The Large Sieve Inequality and Other Sieves

- with Applications

A thesis submitted towards partial fulfillment of

5 year BS-MS dual degree programme

by

Santosh Arvind Adimoolam

Under the guidance of

Dr. R. Thangadurai

Harischandra Research Institute, Allahabad

Local Supervisor - Dr. Anupam Kumar Singh

Department of Mathematics

Indian Institute of Science Education and Research Pune

Certificate

This is to certify that this dissertation entitled " The large sieve inequality and other sieves - with applications " towards the partial fulfillment of the BS-MS duel degree programme at the Indian Institute of Science Education and Research Pune, represents original research carried out by Santosh Arvind Adimoolam (Reg. no. 20061012) at Harischandra Research Institute under the supervision of Dr. R. Thangadurai during the academic year 2010-2011.

Name and signature of the student

Supervisor: Head Mathematics Department:

Date: Date:

Place: Place:

Acknowledgement

I am very thankful to my guide Dr. R. Thangadurai at HRI, who has lent invaluable guidance to my work and supported me through the first comprehensive project work undertook by me. I am thankful to Dr. Anupam Kumar Singh at IISER Pune who gave encouragement in the pursuit of this study and helped me format my thesis. I am thankful to the faculty of mathematics department of IISER Pune for broadening my horizon of knowledge in mathematics which enabled me to understand the reference material of my project work. I am thankful to IISER Pune for making this provision for project work in 5th year and providing me the opportunity to work at HRI with my guide. I am thankful to HRI for accepting me at their institute for this project and the enjoyable stay with all its best facilities. I am thankful to my colleagues and friends who have made this period of project work cherishable.

Abstract

This thesis is the study of Large sieve inequality and other sieves in the context of understanding the Bombieri-Vinogradov inequality and solution of the Titchmarsh divisor problem. In this thesis, I have elaborated on certain proofs and provided detailed understanding of the Gallagher and Turan Sieve with their applications. The framework of development from Large sieve inequality to the Bombieri-Vinogradov theorem has been emphasized, although it does not cover the proof of the Bombieri-Vinogradov theorem. It then leads to the description of Titchmarsh divisor problem with its solution.

Contents

Chapter 1

Some preliminaries

1.0.1 The big $'O'$ and small $'o'$ notation:

Let D be a subset of $\mathbb{C}$ and let $f : D \to \mathbb{C}$ be a complex valued function defined on D. Then we write

$$f(x) = O(g(x))$$

if $g : D \to \Re^+$ and there exists a positive constant A such that $|f(x)| \leq Ag(x)$ for all $x \in D$. Sometimes we will write

$$f(x) \ll g(x) \text{ or } g(x) \gg f(x)$$

to indicate that $f(x) = O(g(x))$. If we have $f(x) \ll g(x)$ and $g(x) \ll f(x)$ then we write

$$f(x) \asymp g(x).$$

In the case D is unbounded, we will write

$$f(x) = o(g(x))$$

if

$$\lim_{\substack{x\to\infty \\ x\in D}} \frac{f(x)}{g(x)} = 0.$$

We will also write

$$f(x) \sim g(x)$$

to mean

$$\lim_{\substack{x\to\infty \\ x\in D}} \frac{f(x)}{g(x)} = 1.$$

1.0.2 The technique of partial summation

Theorem 1.0.1. *[1] Let $c_1, c_2, ...$ be a series of complex numbers and set*

$$S(x) = \sum_{n\leq x} c_n.$$

Let n_0 be a fixed positive integer. If $c_j = 0$ for $j \leq n_0$ and $f : [n_0, \infty) \to \mathbb{C}$ has a continuous derivative in $[n_0, \infty)$, then for x any integer $x > n_0$ we have

$$\sum_{n\leq x} c_n f(n) = S(x)f(x) - \int_{n_0}^{x} S(t)f'(t)dt.$$

Proof. This is easily deduced by writing the left hand side as

$$\sum_{n\leq x}(S(n)-S(n-1))f(x)=\sum_{n\leq x}S(n)f(n)-\sum_{n\leq x-1}S(n+1)f(n)$$

$$=S(x)f(x)-\sum_{n\leq x-1}\int_{n}^{n+1}f'(t)dt$$

$$=S(x)f(x)-\int_{n_0}^{x}S(t)f'(t)dt,$$

because $S(t)$ is a step function that is constant on intervals of the form $[n, n+1]$.

