# Some Properties of Elliptic Modular Forms at the Supercuspidal Primes

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by

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*Dedicated to*

*my grandparents*

# Certificate

Certified that the work incorporated in the thesis entitled submitted by *Tathagata Mandal* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

*Date: November 22, 2018* *Dr. Debargha Banerjee*

Thesis Supervisor

# Declaration

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*Tathagata Mandal*

# Abstract

The Brauer class of the endomorphism algebra attached to a primitive non-CM cusp form of weight two or more is a two torsion element in the Brauer group of some number field. We give a formula for the ramification of that algebra locally for all places lying above **all** supercuspidal primes. For $p = 2$, we also treat the interesting case where the image of the local Weil-Deligne representation attached to that modular form is an exceptional group. We have completed the programme initiated by Eknath Ghate to give a satisfactory answer to a question asked by Ken Ribet.

In a different project, we studied the variance of the local epsilon factor for a modular form with arbitrary nebentypus with respect to twisting by a quadratic character. As an application, we detect the nature of the supercuspidal representation from that information, similar results are proved by Pacetti for modular forms with trivial nebentypus. Our method however is completely different from that of Pacetti and we use representation theory crucially. For ramified principal series (with $p \parallel N$ and $p$ odd, $N$ denote the level of modular forms) and unramified supercuspidal representations of level zero, we relate these numbers with the Morita's $p$-adic Gamma function.

# Introduction

The overview of this thesis is presented here briefly which contains two projects I worked on during pursuing my PhD.

The thesis problem is divided into two parts. The *first part* is based on determining the local Brauer classes of modular endomorphism algebras for **all** supercuspidal primes in terms of traces of adjoint lifts at auxiliary primes. The ramification formulae of local endomorphism algebras depend on the nature (unramified or ramified) of the supercuspidal primes. In the *second part* of the thesis, we determine that nature as an application while studying the variance of local epsilon factor for a modular form with respect to twisting by a quadratic character.

In the next two sections, we briefly discuss my works during my PhD study at IISER Pune.

## Modular endomorphism algebras at supercuspidal primes

Let $f(z) = \sum_{n\geq 1} a_n q^n \in S_k(N, \epsilon)$ be a primitive non-CM Hecke eigenform of weight $k \geq 2$, level $N \geq 1$ and nebentypus $\epsilon$. Consider the number field $E = \mathbb{Q}(\{a_n\})$ obtained by attaching all the Fourier coefficients of $f$. For $k = 2$, let $M_f$ denote the abelian variety attached to $f$ by Shimura [47]. For $k > 2$, we also denote by $M_f$ the Grothendieck motive over $\mathbb{Q}$ with coefficients in $E$ associated to $f$ by Scholl [45]. The $\lambda$-adic realization of this motive produces a $\lambda$-adic Galois representation associated to the modular form $f$ by a well-known theorem of Deligne. Let $X_f$ denote the $\mathbb{Q}$-algebra of endomorphisms of $M_f$ defined by

$$X = X_f := \mathrm{End}_{\overline{\mathbb{Q}}}(M_f) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

An extra twist for $f$ is a pair $(\gamma, \chi_\gamma)$, where $\gamma \in \mathrm{Aut}(E)$ and $\chi_\gamma$ is an $E^\times$-valued Dirichlet character such that $a_p^\gamma = a_p \cdot \chi_\gamma(p)\ \forall\ p \nmid N$. The set of such elements $\gamma$ forms an abelian group, called the group of extra twists for $f$, denoted by $\Gamma$. The subfield of $E$ fixed by the group $\Gamma$ is denoted by $F$. One knows that $F$ is a totally real number field and it is generated by the elements $a_p^2 \epsilon(p)^{-1}$, for all $p \nmid N$.

It is a fundamental fact, due to Momose and Ribet [38] in weight two and Ghate and his collaborators [10], [23], [35] in higher weights, that the full algebra of endomorphisms of this motive has the structure of an explicit crossed product algebra over $F$. The algebra $X$ is a central simple algebra over $F$ and its class $[X] \in {}_2\mathrm{Br}(F)$, the 2-torsion part of the Brauer group of $F$. In [38], Ribet wondered if it is possible to determine the local Brauer classes by pure thought.

We study the Brauer classes of $X$ locally by the following exact sequence:

$$0 \to {}_2\mathrm{Br}(F) \to \oplus_{v|p}\ {}_2\mathrm{Br}(F_v) \to \mathbb{Z}/2 \to 0$$

where $v$ runs over the primes of $F$, and $F_v$ is the completion of $F$ at $v$. One knows that the algebra $X_v = X \otimes_F F_v$ is a 2-torsion element in the Brauer group $\mathrm{Br}(F_v)$, that is, the class $[X_v] \in {}_2\mathrm{Br}(F_v) \cong \mathbb{Z}/2$. We say $X_v$ is unramified if the class of $X_v$ is trivial and ramified if the class is non-trivial.

For a prime $p$, let $G_p := \mathrm{Gal}(\bar{\mathbb{Q}}_p|\mathbb{Q}_p)$ be the local Galois group. Write the level $N$ of the non-CM primitive cusp form as $p^{N_p}N'$, with $p \nmid N'$ and the nebentypus $\epsilon = \epsilon_p \cdot \epsilon'$ as a product of characters of $(\mathbb{Z}/p^{N_p}\mathbb{Z})^\times$ and $(\mathbb{Z}/N'\mathbb{Z})^\times$ of conductors $p^{C_p}$ for some $C_p \leq N_p$ and $C'$ dividing $N'$ respectively. We say $f$ is $p$-minimal, if $N_p$ is the smallest among all twists $f \otimes \psi$ of $f$ by Dirichlet characters $\psi$. The classification of the primes $p$ for a $p$-minimal newform $f$ can be given in terms of the nature of the local component at $p$ in the automorphic representation attached to $f$, which can be further given in terms of $N_p$ and $C_p$ as follows:

- $N_p = C_p = 0$ : $p$ is a prime of good reduction,
- $N_p = C_p \neq 0$ : $p$ is a ramified principal series prime,
- $N_p = 1$ and $C_p = 0$ : $p$ is a Steinberg prime,
- $2 \leq N_p > C_p$ : $p$ is a supercuspidal prime.

In the fourth case the representation can be extraordinary and it happens only when $p = 2$.

The computation of the algebra $X_v$ has been treated by many people depending upon the nature of the primes. At the infinite primes $v$, Momose determined that $X$ is totally indefinite if $k$ is even, and totally definite if $k$ is odd [33, Theorem 3.1]. The local Brauer classes of these algebras is known by a series of papers pioneered by Ghate and his collaborators [3], [4], [10] and [23] for non-supercuspidal primes. We intend to complete the program initiated by Ghate to provide a satisfactory answer to the question raised by Ribet for all places of $\mathbb{Q}$.

**In the first part of this thesis**, we wish to give a formula that precisely determines when $X_v$ is ramified if the corresponding local automorphic representation is supercuspidal in terms of Fourier coefficients at good primes (that do not divide the level of $f$). Unfortunately, if $p$ is a supercuspidal prime, then $a_p = 0$ and the corresponding slope is infinity and it is not possible to talk about the parity of slopes. We give a complete description of the local Brauer classes at the supercuspidal primes in terms of traces of adjoint lifts at auxiliary good primes. That traces of adjoint lifts are important in the study of extra twists is evident in the recent Theses of [15] and [28]. Our results should be useful to study the images of $\lambda$-adic Galois representations following [31].

**Definition 0.0.1.** *We call a supercuspidal prime $p$ to be dihedral for $f$ if the local Galois representation $\rho_f|_{G_p} \sim \mathrm{Ind}_{G_K}^{G_p}\chi$ for some quadratic extension $K|\mathbb{Q}_p$ and some character $\chi$ of $G_K := \mathrm{Gal}(\bar{\mathbb{Q}}_p|K)$. Depending on $K|\mathbb{Q}_p$ is unramified (or ramified), we call the prime $p$ to be an unramified (or ramified) supercuspidal prime for $f$. By the level of an unramified supercuspidal prime $p$, we mean the level of the corresponding local automorphic representation $\pi_p$.*

Choose a prime $p'$ coprime to $N$, with nonzero Fourier coefficients $a_{p'}$, satisfying the following properties:

$$p' \equiv 1 \pmod{p^{N_p}}, \quad p' \equiv p \pmod{N'}. \tag{1}$$

Let $v$ be a valuation on $F$ such that $v(p) = 1$.

**Definition 0.0.2.** *(Companion adjoint slope) We define the "companion adjoint slope" at a place $v$ of $F$ lying above a supercuspidal prime $p$ to be the $v$-adic valuation of the trace of adjoint lift at the good prime $p'$. In other words, we denote by*

$$m_v := [F_v : \mathbb{Q}_p] \cdot v(a_{p'}^2 \epsilon(p')^{-1})$$

*the "companion adjoint slope" at the place $v$.*

We choose the following auxiliary primes with nonzero Fourier coefficients:

- $p'' \equiv 1 \pmod{N'}$ and $p''$ has order $(p-1)$ in $(\mathbb{Z}/p^{N_p}\mathbb{Z})^\times$,
- $p''' \equiv 1 \pmod{N'}$ and $p'''$ has order 2 in $(\mathbb{Z}/2^{N_2}\mathbb{Z})^\times$,
- for all $\gamma \in \Gamma$,

$$\chi_\gamma(p^\dagger) = \begin{cases} -1, & \text{if } \chi_\gamma \text{ is ramified,} \\ 1, & \text{if } \chi_\gamma \text{ is unramified.} \end{cases}$$

There exist infinitely many such primes since $f$ is assumed to be non-CM. The results obtained here are independent of these auxiliary primes chosen. That the primes $p'$ and $p^{''}$ determine the Brauer class for $p$ odd was discovered by Bhattacharya-Ghate [8] using a different method involving local Hilbert symbols and $(p,\ell)$-Galois representation. We generalize their results for **all** supercuspidal primes including $p = 2$ using group cohomology and $(p,p)$-Galois representation.

The ramification formulae of local endomorphism algebras for **all** supercuspidal primes are determined by the following theorem.

**Theorem 0.0.3.**

- *If $p$ is a dihedral supercuspidal prime for a modular form $f$ then the local Brauer class of the modular endomorphism algebra is determined by the parity of the "companion adjoint slope" and an explicitly computable error term that is determined by at most three local symbols involving $\pi_p$ and the trace of adjoint lift of $f$ at one of the auxiliary primes associated to $p$ as listed above.*

- *If $p = 2$ is a non-dihedral supercuspidal prime for the modular form $f$ then the local Brauer class is determined by the trace of adjoint lift of the corresponding complex local Weil-Deligne representation.*

# On quadratic twisting of epsilon factors for modular forms with arbitrary nebentypus

The associated $L$-function to a modular form $f$ satisfies a functional equation with values at $s$ and $1 - s$ differ by a quantity called the root number/epsilon factor of $f$. Ariel Pacetti studied the variance of local root numbers under twisting by a quadratic character for modular forms with trivial nebentypus. We study the same for modular forms with arbitrary nebentypus and determine the nature of the supercuspidal primes.

Consider the quadratic character $\chi$ associated to a quadratic extension of $\mathbb{Q}$ ramified only at $p$. The adelization of $\chi$ gives rise to the characters $\{\chi_q\}_q$. **In this part of my thesis**, we study the changes of the local epsilon factor associated to $f$ while twisting by $\chi$. We denote by $\varepsilon_p$ the variation of the local factor of $f$ at $p$ while twisting by $\chi_p$. On both sides, we choose the same additive character and Haar measure.

For each prime $p$ dividing the level of $f$, let $\pi_{f,p}$ be the local component of the automorphic representation $\pi_f$ of the adèle group $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ attached to $f$. We treat each cases separately depending upon the nature of the local component of $f$ at $p$. Supercuspidal case is the most interesting case.

We call a prime $p$ to be a supercuspidal prime for $f$ if the local component of $f$ at $p$ is supercuspidal. By local Langlands correspondence the representations $\pi_{f,p}$ are in a bijection with (isomorphism classes of) complex two dimensional Frobenius-semisimple Weil-Deligne representations $\rho_p(f)$ associated to a modular form $f$ at $p$. We will be using the information about $\rho_p(f)$ in this case.

The Weil group of any local field $K$ and $\mathbb{Q}_p$ are denoted by $W_K$ and $W_p$ respectively. A supercuspidal prime $p$ is said to be *dihedral* if the local representation is induced by a character $\varkappa$ of an index two subgroup $W_K$ of $W_p$, i.e.,

$$\rho_p(f) = \mathrm{Ind}_{W_K}^{W_p} \varkappa$$

with $K$ a quadratic extension of $\mathbb{Q}_p$ and $\varkappa$ a quasi-character of $W_K^{\mathrm{ab}}$ which does not factor through the norm map with a quasi-character of $W_p^{\mathrm{ab}}$. Depending on $K$ unramified (or ramified), we say $p$ is an unramified (or ramified) supercuspidal prime for $f$.

For an odd supercuspidal prime $p$, the local representation is always dihedral. When $p = 2$, more representations are involved and it can be non-dihedral.

Let $\mathfrak{p}_K$ and $U_K^m$ be the unique maximal ideal and $m$-th principal units of $K$ respectively. We can consider $\varkappa$ as a character of $K^\times$ via the isomorphism $W_K^{\mathrm{ab}} \simeq K^\times$. The conductor of $\varkappa$ is the smallest positive integer $m$ such that $\varkappa|_{U_K^m} = 1$. It is denoted by $a(\varkappa)$. We say $\varkappa$ is minimal if $\varkappa|_{U_K^{a(\varkappa)-1}}$ does not factor through the norm map $N_{K|\mathbb{Q}_p}$.

Let $d$ be the 2-adic valuation of the discriminant of the ramified quadratic extension $K|\mathbb{Q}_2$. We then have $d \in \{2, 3\}$. We give a criterion for a modular form to be $p$-minimal in terms of the parity of $N_p$, the exact power of $p$ that divides the level of the modular form. More precisely, we have proved the following:

**Proposition 0.0.4.** *Let $p$ be a dihedral supercuspidal prime for $f$.*

1. *If $K$ is unramified, then $N_p$ is even.*
2. *Assume that $K|\mathbb{Q}_p$ is ramified with $a(\varkappa) \geq d + 1$ if $p = 2$. Then $N_p$ is odd if and only if $f$ is $p$-minimal.*

The next theorem determines the number $\varepsilon_p$ in the dihedral supercuspidal case.

**Theorem 0.0.5.** *Let $p$ be a dihedral supercuspidal prime for $f$.*

1. *Let $K|\mathbb{Q}_p$ be unramified. For an additive character $\phi$ of $\mathbb{Q}_p$ with $\varkappa(1 + x) = (\phi \circ \mathrm{Tr}_{K|\mathbb{Q}_p})(cx)$ for all $x \in \mathfrak{p}_K^r$ where $2r \geq a(\varkappa) > 1$, we have $\varepsilon_p = 1$.*
2. *Assume that $p$ is odd with $K|\mathbb{Q}_p$ is ramified. We have*
   - *$\varepsilon_p = 1$ if the conductor of $\varkappa$ is odd.*

- *In the case of $a(\varkappa)$ even, the number*

$$\varepsilon_p = \begin{cases} 1, & \textit{if } (p, K|\mathbb{Q}_p) = 1, \\ \left(\frac{-1}{p}\right), & \textit{if } (p, K|\mathbb{Q}_p) = -1. \end{cases} \tag{2}$$

3. *When $p = 2$ with $K|\mathbb{Q}_2$ ramified, we have*

$$\varepsilon_2 = \begin{cases} 1, & \textit{if } K = \mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{3}), \\ -1, & \textit{if } K = \mathbb{Q}_2(\sqrt{6}), \mathbb{Q}_2(\sqrt{-6}) \textit{ with } \varkappa \textit{ minimal}, \\ 1, & \textit{if } K = \mathbb{Q}_2(\sqrt{6}), \mathbb{Q}_2(\sqrt{-6}) \textit{ with } \varkappa \textit{ not minimal.} \end{cases}$$

When $K$ is unramified with $a(\varkappa) = 1$, $\widetilde{\varkappa} := \varkappa^{-1}|_{\mathcal{O}_K^\times}$ can be considered as a character of $\mathcal{O}_K^\times/U_K^1 \cong \mathbb{F}_{p^2}^\times$. Here, we take an additive character $\phi$ of $K$ which induces the canonical additive character $\widetilde{\phi}$ of $\mathbb{F}_{p^2}$. Morita's $p$-adic gamma function is denoted by $\Gamma_p$. Then we prove the following:

**Theorem 0.0.6.** *Let $p$ be an odd unramified supercuspidal prime with $a(\varkappa) = 1$.*

1. *If the order of $\widetilde{\varkappa}$ is even, then $\varepsilon_p = 1$.*
2. *Assume the order $m$ of $\widetilde{\varkappa}$ is odd that divides $p - 1$. Write $p = bm + 1$ for some $b \in \mathbb{N}$. Then*

$$\varepsilon_p = p^{-1/m} \left\{ \Gamma_p\Big(\frac{1}{2m}\Big) / \Gamma_p\Big(\frac{1}{m}\Big) \right\}^2.$$

3. *Assume that the order $m$ of $\widetilde{\varkappa}$ divides $p + 1$. Then*

$$\varepsilon_p = \begin{cases} -1, & \textit{if } m \textit{ odd and } p \equiv 1 \pmod 4, \\ 1, & \textit{if } m \textit{ even and } p \equiv 1 \pmod 4, \\ 1, & \textit{if } m \textit{ even and } p \equiv 3 \pmod 4 \textit{ with } \frac{p+1}{m} \textit{ odd.} \end{cases}$$

*For $p = 2$, we have $\varepsilon_2 = 1$.*

Let $\varepsilon(f)$ be the global $\varepsilon$-factor associated to $f$ and $\varepsilon_p(f)$ be its $p$-part. For the character $\chi_p$ defined before, the newform twisted by $\chi_p$ is denoted by $f \otimes \chi_p$. The following two corollaries determine the nature of a supercuspidal prime.

**Corollary 0.0.7.** *Assume that $\pi_{f,p}$ is supercuspidal. For odd primes, it is always induced by a quadratic extension $K|\mathbb{Q}_p$. If $N_p \geq 2$ is even, then $K$ is the unique unramified quadratic extension of $\mathbb{Q}_p$. In the case of $N_p \geq 3$ odd with $p \equiv 3 \pmod 4$, we have*

- *$K = \mathbb{Q}_p(\sqrt{-p})$ if $\varepsilon(f \otimes \chi_p) = \chi_p(N')\varepsilon(f)$.*
- *$K = \mathbb{Q}_p(\sqrt{-p\zeta_{p-1}})$ if $\varepsilon(f \otimes \chi_p) = -\chi_p(N')\varepsilon(f)$.*

*The same cannot be concluded, when $p \equiv 1 \pmod 4$.*

Consider the quadratic character $\chi$ associated to the quadratic extensions ramified only at 2 by class field theory. We then have the following two relations:

I. $\varepsilon(f \otimes \chi) = \chi(N')\varepsilon(f)$, if $\varepsilon_2 = 1$,

II. $\varepsilon(f \otimes \chi) = -\chi(N')\varepsilon(f)$, if $\varepsilon_2 = -1$.

**Corollary 0.0.8.** *Let $p = 2$ be a dihedral supercuspidal prime for $f$. Then $\pi_{f,2}$ is always induced by a quadratic extension $K|\mathbb{Q}_2$. If $f$ is 2-minimal and $N_2 \geq 2$ is even, then $K$ is the unique unramified quadratic extension of $\mathbb{Q}_2$. In the ramified case, we have the following classifications of $K$:*

| ***Classification of*** $K$ ***for*** $p = 2$ | | |
|---|---|---|
| *$p$-minimality of $f$* | $K = \mathbb{Q}_2(\sqrt{t})$ | *Property* |
| *Yes* | $t = -1, -2, 2, 3$ | *I* |
| | $t = -6, 6$ | *II* |
| *No* | $t = -1, -2, 2, 3, -6, 6$ | *I* |

*Remark* 0.0.9. If $f$ is not 2-minimal, then we cannot distinguish whether the extension $K|\mathbb{Q}_2$ is unramified or ramified (since $N_2$ is even in both cases). Moreover, when $K|\mathbb{Q}_2$ is ramified, the above property $I$ always satisfies for $f$. If $f$ is a 2-minimal newform, then the extension $K|\mathbb{Q}_2$ (from which the local representation is induced from) can be distinguished by the parity of $N_2$ ($K$ is unramified if $N_2$ is even and $K$ is ramified if $N_2$ is odd).

The structure of the thesis is described as follows:

- **Chapter** 1 presents a brief overview of the basic properties of modular forms, Hecke operators and local Hilbert symbols.
- **Chapter** 2 deals with the 2-cocycle that determines the Brauer classes of the endomorphism algebra attached to a non-CM primitive cusp form.
- **Chapter** 3 describes the Galois representations attached to modular forms and local global compatibility.
- **Chapter** 4 contains the structure of the character $\chi$ (from which the local Galois representation is induced) on the inertia group.
- **Chapter** 5 determines the local Brauer classes of endomorphism algebras attached to a non-CM primitive cusp form for **all** supercuspidal primes in terms of traces of adjoint lifts at auxiliary primes. We provide numerical examples using `Sage` and LMFDB supporting some of our theorems in the end of this chapter based on our work [6].
- **Chapter** 6 is the last chapter of this thesis. Here, we study the variance of local epsilon factor for a modular form with arbitrary nebentypus with respect to twisting by a quadratic character. As an application, we detect the type (unramified or ramified) of the supercuspidal representation from that information. This is based on our work [5].

# 1

# Preliminaries

In this chapter we will briefly review some basic properties of modular forms and local Hilbert symbols which will be very useful to read this thesis. The interested readers should consult [19], [21] and [46].

## 1.1 Modular forms

The modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ in the following way:

$$\gamma z = \tfrac{az+b}{cz+d}, \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } z \in \mathbb{H}.$$

For any $N \in \mathbb{N}$, the **principal congruence subgroup** of level $N$ is defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \pmod{N}, b \equiv c \equiv 0 \pmod{N} \right\}.$$

In particular, $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. A subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is called a **congruence subgroup** of level $N$ if $\Gamma(N) \subset \Gamma$. The following are the important examples of congruence subgroups other than principal congruence subgroup and $\mathrm{SL}_2(\mathbb{Z})$:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}$$

and

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N}, a \equiv d \equiv 1 \pmod{N} \right\}.$$

Given a meromorphic function $f : \mathbb{H} \to \mathbb{C}$, for each $k \in \mathbb{Z}$ we define the **weight-$k$ operator** on $f$ by

$$f[\gamma]_k(z) = \det(\gamma)^{k-1}(cz+d)^{-k} f(\gamma z) \ \forall \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q}) \text{ and } z \in \mathbb{H}.$$

Let $D' = \{z \in \mathbb{C} : |z| < 1\} - \{0\}$ be the punctured disc. Consider a meromorphic function $f$ on $\mathbb{H}$ that satisfies $f[\gamma]_k(z) = f(z) \ \forall \ \gamma \in \Gamma$. Since a congruence subgroup $\Gamma \supset \Gamma(N)$ for some $N$, it contains a translation matrix $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ for some minimal $r \in \mathbb{N}$. Thus, the function $f$ is $r\mathbb{Z}$-periodic and defines a holomorphic function $g : D' \to \mathbb{C}$ with $f(z) = g(e^{2\pi iz/r})$. The holomorphy of $f$ on $\mathbb{H}$ implies that $g$ is holomorphic on $D'$. We say $f$ is holomorphic at $\infty$ if $g$ is holomorphic at 0. The expression

$$f(z) = \sum_{n=0}^{\infty} a_n(f) q_r^n, \quad q_r = e^{2\pi iz/r}$$

is called the Fourier expansion of $f$.

A congruence subgroup $\Gamma$ acts on $\mathbb{Q} \cup \{\infty\}$. The cusps of $\Gamma$ is defined to be the $\Gamma$-equivalence classes of $\mathbb{Q} \cup \{\infty\}$. Let $s \in \mathbb{Q} \cup \{\infty\}$ be any cusp. Then we can write $s = \alpha(\infty)$ for some $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. Note that $\alpha^{-1}\Gamma\alpha$ is also a congruence subgroup. The function $h = f[\alpha]_k$ is holomorphic on $\mathbb{H}$ and satisfies $h[\beta]_k = h \ \forall \ \beta \in \alpha^{-1}\Gamma\alpha$. We say $f$ is holomorphic at the cusp $s$ if $f[\alpha]_k$ is holomorphic at $\infty$. We also say $f$ is holomorphic at the cusps if $f[\alpha]_k$ is holomorphic at $\infty$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

**Definition 1.1.1.** *A modular form of weight $k$ with respect to $\Gamma$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying*

- *$f[\alpha]_k = f$ for all $\alpha \in \Gamma$,*
- *$f$ is holomorphic at the cusps of $\Gamma$.*

*If in addition,* $a_0(f[\alpha]_k) = 0$ *for all* $\alpha \in \mathrm{SL}_2(\mathbb{Z})$*, then* $f$ *is called a cusp form of weight* $k$ *with respect to* $\Gamma$*.*

We denote the set of modular forms (resp. cusp forms) of weight $k$ with respect to $\Gamma$ by $M_k(\Gamma)$ (resp. $S_k(\Gamma)$). These are finite dimensional vector spaces over $\mathbb{C}$.

From now on we will only consider the congruence subgroup $\Gamma_1(N)$. For a Dirichlet character $\epsilon \in \widehat{(\mathbb{Z}/N\mathbb{Z})^{\times}}$, the $\epsilon$-eigenspace of $S_k(\Gamma_1(N))$ is defined as follows:

$$S_k(N,\epsilon) = \{f \in S_k(\Gamma_1(N)) : f[\alpha]_k(z) = \epsilon(d)f(z), \text{ for all } \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \text{ and } z \in \mathbb{H}\}.$$

The space $S_k(\Gamma_1(N))$ decomposes as a direct sum of $\epsilon$-eigenspaces

$$S_k(\Gamma_1(N)) = \bigoplus_{\epsilon} S_k(N,\epsilon).$$

The eigenspace $S_k(N,\epsilon)$ is called the space of cusp forms of weight $k$, level $N$ and character $\epsilon$.

## 1.2 Hecke Operators

Let $f(z) = \sum_{n=1}^{\infty} a_n(f)q^n \in S_k(N,\epsilon)$ be a cusp form of weight $k$, level $N$ and character $\epsilon$. For $n \in (\mathbb{Z}/N\mathbb{Z})^{\times}$, we define the Hecke operator $\langle n \rangle$ on $S_k(N,\epsilon)$ as follows:

$$\langle n \rangle f = \epsilon(n)f.$$

This definition can be extended to $\langle n \rangle$ for any $n \in \mathbb{N}$ by setting $\langle n \rangle := \langle d \rangle$ with $n \equiv d \pmod{N}$, if $(n,N) = 1$ and $\langle n \rangle = 0$ if $(n,N) \neq 1$. For $n \in \mathbb{N}$, the Hecke operator

$$T_n : S_k(N,\epsilon) \to S_k(N,\epsilon)$$

is defined in the following way:

$$(T_n f)(z) = \sum_{m=1}^{\infty} \Big( \sum_{t|(m,n)} \epsilon(t)t^{k-1}a_{mn/t^2}(f) \Big) q^m.$$

We now list some properties of Hecke operators.

1. The map $n \mapsto \langle n \rangle$ is multiplicative. In other words, $\langle mn \rangle = \langle m \rangle \langle n \rangle$ for all $m, n \in \mathbb{N}$.

2. For $n \in \mathbb{N}$, the Hecke operators $T_n$ commute with each other, that is,

$$T_m T_n = T_n T_m.$$

   Moreover, if $m$ and $n$ are co-prime, then $T_m T_n = T_{mn}$.

3. One knows that the space $S_k(\Gamma_1(N))$ is equipped with the **Petersson inner product**. With respect to that inner product the Hecke operators $\{\langle n \rangle, T_n : (n, N) = 1\}$ are normal.

4. A Hecke eigenform $f(z) = \sum_{n=0}^{\infty} a_n(f) q^n$ is a nonzero modular form in $M_k(\Gamma_1(N))$ that is an eigenform for the Hecke operators $\{\langle n \rangle, T_n : n \in \mathbb{N}\}$. We say a Hecke eigenform is normalized if $a_1(f) = 1$.

5. Recall that $S_k(\Gamma_1(M)) \subset S_k(\Gamma_1(N))$ if $M|N$. We have a subspace of $S_k(\Gamma_1(N))$ containing lower level cusp forms, called the subspace of **oldforms at level** $N$. The orthogonal complement of this subspace with respect to the Petersson inner product is called the subspace of **newforms at level** $N$. It has an orthogonal basis consisting of normalized Hecke eigenforms, such basis elements are called **newforms**.

## 1.3 Twists of modular forms

Let $f(z) = \sum_{n=0}^{\infty} a_n(f) q^n$ be a modular form in $M_k(N, \epsilon)$. Then the twist of $f$ by a Dirichlet character $\chi \pmod{M}$ is defined as follows:

$$(f \otimes \chi)(z) = \sum_{n=0}^{\infty} \chi(n) a_n(f) q^n.$$

One knows that $f \otimes \chi \in M_k(NM^2, \epsilon\chi^2)$ is a modular form of weight $k$, level $NM^2$ and character $\epsilon\chi^2$ [47, Proposition 3.64].

**Definition 1.3.1.** *[38, Section 3] We say a modular form $f$ has a complex multiplication (CM) by a non-trivial Dirichlet character $\chi$ if*

$$a_p(f) = a_p(f \otimes \chi) = \chi(p)a_p(f)$$

*for a set of primes $p$ of density* 1.

If $f$ does not have complex multiplication by any Dirichlet character, then $f$ is called a non-CM modular form. From now on we will consider non-CM Hecke eigenforms.

## 1.4 Local symbols

Let $F$ be a number field. Let $v$ be a place of $F$ which is prime to 2, and let $\pi_v$ be a fixed uniformizer in $F_v$. For $a, b \in F_v^\times$, write $a = \pi_v^{v(a)} \cdot a'$ and $b = \pi_v^{v(b)} \cdot b'$ where we take the valuation $v$ to be normalized so that $v(\pi_v) = 1$. Then the local symbol $(a, b)_v$ is given by the following equation:

$$(a, b)_v = (-1)^{v(a)v(b)\frac{Nv-1}{2}} \cdot \Big(\frac{b'}{v}\Big)^{v(a)} \cdot \Big(\frac{a'}{v}\Big)^{v(b)}. \tag{1.1}$$

Here $\left(\frac{\cdot}{v}\right)$ is the local quadratic residue symbol in the residue field at $v$.

Now assume $v|2$. We first consider the case $F = \mathbb{Q}$ so that $v = 2$. Let $\epsilon$ and $\omega$ be the homomorphisms of the unit group of $\mathbb{Q}_2^\times$ into $\mathbb{Z}/2\mathbb{Z}$ defined respectively by the following rules:

$$\epsilon(a) = \frac{a-1}{2} \bmod 2, \quad \omega(a) = \frac{a^2-1}{8} \bmod 2$$

where $a$ is a unit in $\mathbb{Q}_2^\times$. Now for units $a, b \in \mathbb{Q}_2^\times$, we have:

$$(2, a)_2 = (-1)^{\omega(a)}, \tag{1.2}$$

$$(a, b)_2 = (-1)^{\epsilon(a)\epsilon(b)}. \tag{1.3}$$

Note that the formulas just defined above will determine the local symbol $(a, b)_2$ explicitly for any $a, b \in \mathbb{Q}_2^\times$.

Now we define Hilbert symbol $(.,.)_v$ for a field $F$ other than $\mathbb{Q}$. We found the only reference [21, p. 247] for this. Consider an arbitrary field $F$, not necessarily equal to $\mathbb{Q}$.

