Quantum Metrology in the Heisenberg Limit

And maximum possible resolutions

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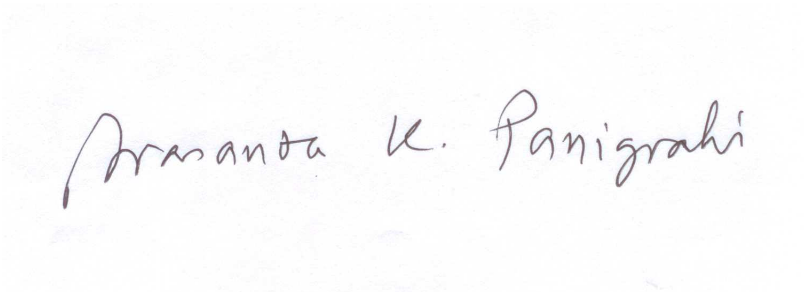
Certificate

This is to certify that this dissertation entitled “Quantum Metrology in the Heisenberg Limit ”towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune, represents original research carried out by ‘Abhijeet Kumar ’at the Indian Institute of Science Education and Research-Kolkata under my supervision during the academic year 2010-2011.



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Acknowledgements

Starting from the most elementary and fundamental definitions and navigating through something as young a field as quantum information, an indispensable need is for a persevere guiding force. Not only this has to be thorough, precise and cooperating, it should have the willingness to take us out of every trouble that we face, be it directly in our work or in the resources that we need.

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Abstract

This study is aimed at quantum measurements with Heisenberg limited sensitivities. The high-precision measurements of length and angle variables have led to the development of many novel techniques, both on theoretical and experimental fronts. Squeezed states which have smaller uncertainties in some variables have long been employed for this use. In many cases, parametric variations translate into the phase variations, which are measured by interferometric means.

It has been demonstrated by Zurek that interference effects in the phase space can profitably be used for the measurements of variables, which were otherwise limited by Heisenberg uncertainty principle. For this purpose certain superposed states,viz. Cat and kitten states have been employed. This way of achieving better sensitivity in parameter estimation have experimentally been verified as well, using Cat-like laser beams and has been termed as sub-Fourier sensitivity.In this work, these branches of quantum metrology have been greatly emphasized. The role of entangled states in quantum metrology has also been explored starting from characterization of entanglement to improving sensitivities in more dimensions through an entangled state comprising superposed cat-like states.

Lastly, the concept of maximum achievable resolution, is explored through a statistical Cramer-Rao bound. This bound has been calculated for several states and it has been found that pair coherent states have a robust nature with regards to interferometry than many of the commonly used states in an interferometer.

Contents

Chapter 1

Some Quantum optics

Theme

This chapter introduces the basic quantum optical framework needed for the work that has been done in the subsequent chapters. It starts with the definition of coherent and squeezed states highlighting the uncertainties of observables. Then, it proceeds to a brief characterization of entanglement and finally it ends with a description of phase. It also forms the basis of some calculations that has been done in chapter two and three.

1.1 Coherent states

The coherent states of the radiation field are the states generated by a classically oscillating current distribution. Hence, they are represented as

$$|\alpha\rangle = e^{(\alpha a^\dagger - \alpha^* a)}|0\rangle \tag{1.1}$$

. They can also be defined as the eigenstates of the annihilation operator $a|\alpha\rangle = \alpha|\alpha\rangle$. In terms of the number states, this can be written as

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{1.2}$$

As seen above, a coherent state is obtained by applying the displacement opertaor$D(\alpha) = e^{(\alpha a^\dagger - \alpha^* a)}$ on the vaccum state. Hence, it will be a displaced form of the harmonic oscillator ground state.
The annihilation and creation operaors for a field are

$$a = \frac{1}{\sqrt{2\hbar\nu}}\left(\nu q + \hbar\frac{\partial}{\partial q}\right), \qquad a^\dagger = \frac{1}{\sqrt{2\hbar\nu}}\left(\nu q - \hbar\frac{\partial}{\partial q}\right) \tag{1.3}$$

Since $a|0\rangle = 0$, this equation reduces to

$$\left(\nu q + \hbar\frac{\partial}{\partial q}\right)\phi_0(q) = 0 \tag{1.4}$$

A normalized solution to this equation is given by

$$\phi_0(q) = \left(\frac{\nu}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{\nu q^2}{2\hbar}} \tag{1.5}$$

Higher order eigenfunctions are represented by

$$\phi_n(q) = \frac{1}{(2^n n!)^{1/2}} H_n \left(\sqrt{\frac{\nu}{\hbar}} q\right) \phi_0(q) \tag{1.6}$$

where H_n are the Hermite polynomials. The uncertainties in generalized coordinate and momentum variables are given by

$$\begin{aligned} \Delta p^2 &= \hbar\nu \left(n + \frac{1}{2}\right) \\ \Delta q^2 &= \frac{\hbar}{\nu} \left(n + \frac{1}{2}\right) \end{aligned} \tag{1.7}$$

The product in their uncertainties are given by

$$\Delta p \Delta q = \left(n + \frac{1}{2}\right) \hbar \tag{1.8}$$

From here, it can be seen that for the ground state wave function, this has the minimum possible value of $\frac{\hbar}{2}$. Now, a wave packet which maintains the same variance Δq while undergoing a simple harmonic motion would correspond most closely to a classical field. Hence, writing a displaced ground state of a harmonic oscillator,

$$\psi(q, 0) = \left(\frac{\nu}{\pi\hbar}\right)^{1/4} e^{\left[-\frac{\nu}{2\hbar}(q-q_0)^2\right]} \tag{1.9}$$

The time evolution of this wave packet would mean that the probability density at a later time is

$$|\psi(q, t)|^2 = \left(\frac{\nu}{\pi\hbar}\right)^{1/2} e^{\left[-\frac{\nu}{\hbar}(q-q_0 cos\nu t)^2\right]} \tag{1.10}$$

It can be seen from here that the wave packet does not change its shape while oscillating. Due to this coherence in its shape it is called a coherent wave packet. This state has the minimum product of uncertianties, which can be separately checked as well. For a coherent state, $\Delta p \Delta q = \frac{\hbar}{2}$. Coherent states are the closest analogue to a free classical, single-mode field.

1.2 Squeezed states

Now that we have understood coherent states as the minimum uncertainty states, it is certainly easy to introduce squeezed states through the uncertainty relation. Consider two Hermitian operator A and B satisfying the commutation relation

$$[A, B] = iC \tag{1.11}$$

Heisenberg uncertainty principle states that the product of uncertainties in these two observables

$$\Delta A \Delta B \geq \frac{1}{2}|\langle C\rangle| \tag{1.12}$$

A state of the system will be called squeezed state, if the uncertainties in one of the observables satisfy the relation

$$(\Delta A)^2 < \frac{1}{2}|\langle C\rangle| \tag{1.13}$$

In addition to this condition, if the two variances satisfy the minimum uncertainty relation,

$$\Delta A \Delta B = \frac{1}{2}|\langle C\rangle| \tag{1.14}$$

then the state would be called an ideal squeezed state. It means that in a squeezed state, the uncertainty in one variable is reduced at the expense of an incrrease in the uncertainty in another.
The squeezed states of the radiation field have a deep connection with the degenerate parametric processes. A two photon hamiltonian can be written as

$$\mathcal{H} = i\hbar(ga^{\dagger 2} - g^* a^2) \tag{1.15}$$

where g is a coupling constant. The state of the field generated will be

$$|\psi(t)\rangle = e^{(ga^{\dagger 2} - g * a^2)t}|0\rangle \tag{1.16}$$

It leads us to the definition of a unitary squeeze operator

$$S(\xi) = e^{(\frac{1}{2}\xi^* a^2 - \frac{1}{2}\xi a^{\dagger 2})} \tag{1.17}$$

where $\xi = re^{i\theta}$ is an arbitrary complex number. We find that

$$S^\dagger(\xi) = S^{-1}(\xi) = S(-\xi) \tag{1.18}$$

It is important to note the unitary transformation properties of the squeeze operator

$$\begin{aligned} S^\dagger(\xi) a S(\xi) &= a\cosh r - a^\dagger e^{i\theta}\sinh r \\ S^\dagger(\xi) a^\dagger S(\xi) &= a^\dagger \cosh r - a e^{-i\theta}\sinh r \end{aligned} \tag{1.19}$$

A squeezed coherent state is a canonical example of a coherent state. It is obtained by first operating the displacement operator on the vacuum state and later acting it with a squeezing operator.

$$|\alpha, \xi\rangle = S(\xi) D(\alpha)|0\rangle \tag{1.20}$$

The difference between this state and the standard coherent state should be noted here, as a coherent state is generated by linear terms a and $a^\dagger$ where as this state has quadratic terms involving a and $a^\dagger$. By making use of transformation properties of displacement and squeezing operators, the operator expectation values of the squeezed coherrent state can be found out. It turns out that the variances of rotated amplitudes Y_1 and Y_2 where $Y_1 + iY_2 = ae^{-i\frac{\theta}{2}}$ can be found out as

$$\Delta(Y_1)^2 = \langle Y_1^2 \rangle - \langle Y_1 \rangle^2 = \frac{1}{4}e^{-2r} \tag{1.21}$$

and

$$\Delta(Y_2)^2 = \frac{1}{4}e^{2r} \tag{1.22}$$

such that $\Delta Y_1 \Delta Y_2 = \frac{1}{4}$ stressing that a squeezed coherent state is indeed an ideal squeezed state. The degree of squeezing is determined by $r = |\xi|$ which is called the squeezing parameter.

