global symmetries in eight and nine dimensions, can also be constructed in a similar fashion. It would also be interesting to perform a systematic study of symmetries in the $d < 4$ models which include the infinite dimensional Kac-Moody algebras [70–74].

7.2 An aside: the Lagrangians

It is also interesting to pause and ponder about the actions of the corresponding theories discussed thus far. We find that these actions can be obtained from the four-dimensional action by'oxidizing' the theory. We introduce the 'generalized derivative' [64] to construct the Lagrangians in six and seven dimensions. These generalized derivatives are defined as

$$\overline{\nabla}^{(6)} \;=\; \bar{\partial} \,+\, \frac{\eta}{16}\,C^{ij}\,\bar{d}_i\,\bar{d}_j\,\frac{\partial^5}{\partial^+} \,+\, \frac{\eta}{16}\,C^{ab}\,\bar{d}_a\,\bar{d}_b\,\frac{\partial^6}{\partial^+}\,, \quad (7.23)$$

$$\overline{\nabla}^{(7)} \;=\; \bar{\partial} \,+\, \frac{\chi}{16}\,C^{ij}\,\bar{d}_i\,\bar{d}_j\,\frac{\partial^5}{\partial^+} \,+\, \frac{\chi}{16}\,C^{ij}\,\bar{d}_{\underline{i}}\,\bar{d}_{\underline{j}}\,\frac{\partial^6}{\partial^+} \,+\, \frac{\chi}{16}\,C^{ij}\,\bar{d}_{\underline{i}}\,\bar{d}_j\,\frac{\partial^7}{\partial^+}\,. \quad (7.24)$$

The constants η and χ are again determined using the requirement of Lorentz invariance.

7.3 Why E_8 does not comply with the universal form

We review some of the aspects of the construction of E_8 generators in light-cone superspace, first done in [62]. The Grassmann variables, θ^m and $\bar{\theta}_m$, form a **16** representation

$$SO(16) \supset SU(8) \times U(1) \,, \qquad \mathbf{16} \,=\, \mathbf{8} + \overline{\mathbf{8}} \,.$$

The $SO(16)$ transformations are obtained as quadratic action of the q^m, $\bar{q}_m$ on ϕ. These split in terms of $SU(8) \times U(1)$ as follows

$$\mathbf{120} \,=\, \mathbf{63_0} \,+\, \mathbf{28_{-1}} \,+\, \overline{\mathbf{28}}_{\mathbf{1}} \,+\, \mathbf{1_0} \tag{7.25}$$

where the subscript indicate $U(1)$ charge. The $SU(8)$ generators are given in (7.5) and $U(1)$ generators are given by

$$T \,=\, \frac{i}{4\sqrt{2}\,\partial^+}\,[\,q^m\,,\,\bar{q}_m\,]\,, \qquad [\,T\,,\,T^m{}_n\,] \,=\, 0\,. \tag{7.26}$$

The **28** and $\overline{\mathbf{28}}$ of $SU(8)$ generate the coset transformations $SO(16)/(SU(8) \times U(1))$

$$T^{mn} \,=\, \frac{1}{2}\frac{1}{\partial^+}q^m q^n\,, \qquad T_{mn} \,=\, \frac{1}{2}\frac{1}{\partial^+}\bar{q}_m\bar{q}_n\,, \tag{7.27}$$

which close on $(SU(8) \times U(1))$

$$[\,T^{mn}\,,\,T_{pq}\,] \,=\, \delta^n{}_p T^m{}_q \,-\, \delta^m{}_p T^n{}_q \,-\, \delta^m{}_q T^n{}_p \,+\, \delta^m{}_q T^n{}_p \,+\, 2\,(\,\delta^n{}_p\delta^m{}_q \,-\, \delta^n{}_q\delta^m{}_p\,)\,T\,.$$

Hence, on the superfield, the complete $SO(16)$ transformations read

$$\delta_{SU_8}\,\varphi \,=\, \omega^n{}_m\,T^m{}_n\,\varphi\,, \qquad \delta_{U(1)}\,\varphi \,=\, T\,\varphi\,,$$

$$\delta_{\mathbf{28}}\,\varphi \,=\, \alpha_{mn}\,\frac{q^m q^n}{\partial^+}\,\varphi\,, \qquad \delta_{\overline{\mathbf{28}}}\,\varphi \,=\, \alpha^{mn}\,\frac{\bar{q}_m\bar{q}_n}{\partial^+}\,\varphi\,, \tag{7.28}$$

where $\omega^n{}_m$, α_{mn}, and α^{mn} are the transformation parameters.