Let $v$ be a place of $F$ lying above the prime $p = 2$, and $F_v$ be the completion of $F$ at $v$. Fix a uniformizer $\pi_v$ of $F_v$. Set $F_0 = F_{v_0} := F_v \cap \mathbb{Q}_2^{\mathrm{ur}}$, where $\mathbb{Q}_2^{\mathrm{ur}}$ is the maximal unramified extension of $\mathbb{Q}_2$, with its ring of integer $\mathcal{O}_0$. Let $\mathcal{R}$ denote the group of multiplicative representatives of the residue field of $F_v$ in $\mathcal{O}_0$. We know that $\mathrm{Gal}(\mathbb{Q}_2^{\mathrm{ur}}|\mathbb{Q}_2)$ is topologically generated by the Frobenius automorphism which will be denoted by $\varphi$. Let $\widehat{\mathcal{O}}$ denote the ring of integers of the completion $\widehat{\mathbb{Q}_2^{\mathrm{ur}}}$ of $\mathbb{Q}_2^{\mathrm{ur}}$ and $\varphi$ the continuous extension of $\varphi$ to $\widehat{\mathbb{Q}_2^{\mathrm{ur}}}$. We define the Frobenius operator $\Delta_X$ as follows:

$$\Delta_X(f) = f^{\Delta_X} = \sum \varphi(\alpha_n) X^{2n}$$

for a formal power series $f(X) = \sum \alpha_n X^n$ over $\widehat{\mathcal{O}}$. This operator $\Delta_X$ is a $\mathbb{Z}_2$-endomorphism of $\widehat{\mathcal{O}}[[X]]$. Note that $\Delta_X$ depends on $X$. For simplicity, write $\Delta$ instead of $\Delta_X$. Let

$$V(X) = \frac{1}{2} + \frac{1}{s(X)} \in \mathcal{O}_0\{\{X\}\},$$

where $s(X) = z(X)^{2^n} - 1$ and $z(X) \in 1 + X\mathcal{O}_0[[X]]$ is such that $z(\pi_v) = \zeta$ is a $2^n$-th primitive root of unity in $F_v$. Consider the homomorphism $l_X$ on $\mathcal{O}_0((X))^\times$ defined by $l_X(f) = \frac{1}{p}\log(f^p/f^\Delta)$. For $p = 2$, the formula of $l_X(f)$ is not defined for an arbitrary series of $\mathcal{O}_0((X))$, but for series $f$ of $Q$ where

$$Q = \{X^m a\epsilon(X) : \epsilon(X) \in 1 + X\mathcal{O}_O[[X]], a \in \mathcal{O}_0^\times, a^\varphi \equiv a^2 \pmod 4, m \in \mathbb{Z}\}.$$

Denote $l_X$ by $l$. Let $\alpha, \beta \in Q$. Put

$$\Phi_{\alpha,\beta} = \alpha^{-1}\alpha' l(\beta) - l(\alpha)\beta^{-1}\beta' + l(\alpha)l(\beta)',$$

$$\Phi^{(1)}_{\alpha,\beta} = \Big(\frac{\Delta}{2}\Big(\frac{\alpha^2 - \alpha^\Delta}{2\alpha^\Delta}\frac{\beta^2 - \beta^\Delta}{2\beta^\Delta}\Big)\Big)'$$

and

$$\Phi^{(2)}_{\alpha,\beta} = X^{-1} v_X(\alpha) v_X(\beta) l_X(1 + s_{n-1}(X))$$

where $v_X$ is the discrete valuation of $\mathcal{O}_0((X))$ corresponding to $X$. Define the pairing

$$< .,. >_X : Q \times Q \to < \zeta >$$

by the formula

$$<\alpha,\beta>_X = \zeta^{\mathrm{Tr}\ \mathrm{res}\left(\Phi_{\alpha,\beta}+\Phi^{(1)}_{\alpha,\beta}+\Phi^{(2)}_{\alpha,\beta}\right)r(X)V(X)}$$

where $\mathrm{Tr} = \mathrm{Tr}_{F_0|\mathbb{Q}_2}, r(X) = 1 + 2^{n-1}\Delta_X r_0(X)$ and $r_0(X) \in X\mathcal{O}_0[X]$ be a polynomial of degree $e_v - 1$ satisfying a certain condition [21, Section 3.4, p. 223]. For elements $\alpha, \beta \in F_v^\times$, let $\alpha(X), \beta(X) \in Q$ be such that $\alpha(\pi_v) = \alpha$ and $\beta(\pi_v) = \beta$. Put $<\alpha,\beta>_{\pi_v} = <\alpha(X),\beta(X)>_X$. The pairing $<.,.>_{\pi_v}$ is invariant with respect to the choice of a uniformizer $\pi_v$ in $F_v$. We will often write $<.,.>_v$ instead of $<.,.>_{\pi_v}$. It is bilinear and antisymmetric. Furthermore,

$$<\alpha,\alpha>_v = 1, \quad <\theta,\alpha>_v = 1 \quad \text{for } \alpha \in F_v^\times \text{ and } \theta \in \mathcal{R}^\times$$

and $<\alpha, 1-\alpha>_v = 1$ for every $\alpha$ different from 0 and 1. The pairing $<a,b>_v$ coincides with the Hilbert symbol $(a,b)_v$ follows from [21, Chapter 7, Section 4, p. 255].

# 2

# Brauer class of X and its co-cycle

In this chapter, we study the Brauer class of the endomorphism algebra $X := X_f$ attached to a non-CM primitive cusp form $f$ of weight $k \geq 2$, level $N \geq 1$, with character $\epsilon$.

## 2.1 The alegrba X and its Brauer class

Let $f(z) = \sum_{n\geq 1} a_n(f)q^n \in S_k(N,\epsilon)$ be a primitive non-CM cusp form of weight $k \geq 2$, level $N \geq 1$ and nebentypus $\epsilon$. By primitive we mean that $f$ is a normalized newform that is a common eigenform of all the Hecke operators. Consider the number field $E = \mathbb{Q}(\{a_n(f)\})$ obtained by attaching all the Fourier coefficients of $f$. For $k = 2$, let $M_f$ denote the abelian variety attached to $f$ by Shimura. For $k > 2$, we also denote by $M_f$ the Grothendieck motive over $\mathbb{Q}$ with coefficients in $E$ associated to $f$ by Scholl. Let $X_f$ denote the $\mathbb{Q}$-algebra of endomorphisms of $M_f$ defined by

$$X = X_f := \mathrm{End}_{\overline{\mathbb{Q}}}(M_f) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

For simplicity we denote the $n$-th Fourier coefficient of $f$ by $a_n$ instead of $a_n(f)$ unless we further mention. We now define a group of extra twists for a non-CM cusp form $f$.

**Definition 2.1.1.** *A pair $(\gamma, \chi_\gamma)$ where $\gamma \in \mathrm{Aut}(E)$ and $\chi_\gamma$ is an $E^\times$-valued Dirichlet character is said to be an extra twist for $f$ if $a_p^\gamma = a_p \cdot \chi_\gamma(p)$, for all primes $p \nmid N$. Such a Dirichlet character $\chi_\gamma$ is unique since $f$ is assumed to be non-CM.*

Here $a_p^\gamma$ means that $\gamma(a_p)$. For a given $\gamma$, the uniqueness of the Dirichlet character $\chi_\gamma$ in the above definition follows from the fact that $a_p^\gamma \neq a_p$ for all primes $p$ but finitely many, since $f$ is assumed to be non-CM [cf. Section 1.3]. Let $\Gamma \subset \mathrm{Aut}(E)$ be the abelian subgroup of extra twists. Let $F$ denote the subfield of $E$ fixed by $\Gamma$. For $\gamma, \delta \in \Gamma$, the relation $\chi_{\gamma\delta} = \chi_\gamma \chi_\delta^\gamma$ shows that $\gamma \mapsto \chi_\gamma(g)$ is a 1-cocycle for a fixed $g \in G_\mathbb{Q} := \mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q})$. By Hilbert's theorem 90, since $H^1(\Gamma, E^\times)$ is trivial, there exists $\alpha(g) \in E^\times$ such that

$$\alpha(g)^{\gamma-1} = \chi_\gamma(g) \tag{2.1}$$

for all $\gamma \in \Gamma$. The element $\alpha(g)$ is well defined modulo $F^\times$. Thus, the map $\tilde{\alpha} : G_\mathbb{Q} \to E^\times/F^\times$, $g \mapsto \alpha(g)$ modulo $F^\times$, is a continuous homomorphism. Now $\alpha : G_\mathbb{Q} \to E^\times$ can be thought of as a lift of $\tilde{\alpha}$. Here we list some properties of any lift $\alpha$ of the homomorphism $\tilde{\alpha}$. For the proof, the interested readers should consult [3, Lemma 1] and [40, Theorem 5.5].

**Proposition 2.1.2.** *Let $\rho_f$ denote the $\lambda$-adic representation attached to $f$, for some prime $\lambda \,|\, \ell$ of $E$. The map $\alpha$ satisfies the following properties:*

1. *$\alpha^2(g) \equiv \epsilon(g)$ modulo $F^\times$, for all $g \in G_\mathbb{Q}$.*
2. *$\alpha(g) \equiv$ trace$(\rho_f(g))$ modulo $F^\times$, for all $g \in G_\mathbb{Q}$, provided that the trace is nonzero.*
3. *$\alpha(\mathrm{Frob}_p) \equiv a_p$ modulo $F^\times$, for all prime $p \nmid N$ with $a_p \neq 0$.*

Comparing (2.1) and the property (1) of the above proposition, we have the following identity:

$$\chi_\gamma^2 = \epsilon^{\gamma-1} \tag{2.2}$$

for all $\gamma \in \Gamma$. According to [39], the Brauer class of $X$ in $\mathrm{Br}(F) = \mathrm{H}^2(G_F, \bar{F}^\times)$ is given by the 2-cocycle

$$c_\alpha(g,h) = \frac{\alpha(g)\alpha(h)}{\alpha(gh)}; \quad g, h \in G_F,$$

for any continuous lift $\alpha$ of $\tilde{\alpha}$ and this class is independent of the lift choosen [4, Section 3.2]. The restriction $[c_\alpha|_{G_v}] \in \mathrm{Br}(F_v) = \mathrm{H}^2(G_v, \bar{F_v}^\times)$ gives the local Brauer class of $X_v$ for any prime $v$ of $F$. Since $\mathrm{inv}_v(c_\alpha|_{G_v}) \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$, the invariant map $\mathrm{inv}_v$ at $v$ completely determines the class $[X_v]$ in $\mathrm{Br}(F_v)$. For the definition of the invariant map, we refer to [46, Ch. XIII, Section 3, p. 193].

## 2.2 Invariant of local 2-cocycle

We now state a lemma [4, Lemma 9] which will be useful to determine the Brauer class of any local 2-cocycle of the form:

$$c_S(g,h) = \frac{S(g)S(h)}{S(gh)};$$

for all $g, h \in G_v$, where $S : G_v \to \bar{F_v}^\times$ is any map.

**Lemma 2.2.1.** *Let $S : G_v \to \bar{F_v}^\times$ be any map and $t : G_v \to \bar{F_v}^\times$ be an unramified homomorphism such that*

1. *$S(i) \in F_v^\times$, for all $i \in I_v$,*
2. *$S(g)^2/t(g) \in F_v^\times$, for all $g \in G_v$.*

*For any arithmetic Frobenius $\mathrm{Frob}_v$, we have then*

$$\mathrm{inv}_v(c_S) = \frac{1}{2} v\Big(\frac{S(\mathrm{Frob}_v)^2}{t(\mathrm{Frob}_v)}\Big) \textit{ mod } \mathbb{Z} \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}.$$

*Here $v : F_v^\times \to \mathbb{Z}$ is the surjective valuation.*

Since $[c_\alpha|_{G_v}]$ determines the algebra $X_v$, the most obvious choice for $S$ in the above lemma would be $\alpha$. When $\alpha$ cannot be taken as $S$ in the above lemma, we have to divide $\alpha$ by a suitable *auxiliary function* (i.e., $S = \alpha/f$) to make the above lemma applicable. Then the cocycle $c_\alpha$ can be decomposed as $c_S c_f$ where $c_S$ and $c_f$ are the cocycles corresponding to $S$ and $f$ respectively. The class of the 2-cocycle $[c_\alpha|_{G_v}]$ and hence, $X_v$ will be determined by

$$\mathrm{inv}_v(c_\alpha) = \mathrm{inv}_v(c_S) + \mathrm{inv}_v(c_f).$$

For odd primes, we will see that there are cases where we do not need any auxilliary function. For $p = 2$, we will always be needed to divide $\alpha$ by (one or more) auxiliary functions unless $N_2 = 2$ and their corresponding cocycles will contribute in the ramification of $X_v$ as *error terms* if they are not trivial.

# 3

# Galois representations attached to modular forms and local global compatibility

Consider a non-CM Hecke eigenform $f = \sum_{n=1}^{\infty} a_n q^n$ of weight $k \geq 2$, level $N \geq 1$ and nebentypus $\epsilon$. Let $M_f$ be the abelian variety attached to $f$ by Shimura [47], when $k = 2$, or the Grothendieck motive associated to $f$ by Scholl [45], when $k > 2$. For all rational prime $\ell$, we consider a prime $\lambda \mid \ell$ of $E = \mathbb{Q}(\{a_n\})$ and let $E_\lambda$ be the completion of $E$ at $\lambda$. For a modular form $f$ as above, Eichler-Shimura-Deligne constructed a Galois representation

$$\rho_f = \rho_{f,\lambda} : G_{\mathbb{Q}} \to \mathrm{GL}_2(E_\lambda),$$

unramified outside the primes dividing $N\ell$, satisfying the following property: for primes $p \nmid \ell N$, we have

$$\mathrm{trace}(\rho_f(\mathrm{Frob}_p)) = a_p, \quad \det(\rho_f) = \chi_\ell^{k-1}\epsilon \tag{3.1}$$

where $\mathrm{Frob}_p$ is an arithmetic Frobenius at $p$, and $\chi_\ell$ is the $\ell$-adic cyclotomic character.

Let $G_p$ be the decomposition group at the prime $p$. In this thesis, we will be using information about the local Galois representation $\rho_{f,\lambda}|_{G_p}$ with $\ell = p$. We refer to it as $(p,p)$ Galois representation. For more details about $(p,p)$-Galois representations, refer to [24].

Let $\mathbb{A}_{\mathbb{Q}}$ denotes the adeles of $\mathbb{Q}$ and $\pi$ be the automorphic representation of the adele group $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ associated to $f$ [36]. Then $\pi$ has a decomposition as a restricted tensor product

$$\pi = \sideset{}{'}\bigotimes_p \pi_p$$

over all places $p$ (including the infinite primes). Each local component $\pi_p$ is an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. By the local Langlands correspondence for $n = 2$, these representations $\pi_p$ are in a bijection with (isomorphism classes of) complex 2-dimensional Frobenius-semisimple Weil-Deligne representations.

The local global compatibility between these two representations was proved by Carayol in [12] if $\ell \neq p$. In this thesis, we will use the local global compatibility even for $\ell = p$ proved by Saito [43] which we describe now. For $p \neq \ell$ (not necessarily $p \nmid N$), the restriction $\rho_{f,p} := \rho_f|_{G_p}$ induces a representation $'\rho_{f,p} : {}'W_p \to \mathrm{GL}_2(\bar{\mathbb{Q}}_\ell)$ of the Weil-Deligne group $'W_p$ of $\mathbb{Q}_p$. Let $'\rho_{f,p}^{ss}$ denote its Frobenius semisimplification and let the isomorphism class of Frobenius semisimple representation of $'W_p$ associated to $\pi_p$ be denoted by $'\rho(\pi_p)$ [17, Section 3]. The representation $'\rho(\pi_p)$ of the Weil-Deligne group of $\mathbb{Q}_p$ is a pair $(\rho_p(f), N)$ with

1. a representation $\rho_p(f) : W_p \to \mathrm{GL}_2(\mathbb{C})$,
2. a nilpotent endomorphism $N$ of $\mathbb{C}^2$ such that

$$wNw^{-1} = \omega_1(w) \cdot N, \ \forall\ w \in W_p.$$

   Here, $\omega_1$ denote the unramified quasi-character giving the action of $W_p$ on the roots of unity (and by local class field theory it corresponds to the norm $||\,.\,||_p$).

In this setting, we have the following diagram:

$$\begin{tikzcd}
\pi_f = \bigotimes'_p \pi_p \arrow[r, leftrightarrow, "\text{Global Langlands}"] \arrow[dd] & \rho(\pi) = \rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_\ell) \arrow[d, "\text{restriction to } G_p \text{ induces}"] \\
& {}'\rho_{f,p}^{ss} : {}'W_p \to \mathrm{GL}_2(\bar{\mathbb{Q}}_\ell) \arrow[d, leftrightarrow, "(*)"] \\
\pi_p \arrow[r, leftrightarrow, "\text{Local Langlands}"] & {}'\rho(\pi_p) = \rho_p(f) : W_p \to \mathrm{GL}_2(\mathbb{C})
\end{tikzcd}$$

The relation $(*)$ is an isomorphism and it is known for $\ell \neq p$ by the work of Deligne-Langlands-Carayol. Saito proved the isomorphism $(*)$ even for $\ell = p$. For $\ell = p$, the representation $'\rho_{f,p}^{ss}$ is de-Rham and by the landmark paper [14] can be studied using filtered $(\phi, N)$ modules. In the context of this thesis, it will be prudent to use the explicit description of the filtered $(\phi, N)$ module for the given elliptic modular form $f$ following [25]. That the $(p,p)$-Galois representations are important in determining the local Brauer classes of the modular endomorphism algebras are discovered by Brown-Ghate [10].

We are interested to determine the local Brauer classes of the modular endomorphism algebras attached to a non-CM primitive cusp form for primes at which the local Galois representation is supercuspidal. Recall that for a dihedral supercuspidal prime for $f$, the local Galois representation is induced from a quadratic extension $K$ of $\mathbb{Q}_p$. In other words,

$$\rho_f|_{G_p} \sim \mathrm{Ind}_{G_K}^{G_p} \chi.$$

For an odd supercuspidal prime with $N_p = 2$, the extension $K|\mathbb{Q}_p$ is always unramified [8, Section 4]. This will also happen for a dihedral supercuspidal prime $p = 2$.

For the dihedral supercuspidal prime $p = 2$ with $N_p = 2$, we will now show that the extension $K|\mathbb{Q}_2$ is always unramified. For the character $\chi$ of $G_K$, the usual conductor $a(\chi) = \min\{n : \chi(U_K^n) = 1\}$. Let $v_2$ be the normalized valuation of $\mathbb{Q}_2^\times$ and $\delta(K|\mathbb{Q}_2)$, $f(K|\mathbb{Q}_2)$ denote the discriminant and the residual degree for $K|\mathbb{Q}_2$ respectively. We now recall the formula [13, Proposition 4($b$), p. 158] which coincides with the formula for the Artin conductor of a 2-dimensional induced representation of a local Galois group: $a(\mathrm{Ind}_{G_K}^{G_2} \chi) = v_2(\delta(K|\mathbb{Q}_2)) + f(K|\mathbb{Q}_2) a(\chi)$. This gives

$$N_2 = \begin{cases} 2a(\chi), & \text{if } K|\mathbb{Q}_2 \text{ is unramified,} \\ 2 + a(\chi), & \text{if } K|\mathbb{Q}_2 \text{ is ramified with discriminant valuation 2,} \\ 3 + a(\chi), & \text{if } K|\mathbb{Q}_2 \text{ is ramified with discriminant valuation 3.} \end{cases} \tag{3.2}$$

If $K|\mathbb{Q}_2$ is unramified, $N_2$ always becomes even. We see that $N_2 = 2$ happens in the following cases:

1. $K|\mathbb{Q}_2$ is unramified with $a(\chi) = 1$ and
2. $K|\mathbb{Q}_2$ is ramified with discriminant valuation 2 and $a(\chi) = 0$.

The second case cannot occur. Since the algebra $X_f$ is invariant with respect to twisting by a Dirichlet character [39, Proposition 3], without loss of generality one can take $f$ to be minimal in the sense that its level is the smallest among all twists $f \otimes \psi$ of $f$ by Dirichlet characters $\psi$. Then by [11, §41.4 Lemma], we have $a(\chi) \geq d = 2$, a contradiction to $a(\chi) = 0$ in the second case.

**Lemma 3.0.1.** *Let $p = 2$ be an unramified dihedral supercuspidal prime for $f$ with $N_2 = 2$. We have $\alpha(j) \in F^\times$ for all $j \in I_W(K)$, the wild inertia group of $K$.*

*Proof.* Since $N_2 = 2$, we have $a(\chi) = 1$, that is, $\chi|_{U_K^1} = 1$. We know that reciprocity map sends wild inertia group of $K$ onto the principal unit group of $K$. Let $\tau = \tau_k \in I_W(K)$ be an element which is mapped to $k \in U_K^1 \subset K^\times$ under the reciprocity map. Hence, using property (2) of Proposition 2.1.2 we obtain $\alpha(\tau) \equiv \chi(k) + \chi^\sigma(k) \equiv 1 \bmod F^\times$. □

We now recall a corollary of Brauer-Nesbitt theorem.

**Corollary 3.0.2.** *Two semi-simple representations $\rho_1$ and $\rho_2$ of $G_K$ on a finite-dimensional vector space over any field are equivalent if and only if for all $g \in G_K$ we have trace$(\rho_1(g))$ = trace$(\rho_2(g))$.*

For $\gamma \in \Gamma$, there is a unique Dirichlet character $\chi_\gamma$ such that $f^\gamma \equiv f \otimes \chi_\gamma$. By restricting to the corresponding decomposition group $G_p$, we deduce that the $p$-adic Galois representations are similar. In other words,

$$\rho_{f^\gamma,p} \sim \rho_{f,p} \otimes \chi_\gamma.$$

By the above corollary, we have $\rho_{f^\gamma,p} \sim \rho_{f,p}^\gamma$. Hence, the property (2) of Proposition 2.1.2, is also true even for $p$-adic Galois representation by comparing the traces of the similar $p$-adic Galois representations associated to $f^\gamma$ and $f \otimes \chi_\gamma$.

One knows that $\det(\rho_f) = \chi_\ell^{k-1}\epsilon$, where $\chi_\ell$ is the $\ell$-adic cyclotomic character [cf. Equation 3.1]. We can realize the nebentypus $\epsilon$ as an idelic character as follows: for $x \in \mathbb{Q}_p^\times$, let $[x]$ denote the corresponding element $(1, \cdots, x, \cdots, 1)$ in $\mathbb{A}_\mathbb{Q}^\times$. The restriction of $\epsilon$ to $\mathbb{Q}_p^\times$ is then given by the formula:

$$\epsilon([p^m u]) = \epsilon'(p)^m \epsilon_p(u)^{-1}, \tag{3.3}$$

for $m \in \mathbb{Z}$ and $u \in \mathbb{Z}_p^\times$. Here $\epsilon'$ and $\epsilon_p$ denote the prime-to-$p$ part and $p$-part of the character $\epsilon$ respectively. From class field theory, we know that norm residue map sends $\mathbb{Q}_p^\times \subseteq \mathbb{A}_\mathbb{Q}^\times$ onto a dense subset of the decomposition group $G_p$ at $p$. The Galois character $\epsilon|_{G_p}$ can also be determined by this fact using the formula above.

# 4

# Inertial Galois representations

In this chapter, we discuss about the structure of the induced character at the inertia groups for a dihedral supercuspidal prime. We call a prime $p$ to be a dihedral supercuspidal prime for a cusp form $f$ if the local Galois representation is induced by a character $\chi$ of an index two subgroup $G_K := \mathrm{Gal}(\bar{\mathbb{Q}}_p|K)$ of the local Galois group $G_p := \mathrm{Gal}(\bar{\mathbb{Q}}_p|\mathbb{Q}_p)$, that is,

$$\rho_f|_{G_p} \sim \mathrm{Ind}_{G_K}^{G_p}\chi.$$

Since the invariant of the 2-cocycle $c_\alpha$ determines the Brauer class of the algebra $X_f$ attached to a non-CM primitive cusp form $f$ [cf. Chapter 2], we need the information of $\alpha$ at the inertia groups to apply Lemma 2.2.1. By [Proposition 2.1.2, property (2)], it is enough to know the structure of the induced character $\chi$ at the inertia groups. We now denote the inertia subgroups of $G_p$ and $G_K$ by $I_p$ and $I_K$ respectively. In this chapter, we assume the familiarity of the reader with [25].

## 4.1 Odd supercuspidal primes

For an odd supercuspidal prime, the local Galois representation is always dihedral, that is, $\rho_f|_{G_p} \sim \mathrm{Ind}_{G_K}^{G_p}\chi$, where $K|\mathbb{Q}_p$ is quadratic. Assume $\sigma$ is the generator of $\mathrm{Gal}(K|\mathbb{Q}_p)$. Choose a finite extension $L|\mathbb{Q}_p$ with the property that $\rho_f$ is crystalline over $L$ (cf. [25] for more details) and $L|\mathbb{Q}_p$ is Galois. Then the inertia type of $\chi$ can be written as follows: $\chi|_{I_p} = \chi|_{I(L|\mathbb{Q}_p)} = \omega_2^l \cdot \chi_1 \cdot \chi_2$ (when $K$ is unramified) [25, Sections 3.3.2] and $\chi|_{I_K} = \chi|_{I(L|K)} = \omega^l \cdot \chi_1 \cdot \chi_2$ (when $K$ is ramified) [25, Sections 3.4.2], where $\omega_2$ is the fundamental character of level 2, $\omega$ is the Teichmüller character and $\chi_m$ is the character having $p$-power order for $m = 1, 2$, and $l$ is an integer. The action of $\sigma$ on these characters is given by the following rule: $\omega_2^\sigma = \omega_2^p, \omega^\sigma = \omega, \chi_1^\sigma = \chi_1$ and $\chi_2^\sigma = \chi_2^{-1}$.

Since $\chi$ does not extend to $G_p$, we have $\chi \neq \chi^\sigma$ on $G_K$ which is equivalent to that $\chi \neq \chi^\sigma$ on $I_K$. The last condition is equivalent to: either $l \not\equiv 0 \pmod{p+1}$ or $\chi_2^\sigma \neq \chi_2^{-1}$ (unramified case) and $\chi_2^\sigma \neq \chi_2^{-1}$ (ramified case).

## 4.2 Dihedral supercuspidal prime $p = 2$

For the dihedral supercuspidal prime $p = 2$, we will see that $\chi|_{I_2}$ can be thought of as a character of an inertia subgroup of a finite Galois extension of $\mathbb{Q}_2$. By a computation similar to [25, Section 3.3.2, Section 3.4.2] for odd primes $p$, we show that $\chi$ restricted to inertia group can be written as $\chi|_{I_2} = \omega_2^l \cdot \chi_1 \cdot \chi_2$ (in the unramified case) and $\chi|_{I_K} = \omega^l \cdot \chi_1 \cdot \chi_2$ (in the ramified case). The results in this section are obtained by generalizing the construction of $\chi$ on the inertia group for $p = 2$ following [25].

Let $W_2$ (respectively $W_K$) be the Weil group of $\mathbb{Q}_2$ (respectively $K$) and $\rho_2(f)$ be the local representation associated to the local representation $\pi_2$ [cf. Chapter 3]. In this case, the inertia group acts reducibly. If it acts irreducibly, then the image of $\rho_2(f)$ becomes an exceptional group. Here, we only concentrate on the dihedral supercuspidal representations. To write down the inertia type $\chi|_{I_2}$ or $\chi|_{I_K}$, we recall the structure of the local Galois representation following [25].

### 4.2.1 The case $K$ unramified

In this case, $\chi$ is a character of $W_K$ which does not extend to $W_2$ and it is finite on $I_2$. Let $\mathrm{Gal}(K|\mathbb{Q}_2)$ be generated by $\sigma$. Let $K = \mathbb{Q}_2(\omega)$ be the unique unramified quadratic extension of $\mathbb{Q}_2$ with $\omega$ a primitive 3-rd root of unity. We choose a finite extension $L|K$ over which $\rho_2(f)$ becomes crystalline and $L|\mathbb{Q}_2$ is Galois. For an integer $m \geq 1$, let $K^m$ be the unique cyclic unramified extension of $K$ of degree $m$. Consider the polynomial $g(X) = \pi X + X^4$, where $\pi$ is a fixed uniformizer of $K$. For a Lubin-Tate module $M$, consider the $\mathcal{O}_K$-module of $\pi^{n+1}$-torsion points

$$W_g^n := \ker([\pi^{n+1}]_M)$$

whose module structure is induced by the formal group attached to $g(X)$. Let $K(W_g^n)$ be its field. By local class field theory, it is a totally ramified abelian extension of $K$ and its Galois group

$$\mathrm{Gal}(K(W_g^n)|K) \cong U_K/U_K^{n+1} = \mathbb{F}_4^\times \times \mathcal{O}_K/\pi^n, \tag{4.1}$$

where $U_K$ and $U_K^{(n+1)}$ denote the units and $(n+1)$-th principal units of $K$ respectively. Furthermore, since $g(X)$ is defined over $\mathbb{Q}_2$ (if the uniformizer $\pi$ is chosen from $\mathbb{Q}_2$), the extension $K(W_g^n)|\mathbb{Q}_2$ is also Galois.

Consider a finite cyclic extension $F|K$ such that $\chi|_{I_F}$ is trivial. By local class field theory the field $F$ is contained in $K^m K(W_g^n)$ for some $m$ and $n$ and so $\rho_2(f)$ restricted to its inertia subgroup is trivial. For this reason, we take $L = K^m K(W_g^n)$ over which $\rho_2(f)$ becomes crystalline as $\rho_2(f)$ is trivial on $I_L$ and if the fixed uniformizer $\pi$ is chosen to be 2, the extension $L|\mathbb{Q}_2$ becomes Galois.

**Description of** $\mathrm{Gal}(L|\mathbb{Q}_2)$

We now describe $\mathrm{Gal}(L|\mathbb{Q}_2)$ in detail. Let $\alpha$ be a root of $g^{(n+1)}(X)$ but not a root of $g^{(n)}(X)$, where $g^{(n)}(X)$ denote the $n$-th iterate of $g(X)$. We have an identification of fields $K(W_g^n) = K(\alpha)$ with $\mathbb{Q}_2(\alpha)|\mathbb{Q}_2$ a totally ramified extension of degree $(2^2-1)\cdot 2^{2n}$ and $L|\mathbb{Q}_2(\alpha)$ is an unramified extension of degree $2m$. Let $\sigma$ be a generator of $\mathrm{Gal}(L|\mathbb{Q}_2(\alpha))$ and its projection to the generator of $\mathrm{Gal}(K|\mathbb{Q}_2)$ is also denoted by $\sigma$ .

Note that the inertia subgroup of $\mathrm{Gal}(L|\mathbb{Q}_2)$ is $\mathrm{Gal}(L|K^m) \simeq \mathrm{Gal}(K(W_g^n)|K)$. Here $K(W_g^0) = K(\beta)$ with $\beta$ a root of $X^3+2=0$. Let $\Delta$ be its Galois group over $K$ which is generated by an element, say $\delta$, of order 3. It is isomorphic to the tame part of the inertia subgroup of $\mathrm{Gal}(L|\mathbb{Q}_2)$. Since the order of $\delta$ and 2 are relatively prime, $\delta$ can be lifted uniquely to an element of order 3 in $\mathrm{Gal}(K(W_g^n)|K)$, again denoted by $\delta$. The wild part of the inertia subgroup of $\mathrm{Gal}(L|\mathbb{Q}_2)$ is isomorphic to $\Gamma = \mathcal{O}_K/2^n \cong \mathbb{Z}/2^n \oplus \mathbb{Z}/2^n =< \gamma_1 > \oplus < \gamma_2 >$ with $\gamma_1, \gamma_2$ each have order $2^n$.