1.3 Entanglement characteristics

Quantum entanglement plays a fundamental role in almost all branches of quantum information. Entanglement in several quantum states is used as a resource in many of the protocols for quantum communication and computaion. Hence, the understanding of entanglement characteristics assumes a central role in quantum information theory. This issue was addressed for the first time by Peres[3], in terms of the negative eigenvalues of the partial transpose of the composite density operator as a measure of entanglement. Later, Horodecki showed that Peres's inseparability criterion was a necessary and sufficient condition for inseperability of 2×2 and 2×3 dimensional states but it ceased to be a necessary condition for higher dimensional states. Duan et.al had proposed an inseparability condition based on the variances of EPR operators. They found a lower bound for separable states based on Heisenberg's uncertainty condition. This bound was said to be exceeded by entangled states, which meant that this provided a sufficient condition for testing entanglement for a continuous variable state. Later in the paper, they also show that for Gaussian states, this reduces to be the necessary and sufficient condition. Considering the EPR-like operators

$$\begin{aligned} \hat{u} &= |a|\hat{x}_1 + \frac{1}{a}\hat{x}_2 \\ \hat{v} &= |a|\hat{p}_1 - \frac{1}{a}\hat{p}_2 \end{aligned} \tag{1.23}$$

To prove the sufficient condition for inseparability, it is needed to be shown that for any separable quantum state characterized by a composite density operator ρ, the sum of the variances of the operators defined above satisfies

$$\langle(\Delta\hat{u})^2\rangle_\rho + \langle(\Delta\hat{v})^2\rangle_\rho \geq a^2 + \frac{1}{a^2} \tag{1.24}$$

The proof for this inequality follows from the calculation of the variance of these operators and then using Cauchy-Schwarz inequality. Then, by means of two lemma

they prove that this is indeed the necessary and sufficient condition for inseparability of a Gaussian state.
However, in the context of quantum information, non-Gaussian states hold the same importance as Gaussian states. Hence, characterization of entanglement for such states is an equally important problem. Agarwal[5] demonstrated the limitations of existing inseparability criteria based on second order correlations. Based on Peres-Horodecki condition, he derived a new set of inseperability inequalities, involving higher order correlations of quadrature variables. Considering the following continuous variable Bell state

$$\psi(x_a, xb) = \sqrt{\frac{2}{\pi}}(\alpha x_a + \beta x_b)e^{\frac{-x_a^2+x_b^2}{2}} \tag{1.25}$$

Here $|\alpha^2| + |\beta|^2 = 1$. It can be seen that this is a non-Gaussian state in coordinate space. The Peres-Horodecki condition states that for this state to be inseperable,the partial transpose of its density matrix must have at least one negative eigenvalue. It is shown that the partial transpose of the density matrix

$$\rho^{PT} = |\alpha|^2|1,0\rangle\langle 1,0| + |\beta|^2|0,1\rangle\langle 0,1| + \alpha^*\beta|0,0\rangle\langle 1,1| + H.c... \tag{1.26}$$

has the eigenvalues $|\alpha|^2$, $|\beta|^2$, $\pm|\alpha||\beta|$. The negative eigenvalue shows that the given state is inseperable or entangled. However, the sum of variances

$$\langle(\Delta\hat{u})^2\rangle_\rho + \langle(\Delta\hat{v})^2\rangle_\rho = |a|^2 + \frac{1}{a^2} + 2\left(|\alpha|^2|m|^2 + \frac{1}{m^2}|\beta|^2\right) \tag{1.27}$$

is clearly greater than $|a|^2 + \frac{1}{a^2}$. Hence it should have been a separable state. Therefore, the inequalities based on second order variances are not able to detect entanglement in this case. Then inequalities based on higher order correlations are derived starting from the Peres- Horodecki condition for inseperability, which finally characterize entanglement in this non-Gaussian state. In second chapter of this report as well, we report our state to violate this inequalty on the same grounds.

1.4 The quantum phase

The concept of phase in quantum mechanics is an intriguing topic and needs a detailed formalism to be understood as an observable. The uncertainties in phase measurement forms the crux of this work and hence having a sound understanding, in this regard is needed. Here, the formalism is briefly illustrated at an introductory level. The electric field, in a single-mode plane wave is represented by[2]

$$\hat{E}(\mathbf{r},t) = i\left(\frac{\hbar\omega}{2\epsilon V}\right)\mathbf{e_x}\left[\hat{a}e^{i(k.\mathbf{r}-\omega t)} - \hat{a}^\dagger e^{-i(k.\mathbf{r}-\omega t)}\right] \tag{1.28}$$

Considering a light wave as being described in classical electromagnetic theory,

$$\begin{aligned}\hat{E}(\mathbf{r},t) &= \mathbf{e_x}E_0 cos(k.\mathbf{r}-\omega t+\Phi) \\ &= \mathbf{e_x}\frac{E_0}{2}\left[e^{i(k.\mathbf{r}-\omega t+\Phi)} + e^{-i(k.\mathbf{r}-\omega t+\Phi)}\right]\end{aligned} \tag{1.29}$$

From these two relations, one may think to write $\hat{a}$ in the polar form. Dirac[6] was the first to write the annihilation and creation operators as

$$\begin{aligned} \hat{a} &= e^{i\hat{\phi}}\sqrt{\hat{n}} \\ \hat{a}^\dagger &= \sqrt{\hat{n}}e^{-i\hat{\phi}} \end{aligned} \tag{1.30}$$

Here, $\hat{\phi}$ is thought to be a hermitian operator for phase. The commutaion relation

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{1.31}$$

gives

$$e^{i\hat{\phi}}\hat{n}e^{-i\hat{\phi}} - \hat{n} = 1 \tag{1.32}$$

or

$$e^{i\hat{\phi}}\hat{n} - \hat{n}e^{i\hat{\phi}} = e^{i\hat{\phi}} \tag{1.33}$$

Expanding the exponentials in above equation, it can be seen that it means

$$[\hat{n}, \hat{\phi}] = i \tag{1.34}$$

The commutation relation means that Heisenberg uncertainty principle would lead us to $\Delta\hat{\phi}\Delta\hat{n} \geq \frac{1}{2}$. However, this decomposition and the following conclusion are not correct . The Dirac approach fails because of the assumption that $\hat{\phi}$ can be interpreted as a hermitian operator. If that were so, then $e^{i\phi}$ would have been a unitary operator. It can be checked through a straightforward calculation,

$$e^{i\hat{\phi}}e^{-i\hat{\phi}} = \hat{a}\hat{n}^{\frac{-1}{2}}\hat{n}^{\frac{-1}{2}}\hat{a}^\dagger = \frac{\hat{a}\hat{a}^\dagger}{\hat{n}} \neq 1 \tag{1.35}$$

that $e^{i\hat{\phi}}$ is not a unitary operator and thereby $\hat{\phi}$ is not hermitian. The main cause of this problem is the restriction on $\hat{n}$, the number operator to have only positive integers as its eigenstates. Hence a suitable extension of number operators, so as to include negative integers as well can help us get rid of this anomaly.
Another problem with this formalism is the belief that $\hat{\phi}$ is an angle operator. This is also circumvented in an elegant manner[7] by introducing a periodic coordinate $\Phi(\phi)$ behaving in a dicontinuous manner. There have been numerous attempts to create a formalism which can overcome these shortcomings.