We decompose the non-linearly realized coset $E_{8(8)}/SO(16)$ in terms of $SU(8) \times U(1)$

representations

$$\mathbf{128} = \mathbf{1}'_{\mathbf{2}} + \mathbf{28}'_{\mathbf{1}} + \mathbf{70}_{\mathbf{0}} + \overline{\mathbf{28}}'_{\mathbf{-1}} + \overline{\mathbf{1}}'_{\mathbf{-2}} \quad . \tag{7.29}$$

All the bosonic components of the superfield are essentially scalars and this is the main distinction between three and higher dimensions. These transformations shift each of these fields by a constant. At the lowest order [62]

$$\delta_{E_{8(8)}/SO(16)}\,\phi \;=\; \frac{1}{\kappa}\,F$$

where

$$\begin{aligned} F \;&=\; \frac{1}{\partial^{+2}}\,\beta\,(y^-) \;+\, i\,\theta^{mn}\,\frac{1}{\partial^{+}}\,\overline{\beta}_{mn}\,(y^-) - \theta^{mnpq}\,\overline{\beta}_{mnpq}\,(y^-) + \\ &\quad + i\widetilde{\theta}_{\;mn}\,\partial^{+}\,\beta^{mn}\,(y^-) + 4\,\widetilde{\theta}\,\partial^{+2}\,\bar{\beta}\,(y^-)\,, \end{aligned}$$

Therefore, we find that at the lowest order one needs to add terms proportional to all even powers of θ. Therefore the E_8 transformations do not conform with the universal form (7.4) and are more complicated to derive.

It would be interesting to find another form for this symmetry where one directly works with the spinors of $SO(16)$ instead of the embeddings of $SU(8)$ in $SO(16)$.

7.4 Towards an all-order expression for E_7

Throughout this thesis, we have always adopted a perturbative approach and one limitation is that one needs to derive results at each new order in the coupling constant. This gets incredibly complicated at higher-orders. Therefore, it is very important to understand how one can arrive at such results and if possible generalize it to all-orders. In this section, we sketch the derivation of all orders result for E_7 symmetry in $d = 4$. One can easily extend these results to higher dimensions, since it is clear from analysis that symmetries in higher dimensions mimic the structure of E_7.

The $SU(8)$ transformations are always linear while the coset transformations $E_{7(7)}/SU(8)$ are given by a series in odd powers of the coupling constant. We find the higher contributions in this series by requiring the commutator of two such generators to produce a $SU(8)$ generator. We demonstrate this specifically for order κ^3 by writing all the relevant contributions

$$\{[\,\boldsymbol{\delta}_1^{\kappa^{-1}}\,,\,\boldsymbol{\delta}_2^{\kappa^3}\,]\,+\,[\,\boldsymbol{\delta}_1^{\kappa^3}\,,\,\boldsymbol{\delta}_2^{\kappa^{-1}}\,]\,+\,[\,\boldsymbol{\delta}_1^{\kappa}\,,\,\boldsymbol{\delta}_2^{\kappa}\,]\}\phi\,=\,0. \tag{7.30}$$

where $\boldsymbol{\delta}$ denotes the coset variation. The third term in the above equation is given by

$$\kappa^2\,(\Xi_1^{klmn}\,\Xi_2^{pqrs}\,-\,\Xi_2^{klmn}\,\Xi_1^{pqrs})\,\phi_{klmnpqrs}, \tag{7.31}$$

where

$$\begin{aligned}\phi_{klmnpqrs} \quad = \quad & \frac{1}{4!}\frac{1}{\partial^{+\,2}}\bigg(\bar{d}_{klmn}\frac{1}{\partial^+}\phi_{pqrs}\,\partial^{+\,3}\phi\,+\,\bar{d}_{klmn}\frac{1}{\partial^+}\phi\,\partial^{+\,3}\phi_{pqrs}\\ & -4\,\bar{d}_{klm}\,\phi_{pqrs}\,\bar{d}_n\,\partial^{+\,2}\phi\,-\,4\,\bar{d}_{klm}\,\phi\,\bar{d}_n\,\partial^{+\,2}\phi_{pqrs}\\ & +3\,\bar{d}_{kl}\,\partial^+\phi_{pqrs}\,\bar{d}_{mn}\,\partial^+\phi\,+\,3\,\bar{d}_{kl}\,\partial^+\phi\,\bar{d}_{mn}\,\partial^+\phi_{pqrs}\bigg).\end{aligned} \tag{7.32}$$