The full inertia subgroup of $\mathrm{Gal}(L|\mathbb{Q}_2)$ is $\Delta \times \Gamma$. This is a normal subgroup and it is generated by the elements $\delta, \gamma_1$ and $\gamma_2$. These generators are characterized by the Equ. (4.4) below. Since $\sigma^2$ fixes $K(W_g^n)$ the action of $\sigma$ on $\mathrm{Gal}(L|K^m)$ by conjugation is an involution. Indeed, if $h \in \mathrm{Gal}(L|K^m)$ and $x \in K^m$, then we have $\sigma^2 \cdot h(x) = \sigma^2 h \sigma^{-2}(x) = \sigma^2(\sigma^{-2}(x)) = x = h(x)$ and if $x \in K(W_g^n)$, we have $\sigma^2 \cdot h(x) = \sigma^2 h \sigma^{-2}(x) = \sigma^2(h(x)) = h(x)$. This action coincides with the action of $\mathrm{Gal}(K|\mathbb{Q}_2)$ on $\mathrm{Gal}(K(W_g^n)|K))$ by conjugation. The group $\mathrm{Gal}(K|\mathbb{Q}_2)$ acts on $\mathcal{O}_K^\times/U_K^{n+1}$ in a natural way. Note that $\mathcal{O}_K^\times/U_K^{n+1} = \mathcal{O}_K^\times/U_K^1 \times U_K^1/U_K^{n+1} \cong \left(\mathcal{O}_K/2\right)^\times \times \mathcal{O}_K/2^n \cong \mathbb{F}_4^\times \times \mathcal{O}_K/2^n$ and

$$\mathcal{O}_K/2^n = \mathbb{Z}_2[\alpha]/2^n = (\mathbb{Z}_2 \oplus \mathbb{Z}_2 \cdot \alpha)/2^n \cong \mathbb{Z}/2^n \oplus \mathbb{Z}/2^n \cdot \bar{\alpha} \text{ with } \alpha^\sigma = -\alpha.$$

Let $\rho_K : K^\times \to \mathrm{Gal}(K^{ab}|K)$ be the norm residue map. We again denote the restriction $\rho_K|_{\mathcal{O}_K^\times}$ modulo $U_K^{n+1}$ by $\rho_K$. This is the isomorphism (4.1). We now consider the following commutative diagram [26, Thoerem 6.11]:

$$\begin{array}{ccc}
\mathcal{O}_K^\times/U_K^{n+1} = \mathbb{F}_4^\times \times \mathcal{O}_K/2^n & \xrightarrow{\rho_K} & \mathrm{Gal}(F_1|K) \\
\Big\downarrow{\scriptstyle \sigma} & & \Big\downarrow{\scriptstyle \sigma^*} \\
\mathcal{O}_K^\times/U_K^{n+1} = \mathbb{F}_4^\times \times \mathcal{O}_K/2^n & \xrightarrow{\rho_K} & \mathrm{Gal}(F_1|K)
\end{array}$$

where $F_1 = K(W_g^n)$ and the map $\sigma^*$ is obtained by the conjugated action of $\sigma$ on $\mathrm{Gal}(F_1|K)$. Using the commutativity of the above diagram, we have $\rho_K(\sigma(x)) = \sigma^{-1}\rho_K(x)\sigma$, for all $x \in \mathcal{O}_K^\times/U_K^{n+1}$. This gives us the following relations:

$$\sigma^{-1}\delta\sigma = \delta^2, \quad \sigma^{-1}\gamma_1\sigma = \gamma_1 \quad \text{and} \quad \sigma^{-1}\gamma_2\sigma = \gamma_2^{-1}. \tag{4.2}$$

### Action of $\sigma$

By the action (2.1) of [25], the character $\chi$ on $I_2$ can be thought of a character $\chi$ on the inertia subgroup of $\mathrm{Gal}(L|\mathbb{Q}_2)$ which is $\Delta \times \Gamma$ [25, Section 3.3.2]. Write

$$\chi|_{I_2} = \chi|_{I(L|\mathbb{Q}_2)} = \omega_2^l \cdot \chi_1 \cdot \chi_2, \tag{4.3}$$

where $\omega_2$ is the fundamental character of level 2, $l$ is an integer and $\chi_m$ is the character taking $\gamma_m$ to a $2^n$-th root of unity $\zeta_m$ for $m = 1, 2$. Let us assume that $\chi_1$ takes $\gamma_1$ to $\zeta_{2^r}$

and $\chi_2$ takes $\gamma_2$ to $\zeta_{2^s}$. Here, we denote by $\zeta_{2^r}$ and $\zeta_{2^s}$ a primitive $2^r$-th root of unity and a primitive $2^s$-th root of unity respectively, so $r, s \leq n$. Let $\sigma$ be the non-trivial element of the Galois group of $K|\mathbb{Q}_2$ and it acts on the above characters in the following way:

$$\omega_2^\sigma = \omega_2^2, \quad \chi_1^\sigma = \chi_1, \quad \chi_2^\sigma = \chi_2^{-1}. \tag{4.4}$$

The condition that $\chi$ does not extend to $W_2$, we have $\chi \neq \chi^\sigma$ on $W_K$ which is further equivalent to that $l \not\equiv 0 \pmod 3$ or $\zeta_{2^s} \neq \zeta_{2^s}^{-1}$. Since $\zeta_{2^r}^\sigma = \zeta_{2^r}$ and $\zeta_{2^s}^\sigma = \zeta_{2^s}^{-1}$, one can deduce that $r < s$.

### 4.2.2 The case $K$ ramified

Let us now assume that $K|\mathbb{Q}_2$ is a ramified quadratic extension with $\chi$ finite on $I_K$ such that $\chi|_{I_K}$ does not extend to $I_2$. Let us denote the $\mathrm{Gal}(K|\mathbb{Q}_2)$ by $< \iota >$. Similar to the unramified case, we find out a Galois extension $L|\mathbb{Q}_2$ such that $\rho_2(f)|_{I_L}$ is trivial.

**Description of** $\mathrm{Gal}(L|\mathbb{Q}_2)$

For an integer $m \geq 1$, let $K^m$ be the unramified extension of $K$ of degree $m$. For a uniformizer $\pi$ of $K$, let $g(X) = \pi X + X^2$ and as before let

$$W_g^n = \{\alpha \in \mathcal{M}_g \mid \pi^{n+1} \cdot \alpha = 0\}.$$

Here, $\mathcal{M}_g$ denote the formal $\mathcal{O}_K$-module whose underlying set is the ring of integers of the completion of $\bar{K}$ and its module structure is induced by the formal group attached to $g$. The field $K_\pi^n = K(W_g^n)$ is a totally ramified abelian extension of $K$ with Galois group isomorphic to $U_K/U_K^{n+1} = \{\pm 1\} \times \mathcal{O}_K/\pi^n$, where $U_K$ and $U_K^{n+1}$ denote the units and $(n+1)$-th principal units of $K$ respectively. Note that $\mathcal{O}_K/\pi^{2n} \cong \mathbb{Z}/2^n \oplus \mathbb{Z}/2^n$.

We now choose a finite cyclic extension $F|K$ such that $\chi|_{I_F} = 1$. As every abelian extension of $K$ is contained in $K^m K_\pi^n$ for some $m, n$ by class field theory, we have an inclusion of fields $F \subset K^m K_\pi^n$, for some $m, n$. We take a uniformizer $\pi$ of $K$ such that $\pi^\iota = -\pi$ (for any lift of $\iota$, again call $\iota$). The polynomial $g_{-\pi}(X) = -\pi X + X^2$ gives rise to the Lubin-Tate extension $K_{-\pi}^n$ with $(K_\pi^n)^\iota = K_{-\pi}^n$ which is same as $K_\pi^n$. Indeed, if $K_\pi^n = K(\alpha)$ then $K_{-\pi}^n = K(-\alpha)$. Thus, the field $K_\pi^n$ is preserved by (a lift of)

$\iota \in \mathrm{Gal}(K|\mathbb{Q}_2)$ and so $K_\pi^n$ is Galois over $\mathbb{Q}_2$.

For our convenience, set $L = K^{2m} K_\pi^{2n}$. Then $L|\mathbb{Q}_2$ is a Galois extension containing $F$. In particular $\rho_2(f)|_{I_L} = 1$ and $\rho_2(f)$ becomes crystalline over $L$. The description of the Galois group of $L|\mathbb{Q}_2$ is given using the following exact sequence:

$$1 \to \mathrm{Gal}(L|K) \to \mathrm{Gal}(L|\mathbb{Q}_2) \to \mathrm{Gal}(K|\mathbb{Q}_2) \to 1$$

where $\mathrm{Gal}(L|K) = \mathrm{Gal}(L|K_\pi^{2n}) \times \mathrm{Gal}(L|K^{2m}) =< \sigma > \times (\Delta \times \Gamma), \mathrm{Gal}(K|\mathbb{Q}_2) =< \iota >$, with $\sigma^{2m} = 1$ and

$$\Delta = \{\pm 1\},\ \Gamma = \mathcal{O}_K/\pi^{2n} =< \gamma_1 > \times < \gamma_2 > \text{ with } \gamma_i^{2^n} = 1 \text{ for } i = 1, 2.$$

Here, $\Delta$ and $\Gamma$ are the tame and wild parts of the inertia group $I(L|K) = \mathrm{Gal}(L|K^{2m})$ respectively. The full inertia subgroup of $\mathrm{Gal}(L|K)$ is $\Delta \times \Gamma$.

### Action of $\iota$

The inertia $I(L|K)$ is a normal subgroup of $I(L|\mathbb{Q}_2)$ and the conjugation action of $\iota$ is given by

$$\iota^{-1}\{\pm 1\}\iota = \{\pm 1\}, \quad \iota^{-1}\gamma_1\iota = \gamma_1, \quad \iota^{-1}\gamma_2\iota = \gamma_2^{-1}, \tag{4.5}$$

which can be checked as in the unramified supercuspidal case. As in the previous case we can think of $\chi|_{I_K}$ as a character of $I(L|K) \cong \Delta \times \Gamma$. Write

$$\chi|_{I_K} = \chi|_{I(L|K)} = \omega^l \cdot \chi_1 \cdot \chi_2, \tag{4.6}$$

where $\omega$ is the fundamental character of level 1, $l$ is an integer and $\chi_m$ is the character taking $\gamma_m$ to a $2^n$-th root of unity $\zeta_m$ for $m = 1, 2$. Let us assume that $\chi_1$ takes $\gamma_1$ to $\zeta_{2^r}$ and $\chi_2$ takes $\gamma_2$ to $\zeta_{2^s}$. Here, $\zeta_{2^r}$ and $\zeta_{2^s}$ denote a primitive $2^r$-th root of unity and a primitive $2^s$-th root of unity respectively and so $r, s \le n$. The element $\iota$ acts on the above characters in the following way:

$$\omega^\iota = \omega, \quad \chi_1^\iota = \chi_1, \quad \chi_2^\iota = \chi_2^{-1}. \tag{4.7}$$

The condition $\chi|_{I_K}$ does not extend to $I_2$ is equivalent to $\zeta_{2^s} \neq \zeta_{2^s}^{-1}$ and hence $r < s$. Note that there are seven quadratic extensions $\mathbb{Q}_2(\sqrt{d})$ of $\mathbb{Q}_2$ with $d = -3, -1, 3, 2, -2, 6, -6$. Among them $\mathbb{Q}_2(\sqrt{-3})$ is unramified and rest of them are ramified.

*Remark* 4.2.1. The above generators $\gamma_1$ and $\gamma_2$ are characterized by the Equ. (4.7). Note that the above characters $\omega_2^l, \omega, \chi_1$ and $\chi_2$ are canonically determined by the modular form $f$ (more precisely, the actions 4.4 and 4.7) as we started with the local representation canonically attached to $f$

**Definition 4.2.2.** *($\gamma_1$-element and $\gamma_2$-element) An element of $I_W(K)$ (the wild inertia part of $K$) is called a $\gamma_1$-element (resp. $\gamma_2$-element) if its projection to $I_W(L|K)$ is $\gamma_1$ (resp. $\gamma_2$).*

# 5

# Modular endomorphism algebras at supercuspidal primes

In this chapter, we determine the local modular endomorphism algebras (attached to a non-CM primitive cusp form) at certain primes of infinite slope, called *supercuspidal primes*.

## 5.1 Set up

Let $M_f$ be the motive attached to a non-CM primitive cusp form $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(N, \epsilon)$ of weight $k \geq 2$, level $N \geq 1$ and nebentypus $\epsilon$. Consider the $\mathbb{Q}$-algebra of endomorphism of $M_f$ defined by

$$X = X_f := \mathrm{End}_{\bar{\mathbb{Q}}}(M_f) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

For a prime $\lambda \mid \ell$ of the Hecke field $E = \mathbb{Q}(\{a_n\})$, Deligne [16] associated a Galois representation to $f$ which is a continuous homomorphism

$$\rho_f = \rho_{f,\lambda} : G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q}) \to \mathrm{GL}_2(E_\lambda).$$

It is unramified outside the primes dividing $N\ell$ and its determinant is $\chi_\ell^{k-1}\epsilon$, where $\chi_\ell$ is the $\ell$-adic cyclotomic character. Let $G_p$ be the decomposition group at $p$. We will be using information about the local Galois representation $\rho_f|_{G_p}$ with $\ell = p$, called $(p,p)$-Galois representation.

Inside the automorphism group of $E$, we have an abelian subgroup $\Gamma$, called the group of extra twists for $f$:

$$\Gamma = \{\gamma \in \mathrm{Aut}(E) : \ \exists \text{ a Dirichlet character } \chi_\gamma \text{ such that } a_p^\gamma = a_p \cdot \chi_\gamma(p) \text{ with } (p,N)=1\}.$$

The fixed field of $E$ by $\Gamma$ is denoted by $F$. It is well known that $E$ is either a CM field or a totally real field. Thus, the field $F$ is a totally real number field, since if $E$ is a CM field then complex conjugation is always an element of $\Gamma$ [38, Example (3.7)]. One knows that $X$ is a central simple algebra over $F$ and its class $[X] \in {}_2\mathrm{Br}(F)$, the 2-torsion part of the Brauer group of $F$. For a prime $v \mid p$, let $F_v$ be the completion of $F$ at $v$.

Using the exact sequence below one can study the Brauer class of $X$ locally:

$$0 \to {}_2\mathrm{Br}(F) \to \oplus_v \ {}_2\mathrm{Br}(F_v) \to \mathbb{Z}/2 \to 0$$

where $v$ runs over the primes of $F$. It is known that the algebra $X_v = X \otimes_F F_v$ is a 2-torsion element in $\mathrm{Br}(F_v)$, that is, the class $[X_v] \in {}_2\mathrm{Br}(F_v) \cong \mathbb{Z}/2\mathbb{Z}$. If the class $[X_v]$ is trivial, the algebra $X_v$ is a matrix algebra over $F_v$; otherwise it is a quaternion division algebra over $F_v$.

We treat the case to find out the algebra $X_v$, when the local automorphic representation at $p$ (attached to $f$) is supercuspidal. These primes are called *supercuspidal* primes. Let $N_p$ and $C_p$ be the exact power of $p$ that divides the level $N$ and the conductor of the $p$-part of $\epsilon$ respectively. Supercuspidal primes can be characterized by the conditions: $2 \leq N_p > C_p$.

**Definition 5.1.1.** *We call a supercuspidal prime $p$ to be dihedral for $f$ if the local Galois representation $\rho_f|_{G_p} \sim \mathrm{Ind}_{G_K}^{G_p} \chi$ for some quadratic extension $K|\mathbb{Q}_p$ and some character $\chi$ of $G_K := \mathrm{Gal}(\bar{\mathbb{Q}}_p|K)$. Depending on $K|\mathbb{Q}_p$ is unramified (or ramified), we call the prime $p$ to be an unramified (or ramified) supercuspidal prime for $f$. By the level of an unramified supercuspidal prime $p$, we mean the level of the corresponding local automorphic representation $\pi_p$.*

If $p$ is a supercuspidal prime, then $a_p = 0$ and the corresponding slope is infinity and it is not possible to talk about the parity of slopes. In this chapter, we will give a formula that precisely determines the Brauer class of $X_v$ in $\mathrm{Br}(F_v)$ in terms of traces of adjoint lifts at auxiliary good primes (that does not divide the level of $f$) for a prime $v$ of $F$ lying above a supercuspidal prime $p$. We choose a prime $p'$ coprime to $N$, with non-zero Fourier coefficients $a_{p'}$, satisfying the following properties:

$$p' \equiv 1 \pmod{p^{N_p}}, \quad p' \equiv p \pmod{N'}. \tag{5.1}$$

**Definition 5.1.2.** *Let $v$ be a valuation on $F$ such that $v(p) = 1$. We define the* "companion adjoint slope" *at a place $v$ of $F$ lying above a supercuspidal prime $p$ to be the $v$-adic valuation of the trace of adjoint lift at $p'$. In other words,*

$$m_v := [F_v : \mathbb{Q}_p] \cdot v(a_{p'}^2 \epsilon(p')^{-1})$$

*denote the "companion adjoint slope" at $v$.*

We also choose the following auxiliary primes with non-zero Fourier coefficients:

- $p'' \equiv 1 \pmod{N'}$ and $p''$ has order $(p-1)$ in $(\mathbb{Z}/p^{N_p}\mathbb{Z})^\times$,
- $p''' \equiv 1 \pmod{N'}$ and $p'''$ has order 2 in $(\mathbb{Z}/2^{N_2}\mathbb{Z})^\times$,
- for all $\gamma \in \Gamma$,

$$\chi_\gamma(p^\dagger) = \begin{cases} -1, & \text{if } \chi_\gamma \text{ is ramified,} \\ 1, & \text{if } \chi_\gamma \text{ is unramified.} \end{cases}$$

There exist infinitely many such primes since $f$ is assumed to be non-CM. If the Brauer class of $X_v$ is determined by the parity of an integer $a$, we write $X_v \sim a$ or $X_v \sim (-1)^a$. The computation of $X_v$ uses the technique from group cohomology and $(p,p)$-Galois representation.

We consider two cases separately depending upon $p$ odd or $p = 2$.

## 5.2 Odd supercuspidal primes

For an odd supercuspidal prime $p$, the local automorphic representation attached to $f$, or equivalently (via Local Langlands Correspondence for $\mathrm{GL}_2$) the local Galois representation is always dihedral and hence induced by a character $\chi$ of an index two subgroup $G_K := \mathrm{Gal}(\bar{\mathbb{Q}}_p|K)$ of the local Galois group $G_p$; namely

$$\rho_f|_{G_p} \sim \mathrm{Ind}_{G_K}^{G_p}\chi$$

with $K$ a quadratic extension of $\mathbb{Q}_p$. The structure of $\chi$ on the inertia group is given in the chapter 4. Recall that $l$ is an integer as in Section 4.1.

**Definition 5.2.1.** *We call an odd unramified supercuspidal prime $p$ of level zero to be "good" if*

$$\textbf{\textit{(H)}}\ l \textit{ is not an odd multiple of } (p+1)/2.$$

*If such a prime $p$ is not "good", we call it a "bad" level zero unramified supercuspidal prime.*

We remark that for unramified supercuspidal primes $p$ of level zero, $N_p = 2$ and when $p$ is of positive level, we have $N_p(> 2)$ is even [8, Section 4]. In Lemma 5.2.6, we prove that level zero unramified supercuspidal primes $p \equiv 1 \pmod 4$ with $C_p = 0$ and $p \equiv 3 \pmod 4$ with $C_p = 1$ are always "good".

For a fixed Frobenius $g_\pi$ in $G_K$, assume that $\alpha(g_\pi) \equiv b \bmod F_v^\times$ and consider the field $F'_v = F_v(b)$. Let us now define the following error terms:

$$(-1)^{n_v} = \begin{cases} (\pi^2, a_{p''}^2)_v, & \text{if } p \text{ is an odd ramified supercuspidal prime with } KF_v|F_v \text{ ramified quadratic,} \\ (t, c)_v, & \text{if } p \text{ is an odd unramified "bad" supercuspidal prime,} \end{cases} \tag{5.2}$$

where $\pi$ is a uniformizer in $K$, $c \in F_v^\times$ is given by $\alpha(i) \equiv \sqrt{c} \bmod F_v^\times\ \forall\ i \in I_T(F_v)$ and $t \in F_v^\times$ is the quantity given by the quadratic extension $F_v(\sqrt{t})|F_v$ cut out by the quadratic character $\psi$ defined as follows: $\psi(g) = 1,$ if $\alpha(g) \in F_v'^\times$ and $\psi(g) = -1,$ if $\alpha(g) \notin F_v'^\times$.

Note that $KF_v = F_v$ if and only if $K \subseteq F_v$. Therefore, if $K \nsubseteq F_v$, then the field extension $KF_v|F_v$ has degree 2. The extension $KF_v|F_v$ turns out to be an unramified quadratic extension in the following cases [8]:

1. $p$ is an odd unramified supercuspidal prime,

2. $p \equiv 1 \pmod 4$ is a ramified supercuspidal prime,

3. $p \equiv 3 \pmod 4$ is a ramified supercuspidal prime with $e(F_v|\mathbb{Q}_p) = e_v$ even.

In the remaining case, that is, when $p \equiv 3 \pmod 4$ is a ramified supercuspidal prime with $e_v$ odd, the extension $KF_v|F_v$ becomes ramified quadratic.

The two theorems below determine the local endomorphism algebra $X_v$ for odd supercuspidal primes.

**Theorem 5.2.2.** *Let $v$ be a prime of $F$ lying above an odd supercuspidal prime $p$ for $f$ satisfying one of the following properties:*

1. *$p$ is an unramified supercuspidal prime of positive level or it is a "good" level zero unramified supercuspidal prime,*

2. *$p$ is a ramified supercuspidal prime with $K \subseteq F_v$ or $KF_v|F_v$ unramified quadratic extension.*

*The local endomorphism algebra $X_v$ is a matrix algebra if and only if $m_v$ is even.*

We wish to emphasis that the above result is exactly the same as [8]. For a "good" level zero unramified supercuspidal prime $p$, the hypothesis here is exactly the same as the condition of the Theorem 6.1 of [8] [cf. Lemma 5.2.5]. Observe that this hypothesis of the Theorem 6.1 of [8] is not required for level zero unramified supercuspidal primes $p \equiv 1 \pmod 4$ with $C_p = 0$ and $p \equiv 3 \pmod 4$ with $C_p = 1$ [cf. Lemma 5.2.6]. In the case of odd unramified supercuspidal primes for $f$ of level zero without the hypothesis, we predict the ramifications of endomorphism algebras using the following theorem:

**Theorem 5.2.3.** *Let $v \mid p$ be a prime of $F$ with $p$ a "bad" level zero unramified supercuspidal prime or $KF_v|F_v$ is a ramified quadratic extension. The ramification of $X_v$ is determined by the parity of $m_v + n_v$.*

By Lemma 5.2.21, we note that the result obtained here when $KF_v|F_v$ is a ramified quadratic extension in a different method is also exactly the same as [8].

### 5.2.1 Ramification of endomorphism algebras for odd supercuspidal primes

In this subsection, we give the proof of the results stated above. Let $K$ be an unramified quadratic extension. For $i \in I_K$, let $\bar{i}$ be the projection to the inertia group $I(L|\mathbb{Q}_p)$ whose tame part is generated by $\delta$ [cf. Chapter 4]. We call $\epsilon$ to be tame at $p$ if the order of $\epsilon_p$ divides $p-1$.

**Lemma 5.2.4.** *If $\epsilon$ is tame at $p$, then $\alpha(j) \in F_v^\times$ for all $j \in I_W(K)$.*

*Proof.* Note that $j$ is an element of a pro $p$-group and $p$ is odd. Since $\epsilon$ is tame at $p$, we must have $\epsilon(j) = 1$ and so $\chi_\gamma^2(j) = \epsilon^{\gamma-1}(j) = 1$ for all $\gamma \in \Gamma$. By the nature of $j$ and $p$ is odd, we have $\chi_\gamma(j) = 1$ for all $\gamma \in \Gamma$. This implies that $\alpha(j)^{\gamma-1} = 1$, for all $\gamma \in \Gamma$ [cf. Equ. (2.1)]. Hence, we obtain $\alpha(j) \in F^\times$. □

Let $s$ be a fixed $(p^2-1)$-th primitive root of unity as in [8] and $K = \mathbb{Q}_p(s)$ is unramified. Recall that $g_s \in \mathrm{Gal}(\bar{\mathbb{Q}}_p|K)$ is an element which is mapped to $s \in K^\times$ under the reciprocity map.

The next lemma shows that the hypothesis **(H)** is same as the condition of [8, Theorem 6.1]. First observe that this condition depends on the choice of $s$. By the structure theorem of the local field $K^\times$, we have $K^\times =< p > \times < s > \times U_K^{(1)}$ . Let $L$ be the field as in Chapter 4. By class field theory, the elements of $< s >$ corresponds to the tame part of the inertia group $I(L|\mathbb{Q}_p)$ under the norm residue map. Let $\delta$ be a $p^2-1$-th root of unity as in [25, Equation 3.3] (see also Equation 4.2). Observe that $\delta$ is also a valid choice of $s$.

**Lemma 5.2.5.** *The assumption $Tr(\rho_f(g_s)) \neq 0$ in the [8, Theorem 6.1] is same as **(H)**.*

*Proof.* Note that $\mathrm{Tr}(\rho_f(g_s)) = \chi(s) + \chi(s)^p$. For the choice of $s = \delta$, this is equivalent to $\chi(\delta) + \chi(\delta)^p \neq 0$. In other words, $\omega_2^l(\delta) + \omega_2^{lp}(\delta) \neq 0$ [cf. Chapter 4]. Since $\omega_2$ takes value in the $(p^2-1)$-th roots of unity, the last condition is same as the condition $l$ is not an odd multiple of $(p+1)/2$. □

**Lemma 5.2.6.** *Let $p$ be an odd unramified supercuspidal prime for $f$ and satisfying one of the following conditions:*

1. $p \equiv 1 \pmod 4$ *with* $C_p = 0$

2. $p \equiv 3 \pmod 4$ *with* $C_p = 1$.

*Then, the condition* ***(H)*** *is satisfied for* $p$.

*Proof.* Note that the condition $\mathrm{Tr}(\rho_f(g_s)) = 0$ is equivalent to $\chi(s) + \chi(s)^p = 0$, that is,

$$\chi(s)^{p-1} = -1. \tag{5.3}$$

First consider the case (1). Write $p = 4k+1$, for some $k \in \mathbb{N}$. Since $s^{p+1} \in \mathbb{Z}_p^\times$, using [8, Equ. (4)] and $C_p = 0$, we have $\chi(s)^{p+1} = \epsilon_p(s^{p+1})^{-1} = 1$. Combining it with (5.3), we get $\chi(s)^2 = -1$. Hence, we obtain $\chi(s) = \pm i$. On the other hand, using (5.3) we have that $\chi(s)^{4k} = -1$, a contradiction.

We now consider the case (2). By the same equation of [8], we have $\chi(s)^{p+1} = \epsilon_p(s^{p+1})^{-1} = \eta$, where $\eta$ is a $(p-1)$-th root of unity. Combining it with (5.3), we get $\chi(s)^2 = -\eta$.

First assume that $p = 3$. Since $C_3 = 1$ and $s^4 = -1$, we must have $\epsilon_3(s^4) = -1$ and so $\eta = -1$. Thus, we deduce $\chi(s)^4 = -1$, a contradiction to $\chi(s)^2 = -1$.

Now suppose that $p > 3$. Write $p = 4k+3$, for some $k \in \mathbb{N} \setminus \{0\}$. Since $\chi(s)^2 = -\eta$, we have $\chi(s) = \pm i \cdot \sqrt{\eta}$. Again since $p > 3$ and $\chi(s)$ is a primitive $2(p-1)$-th root of unity, we must have that $\sqrt{\eta}$ is a primitive $2(p-1)$-th root of unity, say $\zeta_{2(p-1)}$. Thus, we get $\chi(s) = \pm i \cdot \zeta_{2(p-1)}$. From the equation (5.3), we have $(\pm i \cdot \zeta_{8k+4})^{4k+2} = -1$. We arrive at a contradiction $\zeta_{8k+4}^{4k+2} = 1$. □

Hence, the assumption of [8, Theorem 6.1] is not needed for primes stated in the above lemma.

Note that $\omega_2^{p^2-1}(\delta) = 1$, i.e., $\omega_2^{(p-1)(p+1)/2}(\delta) = -1$. Without **(H)** we have $\omega_2^{l(p-1)} = -1$ and it is equivalent to $\omega_2^l(\delta) + \omega_2^{lp} = 0$. Then for $i \in I_T(K)$ with $\bar{i} = \delta$, the last condition is further equivalent to $\mathrm{trace}(\rho_f(i)) = (\chi + \chi^\sigma)(i) = \omega_2^l(\delta) + \omega_2^{lp}(\delta) = 0$, i.e., $\omega_2^l(\delta)$ is a primitive $2(p-1)$-th root of unity, say $a$.

**Lemma 5.2.7.** *Let $p$ be an odd unramified supercuspidal prime for $f$ without **(H)**. Suppose that $N_p \geq 3$ and $\epsilon$ is tame at $p$. For all $i \in I_T(K)$, we have:*

$$\alpha(i) \equiv \begin{cases} 1 \textit{ mod } F_v^\times, & \textit{if } \bar{i} \textit{ is an even power of } \delta, \\ a(\zeta_p - \zeta_p^{-1}) \textit{ mod } F_v^\times, & \textit{otherwise.} \end{cases} \tag{5.4}$$

*Proof.* Let $i \in I_T(K)$ be such that $\bar{i} = \delta$. By above, we deduce that $\text{trace}(\rho_f(i)) = \omega_2^l(\delta) + \omega_2^{lp}(\delta) = 0$. For even $n$, we have $\alpha(i^n) \equiv \omega_2^l(\delta^n) + \omega_2^{lp}(\delta^n) \equiv \text{Tr}_{K|\mathbb{Q}_p}(\delta^n) \equiv 1$ mod $F_v^\times$.