The Susskind-Glogower[8] approach is fundamental in this regard and has undergone many improvements over the years. The Susskind-Glogower(SG) operators are defined as

$$\hat{E} = (\hat{n}+1)^{-\frac{1}{2}}\hat{a} = (\hat{a}\hat{a}^\dagger)^{-\frac{1}{2}}\hat{a}$$

$$\hat{E}^\dagger = \hat{a}^\dagger(n+1)^{-\frac{1}{2}} = \hat{a}^\dagger(\hat{a}\hat{a}^\dagger)^{-\frac{1}{2}}$$

These are called exponential operators due to their analogy with phase factors $e^{\pm i\phi}$. Applied on a Fock state

$$\begin{aligned} \hat{E}|n\rangle &= |n-1\rangle \qquad n \neq 0 \\ &= 0 \end{aligned} \tag{1.36}$$

for $n = 0$.

$$\hat{E}^{\dagger}|n\rangle = |n+1\rangle \tag{1.37}$$

. These two operators can also be expresssed as

$$\hat{E} = \sum_{n=0}^{n=\infty} |n\rangle\langle n+1| \qquad \hat{E}^{\dagger} = \sum_{n=0}^{n=\infty} |n+1\rangle\langle n| \tag{1.38}$$

. It can be shown that

$$\hat{E}\hat{E}^{\dagger} = \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} |n\rangle\langle n+1|n^{'}+1\rangle|n^{'}\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n| = 1 \tag{1.39}$$

However,

$$\hat{E}^{\dagger}\hat{E} = \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} |n+1\rangle\langle n|n^{'}\rangle|n^{'}+1\rangle = \sum_{n=0}^{\infty} |n+1\rangle\langle n+1| = 1 - |0\rangle\langle 0| \tag{1.40}$$

means that $\hat{E}$ is **not** an unitary operator. Also, it should be noted that E and $E^{\dagger}$ are not observables, but operators. Hence, from these two operaors two Hermitian operators are constructed.

$$\hat{C} = \frac{1}{2}(E + E^{\dagger}) \qquad \hat{S} = \frac{1}{2i}(E - E^{\dagger}) \tag{1.41}$$

Their commutation relations

$$\begin{aligned}
[C, S] &= \frac{1}{2}i|0\rangle\langle 0| \\
\left[\hat{C}, \hat{n}\right] &= i\hat{S} \\
\left[\hat{S}, \hat{n}\right] &= -i\hat{C}
\end{aligned} \tag{1.42}$$

leads to uncertainty relations

$$(\Delta n)(\Delta C) \geq \frac{1}{2}|\langle\hat{S}\rangle| \tag{1.43}$$

$$(\Delta n)(\Delta S) \geq \frac{1}{2}|\langle\hat{C}\rangle| \tag{1.44}$$

It should be observed from here that in the case of number states, Δn would be zero. Therefore, $|\langle\hat{S}\rangle| = |\langle\hat{C}\rangle| = 0$. For $n \geq 1$, the uncertianties in $\hat{C}$ and $\hat{S}$ is calculated to be

$$\Delta\hat{C} = \Delta\hat{S} = \frac{1}{\sqrt{2}}. \tag{1.45}$$

The eigenstates $|\phi\rangle$ satisfying

$$\hat{E}|\phi\rangle = e^{i\phi}|\phi\rangle \tag{1.46}$$

are given by

$$|\phi\rangle = \sum_{n=0}^{\infty} e^{in\phi}|n\rangle \tag{1.47}$$

The scalar product $\langle\phi_1|\phi_2\rangle$ does not result in a delta function which suffices that these states are not orthogonal. However,

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi|\phi\rangle\langle\phi| = 1 \tag{1.48}$$

as

$$\int_0^{2\pi} e^{i(n-m)\phi}d\phi = 2\pi\delta_{nm} \tag{1.49}$$

The phase distribution $P(\phi)$ of an arbitrary phase state can accordingly be defined as

$$P(\phi) = \frac{1}{2\pi}|\langle\phi|\psi\rangle|^2 \tag{1.50}$$

Writing $|\psi\rangle = \sum_{n=0}^{\infty} C_n|n\rangle$, where $\sum_{n=0}^{\infty}|C_n|^2$ it reduces to be

$$P(\phi) = \frac{1}{2\pi}|\sum_{n=0}^{\infty} e^{-in\phi}C_n|^2 \tag{1.51}$$

The uncertainty in phase measurements holds a significant value for this work, due to which the concept of phase as an eigenstate is illustrated in detail. Here, the uncertainty can be evaluated simply by calculating the expectaion value of the phase using the given distribution function as

$$\Delta\phi = \sqrt{\langle\hat{\phi}^2\rangle - \langle\hat{\phi}\rangle^2} \tag{1.52}$$

where

$$\langle f(\phi)\rangle = \int_0^{2\pi} f(\phi)P(\phi)d\phi \tag{1.53}$$

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Chapter 2

Heisenberg Limited Metrology with Pair Coherent states

Theme

The efficacy of Cat state in Heisenberg limited measurements is explored,in close contact with available experimental setups. Subsequently, it is shown that a specific entangled Cat state can substantially improve the accuracy of the desired quantum metrology.The nature of the entanglement is quantified after which we demonstrate the possibility of physically realizing this state in dissipative systems.

2.1 Introduction

Historically, the partition function, in classical statistical mechanics was first to showcase a constant which had the dimensions of angular momentum. This was expressed to be

$$Z = \int \frac{dp\ dq}{h} e^{-\frac{E}{k_B T}}, \tag{2.1}$$

It was identified to be dimensionally equivalent to $dp\ dq$. Hence, it was related to the area in the phase space. However, Heisenberg uncertainty principle, in terms of uncertainties associated with non-commuting observables introduced a question whether or not, structures in phase space having area less $\frac{\hbar}{2}$ are actually possible Coherent states, characterizing lasers are well known to be classical [9, 10]. However, the superposition of multiple coherent states can show counter-intuitive behavior [11, 12]. It has been recently shown by Zurek that Cat and kitten states given respectively by,

$$\mid \psi\rangle = \frac{\mid \alpha\rangle + \mid -\alpha\rangle}{\sqrt{2}} \quad \text{and} \quad \mid \phi\rangle = \frac{\mid \alpha\rangle + \mid -\alpha\rangle + \mid i\alpha\rangle + \mid -i\alpha\rangle}{2}, \tag{2.2}$$

reveal sub-planck structures, having immense potential for problems in high-precision measurements. Here, it has been shown that subtle changes in the carrier frequencies and other parameters, for example phase, which has recently been a hot topic in quantum metrology due to its applications in gravitational wave detection can be measured, with Heisenberg limited accuracies. The experimental realization of the above proposal was carried in a recent experiment[15] involving superposed laser beams given by

$$\psi(t) = \left(e^{-\frac{(t-t_0)^2}{4\sigma^2}} + e^{-\frac{(t+t_0)^2}{4\sigma^2}}\right) e^{-i\omega_c t} \text{ and } \psi_\delta(t) = \left(e^{-\frac{(t-t_0)^2}{4\sigma^2}} + e^{-\frac{(t+t_0)^2}{4\sigma^2}}\right) e^{-i\omega_c x}\ e^{i\delta t} \tag{2.3}$$

Now, their overlap function being

$$\langle\psi|\psi_\delta\rangle = \int_{-\infty}^{+\infty} \psi^*(t)\psi_\delta(t)dt = 2\sigma\sqrt{2\pi}e^{-\left(\frac{\sigma^2\delta^2}{2}\right)}\left(\cos(\delta t_0) + e^{-\frac{t_0^2}{2\sigma^2}}\right) \tag{2.4}$$

suggests that we can have an orthogonal state corresponding to the original superposition, for appropriate displacements in carrier frequencies,

This experiment had originally been done in the time-frequency domain with laser beams being used as Gaussian profiles. The separation of two such pulses i.e., $2t_0$ had been taken as 305 fs and 309 fs in two different experiments The measured uncertainty in time i.e., Δt, was found to be 20.1 ± 0.5 fs and hence the minimum value for $\Delta\nu$ would be restricted to 4.0 ± 0.1 THz. Now in accordance with the above equation, for sufficiently large values of the separation between two pulses, one can have arbitarily small δ, the shift in carrier frequency or the parameter to be estimated here. However,experimental conditions have a restricting role on the minimum value of δ that can be inferred from the orthogonality of $|\ \psi\rangle$ and $|\ \psi_\delta\rangle$. Nevertheless, the value of δ for this experiment comes out to be 3.3 THz, which notably is less than the uncertainty calculated.