The function ϕ_{pqrs} is chiral and is given by

$$\phi_{pqrs}\,=\,\frac{1}{4!}\frac{1}{\partial^{+\,2}}\left(\bar{d}_{pqrs}\frac{1}{\partial^+}\phi\,\partial^{+\,3}\phi-4\,\bar{d}_{pqr}\,\phi\,\bar{d}_s\,\partial^{+\,2}\phi+3\,\bar{d}_{pq}\,\partial^+\phi\,\bar{d}_{rs}\,\partial^+\phi\right), \tag{7.33}$$

implying that the function $\phi_{pqrsklmn}$ is also chiral. We conjecture the following form at κ^3

$$\boldsymbol{\delta}^{\kappa^3}\phi\,=\,\kappa^3\,\Xi^{klmn}\,\epsilon^{pqrsabcd}\,\phi_{klmnpqrs}\,\bar{d}_{abcd}\phi+\rho(\phi). \tag{7.34}$$

We substitute this expression in (7.30), to find that when the $\boldsymbol{\delta}^{\kappa^{-1}}$ acts on the last superfield of the first term in (7.34), it cancels piece (7.31). Since $\boldsymbol{\delta}^{\kappa^{-1}}\,\phi_{klmnpqrs}\neq 0$, it allows us to fix $\rho(\phi)$ using the equation (7.30). One interesting finding here is that the chiral function that appears at order κ in (4.29) appears again at order κ^3 in (7.34) in a nested form. This suggests that it may be possible to write an all order expression for the E_7 transformations in a compact manner. These transformations relate interaction vertices at different orders in the coupling constant and will therefore play an important role in determining the ultra-violet properties of the theory.

Chapter 8

Conclusion and Outlook

8.1 Summary of key results

In this thesis, we studied pure gravity and maximal supergravity theories in the light-cone gauge, leveraging the power of space-time and exceptional symmetries. The central mathematical and conceptual results derived in this work are summarized below.

- **Non-MHV vertices in pure gravity:**
  We derived the perturbative expansion of the closed-form light-cone gravity action to order κ^4. We established that the first genuinely non-MHV interaction vertices emerge at the six-point level with the helicity structure $h^3\bar{h}^3$:

$$\mathcal{L}_{\kappa^4} = 4\kappa^4 L_6(h^3, \overline{h}^3). \tag{8.1}$$

- **Helicity selection rules to all orders:**
  By demanding the closure of the Poincaré algebra, we derived the helicity selection rules for an arbitrary spin-λ theory. For an n-point interaction vertex of the form $\phi^c\overline{\phi}^d$ (where $c + d = n$), the bounds on the transverse derivatives dictate:

$$\sum_{i=1}^{n-1} \alpha_i = (\lambda - 1)c - d - \lambda + 3 \geq 0, \qquad \sum_{i=1}^{n-1} \beta_i = (\lambda - 1)d - c - \lambda + 3 \geq 0. \tag{8.2}$$

  For pure gravity ($\lambda = 2$), this uniquely fixes the allowed helicity configurations at any perturbative order, proving the absence of MHV vertices beyond the quintic order in the action. An even point vertex $n = 2m$ is always of the form $h^m\,\bar{h}^m$ and an odd point vertex $n = 2m + 1$ is always of the form $h^{m+1}\,\bar{h}^m$ and its complex conjugate.

- **The exceptional bootstrap in four dimensions:**
  We introduced a novel technique to derive interaction vertices in $\mathcal{N} = 8$ supergravity by exploiting the non-linearly realized $E_{7(7)}$ symmetry instead of the traditional dynamical

Poincaré generators. This relates higher-point vertices to lower-point ones via an inhomogeneous differential equation:

$$\delta_{1/\kappa}(\mathcal{V}_4) = -\delta_{\kappa}(\mathcal{V}_2). \tag{8.3}$$

This allowed us to determine the inhomogeneous pieces of the quartic and quintic interaction vertices in a highly compact form. The homogeneous pieces were found by demanding invariance of the vertex under kinematical generators of the Poincaré algebra. This method was also extended to five-dimensions.

- **A universal form for exceptional symmetries ($d \geq 4$):**
  We established that the non-linear coset transformations $K = G/H$ for maximally supersymmetric theories of gravity have a universal form in light-cone superspace for all dimensions $d \geq 4$:

$$\delta_K \phi = -\frac{2}{\kappa}\theta^{klmn}_{\alpha\beta\gamma\delta}\Sigma_{klmn} + \xi(d)\frac{\kappa}{4!}\Sigma^{klmn}\frac{1}{\partial^{+2}}\left(\mathcal{C}\left[\overline{d}^{\alpha\beta\gamma\delta}_{klmn}\frac{1}{\partial^{+}}\phi\partial^{+3}\phi\right]\right) + \mathcal{O}(\kappa^3). \tag{8.4}$$

  This universal form successfully captures the $E_{7(7)}$, $E_{6(6)}$, $SO(5,5)$, and $SL(5,\mathbb{R})$ symmetries, hinting at a unified description of these theories.