We now consider odd $n$. By [8, Lemma 4.1], there exists an element $\tau \in I_W(K)$ such that $\chi(\tau) = \zeta_p$ and $\chi^\sigma(\tau) = \zeta_p^{-1}$, for some primitive $p$-th root of unity $\zeta_p$ and $\alpha(\tau) \equiv 1$ mod $F^\times$. Thus, we deduce that $\alpha(i) \equiv \alpha(i\tau) \equiv (\chi + \chi^\sigma)(i\tau) \equiv \omega_2^l(\delta)(\zeta_p - \zeta_p^{-1})$ mod $F_v^\times$. Notice that $\tilde{\alpha}$ is a homomorphism and $a^2(\zeta_p - \zeta_p^{-1})^2 \in F_v^\times$ by the same lemma. Hence, we obtain $\alpha(i^n) \equiv \alpha(i^{m+1}) \equiv \alpha(i) \equiv a(\zeta_p - \zeta_p^{-1})$ mod $F_v^\times$ with $m$ even. □

Consider the field $F_v^{'} = F_v(b)$ as in the beginning of Section 5.2 . We have:

**Lemma 5.2.8.** *Let $p$ be an odd unramified supercuspidal prime for $f$ with $N_p \geq 3$. Assume the hypothesis **(H)** and that $\epsilon$ is tame at $p$. If $g \in G_K$ and $\alpha(g) \notin (F_v^{'})^\times$, then $\alpha(g) \equiv a(\zeta_p - \zeta_p^{-1})$ mod $(F_v^{'})^\times$.*

*Proof.* For an unformizer $\pi$ of $K$, let $g_\pi$ be the image of $\pi$ under Norm residue map. Note that every element $g \in G_K$ can be written as $g_\pi^n i$ for some $n \in \mathbb{Z}$ and $i \in I_K$. We use Lemma 5.2.7 and the homomorphism $\tilde{\alpha}$ to obtain the result. □

**Lemma 5.2.9.** *Let $p$ be an odd unramified supercuspidal prime for $f$ with $K \subseteq F_v$. Define a function $f$ on $G_v(\subseteq G_K)$ by*

$$f(g) = \begin{cases} 1, & \textit{if } \alpha(g) \in (F_v^{'})^\times, \\ a(\zeta_p - \zeta_p^{-1}), & \textit{if } \alpha(g) \notin (F_v^{'})^\times. \end{cases} \tag{5.5}$$

*The cocycle class of $c_f$ is trivial.*

*Proof.* We call an element $g$ type 1 if $\alpha(g) \in (F_v^{'})^\times$, otherwise we call it type 2. If $\epsilon$ is tame at $p$, then we use the fact $a^2(\zeta_p - \zeta_p^{-1})^2 \in F_v^\times$ and Lemma 5.2.8. We see that if $g$

and $h$ both are type 1 elements then $gh$ is also so, but if one of them is of type 1 and the other one is of type 2 then their product is an element of type 2. The product of two type 2 elements is an element of type 1. Thus, we can and do replace the conditions which define the function $f$ by a quadratic character $\psi$ in the following way: $\psi(g) = 1$, if $\alpha(g) \in (F_v^{'})^\times$ and $\psi(g) = -1$, otherwise. The function $f$ can be seen alternatively as:

$$f(g) = \begin{cases} 1, & \text{if } \psi(g) = 1, \\ a(\zeta_p - \zeta_p^{-1}), & \text{if } \psi(g) = -1. \end{cases} \tag{5.6}$$

The quadratic character $\psi$ on $G_v$ cut out a quadratic extension of $F_v$, namely $F_v(\sqrt{t})$, for some $t \in F_v^\times$. To compute $\mathrm{inv}_v(c_f)$, let $\sigma$ be the non-trivial element of $\mathrm{Gal}(F_v(\sqrt{t})|F_v)$. The cocycle table of the 2-cocycle $c_f$ is given by:

| | 1 | $\sigma$ |
|---|---|---|
| 1 | 1 | 1 |
| $\sigma$ | 1 | $a^2(\zeta_p - \zeta_p^{-1})^2$ |

which gives the symbol $(t, a^2(\zeta_p - \zeta_p^{-1})^2)_v$. Note that he element $t$ has no square root in $F_v^\times$ and it is fixed by the kernel of $\psi$. We claim that $t = a^2(\zeta_p - \zeta_p^{-1})^2$. Since $\mathrm{Frob}_v = \mathrm{Frob}_K^{f(F_v|K)}$, let $g_v = g_x^{f(F_v|K)}$ be a fixed Frobenius in $G_v$. Hence, we deduce that $\alpha(g_v) \in F_v'^\times$.

Consider now the elements of the kernel of $\psi$. Let $i$ denote the elements of $I_T(F_v)$ such that $\bar{i} = \delta$. Let $H$ denote the subgroup of $G_v$ generated by the elements of $I_W(F_v)$, even power of $i$ and $g_v$. We first show that, $H = \ker(\psi)$. Note that

$$\ker(\psi) = \{g \in G_v \mid \alpha(g) \in (F_v^{'})^\times\}.$$

Since $\tilde{\alpha}$ is a homomorphism, by Lemma 5.2.4 and 5.2.7 we obtain $H \subseteq \ker(\psi)$. Using the homomorphism $\tilde{\alpha}$ again and Lemma 5.2.7, we have $\alpha(i^n) \notin (F_v^{'})^\times$, for all $n \in \mathbb{Z}$ odd and hence it cannot belong to $\ker(\psi)$. Since every element $g \in G_v$ has the form $g = g_v^n i$ for some $i \in I_v$ and $n \in \mathbb{Z}$, we have $\alpha(g) \equiv \alpha(i) \bmod (F_v^{'})^\times$. Since $I_v$ is a product of its tame part and wild part, we have shown $\ker(\psi) \subseteq H$ and hence $\ker(\psi) = H$.

We now show that $t$ is fixed by all generators of $H$. For all $g \in G_p$, we have $g(a) = a$ or $a^2$ and $g(\zeta_p - \zeta_p^{-1}) = \zeta_p - \zeta_p^{-1}$ or $-(\zeta_p - \zeta_p^{-1})$. Let $j \in I_W(F_v)$ be an element of the wild inertia group of $F_v$. Since it is an element of a pro-$p$ group and $p$ is odd, we must have $j(a) = a$ and $j(\zeta_p - \zeta_p^{-1}) = \zeta_p - \zeta_p^{-1}$. For all even $n \in \mathbb{N}$, the elements $i^n$ acts on $a$ and $\zeta_p - \zeta_p^{-1}$ in a similar way. Since $\mathrm{Frob}_K = \mathrm{Frob}_p^2$, the action of $\mathrm{Frob}_K$ and hence the action of $g_v$ on them is exactly the same as above. Hence, we deduce that $t = a^2(\zeta_p - \zeta_p^{-1})^2$. □

Suppose that $K \not\subseteq F_v$ (i.e., $G_v \not\subseteq G_K$) with $K|\mathbb{Q}_p$ unramified quadratic. For a fixed Frobenius $g_v \in G_v$, the element $\bar{g}_v \in G_v/G_{KF_v}$ is nontrivial. Thus, every element $g \in G_v$ can be written as

$$g = g_v^n h, \text{ for some } h \in G_{KF_v} \text{ and } n \in \{0, 1\}. \tag{5.7}$$

Note that $n = 0$ when $g \in G_{KF_v} (\subseteq G_K)$. Using this decomposition, we extend the function $f$ (5.5) (defined on $G_{KF_v} \subseteq G_K$) uniquely to $G_v$, call it $F$, as follows: $F(g) = f(h)$. The inflation map $\mathrm{Inf} : {}_2\mathrm{H}^2(G_{KF_v}, (\bar{F_v}^{\times})^{\mathrm{Gal}(KF_v|F_v)}) \hookrightarrow {}_2\mathrm{H}^2(G_v, \bar{F_v}^{\times})$ sends the cocycle $c_f$ to $c_F$. Since the inflation map is injective and the class of $c_f$ is trivial, the cocycle class of $c_F$ is trivial.

### The case $K \subseteq F_v$ or $KF_v|F_v$ unramified quadratic extension

First we determine the value of $\alpha$ at the inertia groups.

**Lemma 5.2.10.** *Let $p$ be an odd supercuspidal prime with $K \subseteq F_v$. Assume that $\epsilon$ is tame at $p$. When $p$ is an unramified supercuspidal prime, we also assume* ***(H)***. *For all $\iota \in I_v$, we have $\alpha(\iota) \in F_v^{\times}$.*

*Proof.* In this case, we have $KF_v = F_v$ and $I_v \subseteq I_K$. Every element $\iota \in I_v$ has the form $\iota = ij$ for some element $i$ of the tame part and some element $j$ of the wild part of the inertia group $I_v$.

In the unramified case, we deduce that:

$$\begin{aligned}
\alpha(\iota) = \alpha(ij) &\equiv \alpha(i) \bmod F_v^{\times} \\
&\equiv \chi(i) + \chi^{\sigma}(i) \bmod F_v^{\times} \\
&\equiv \omega_2^l(i) + (\omega_2^l)^{\sigma}(i) \bmod F_v^{\times} \\
&\equiv \omega_2^l(i) + \omega_2^{lp}(i) \bmod F_v^{\times}.
\end{aligned}$$

In the case of ramified supercuspidal primes, we obtain:

$$\begin{aligned}\alpha(\iota) = \alpha(ij) &\equiv \alpha(i) \bmod F_v^\times \\ &\equiv \omega^l(i) + (\omega^l)^\sigma(i) \bmod F_v^\times \\ &\equiv \omega^l(i) + \omega^l(i) = 2\omega^l(i) \bmod F_v^\times.\end{aligned}$$

The first congruence relation in both cases follows from Lemma 5.2.4 and the definition of the homomorphism $\tilde{\alpha}$, and the second one follows from [Proposition 2.1.2, property (2)]. Since $\omega_2^l(i)$ belongs to $K = \mathbb{Q}_{p^2}$, we obtain $\omega_2^l(i) + \omega_2^{lp}(i) = \mathrm{Tr}_{K|\mathbb{Q}_p}\omega_2^l(i) \in \mathbb{Q}_p^\times \subseteq F_v^\times$. Again since $\omega^l$ takes values in the multiplicative group of $(p-1)$-th roots of unity, in both cases, we conclude that $\alpha(\iota) \in F_v^\times$, for all $\iota \in I_v$. □

We now prove Theorem 5.2.2 when $K \subseteq F_v$ or $KF_v|F_v$ is unramified quadratic.

**Theorem 5.2.11.** *Let $p$ be an odd supercuspidal prime with $K \subseteq F_v$ or $KF_v|F_v$ is unramified quadratic. If $p$ is an unramified supercuspidal prime, we assume **(H)** unless $N_p \geq 3$. Then $X_v \sim m_v$ for $v \mid p$.*

*Proof.* Since the endomorphism algebra is invariant under twisting [39, Proposition 3], without loss of generality one can assume that $\epsilon$ is tame at $p$.

1. Consider $K \subseteq F_v$ with the hypothesis **(H)**. By the lemma above, $\alpha(\iota) \in F_v^\times$ for all $\iota \in I_v$. Using [Proposition 2.1.2, part (1)] and $\epsilon_p(g) \in \mathbb{Q}_p^\times$ (as $\epsilon$ is tame at $p$), we obtain $\frac{\alpha^2}{\epsilon'}(g) \in F_v^\times$, for all $g \in G_v$. By Lemma 2.2.1 applied to $S = \alpha$ and $t = \epsilon'$, we get
$$\mathrm{inv}_v(c_\alpha) = \frac{1}{2}v\Big(\frac{\alpha^2}{\epsilon'}(\mathrm{Frob}_v)\Big) \bmod \mathbb{Z}.$$

2. Assume $K \subseteq F_v$ with $K$ unramified and $N_p \geq 3$. The previous computation works with **(H)**. Thus, we consider this case without the hypothesis **(H)**.

   Note that $G_v \subseteq G_K$ and $I_v \subseteq I_K$. Set $S = \frac{\alpha}{f}$ on $G_v$ with $f$ as in (5.5). Since $\alpha(i) \equiv a(\zeta_p - \zeta_p^{-1}) \bmod F_v^\times\ \forall\, i \in I_v$ with $\alpha(i) \notin F_v^\times$ (by Lemmas 5.2.4 and 5.2.7), we get $S(i) \in F_v^\times\ \forall\, i \in I_v$. Since $a^2(\zeta_p - \zeta_p^{-1})^2$ and $\frac{\alpha^2}{\epsilon'}(g) \in F_v^\times$, we obtain $\frac{S^2}{\epsilon'}(g) \in F_v^\times\ \forall\, g \in G_v$. Then by Lemma 2.2.1, $\mathrm{inv}_v(c_S) = \frac{1}{2}v\Big(\frac{S^2}{\epsilon'}(\mathrm{Frob}_v)\Big) \bmod \mathbb{Z} = \frac{1}{2}v\Big(\frac{\alpha^2}{\epsilon'}(\mathrm{Frob}_v)\Big) \bmod \mathbb{Z}$. The cocycle $c_\alpha$ can be decomposed as $c_s c_f$ with $c_S, c_f$ are the cocycles corresponding to $S$ and $f$ respectively. Note that the cocycle class of $c_f$ is trivial by Lemma 5.2.9. Hence, we obtain

$$\mathrm{inv}_v(c_\alpha) = \mathrm{inv}_v(c_S) + \mathrm{inv}_v(c_f) = \mathrm{inv}_v(c_S) = \frac{1}{2} v\Big(\frac{\alpha^2}{\epsilon'}(\mathrm{Frob}_v)\Big) \bmod \mathbb{Z}.$$

3. Next assume that $KF_v|F_v$ is unramified quadratic. In this case, we get $I_v = I_{KF_v} \subseteq I_K$. The same computation in (1) works here with **(H)**. So assume this case without the hypothesis **(H)**.

   Define $S = \frac{\alpha}{F}$ on $G_v$ with $F$ as in the previous paragraph of Section 5.2.1. Since $I_v = I_{KF_v}$, in the decomposition (5.7) for any element of $I_v$, we must have $n = 0$. By writing the definition of $F$, we deduce that $\frac{\alpha}{F} = \frac{\alpha}{f}$ on $I_v$. By the same argument as in (2), we see that two conditions of Lemma 2.2.1 are satisfied by $S$ and $t = \epsilon'$. Hence, we obtain $\mathrm{inv}_v(c_S) = \frac{1}{2} v\Big(\frac{S^2}{\epsilon'}(\mathrm{Frob}_v)\Big) \bmod \mathbb{Z} = \frac{1}{2} v\Big(\frac{\alpha^2}{\epsilon'}(\mathrm{Frob}_v)\Big) \bmod \mathbb{Z}$. Since the cocycle class of $c_F$ is trivial, we deduce that

$$\mathrm{inv}_v(c_\alpha) = \mathrm{inv}_v(c_S) + \mathrm{inv}_v(c_F) = \mathrm{inv}_v(c_S) = \frac{1}{2} v\Big(\frac{\alpha^2}{\epsilon'}(\mathrm{Frob}_v)\Big) \bmod \mathbb{Z}.$$

For a prime $p'$ introduced before, we have that $\chi_\gamma(\mathrm{Frob}_p) = \chi_\gamma([p]) \overset{(3.3)}{=} \chi'_\gamma(p) = \chi_\gamma(p')$, where $\chi'_\gamma$ denote the prime-to-$p$ part of $\chi_\gamma$. By a similar computation, we deduce $\epsilon'(\mathrm{Frob}_p) = \epsilon'(p)$. Thus, using (2.1) we have $\alpha(\mathrm{Frob}_p) \equiv \alpha(\mathrm{Frob}_{p'}) \equiv a_{p'} \bmod F^\times$, where $\mathrm{Frob}_p$ and $\mathrm{Frob}_{p'}$ denote the Frobenii at the primes $p$ and $p'$ respectively. Hence, we deduce that $\alpha(\mathrm{Frob}_v) = \alpha(\mathrm{Frob}_p^{f_v}) \equiv a_{p'}^{f_v} \bmod F_v^\times$. On the other hand, we have $\epsilon'(p) = \epsilon'(p') = \epsilon(p')$. Hence, in all of the above cases we obtain

$$\mathrm{inv}_v(c_\alpha) = \frac{1}{2} v\Big(\frac{\alpha^2}{\epsilon'}(\mathrm{Frob}_v)\Big) = \frac{1}{2} \cdot f_v \cdot v\Big(\frac{\alpha^2(\mathrm{Frob}_p)}{\epsilon'(p)}\Big) = \frac{1}{2} \cdot f_v \cdot v(a_{p'}^2 \epsilon(p')^{-1}) \bmod \mathbb{Z}.$$

□

*Remark* 5.2.12. Writing multiplicatively the above formula, we obtain the same result as in [8, Theorems 6.1, 6.2 and 7.1]. When $p \equiv 3 \pmod 4$ is a ramified supercuspidal prime with $e_v$ even and $K \not\subseteq F_v$, we have $[X_v] \sim (-1)^{f_v \cdot v(a_{p'}^2 \epsilon(p')^{-1})}$. Thus, when $f_v$ is even, we deduce $X_v$ is a matrix algebra over $F_v$ which also follows from the formula (10) of [8] as $(p^{f_v}-1)/2 \equiv f_v \bmod 2$. We now consider the case where $f_v$ is odd. Since $(p-1)/2$ is odd and it divides $e_v$ [8, Lemma 4.1] which is even, using [8, Lemma 7.2] we get

$$[X_v] \sim (p, K|\mathbb{Q}_p)^{f_v \cdot v(a_{p''}^2 \epsilon(p'')^{-1})} \sim (p, K|\mathbb{Q}_p)^{v(a_{p''}^2 \epsilon(p'')^{-1})}.$$

Hence, $X_v$ is unramified, when $(p, K|\mathbb{Q}_p) = 1$ which we cannot conclude from the result obtained in [8]. When $(p, K|\mathbb{Q}_p) = -1$, our result matches up with [8, Theorem 7.6].

*Remark* 5.2.13. The Brauer class of $X_v$ is essentially determined by the parity of $m_v$ in which the auxiliary prime $p'$ involved. One can easily check that it is independent of the choice of $p'$. For two distinct primes $p'$ and $q'$ satisfying the congruence relation (6.12), one has $\epsilon(p') = \epsilon(q')$. Also, $\chi_\gamma(p') = \chi_\gamma(q')\ \forall\ \gamma \in \Gamma$. Using (2.1) and [Proposition 2.1.2, part (3)], we have $a_{p'}^{\gamma-1} = a_{q'}^{\gamma-1}\ \forall\ \gamma \in \Gamma$. Thus, we get $a_{p'} \equiv a_{q'} \bmod F^\times$ and so $a_{p'}^2\epsilon(p')^{-1} \equiv a_{q'}^2\epsilon(q')^{-1} \bmod (F^\times)^2$. Hence, they have the same $v$-adic valuation modulo 2.

**Corollary 5.2.14.** *Assume that $K \subseteq F_v$. If $p$ is an odd unramified supercuspidal prime (the hypothesis **(H)** is needed if necessary) or $p \equiv 3 \pmod 4$ is a ramified supercuspidal prime, then $X_v$ is a matrix algebra over $F_v$ .*

*Proof.* For such primes $p$, we have proved that $X_v \sim (-1)^{f_v \cdot v(a_{p'}^2 \epsilon(p')^{-1})}$, for $v \mid p$. When $p$ is an odd unramified supercuspidal prime, the containment $K = \mathbb{Q}_{p^2} \subseteq F_v$ implies that $f_v$ is even and the result follows.

When $p \equiv 3 \pmod 4$ is a ramified supercuspidal prime with $K \subseteq F_v$, we have $e_v$ is even. On the other hand, $K \nsubseteq F_v$ if $e_v$ is odd. If $(p, K|\mathbb{Q}_p) = 1$, we get the result by using [8, Lemma 7.2] and the fact $(p-1)/2$ is odd and it divides $e_v$ [8, Lemma 4.1]. If $(p, K|\mathbb{Q}_p) = -1$, then we have $\sqrt{p} \in K \subseteq F_v$ [cf. Lemma 5.2.19]. Let $g_{\sqrt{p}}$ be an element which is mapped to $\sqrt{p} \in K^\times$ and $g_p \in G_p$ be an element which is mapped to $p \in \mathbb{Q}_p^\times$ under the reciprocity map. We deduce that $g_p = g_{\sqrt{p}}^2$. Thus, using [Proposition 2.1.2, property (1)] we obtain that $\frac{\alpha^2}{\epsilon'}(g_{\sqrt{p}}) \in F_v^\times$ and so $\frac{\alpha^2}{\epsilon'}(g_p) \in (F_v^\times)^2$. Note that $g_p$ is one of the Frobenius at $p$ and $g_v = g_p^{f_v}$ is a Frobenius at $v$. Thus, the valuation $v\left(\frac{\alpha^2}{\epsilon'}(g_v)\right)$ is even. This completes the proof. $\square$

**Corollary 5.2.15.** *Let $p$ be an odd unramified supercuspidal prime of positive level or it is a "good" level zero unramified supercuspidal prime or $p \equiv 1 \pmod 4$ be a ramified* *supercuspidal prime for $f$. We have the following:*

1. *If $X_v$ is ramified then $v$ must divide the discriminant of the field $E$.*

2. *Let the level $N$ be such that*

   - *$N$ is a prime power (that is, $N' = 1$), or $N' = 2$, or*

   - *$p$ has odd order in $(\mathbb{Z}/N'\mathbb{Z})^\times$,*

   *then $X_v$ is unramified.*

*Proof.* For such primes $p$, we have proved that $[X_v] \sim (-1)^{f_v \cdot v(a_{p'}^2 \epsilon(p')^{-1})}$, for any prime $v \mid p$.

1. Since $X_v$ is ramified, we must have $v(a_{p'}^2 \epsilon(p')^{-1})$ is odd. Then $v(a_{p'})$ cannot be an integer if we extend $v$ to be a valuation on $E^\times$. Thus, $v$ ramifies in $E$. This proves (1).

2. We decompose any Dirichlet character $\chi_\gamma = \chi'_\gamma \cdot \chi_{\gamma,p}$ into its prime-to-$p$ and $p$ parts. If $N' = 1$ or $2$ then with a prime $p'$ chosen before we have $\chi_\gamma(p') = \chi'_\gamma(p') \cdot \chi_{\gamma,p}(p') = 1 \ \forall\ \gamma \in \Gamma$ and so $a_{p'} \in F^\times$ using (2.1).

   If $N' \neq 1, 2$ then by hypothesis $p^n \equiv 1 \pmod{N'}$ for some odd $n$. Write $n = 2k+1$. Since $\chi_\gamma(p') = \chi'_\gamma(p)$, we have $\chi_\gamma(p')^n = 1 \ \forall\ \gamma \in \Gamma$. Now $\alpha(\mathrm{Frob}_{p'})^{\gamma-1} = \chi_\gamma(p') = \chi_\gamma(p')^{n-2k} = \left(\chi_\gamma(p')^2\right)^{-k} = \left(\epsilon(p')^{-k}\right)^{\gamma-1} \ \forall\ \gamma \in \Gamma$, by the formula (2.2). This implies that $a_{p'} \equiv \epsilon(p')^{-k} \bmod F^\times$. Using the above formula we get the result in both cases.

□

We now give the formula for the ramification of $X_v$ for unramified supercuspidal primes of level zero without the hypothesis **(H)**. For such primes without the hypothesis, no element (like $\tau$ in the positive level case [cf. Lemma 5.2.7]) will help us to determine the exact value of $\alpha(i)$ mod $F_v^\times$, where $i \in I_T(F_v)$ whose projection in $\mathrm{Gal}(L|\mathbb{Q}_p)$ is $\delta$.

Let $p$ be an odd unramified supercuspidal prime for $f$ of level 0 without **(H)**, that is, $p$ is a bad prime. In this case, for $i \in I_T(K)$ with $\bar{i} = \delta$, we have $\omega_2^{l(p-1)}(\delta) = -1$ and so $\mathrm{trace}(\rho_f(i)) = 0$ as before. By Lemma 5.2.7, we have $\alpha(i^2) \in F_v^\times$ and we write

$\alpha(i) = \sqrt{t(i)}c(i)$ for some $t(i), c(i) \in F_v^\times$. Consider two elements $i, j \in I_T(K)$ with $\bar{i} = \bar{j} = \delta$. Since $c_\alpha(i,j) \in F_v^\times$ and $\alpha(ij) \in F_v^\times$ (cf. Lemma 5.2.7), we must have $\sqrt{t(i)} \equiv \sqrt{t(j)} \bmod F_v^\times$. For some fixed $c \in F_v^\times$, we have $\sqrt{t(i)} \equiv \sqrt{c} \bmod F_v^\times$. By Lemmas 5.2.4 and 5.2.7, we have $\alpha(i) \in F_v^\times$ for all $i \in I_K$ except those for which $\bar{i}$ is an odd power of $\delta$. For $i \in I_T(K)$ with $\bar{i} = \delta$, let us assume:

$$\alpha(i) \equiv \sqrt{c} \bmod F_v^\times, \quad c \in F_v^\times. \tag{5.8}$$

Consider an integer $n_v$ modulo 2 as in (5.2) when $p$ is a bad prime.

**Theorem 5.2.16.** *Let $v \mid p$ be a place of $F$ with $p$ a "bad" level zero unramified supercuspidal prime. The ramification of $X_v$ is determined by the parity of $m_v + n_v$.*

*Proof.* We will proceed the same way as before. When $K \subseteq F_v$, consider the function on $G_v$ defined by: $f(g) = 1$, if $\alpha(g) \in (F_v^{'})^\times$ and $f(g) = \sqrt{c}$, if $\alpha(g) \notin (F_v^{'})^\times$. Consider the extension $F_v(\sqrt{t})|F_v$ cut out by the quadratic character obtained from the conditions that define $f$. By a computation as in the previous cases, the cocycle class of $c_f$ is determined by the symbol $(t,c)_v$,

If $K \nsubseteq F_v$, we extend this function $f$ uniquely to $G_v$, call it $F$. As above, observe that both $c_f$ and $c_F$ have the same Brauer class. Define a function $\alpha'$ on $G_v$ as follows:

$$\alpha' = \begin{cases} \frac{\alpha}{f}, & \text{if } K \subseteq F_v, \\ \frac{\alpha}{F}, & \text{if } K \nsubseteq F_v. \end{cases} \tag{5.9}$$

Then the assumptions of Lemma 2.2.1 will be satisfied by $S = \alpha'$ and $t = \epsilon'$ and we get the result. □

### The case $KF_v|F_v$ is ramified

This case will happen only if $p$ is an odd ramified supercuspidal prime with $p \equiv 3 \pmod 4$ and $e_v$ odd. For any quadratic extension $L_1|L_2$ and $x \in L_2^\times$, the symbol $(x, L_1|L_2) = 1$ or $-1$ according as $x$ is a norm of an element of $L_1$ or not.

In the ramified case, the possibilities for $K$ are $\mathbb{Q}_p(\sqrt{-p})$ and $\mathbb{Q}_p(\sqrt{-p\zeta_{p-1}})$ depending on $(p, K|\mathbb{Q}_p) = 1$ or $-1$ respectively. We can choose $\pi = \sqrt{-p}$ or $\sqrt{-p\zeta_{p-1}}$ as a uniformizer of $K$ and write $K = \mathbb{Q}_p(\pi)$. For any lift $\sigma$ of the generator of $\mathrm{Gal}(K|\mathbb{Q}_p)$ to $G_p$, we have $\pi^\sigma = -\pi$ and $N_{K|\mathbb{Q}_p}(\pi) = -\pi^2$.

For a field $L$, let $\mathcal{O}_L^\times$ be the ring of units inside $\mathcal{O}$. Since $KF_v|F_v$ is a ramified quadratic extension, $N_{KF_v|F_v}(\mathcal{O}_{KF_v}^\times) = \mathcal{O}_{F_v}^{\times 2}$. Let $\pi_v$ be a fixed uniformizer in $N_{KF_v|F_v}((KF_v)^\times) \subseteq F_v^\times$. Writing $a = \pi_v^{v(a)} \cdot a' \in F_v^\times$, we have $\left(\frac{a'}{v}\right) = (a, KF_v|F_v)$.

Note that $f_v$ is odd in our case. For these primes, we have

$$(-1, KF_v|F_v) = \left(\frac{-1}{v}\right) = \left(\frac{-1}{p}\right)^{f_v} = (-1)^{f_v} = -1. \tag{5.10}$$

Otherwise, $X_v$ is a matrix algebra over $F_v$ [cf. Remark 5.2.22]. Since $N_{KF_v|F_v}(\sqrt{-p\zeta_{p-1}}) = p\zeta_{p-1}$ and $N_{KF_v|F_v}(\sqrt{-p}) = p$, we deduce that

$$\left(\frac{(\pi^2)'}{v}\right) = \begin{cases} \left(\frac{(-p)'}{v}\right) = \left(\frac{-1}{v}\right)(p, KF_v|F_v) = \left(\frac{-1}{v}\right), & \text{if } (p, K|\mathbb{Q}_p) = 1, \\ \left(\frac{(-p\zeta_{p-1})'}{v}\right) = \left(\frac{-1}{v}\right)(p\zeta_{p-1}, KF_v|F_v) = \left(\frac{-1}{v}\right), & \text{if } (p, K|\mathbb{Q}_p) = -1. \end{cases} \tag{5.11}$$

**Lemma 5.2.17.** *For all $\iota \in I_v \setminus I_{KF_v}$ and $\alpha(\iota) \notin F_v^\times$, we have $\alpha^2(\iota) \equiv d \bmod F_v^{\times^2}$, for some fixed $d \in F_v^\times$.*

*Proof.* Let us consider an element $i \in I_v \setminus I_{KF_v}$ which we fix now. Since $i^2 \in I_{KF_v} \subseteq I_K$, we have $\alpha^2(i) \in F_v^\times$ [cf. Lemma 5.2.10]. Hence, $\alpha(i) \equiv \sqrt{d} \bmod F_v^\times$ for some $d \in F_v^\times$. Any element $\iota \in I_v \setminus I_{KF_v}$ can be written as $\iota = ij$ for some $j \in I_{KF_v}$. Using Lemma 5.2.10 and the homomorphism $\tilde{\alpha}$, we have $\alpha(\iota) \equiv \alpha(i)\alpha(j) \equiv \alpha(i) \equiv \sqrt{d} \bmod F_v^\times$. □

We show that the value of the constant $d$ is $a_{p''}^2$. Let $[\ ]_v : F_v^\times \to G_v^{\mathrm{ab}}$ be the usual norm residue map.

**Lemma 5.2.18.** *As an element of the Galois group, we have $i = [-1]_v \in G_v \setminus G_{KF_v}$. Moreover, the value of the map $\alpha$ at $i$ is given by: $\alpha(i) \equiv a_{p''} \pmod{F_v^\times}$.*

*Proof.* As the norm residue map is surjective, we need to show that $[-1]_v \neq [x]_{KF_v}$ for any $x \in (KF_v)^\times$. Suppose towards a contradiction that $[-1]_v = [x]_{KF_v}$, for some $x \in (KF_v)^\times$. Let $\rho_{KF_v} = [\ ]_{KF_v}$ and $\rho_v = [\ ]_v$ be the norm reciprocity maps. Recall, the following commutative diagram from the class field theory:

$$\begin{tikzcd} (KF_v)^\times \arrow[r, "\rho_{KF_v}"] \arrow[d, "N_{KF_v|F_v}"] & G^{\mathrm{ab}}_{KF_v} \arrow[d, "\mathrm{incl}"] \\ F_v^\times \arrow[r, "\rho_v"] & G_v^{\mathrm{ab}}. \end{tikzcd} \tag{5.12}$$

From the above diagram, we have $[x]_{KF_v} = [N(x)]_v$ and so $[-1]_v = [N(x)]_v$. We write $-1 = N(x)y$ for some $y \in N_{\bar{F}_v|F_v}(\bar{F}_v^\times) = \bigcap_{F_v \subset L \text{ finite}} N_{L|F_v}(L^\times)$. As $KF_v$ is a finite extension of $F_v$, we deduce $-1$ is a norm of some element of $(KF_v)^\times$, a contradiction to (5.10). Since the norm residue map sends the unit group of $F_v$ onto the inertia subgroup of $G_v$, the element $i = [-1]_v \in I_v \backslash I_{KF_v}$.