□

In mathematical literature the phrase 'by partial summation' often refers to a use of above lemma with appropriate choice of c_n and $f(t)$. For instance we can apply it with $c_n = 1$ and $f(t) = \log t$ to deduce that,

$$\sum_{n\leq x}\log n=[x]\log x-\int_{1}^{x}\frac{[t]}{t}dt$$

where $[x]$ is the greatest integer less than or equal to x. As

$$[x]=x+O(1),$$

we can deduce that

Proposition 1.0.1.

$$\sum_{n \leq x} \log n = x \log x - x + O(\log x).$$

Similarly we deduce

Proposition 1.0.2.

$$\sum_{n \leq x} \frac{1}{n} = \log x + O(1).$$

1.0.3 Chebysheff's theorem

From here on we use the notation $p, (q \text{ or } l)$ to denote prime. Let $\pi(x)$ denote the number of primes upto x. In 1850, Chebysheff proved, by an elementary method, that

$$\pi(x) = O\left(\frac{x}{\log x}\right).$$

In fact if we define $\theta(x)$ as

$$\sum_{p \leq x} \log p$$

then Chebysheff proved:

Theorem 1.0.2. *(Chebysheff's theorem) [1] There exists positive constants A and B such that*

$$Ax \leq \theta(x) \leq Bx.$$

By partial summation this implies the bound on $\pi(x)$ as $O\left(\frac{x}{\log x}\right)$. From it is clear that there is always a prime number between x and $\frac{Bx}{A}$. By obtaining bounds for A and B such that $B/A \leq 2$ Chebysheff was able to deduce further from this theorem:

Theorem 1.0.3. *[8] (Bertrand's postulate)*

There is always a prime number between n and $2n$ for $n > 1$.

Proof. **Chebysheff's proof of theorem 1.0.2** The key observation is that

$$\prod_{n \leq p \leq 2n} p \left| \binom{2n}{n} \right..$$

We find upon taking logartithms

$$\theta(2n) - \theta(n) \leq 2n \log 2.$$

By writing succesively

$$\theta(n) - \theta(n/2) \leq n \log 2$$

$$\theta(n/2) - \theta(n/4) = n/2 \log 2...$$

and summing up the inequalities we find,

$$\theta(2n) \leq 4n \log 2.$$

In other words $\theta(x) = O(x)$. Hence

$$x \gg \theta(x) \geq \sum_{\sqrt{x} \leq p \leq x} \log p$$

$$\geq \frac{1}{2} \log x(\pi(x) - \pi(\sqrt{x}))$$

$$\geq \frac{1}{2} \log x\pi(x) + O(\sqrt{x} \log x)$$

which gives

$$\pi(x) = O\left(\frac{x}{\log x}\right).$$

□

Theorem 1.0.4. *[8]*

$$\sum_{p \leq n} \frac{\log p}{p} = \log n + O(1)$$

.

Proof. We study the prime factorization of $n!$. We write

$$n! = \prod_{p \leq n} p^{e_p}$$

since only $p \leq n$ divide $n!$. The number of multiples of p that are $\leq n$ are $\left[\frac{n}{p}\right]$. The number of multiples of p^2 that are $\leq n$

are $\left[\frac{n}{p^2}\right]$ and so on. Hence it is easily seen that

$$e_p = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + ...$$

where the sum is finite. We therefore deduce

$$\log n! = \sum_{p \leq n} \left(\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + ... \right) \log p.$$

Since we also have

$$\log n! = \sum_{k \leq n} \log k = n \log n - n + O(\log n)$$

and

$$\left(\left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + ... \right) \log p \leq \sum_{p} \frac{\log p}{p(p-1)} \ll n.$$

We find

$$\sum_{p \leq n} \left[\frac{n}{p}\right] \log p = n \log n + O(n).$$

□

Setting $c_n = (\log p)/p$ when $n = p$ and zero otherwise, we apply partial summation with $f(t) = (\log t)^{-1}$ to deduce theorem 1.0.5:

Theorem 1.0.5.

$$\sum_{p \leq n} \frac{1}{p} = \log \log n + O(1)$$

Remark 1.0.1. *The lower bound*

$$\sum_{p \leq n} \frac{1}{p} \geq \log \log n + O1$$

can be derived by partial summation from

$$\sum_{p \leq n} \frac{\log p}{p} \geq \log n + O(1)$$

which in turn can be derived from Chebyshev's upper bound.

Chapter 2

Gallagher's sieve

Though relatively recent in origin, sieve techniques are simple enough to be treated first, especially from didactic perspective.