8.2 Outlook and future directions

The formalism and results developed in this thesis open several promising avenues for future research in various aspect of quantum field theory and supersymmetry.

- **Color-kinematics duality and off-shell amplitudes:**
  A particularly interesting direction is investigating color-kinematics duality directly within the light-cone action of Yang-Mills theory. Establishing this could provide a systematic way to simplify the perturbation theory of pure gravity via double-copy relations. Because we are dealing with the Lagrangian rather than strictly on-shell scattering amplitudes, this would constitute a powerful off-shell result.

  There have been interesting works in this direction, particularly in the chiral sectors of the theory [75–77]. The color Jacobi identity is crucial for the closure of the Poincaré algebra at the quartic level in the full Yang-Mills theory [78, 79]. It is worth exploring this approach for the study of quartic and higher point interactions in pure gravity using the dual kinematic algebra.

It would also be very interesting to study interaction vertices in curved spaces such as dS/AdS spaces within the light-cone approach [80, 81]. Specifically, it may be important to define 'helicity'-like good variables in these spaces, that produce simpler and compact expressions for interaction vertices, also useful for studying the flat space-limits.

- **All-order symmetry generators and counterterms:**
  Although we have sketched the derivation of the $\mathcal{O}(\kappa^3)$ contributions to the $E_{7(7)}$ generators, finding an exact, closed-form expression to all orders remains an open problem. Such an expression would be extremely powerful in deriving higher-point interaction vertices and ultimately determining an all-order action for maximal supergravity. With this in hand, the next critical step is to analyze Feynman supergraphs and adapt the finiteness proof of $\mathcal{N}=4$ super Yang-Mills to maximal supergravity. This requires a precise understanding of how exceptional symmetries relate different supergraphs to count the degree of divergence. Closely related to this, is the derivation and classification of counterterms in light-cone superspace, specifically to determine whether a valid counterterm exists at seven loops (at which the four-dimensional $\mathcal{N}=8$ supergravity is expected to diverge).

- **Supersymmetry without anti-commuting variables:**
  An alternative approach to simplifying the computation of amplitudes is the Nicolai map, which maps the full non-linear interacting action of a supersymmetric theory into a free bosonic theory [82, 83]. It has been recently demonstrated that a particularly simple Nicolai map exists for Yang-Mills theories in the light-cone gauge [84, 85]. The natural first step is to understand the complete construction of this map for $\mathcal{N}=4$ super Yang-Mills in light-cone superspace. Subsequently, we plan to extend this framework to $\mathcal{N}=8$ supergravity, investigating how exceptional symmetries are realized under the map and also simplify the calculation of supergravity amplitudes.

- **Infinite-dimensional symmetries and $d<3$:**

Descending to lower dimensions, the reduction of maximal supergravity to $d = 2$ and $d = 1$ reveals infinite-dimensional Kac-Moody algebras, such as the affine E_9 and the hyperbolic E_{10}. A comprehensive study of these symmetries using the light-cone approach would be very interesting, beginning with the $E_{8(8)}$ symmetry constructed in three dimensions. A good starting point for this exercise is the reduction of four-dimensional pure gravity to two dimensions to study the Geroch group [86], which is mathematically defined as the affine extension of the $SL(2,\mathbb{R})$ Ehlers symmetry [87] present in three dimensions.

Finally, we make come concluding remarks. The derivation of a universal form for exceptional symmetries in light-cone superspace hints at a natural intersection with frameworks like Exceptional Field Theory (ExFT), where these hidden symmetries are made manifest by extending space-time coordinates. The $\mathcal{N} = 8$ supergravity remains as one of the promising candidate for an ultraviolet-finite, point-particle quantum theory of gravity. If the theory fails the finiteness test, it would be an indication that we must consider theories with extended objects such as string theory and M-theory. It will be fascinating to see if nature makes use of exceptional groups, even if supersymmetry may never be observed.