Note that $i = [-1]_v \in G_v (\subseteq G_p)$ is one of the several elements that maps to $-1 \in \mathbb{Q}_p^\times$ under the reciprocity map. For all $\gamma \in \Gamma$, we obtain

$$\alpha(i)^{\gamma-1} = \chi_\gamma(i) = \chi_\gamma([-1]) \overset{(3.3)}{=} \chi_{\gamma,p}(-1)^{-1} = \chi_{\gamma,p}(-1) = \chi_{\gamma,p}(p'')^{\frac{p-1}{2}} = \left(\alpha(\mathrm{Frob}_{p''})^{\gamma-1}\right)^{\frac{p-1}{2}}.$$

We deduce that $\alpha(i) \equiv a_{p''}^{\frac{p-1}{2}} \bmod F_v^\times$. Using the property (1) of Proposition 2.1.2 and $\alpha(\mathrm{Frob}_{p''}) \equiv a_{p''} \pmod{F_v^\times}$, we have $\epsilon(p'') = \epsilon_p(p'') \equiv a_{p''}^2 \pmod{F_v^\times}$. Since $p''$ has order $(p-1)$ in $(\mathbb{Z}/p^{N_p}\mathbb{Z})^\times$, we have $a_{p''}^2 \in F_v^\times$. As $p \equiv 3 \pmod 4$ and $(p-1)/2$ is odd, we deduce that $\alpha(i) \equiv a_{p''} \bmod F_v^\times$. □

**Lemma 5.2.19.** *If $p \equiv 3 \pmod 4$ and $(p, K|\mathbb{Q}_p) = -1$, then we have $a_{p''}^2 u \in F_v^{\times 2}$, for some unit $u \in \mathcal{O}_v^\times$.*

*Proof.* For an odd prime $p$, the two ramified quadratic extensions of $\mathbb{Q}_p$ are $\mathbb{Q}_p(\sqrt{-p})$ and $\mathbb{Q}_p(\sqrt{-p\zeta_{p-1}})$ up to an isomorphism. Note that $\mathbb{Q}_p(\sqrt{p})$ is always a ramified quadratic extension of $\mathbb{Q}_p$ and $-1$ has no square root modulo $p$ for primes $p \equiv 3 \pmod 4$. Since when $(p, K|\mathbb{Q}_p) = -1$, the only possibility for $K$ is $\mathbb{Q}_p(\sqrt{p}) = \mathbb{Q}_p(\sqrt{-p\zeta_{p-1}})$.

We obtain $s' = \sqrt{-\zeta_{p-1}} \in K$ and let $g_{s'} \in G_K$ be an element which is mapped to $s' \in K^\times$ under the reciprocity map. We have a following equality:

$$\chi_\gamma(g_{s'}) = \chi_\gamma([N_{K|\mathbb{Q}_p}(s')]) = \chi_\gamma([\zeta_{p-1}]) \overset{(3.3)}{=} \chi_{\gamma,p}(p'')^{-1} = \chi_\gamma(p'')^{-1}.$$

Using (2.1), we deduce that $\alpha(g_{s'})\alpha(\mathrm{Frob}_{p''}) \in F_v^\times$ and hence $\alpha(g_{s'}) \cdot a_{p''} \in F_v^\times$. We now claim that $\alpha(g_{s'}) \equiv u \bmod F_v^\times$, for some unit $u \in \mathcal{O}_v^\times$. Since $s'$ is a root of unity, the element $g_{s'} \in I_T(K)$ by class field theory. For an element $i \in I_T(K)$, we know that $\alpha(i) \in F_v^\times$ and $\alpha(i) \equiv \omega(i) \bmod F_v^\times$ [cf. Lemma 5.2.10]. Since $\omega$ takes values in the $(p-1)$-th roots of unity, we get the result. $\square$

**Theorem 5.2.20.** *Let $v \mid p$ be a prime of $F$ with $KF_v|F_v$ is a ramified quadratic extension. Then the ramification of $X_v$ is determined by the parity of $m_v + n_v$.*

*Proof.* Define a function $f$ on $G_v$ by

$$f(g) = \begin{cases} 1, & \text{if } g \in G_{KF_v}, \\ a_{p''}, & \text{if } g \in G_v \backslash G_{KF_v}. \end{cases} \tag{5.13}$$

Note that $KF_v = F_v(\pi)$. Denote the image of $g \in G_v$ under the projection in $G_v/G_{KF_v} = \mathrm{Gal}(F_v(\pi)|F_v)$ by $\bar{g}$. We now consider the function $F$ on $\mathrm{Gal}(KF_v|F_v)$:

$$F(g) = \begin{cases} 1, & \text{if } \bar{g} = 1, \\ a_{p''}, & \text{if } \bar{g} \neq 1. \end{cases} \tag{5.14}$$

Using equations (5.13) and (5.14), one can check that $c_F(\bar{g}, \bar{h}) = c_f(g, h)$. In other words, we deduce that the inverse of the inflation map $\mathrm{Inf} : \mathrm{H}^2(F_v(\pi)|F_v) \hookrightarrow \mathrm{H}^2(\bar{F}_v|F_v)$ sends $c_f$ to $c_F$. Let $\sigma$ be the non-trivial element of $\mathrm{Gal}(F_v(\pi)|F_v)$. The cocycle table of $C_F$ is given by:

| | 1 | $\sigma$ |
|---|---|---|
| 1 | 1 | 1 |
| $\sigma$ | 1 | $a_{p''}^2$ |

which gives the symbol $(\pi^2, a_{p''}^2)_v$. Using the above inflation map we get both $c_f$ and $c_F$ have same class in their respective Brauer groups. Define an integer $n_v$ mod 2 by $(-1)^{n_v} = (\pi^2, a_{p''}^2)_v$.

Let $\alpha'(g) = \frac{\alpha(g)}{f(g)}$ on $G_v$. Then the cocycle $c_\alpha$ can be decomposed as $= c_{\alpha'} c_f$. The two conditions of Lemma 2.2.1 are satisfied by $S = \alpha'$ and $t = \epsilon'$. Thus, we obtain:

$$\mathrm{inv}_v(c_{\alpha'}) = \frac{1}{2} \cdot v\Big(\frac{\alpha'^2}{\epsilon'}(\mathrm{Frob}_v)\Big) = \frac{1}{2} \cdot v\Big(\frac{\alpha^2}{\epsilon'}(\mathrm{Frob}_v)\Big) = \frac{1}{2} \cdot f_v \cdot v(a_{p'}^2 \epsilon(p')^{-1}) \bmod \mathbb{Z}$$

as before. Hence, we deduce that:

$$\mathrm{inv}_v(c_\alpha) = \mathrm{inv}_v(c_{\alpha'}) + \mathrm{inv}_v(c_f) = \frac{1}{2} \cdot \Big( f_v \cdot v(a_{p'}^2 \epsilon(p')^{-1}) + n_v \Big) \mathrm{mod}\ \mathbb{Z}.$$

□

Multiplicatively, we can write the above as $[X_v] \sim (-1)^{f_v \cdot v(a_{p'}^2 \epsilon(p')^{-1})} \cdot (\pi^2, a_{p''}^2)_v$. The following lemma will complete the proof which is a simplification of this product depending upon the value of $(p, K|\mathbb{Q}_p)$.

**Lemma 5.2.21.** *If $KF_v|F_v$ is a ramified quadratic extension, then the ramification formula is given by:*

$$[X_v] \sim \Big( (-1)^k a_{p''}^2 \epsilon(p'')^{-1}, KF_v|F_v \Big).$$

*Proof.* Since $p \equiv 3 \pmod 4$, we get $(Nv - 1)/2 = (p^{f_v} - 1)/2 \equiv f_v \pmod 2$. Recall that both $v(\pi^2) = e_v$ and $(p-1)/2$ is odd that divides $e_v$ [8, Lemma 4.1]. We have an equality of symbols: $(\pi^2, a_{p''}^2)_v \overset{(1.1)}{=} (-1)^{v(\pi^2) v(a_{p''}^2)(p^{f_v}-1)/2} \Big( \frac{(a_{p''}^2)'}{v} \Big)^{v(\pi^2)} \Big( \frac{(\pi^2)'}{v} \Big)^{v(a_{p''}^2)} \overset{(6.18)}{=} (-1)^{f_v \cdot v(a_{p''}^2)} \Big( \frac{(a_{p''}^2)'}{v} \Big) \Big( \frac{-1}{v} \Big)^{v(a_{p''}^2)} = \Big( \frac{(a_{p''}^2)'}{v} \Big)$.

When $(p, K|\mathbb{Q}_p) = 1$, using [8, Lemma 7.2] we deduce that:

$$\begin{aligned}
[X_v] &\sim (-1)^{f_v \cdot v(a_{p'}^2 \epsilon(p')^{-1})} \cdot (\pi^2, a_{p''}^2)_v \\
&= (-1)^{e_v f_v (k-1)} (-\epsilon_p(-1))^{2e_v f_v/(p-1)} \cdot \Big( \frac{(a_{p''}^2)'}{v} \Big) \\
&= (-1)^{(k-1) f_v} (-\epsilon_p(-1))^{f_v} \cdot \Big( \frac{(a_{p''}^2)'}{v} \Big) \\
&= (-1)^{k f_v} (\epsilon_p(p'')^{(p-1)/2})^{f_v} \cdot \Big( \frac{(a_{p''}^2)'}{v} \Big) \\
&= (-1)^{k f_v} \Big( \frac{\epsilon(p'')^{(p-1)/2}}{p} \Big)^{f_v} \cdot \Big( \frac{(a_{p''}^2)'}{v} \Big) \\
&= (-1)^{k f_v} \Big( \frac{\epsilon(p'')}{p} \Big)^{f_v} \cdot \Big( \frac{(a_{p''}^2)'}{v} \Big) \\
&= (-1)^{k f_v} \Big( \frac{(\epsilon(p''))'}{v} \Big) \cdot \Big( \frac{(a_{p''}^2)'}{v} \Big) \sim \Big( (-1)^k \epsilon(p'')^{-1}, KF_v|F_v \Big) \cdot \Big( a_{p''}^2, KF_v|F_v \Big) \\
&= \Big( (-1)^k a_{p''}^2 \epsilon(p'')^{-1}, KF_v|F_v \Big).
\end{aligned}$$

On the other hand for $(p, K|\mathbb{Q}_p) = -1$, using [8, Lemma 7.2] we have:

$$
\begin{aligned}
[X_v] &\sim (-1)^{f_v \cdot v(a_{p'}^2 \epsilon(p')^{-1})} \cdot (\pi^2, a_{p''}^2)_v \\
&= (-1)^{e_v f_v (k-1)} (-\epsilon_p(-1))^{2e_v f_v/(p-1)} \cdot (-1)^{f_v \cdot v(a_{p''}^2 \epsilon(p'')^{-1})} \cdot \left(\frac{(a_{p''}^2)'}{v}\right) \\
&= \Big((-1)^k a_{p''}^2 \epsilon(p'')^{-1}, KF_v|F_v\Big) \cdot (-1)^{f_v \cdot v(a_{p''}^2 \epsilon(p'')^{-1})} \\
&\overset{(5.2.19)}{=} \Big((-1)^k a_{p''}^2 \epsilon(p'')^{-1}, KF_v|F_v\Big).
\end{aligned}
$$

□

*Remark* 5.2.22. In general, when $KF_v|F_v$ is ramified quadratic, we have $[X_v] \sim (-1)^{f_v \cdot v(a_{p'}^2 \epsilon(p')^{-1})} \cdot (\pi^2, d)_v$ with $d$ as in Lemma 5.2.17. If $f_v$ is even, then by (1.1), the symbol $(\pi^2, d)_v = 1$ as $(p^{f_v} - 1)/2 \equiv f_v \pmod 2$. Hence, $X_v$ is unramified.

## 5.3 Dihedral supercuspidal prime $p = 2$

We now consider the case $p = 2$. In this case, the ramification cannot be determined always by the "companion adjoint slope" but it can be predicted together with explicitly computable error terms that can be obtained from the information about the local automorphic representation $\pi_p$ given by $p$-adic Hodge theory.

In the dihedral supercuspidal case, the local representation is induced by a character $\chi$ of an index two subgroup $W_K$ of $W_2$:

$$\rho_2(f) \sim \mathrm{Ind}_{W_K}^{W_2} \chi \tag{5.15}$$

where $K$ is a quadratic extension of $\mathbb{Q}_2$. We show that the inertia type $\chi|_{I_2}$ or $\chi|_{I_K}$ can be written as follows [cf. Chapter 4]: $\chi|_{I_2} = \chi|_{I(L|\mathbb{Q}_2)} = \omega_2^l \cdot \chi_1 \cdot \chi_2$ (when $K$ is unramified) and $\chi|_{I_K} = \chi|_{I(L|K)} = \omega^l \cdot \chi_1 \cdot \chi_2$ (when $K$ is ramified), where $\omega_2$ is the fundamental character of level 2, $\omega$ is the fundamental character of level 1, $l$ is an integer, and $\chi_1$ and $\chi_2$ are characters of a cyclic group of 2-power order generated by $\gamma_1$ and $\gamma_2$ respectively. Assume that $\chi_1$ takes $\gamma_1$ to $\zeta_{2^r}$ and $\chi_2$ takes $\gamma_2$ to $\zeta_{2^s}$. Here, we denote by $\zeta_{2^n}$ a primitive $2^n$-th root of unity.

We now fix a uniformizer $\pi$ in $K$ and let $g_\pi \in G_K$ be an element which is mapped to $\pi \in K^\times$ under the reciprocity map. Note that $g_\pi$ is a Frobenius element in $G_K$. Assume that

$$\alpha(g_\pi) \equiv b \bmod F_v^\times. \tag{5.16}$$

Consider two fields $F_v^{'} = F_v(b, \zeta_{2^s} + \zeta_{2^s}^{-1})$ and $F_v^{''} = F_v(b, \zeta_{2^r})$. We now define two characters $\psi_1, \psi_2$ on $G_v$ as follows:

$$\psi_1(g) = \begin{cases} 1 & \text{if } \alpha(g) \in (F_v^{'})^\times \\ -1 & \text{if } \alpha(g) \notin (F_v^{'})^\times \end{cases} \quad \text{and} \quad \psi_2(g) = \begin{cases} 1 & \text{if } \alpha(g) \in (F_v^{''})^\times \\ -1 & \text{if } \alpha(g) \notin (F_v^{''})^\times. \end{cases}$$

We denote by $F_v(\sqrt{t_1}), F_v(\sqrt{t_2})$, the quadratic extensions of $F_v$ cut out by the characters $\psi_1$ and $\psi_2$. Define an integer $n_v$ modulo 2 as follows:

$$(-1)^{n_v} = \begin{cases} (t_1, \zeta_{2^{r-1}})_v \cdot (t_2, (\zeta_{2^s} + \zeta_{2^s}^{-1})^2)_v, & \text{if } p = 2 \text{ and } s \neq 2, \\ (t_2, a_{p^\dagger}^2)_v, & \text{if } p = 2 \text{ and } s = 2. \end{cases} \tag{5.17}$$

Consider an element $d_0 \in F_v^\times$ given by $\alpha(i) \equiv \sqrt{d_0} \bmod F_v^\times \ \forall\, i \in I_T(F_v) \setminus I_T(KF_v)$. An easy check using Lemma 5.4.1 shows that $d_0$ is well-defined. We also define two integers $n_v^{'}, n_v^{''}$ mod 2 by $(-1)^{n_v^{'}} = (t_1, \zeta_{2^{r-1}})_v \cdot (t_2, (\zeta_{2^s} + \zeta_{2^s}^{-1})^2)_v \cdot (t, d_0)_v$ and $(-1)^{n_v^{''}} = (t_2, a_{p^\dagger}^2)_v \cdot (t, d_0)_v$, where $K = \mathbb{Q}_2(\sqrt{t})$ is ramified. Note that these error terms can be explicitly computed from the information about a given modular form following [21].

The following is the main theorem for dihedral supercuspidal prime $p = 2$.

**Theorem 5.3.1.** *Let $p = 2$ be a dihedral supercuspidal prime for $f$ and $v$ be a place of $F$ lying above prime $p$. The ramification of $X_v$ is determined by the parity of $m_v + r_v$.*

1. *If $K \subseteq F_v$ or $KF_v|F_v$ is an unramified quadratic extension, then the error term is $r_v = n_v$.*

2. *Assume $KF_v|F_v$ is a ramified quadratic extension.*

   - *If $\zeta_{2^s} + \zeta_{2^s}^{-1} \neq 0$, then the error term $r_v = n_v^{'}$.*
   - *For $\zeta_{2^s} + \zeta_{2^s}^{-1} = 0$ the error term is given by $r_v = n_v^{''}$.*

In case (2) of the above theorem with $F = \mathbb{Q}$, we will prove that the quantity $d_0$ in the error term is equal to $a_{p'''}^2$ except $K = \mathbb{Q}_2(\sqrt{t})$ with $t = 2, -6$ [cf. Lemma 5.4.9]. We derive he following corollary that determines the situation of the above theorem when the local algebra $X_v$ is determined by the parity of $m_v$ itself.

**Corollary 5.3.2.** *Let $p = 2$ be a dihedral supercuspidal prime for $f$ with $N_2 = 2$. The ramification of the local Brauer class of $X_v$ is determined by the parity of $m_v$, for any $v \mid 2$.*

## 5.4 Ramifications for primes lying above dihedral supercuspidal prime $p = 2$

In this section, we will provide the proof of the results stated above for the dihedral supercuspidal prime $p = 2$. For $i \in I_K$, let $\bar{i}$ denote the projection to the inertia subgroup $I(L|K)$. The following lemma will give the information about $\alpha$ on the inertia group $I_K$.

**Lemma 5.4.1.** *Let $p = 2$ be a dihedral supercuspidal prime for $f$. For all $i \in I_K$, we have*

$$\alpha(i) \equiv \begin{cases} 1 \ \mathit{mod}\ F_v^\times, & \mathit{if}\ i \in I_T(K), \\ \zeta_{2^r} \ \mathit{mod}\ F_v^\times, & \mathit{if}\ \bar{i}\ \mathit{is\ an\ odd\ power\ of}\ \gamma_1, \\ \zeta_{2^s} + \zeta_{2^s}^{-1} \ \mathit{mod}\ F_v^\times, & \mathit{if}\ \bar{i}\ \mathit{is\ an\ odd\ power\ of}\ \gamma_2\ \mathit{and}\ \zeta_{2^s} + \zeta_{2^s}^{-1} \neq 0. \end{cases}$$

*Furthermore, $\alpha(i^{2k}) \equiv 1$ mod $F_v^\times$ for all $k \in \mathbb{Z}$.*

*Proof.* Consider the case $K$ is unramified. By the part (2) of Proposition 2.1.2 and using (4.3), (4.4) we have $\alpha(i) \equiv \omega_2^l(\bar{i}) + \omega_2^{2l}(\bar{i})$ mod $F_v^\times$ for all $i \in I_T(K)$. Since $\omega_2$ takes values in the third roots of unity, we have $\alpha(i) \equiv 1$ mod $F_v^\times$.

In the ramified case, we obtain $\alpha(i) \equiv \omega(\bar{i}) + \omega(\bar{i}) \equiv 1$ mod $F_v^\times$. Let $j_1$ and $j_2$ be a $\gamma_1$-element and a $\gamma_2$-element respectively [cf. Definition 4.2.2]. Then $\alpha(j_1) \equiv \chi_1(\gamma_1) + \chi_1^\sigma(\gamma_1) \equiv \zeta_{2^r}$ mod $F_v^\times$ and $\alpha(j_2) \equiv \chi_2(\gamma_2) + \chi_2^\sigma(\gamma_2) \equiv \zeta_{2^s} + \zeta_{2^s}^{-1}$ mod $F_v^\times$.

Since $\alpha(j_2) \equiv \zeta_{2^s} + \zeta_{2^s}^{-1}$ mod $F^\times$, using [Proposition 2.1.2, property (1)] we have that $\epsilon(j_2) = a(\zeta_{2^s} + \zeta_{2^s}^{-1})^2$ for some $a \in F^\times$. Recall that $\epsilon(j_2)$ is a root of unity. Thus, we obtain $\epsilon(j_2) \in \{\pm 1\}$ and so $(\zeta_{2^s} + \zeta_{2^s}^{-1})^2 \in F^\times$. In turn, this implies $\zeta_{2^{s-1}} + \zeta_{2^{s-1}}^{-1} \in F^\times$. As $r < s$, we must have $\zeta_{2^r} + \zeta_{2^r}^{-1} \in F^\times$. Note that the field $F(\zeta_{2^r})$ inside $E$ has degree 2 over both the fields $F = F(\zeta_{2^r} + \zeta_{2^r}^{-1})$ and $F(\zeta_{2^{r-1}})$. Since $F \subseteq F(\zeta_{2^{r-1}})$, we conclude that $\zeta_{2^{r-1}} \in F_v^\times$. We get the desired result using the homomorphism $\tilde{\alpha}$. The last statement follows from the observation $\zeta_{2^{r-1}}, (\zeta_{2^s} + \zeta_{2^s}^{-1})^2 \in F_v^\times$. □

**Lemma 5.4.2.** *If $p = 2$ is a dihedral supercuspidal prime for $f$, then $\epsilon$ is $F^\times$-valued on $I_W(K)$.*

*Proof.* For an extra twist $(\gamma, \chi_\gamma)$, we have $\rho_{f^\gamma} \sim \rho_f \otimes \chi_\gamma$. By restricting the representation to $G_2$, we obtain:

$$\begin{pmatrix} \chi^\gamma & 0 \\ 0 & (\chi^\sigma)^\gamma \end{pmatrix} \sim \begin{pmatrix} \chi\chi_\gamma & 0 \\ 0 & \chi^\sigma\chi_\gamma \end{pmatrix}.$$

Equating deteminants on both sides, we get $(\chi\chi^\sigma)^\gamma = \chi\chi^\sigma\chi_\gamma^2$ for all $\gamma \in \Gamma$. Since $\chi_\gamma^2 = \epsilon^{\gamma-1}$, the quantity $\frac{\chi\chi^\sigma}{\epsilon} \in F^\times$ and so $\frac{\chi_1^2}{\epsilon} \in F^\times$ on the wild inertia group $I_W(K)$. We get the lemma as $\chi_1^2 \in F^\times$. □

**Lemma 5.4.3.** *Let $p = 2$ be a dihedral supercuspidal prime for $f$ and $\zeta_{2^r} + \zeta_{2^r}^{-1} \neq 0$.*

- *If $g \in G_K$ and $\alpha(g) \notin (F_v^{'})^\times$, then $\alpha(g) \equiv \zeta_{2^r} \mod (F_v^{'})^\times$.*
- *If $g \in G_K$ and $\alpha(g) \notin (F_v^{''})^\times$, then we have $\alpha(g) \equiv (\zeta_{2^s} + \zeta_{2^s}^{-1}) \mod (F_v^{''})^\times$.*

*Proof.* Recall that every element $g \in G_K$ can be written as $g_\pi^n i$, for some $n \in \mathbb{Z}$ and $i \in I_K$. Using the homomorphism $\tilde{\alpha}$ and Lemma 5.4.1, we get the result. □

### 5.4.1 The case $s \neq 2$

In this case, we must have that $\zeta_{2^s} + \zeta_{2^s}^{-1}$ is non-zero. As a result, we can apply Lemma 5.4.1 and Lemma 5.4.3.

**Auxillary functions for $s \neq 2$.** First assume that $K \subseteq F_v$ with $\zeta_{2^s} + \zeta_{2^s}^{-1} \neq 0$. Define two functions on $G_v$ by

$$f_1(g) = \begin{cases} 1 & \text{if } \alpha(g) \in (F_v^{'})^\times \\ \zeta_{2^r} & \text{if } \alpha(g) \notin (F_v^{'})^\times \end{cases} \quad \text{and} \quad f_2(g) = \begin{cases} 1 & \text{if } \alpha(g) \in (F_v^{''})^\times \\ \zeta_{2^s} + \zeta_{2^s}^{-1} & \text{if } \alpha(g) \notin (F_v^{''})^\times. \end{cases} \tag{5.18}$$

We call an element $g$ type 1 if $\alpha(g) \in (F_v^{'})^{\times}$, otherwise we call it type 2. Here, we will use Lemma 5.4.3 and the fact that $\zeta_{2^{r-1}}, (\zeta_{2^s} + \zeta_{2^s}^{-1})^2 \in F_v^{\times}$. We see that if $g$ and $h$ both are type 1 elements then $gh$ is also so, but if one of them is of type 1 and the other one is of type 2, then their product is an element of type 2. The product of two type 2 elements is an element of type 1. Thus, we can and do replace the conditions which define the function $f_1$ by a quadratic character $\psi_1$ in the following way: $\psi_1(g) = 1$, if $\alpha(g) \in (F_v^{'})^{\times}$ and $\psi_1(g) = -1$, otherwise. In a similar way, we can replace the conditions for $f_2$ by a quadratic character $\psi_2$ defined by $\psi_2(g) = 1$, if $\alpha(g) \in (F_v^{''})^{\times}$ and $\psi_2(g) = -1$, otherwise. Then the functions $f_1$ and $f_2$ can be seen alternatively as:

$$f_1(g) = \begin{cases} 1 & \text{if } \psi_1(g) = 1 \\ \zeta_{2^r} & \text{if } \psi_1(g) = -1 \end{cases} \quad \text{and} \quad f_2(g) = \begin{cases} 1 & \text{if } \psi_2(g) = 1 \\ \zeta_{2^s} + \zeta_{2^s}^{-1} & \text{if } \psi_2(g) = -1. \end{cases} \tag{5.19}$$

We denote the quadratic extensions of $F_v$ cut out by the characters $\psi_1$ and $\psi_2$ by $F_v(\sqrt{t_1}), F_v(\sqrt{t_2})$ for $t_1, t_2 \in F_v^{\times}$ respectively. The functions $f_1$ and $f_2$ will induce 2-cocycles $c_{f_1}$ and $c_{f_2}$ in $\mathrm{Br}(F_v)$. To compute $\mathrm{inv}_v(c_{f_1})$, consider the non-trivial element $\sigma$ of the Galois group $\mathrm{Gal}(F_v(\sqrt{t_1})|F_v)$. The cocycle table of the 2-cocycle $c_{f_1}$ is given by

| | $1$ | $\sigma$ |
|---|---|---|
| $1$ | $1$ | $1$ |
| $\sigma$ | $1$ | $\zeta_{2^{r-1}}$ |

which gives the symbol $(t_1, \zeta_{2^{r-1}})_v$. Similarly, the cocycle table of $c_{f_2}$ gives the symbol $(t_2, (\zeta_{2^s} + \zeta_{2^s}^{-1})^2)_v$.

Define two integers $n_{1,v}, n_{2,v}$ mod 2 by $(-1)^{n_{1,v}} = (t_1, \zeta_{2^{r-1}})_v$ and $(-1)^{n_{2,v}} = (t_2, (\zeta_{2^s}+\zeta_{2^s}^{-1})^2)_v$. Let us consider an integer $n_v$ mod 2 defined by $(-1)^{n_v} = (-1)^{n_{1,v}+n_{2,v}} = (t_1, \zeta_{2^{r-1}})_v \cdot (t_2, (\zeta_{2^s} + \zeta_{2^s}^{-1})^2)_v$.

We now assume the case $K \not\subseteq F_v$ with $\zeta_{2^s} + \zeta_{2^s}^{-1} \neq 0$. Consider a non-trivial element $\bar{\sigma}_v \in G_v/G_{KF_v}$ for some $\sigma_v \in G_v$. Then any element $g \in G_v$ can be written as:

$$g = \sigma_v^n h \text{ for some } h \in G_{KF_v} \text{ and } n \in \{0, 1\}. \tag{5.20}$$

Note that $n = 0$ when $g \in G_{KF_v}$. Using this decomposition one can extend $f_1$ and $f_2$

to $G_v$, call it $F_1$ and $F_2$, by defining $F_1(g) = f_1(h)$ and $F_2(g) = f_2(h)$. The inflation map $\mathrm{Inf} : {}_2\mathrm{H}^2(G_{KF_v}, (\bar{F_v}^\times)^{\mathrm{Gal}(KF_v|F_v)}) \hookrightarrow {}_2\mathrm{H}^2(G_v, \bar{F_v}^\times)$ sends $c_{f_1}$ and $c_{f_2}$ to $c_{F_1}$ and $c_{F_2}$ respectively. Since the 2-cocycles $c_{f_1}$ and $c_{f_2}$ are trivial, so are $c_{F_1}$ and $c_{F_2}$. Thus, their classes are same in their respective Brauer groups. We now prove Theorem 5.3.1 for $s \neq 2$.

**Theorem 5.4.4.** *Let $p = 2$ be a dihedral supercuspidal prime for $f$ with $s \neq 2$ and $v$ be a place of $F$ lying above prime $p$.*

1. *If $K \subseteq F_v$ or $KF_v|F_v$ is an unramified quadratic extension, then $[X_v] \sim m_v + n_v$.*
2. *Assume $KF_v|F_v$ is a ramified quadratic extension. We then have $[X_v] \sim m_v + n_v^{'}$.*

*Proof.* Let $p = 2$ be a dihedral supercuspidal prime for $f$ and $v$ be a prime of $F$ lying above $p$. Suppose that $s \neq 2$, that is, $\zeta_{2^s} + \zeta_{2^s}^{-1}$ is non-zero.

1. First assume that $K \subseteq F_v$ or $KF_v|F_v$ is unramified quadratic. In this case, we have $I_v = I_{KF_v} \subseteq I_K$. Hence, we can apply Lemma 5.4.1 to determine $\alpha(i)\ \forall\ i \in I_v$. Consider a function $\alpha'$ on $G_v$ defined as follows:

$$\alpha' = \begin{cases} \frac{\alpha}{f_1 f_2}, & \text{if } K \subseteq F_v, \\ \frac{\alpha}{F_1 F_2}, & \text{if } KF_v|F_v \text{ unramified quadratic.} \end{cases}$$

   When $KF_v|F_v$ is an unramified quadratic extension, we have $I_v = I_{KF_v}$ and hence, in the decomposition (5.20) for any element of $I_v$, we must have $n = 0$. As a map on $I_v$, we have $\frac{\alpha}{F_1F_2} = \frac{\alpha}{f_1f_2}$.

   To check $\epsilon_2$ is $F^\times$-valued on $G_v$, it is enough to compute $\epsilon_2$ on $I_v$, since $\epsilon_2(G_v) = \epsilon_2(I_v)$ [2, Lemma 7.3.4]. Let $\tau = ij \in I_v$ with $i \in I_T(F_v)$ and $j \in I_W(F_v)$. Then $\epsilon_2(\tau) = \epsilon_2(ij) = \epsilon_2(j) = \epsilon(j) \equiv 1 \bmod F^\times$ by Lemma 5.4.2.

   Now by unravelling the definition of $\alpha'$ and using Lemma 5.4.1, [Proposition 2.1.2, part (1)] and the fact $\zeta_{2^{r-1}}, (\zeta_{2^s} + \zeta_{2^s}^{-1})^2 \in F_v^\times$, we see that the two conditions of Lemma 2.2.1 are satisfied by $S = \alpha'$ and $t = \epsilon'$. Thus, we obtain

$$\mathrm{inv}_v(c_{\alpha'}) = \frac{1}{2} \cdot v\Big(\frac{\alpha'^2}{\epsilon'}(\mathrm{Frob}_v)\Big) = \frac{1}{2} \cdot v\Big(\frac{\alpha^2}{\epsilon'}(\mathrm{Frob}_v)\Big) = \frac{1}{2} \cdot f_v \cdot v(a_{p'}^2 \epsilon(p')^{-1}) \bmod \mathbb{Z}$$

as before. Since $\mathrm{inv}_v(c_\alpha) = \mathrm{inv}_v(c_{\alpha'}) + \mathrm{inv}_v(c_{f_1}) + \mathrm{inv}_v(c_{f_2})$ and the invariant of the 2-cocycles $c_{f_1}$ and $c_{f_2}$ are determined by the integers $n_{1,v}$ and $n_{2,v}$ modulo 2 respectively, we obtain that $X_v \sim m_v + n_v$.

2. Since $KF_v|F_v$ is a ramified quadratic extension, we have $N_{KF_v|F_v}(\mathcal{O}_{KF_v}^\times) = \mathcal{O}_{F_v}^{\times 2}$, where $\mathcal{O}^\times$ denotes units. We choose an element $a_0 \in \mathcal{O}_{F_v}^\times$ which is not a norm of the extension. Let $i_0 \in G_v$ be an element which is mapped to $a_0$ under the reciprocity map. Assume that $i_0$ is an element of the tame inertia part. Since $I_T(F_v)^2 = I_T(KF_v)$, we have that $i_0 \in I_T(F_v) \setminus I_T(KF_v)$. Since $i_0^2$ belongs to the tame inertia part of $K$, we deduce that $\alpha(i_0) \equiv \sqrt{d_0} \pmod{F_v^\times}$ for some $d_0 \in F_v^\times$ [cf. Lemma 5.4.1].