2.1 Generalities

Let A be a finite set of objects and $\wp$ be an index set of primes such that for each prime p a subset A_p of A is assigned. The sieve problem is to estimate from above and below the size of the set

$$S(A, \wp) = A \setminus \bigcup_{p \in \wp} A_p.$$

This is the formulation of the problem in the most general context. Of course the general answer is given by the familiar inclusion exclusion principle in combinatorics. More precisely, for each subset J of $\wp$, denote

$$A_J := \bigcap_{p \in J} A_p.$$

Thus inclusion exclusion principle gives us

$$\sharp S(A, \wp) = \sum_{I \in \wp} (-1)^{\sharp J} \sharp J.$$

2.2 The larger sieve

Let B be a (non empty) finite set of integers and Γ be a set of prime powers. Suppose for each $t \in \Gamma$ we have

$$\sharp B(mod\ t) \leq u(t)$$

for some $u(t)$. Thus B represents at most $u(t)$ residue classes $mod(t)$.

Theorem 2.2.1. *[3] (Gallagher's large sieve) We keep the above setting and let*

$$X = \max_{b \in B} |b|.$$

If

$$\sum_{t\in\Gamma}\frac{\Lambda(t)}{u(t)}-\log(2X)>0,$$

then

$$\sharp B\leq\frac{\sum_{t\in\Gamma}\Lambda(t)-\log(2X)}{\sum_{t\in\Gamma}\frac{\Lambda(t)}{u(t)}-\log(2X)},$$

where $\Lambda(.)$ *is the Mangoldt function.*

Proof. Let $t\in\Gamma$ and for each residue class r(mod t) define

$$Z(B,t,r)=\sharp\{b\in B:\ b\equiv r(mod\ t)\}.$$

Then

$$\sharp B=\sum_{r(mod\ t)}Z(B,t,r).$$

By Cauchy-Schwartz inequality this is

$$\leq u(t)^{1/2}\left(\sum_{r(mod\ t)}Z(B,t,r)^2\right)^{1/2}.$$

Hence

$$\frac{(\sharp B)^2}{u(t)}\leq\sum_{r(mod\ t)}\sum_{\substack{b,b'\in B\\ b,b'\equiv r(mod\ t)}}1\leq\sharp B+\sum_{\substack{b,b'\in B\\ b\neq b'}}\sum_{t|b-b'}1.$$

We multiply this inequality by $\Lambda(t)$ and we sum over $t\in\Gamma$. Using

$$\sum_{t|n}\log t=\log n$$

we obtain

$$\sum_{t\in\Gamma}\frac{(\sharp B)^2}{u(t)}\Lambda(t) \leq \sharp B\sum_{t\in\Gamma}\Lambda(t) + (\log 2X)((\sharp B)^2 - \sharp B).$$

By cancelling $\sharp B$ and rearranging we establish the inequality.

□

Following Gallagher, we apply the larger sieve to prove:

Theorem 2.2.2. *[8] Let a, b be integers having the property that for any prime power t, there exists an integer $v(t)$ such that*

$$b \equiv a^{v(t)}(mod\ t).$$

Then there exists an integer v such that

$$b = a^v.$$

Before going through the proof of the theorem, let us recall that if $(a, n) = 1$ then order of a modulo n given by $f_a(n)$ is d if and only if $n|\Phi_d(a)$ where $\Phi(.)$ is the cyclotomic polynomial.

Proof. Let a and b be as in the statement of the theorem. We note that a and b are positive and $a \geq 3$. Let

$$B := \{n \leq x :\ n = a^i b^j \text{ for all } i, j\}$$

$$\Gamma := \{t : t \text{ is a prime power }, \ f_a(t) \leq y\},$$

where $y = y(x)$ is some parameter to be choosen later. We keep the notation of theorem 2.2.1. If for every prime power t we have that b is a power of a modulo t, then

$$u(t) \leq f_a(t).$$

Thus theorem 2.2.1 implies that

$$\sharp B \leq \frac{\sum_{t\in\Gamma} \Lambda(t) - 2x}{\sum_{t\in\Gamma} \frac{\Lambda(t)}{f_a(t)} - 2x}, \tag{2.2.1}$$

provided that the denominator is positive. We have

$$\begin{aligned}\sum_{t\in\Gamma} \Lambda(t) &= \sum_{d\leq y} \sum_{f_a(t)\leq d} \Lambda(t) \\ &= \sum_{d\leq a} \sum_{t|\Phi_d(a)} \Lambda(t) \\ &= \sum_{d\leq y} \log \Phi_d(a),\end{aligned}$$

. But

$$(a-1)^{\phi(d)} \leq \Phi(d) \leq (a+1)^{\phi(d)},$$

so that

$$\sum_{t\in\Gamma} \Lambda(t) = \sum_{d\leq y} \log|\Phi_d(a)| \asymp \sum_{d\leq y} \phi(d) \asymp y^2.$$