Appendix

A Scattering amplitudes in higher-derivative theories

We would like to compute amplitudes using the interaction vertices derived in eqn (3.46) of section 3.3. In principle, one can construct the n-point tree-level amplitudes. However, the calculation becomes mathematically tedious as the number of Feynman diagrams and the number of terms in the diagram increases exponentially. For example, to calculate a 4-point tree-level amplitude, we need to sum over contributions from the exchange diagrams and the contact diagram. The contact term may be derived by the closure of Poincaré algebra in special cases which in itself requires a lot of work. We therefore use a complementary technique, the inverse soft method, in the next section to derive higher-point tree-level amplitudes for the higher derivative theories.

Inverse soft construction in higher derivative theories

Soft factors in the light-cone gauge and their use in the construction of higher-point interaction vertices were the focus of [88] (this includes a review of the light-cone realization of many covariant results from [89, 90, 92]). In this section, we explore these methods in the context of the higher-derivative theories considered in this paper.

The idea that higher-point tree level amplitudes can be constructed from the lower-point ones by using a multiplicative universal factor, associated with the emission of a soft boson was first presented in [89] and then subsequently developed in [90, 91]. It was shown in [90], that inverse soft construction with the Britto-Cachazo-Feng-Witten (BCFW) technique [2, 3] as a guide, can be used to construct gauge theory and gravity amplitudes.

The inverse soft recursion relation for a n-point tree-level gauge amplitude is

$$A_n(1, 2, ..., n-1, n) = S(n-1, n, 1)\, A_{n-1}(1', 2.....n-1')\,. \tag{8.5}$$

The prime above indicates that momentum conservation on the right-hand side requires a shift in the momenta of adjacent particles p_{n-1} and p_1. For a positive helicity soft particle n, the shift is [90]

$$
\begin{aligned}
|1'\rangle &= |1\rangle\,, \qquad\qquad |1'] = |1] + |n]\frac{\langle n, n-1\rangle}{\langle 1, n-1\rangle}\,,\\
|n-1'\rangle &= |n-1\rangle\,, \qquad\qquad |n-1'] = |n-1] + |n]\frac{\langle n1\rangle}{\langle n-1, 1\rangle}\,.
\end{aligned}
\tag{8.6}
$$

Here only the neighbouring particles are shifted because only they are affected by the soft limit. For the case of gravity, the soft factor depends on all legs thus inverse soft expression (8.5) will involve sum over all particles.

We will now employ the inverse soft method to construct higher-point amplitudes for theories with higher-dimensional operators. We use the three-point amplitudes found in eqns (3.50) and (3.51) as seed amplitudes to systematically construct different classes of higher-point amplitudes.

F^3 operator

We now consider a gauge invariant higher-dimensional operator involving purely spin-1 fields. The simplest of such operators is 6-dimensional given by

$$
O^{\alpha} = \alpha\, f^{abc} F^{a\,\nu}_{\mu}\, F^{b\,\rho}_{\nu}\, F^{c\,\mu}_{\rho}\,,
\tag{8.7}
$$

where $F^{a\,\nu}_{\mu} = \partial_\mu A^{a\,\nu} - \partial^\nu A^a_\mu + g\, f^{abc} A^b_\mu A^{c\,\nu}$ is the gluon field strength and f^{abc}, the structure constant of the gauge group.

The cubic interaction vertex can be obtained from (3.46) by setting $\lambda_1 = \lambda_2 = \lambda_3 = 1$ so $\lambda = 3$, and the Hamiltonian is

$$
\begin{aligned}
H^{\alpha} \;&=\; \alpha \int d^3x\, f^{abc} \left\{ \frac{1}{\partial^+}\bar{A}^a \frac{\partial^3}{\partial^+}\bar{A}^b \partial^{+\,2}\bar{A}^c - \frac{1}{\partial^+}\bar{A}^a \partial^{+\,2}\bar{A}^b \frac{\partial^3}{\partial^+}\bar{A}^c - 3\frac{1}{\partial^+}\bar{A}^a \partial^2 \bar{A}^b \partial\partial^+ \bar{A}^c \right.\\
&+\; \left. 3\,\frac{1}{\partial^+}\bar{A}^a \partial\partial^+ \bar{A}^b \partial^2 \bar{A}^c \right\} + c.c.\,.
\end{aligned}
\tag{8.8}
$$

We see that the operator (8.7) decomposes into holomorphic and anti-holomorphic parts in the light-cone gauge. This decomposition was first presented in [93, 94]. So the full F^3 amplitude can be constructed by taking the sum of the holomorphic and anti-holomorphic amplitude.

In this theory, holomorphic amplitudes with exactly three negative helicity gluons and an arbitrary number of positive helicity gluons are referred as MHV amplitudes and are denoted by A^{F_+}. The anti-holomorphic amplitudes with exactly three positive helicity gluons and an arbitrary number of negative helicity gluons are referred as anti-MHV and are denoted by A^{F_-}.