   Define a function $f$ on $G_v$ by

$$f(g) = \begin{cases} 1, & \text{if } g \in G_{KF_v}, \\ \sqrt{d_0}, & \text{if } g \in G_v \backslash G_{KF_v}. \end{cases} \tag{5.21}$$

   Note that $KF_v = F_v(\sqrt{t})$. Denote the image of $g \in G_v$ under the projection in $G_v/G_{KF_v} = \mathrm{Gal}(F_v(\sqrt{t})|F_v)$ by $\bar{g}$. We now consider the function $F$ on $\mathrm{Gal}(KF_v|F_v)$:

$$F(g) = \begin{cases} 1, & \text{if } \bar{g} = 1, \\ \sqrt{d_0}, & \text{if } \bar{g} \neq 1. \end{cases} \tag{5.22}$$

   Using equations (5.21) and (5.22), one can check that $c_F(\bar{g}, \bar{h}) = c_f(g, h)$. In other words, we deduce that the inverse of the inflation map $\mathrm{Inf} : \mathrm{H}^2(F_v(\pi)|F_v) \hookrightarrow \mathrm{H}^2(\bar{F}_v|F_v)$ sends $c_f$ to $c_F$. Let $\sigma$ be the non-trivial element of $\mathrm{Gal}(F_v(\sqrt{t})|F_v)$. The cocycle table of $c_F$ is given by:

| | 1 | $\sigma$ |
|---|---|---|
| 1 | 1 | 1 |
| $\sigma$ | 1 | $d_0$ |

   which gives the symbol $(t, d_0)_v$. Using the above inflation map, both $c_f$ and $c_F$ have the same class in their respective Brauer groups. We now define an integer $n_v$ mod 2 by $(-1)^{n_v'} = (t_1, \zeta_{2^{r-1}})_v \cdot (t_2, (\zeta_{2^s} + \zeta_{2^s}^{-1})^2)_v \cdot (t, d_0)_v$.

Let $F_1, F_2$ be the functions as before. We apply Lemma 2.2.1 to compute the invariant of the cocycle $[c_S]$ with

$$S(g) = \tfrac{\alpha(g)}{F_1(g)F_2(g)f(g)} \text{ on } G_v \text{ and } t = \epsilon'.$$

We now deduce that $S(i) \in F_v^\times$, for all $i \in I_v$. Consider the element $i_0 \in G_v \setminus G_{KF_v}$ and hence $\bar{i_0}$ is a non-trivial element in $G_v/G_{KF_v}$. We will consider the decomposition (5.20) with respect to the element $i_0$ instead of $\sigma_v$. For $i \in I_v \setminus I_{KF_v}$, we have $i = i_0\tilde{i}$, for some $\tilde{i} \in I_{KF_v} (\subseteq I_K)$. We obtain

$$S(i) = \frac{\alpha(i)}{F_1(i)F_2(i)f(i)} \equiv \frac{\alpha(i_0)\alpha(\tilde{i})}{F_1(i)F_2(i)f(i)} \equiv \frac{\alpha(\tilde{i})}{F_1(i)F_2(i)} \equiv \frac{\alpha(\tilde{i})}{f_1(\tilde{i})f_2(\tilde{i})} \equiv 1 \bmod F_v^\times.$$

The last equality follows from the fact that $\alpha(i) \equiv f_1(i)f_2(i) \pmod{F_v^\times}$ for all $i \in I_{KF_v}$.

Note that $\epsilon_2$ is $F^\times$-valued on $I_{KF_v} \subseteq I_K$. Now for $i \in I_v \setminus I_{KF_v}$ as above, we have $\epsilon_2(i) = \epsilon_2(i_0\tilde{i}) = \epsilon_2(\tilde{i})$ with $\tilde{i} \in I_{KF_v}$ and so it is $F^\times$-valued. Thus, $\epsilon_2$ is $F^\times$-valued on $I_v$ and so is on $G_v$, since $\epsilon_2(G_v) = \epsilon_2(I_v)$. By the same argument as in (1), we see that two conditions of Lemma 2.2.1 are satisfied by $S$ and $t = \epsilon'$. Hence, we obtain

$$\mathrm{inv}_v(c_S) = \frac{1}{2} \cdot v\Big(\frac{S^2}{\epsilon'}(\mathrm{Frob}_v)\Big) = \frac{1}{2} \cdot v\Big(\frac{\alpha^2}{\epsilon'}(\mathrm{Frob}_v)\Big) = \frac{1}{2} \cdot f_v \cdot v(a_{p'}^2 \epsilon(p')^{-1}) \bmod \mathbb{Z}.$$

As usual, the cocycle $c_S$ can be decomposed as $c_\alpha = c_S c_{F_1} c_{F_2} c_f$. Applying the invariant map to the 2-cocycle $c_\alpha$, we conclude that:

$$\mathrm{inv}_v(c_\alpha) = \mathrm{inv}_v(c_S) + \mathrm{inv}_v(c_{F_1}) + \mathrm{inv}_v(c_{F_2}) + \mathrm{inv}_v(c_f) = \frac{1}{2} \cdot \Big(f_v \cdot v(a_{p'}^2 \epsilon(p')^{-1}) + n_v\Big) \bmod \mathbb{Z}.$$

$\square$

### 5.4.2 The case $s = 2$

In this case, we have $\zeta_{2^s} + \zeta_{2^s}^{-1} = 0$.

**Lemma 5.4.5.** *Assume that $K \subseteq F_v$ with $\zeta_{2^s} + \zeta_{2^s}^{-1} = 0$. Let $j_2$ be a $\gamma_2$-element. Then $\chi_\gamma$ becomes unramified (or ramified) depending on $\chi_\gamma(j_2) = 1$ (or $-1$), for all $\gamma \in \Gamma$.*

*Proof.* By assumption, we have $s = 2$. Since $\epsilon$ is $F^\times$-valued on $I_W(K)$ by Lemma 5.4.2, we have $\chi_\gamma^2(j_2) = \epsilon(j_2)^{\gamma-1} = 1$, for all $\gamma \in \Gamma$. Thus, we obtain $\chi_\gamma(j_2) = \pm 1 \ \forall \ \gamma \in \Gamma$. Depending on the value, the Dirichlet character $\chi_\gamma$ is unramified or ramified. For elements $i \in I_T(F_v)$, we have $\chi_\gamma(i) = 1$ as $\alpha(i) \in F_v^\times$. This is also the case for the elements of $\Gamma_1$, one part of the wild inertia group, as $r = 0, 1 (r < s)$ [cf. Lemma 5.4.1]. $\square$

Note that the above is true for any $\gamma_2$-element. Since $f$ is non-CM, we can and do choose an auxiliary prime $p^\dagger$ imitating a similar construction of [4, Section 6.2.3] as follows:

$$\chi_\gamma(p^\dagger) = \begin{cases} -1, & \text{if } \chi_\gamma \text{ is ramified,} \\ 1, & \text{if } \chi_\gamma \text{ is unramified,} \end{cases}$$

for all $\gamma \in \Gamma$. Since $\epsilon^{-1}$ is an extra twist, $\epsilon(p^\dagger) = -1$ and we obtain $a_{p^\dagger}^2 = -a_{p^\dagger}^2 \epsilon(p^\dagger)^{-1} \in F^\times$. We deduce that $\alpha(j_2) \equiv \alpha(\mathrm{Frob}_{p^\dagger}) \equiv a_{p^\dagger} \bmod F_v^\times$.

**Lemma 5.4.6.** *Let $p = 2$ be a dihedral supercuspidal prime for $f$ and $\zeta_{2^s} + \zeta_{2^s}^{-1} = 0$. If $g \in G_K$ and $\alpha(g) \notin (F_v'')^\times$, then we have $\alpha(g) \equiv a_{p^\dagger} \bmod (F_v'')^\times$.*

*Proof.* Since $r \in \{0, 1\}$ (as $r < s = 2$), we must have $\alpha(i) \in F_v^\times$, for all $i \in I_K$ with $\bar{i} \in < \gamma_1 >$ [cf. Lemma 5.4.1]. Then as in Lemma 5.4.3, we get the result. $\square$

**Definition 5.4.7.** *(Auxiliary functions for $s = 2$) Assume that $\zeta_{2^s} + \zeta_{2^s}^{-1} = 0$. If $K \subseteq F_v$, we define a function $f$ on $G_v (\subseteq G_K)$ by*

$$w(g) = \begin{cases} 1, & \text{if } \psi_2(g) = 1, \\ a_{p^\dagger}, & \text{if } \psi_2(g) = -1, \end{cases} \tag{5.23}$$

*with $\psi_2$ defined in Section 5.4.1. When $KF_v | F_v$ is quadratic, we use the above decomposition (5.20) to extend $w$ to $G_v$, call it $W$.*

As before, both $c_w$ and $c_W$ have the same cocycle class in their respective Brauer groups and determined by the symbol $(t_2, a_{p^\dagger}^2)_v$ with $t_2$ as above. Recall that the integer

$n_v$ be as in (5.17). We now define an integer $n_v''$ modulo 2 by $(-1)^{n_v''} = (t_2, a_{p^\dagger}^2)_v \cdot (t, d_0)_v$ with $t$ and $d_0$ as before.

**Theorem 5.4.8.** *Let $p = 2$ be a dihedral supercuspidal prime for $f$ with $s = 2$ and $v \mid p$.*

1. *Assume $K \subseteq F_v$ or $KF_v|F_v$ is an unramified quadratic extension. We then have $[X_v] \sim m_v + n_v$.*

2. *When $KF_v|F_v$ is a ramified quadratic extension, we have $[X_v] \sim m_v + n_v''$.*

*Proof.* Let $p = 2$ be a dihedral supercuspidal prime for $f$ and $v \mid p$. Suppose that $s = 2$, that is, $\zeta_{2^s} + \zeta_{2^s}^{-1}$ is zero.

1. Define a map $\alpha'$ on $G_v$ by:

$$\alpha' = \begin{cases} \frac{\alpha}{w}, & \text{if } K \subseteq F_v, \\ \frac{\alpha}{W}, & \text{if } KF_v|F_v \text{ unramified quadratic.} \end{cases}$$

   In this case, we have $I_v = I_{KF_v} \subseteq I_K$. As before, since $\frac{\alpha}{W} = \frac{\alpha}{w}$ on $I_v$, using Lemma 5.4.6 and the same argument as in the proof of Theorem 5.3.1, the two conditions of Lemma 2.2.1 are satisfied by $S = \alpha'$ and $t = \epsilon'$. Thus, we obtain $\mathrm{inv}_v(c_{\alpha'}) = \frac{1}{2} \cdot f_v \cdot v(a_{p'}^2 \epsilon(p')^{-1}) \bmod \mathbb{Z}$ and so

$$\mathrm{inv}_v(c_\alpha) = \frac{1}{2} \cdot \Big(f_v \cdot v(a_{p'}^2 \epsilon(p')^{-1}) + n_v\Big) \bmod \mathbb{Z}.$$

2. Since $KF_v|F_v$ is a ramified quadratic extension, we have $I_v^2 = I_{KF_v}$. Let $\alpha'(g) = \frac{\alpha(g)}{W(g)f(g)}$ on $G_v$ with the function $f$ as in (5.21). Then the cocycle $c_\alpha$ can be decomposed as $c_\alpha = c_{\alpha'} c_W c_f$. By the argument used in the proof of [Theorem 5.3.1, part (2)], we see that the two conditions of Lemma 2.2.1 are satisfied by $S = \alpha'$ and $t = \epsilon'$. Hence, we have $\mathrm{inv}_v(c_{\alpha'}) = \frac{1}{2} \cdot f_v \cdot v(a_{p'}^2 \epsilon(p')^{-1}) \bmod \mathbb{Z}$. Thus,

$$\mathrm{inv}_v(c_\alpha) = \mathrm{inv}_v(c_{\alpha'}) + \mathrm{inv}_v(c_W) + \mathrm{inv}_v(c_f) = \frac{1}{2} \cdot \Big(f_v \cdot v(a_{p'}^2 \epsilon(p')^{-1}) + n_v''\Big) \bmod \mathbb{Z}.$$

□

The next lemma will explicitly determine the value of $d_0$ involved in the error term of Theorem 5.3.1.

**Lemma 5.4.9.** *Let $p = 2$ be the dihedral supercuspidal prime satisfying the second condition of Theorem 5.3.1. Assume that $-1$ is not a norm of $KF_v|F_v$. Then the quantity $d_0$ is equal to $a_{p'''}^2$.*

*Proof.* Since $-1$ is not a norm of the extension $KF_v|F_v$, we can choose $a_0 = -1$ in the above theorem. For a prime $p'''$ chosen before, by local class field theory we have $\alpha(i_0)^{\gamma-1} = \chi_\gamma(i_0) \overset{(3.3)}{=} \chi_{\gamma,2}(-1)^{-1} = \chi_\gamma(p''') = \alpha(\mathrm{Frob}_{p'''})^{\gamma-1}$, for all $\gamma \in \Gamma$ and so $\alpha(i_0) \equiv a_{p'''} \bmod F_v^\times$. Hence, we deduce that $d_0 = a_{p'''}$. □

*Remark* 5.4.10. Assume $F = \mathbb{Q}$ and the dihedral supercuspidal prime $p = 2$ satisfies the second condition of Theorem 5.3.1. In this case, the ramified quadratic extension $KF_v|F_v$ becomes $K|\mathbb{Q}_2$. We have that $-1$ is not a norm of the extension $K|\mathbb{Q}_2$ except $K = \mathbb{Q}_2(\sqrt{t})$ with $t = 2, -6$, see [48, p. 34]. Then by above lemma, the quantity $d_0 = a_{p''}^2$ except $K = \mathbb{Q}_2(\sqrt{2})$ and $\mathbb{Q}_2(\sqrt{-6})$.

We now prove Corollary 5.3.2.

**Corollary 5.4.11.** *Let $p = 2$ be a dihedral supercuspidal prime for $f$ with $N_2 = 2$. The ramification of the local Brauer class of $X_v$ is determined by the parity of $m_v$, for any $v \mid 2$.*

*Proof.* If $N_2 = 2$, then the extension $K|\mathbb{Q}_2$ is unramified [cf. Chapter 3]. By the Lemmas 3.0.1 and 5.4.1, we have $\alpha(\iota) \in F_v^\times$, for all $\iota \in I_K$. Since $I_v = I_{KF_v} \subseteq I_K$, this is true for all $\iota \in I_v$. We choose $S = \alpha$ and $t = \epsilon'$ in Lemma 2.2.1 to complete the proof. □

## 5.5 Non-dihedral supercuspidal prime $p = 2$

Let $\rho_2(f)$ be the local representation of the Weil-Deligne group of $\mathbb{Q}_2$ associated to $f$ at the prime $p = 2$. When inertia acts irreducibly, the projective image of $\rho_2(f)$ is isomorphic to one of the two "exceptional" groups $A_4$ and $S_4$. Consider the natural projection $P : \widetilde{S_4} \twoheadrightarrow S_4$, where $\widetilde{S_4}$ is the double cover of $S_4$. Since Weil [52] proved that there are no extensions of $\mathbb{Q}_2$ having Galois group $\widetilde{A_4} = P^{-1}(A_4) \cong \mathrm{SL}(2, \mathbb{F}_3)$, the only possibility for the projective image of $\rho_2(f)$ is $S_4$.

For $D = \det(\rho_2(f))$ and $d = \frac{\alpha^2}{D}$, we have a cocycle class decomposition $[X] = [c_D]\cdot[c_d]$, where the cocycles $c_D$ and $c_d$ are given by

$$c_D(g,h) = \frac{\sqrt{D(g)}\sqrt{D(h)}}{\sqrt{D(gh)}}, \quad c_d(g,h) = \frac{\sqrt{d(g)}\sqrt{d(h)}}{\sqrt{d(gh)}},$$

respectively. In this section, we find the local Brauer class $[X_v]$ which is $[c_D]_v \cdot [c_d]_v$, for any $v \mid 2$. The following is the main result for non-dihedral supercuspidal prime $p = 2$.

**Theorem 5.5.1.** *Let $p = 2$ be a non-dihedral supercuspidal prime for a modular form $f$ and $v \mid 2$. The class of $X_v$ in $Br(F_v)$ is given by the symbol*

$$D(-1)^{[F_v:\mathbb{Q}_2]} \cdot (2, D_K)_v,$$

*where $D_K$ is the discriminant of the field $K$ cut out by* $\ker(d)$.

Here, $D(-1)$ denotes the value of the Galois character $D$ at the complex conjugation. If $k$ is odd, we can predict ramification in terms of nebentypus $\epsilon$. More precisely, we have the following result for exceptional prime $p = 2$ of odd weight.

**Corollary 5.5.2.** *If $p = 2$ is a non-dihedral supercuspidal prime for a modular form $f$ of odd weight then we have*

$$[X_v] \sim \epsilon(-1)^{[F_v:\mathbb{Q}_2]} \cdot (2, D_K)_v,$$

*for any prime $v \mid 2$.*

The following Lemma is a straightforward adaptation in our setting of Lemma 7.3.17 of [2].

**Lemma 5.5.3.** *The 2-cocycle $[c_D]_v = 1$ if and only if $D(-1) = 1$.*

We will show that $d(g) \in F_v^\times$, for all $g \in W_2$. Now, $f^\gamma \equiv f \otimes \chi_\gamma$ implies that $\rho_2(f)^\gamma \sim \rho_2(f) \otimes \chi_\gamma$ and taking determinant gives $\det(\rho_2(f))^\gamma = \det(\rho_2(f))(\chi_\gamma)^2$. Thus, we obtain $d(g)^{\gamma - 1} = 1$ and so $d(g) \in F_v^\times$. Then $d : G_v \to F_v^\times/F_v^{\times 2}$ is a continuous homomorphism, where $F_v^\times/F_v^{\times 2}$ has the discrete topology. Thus $G_v/\ker(d) \cong \mathrm{Gal}(K|F_v) \cong (\mathbb{Z}/2\mathbb{Z})^m$, for some elementary 2-extension $K$ of $F_v$. Let $\sigma_i \in \mathrm{Gal}(K|F_v)$ be the element corresponding to $(0, \cdots, 1, \cdots, 0) \in (\mathbb{Z}/2\mathbb{Z})^m$ (1 in the $i$-th position), for each $i = 1, \cdots, m$. Define $t_j \in F_v^\times (1 \leq j \leq m)$ as follows:

$$\sigma_i(\sqrt{t_j}) = \delta_{ij}\sqrt{t_j}.$$

We lift $\sigma_i$ to an element of $G_v$, also denoted $\sigma_i$ and set $d_i := d(\sigma_i) \in F_v^\times/F_v^{\times 2}$. Hence, as in [35], the class of $c_d$ is given by $[c_d]_v = (t_1, d_1)_v \cdots (t_m, d_m)_v$.

**Proposition 5.5.4.** *The local symbol can be determined completely by the discriminant of the field cut out by* $\ker(d)$*:*

$$[c_d]_v = (2, D_K)_v.$$

*Proof.* We first prove that $\ker(\tilde{\rho}_2) \subset \ker(d)$. Suppose $g \in \ker(\tilde{\rho}_2)$, then $\rho_2(g)$ is a scalar matrix $\rho_2(g) = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$. Thus, the quantity $\mathrm{trace}(\rho_2(g))^2/\det(\rho_2(g))$ is 4. As trace is nonzero, using the part (2) of Proposition 2.1.2, this quantity is equal to $d(g)$ up to an element of $F^{\times 2}$ and so $d(g) \equiv 1 \bmod F_v^{\times 2}$. Hence, we obtain the result.

Thus, there is an onto map $S_4 = G_v/\ker(\tilde{\rho}_2) \to G_v/\ker(d)$. Only 2-subgroup that can be quotient of $S_4$ is the trivial group or $\mathbb{Z}/2\mathbb{Z}$. Thus $m$ is either 0 or 1. Now since every element of the projective image of $\rho_2$ (which is $S_4$) has order $1, 2, 3$ or 4, we have that for every $g \in G_v$ the quantity $\mathrm{trace}\ \rho_2(g)^2/\det \rho_2(g)$ is $4, 0, 1$ or 2. Thus, the possible values of $d(g)$ are $4, 0, 1$ or 2 up to an element of $F_v^{\times 2}$, for all $g \in G_v$ [37, p. 264] and we can say that either $d(g) \in F_v^{\times 2}$ or $d(g) \equiv \sqrt{2} \bmod F_v^\times$, for all $g \in G_v$. The value of $m(0$ or $1)$ depends on this fact. Since the projective image of $\rho_2$ is $S_4$, there is an element $g \in G_v$ whose projective image in $S_4$ is a 4-cycle. For such an element $g$, we have $d(g) \equiv 2 \bmod F_v^{\times 2}$ and so we conclude that $m = 1$. Thus, the field $K$ cut out by the kernel of the homomorphism $d$ must be a quadratic field and the class $[c_d]_v$ is determined by the symbol $(2, D_K)_v$ as $t_1 = D_K$ (the discriminant of $K$) and $d_1 = 2$ up to an element of $F_v^{\times 2}$. Hence, we obtain the result. $\square$

We now prove Theorem 5.5.1 and Corollary 5.5.2.

*Proof.* Since $[X_v] \sim [c_d]_v \cdot [c_D]_v$, we obtain the Theorem 5.5.1 for a non-dihedral prime $p = 2$.

By the isomorphism $(*)$ [cf. Chapter 3] we can replace $D$ by $\det(\rho_{f,2}) = \chi_2^{k-1}\epsilon$, where $\chi_2$ is the 2-adic cyclotomic character. When $k$ is odd, the cocycle class of $c_D$ is same as the cocycle class of $c_\epsilon$, that is, $c_D \sim c_\epsilon$, where the 2-cocycle $c_\epsilon$ is defined as follows:

$$c_\epsilon(g,h) = \frac{\sqrt{\epsilon(g)}\sqrt{\epsilon(h)}}{\sqrt{\epsilon(gh)}}.$$

Apply Lemma 5.5.3 and observe that $[c_D] = [c_\epsilon]$ for odd $k$, we obtain Corollary 5.5.2. $\square$

## 5.6 Numerical Examples

For an odd prime $p$, our results are concurrent with the theorems proved in [8]. However, the example (5) of loc. cit. shows that $X_v$ is not determined by $m_v$ if $p$ is an unramified "bad" level zero supercuspidal prime. This example corroborates our Theorem 5.2.3.

The following result will help us to determine what kind of supercuspidal prime $p = 2$ [34, Corollary 4.1]. If $p = 2$ is a dihedral supercuspidal prime for $f \in S_k(\Gamma_0(N))$, then $v_2(N) \geq 2$. Furthermore, depending on the different extensions we have:

1. If $K|\mathbb{Q}_2$ is unramified then $v_2(N)$ is even and greater or equal to 2.

2. If $K|\mathbb{Q}_2$ is ramified with valuation 2, then $v_2(N) = 5$ or it is even and greater or equal to 6.

3. If $K|\mathbb{Q}_2$ is ramified with valuation 3, then $v_2(N) = 8$ or it is odd and greater or equal to 9.

Also, if $p = 2$ is a non-dihedral supercuspidal prime for $f$ then $v_2(N) \in \{3, 4, 6, 7\}$ [41, Section 6]. To support our results, we give numerical examples about local ramifications at supercuspidal prime $p = 2$. The examples are provided in the table of [23]. Using `Sage` and $L$-function and modular forms database (LMFDB), we determine the $v$-adic valuation of the trace of adjoint lift at the prime $p'$.

1. $f \in S_3(20, [0,3])$ with $E = \mathbb{Q}(\sqrt{-1})$ and $F = \mathbb{Q}$. Since $N_2 = 2$ the prime $p = 2$ is an unramified dihedral supercuspidal prime. We choose $p' = 17$. Using `Sage` we check that $a_{p'} = 1 - i$ and hence $v_2(a_{p'}^2\epsilon(p')^{-1}) = 1$, so $X_v$ is ramified.

2. $f \in S_5(36, [0,3]), E = \mathbb{Q}(\sqrt{-2})$ and $F = \mathbb{Q}$. Here $p = 2$ is an unramified dihedral supercuspidal prime as $N_2 = 2$. We choose $p' = 29$ and compute that $a_{p'}^2 = a_{29}^2 = -421362 = -2 \cdot 459^2$. Hence $v_2(a_{29}^2 \epsilon(29)^{-1}) = 1$, so $X_v$ is ramified.

3. $f \in S_3(24, [0,0,1]), E = \mathbb{Q}(\sqrt{-2}), F = \mathbb{Q}$. Here $p = 2$ is a non-dihedral supercuspidal prime for $f$. We have $D_K = 64$ [41, Section 6] and $\epsilon(-1) = -1$. Hence $X_v$ is ramified.

# 6

# Quadratic twisting of root numbers of modular forms

Pacetti [34] studied the variance of local root numbers in the context of twisting by a quadratic character for modular forms with *trivial nebentypus* and determined the type of local automorphic representations at $p$ as an application. In this thesis, we explore the same and investigate what properties of modular forms with *arbitrary nebentypus* are encoded therein. In particular, we determine the type of the local component $\pi_{f,p}$ of the automorphic representation $\pi_f$ attached to $f$ from that [cf. Corollary 6.3.12]. We also give a criterion for a modular form to be $p$-minimal [cf. Definition 6.1.6] in terms of the parity of $N_p$ (the exact power of $p$ that divides $N$) [cf. Proposition 6.3.4 and Proposition 6.3.14]

The ramification formulae of the endomorphism algebras of motives attached to non-CM Hecke eigenforms for all supercuspidal primes are given in [6]. The statement of the main theorem in the above mentioned article depends on the nature of the supercuspidal primes [cf. §6.1.2] . We endeavor to determine the same that appear as local components of elliptic Hecke eigenforms by analyzing the variance of the local factors by twisting. In another direction, Pacetti and his co-authors found applications of the computation in the context of Heegner/ Darmon points [27]. Our results will be useful in a similar context for modular forms with arbitrary nebentypus. Following [32], we will be mostly interested in the case when $\pi_{f,p}$ is supercuspidal.

The local $\varepsilon$-factors depend on additive characters chosen [cf. Section 6.1]. Pacetti used an additive character of conductor zero. In this present thesis, we choose an additive character of conductor $-1$ for non-supercuspidal representations, and of conductor zero (or any suitable with our situation, for example see the case where $p$ is a dihedral unramified supercuspidal prime [cf. Definition 6.1.2]) for supercuspidal representations. Note that the global $\varepsilon$-factor which is a product of all local $\varepsilon$-factors does not depend on additive characters [49, Section 3.5].

We classify the quadratic extensions $K$ of $\mathbb{Q}_p$ in the dihedral [cf. §6.1.2] supercuspidal primes in terms of the variation of global $\varepsilon$-factor with respect to twisting by a quadratic character [cf. Corollary 6.3.12 and Corollary 6.3.16]. Our method is completely different from that of Pacetti as we relied on a theorem due to Deligne [cf. Theorem 6.1.5]. The above mentioned theorem is not applicable for unramified dihedral supercuspidal prime $p$ with $a(\varkappa) = 1$. Here, $\varkappa$ denote the character from which the local representation is induced from in the dihedral supercuspidal case. In this situation and principal series representation with $p \parallel N$ and $p$ odd, we relate the variance of the local $\varepsilon$-factor with Morita's $p$-adic Gamma function [cf. §6.1.3].

Using the local inverse theorem [11, Section 27], it is possible to determine the case where $\pi_{f,p}$ is supercuspidal by the variation of the local $\varepsilon$-factors by twisting with respect to a certain set of characters and it is less convenient from computational perspective [32]. We manage to do the same by a suitable *quadratic* twist. We also consider the case where corresponding local Weil-Deligne representations are non-dihedral for $p = 2$.

## 6.1 Preliminaries

### 6.1.1 Notation

For a non-archimedian local field $F$ of characteristic zero, let $\mathcal{O}_F$ denote the ring of integers in $F$, $\mathfrak{p}_F$ the maximal ideal in $\mathcal{O}_F$ and $k_F = \mathcal{O}_F/\mathfrak{p}_F$ the residue field of $F$. The $m$-th principal units of $F$ is denoted by $U_F^m = 1 + \mathfrak{p}_F^m$. Let $v_F$ be a valuation on $F$. For the local field $\mathbb{Q}_p$, we will denote them by $\mathbb{Z}_p, \mathfrak{p}_p, k_p, U_p^m$ and $v_p$ respectively. The norm and trace maps from $F$ to $\mathbb{Q}_p$ are denoted respectively by $N_{F|\mathbb{Q}_p}$ and $\mathrm{Tr}_{F|\mathbb{Q}_p}$. We denote the set of multiplicative (respectively additive) characters of $F$ by $\widehat{F^\times}$ (respectively $\widehat{F}$).

For any quadratic extension $K|F$ and $x \in F^\times$, the symbol $(x, K|F) = 1$ or $-1$ according as $x$ is a norm of an element of $K$ or not.

**Definition 6.1.1.** *The level $l(\chi)$ of a non-trivial quasi-character $\chi$ of $F^\times$ is the smallest positive integer $m \geq 0$ such that $\chi(U_F^{m+1}) = 1$. We say the conductor of $\chi$ to be the smallest positive integer $m \geq 0$ such that $\chi(U_F^m) = 1$ and it is denoted by $a(\chi)$.*

It follows from the definition that $a(\chi) = l(\chi) + 1$ when $a(\chi) \geq 1$. A character $\chi$ is called *unramified* if the conductor is zero, *tamely ramified* if it has conductor 1 and *wildly ramified* if its conductor is greater or equal to 2. For $\chi_1, \chi_2 \in \widehat{F^\times}$, we have $a(\chi_1\chi_2) \leq \max(a(\chi_1), a(\chi_2))$. The equality holds if $a(\chi_1) \neq a(\chi_2)$. For a non-trivial additive character $\phi$ of $F$, the conductor $n(\phi)$ is the smallest integer such that $\phi$ is trivial on $\mathfrak{p}_F^{-n(\phi)}$.

Let $\mathbb{F}_q$ denote a finite field of order $q = p^r$. The classical Gauss sum $G(\chi, \phi)$ associated to a multiplicative character $\chi$ of $\mathbb{F}_q^\times$ and an additive character $\phi$ of $\mathbb{F}_q$ is defined by

$$G(\chi, \phi) = \sum_{x \in \mathbb{F}_q^\times} \chi(x)\phi(x).$$

We will denote it by $G_r(\chi)$ as the additive character $\phi$ is fixed. For $\chi \in \widehat{\mathbb{F}_q^\times}$ and $\phi \in \widehat{\mathbb{F}_q}$, let $\chi' = \chi \circ N_{\mathbb{F}_q|\mathbb{F}_p}$ and $\phi' = \phi \circ \mathrm{Tr}_{\mathbb{F}_q|\mathbb{F}_p}$ denote their lifts to $\mathbb{F}_q$. Then by the Davenport-Hasse theorem [7, Theorem 11.5.2], we have $G(\chi', \phi') = (-1)^{r-1} G(\chi, \phi)^r$. In our notation, we simply write it as $G_r(\chi') = (-1)^{r-1} G_1(\chi)^r$.

### 6.1.2 Dihedral and non-dihedral supercuspidal representations

As mentioned in the introduction, the main technical novelty of this chapter are in the case when $\pi_{f,p}$ is supercuspidal. Let $\rho_p(f)$ be the local representation of the Weil-Deligne group associated to $\pi_{f,p}$ via Local Langlands Correspondence. The Weil groups of $\mathbb{Q}_p$ and $K$ are denoted by $W_p$ and $W_K$ respectively.

**Definition 6.1.2.** *In the supercuspidal case, we call a prime $p$ to be* dihedral *for the modular form $f$ if the representation is induced by a character $\varkappa$ of an index two subgroup $W_K$ of $W_p$.*

Depending on $K$ unramified (or ramified), we say $p$ is an unramified (or ramified) supercuspidal prime for $f$.