This sub-fourier resolution opens up the possibility of tremendous precisions in quantum parameter estimations

It needs to be mentioned that the Cat state provides sensitivity along only one direction in phase space where as the kitten state achieves the both in time and frequency domain. This is apparent from the representation of kitten state

$$\psi(x) = \left(e^{-\frac{(x-\alpha)^2}{4\sigma^2}} + e^{-\frac{(x+\alpha)^2}{4\sigma^2}} + e^{-\frac{(x+i\alpha)^2}{4\sigma^2}} + e^{-\frac{(x-i\alpha)^2}{4\sigma^2}}\right), \tag{2.5}$$

as we compare this with the cat state.

The possibility of experimentally realizing the kitten state and their decoherence properties have been thoroughly investigsted[16]. The fact that some of these states naturally manifest in dissipative systems may aid in their preparation and use[11, 16] It was found that an entangled Cat state involving two particles can carry out parameter estimation, with better accuracy[19].The use of entangled states for precision lithography and other applications have generated considerable interest in them[**?**].

In the following , we start with the odd Cat state $|\ \alpha\rangle - |\ -\alpha\rangle$ which comprises all the odd oscillator states and compare it with $|\ \alpha\rangle + |\ -\alpha\rangle$, involving the even states. We study it's sensitivity in parameter estimation and contrast it with the even state $|\ \alpha\rangle + |\ -\alpha\rangle$. We then characterize the entanglement properties of entangled Cat state of the type $|\ \pm\alpha^-\rangle_1\ |\ \pm i\alpha^+\rangle_2 + |\ \pm i\alpha^+\rangle_1\ |\ \pm\alpha^-\rangle_2$ and show it's efficacy in improving the sensitivity in the above parameter estimation. Here, $| \pm\alpha^{\pm}\rangle$ is griven by

$$| \pm\alpha^{\pm}\rangle = \frac{| \alpha\rangle \pm | -\alpha\rangle}{\sqrt{2}} \tag{2.6}$$

2.2 The odd Cat state

The odd cat state, is obtained after accumulating odd terms in the expansion of the coherent state unlike the popular Cat state which is obtained with even terms.The importance of this state, in this work will become evident when we examine the entanglement characteristics of the compass state. It is represented by a wavefunction

$$\Phi_{\alpha}(x) = \Psi_{\alpha}(x) - \Psi_{-\alpha}(x) = \sqrt{\frac{1}{\sigma\sqrt{2\pi}(1 - e^{\frac{-\alpha^2}{2\sigma^2}})}}(e^{-\frac{(x\ -\alpha)^2}{4\sigma^2}}\ -\ e^{-\frac{(x\ +\ \alpha)^2}{4\sigma^2}}) \tag{2.7}$$

The quantum chararacter of this state is characterized by the Wigner function being defined as

$$W(x,p) = \frac{1}{h}\int \Psi^{*}(x-a)\ \Psi(x+a)\ e^{\frac{2i\pi pa}{h}}\ da, \tag{2.8}$$

. The Wigner function describes quantum mechanics in a manner, resembling the classical statistical description in the phase space. Since it is not a probability distribution,it can be negative as well. The negativity of the Wigner function implies the quantum character of the given state. On calculating for the given state, this comes out to be

$$W = \frac{\sigma\sqrt{2\pi}}{h}\{e^{-\left(\frac{\sigma^2 k^2}{2}\right)}\left(e^{-\frac{(\frac{x}{2}+\alpha)^2}{2\sigma^2}} + e^{-\frac{(\alpha-\frac{x}{2})^2}{2\sigma^2}}\right) - e^{-\frac{\sigma^4 k^2+\frac{x^2}{4}}{2\sigma^2}}\left(2\cos\left(\frac{\alpha^2 k^2}{\sigma^2}\right)\right)\} \tag{2.9}$$

This form of the Wigner function which essentially is oscillatory in nature asserts that it would be negative in certain regions, establishing the quantum character of the above state. It is worth mentioning here that the Wigner function for a Gaussian state, being the fourier transform of a Gaussian, is positive everywhere, supporting our earlier assertion that Gaussian states are 'classical'. The oscillaotry strucure has significant implications for quantum metrology as they correspond to the presence of sub-Planck structures.

The uncertainty in position and momentum operators for this state can be obtained as

$$\Delta\text{x} \quad = \quad \sqrt{\langle x^2\rangle - < x >^2} = \sqrt{\left(\sigma^2 + \frac{\alpha^2}{1 - e^{-\frac{\alpha^2}{2\sigma^2}}}\right)}$$

$$\text{and} \quad \Delta\text{p} \quad = \quad \sqrt{< p^2 > - < p >^2} = \frac{\hbar}{2\sigma^2}\sqrt{\left(\frac{\sigma^2\left(1 - e^{-\frac{\alpha^2}{2\sigma^2}}\right) + \alpha^2 e^{\frac{-\alpha^2}{2\sigma^2}}}{\left(1 - e^{-\frac{\alpha^2}{2\sigma^2}}\right)}\right)} \tag{2.10}$$

The negative exponential term in the denominator is the only variation from the general Cat state. Also from here, we can infer that the superposed cat state is not the minimum uncertainty state unlike the coherent states as the product of the squares of the uncertainties comes to be

$$\langle \Delta x^2 \rangle \langle \Delta p^2 \rangle = \frac{\hbar^2}{4\sigma^2}(\sigma^2 + \frac{\alpha^2}{1 - e^{\frac{-\alpha^2}{2\sigma^2}}})(1 + \frac{e^{\frac{-\alpha^2}{2\sigma^2}} \frac{\alpha^2}{\sigma^2}}{1 - e^{\frac{-\alpha^2}{2\sigma^2}}}) \tag{2.11}$$

It is evident from here that the values of uncertainties will be slightly different for this case as compared with the cat state. However,the quantum parameter estimation done by using the odd superposition in the same way as done in [15] will fetch us

$$\langle \psi | \psi_\delta \rangle = \int_{-\infty}^{+\infty} \psi_\alpha^*(x) \psi_\alpha^\delta(x) dx = 2\sigma\sqrt{2\pi} e^{-\left(\frac{\sigma^2\delta^2}{2}\right)} \left(\cos(\alpha\delta) - e^{-\frac{\alpha^2}{2\sigma^2}} \right) \tag{2.12}$$

It is quite obvious to see that for the same α, i.e, the separation between two gaussian wavepackets we would obtain the same δ which is the interval between two zeros in the intensity plot, the parameter we wish to estimate here for both the states.

2.3 The entangled Compass state

An entangled, bipartite system where the two constituents are themselves characterized by Cat states, can be represented as

$$| \psi \rangle_c = \frac{1}{\sqrt{2}}(A \mid \pm\alpha^+ \rangle_1 \mid \pm i\alpha^+ \rangle_2 + B \mid \pm i\alpha^+ \rangle_1 \mid \pm\alpha^+ \rangle_2) \tag{2.13}$$

where $A = A_1 + iA_2 \text{and} B = B_1 + iB_2$ are complex parameters which control the entanglement. Here, the two constituents or the two modes of the field are described by the Cat states and the combined state is an entangled system.