It was shown in [95, 96] that for the higher derivative theory of massless particles in four dimensions, the tree-level soft photon and graviton theorems receive modifications at subleading and subsubleading orders. However, the leading soft factors are not altered for these theories because these interactions (F^3, R^3) are generically suppressed in the soft-limit [96]. Therefore, the inverse soft recursion relation for this operator is similar to the Yang-Mills case [90]

$$A_{n+1}(\phi, 1, 2, \ldots\ldots n) = S_{YM} \times A_n(\phi, 1', 2, \ldots\ldots n-1')\,, \tag{8.9}$$

where

$$S_{YM} = \frac{\langle n-1, 1\rangle}{\langle n-1, n\rangle\langle n1\rangle}\,, \tag{8.10}$$

is the soft factor for the Yang-Mills theory.
The three-point amplitudes extracted from (8.8) are

$$A_3^{F^3}(p^-,\, k^-,\, l^-) \;\equiv\; A_3^{F_+}(p^-,\, k^-,\, l^-) = \langle kl\rangle\langle lp\rangle\langle pk\rangle\,,$$

$$A_3^{F^3}(p^+,\, k^+,\, l^+) \;\equiv\; A_3^{F_-}(p^+,\, k^+,\, l^+) = [kl][lp][pk]\,. \tag{8.11}$$

A four-point tree-level MHV amplitude corresponding to a single insertion of the F^3 operator can be constructed using this as follows (we attach a soft gluon between the adjacent legs 1 and

3).

$$\begin{aligned} A_4^{F^3}(1^-,2^-,3^-,4^+) &= A_3^{F_+}(1^-,2^-,3^-)\times S_{YM}\,, \\ &= \langle 12\rangle\langle 23\rangle\langle 31\rangle \times \frac{\langle 31\rangle}{\langle 34\rangle\langle 41\rangle}\,, \\ &= \frac{\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,. \end{aligned} \tag{8.12}$$

The above construction is valid because under the following shift $[1^-,4^+\rangle$, the amplitude has no pole at infinity. So the inverse soft construction is valid for this class of amplitudes. The 5-point MHV amplitude can also be constructed similarly. At the 6-point level, the full F^3 amplitude will get contributions from both holomorphic and anti-holomorphic parts

$$\begin{aligned} A_6^{F^3}(1^-,2^-,3^-,4^+,5^+,6^+) &= A_6^{F_+}(1^-,2^-,3^-,4^+,5^+,6^+) + A_6^{F_-}(1^-,2^-,3^-,4^+,5^+,6^+)\,, \\ &= \frac{\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle} + \frac{[45]^2[56]^2[64]^2}{[12][23][34][45][56][61]}\,. \end{aligned} \tag{8.13}$$

Beyond this order, the calculation for the complete F^3 amplitude becomes complicated since not *all* relevant seed amplitudes are constructible within this formalism.

To any order, the MHV holomorphic amplitudes may be constructed recursively - within this formalism - by attaching soft gluon legs to lower point amplitudes.

$$A_n^{F_+}(1^-,2^-,3^-,4^+,......,n^+) = \frac{\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle.........\langle n1\rangle}\,. \tag{8.14}$$

Note that for $n \leq 6$, this is *also* the full MHV amplitude. Similarly $A_n^{F_-}(1^+,2^+,3^+,4^-,......,n^-)$ can be obtained from (8.14) by parity. The above expression matches with the results in [93**?**] obtained using Feynman rules.

Another class of amplitudes that may be built in this formalism are the NMHV amplitudes. For these, we start with the seed amplitude $A_4^{F_+}(1^-,2^-,3^-,4^-)$ [94, 97]

$$A_4^{F_+}(1^-,2^-,3^-,4^-) = 2\frac{stu}{[12][23][34][41]}\,, \tag{8.15}$$

where s, t, u are the usual Mandelstam variables. For example, we can compute the 5-point amplitude

$$\begin{aligned} A_5^{F+}(1^-, 2^-, 3^-, 4^-, 5^+) &= S(5,1,2)A_4(2'^-, 3^-, 4^-, 5'^+) + S(3,4,5)A_4(1^-, 2^-, 3'^-, 5'^+) \\ &\quad + S(4,5,1)A_4(1'^-, 2^-, 3^-, 4'^-)\,, \\ &= \frac{\langle 34\rangle^2[35]^2[45]}{[12][23][34][15]} + \frac{\langle 12\rangle^2[25]^2[15]}{[12][23][34][45]} + 2\frac{\langle 14\rangle^2\langle 13\rangle\langle 24\rangle}{[23]\langle 45\rangle\langle 15\rangle}\,. \end{aligned} \tag{8.16}$$

This matches with the result in [94]. As pointed out in [90] this technique will only work up to $n \leq 8$ for gauge theory and for some classes of NMHV amplitudes with arbitrary number of legs.