If $p = 2$, there are supercuspidal representations that are not induced by a character; we call it *non-dihedral* supercuspidal representations.

### 6.1.3 $p$-adic Gamma function

For an odd prime $p$ and $z \in \mathbb{Z}_p$, define the $p$-adic gamma function [42, Chapter 7] to be

$$\Gamma_p(z) = \lim_{n \to z} (-1)^n \prod_{0<j<n,(p,j)=1} j,$$

where $n$ tends to $z$ $p$-adically through positive integers. Let $\chi$ be a multiplicative character of $\mathbb{F}_p$ of order $k$. Using the Gross-Koblitz formula, we deduce that [53, Corollary 3.1]:

$$G_1(\chi^r) = (-p)^{r/k} \Gamma_p\Big(\frac{r}{k}\Big). \tag{6.1}$$

For a given non-trivial additive character $\phi$ of $F$ and a Haar measure $dx$ on $F$, the $L$-function corresponding to a quasi-character $\chi$ of $F^\times$ satisfies a functional equation. This defines a number $\varepsilon(\chi, \phi, dx) \in \mathbb{C}^\times$ [49, Section 3].

### 6.1.4 Local $\varepsilon$-factors

The local $\varepsilon$-factor associated to a non-trivial character $\chi$ of $F^\times$ and a non-trivial character $\phi$ of $F$ is defined as follows [29, p. 5]:

$$\varepsilon(\chi, \phi, c) = \chi(c) \frac{\int_{U_F} \chi^{-1}(x)\phi(\frac{x}{c})dx}{|\int_{U_F} \chi^{-1}(x)\phi(\frac{x}{c})dx|},$$

where $c \in F^\times$ has valuation $a(\chi)+n(\phi)$. Here, we consider the normalized Haar measure $dx$ on $F$. The above formula can be simplified as [50, p. 94]:

$$\varepsilon(\chi, \phi, c) = q^{-\frac{a(\chi)}{2}} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}\Big(\frac{x}{c}\Big)\phi\Big(\frac{x}{c}\Big) = q^{-\frac{a(\chi)}{2}} \chi(c)\tau(\chi, \phi), \tag{6.2}$$

where $\tau(\chi,\phi)=\sum_{x\in\frac{U_F}{U_F^{a(\chi)}}}\chi^{-1}(x)\phi(\frac{x}{c})$. This is called the local Gauss sum associated to the characters $\chi$ and $\phi$. It is independent of the coset representatives $x$ chosen. The element $c$ in the formula (6.2) is determined by its valuation up to a unit $u$. It can be shown that $\varepsilon(\chi,\phi,c)=\varepsilon(\chi,\phi,cu)$. Thus, for simplicity we write $\varepsilon(\chi,\phi,c)=\varepsilon(\chi,\phi)$.

If $\chi$ is unramified, then the valuation $v_F(c)=n(\phi)$ and thus we have $\varepsilon(\chi,\phi,c)=\chi(c)$. When $a(\chi)=1$, the local Gauss sum turns out to be the classical Gauss sum. Since $\chi$ is tamely ramified, $\widetilde{\chi}:=\chi^{-1}|_{\mathcal{O}_F^\times}$ can be considered as a character of $\mathcal{O}_F^\times/U_F^1\cong k_F^\times$. If we take an additive character $\phi$ of $F$ with $n(\phi)=-1$, then we can choose $c=1$. In this settings, the local Gauss sum coincides with the well-known classical Gauss sum.

We now list some basic properties of local $\varepsilon$-factors which can be found in [49].

1. $\varepsilon(\chi,\phi_a)=\chi(a)|a|_F^{-1}\varepsilon(\chi,\phi)$, where $a\in F^\times$ and $\phi_a(x)=\phi(ax)$. Here, $|\ |_F$ denote the absolute value of $F$.

2. $\varepsilon(\chi\theta,\phi)=\theta(\pi_F)^{a(\chi)+n(\phi)}\varepsilon(\chi,\phi)$, where $\theta$ is an unramified character of $F^\times$. The element $\pi_F$ is a uniformizer of $F$.

3. $\varepsilon\Big(\mathrm{Ind}_{W(F)}^{W(\mathbb{Q}_p)}\rho,\phi\Big)=\varepsilon\Big(\rho,\phi\circ\mathrm{Tr}_{F|\mathbb{Q}_p}\Big)$, where $\rho$ denote a virtual zero dimensional representation of a finite extension $F|\mathbb{Q}_p$.

The local $\varepsilon$-factor $\varepsilon(\chi,\phi)$ depends on the additive character $\phi$ chosen which follows from the property (1) above.

Let $\chi$ denote the quadratic character attached to the quadratic extension of $\mathbb{Q}$ ramified only at $p$. For an odd prime $p$, we consider the element $p^*=\left(\frac{-1}{p}\right)\cdot p$. Then the quadratic extension $\mathbb{Q}(\sqrt{p^*})$ of $\mathbb{Q}$ is ramified only at $p$. Note that, for $p=2$ there are three quadratic extensions of $\mathbb{Q}$ ramified only at 2 (having absolute discriminant a power of 2), namely $\mathbb{Q}(i),\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-2})$. We denote their corresponding quadratic characters by $\chi_{-1},\chi_2$ and $\chi_{-2}$ respectively. Let $\chi$ denote one of these characters $\chi_j$ for $j=-1,2,-2$. By class field theory, the character $\chi$ can be identified with a character of the idèle group, that is, characters $\{\chi_q\}_q$ with $\chi_q:\mathbb{Q}_q^\times\to\mathbb{C}^\times$ satisfying the following conditions:

1. For distinct primes $q\neq p$, the character $\chi_q$ is unramified and $\chi_q(q)=\left(\frac{q}{p}\right)$.

2. $\chi_p$ is ramified with conductor $p$ and the restriction $\chi|_{\mathbb{Z}_p^\times}$ factors through the unique quadratic character of $\mathbb{F}_p^\times$ with $\chi_p(p)=1$.

By definition, we say that $\chi_p$ is tamely ramified. In this thesis, we study the changes of the local factors associated to $f$ while twisting by $\chi$. Let $\varepsilon_q$ denote the variation of the local factor of $f$ at $q$ while twisting by $\chi_q$. On both sides, we choose the same additive character and Haar measure. For $p \neq q$, the number $\varepsilon_q$ is determined by [34, Theorem 3.2, part (1)] that works for modular forms with arbitrary nebentypus and we find it useful in Corollary 6.3.12. Thus, we will be interested to compute the number $\varepsilon_p$.

**Lemma 6.1.3.** *Let $\chi_p$ be the quadratic character of $\mathbb{Q}_p^\times$ as above. Then for an additive character $\phi$ of $\mathbb{Q}_p$ of conductor $-1$, we have*

$$\varepsilon(\chi_p, \phi) = \begin{cases} 1, & \text{if } p \equiv 1 \pmod 4, \\ i, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

*For $p = 2$, we have $\varepsilon(\chi_2, \phi) = 2^{-1/2}$.*

*Proof.* Since $\chi_p$ is tamely ramified, $\widetilde{\chi}_p := \chi_p^{-1}|_{\mathbb{Z}_p^\times}$ becomes a character of $\mathbb{F}_p^\times$. Let $\phi_p, \widetilde{\phi}_p$ denote the canonical additive character of $\mathbb{Q}_p, \mathbb{F}_p$ respectively. Note that $\phi$ can be written as $a \cdot \phi_p$, for some $a \in \mathbb{Q}_p^\times$ [11, §1.7 Proposition, p. 11]. We will find out a proper element $a$ such that $\phi|_{\mathbb{Z}_p} = a \cdot \phi_p|_{\mathbb{Z}_p}$ induces the canonical additive character of $\mathbb{F}_p$. [9, Lemma 3.1] ensures us that $1/p$ can be taken as a value of $a$. By the property (1) of local $\varepsilon$-factors, we have

$$\varepsilon(\chi_p, \phi) = \varepsilon(\chi_p, \frac{1}{p}\phi_p) = \chi_p(\frac{1}{p})\varepsilon(\chi_p, \phi_p) \overset{(6.2)}{=} p^{-1/2}\tau(\chi_p, \phi_p).$$

Now $\tau(\chi_p, \phi_p)$ turns out to be the classical Gauss sum $G(\widetilde{\chi}_p, \widetilde{\phi}_p)$. Using [30, Theorem 5.15], we have

$$G(\widetilde{\chi}_p, \widetilde{\phi}_p) = \begin{cases} p^{1/2}, & \text{if } p \equiv 1 \pmod 4, \\ ip^{1/2}, & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

where $i$ is a fourth primitive root of unity. Combining all above, we obtain the required result for odd primes. For $p = 2$, the Gauss sum $G(\widetilde{\chi}_2, \widetilde{\phi}_2) = 1$. Therefore, we get that $\varepsilon(\chi_2, \phi) = 2^{-1/2}$. $\square$

For an additive character $\phi$ of $\mathbb{Q}_p$, the induced character on $F$ is denoted by $\phi_F = \phi \circ \mathrm{Tr}_{F|\mathbb{Q}_p}$. For all $c \in F$, consider the additive character $\phi_{F,c}(x) = \phi_F(cx)$.

**Lemma 6.1.4.** *Let $\chi \in \widehat{F^\times}$ and $\phi_F \in \widehat{F}$ be two non-trivial characters. Let $r \in \mathbb{N}$ be such that $2r \geq a(\chi)$. Then there is an element $c \in F^\times$ with valuation $-(a(\chi) + n(\phi_F))$ such that*

$$\chi(1 + x) = \phi_F(cx) \; \forall \; x \in \mathfrak{p}_F^r. \tag{6.3}$$

*Proof.* Since $2r \geq a(\chi)$, the character $\chi$ satisfy the relation: $\chi(1+x)\chi(1+y) = \chi(1+x+y)$, for all $x, y \in \mathfrak{p}_F^r$. This is same to having that the map $x \mapsto \chi(1+x)$ is an additive character on $\mathfrak{p}_F^r$ which can be extended to $F$.

By the property of local additive duality [11, §1.7 Proposition, p. 11], the set $\{\phi_{F,c} : c \in F\}$ is the group of all characters of $F$. Hence, there exists an element $c \in F^\times$ such that

$$\chi(1 + x) = \phi_F(cx) \; \forall \; x \in \mathfrak{p}_F^r.$$

Using the same proposition, the conductor of the character $\phi_{F,c}$ is $-(n(\phi_F) + v_F(c))$. From the equality of the conductors of both sides, we get the desired valuation of $c$. □

We now recall a fundamental result of Deligne about the behavior of local factors while twisting.

**Theorem 6.1.5.** *[18, Lemma 4.1.6] Let $\alpha$ and $\beta$ be two quasi-characters of $F^\times$ such that $a(\alpha) \geq 2a(\beta)$. If $\alpha(1 + x) = \phi_F(cx)$ for $x \in \mathfrak{p}_F^r$ with $2r \geq a(\alpha)$ (if $a(\alpha) = 0$, then $c = \pi_F^{-n(\phi_F)}$), then*

$$\varepsilon(\alpha\beta, \phi_F) = \beta^{-1}(c)\varepsilon(\alpha, \phi_F).$$

Note that the valuation of $c$ is $-(a(\alpha) + n(\phi_F))$ in the above theorem.

We write the level $N$ of the newform $f$ as $p^{N_p}N'$, with $p \nmid N'$ and the nebentypus $\epsilon = \epsilon_p \cdot \epsilon'$ as a product of characters of $(\mathbb{Z}/p^{N_p}\mathbb{Z})^\times$ and $(\mathbb{Z}/N'\mathbb{Z})^\times$ of conductors $p^{C_p}$ for some $C_p \leq N_p$ and $C'$ dividing $N'$ respectively.

**Definition 6.1.6.** *[1, p. 236] We say $f$ is $p$-minimal, if the $p$-part of its level is the smallest among all twists $f \otimes \psi$ of $f$ by Dirichlet characters $\psi$.*

## 6.2 Non-supercuspidal representations

We denote the group $\mathrm{GL}_2(\mathbb{Q}_p)$ by $G$. Let $\mu_1, \mu_2$ be two quasi-characters of $\mathbb{Q}_p^\times$ and $V(\mu_1, \mu_2)$ be the space of locally constant functions $\psi : G \to \mathbb{C}$ with the following property:

$$\psi\Big(\begin{bmatrix} a & * \\ 0 & d \end{bmatrix} g\Big) = \mu_1(a)\mu_2(d)|a/d|^{1/2}\psi(g),$$

for all $a, d \in \mathbb{Q}_p^\times$ and $g \in G$. The induced representation of $G$ by its action on $V(\mu_1, \mu_2)$ through right translation is denoted by $\rho(\mu_1, \mu_2)$.

One knows that $\rho(\mu_1, \mu_2)$ is irreducible except when $\mu_1\mu_2^{-1} = |\ |^{\pm 1}$. In this case, the representation $\rho(\mu_1, \mu_2)$ is called a **principal series representation**, denoted by $\pi := \pi(\mu_1, \mu_2)$.

The induced representation $\rho(\mu_1, \mu_2)$ is not irreducible if and only if $\mu_1\mu_2^{-1} = |\ |^{\pm 1}$. The unique irreducible sub-representation of $\rho(|\ |^{1/2}, |\ |^{-1/2})$ is the **Steinberg representation**, denoted by St. More generally, we may assume that

$$\mu_1 = \mu|\ |^{1/2}, \quad \mu_2 = \mu|\ |^{-1/2}, \tag{6.4}$$

for some character $\mu$ of $\mathbb{Q}_p^\times$. In this case $\rho(\mu|\ |^{1/2}, \mu|\ |^{-1/2})$ contains a unique irreducible sub-representation which is the twist $\mu \cdot \mathrm{St}$ of the Steinberg representation. This representation $\mu \cdot \mathrm{St}$ is called a **special representation**, again denoted by $\pi := \pi(\mu_1, \mu_2)$. The resulting factor is the one dimensional representation $\mu \circ \det$ of $G$.

The local $\varepsilon$-factor of a special representation $\pi$ is given by $\varepsilon_T(\pi, s, \phi) = \varepsilon_T(\mu_1, s, \phi)\varepsilon_T(\mu_2, s, \phi)E(\mu_1, \mu_2, s)$ [22, Table, p. 113] with

$$E(\mu_1, \mu_2, s) = \begin{cases} 1, & \text{if } \mu_1 \text{ is ramified,} \\ -\mu_2(p)p^{-s}, & \text{otherwise.} \end{cases} \tag{6.5}$$

For a ramified character $\mu$ of $\mathbb{Q}_p$ of level $n \geq 0$, $s \in \mathbb{C}$ and an additive character $\phi$ of $F$ with $n(\phi) = -1$, the local rational function $\varepsilon_T(\mu, s, \phi) \in \mathbb{C}(p^{-s})$ is defined by [11, Equ. 23.6.2]

$$\varepsilon_T(\mu, s, \phi) = p^{n(\frac{1}{2}-s)}\mu(c)\tau(\mu, \phi)/p^{(n+1)/2}.$$

This function is called the *Tate local constant* of $\mu$ associated to $\phi$. The local epsilon factors and Tate local constants are related by the following relation: $\varepsilon_T(\mu, \frac{1}{2}, \phi) = \varepsilon(\mu, \phi)$.

Let $\omega_p$ be the $p$-part of the central character of $\pi_f$, $a_p(f)$ be the $p$-th Fourier coefficient of $f$ and $\mu_1$ be an unramified character with $\mu_1(p) = a_p(f)/p^{(k-1)/2}$. We also consider a character $\mu_2$ with $\mu_1\mu_2 = \omega_p$. In the ramified principal series case, $\omega_p$ has conductor $p^{N_p}$ with $N_p \geq 1$. When $f$ is a $p$-minimal form with $\pi_{f,p}$ is a ramified principal series representation, we have $\pi_{f,p} \cong \pi(\mu_1, \mu_2)$ [32, prop. 2.8]. In this case, choose an additive character $\phi$ of conductor $-1$ satisfying

$$\omega_p(1+x) = \phi(cx) \ \forall \ x \in \mathfrak{p}_p^r, \tag{6.6}$$

with $2r \geq a(\omega_p)$.

If $N_p = 1$, let $m$ be the order of $\widetilde{\omega}_p := \omega_p^{-1}|_{\mathbb{Z}_p}^{\times}$. Consider the quantity that depends on $p$ and $m$:

$$c_p = \begin{cases} (-p)^{-1/2m}\{\Gamma_p(\frac{1}{2m})/\Gamma_p(\frac{1}{m})\}, & \text{if } p \text{ odd with } N_p = 1 \text{ and } m \text{ even}, \\ 1, & \text{otherwise.} \end{cases}$$

With the choice of an additive character as in equation 6.6, we prove the following theorem:

**Theorem 6.2.1.** *Let $\pi_{f,p}$ be a ramified principal series representation. Choose an additive character $\phi$ as above. For odd primes $p$, the number*

$$\varepsilon_p = \begin{cases} p^{\frac{1-k}{2}} a_p(f) c_p, & \text{if } p \equiv 1 \pmod 4, \\ i p^{\frac{1-k}{2}} a_p(f) c_p, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

*For $p = 2$, we have $\varepsilon_2 = 2^{-\frac{k}{2}} a_2(f)$.*

*Proof.* By [22, Table, p. 113], the local factor associated to $f$ is $\varepsilon(\mu_1, \phi)\varepsilon(\mu_2, \phi)$ and the local $\varepsilon$-factor corresponding to $f \otimes \chi_p$ is $\varepsilon(\mu_1\chi_p, \phi)\varepsilon(\mu_2\chi_p, \phi)$. Since $\mu_1$ is unramified, the local $\varepsilon$-factors are computed as follows:

1. $\varepsilon(\mu_1, \phi) = \mu_1(c) = \mu_1(\frac{1}{p}) = \frac{1}{\mu_1(p)}$.

2. $\varepsilon(\mu_2, \phi) = \varepsilon(\mu_1^{-1}\omega_p, \phi) = \mu_1^{-1}(p)^{a(\omega_p)-1}\varepsilon(\omega_p, \phi)$, by property (2) of local $\varepsilon$-factors.

3. $\varepsilon(\mu_1\chi_p, \phi) = \mu_1(p)^{a(\chi_p)-1}\varepsilon(\chi_p, \phi) = \varepsilon(\chi_p, \phi)$ [Again, by property (2) of local $\varepsilon$-factors and since the conductor of $\chi_p$ is 1].

4. For $N_p > 1$, note that $a(\omega_p) \geq 2a(\chi_p)$. In this case, we compute that

$$\begin{aligned}
\varepsilon(\mu_2\chi_p,\phi) &= \varepsilon(\mu_1^{-1}\omega_p\chi_p,\phi)\\
&= \mu_1^{-1}(p)^{a(\omega_p)-1}\varepsilon(\omega_p\chi_p,\phi) \quad \text{(by property (2) of local } \varepsilon-\text{factors)}\\
&\overset{\text{Theorem (6.1.5)}}{=} \mu_1^{-1}(p)^{a(\omega_p)-1}\chi_p^{-1}(c)\varepsilon(\omega_p,\phi)\\
&= \mu_1^{-1}(p)^{a(\omega_p)-1}\varepsilon(\omega_p,\phi) \quad (\text{since } \chi_p(c)=1).
\end{aligned}$$

Thus, we deduce that $\varepsilon(\mu_2,\phi) = \varepsilon(\mu_2\chi_p,\phi)$.

Now assume that $N_p = 1$. Both $\widetilde{\omega}_p := \omega_p^{-1}|_{\mathbb{Z}_p}^\times$ and $\widetilde{\chi}_p$ can be thought of as a character of $\mathbb{F}_p^\times$. Notice that $\widehat{\mathbb{F}_p^\times}$ is cyclic, say $\widehat{\mathbb{F}_p^\times} = \langle\chi_1\rangle$.

If $\widetilde{\omega}_p$ has even order, then both $\widetilde{\omega}_p$ and $\widetilde{\omega}_p\widetilde{\chi}_p$ have same order. Hence, we can write $\widetilde{\omega}_p = \widetilde{\omega}_p\widetilde{\chi}_p = \chi_1^a$ for some $a$. As a result, we obtain $\varepsilon(\omega_p,\phi) = \varepsilon(\omega_p\chi_p,\phi)$.

If $\widetilde{\omega}_p$ has odd order $m$, then $\widetilde{\omega}_p\widetilde{\chi}_p$ has order $2m$. Write $p = bm+1$ for some $b \in \mathbb{N}$. Thus, we have $\widetilde{\omega}_p = \chi_1^b$ and $\widetilde{\omega}_p\widetilde{\chi}_p = \chi_1^{\frac{b}{2}}$. By the formula (6.1), we obtain $G_1(\chi_1^b) = (-p)^{\frac{b}{p-1}}\Gamma_p\left(\frac{b}{p-1}\right) = (-p)^{\frac{1}{m}}\Gamma_p\left(\frac{1}{m}\right)$ and $G_1(\chi_1^{\frac{b}{2}}) = (-p)^{\frac{1}{2m}}\Gamma_p\left(\frac{1}{2m}\right)$. Therefore, we deduce that $\varepsilon(\omega_p\chi_p,\phi) = \varepsilon(\omega_p,\phi)\cdot(-p)^{-1/2m}\{\Gamma_p(\frac{1}{2m})/\Gamma_p(\frac{1}{m})\}$. For $p=2$, we have $\varepsilon(\omega_p\chi_p,\phi) = \varepsilon(\omega_p,\phi)$.

From above, we compute that

$$\varepsilon_p = \frac{\varepsilon(\mu_1\chi_p,\phi)\varepsilon(\mu_2\chi_p,\phi)}{\varepsilon(\mu_1,\phi)\varepsilon(\mu_2,\phi)} = c_p\cdot\frac{\varepsilon(\mu_1\chi_p,\phi)}{\varepsilon(\mu_1,\phi)} = c_p\cdot\mu_1(p)\varepsilon(\chi_p,\phi).$$

Using Lemma 6.1.3, we get the required result. □

We now consider the case where $\pi_{f,p}$ is a special representation. If $f$ is a $p$-primitive newform, then by [32, prop. 2.8], $\pi_{f,p} \cong \mu\cdot\mathrm{St}$, where $\mu$ is unramified and $\mu(p) = a_p(f)/p^{(k-2)/2}$. Hence, for $j=1,2$, the character $\mu_j$ in (6.4) is unramified. With an additive character $\phi$ of conductor $-1$, our result in this case is as follows:

**Theorem 6.2.2.** *If $\pi_{f,p}$ is a special representation, then the number $\varepsilon_p$ is*

$$\varepsilon_p = \begin{cases} -p^{\frac{3-k}{2}}a_p(f), & \textit{if } p\equiv 1 \pmod 4,\\ p^{\frac{3-k}{2}}a_p(f), & \textit{if } p\equiv 3 \pmod 4.\end{cases}$$

*For $p = 2$, we have $\varepsilon_2 = -2^{\frac{1-k}{2}} a_2(f)$.*

*Proof.* Note that both $\mu_1, \mu_2$ are unramified characters. Thus, we have that $\varepsilon(\mu_j, \phi) = \mu_j(c) = \mu_j(\frac{1}{p}) = 1/\mu_j(p)$, for all $j = 1, 2$. Also, for $j = 1, 2$, we have $\varepsilon(\mu_j \chi_p, \phi) = \mu_j(p)^{a(\chi_p)-1} \varepsilon(\chi_p, \phi) = \varepsilon(\chi_p, \phi)$ as $a(\chi_p) = 1$. By Lemma 6.1.3, we have

$$\varepsilon(\mu_j \chi_p, \phi) = \begin{cases} 1, & \text{if } p \equiv 1 \pmod 4, \\ i, & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

for all $j = 1, 2$. Since the number $\varepsilon_p$ is

$$\varepsilon_p = \frac{\varepsilon(\mu_1 \chi_p, \phi) \varepsilon(\mu_2 \chi_p, \phi) E(\mu_1 \chi_p, \mu_2 \chi_p, \frac{1}{2})}{\varepsilon(\mu_1, \phi) \varepsilon(\mu_2, \phi) E(\mu_1, \mu_2, \frac{1}{2})},$$

using the relation (6.5), we get the result. □

## 6.3 Supercuspidal representations

Every irreducible admissible representation of $G$ that is not a sub-representation of some $\rho(\mu_1, \mu_2)$ is called a **supercuspidal representation**. Note that supercuspidal cases are the most interesting cases in the computation of the local data of a modular form [32, Section 2].

By local Langlands correspondence the representations $\pi_{f,p}$ are in a bijection with (isomorphism classes of) complex two dimensional Frobenius-semisimple Weil-Deligne representations $\rho_p(f)$ associated to a modular form $f$ at $p$. For more details of Weil-Deligne representations, we refer to [17, Section 3]. We will be using the information about $\rho_p(f)$ in this case.

### 6.3.1 The case $p$ odd

Let $\rho_p(f)$ denote the local representation of the Weil-Deligne group attached to $f$ at a prime $p$. In the supercuspidal case,

$$\rho_p(f) = \mathrm{Ind}_{W_K}^{W_p} \varkappa$$

with $K$ a quadratic extension of $\mathbb{Q}_p$ and $\varkappa$ a quasi-character of $W_K^{\mathrm{ab}}$ which does not factor through the norm map with a quasi-character of $W_p^{\mathrm{ab}}$. We can consider $\varkappa$ as a character of $K^\times$ via the isomorphism $W_K^{\mathrm{ab}} \simeq K^\times$ and say that $(K, \varkappa)$ is an admissible pair attached to $f$ at $p$. It satisfies the following conditions:

1. $\varkappa$ does not factor through the norm map $N_{K|\mathbb{Q}_p} : K^\times \to \mathbb{Q}_p^\times$ and
2. if $K|\mathbb{Q}_p$ is ramified, then the restriction $\varkappa|_{U_K^1}$ does not factor through $N_{K|\mathbb{Q}_p}$.

The pair $(K, \varkappa)$ is said to be minimal if $\varkappa|_{U_K^{l(\varkappa)}}$ does not factor through the norm map $N_{K|\mathbb{Q}_p}$. If $\varkappa$ is minimal over $\mathbb{Q}_p$, then we have $a(\varkappa) \leq a(\theta_K \varkappa)$ for all characters $\theta$ of $\mathbb{Q}_p^\times$. The induced character on $K$ is denoted by $\theta_K = \theta \circ N_{K|\mathbb{Q}_p}$. Clearly, $f$ is $p$-minimal if and only if its associated admissible pair is minimal. For more details of an admissible pair, we refer to [11, Section 18].

By the properties of local $\varepsilon$-factors, recall the formula for the conductor of the supercuspidal representations [44, Theorem 2.3.2]:

$$a(\mathrm{Ind}_{W_K}^{W_p} \varkappa) = v_p(\delta(K|\mathbb{Q}_p)) + f(K|\mathbb{Q}_p)a(\varkappa). \tag{6.7}$$

Here, the normalized valuation of $\mathbb{Q}_p^\times$ is denoted by $v_p$ and $\delta(K|\mathbb{Q}_p), f(K|\mathbb{Q}_p)$ denote the discriminant and the residual degree of $K|\mathbb{Q}_p$ respectively. The above formula is same as the formula for the Artin conductor of a 2-dimensional induced representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$ [13, Proposition 4$(b)$, p. 158].

**Definition 6.3.1.** *[11, §13.4 Definition] An element $\alpha \in K^\times$ with $v_K(\alpha) = -n$ is said to be minimal over $\mathbb{Q}_p$ if one of the following holds:*

1. *$K|\mathbb{Q}_p$ is ramified and $n$ is odd;*
2. *$K|\mathbb{Q}_p$ is unramified and the field extension $k_K|k_p$ is generated by the coset $p^n\alpha + \mathfrak{p}_K$.*

**Lemma 6.3.2.** *[11, §18.2 Proposition] Let $K|\mathbb{Q}_p$ be a tamely ramified quadratic extension with $l(\varkappa) = m \geq 1$. If $\alpha \in \mathfrak{p}_K^{-m}$ be such that $\varkappa(1 + x) = \phi_K(\alpha x)$ for $x \in \mathfrak{p}_K^m$, then $(K, \varkappa)$ is a minimal admissible pair if and only if $\alpha$ is minimal over $\mathbb{Q}_p$.*

Applying the above lemma when $K|\mathbb{Q}_p$ is ramified quadratic, we see that if $(K, \varkappa)$ is a minimal admissible pair, then $m$ is odd and so $a(\varkappa) = l(\varkappa) + 1$ is even [32, Theorem 3.3].

**Lemma 6.3.3.** *[51, Lemma 1.8] Let $E|K$ be a quadratic separable extension with residue degree $f$. If $\eta$ is a quasi-character of $K^\times$, then $fa(\eta\circ N_{E|K}) = a(\eta)+a(\eta\omega_{E|K})-a(\omega_{E|K})$. If $\psi$ is a non-trivial additive character of $K$, then $n(\psi\circ \mathrm{Tr}_{E|K}) = (2/f)n(\psi)+d(E|K)$. Here, $\omega_{E|K}$ denote the non-trivial character of $K^\times$ with kernel equal to the group of norms from $E^\times$ to $K^\times$.*

We can find the valuation of the level of the modular form with arbitrary nebentypus from the following proposition (see also [34, Corollary 3.1] for $\Gamma_0(N)$). The local factor in the ramified case can be computed from that.

**Proposition 6.3.4.** *The pair $(K, \varkappa)$ can be characterized in terms of $N_p$.*

1. *If $K|\mathbb{Q}_p$ is unramified, then $N_p$ is even and $\varkappa$ is ramified.*
2. *Assume that $K|\mathbb{Q}_p$ is ramified. We have $N_p$ is odd if $(K,\varkappa)$ is minimal; otherwise it is even.*

*Proof.* Using the relation (6.7), we get that

$$N_p = \begin{cases} 2a(\varkappa), & \text{if } K|\mathbb{Q}_p \text{ is unramified,} \\ 1+a(\varkappa), & \text{if } K|\mathbb{Q}_p \text{ is ramified.} \end{cases} \tag{6.8}$$

When $K|\mathbb{Q}_p$ is unramified, we have $N_p$ is even from above. If $\varkappa$ is unramified, then it would factor through the norm map. Hence, the character $\varkappa$ is ramified.

In the ramified case, if $(K,\varkappa)$ is a minimal admissible pair, then the result follows from paragraph after Lemma 6.3.2.

We now prove that $N_p$ is even if $(K,\varkappa)$ is not minimal. Consider a non-minimal pair $(K,\varkappa)$. We can write $\varkappa = \varkappa'\theta_K$ [11, Section 18.2] for a character $\theta$ of $\mathbb{Q}_p^\times$ and a minimal admissible pair $(K,\varkappa')$. Since $\varkappa'$ is minimal over $\mathbb{Q}_p$, we have $a(\varkappa') \leq a(\varkappa'\theta_K) = a(\varkappa)$ and $l(\varkappa')$ is odd. As a result, we obtain that $a(\varkappa')\geq 2$ is even.

If $a(\theta) = 0$, then we have $a(\theta_K) = 0$ by Lemma 6.3.3. Thus, we get that $a(\varkappa) = a(\varkappa'\theta_K) = a(\varkappa')$ and hence $\varkappa|_{U_K^{l(\varkappa)}} = (\varkappa'\theta_K)|_{U_K^{l(\varkappa)}} = \varkappa'|_{U_K^{l(\varkappa')}}$ does not factor through the norm map, contradicts the non-minimality of $\varkappa$. Therefore, $a(\theta)\geq 1$ so that $a(\theta\omega_{K|\mathbb{Q}_p}) = a(\theta)$ [51, Proof of Proposition 2.6]. By Lemma 6.3.3 and $a(\omega_{K|\mathbb{Q}_p}) = 1$ [11, Proposition-Definition (1), p. 217], we deduce that $a(\theta_K) = a(\theta)+a(\theta\omega_{K|\mathbb{Q}_p}) - a(\omega_{K|\mathbb{Q}_p}) = 2a(\theta)-1$
.