The system as described by $| \psi \rangle_c$, is known as Compass state in literature. The rationale behind the use of odd cat state is now evident with the relation

$$\langle \pm\alpha^- \mid \pm i\alpha^+ \rangle = 0 \tag{2.14}$$

highlighting the orthogonality of the even and odd cat states and hence we modify our state in Eq. (2.13) to be

$$| \psi \rangle_c = \frac{1}{\sqrt{2}}(A \mid \pm\alpha^- \rangle_1 \mid \pm i\alpha^+ \rangle_2 + B \mid \pm i\alpha^+ \rangle_1 \mid \pm\alpha^- \rangle_2) \tag{2.15}$$

such that it resembles more closely to the EPR state $\frac{|01\rangle + |10\rangle}{\sqrt{2}}$. As can be noted, these states are the eigenstates of a^2b^2 which imparts them a cubic algebra represented by

$$K_- = a^2b^2; K_+ = a^{\dagger^2} b^{\dagger^2} \tag{2.16}$$

such that

$$[K_0, K_\pm] = \pm K_\pm, [K_+, K_-] = 2cK_0 + 4hK_0^3 \tag{2.17}$$

This is also known as Higg's algebra. Taking K_+ and K_- as the general creation and annihilation operators we find these states to be the steady state solutions of the master equation governing the evolution of a general Hamiltonian [20, 21].

$$\frac{\partial \rho}{\partial t} = -ig(K_-\rho - \rho K_- + K_+\rho - \rho K_+) + \frac{\kappa}{2}(2K_-\rho K_+ - K_+K_-\rho - \rho K_+K_-) \quad (2.18)$$

The state in Eq.) does not satisfy the criterion for separability based on the variances of two EPR operators as given in [23, 24]

Sensitivity

These entangled compass states can be represented by

$$\Psi_c(x_1, x_2) = N(A\psi(x_1)\phi(x_2) + B\psi(x_2)\phi(x_1)) \quad (2.19)$$

where $\psi(x)$ and $\phi(x)$ themselves are the wavefunctions representing the superposed states in $Eq(13)$ such that

$$\langle x \mid \pm\alpha^- \rangle = \psi(x) = \frac{e^{-\frac{(x-x_0)^2}{2\sigma^2}} - e^{-\frac{(x+x_0)^2}{2\sigma^2}}}{\sqrt{2}\pi^{\frac{1}{4}}\sigma^{\frac{1}{2}}[1 - e^{-\frac{x_0^2}{\sigma^2}}]^{\frac{1}{2}}} \quad (2.20)$$

represents the odd cat state and

$$\langle x \mid \pm i\alpha^+ \rangle = \phi(x) = \frac{e^{-\frac{x^2}{2\sigma^2} + ip_0\frac{x}{\hbar}} + e^{-\frac{x^2}{2\sigma^2} - ip_0\frac{x}{\hbar}}}{\sqrt{2}\pi^{\frac{1}{4}}\sigma^{\frac{1}{2}}[1 + e^{\frac{-p_0^2\sigma^2}{\hbar^2}}]^{\frac{1}{2}}} \quad (2.21)$$

represents the even momentum state.

Now as pointed out by Toscano *et al.* [14], we consider two displacement operators $D_1(\alpha)$ and $D_2(\beta)$ such that they displace particles 1 and 2 by amounts α and β. This leads our system to be in a new state $|\psi\rangle_{per} = D_1(\alpha)D_2(\beta) \mid \psi\rangle_c$. For equal amount of displacements of both the particles i.e, $\alpha = \beta = \frac{x_0}{|x_0|}$, the overlap function, is found to be proportional to

$$|\langle\psi_c|\psi\rangle_{per}|^2 \propto 1 + \cos(4x_0(s + \theta)) \quad (2.22)$$

The phase difference of θ underlies the difference due to the odd Cat state. Clearly, one can see that for displacements $x \sim \frac{\pi}{4x_0} - \theta$, the overlap function attains a minimum. This should be contrasted with the result for one particle Cat state discussed in detail above. Henceforth, the role of this state in carrying out the Heisenberg limited measurements gets emphasized.

To study the sub-planck structure in the phase space, we set out to compute the Wigner function for the state in Eq(2.13) as has been done in [19] .

The correlation function is first obtained as

$$c(x_1, a_1, x_2, a_2) = \Psi^\dagger(x_1 + \frac{a}{2}, x_2 + \frac{b}{2})\Psi(x_1 - \frac{a}{2}, x_2 - b/2) \quad (2.23)$$

The Wigner function, can then be computed as

$$W(x_1,p_1;x_2,p_2) = \frac{1}{(2\pi\hbar)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(x_1,a_1,x_2,a_2)e^{i\frac{(p_1a+p_2b)}{\hbar}}dadb, \tag{2.24}$$

which yields,

$$W(x_1,p_1;x_2,p_2) = \frac{2\sigma^2|N|^2}{\pi\hbar^2}e^{\frac{x_1^2+x_2^2}{\sigma^2}}e^{\frac{(p_1^2+p_2^2)\sigma^2}{\hbar^2}}((W_{D1}+W_{D2})+e^{-\frac{x_0^2}{2\sigma^2}-\frac{p_0^2\sigma^2}{2\hbar^2}}(W_{OD1}+W_{OD2})). \tag{2.25}$$

Here, W_{D1}, W_{D2} are the diagonal terms and W_{OD1} and W_{OD2}, the off-diagonal terms in the evaluation of Wigner Integral. They have been computed as

$$\begin{aligned} W_{D1} &= 2|A|^2(e^{-\frac{x_0^2}{\sigma^2}-\frac{p_0^2\sigma^2}{\hbar^2}}\cosh\left(\frac{2p_0p_2\sigma^2}{\hbar^2}\right)\cosh\left(\frac{2x_0x_1}{\sigma^2}\right)+e^{-\frac{x_0^2}{\sigma^2}}\cosh\left(\frac{2x_0x_1}{\sigma^2}\right)\cos\left(\frac{2p_0x_2}{\hbar}\right) \\ &+e^{-\frac{p_0^2\sigma^2}{\hbar^2}}\cos\left(\frac{2x_0p_1}{\hbar}\right)\cosh\left(\frac{2p_0p_2\sigma^2}{\hbar^2}\right)+2\cos\left(\frac{2p_0x_2}{\hbar}\right)\cos\left(\frac{2x_0p_1}{\hbar}\right)) \end{aligned} \tag{2.26}$$

It can be seen that the first three terms are multiplied with Gaussian factors which for large values of the arguments will render them negligible The only term that would contribute significantly is the last oscillatory term. The zeros of this term are spaced between intervals of length $\frac{\pi\hbar}{2p_0}$ in each x_2 and p_1 directions. This clearly indicates the presence of sub-Planck structures in x_2, p_1. A similar inference can be drawn from the following expression.

$$\begin{aligned} W_{D2} = 2|B|^2(e^{-\frac{x_0^2}{\sigma^2}-\frac{p_0^2\sigma^2}{\hbar^2}}\cosh\left(\frac{2p_0p_1\sigma^2}{\hbar^2}\right)\cosh\left(\frac{2x_0x_2}{\sigma^2}\right)+e^{-\frac{x_0^2}{\sigma^2}}\cosh\left(\frac{2x_0x_2}{\sigma^2}\right)\cos\left(\frac{2p_0x_1}{\hbar}\right) \\ +e^{-\frac{p_0^2\sigma^2}{\hbar^2}}\cos\left(\frac{2x_0p_2}{\hbar}\right)\cosh\left(\frac{2p_0p_1\sigma^2}{\hbar^2}\right)+2\cos\left(\frac{2p_0x_1}{\hbar}\right)\cos\left(\frac{2x_0p_2}{\hbar}\right)) \end{aligned} \tag{2.27}$$

The off-diagonal terms can be obtained as

$$\begin{aligned} W_{OD1} = ((A_1B_1+A_2B_2)-(A_1B_2-A_2B_1))(e^{\frac{ip_0x_0}{\hbar}}(\cosh\left((\frac{x_0}{\sigma^2}-\frac{ip_0}{\hbar})(x_1+x_2)+(\frac{ix_0}{\hbar}-\frac{p_0\sigma^2}{\hbar^2})(p_1-p \right. \\ +\cosh\left((\frac{x_0}{\sigma^2}-\frac{ip_0}{\hbar})(x_1-x_2)+(\frac{ix_0}{\hbar}-\frac{p_0\sigma^2}{\hbar^2})(p_1+p_2)\right)) \\ +e^{\frac{-ip_0x_0}{\hbar}}(\cosh\left((\frac{x_0}{\sigma^2}+\frac{ip_0}{\hbar})(x_1+x_2)+(\frac{ix_0}{\hbar}+\frac{p_0\sigma^2}{\hbar^2})(p_1-p_2)\right) \\ +\cosh\left((\frac{x_0}{\sigma^2}+\frac{ip_0}{\hbar})(x_1-x_2)+(\frac{ix_0}{\hbar}+\frac{p_0\sigma^2}{\hbar^2})(p_1+p_2)\right)) \\ +2(\cos\left(p_0(\frac{(x_1-x_2)}{\hbar}-i\frac{(p_1+p_2)\sigma^2}{\hbar^2})\right)\cosh\left(x_0(\frac{(x_1+x_2)}{\sigma^2}+i\frac{(p_1-p_2)}{\hbar})\right) \\ +\cos\left(p_0(\frac{(x_1+x_2)}{\hbar}-i\frac{(p_1-p_2)\sigma^2}{\hbar^2})\right)\cosh\left(x_0(\frac{(x_1-x_2)}{\sigma^2}+i\frac{(p_1+p_2)}{\hbar})\right))) \end{aligned} \tag{2.28}$$