R^3 operator

We now consider a higher-dimensional operator in the case of gravity denoted by R^3 . This operator also decomposes into holomorphic and anti-holomorphic part as explained before. In this section, we focus only on the holomorphic amplitude denoted by M^{R_+}. The construction for anti-holomorphic part follows similarly. The full amplitude is again a sum of contributions from both holomophic and anti-holomorphic parts. We start with (3.49) and set $\lambda_1 = \lambda_2 = \lambda_3 = 2$, so the three-point holomorphic amplitude is

$$M_3^{R+}(p^-,\, k^-,\, l^-) = \langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2\,. \tag{8.17}$$

The inverse soft recursion relation for gravity, for a n-point amplitude, reads [90]

$$M_{MHV}(1, \ldots\ldots, n^+) = \sum_{i=3}^{n-1} \frac{[in]\langle 1i\rangle\langle 2i\rangle}{\langle ni\rangle\langle 1n\rangle\langle 2n\rangle} M_{MHV}(1', \ldots., i', \ldots., (n-1))\,. \tag{8.18}$$

The prime indicates the momentum shift, mentioned earlier. For a positive helicity soft graviton k, the shift is

$$\begin{aligned} |i'\rangle &= |i\rangle\,, \qquad & |i'] &= |i] + |k]\frac{\langle kj\rangle}{\langle ij\rangle}\,, \\ |j'\rangle &= |j\rangle\,, \qquad & |j'] &= |j] + |k]\frac{\langle ki\rangle}{\langle ji\rangle}\,. \end{aligned} \tag{8.19}$$

Using the inverse soft method, the four-point tree-level amplitude is

$$\begin{aligned} M_4^{R+}(1^-,2^-,3^-,4^+) &= \langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2 \times \frac{[34]\langle 13\rangle\langle 23\rangle}{\langle 43\rangle\langle 14\rangle\langle 24\rangle}\,, \\ &= \frac{\langle 12\rangle^3\langle 23\rangle^3\langle 31\rangle^3\,[12]}{\langle 34\rangle^2\langle 14\rangle\langle 24\rangle}\,. \end{aligned} \tag{8.20}$$

For the case of gravity and R^3 theories, the higher-point amplitudes are functions of both the holomorphic and anti-holomorphic spinors that makes the construction of amplitudes using the inverse soft approach more involved. We show below an example of the construction of the five-point amplitude in the R^3 theory.

$$\begin{aligned} M_5^{R+}(1^-,2^-,3^-,4^+,5^+) &= \sum_{i=3}^{4}\frac{[i5]\langle 1i\rangle\langle 2i\rangle}{\langle 5i\rangle\langle 15\rangle\langle 25\rangle}M_4^{R+}(1'^-,2^-,...,i')\,, \\ &= \frac{[35]\langle 13\rangle\langle 23\rangle}{\langle 53\rangle\langle 15\rangle\langle 25\rangle}M_4^{R+}(1'^-,2^-,3'^-,4^+) \\ &\quad +\frac{[45]\langle 14\rangle\langle 24\rangle}{\langle 54\rangle\langle 15\rangle\langle 25\rangle}M_4^{R+}(1'^-,2^-,3^-,4'^+)\,. \end{aligned}$$

Using the appropriate shifts, the amplitude reads

$$\begin{aligned} M_5^{R+}(1^-,2^-,3^-,4^+,5^+) &= \frac{\langle 12\rangle^4\langle 23\rangle^4\langle 31\rangle^4}{\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 34\rangle\langle 15\rangle\langle 25\rangle}\left[\frac{[35][24]}{\langle 35\rangle\langle 24\rangle}-\frac{[45][23]}{\langle 45\rangle\langle 23\rangle}\right]\,, \\ &= \frac{\langle 12\rangle^4\langle 23\rangle^4\langle 31\rangle^4}{\langle 12\rangle^8}\,M_5^{R}(1^-,2^-,3^+,4^+,5^+)\,, \end{aligned} \tag{8.21}$$

where M_5^R is the five-point amplitude for pure gravity (the proportionality to gravity ensures 'constructibility'). The n-point holomorphic MHV amplitude for R^3 operator can be written as

$$M_n^{R+}(1^-,2^-,3^-,4^+,.......,n^+) = \frac{\langle 12\rangle^4\langle 23\rangle^4\langle 31\rangle^4}{\langle 12\rangle^8}\,M_n^{R}(1^-,2^-,3^+,.......,n^+)\,. \tag{8.22}$$

The anti-holomorphic part corresponding to R^3 operator can be similarly obtained. This is consistent with the results derived in [94] using color-kinematic duality.