Now, we claim that $a(\theta_K) > a(\varkappa')$. If not, then we have $a(\theta_K) < a(\varkappa')$ (since the equality is not possible as $a(\varkappa')$ is even and $a(\theta_K)$ is odd). Hence, both $\varkappa = \varkappa'\theta_K$ and $\varkappa'$ have same conductor, which contradicts the fact that $(K, \varkappa)$ is not minimal. Thus, we have proved that $a(\varkappa) = a(\varkappa'\theta_K) = a(\theta_K)$ is odd. Hence, we deduce that $N_p$ is even. $\square$

The above proposition gives the criterion for the associated admissible pair to be minimal in the ramified case.

*Remark* 6.3.5. In the case where $K|\mathbb{Q}_p$ is ramified quadratic, $a(\varkappa) = 1$ is not possible by the part (2) of the definition of an admissible pair. Thus, $N_p = 2$ does not occur.

Note that $(\mathrm{Ind}_{W_K}^{W_p} \varkappa)\chi_p = \mathrm{Ind}_{W_K}^{W_p}(\varkappa\chi_p')$ with $\chi_p' = \chi_p \circ N_{K|\mathbb{Q}_p}$. Thus, by the property (3) of local $\varepsilon$-factors, we only need to treat the one dimensional cases. We now consider two cases depending on $K$ unramified or ramified.

### The case $K|\mathbb{Q}_p$ unramified

Since $\varkappa|_{\mathbb{Z}_p} = \epsilon_p^{-1}$ [8, Equ. 4] and $\epsilon_p$ is a trivial character when $C_p = 0$, we get the following corollary [34, part (3) of Theorem 3.2]:

**Corollary 6.3.6.** *Assume that $C_p = 0$. If $p$ is an unramified supercuspidal prime, then $\varepsilon_p = -\left(\frac{-1}{p}\right)$.*

Since $K|\mathbb{Q}_p$ is unramified quadratic, we can take $\pi = p$ as a uniformizer of $K$ which we fix now.

**Theorem 6.3.7.** *Let $p$ be a dihedral supercuspidal prime for $f$. Assume that $K|\mathbb{Q}_p$ is unramified. For an additive character $\phi$ of $\mathbb{Q}_p$ with*

$$\varkappa(1 + x) = \phi_K(cx) \ \forall \ x \in \mathfrak{p}_K^r, \tag{6.9}$$

*where $2r \geq a(\varkappa) > 1$, we have $\varepsilon_p = 1$.*

*Proof.* We apply Theorem 6.1.5 with $\alpha = \varkappa$ and $\beta = \chi_p'$ and get the number $\varepsilon_p = \chi_p'(p)^{-1} = 1$. $\square$

Note that the above theorem does not work when $a(\varkappa) = 1$. We give a different proof of the theorem above when $a = a(\varkappa) = 2$ in the unramified case.

*Proof.* Choose an additive character $\phi$ of $\mathbb{Q}_p$ of conductor zero. Using Lemma 6.3.3, we get $n(\phi_K) = 0$. Assume that $\varkappa(1+x) = \phi_K(cx)$, for all $x \in \mathfrak{p}_K$. Every element $x \in \mathcal{O}_K^\times/1+\mathfrak{p}_K^2$ has the form $b_0 + b_1 p$ with $b_0 \neq 0$ and $b_i \in \mathbb{F}_{p^2}\ \forall\ i$. Now

$$\begin{aligned}
\tau(\varkappa, \phi_K) &= \sum_{x \in \mathcal{O}_K^\times/1+\mathfrak{p}_K^2} \varkappa^{-1}(x)\phi_K(\frac{x}{p^2}) \\
&= \sum_{b_i} \varkappa^{-1}\Big(b_0 + b_1 p\Big)\phi_K\Big(\frac{b_0}{p^2} + \frac{b_1}{p}\Big) \\
&= \sum_{b_0 \in \mathbb{F}_{p^2}^\times} \varkappa^{-1}(b_0) \sum_{b_1 \in \mathbb{F}_{p^2}} \varkappa^{-1}\Big(1 + \frac{b_1}{b_0}p\Big)\phi_K\Big(\frac{b_0}{p^2} + \frac{b_1}{p}\Big).
\end{aligned}$$

With the choice of the additive character, we obtain that

$$\begin{aligned}
\tau(\varkappa, \phi_K) &= \sum_{b_0 \in \mathbb{F}_{p^2}^\times} \varkappa^{-1}(b_0) \sum_{b_1 \in \mathbb{F}_{p^2}} \phi_K\Big(-\frac{b_1}{b_0}\frac{1}{p}\Big)\phi_K\Big(\frac{b_0}{p^2} + \frac{b_1}{p}\Big) \\
&= \sum_{b_0 \in \mathbb{F}_{p^2}^\times} \varkappa^{-1}(b_0)\phi_K\Big(\frac{b_0}{p^2}\Big) \sum_{b_1 \in \mathbb{F}_{p^2}} \phi_K\Big(\Big(1 - \frac{1}{b_0}\Big)\frac{b_1}{p}\Big)
\end{aligned}$$

Since the sum of a non-trivial character over a group vanishes, the inner sum is zero unless $b_0 = 1$. As a result, we obtain that $\tau(\varkappa, \phi_K) = p^2\phi_K\big(\frac{1}{p^2}\big)$. Since $K|\mathbb{Q}_p$ is unramified, $a(\chi_p') = 1$ by Lemma 6.3.3. Thus, in a similar way we get that $\tau(\varkappa\chi_p', \phi_K) = p^2\phi_K\big(\frac{1}{p^2}\big)$, completing the proof. $\square$

**The case $a(\varkappa) = 1$.** The above theorem is not valid for $a(\varkappa) = 1$. In that case, $\widetilde{\varkappa} := \varkappa^{-1}|_{\mathcal{O}_K^\times}$ can be considered as a character of $\mathcal{O}_K^\times/U_K^1 \cong \mathbb{F}_{p^2}^\times$ and the associated local Gauss sum turns out to be the classical Gauss sum. Here, we take an additive character $\phi$ of $K$ which induces the canonical additive character $\widetilde{\phi}$ of $\mathbb{F}_{p^2}$.

**Theorem 6.3.8.** *Let $p$ be an odd unramified supercuspidal prime with $a(\varkappa) = 1$.*

1. *If the order of $\widetilde{\varkappa}$ is even, then $\varepsilon_p = 1$.*
2. *Assume the order $m$ of $\widetilde{\varkappa}$ is odd that divides $p-1$. Write $p = bm + 1$ for some $b \in \mathbb{N}$. Then*

$$\varepsilon_p = p^{-1/m}\Big\{\Gamma_p\Big(\frac{1}{2m}\Big)/\Gamma_p\Big(\frac{1}{m}\Big)\Big\}^2. \tag{6.10}$$

*Proof.* 1. We know that, as a group $\widehat{\mathbb{F}_{p^2}^{\times}} \simeq \mathbb{F}_{p^2}^{\times}$. Thus, $\widehat{\mathbb{F}_{p^2}^{\times}}$ is cyclic, say $\widehat{\mathbb{F}_{p^2}^{\times}} = \langle \chi_2 \rangle$. Since $\widetilde{\varkappa}$ has even order, both $\widetilde{\varkappa}$ and $\widetilde{\varkappa}\widetilde{\chi}'_p$ have same order. Hence, we can write $\widetilde{\varkappa} = \widetilde{\varkappa}\widetilde{\chi}'_p = \chi_2^a$ for some $a \in \{1, \cdots, p^2 - 1\}$. Thus, we get $\varepsilon_p = 1$.

2. Since $o(\widetilde{\varkappa}) \mid (p-1)$, the character $\widetilde{\varkappa}$ can be thought of as a lift of some character $\widetilde{\varkappa}^*$ on $\mathbb{F}_p$ and both $\widetilde{\varkappa}, \widetilde{\varkappa}^*$ have same order [7, Theorem 11.4.4]. Using the Davenport-Hasse theorem [cf. Section 6.1], we have

$$\frac{G_2\left(\widetilde{\varkappa}\widetilde{\chi}'_p\right)}{G_2(\widetilde{\varkappa})} = \frac{G_1\left(\widetilde{\varkappa}^*(\widetilde{\chi}'_p)^*\right)^2}{G_1(\widetilde{\varkappa}^*)^2}. \tag{6.11}$$

Suppose that $\widehat{\mathbb{F}_p^{\times}} = \langle \chi_1 \rangle$, the group of multiplicative characters of $\mathbb{F}_p$. Note that $\widetilde{\varkappa}^*$ has odd order $m$ and $\widetilde{\varkappa}^*(\widetilde{\chi}'_p)^*$ has order $2m$. We have $\widetilde{\varkappa}^* = \chi_1^b$ and $\widetilde{\varkappa}^*(\widetilde{\chi}'_p)^* = \chi_1^{\frac{b}{2}}$ for some $b$. Using the formula (6.1), we get that $G_1(\chi_1^b) = (-p)^{\frac{b}{p-1}}\Gamma_p\left(\frac{b}{p-1}\right) = (-p)^{\frac{1}{m}}\Gamma_p\left(\frac{1}{m}\right)$ and $G_1(\chi_1^{\frac{b}{2}}) = (-p)^{\frac{1}{2m}}\Gamma_p\left(\frac{1}{2m}\right)$. Hence, the desired result is obtained by equation (6.11).

□

**Corollary 6.3.9.** *The quantity in (6.10) that determines $\varepsilon_p$ in the above theorem can be simplified as*

$$\left\{\frac{(x_0-1)!}{\left[\frac{x_0-1}{p}\right]!} \times \frac{\left[\frac{2x_0-1}{p}\right]!}{(2x_0-1)!}\right\}^2 \pmod p,$$

*where $x_0$ is a solution of $2mx \equiv 1 \pmod p$.*

*Proof.* Consider the following two congruence equations:

$$2mx \equiv 1 \pmod p \quad \text{and} \tag{6.12}$$

$$my \equiv 1 \pmod p. \tag{6.13}$$

Both the congruence equations have an integer solution. Note that $2x_0$ is a solution of (6.13). By the property of $p$-adic gamma function, we have

$$\Gamma_p\left(\frac{1}{2m}\right) \equiv \Gamma_p\left(x_0\right) \pmod p \quad \text{and} \quad \Gamma_p\left(\frac{1}{m}\right) \equiv \Gamma_p\left(2x_0\right) \pmod p.$$

Using the values of $p$-adic gamma function at integer points [42, Chapter 7, §1.2], we obtain:

$$\begin{aligned}\left\{\Gamma_p\Big(\frac{1}{2m}\Big)/\Gamma_p\Big(\frac{1}{m}\Big)\right\}^2 &\equiv \left\{\Gamma_p\big(x_0\big)/\Gamma_p\big(2x_0\big)\right\}^2 \pmod p\\ &\equiv \left\{\frac{(-1)^{x_0}(x_0-1)!}{\left[\frac{x_0-1}{p}\right]!p^{[(x_0-1)/p]}} \times \frac{\left[\frac{2x_0-1}{p}\right]!p^{[(2x_0-1)/p]}}{(-1)^{2x_0}(2x_0-1)!}\right\}^2 \pmod p\\ &\equiv \left\{\frac{(x_0-1)!}{\left[\frac{x_0-1}{p}\right]!} \times \frac{\left[\frac{2x_0-1}{p}\right]!}{(2x_0-1)!}\right\}^2 \pmod p.\end{aligned}$$

In a special case where $2x_0 < p+1$, the above quantity is same as $\left\{\frac{1}{(2x_0-1)\cdots(x_0-2)}\right\}^2$ $\pmod p$. ◻

If the order of $\widetilde{\varkappa}$ is even and that divides $p+1$, then we also get the same result as part (1) of Theorem 6.3.8 using the Stickelberger's theorem, which shows the consistency of our result.

**Theorem 6.3.10.** *If $p$ is an odd unramified supercuspidal prime with $a(\varkappa) = 1$ and the order $m$ of $\widetilde{\varkappa}$ divides $p+1$, then*

$$\varepsilon_p = \begin{cases} -1, & \text{if } m \text{ odd and } p \equiv 1 \pmod 4,\\ 1, & \text{if } m \text{ even and } p \equiv 1 \pmod 4,\\ 1, & \text{if } m \text{ even and } p \equiv 3 \pmod 4 \text{ with } \frac{p+1}{m} \text{ odd}.\end{cases} \tag{6.14}$$

*For $p = 2$, we have $\varepsilon_2 = 1$.*

*Proof.* We split the proof into two cases. First assume that $m$ is odd. We now consider primes $p \equiv 1 \pmod 4$. Using Stickelberger's theorem [30, Theorem 5.16], we have $G(\widetilde{\varkappa}, \widetilde{\phi}) = p$. Since $\widetilde{\chi}'_p$ has order 2, the order of $\widetilde{\varkappa}\widetilde{\chi}'_p$ is $2m$. Write $p = 4k+1$ for some $k \in \mathbb{N}$. Thus, we obtain that $(p+1)/2m = (2k+1)/m$ is odd. Hence, by the same theorem we have $G(\widetilde{\varkappa}\widetilde{\chi}'_p, \widetilde{\phi}) = -p$, as required. When $p \equiv 3 \pmod 4$ with $m$ odd, we cannot apply the Stickelberger's theorem to find $G(\widetilde{\varkappa}\widetilde{\chi}'_p, \widetilde{\phi})$ as the quantity $(p+1)/2m$ is not odd.

Next we deal with the case where $m$ is even. Thus, the order of $\widetilde{\varkappa}\widetilde{\chi}'_p$ is $m$. For primes $p \equiv 1 \pmod 4$, we have $(p+1)/m = 2(2k+1)/m$ is odd. Hence, by Stickelberger's theorem we obtain that $G(\widetilde{\varkappa}, \widetilde{\phi}) = G(\widetilde{\varkappa}\widetilde{\chi}'_p, \widetilde{\phi}) = -p$, as desired. In a similar way we can show that $\varepsilon_p = 1$, when $p \equiv 3 \pmod 4$ with $(p+1)/m$ odd. □

**The case $K|\mathbb{Q}_p$ ramified**

As $p$ is odd, the possibilities for $K$ are $\mathbb{Q}_p(\sqrt{-p})$ and $\mathbb{Q}_p(\sqrt{-p\zeta_{p-1}})$ depending on $(p, K|\mathbb{Q}_p) = 1$ or $-1$ respectively. We can choose $\pi = \sqrt{-p}$ or $\sqrt{-p\zeta_{p-1}}$ as a uniformizer of $K$ and write $K = \mathbb{Q}_p(\pi)$.

Since $K|\mathbb{Q}_p$ is ramified quadratic, we have $N_{K|\mathbb{Q}_p}(\mathcal{O}_K^\times) = \mathbb{Z}_p^{\times 2}$. In this case, we have $\chi'_p|_{\mathcal{O}_K^\times} = 1$ (i. e., $\chi'_p$ is unramified). Choose an additive character $\phi$ of conductor 0 and a normalized Haar measure $dx$. Since $K|\mathbb{Q}_p$ is ramified, the conductor of $\phi_K = \phi \circ \mathrm{Tr}_{K|\mathbb{Q}_p}$ is 1 [cf. Lemma 6.3.3].

**Theorem 6.3.11.** *Let $p$ be a dihedral supercuspidal prime for $f$ with $K|\mathbb{Q}_p$ ramified. We then have*

- $\varepsilon_p = 1$ *if the conductor of $\varkappa$ is odd.*
- *In the case of $a(\varkappa)$ even, the number*

$$\varepsilon_p = \begin{cases} 1, & \text{if } (p, K|\mathbb{Q}_p) = 1, \\ \left(\frac{-1}{p}\right), & \text{if } (p, K|\mathbb{Q}_p) = -1. \end{cases} \tag{6.15}$$

*Proof.* Since $\chi'_p$ is unramified, using the property (2) of local $\varepsilon$-factors, we get that

$$\varepsilon(\varkappa\chi'_p, \phi_K, dx) = \chi_p(N_{K|\mathbb{Q}_p}(\pi))^{a(\varkappa)+1} \cdot \varepsilon(\varkappa, \phi_K, dx). \tag{6.16}$$

If the conductor of $\varkappa$ is odd we get that the number $\varepsilon_p = 1$. So assume that $a(\varkappa)$ is even. The epsilon factor is thus given by the quantity:

$$\varepsilon_p = \chi_p(N_{K|\mathbb{Q}_p}(\pi)) = \left(\frac{N(\pi)/p}{p}\right). \tag{6.17}$$

Since $N_{K|\mathbb{Q}_p}(\pi) = -\pi^2$, we obtain that $\varepsilon_p = \left(\frac{-\pi^2/p}{p}\right)$. Therefore when $p$ is odd, we deduce that:

$$\varepsilon_p = \begin{cases} \left(\frac{1}{p}\right) = 1, & \text{if } (p, K|\mathbb{Q}_p) = 1, \\ \left(\frac{\zeta_{p-1}}{p}\right) = \left(\frac{-1}{p}\right), & \text{if } (p, K|\mathbb{Q}_p) = -1. \end{cases} \tag{6.18}$$

□

Let $\varepsilon(f)$ be the global $\varepsilon$-factor associated to $f$ and $\varepsilon_p(f)$ be its $p$-part. For the character $\chi_p$ defined before, the newform twisted by $\chi_p$ is denoted by $f \otimes \chi_p$. Then we have the following classification of the local data of a newform.

**Corollary 6.3.12.** *Let $\pi_p = \pi_{f,p}$ be the local component at $p$ of a $p$-minimal newform $f$. We have*

1. *$\pi_p$ is Steinberg if $N_p = 1$ and $C_p = 0$.*
2. *$\pi_p$ is principal series if $N_p \geq 1$ with $N_p = C_p$.*
3. *If $\pi_p$ is not of the above type, then it is supercuspidal. In this case, we have $N_p > C_p$. For odd $p$, it is always induced by a quadratic extension $K|\mathbb{Q}_p$. If $N_p \geq 2$ is even, then $K$ is the unique unramified quadratic extension of $\mathbb{Q}_p$. In the case of $N_p \geq 3$ odd with $p \equiv 3 \pmod 4$, we have*

   *(a) $K = \mathbb{Q}_p(\sqrt{-p})$ if $\varepsilon(f \otimes \chi_p) = \chi_p(N')\varepsilon(f)$.*

   *(b) $K = \mathbb{Q}_p(\sqrt{-p\zeta_{p-1}})$ if $\varepsilon(f \otimes \chi_p) = -\chi_p(N')\varepsilon(f)$.*

   *The same cannot be concluded, when $p \equiv 1 \pmod 4$.*

*Proof.* Depending upon $N_p$ and $C_p$, the local component at $p$ of a newform has been classified in [32, prop. 2.8]. We just need to prove the relation $\varepsilon(f \otimes \chi_p) = \chi_p(N')\varepsilon(f)\varepsilon_p$ to complete the proof. Note that the number $\varepsilon_q (p \neq q)$ is determined by $\varepsilon_q = \left(\frac{q}{p}\right)^{N_q}$ [34, Theorem 3.2, part (1)], where $N_q$ denote the exact power of $q$ that divides $N$. By hypothesis, we have $\varepsilon_p(f \otimes \chi_p) = \varepsilon_p(f)\varepsilon_p$. Running over all primes $p$, we get that $\varepsilon(f \otimes \chi_p) = \varepsilon(f) \prod_p \varepsilon_p = \varepsilon(f)\varepsilon_p \prod_{q \neq p, q|N} \left(\frac{q^{N_q}}{p}\right) = \chi_p(N')\varepsilon(f)\varepsilon_p$.

For primes $p \equiv 1 \pmod 4$, since the number $\varepsilon_p = 1$ for both fields $K = \mathbb{Q}_p(\sqrt{-p})$ and $K = \mathbb{Q}_p(\sqrt{-p\zeta_{p-1}})$, we cannot distinguish from which quadratic extension the local representations is induced from. □

*Remark* 6.3.13. The classification of the local data at $p$ of a newform determined by $N_p$ and $C_p$ does not distinguish the quadratic extensions of $\mathbb{Q}_p$, the local component $\pi_p$ at $p$ is induced from in the supercuspidal case. The above corollary does that in terms of the variation of $\varepsilon$-factor of $f$.

### 6.3.2 The case $p = 2$

For $p = 2$, more representations of the Weil group are involved and it can be non-dihedral. When inertia acts reducibly, the local representation $\rho_2(f)$ is dihedral; otherwise it has projective image isomorphic to one of three "exceptional" groups $A_4, S_4$ or $A_5$. The $A_5$ case cannot occur since the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_2|\mathbb{Q}_2)$ is solvable. Weil proved in [52] that over $\mathbb{Q}$, the $A_4$ case also does not occur, so $S_4$ is the only possibility of the projective image of $\rho_2(f)$. In this case, the corresponding field extension of $\mathbb{Q}_2$ is obtained by adjoining the coordinates of the 3-torsion points of the following elliptic curves [41, Section 6]:

$$E_+^{(r)} : rY^2 = X^3 + 3X + 2 \quad \text{and } E_-^{(r)} : rY^2 = X^3 - 3X + 2 \quad \text{with } r \in \{\pm 1, \pm 2\}.$$

In local cit., the 2-adic valuation of the level of the modular form is given as follows: 7 for the curve $E_+^r$ with $r$ as above, 3 for the curve $E_-^{(-1)}$, 4 for the curve $E_-^{(1)}$ and 6 when the curve is $E_-^{(\pm 2)}$. Hence, if $p = 2$ is a non-dihedral supercuspidal prime, then we have $N_2 \in \{3, 4, 6, 7\}$.

From above we see that non-dihedral supercuspidal case can occur only in 8 cases. In all such cases local root number can be computed and it can be found in [34, Remark 11] and [20, Remark 22].

From now on we assume that $p = 2$ is a dihedral supercuspidal prime. In this case, the representation $\rho_2(f)$ is induced from a quadratic extension $K|\mathbb{Q}_2$. Note that there are seven quadratic extensions $\mathbb{Q}_2(\sqrt{t})$ of $\mathbb{Q}_2$ with $t = -3, -1, 3, 2, -2, 6, -6$. Among them $\mathbb{Q}_2(\sqrt{-3})$ is unramified and rest of them are ramified. Among ramified extensions, two of them (corresponding to $t = -1, 3$) have discriminant with valuation 2 and rest of them have discriminant with valuation 3. Let $d$ denote the valuation of the discriminant of $K|\mathbb{Q}_2$. Thus, we have $d \in \{2, 3\}$. The following is the analogue of Proposition 6.3.4 for $p = 2$.

**Proposition 6.3.14.** *Let $p = 2$ be a dihedral supercuspidal prime.*

1. *If $K$ is unramified, then $N_2$ is even.*
2. *Assume that $K|\mathbb{Q}_2$ is ramified with $l(\varkappa) \geq d$. We have $N_2$ is odd if $\varkappa$ is minimal; otherwise $N_2$ is even.*

*Proof.* The relation (6.7) for $p = 2$ gives us that

$$N_2 = \begin{cases} 2a(\varkappa), & \text{if } K|\mathbb{Q}_2 \text{ is unramified,} \\ 2 + a(\varkappa), & \text{if } K|\mathbb{Q}_2 \text{ is ramified with discriminant valuation 2,} \\ 3 + a(\varkappa), & \text{if } K|\mathbb{Q}_2 \text{ is ramified with discriminant valuation 3.} \end{cases} \tag{6.19}$$

Thus, $N_2$ is even when $K$ is unramified.

In the ramified case, we first assume that $\varkappa$ is minimal. Using [11, §41.4 Lemma], we have $l(\varkappa) \geq d-1$. Moreover, if $l(\varkappa) \geq d$, then applying same lemma we get $l(\varkappa) \not\equiv d-1 \pmod 2$. If $d = 2$, then we conclude that $l(\varkappa)$ is even and so $a(\varkappa) = l(\varkappa) + 1$ is odd. When $d = 3$, we conclude that $l(\varkappa)$ is odd and so $a(\varkappa) = l(\varkappa) + 1$ is even.

We now prove that if $\varkappa$ is not minimal, then $N_2$ is even. To prove that, consider a non-minimal character $\varkappa$. As usual, we write $\varkappa = \theta_K \varkappa'$ [11, Section 41.4] with $\theta$ a character of $\mathbb{Q}_2^\times$ and $\varkappa'$ minimal over $\mathbb{Q}_2$. As in the proof of Proposition 6.3.4, we deduce that $a(\varkappa') < a(\varkappa)$ and $a(\theta) \geq 1$. Observe that $a(\varkappa) = a(\theta_K)$. This follows from $a(\varkappa) = a(\varkappa'\theta_K) \leq \max(a(\varkappa'), a(\theta_K))$ with equality if $a(\varkappa') \neq a(\theta_K)$.

Since $K$ is wildly ramified, $\omega_{K|\mathbb{Q}_2}$ has conductor $d$ [11, Section 41.3], we conclude that $a(\omega_{K|\mathbb{Q}_2}) = 2$ or $3$. Consider first the case $d = 2$. If $a(\theta) \leq d = 2$, then by Lemma 6.3.3, we have $a(\theta_K) \leq a(\theta) \leq 2$. Since $\varkappa'$ is minimal over $\mathbb{Q}_2$, we obtain that $l(\varkappa') \geq d-1$ [11, §41.4 Lemma].

Let us first assume $l(\varkappa') = d - 1 = 1$. We then have $a(\varkappa') = 2$. Both $a(\varkappa') = 2$ and $a(\theta) \leq 2$ (i.e, $a(\theta_K) \leq 2$) cannot occur simultaneously as it will contradict the fact $a(\varkappa') < a(\varkappa) = a(\varkappa'\theta_K)$. Hence, we conclude that $a(\theta) > 2$.

We now assume that $l(\varkappa') \geq d = 2$. As above, the minimality of $\varkappa'$ gives us that $l(\varkappa') \geq 2$ is even and so $a(\varkappa') \geq 3$ is odd. In this case, if $a(\theta) \leq 2$, that is, $a(\theta_K) \leq 2$, then we obtain that $a(\varkappa) = a(\varkappa'\theta_K) = a(\varkappa')$ as $a(\varkappa') \geq 3$, contradicts the non-minimality of $\varkappa$. Hence, in this case also $a(\theta) > 2$.

In both cases we obtain $a(\theta) > 2 = a(\omega_{K|\mathbb{Q}_2})$ so that $a(\theta\omega_{K|\mathbb{Q}_2}) = a(\theta)$. By Lemma 6.3.3, we get that $a(\theta_K) = a(\theta) + a(\theta\omega_{K|\mathbb{Q}_p}) - a(\omega_{K|\mathbb{Q}_p}) = 2(a(\theta) - 1)$ and so $a(\varkappa) = a(\theta_K)$ is even. From equation 6.19, we conclude that if $\varkappa$ is non-minimal then $N_2$ is even.

When $d = 3$, we can prove similarly that $a(\theta_K) = 2a(\theta) - 3$ and hence $a(\varkappa) = a(\theta_K)$ is odd. □

**Theorem 6.3.15.** *Let $p = 2$ be a dihedral supercuspidal prime for $f$.*

1. *Assume that $K|\mathbb{Q}_2$ is unramified. Choose an additive character as in (6.9) if $a(\varkappa) > 1$. Then the number $\varepsilon_2 = 1$.*

2. *When $K|\mathbb{Q}_2$ is ramified, the number*

$$\varepsilon_2 = \begin{cases} 1, & \text{if } K = \mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{3}), \\ -1, & \text{if } K = \mathbb{Q}_2(\sqrt{6}), \mathbb{Q}_2(\sqrt{-6}) \text{ with } \varkappa \text{ minimal}, \\ 1, & \text{if } K = \mathbb{Q}_2(\sqrt{6}), \mathbb{Q}_2(\sqrt{-6}) \text{ with } \varkappa \text{ not minimal}. \end{cases}$$

*Proof.* The proof of Theorem 6.3.7 is valid for the unramified supercuspidal prime $p = 2$ also. Thus, we have $\varepsilon_2 = 1$, when $a(\varkappa) > 1$. When $a(\varkappa) = 1$, note that $\widetilde{\varkappa}$ is a character of $\mathbb{F}_{2^2}^\times$ of order 3. Since $\widetilde{\chi}_2$ has order 2, the order of $\widetilde{\varkappa}\widetilde{\chi}_2$ is also 3. Using Stickelberger's theorem [30, Theorem 5.16], we have $G(\widetilde{\varkappa}, \widetilde{\phi}) = G(\widetilde{\varkappa}\widetilde{\chi}_2, \widetilde{\phi}) = 2$ and hence $\varepsilon_2 = 1$.

In the ramified case. by proposition above we see that if $\varkappa$ is minimal, then $a(\varkappa)$ is odd, when $d = 2$, and it is even, when $d = 3$. If $\varkappa$ is not minimal, then $a(\varkappa)$ is even, when $d = 2$, and it is odd, when $d = 3$. As odd primes [cf. Equations 6.16 and 6.17], we have $\varepsilon_2 = 1$, if $a(\varkappa)$ is odd; otherwise we get $\varepsilon_2 = \chi_2(N_{K|\mathbb{Q}_2}(\pi))$. Here, $\pi$ is a uniformizer of $K$. Hence, we get the number $\varepsilon_2$ as desired. □

As before [cf. Corollary 6.3.12], the type of the local component $\pi_{f,2}$ can be classified by $N_2$ and $C_2$ ( with $N_2 \in \{3, 4, 6, 7\}$ in the non-dihedral supercuspidal case). We now classify the quadratic extensions $K|\mathbb{Q}_2$, the local component $\pi_{f,2}$ is induced from in the dihedral supercuspidal case. Note that $\mathbb{Q}$ has three quadratic extensions ramified only at 2 (having absolute discriminant a power of 2), namely $\mathbb{Q}(i), \mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-2})$. Their corresponding quadratic characters will be denoted by $\chi_{-1}, \chi_2$ and $\chi_{-2}$ respectively. Let

$\chi$ be the quadratic character associated by class field theory to any of these characters $\chi_j$ for $j = -1, 2, -2$. Using [34, Theorem 4.2, part (1)], we have the following two relations as odd primes [cf. Corollary 6.3.12]:

I. $\varepsilon(f \otimes \chi) = \chi(N')\varepsilon(f)$, if $\varepsilon_2 = 1$,

II. $\varepsilon(f \otimes \chi) = -\chi(N')\varepsilon(f)$, if $\varepsilon_2 = -1$.

**Corollary 6.3.16.** *Let $p = 2$ be a dihedral supercuspidal prime for $f$. Then $\pi_{f,2}$ is always induced by a quadratic extension $K|\mathbb{Q}_2$. If $f$ is 2-minimal and $N_2 \geq 2$ is even, then $K$ is the unique unramified quadratic extension of $\mathbb{Q}_2$. In the ramified case, we have the following classifications of $K$:*

| ***Classification of $K$ for $p = 2$*** | | |
|---|---|---|
| *$p$-minimality of $f$* | $K = \mathbb{Q}_2(\sqrt{t})$ | *Property* |
| *Yes* | $t = -1, -2, 2, 3$ | *I* |
| | $t = -6, 6$ | *II* |
| *No* | $t = -1, -2, 2, 3, -6, 6$ | *I* |

*Remark* 6.3.17. If $f$ is not 2-minimal, then we cannot distinguish whether the extension $K|\mathbb{Q}_2$ is unramified or ramified (since $N_2$ is even in both cases). Moreover, when $K|\mathbb{Q}_2$ is ramified, the above property $I$ always satisfies for $f$. If $f$ is a 2-minimal newform, then the extension $K|\mathbb{Q}_2$ (from which the local representation is induced from) can be distinguished by the parity of $N_2$ ($K$ is unramified if $N_2$ is even and $K$ is ramified if $N_2$ is odd).

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