We also note that this implies

$$\sum_{t\in\Gamma}\frac{\Lambda(t)}{f_a(t)}\geq\frac{1}{y}\sum_{t\in\Gamma}\Lambda(t)\gg y^2.$$

Now we choose $y=100\log 2x$. From 2.2.1 we deduce that

$$\sharp B\ll\log x.$$

To this end, let us remark that if all the powers of a are distinct, then the set B has cardinality

$$\asymp(\log x)^2.$$

This contradicts the previous equation, hence we conclude that for some i_0, j_0 we have

$$a^{i_0}=b^{j_0}.$$

We may even suppose that $(i_0, j_0)=1.$, for otherwise we can take $(i_0, j_0)-$th roots of both sides of the inequality. Let us write

$$n=\prod_p p^{v_p(n)}.$$

We deduce

$$i_0v_p(a)=j_0v_p(b)$$

for all primes p. As $(i_0, j_0) = 1$, this means $i_0 | v_p(a)$ and $j_0 | v_p(b)$ for all primes p. This implies that a is the $j-$th and b is the $i-$th power of some integer c. The hypothesis now implies that for any prime q there exists an integer v_q such that

$$c^{j_0 v_q} = c^{i_0} mod\ q$$

which is equivalent to $f_c(q) | (j_0 v_q - i_0)$ if $(q, c) = 1$. Now take a prime divisor q of $\Phi_{j_0 t}(c)$ for any t. We deduce $f_c(q) = 0 (mod\ j_0)$. Thus $j_0 | i_0$ and so a is a power of b.

□

Chapter 3

The Turan sieve

In 1934 Paul Turan(1910-1970) gave an extremely simple proof of a classical theorem of Hardy and Ramanujan of the normal number of prime factors of a given natural number. Inherent in his work is the basic sieve method, which was called Turan's sieve. In this section we will see how Turan's sieve can be applied for questions like counting the number of irreducible polynomials of a given degree. Turan's sieve is the fundamental sieve used in the famous Bombieri-Davenport inequality.

3.1 The basic inequality

Let A be an arbitrary finite set and $\wp$ be a set of prime numbers. For every prime p we assign a set $A_p \subseteq A$. Let $A_1 = A$ and for every squarefree integer d composed of product of primes of $\wp$, let

$$A_d = \bigcap_{p|d} A_p.$$

Fix a positive integer z and set

$$P(z) = \prod_{\substack{p \in \wp \\ p<z}} p.$$

We will be interested in estimating

$$S(A, \wp, z) := A \setminus \cup_{p|\wp(z)} A_p.$$

We write for each prime $p \in \wp$,

$$\sharp A_p = \delta_p X + R_p$$

and for distinct primes $p, q \in \wp$

$$\sharp A_{pq} = \delta_p \delta_q X + R_{pq},$$

where $X = \sharp A$ and $0 < \delta_p \leq 1$.

For notational convienience we interpret $R_{pp} = R_p$. Heuristically we usually think of δ_p as the proportion of elements of A

lying in A_p, and of R_p as the error term in the estimation. The same interpretation can be given to δ_{pq} and R_{pq}.

Theorem 3.1.1. *[6](The Turan sieve) We keep the above setting. Let*

$$U(z) := \sum_{p|P(z)} \delta_p.$$

Then

$$S(A, \wp, z) \leq \frac{X}{U(z)} + \frac{2}{U(z)} \sum_{p|P(z)} |R(p)| + \frac{1}{U(z)^2} \sum_{p,q|P(z)} |R(p,q)|.$$

Proof. For each element a in A, let $N(a)$ be the number of primes $p|P(z)$ such that $a \in A_p$.

$$S(A, \wp, z) = \sharp\{a \in A : \ N(a) = 0\} = \frac{1}{U(Z)^2} \sum_{a \in A} (N(a) - U(z))^2.$$