(2.29)

The off-diagonal terms can be computed as

$$W_{OD2} = ((A_1B_1 + A_2B_2) + (A_1B_2 - A_2B_1))(e^{\frac{ip_0x_0}{\hbar}}(\cosh\left((\frac{x_0}{\sigma^2} - \frac{ip_0}{\hbar})(x_1 + x_2) - (\frac{ix_0}{\hbar} - \frac{p_0\sigma^2}{\hbar^2})(p_1 - p\right.$$

$$+ \cosh\left((\frac{x_0}{\sigma^2} - \frac{ip_0}{\hbar})(x_1 - x_2) - (\frac{ix_0}{\hbar} - \frac{p_0\sigma^2}{\hbar^2})(p_1 + p_2)\right))$$

$$+ e^{\frac{-ip_0x_0}{\hbar}}(\cosh\left((\frac{x_0}{\sigma^2} + \frac{ip_0}{\hbar})(x_1 + x_2) - (\frac{ix_0}{\hbar} + \frac{p_0\sigma^2}{\hbar^2})(p_1 - p_2)\right)$$

$$+ \cosh\left((\frac{x_0}{\sigma^2} + \frac{ip_0}{\hbar})(x_1 - x_2) - (\frac{ix_0}{\hbar} + \frac{p_0\sigma^2}{\hbar^2})(p_1 + p_2)\right))$$

$$+ 2(\cos\left(p_0(\frac{(x_1 - x_2)}{\hbar} + i\frac{(p_1 + p_2)\sigma^2}{\hbar^2})\right)\cosh\left(x_0(\frac{(x_1 + x_2)}{\sigma^2} - i\frac{(p_1 - p_2)}{\hbar})\right)$$

$$+ \cos\left(p_0(\frac{(x_1 + x_2)}{\hbar} + i\frac{(p_1 - p_2)\sigma^2}{\hbar^2})\right)\cosh\left(x_0(\frac{(x_1 - x_2)}{\sigma^2} - i\frac{(p_1 + p_2)}{\hbar})\right))) \tag{2.30}$$

It can be seen from here that unlike the diagonal elements, the ocillatory terms in off-diagonal elements are severely damped for large values of x_0 and p_0. Hence collecting the significant terms in the mesoscopic limit, we can approximate the Wigner function to be:

$$W(x,p) = 4|A|^2 \cos\left(\frac{2p_0x_2}{\hbar}\right)\cos\left(\frac{2x_0p_1}{\hbar}\right) + 4|B|^2 \cos\left(\frac{2p_0x_1}{\hbar}\right)\cos\left(\frac{2x_0p_2}{\hbar}\right) \tag{2.31}$$

As has been analyzed earlier, from this expression, one can find the zeros in (x_1, p_1) as well as (x_2, p_2) planes. This again corresponds to the presence of sub-Planck structures in these planes.

2.4 Conclusion

The Cat state $\mid \alpha\rangle - \mid -\alpha\rangle$ has been found to be well suited for quantum metrology and can be generated in dissipative systems.The bipartite entangled state $\mid \pm\alpha^-\rangle_1 \mid \pm i\alpha^+\rangle_2 + \mid \pm i\alpha^+\rangle_1 \mid \pm\alpha^-\rangle_2$ provides a uniformity in the directions in phase space in the context of sub-planck structures which highlights it's significance in the metrological processes.

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Chapter 3

Heisenberg Limited Metrology with Pair Coherent states

Theme

We explore the maximum resolution for phase measurement in a Mach-Zehnder interferometer. The input state that has been employed is the pair coherent state and we evaluate the Quantum Cramer-Rao Bound (QCRB) to investigate the maximum resolution possible in the Mach-Zehnder interferometer. It is found that the uncertianty in phase measurements using certain special class of states are less than the Heisenberg limit.

3.1 Introduction

Precise phase estimation has always been a significant problem for many applications, the most notable of them being gravitational wave detection. In conventional processes of phase measurement, a Mach-Zehnder Interferometer is used. As is detailed in the following sections, the determination of phase is dependent on the input light state. For an input state having one mode as coherent state with average number of photons, $\bar{N}$ and other as vacuum, the uncertainty in phase measurements is limited by shot noise, $\frac{1}{\sqrt{N}}$. This limitation was overcome by using quantum properties like entanglement and in some cases, superposing more number of states. The uncertainty for these states is restricted by Heisenberg limit, which scales as $\frac{1}{N}$. In this chapter, we investigate the minimum possible phase measurements obtained from statistical Cramer-Rao bound. We witness that for a special class of state, called pair coherent state, the Cramer-Rao bound comes out to be in sub-Heisenberg regime.

3.2 The Schwinger representation

J. Schwinger's notes on *Quantum Theory of Angular Momentum* establishes an interesting connection between the algebra of angular momentum operators and the algebra of uncoupled oscillators, or in the present context that of electromagnetic field. Here, in this section, this formalism is briefly described. Writing the annihilation operators of a two-mode field as a and b, we construct the operators

$$\begin{aligned} J_x &= \frac{a^\dagger b + ab^\dagger}{2} \\ J_y &= \frac{a^\dagger b - ab^\dagger}{2i} \\ J_z &= \frac{a^\dagger a - b^\dagger b}{2} \end{aligned} \tag{3.1}$$

These equations satisfy

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{3.2}$$

Writing

$$J_\pm = J_x \pm iJ_y \tag{3.3}$$

we observe that they have an underlying SU[2] algebra with

$$\begin{aligned} [J_+, J_-] &= 2J_z \\ [J_z, J_\pm] &= \pm J_\pm \end{aligned} \tag{3.4}$$

An important point to be noted here is that during the entire process

$$J^2 = \frac{N}{2}(\frac{N}{2} + 1) \tag{3.5}$$

remains invariant, where $N = \frac{a^\dagger a + b^\dagger b}{2}$

J_z denotes the difference in the number of photons between the two modes.J_x and J_y are the quadrature interference terms of the two fields and hence they signify the phase difference between the two fields. In this manner, the standard Heisenberg uncertainty relation

$$\Delta J_z \Delta J_x \geq \frac{1}{2}|\langle J_y \rangle| \tag{3.6}$$

reduces to

$$\Delta(N_a - N_b)\Delta(\phi_a - \phi_b) \geq 1 \tag{3.7}$$

3.3 The Mach-Zehnder Interferometer

A four-port optical lossless device such as a Mach-Zehnder Interferometer, can be described by a unitary operator for rotation matrices, in terms of Euler angles.

$$U = \begin{pmatrix} \cos\frac{\beta}{2} e^{i(\alpha+\gamma)/2} & \sin\frac{\beta}{2} e^{i(\alpha-\gamma)/2} \\ -\sin\frac{\beta}{2} e^{-i(\alpha-\gamma)/2} & \cos\frac{\beta}{2} e^{-i(\alpha+\gamma)/2} \end{pmatrix} \tag{3.8}$$

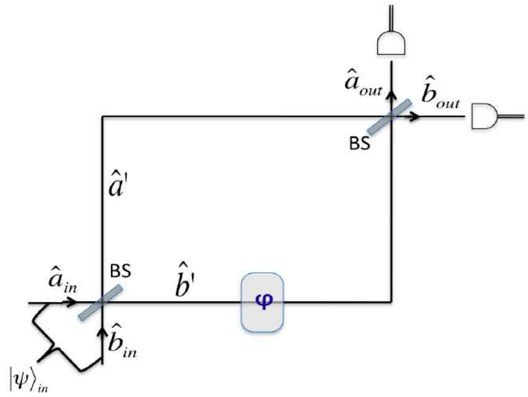

Figure 3.1: **Mach-Zehnder Inferometer.**

This can be seen from the construction of the MZI, as is shown in the following schematic diagram. It consists of two beam splitters, two reflecting mirrors and a phase shifter, which lets it to function as an interferometer. The light field enters the apparatus through the input ports A and B and exits through the output ports C and D. The objective is to measure the phase difference between the two arms of the interferometer, by measuring the intensity difference at the output ports.