B Magic identities in light-cone superspace

In this appendix, we provide a proof of some magic identities in light-cone superspace.

1. The original $\mathcal{N}=8$ supergravity three-point vertex derived in [59] was simplified in [64]. We begin with the relevant terms from the Lagrangian

$$\int \left(\frac{1}{\partial^{+4}} \bar{\phi} \bar{\partial}^2 \phi \partial^{+2} \phi - \frac{1}{\partial^{+4}} \bar{\phi} \bar{\partial} \partial^+ \phi \bar{\partial} \partial^+ \phi \right) \tag{8.23}$$

First, we partially integrate the first term with respect to $\bar{\partial}$. We then use the inside-out relation $\phi = \frac{1}{2 \cdot 8!} (d)^8 \frac{1}{\partial^{+4}} \bar{\phi}$ to convert the anti-chiral superfields into chiral one and also perform a partial integration wrt ∂^+ to obtain:

$$\int \left(-\frac{1}{\partial^{+2}} \bar{\phi} \bar{\partial} \phi \bar{\partial} \phi + \frac{1}{\partial^{+3}} \bar{\phi} \bar{\partial} \phi \bar{\partial} \partial^+ \phi + \frac{1}{\partial^{+4}} \bar{\phi} \bar{\partial} \partial^+ \phi \bar{\partial} \partial^+ \phi \right) \tag{8.24}$$

Observe that the third term in this expansion exactly cancels the second term in our original expression. This cancellation leaves us with only two terms

$$\int \left(-\frac{1}{\partial^{+2}} \bar{\phi} \bar{\partial} \phi \bar{\partial} \phi + \frac{1}{\partial^{+3}} \bar{\phi} \bar{\partial} \phi \bar{\partial} \partial^+ \phi \right) \tag{8.25}$$

To evaluate the second term, we define it as an integral I and partially integrate the single ∂^+ in the numerator. This yields

$$I = \int \frac{1}{\partial^{+3}} \bar{\phi} \bar{\partial} \phi \bar{\partial} \partial^+ \phi = -\int \frac{1}{\partial^{+2}} \bar{\phi} \bar{\partial} \phi \bar{\partial} \phi - I \tag{8.26}$$

$$\implies I = -\frac{1}{2} \int \frac{1}{\partial^{+2}} \bar{\phi} \bar{\partial} \phi \bar{\partial} \phi \tag{8.27}$$

Substituting this result back, the entire vertex collapses into a single term

$$-\frac{3}{2} \int \frac{1}{\partial^{+2}} \bar{\phi} \bar{\partial} \phi \bar{\partial} \phi \tag{8.28}$$

2. We first state the identity

$$\int \frac{1}{\partial^{+3}} \bar{\phi} \bar{\partial} \phi \partial^m \partial^+ \phi = \int \frac{1}{\partial^{+3}} \bar{\phi} \partial^m \phi \bar{\partial} \partial^+ \phi \tag{8.29}$$

We begin by partially integrating $\overline{\partial}$ (and subsequently applying partial integration wrt ∂^m for the resulting second term) to expand the expression into three parts

$$\int \frac{1}{\partial^{+3}}\overline{\phi}\bar{\partial}\phi\partial^m\partial^+\phi = \int\left(-\frac{\overline{\partial}}{\partial^{+3}}\overline{\phi}\phi\partial^m\partial^+\phi + \frac{1}{\partial^{+3}}\overline{\phi}\partial^m\phi\overline{\partial}\partial^+\phi + \frac{\partial^m}{\partial^{+3}}\overline{\phi}\phi\overline{\partial}\partial^+\phi\right) \tag{8.30}$$

Next, we use the inside-out relations in the first term to convert the first $\overline{\phi}$ into ϕ and the third ϕ into $\overline{\phi}$. This reveals that the first and third terms perfectly cancel each other out, to obtain

$$\int \frac{1}{\partial^{+3}}\overline{\phi}\bar{\partial}\phi\partial^m\partial^+\phi = \int \frac{1}{\partial^{+3}}\overline{\phi}\partial^m\phi\overline{\partial}\partial^+\phi \tag{8.31}$$

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