Thus the goal is to derive an upper bound for

$$(N(a) - U(z))^2,$$

Squaring out the summand and expanding we must consider

$$\sum_{a \in A} N(a)^2 - 2U(z) \sum_{a \in A} N(a) + XU(z)^2.$$

For the first sum we have

$$\sum_{a\in A} N(a)^2 = \sum_{a\in A}\left(\sum_{\substack{p|P(z)\\ a\in A_p}} 1\right)^2$$

$$= \sum_{p,q|P(z)} \sharp A_p \cap A_q$$

$$= \sum_{\substack{p,q|P(z)\\ p\neq q}} \sharp A_{pq} + \sum_{p|P(z)} \sharp A_p$$

$$= X \sum_{\substack{p,q|P(z)\\ p\neq q}} \delta_p\delta_q + X\sum_{p|P(z)} \delta_p + \sum_{p,q|P(z)} R(p,q)$$

$$= X\left(\sum_{p|P(z)} \delta_p\right)^2 - X\sum_{p|P(z)} \delta_p^2 + X\sum_{p|P(z)} \delta_p + \sum_{p,q|P(z)} R(p,q),$$

and similarly

$$\sum_{a\in A} N(a) = x\sum_{p|P(z)} \delta_p + \sum_{p|P(z)} R(p).$$

It follows that

$$\sum_{a\in A}(N(a)-U(z))^2 = X\sum_{p|P(z)} \delta_p(1-\delta_p) + \sum_{p,q|P(z)} |R(p,q)| - 2U(z)\sum_{p|P(z)} R(p).$$

Since $(1-\delta_p) \leq 1$ we immediately deduce the upper bound stated in theorem. □

Remark 3.1.1. *In order to use the above theorem, one needs to have upper bound for δ_p, R_p and lower bound for $U(z)$.*

3.2 Counting irreducible polynomials in $F_p[x]$

Let F_p denote the finite field of p elements. Fix a natural number $n > 1$ and let N_n be the number of monic irreducible polynomials in $F_p[x]$ of degree n. One of the ways of obtaining an exact formula for N_n is via the technique of zeta functions. Consider the power series

$$\sum_f T^{deg\ f},$$

where the summation is over all the monic polynomials in $F_p[x]$. Since the total number of monic polynomials $f \in F_p[x]$ of degree n is p^n, the power series is easily seen to be

$$\sum_0^\infty p^n T^n = \frac{1}{1-pT}.$$

On the other hand $F_p[x]$ has a Euclidean domain and it has unique factorization. Thus we can write an Euler product expression for the power series above as

$$\sum_v (1 - T^{deg\ v})^{-1} = \sum_{n=1}^\infty (1 - T^n)^{-N_n},$$

where v runs over monic irreducible polynomials of $F_p[x]$. We therefore obtain

$$(1-pT)^{-1} = \sum_{d=1}^{\infty}(1-T^d)^{-N_d}.$$

By using that

$$-\log(1-pT) = \prod_{n=1}^{\infty}\frac{p^nT^n}{n}$$

and taking logarithms in the previous equation, we get

$$\sum_{n=1}^{\infty}\frac{p^nT^n}{n} = -\log(1-pT) = -\sum_{d=1}^{\infty}N_d\log(1-T^d)$$

$$= \sum_{d=1}^{\infty}\sum_{e=1}^{\infty}dN_d\frac{T^{de}}{de} = \sum_{n=1}^{\infty}\frac{T^n}{n}\left(\sum_{de=n}dN_d\right).$$

This proves:

Theorem 3.2.1. *[8] Let N_d denote the number of monic irreducible polynomials of $F_p[x]$ of degree d. Then*

$$\sum_{d|n}dN_d = p^n.$$

Observe that an immediate consequence is

$$N_n \leq \frac{p^n}{n}.$$

Let N_n denote the number of monic irreducible polynomials of $F_p[x]$ of degree n. Then

$$N_n = \frac{1}{n} \sum_{d|n} \mu_d p^{n/d}.$$

3.3 Counting irreducible polynomials in $Z[x]$

Fix a natural number H and $n > 1$. We will apply Turan sieve to count the number of irreducible polynomials

$$x^n + a_{n-1}x^{n-1} + ..a_1x + a_0$$

with $0 \leq a_i \leq H,\ a_i \in \mathbb{Z}$. We will prove that this number is

$$H^n + O(H^{n-1/3} \log^{2/3} H.$$

Let

$$A := \{(a_{n-1}, a_{n-2}, .., a_o) \in \mathbb{Z}^\ltimes : 0 \leq a_i \leq H\}.$$