3.3.1 Beam Splitter Matrix

A beam splitter can be identified as a two state quantum system with two input and output ports. It does a transformation given by the matrix

$$R = \begin{pmatrix} r & t^{'} \\ t & r^{'} \end{pmatrix} \tag{3.9}$$

where r,t and $r^{'}$, $t^{'}$ are the reflection and tranmission coefficients for the input port 1 and 2 respectively. The assumption that this beam splitter is idealized and lossless imposes unitarity on this matrix, thus giving us a relation between the coefficients for the two input ports.

$$r^{'} = r^{*} \qquad t^{'} = -t^{*}, \tag{3.10}$$

Writing the coefficients in terms of their modulus and phase, for ex- $r = |r|e^{i\delta_r}$ and so on, one obtains

$$(\delta_{r^{'}} - \delta_{t^{'}}) + (\delta_r - \delta_t) = \pi \tag{3.11}$$

Considering a symmetric beam splitter,i.e one which does not discriminate between the two input ports, we find that

$$(\delta_{r^{'}} - \delta_{t^{'}}) = (\delta_r - \delta_t) = \frac{\pi}{2} \tag{3.12}$$

This phase difference of $\frac{\pi}{2}$ introduces a factor of i with the transmission coefficients. Hence, our beam splitter matrix becomes

$$R = \begin{pmatrix} r & \pm it \\ \pm it & r \end{pmatrix} \tag{3.13}$$

This operaion is equivalent to $\alpha = -\gamma = \frac{\pi}{2}$,$\beta = \pm\phi$ in Eq.1, which gives us

$$U = \begin{pmatrix} \cos\frac{\phi}{2} & i\sin\frac{\phi}{2} \\ i\sin\frac{\beta}{2} & \cos\frac{\phi}{2} \end{pmatrix} \tag{3.14}$$

Its action on a bidimensional vector with its components as two field amplitudes is represented as

$$\begin{pmatrix} a^{out} \\ b^{out} \end{pmatrix} = \begin{pmatrix} \cos\frac{\phi}{2} & i\sin\frac{\phi}{2} \\ i\sin\frac{\beta}{2} & \cos\frac{\phi}{2} \end{pmatrix} \begin{pmatrix} a^{in} \\ b^{in} \end{pmatrix} \tag{3.15}$$

In terms of the Schwinger representation, the corresponding SO(3) rotation will be

$$\begin{pmatrix} J_x^{out} \\ J_y^{out} \\ J_z^{out} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \pm\sin\phi \\ 0 & \pm\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} J_x^{in} \\ J_y^{in} \\ J_z^{in} \end{pmatrix} \tag{3.16}$$

This expression is achieved by examining the role of Schwinger operators in the light of rotation of the field amplitudes. Now, it can be seen that this operation is equivlent to a rotation of the light state by $\pm\phi$ around x axis. For the symmetric beam splitter described above with $r = t = \frac{1}{\sqrt{2}}$, so that it becomes a $50-50$ beam splitter, we witness a rotation of $\frac{\pi}{2}$ about x axis. Hence,

$$J_y^{out} = e^{\pm i\frac{\pi}{2}J_x} J_y^{in} e^{\pm i\frac{\pi}{2}J_x} = \pm J_z^{in} \tag{3.17}$$

and similarly,

$$J_z^{out} = e^{\pm i\frac{\pi}{2}J_x} J_z^{in} e^{\pm i\frac{\pi}{2}J_x} = \mp J_y^{in} \tag{3.18}$$

The action of this beam splitter on the eigen state of J^2 and J_z operator is given by

$$|j\mu\rangle_z^{out} = e^{\pm i\frac{\pi}{2}J_x}|j\mu\rangle_z^{in} = |j \pm \mu\rangle_y. \tag{3.19}$$

To illustrate this further, let us see the action of this beam spliiter in the Schwinger representation in case of a twin photon input$|11\rangle$=$|10\rangle_z$ Now,

$$\begin{aligned} e^{i\frac{\pi}{2}J_x}|10\rangle_z &= \sum_{\mu=-1}^{1} i^\mu d_{0\mu}^1\left(\frac{\pi}{2}\right)|1\mu\rangle_z \\ &= \frac{i}{\sqrt{2}}\left(|11\rangle_z + |1-1\rangle_z\right) \\ &= \frac{i}{\sqrt{2}}\left(|20\rangle_z + |02\rangle_z\right) \end{aligned} \tag{3.20}$$

This is a popular result in case of a $50-50$ beam splitter that the twin photon state emerges together at the same output port as ($|11\rangle_z$) is supressed by the destructive quantum interference.

3.3.2 The phase shift

In an MZI, the phase difference between the two arms of the interferometer is measured by measuring the intensity difference at the output ends. Now this difference may be introduced by a sample which is placed along one arm of the interferometer or motion of one of the mirrors or due to any other source. Without this difference in phase the beams along both the arms of the interferometer will interfere constructively on one of the output ports and destructively, on another. This action can be represented by

$$\begin{pmatrix} a^{out} \\ b^{out} \end{pmatrix} = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{i-\phi/2} \end{pmatrix} \tag{3.21}$$

which is nothing but a rotation of ϕ around z. Hence, a phase difference of ϕ will amount to

$$|\psi\rangle^{out} = e^{-i\phi J_z}|\psi\rangle^{in} \tag{3.22}$$

This makes sense with our earlier understanding of Schwinger's representation that a phase difference between the fields a and b should affect J_x and J_y while leaving the intensity difference J_z unchanged.

3.3.3 The MZI in this framework

Based on this construction, we can club the rotation operators for an input beam splitter, a phase shifter and an output beam splitter to give us the resultant product operator for an MZI. Mathematically, this combined rotation is represented by

$$|\psi\rangle_{out} = e^{-i\frac{\pi}{2}J_x}e^{i\phi J_z}e^{i\frac{\pi}{2}J_x}|\psi\rangle_{in} = e^{-i\phi J_y}|\psi\rangle_{in} \tag{3.23}$$

which comes out to be a rotation of ϕ about the y axis. Hence, the rotation operators will be transformed as

$$\begin{pmatrix} J_x^{out} \\ J_y^{out} \\ J_z^{out} \end{pmatrix} = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} J_x^{in} \\ J_y^{in} \\ J_z^{in} \end{pmatrix} \tag{3.24}$$

3.4 Quantum Cramer-Rao Bound

It has been seen that the variance of an unknown parameter or the uncertainty in its measurement is obtained by the standard linear error propagation formula

$$\delta\phi^2 = \frac{\Delta\hat{A}^2}{|\partial_\phi\langle A\rangle|} \tag{3.25}$$

This expression is more suited for peaked distributions like Gaussians. In a more general theory of estimation, the minimum estimate of an unknown parameter is limited by the Cramer-Rao inequality, which relates the variance to the Fisher information.

$$F(X) = \int d\zeta p(\zeta/X) \left(\frac{\partial \ln p(\zeta/X)}{\partial x}\right)^2 = \int d\zeta \frac{1}{p(\zeta/X)} \left(\frac{\partial p(\zeta/X)}{\partial x}\right)^2 \quad (3.26)$$

Here, $p(\zeta/X)$, the probability density for result ζ given the parameter X, is given by

$$p(\zeta/X) = tr\left(\hat{E}(\zeta)\hat{\rho}(X)\right) \quad (3.27)$$

It should be noted from here that the Fisher information is the expected value of $\left(\frac{\partial \ln p(\zeta/X)}{\partial x}\right)^2$ operator. The Cramer-Rao inequality, states

$$\delta\phi^2 \geq \frac{1}{F(X)} \quad (3.28)$$

For a pure state under evolution of phase, the notion of Fisher Information is demonstrated by Durking and Dowling[31]. It states that for an input state $|\psi\rangle_{in}$ undergoing a phase evolution in the follwing manner,

$$|\psi\rangle_{out} = e^{i\theta\hat{G}}|\psi\rangle_{in} \quad (3.29)$$

the quantum Fisher information is given by

$$F_Q = 4\Delta\hat{G}^2 \quad (3.30)$$

Again it should be noted from here that this is the Quantum Fisher information unlike that in Eq which is classical. It has also been shown by Braunstein and Caves [32] that the classical Fisher Information is an upper bound to the quantum version. Hence, the variance or uncertainty

$$\delta\phi^2 \geq \frac{1}{F_Q} \quad (3.31)$$

is the lowest possible estimate in phase estimation. Also, it is worth observing that it depends entirely on the input state.

3.4.1 Calculation of QCRB

The calculation of Quantum Cramer Rao Bound is now straightforward since we have already derived the operator framework for an MZI in the previous section. As we have seen here, the F_Q for an input state $|\psi\rangle_{in}$ in an MZI will be

$$F_Q = 4\Delta J_y^2 \quad (3.32)$$

where

$$J_y = \frac{a^\dagger b - ab^\dagger}{2i} \quad (3.33)$$

so to calulate the minimum of phase uncertainty all we need to do is to evaluate the variance of this operator for different input states. For a simple input state $|\alpha\rangle_a|0\rangle_b$, the minimum of phase, that can be measured as given by the above bound will be

$$\begin{aligned} F_Q &= 4\Delta J_y^2 \\ &= 4\langle\alpha|_a\langle 0|_b J_y^2|\alpha\rangle_a|0\rangle_b \\ &= |\alpha|^2 \end{aligned} \tag{3.34}$$

Now, we know that the number of photons for this input state is proportional to $|\alpha|^2$. Hence, the minimum attainable limit in case of coherent state is given by

$$\phi_{min} = \frac{1}{\sqrt{\bar{n}}} \tag{3.35}$$

often called shot noise limit in literature. We do the same calculation on the Twin Mode Squeezed Vaccum (TMSV) represented by

$$|\psi_{\bar{n}}\rangle = \sum_{n=0}^{\infty} \sqrt{p_n(\bar{n})}|n, n\rangle \tag{3.36}$$

This is a superposition of twin Fock states with $P_{\bar{n}}(n)$ representing the probabilities to be present in a particular twin photon state. It depends on the average number of photons in that state. The bound in this case comes out to be

$$\phi_{min} \propto \frac{1}{\bar{n}} \tag{3.37}$$

This is known as Heisenberg's limit due to it's resemblance with the standard Heisenberg uncertainty principle. However, in actual terms this limit is slightly less than Heisenberg's limit thus allowing the sub- Heisenberg regime. In actual terms, the limit obtained for a twin photon state is

$$\phi_{min} = \sqrt{\frac{2}{(n)(n+2)}} \tag{3.38}$$

For, the cat state discussed in the previous chapter as well, we find the limit to be

$$\phi_{min} \propto \frac{1}{\bar{n}} \tag{3.39}$$

We can infer from here, as well as from the results on the entangled twin photon states that the improvement of $\sqrt{n}$ is attained due to the presence of entanglement. The increase in the number of modes also helps to improve the lowest bound on phase measurement.

3.5 The Pair coherent state

Now, we would investigate the Cramer-Rao bound for a pair coherent state, a state of a two-mode radiation field satisfying

$$\begin{aligned} ab|\zeta, q\rangle &= \zeta|\zeta, q\rangle \\ a^\dagger a - b^\dagger b|\zeta, q\rangle &= q|\zeta, q\rangle \end{aligned} \tag{3.40}$$

Here ζ is a complex number and q is the degeneracy parameter. It can be noticed that a simple product state $|\alpha\rangle_a|\beta\rangle_b$ of a two-mode radiation field may satisfy the first of these equation as they are the eigenstates of the product of the lowering operators but they also have to satisfy the second equation which says the difference in the number of photons should be q. These states were introduced in the quantum optics literature by Agarwal[35] and are generated by nondegenerate parametric oscillators. These are non-Gaussian states unlike most of the states discussed in this report.These states have been studied in detail because of their non-classical properties which stems from their Glauber-Sudarshan P-function and entanglement. Also, they are quite popular because of their violation of Bell's inequalities. Let us first consider the state with $q = 0$ i.e, having the same number of photons in both modes. Clearly, it will be a superposition of twin Fock states.

$$|\zeta, 0\rangle = N_0 \sum_{n=0}^{\infty} \frac{\zeta^n}{n!}|n, n\rangle \tag{3.41}$$

Here $N_0 = \sqrt{\frac{1}{I_0(2|\zeta|)}}$ and $I_0(2|\zeta|)$ is the modified Bessel function given by

$$I_0(2|\zeta|) = J_0(i2|\zeta|) = \sum_{n=0}^{\infty} \left(\frac{|\zeta|^n}{n!}\right)^2. \tag{3.42}$$

In applications involving the creation and annihilation of photons pairs, the difference in the number of photons will remain constant. Hence, in case of processes starting from vacuum, we can see the existence of such states. Evaluating the uncertainty in J_y, we obtain,

$$\langle\zeta, 0|J_y^2|\zeta, 0\rangle^2 = \frac{N_0^2}{2} \sum_{n=0}^{\infty} \left(\frac{\zeta^n}{n!}\right)^2 n(n+1) \tag{3.43}$$

one finds,

$$\begin{aligned} F_Q &= 2N_0^2 \sum_{n=0}^{\infty} \left(\frac{\zeta^n}{n!}\right)^2 n(n+1) \\ &= 2\left(|\zeta|^2 + \frac{|\zeta| I_1(2\zeta)}{I_0(2|\zeta|)}\right). \end{aligned} \tag{3.44}$$

where, $I_1(2|\zeta|)$ and $I_0(2|\zeta|)$ are the modified Bessel function of the order 1 and 0 respectively. The Cramer-Rao bound on the uncertainty in phase, is inversely proportional to F_Q.

$$\Delta\phi_{min} = \frac{1}{\sqrt{F_Q}} \text{ and} \tag{3.45}$$

hence, the minimum possible bound on uncertainty in phase will be

$$\Delta\phi_{min} = \frac{1}{\sqrt{2|\zeta|\left(|\zeta| + \frac{I_1(2|\zeta|)}{I_0(2|\zeta|)}\right)}}. \tag{3.46}$$

This should be compared with the average number of photons present in this state

$$\bar{N} = |\zeta|\frac{I_1(2|\zeta|)}{I_0(2|\zeta|)}. \tag{3.47}$$

By comparing these two expressions, we see that the QCRB for the pair coherent state shows a remarkable improvement over the shot-noise limit and is very close to the Heisenberg's limit. For a relatively smaller average number of photons, the quantum Cramer-Rao bound for pair coherent state is below the Heisenberg limit. This means that even in the sub-Heisenberg regime smaller phase measurements can be done with pair coherent states as compared to other input states in a Mach-Zehnder Interferometer.

Again, the general solution for the above eigenvalue problem, assuming q to be positive, is given by

$$|\zeta, q\rangle = N_q \sum_{n=0}^{\infty} \frac{\zeta^n}{[n!(n+q)!]^{\frac{1}{2}}} |n+q, n\rangle \tag{3.48}$$

where N_q, the normalization constant is given by

$$N_q = \left[\sum_{n=0}^{\infty} \frac{|\zeta|^n}{(n!)(n+q)!}\right]^{-\frac{1}{2}} = [(i|\zeta|)^{-q} J_q(2i|\zeta|)]^{-\frac{1}{2}} \tag{3.49}$$

This is the general pair coherent state with a constant difference betweeen the number of photons in both modes. These states can be generated from vacuum in the following manner

$$|\zeta, q\rangle = M_q(a^\dagger b^\dagger \zeta) a^{\dagger q} |0, 0\rangle \tag{3.50}$$

where $M_q = q!(-z)^{\frac{-q}{2}} [J_q(2i(z)^{\frac{1}{2}}]$

While writing this general form of pair coherent state, it is worth mentioning again, that these states are different from coherent states, which is evident, in terms of their probabilities of being in a state having a particular composition of photons. The probability of finding n photons in mode b and n+q photons in mode a will be given by

$$p(n+q, n) = |\langle n+q, n|\zeta, q\rangle|^2 = N_q^2 \frac{|\zeta|^{2n}}{n!(n+q)!} \tag{3.51}$$

Clearly, unlike coherent states, this is not a Poissonian distribution. In fact Agarwal[36] had shown this to be sub- Poissonian. Evaluating QCRB for this general state, we see

$$\Delta J_y^2 = \frac{|N_q|^2}{2} \sum_{n=0}^{\infty} \frac{|\zeta|^{2n}}{n!(n+q)!} \left[(n+q)(n+1) + (n+q+1)n\right] \tag{3.52}$$

The average number of photons for this state could similarly be found out as

$$\bar{N} = |N_q|^2 (n + \frac{q}{2}) (\frac{|\zeta|^{2n}}{n!(n+q)!}) \tag{3.53}$$

Again from these two expressions, we see that the Quantum Cramer-Rao Bound for pair coherent state is smaller than the Heisenberg limit as well as that of other Gaussian states, thus offering an immense potential for the application of pair coherent states in sub- Heisenberg Quantum Metrology.

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