We will think of the n tuples as corresponding to the monic polynomials

$$x^n + a_{n-1}x^{n-1} + ... + a_1x + a_o.$$

We want to count the number of tuples in A that correspond to the irreducible polynomials in $\mathbb{Z}[x]$. So, let $\wp$ consist of all primes and for each prime $p \in \wp$, let A_p subset of tuples corresponding to irreducible polynomials modulo p. Let $z = z(H)$ be a positive real number to be choosen later. Then $S(A, \wp, z)$ represents the upper bound for number of reducible polynomials in $\mathbb{Z}[x]$, because if a polynomial belongs to A_p for some prime p then it is irreducible. We observe that A has H^n elements. If we specify a monic polynomials $g(x) \in F_p[x]$, then the number of elements of A that, reduced modulo p, are congruent to $g(x) mod\ p$, is

$$\left(\frac{H}{p} + O(1)\right)^n.$$

We will choose z satisfying $z^2 < H$ so that for primes $p < z$, this expression can be written as

$$\frac{H^n}{p^n} + O\left(\frac{H^{n-1}}{p^{n-1}}\right).$$

From our previous discussion the number of monic irreducible polynomials of degree n is

$$N_n = \frac{p^n}{n} + O(p^{n/2}),$$

where implied constant depends on n. Thus the total number of irreducible polynomials in A corresponding to polynomials in $F_p[x]$ is

$$\left(\frac{H^n}{p^n}+O\left(\frac{H^{n-1}}{p^{n-1}}\right)\right)\left(\frac{p^n}{n}+O(p^{n/2})\right)=\frac{H^n}{n}+O\left(\frac{H^n}{p^{n/2}}\right)+O(H^{n-1}p).$$

This implies that

$$\sharp A_p=\frac{1}{n}H^n+O(H^{n-1}p)+O(H^n/p^{n/2})$$

and similarly for $p\neq q$

$$\sharp A_p\cap A_q=\frac{1}{n^2}H^n+O(H^{n-1}pq)+O(H^n/p^{n/2})+O(H^n/q^{n/2}).$$

Now we apply Turan's sieve with $\delta_p=\frac{1}{n}$ and

$$R_{pq}=O(H^{n-1}pq)+O(H^n/p^{n/2})+O(H^n/q^{n/2}).$$

By using Chebycheff's bound we deduce

Theorem 3.3.1. *[8] For A as above, $n\geq 3$ and $z^2<H$, we have*

$$S(A,\wp,z)\ll\frac{H^n\log z}{z}+H^{n-1}z^2.$$

By choosing

$$z=H^{1/3}(\log H)^{1/3}.$$

we obtain

Theorem 3.3.2. *[8] Let $n \geq 3$. The number of reducible polynomials*

$$x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 \ , 0 \leq a_i \leq H, \ a_i \in H,$$

is $O(H^{n-1/3}(\log H)^{2/3})$.

Gallagher obtained a sharper estimate by using a higher dimensinonal version of Large sieve inequality. One conjectures that the optimal exponent should be $n-1$, and this is actually the best possible.

Chapter 4

Large Sieve

We will describe the large sieve, which was introduced by Yuri Linnik (1915-72), in 1941 and was subsequently improved by Renyi (1950), Roth (1965), Davenport and Halberstam (1966), Gallagher (1967), and others. The sieve can be deduced from an inequality, known as large sieve inequality, which has wide range of applications. Linnik's original motivation was to attack Vinogradov's hypothesis concerning the size of the least non quadratic residue $n_p(mod\ p)$. Vinogradov conjectured that

$$n_p = O(p^{\epsilon})$$

for any $\epsilon > 0$. J.Osterle proved under the unproven generalized Riemann hypothesis that

$$n_p \leq 70(\log p)(\log\log p)$$

for all $p \geq 3$. Linnik proved, using his large sieve, that the number of primes $p \leq x$ for which $n_p > p^\epsilon$ is $O(\log\log x)$.

4.1 The large sieve inequality

We begin with a preliminary lemma.

Lemma 4.1.1. *[8] Let $F : [0,1] \to \mathbb{C}$ be a differentiable function with continuous derivative extended by periodicity to all $\mathbb{R}$ with a period 1. Let z be a positive integer. Then*

$$\sum_{d\leq z} \sum_{1\leq a\leq d,\ (a,d)=1} |F\left(\frac{a}{d}\right)| \leq z^2 \int_0^1 |F(\alpha)|d\alpha + \int_0^1 |F'(\alpha)|d\alpha.$$

Proof. Let $d \leq z$, $a \in [1,d] \cap \mathbb{N}$ with $(a,d) = 1$, and $\alpha \in [0,1]$. Then

$$-F\left(\frac{a}{d}\right) = -F(\alpha) + \int_{a/d}^{\alpha} F'(t)dt.$$

By taking absolute value on both sides, this implies

$$\left|F\left(\frac{a}{d}\right)\right| \leq |F(\alpha)| + \int_{a/d}^{\alpha} F'(t)dt.$$

Now let us fix $\delta > 0$ (to be choosen later) so that the intervals

$$I\left(\frac{a}{d}\right) := (a/d - \delta, a/d + \delta)$$

are contained in $[0, 1]$. We integrate the previous equation over $I(\frac{a}{d})$, with respect to α, and obtain

$$2\delta\left|F\left(\frac{a}{d}\right)\right| \leq \int_{I(a/d)} |F(\alpha)|d\alpha + \int_{I(a/d)} \int_{a/d}^{\alpha} |F'(t)|dtd\alpha.$$

Since $\alpha \in I(a/d)$ and $t \in [a/d, \alpha]$, we obtain that $t \in I(a/d)$. Hence the right hand side of the above inequality is

$$\begin{aligned}
&\leq \int_{I(a/d)} |F(\alpha)|d\alpha + \int_{I(a/d)} \int_{I(a/d)} |F'(t)dtd\alpha \\
&= \int_{I(a/d)} |F(\alpha)|d\alpha + 2\delta \int_{I(a/d)} |F'(t)|dt \\
&= \int_{I(\alpha)} |F(\alpha)|d\alpha + 2\delta \int_{I(a/d)} |F'(\alpha)|d\alpha.
\end{aligned}$$

In other words,

$$2\delta\left|F\left(\frac{a}{d}\right)\right| \leq \int_{I(a/d)} |F(\alpha)|d\alpha + 2\delta \int_{I(a/d)} |F'(\alpha)|d\alpha.$$

Now we choose

$$\delta = \frac{1}{2z^2}.$$

With this choice, the intervals $I(a/d)$, $1 \leq a \leq d$, $d \leq z$ do not intersect (modulo 1). Indeed, let $x \in I(a/d) \cap I(a'/b')$ for $a/d \neq a'/b'$. Then

$$|x - a/d| < \delta, \ |x - a'/b'| < \delta,$$

and so

$$|a/d - a'/b'| < 2\delta = \frac{1}{2z^2}.$$

On the other hand, we have

$$|a/d - a'/b'| = \frac{|ad' - a'd|}{|dd'|} \neq 0,$$

since if $ad' = a'd$, then, recalling that $(a, d) = (a', b') = 1$, we obtain $d = d'$, which is false. Thus

$$|a/d - a'/b'| \geq \frac{1}{dd'} \geq \frac{1}{z^2}.$$

Putting together the previous two equations we are led to a contradiction. We sum the previous integral over all intervals $I(a/d)$ and get

$$\frac{1}{z^2} \sum_{d \leq z} \sum_{1 \leq a \leq d, \ (a,d)=1} |F(a/d)| \leq \sum_{I(a/d)} \int_{I(a/d)} |F(\alpha)| d\alpha + \frac{1}{z^2} \sum_{I(a/d)} \int_{I(a/d)} |F'(\alpha)| d\alpha$$

$$\leq \int_0^1 |F(\alpha)|d\alpha + \frac{1}{z^2}\int_0^1 |F'(\alpha)|d\alpha.$$

This completes the proof of the lemma.

□

Now let us choose

$$F(\alpha) = \left(\sum_{n\leq x} a_n e(n\alpha)\right)^2,$$

where $(a_n)_{n\geq 1}$ is an arbitrary sequence of complex numbers, x is a positive integer and for a rational number t, $e(t) = exp(2\pi\iota t)$. For simplicity of notation, set

$$S(\alpha) = \sum_{n\leq x} a_n e(n\alpha),$$

hence

$$F(\alpha) = S(\alpha)^2,\ F'(\alpha) = 2S(\alpha)S'(\alpha).$$

By the previous lemma we obtain

$$\sum_{d\leq z}\ \sum_{1\leq a\leq d\ (a,d)=1} \left|S\left(\frac{a}{d}\right)\right|^2 \leq z^2\int_0^1 |S(\alpha)|^2 d\alpha + 2\int_0^1 |S(\alpha)S'(\alpha)|d\alpha.$$

We now state Parsevals identity:

$$\int_0^1 \left| \sum_{n \leq x} a_n e(n\alpha) \right|^2 d\alpha = \sum_{n \leq x} |a_n|^2,$$

thus

$$\int_0^1 |S(\alpha)|^2 d\alpha = \sum_{n \leq x} |a_n|^2.$$

This implies that

$$\sum_{d \leq z} \sum_{1 \leq a \leq z \ (a,d)=1} \left| S\left(\frac{a}{d}\right) \right|^2 \leq z^2 \sum_{n \leq x} |a_n|^2 + 2 \int_0^1 |S(\alpha) S'(\alpha)| d\alpha.$$

For the second term on the right-hand side of the above inequality we use the Cauchy-Schwartz inequality and once again, Parseval's identity. We obtain

$$\begin{aligned} \int_0^1 |S(\alpha) S'(\alpha)| d\alpha &\leq \left(\int_0^1 |S(\alpha)|^2 d\alpha \right)^{1/2} \left(\int_0^1 |S'(\alpha)|^2 d\alpha \right)^{1/2} \\ &\leq \left(\sum_{n \leq x} |a_n|^2 \right)^{1/2} \left(\sum_{n \leq x} 4\pi^2 n^2 |a_n|^2 \right)^{1/2} \\ &\leq 2\pi x \sum_{n \leq x} |a_n|^2. \end{aligned}$$

We record this result as:

Theorem 4.1.1. *: (The large sieve inequality)*
Let $(a_n)_{n\geq 1}$ *be a sequence of complex numbers and let* x, z *be positive integers. Then*

$$\sum_{d\leq z} \sum_{1\leq a\leq d \ (a,d)=1} |\sum_{n\leq x} a_n e(\frac{na}{d})|^2 \leq (z^2 + 4\pi x) \sum_{n\leq x} |a_n|^2,$$

where for a rational number α, $e(\alpha) = exp(2\pi\iota\alpha)$.

Montgomery and Vaughan, and Selberg have independently shown that $z^2 + 4\pi x$ can be replaced by $z^2 + x$.

4.2 The large sieve

We want to deduce a sieve method from the inequality described in the previous Large sieve inequality. Let A be a set of positive integers $n \leq x$ and let P be a set of primes. For each $p \in P$, suppose that we are given a set $\{w_{1p}, w_{2p}, ..., w_{w(p)p}\}$ of $w(p)$ residue classes modulo p. Let z be a positive real number and denote by $P(z)$ the product of primes $p \in P,\ p < z$. We set

$$S(A,P,z) = \{n \in A : n \neq w_{ip}(mod\ p) \forall\ 1 \leq i \leq w(p),\ \forall\ p|P(z)\}$$

and we denote by *S(A,P,z)* the cardinality of this set.

Theorem 4.2.1. *(The large sieve)[4]*

With the above notation we have

$$S(A,P,z) \leq \frac{z^2 + 4\pi x}{L(z)},$$

where

$$L(z) = \sum_{d \leq z} \mu^2(d) \prod_{p|d} \frac{w(p)}{p - w(p)}.$$

The idea of the proof of the above theorem is to use sums of the form

$$c_d(n) = \sum_{\substack{1 \leq a \leq d \\ (a,d)=1}} e(\frac{na}{d}),$$

where $n, d \in \mathbf{N}$, called Ramanujan sums. They have the following interesting properties:

Proposition 4.2.1. *Let d, d' be positive integers. Then*

1. *if $(d, d') = 1$ we havethat $c_{dd'}(n) = c_d(n)c_{d'}(n)$;*

2. $$c_d(n) = \sum_{D|(d,n)} \mu(d/D)D;$$

3. if $(d,n)=1$, *we have that* $c_d(n)=\mu(d)$, *that is,*

$$\mu(d)=\sum_{\substack{1\le a\le d\\(a,d)=1}} e(\frac{na}{d}).$$

Proof. Parts 1 and 3 of the proposition are easy . We now prove part 2. Let

$$\tilde{c}_d(n)=\sum_{1\le a\le d} e(\frac{na}{d}).$$

On the other hand we can write

$$\tilde{c}_d(n)=e(n/d)\sum_{0\le a\le d-1} e(\frac{na}{d}),$$

and we see that this is $e(n/d)\frac{e(n)-1}{e(n/d)-1}$ if $d\dagger n$ and $e(n/d)$ if $d|n$. In other words,

$$\tilde{c}_d(n)=\left\{\begin{matrix}0\ if\ d\dagger n\\ d\ if\ d|n\end{matrix}\right\}.$$

On the other hand we can write

$$\tilde{c}_d(n)=\sum_{D|d}\sum_{\substack{1\le a\le d\\(a_1,\frac{d}{D})}} e(\frac{na}{d})$$

$$=\sum_{D|d}\sum_{\substack{1\le a\le d\\(a_1,\frac{d}{D})=1}} e\left(\frac{nDa_1}{d}\right)=\sum_{D|d} c_{\frac{d}{D}}(n).$$

By using the Mobius inversion formula we deduce that

$$c_d(n) = \sum_{D|d} \mu(D)\tilde{c}_{\frac{d}{D}}(n) = \sum_{D|d} \mu\left(\frac{d}{D}\right)\tilde{c}_D(n).$$

which be a former deduction is

$$\sum_{D|(d,n)} \mu\left(\frac{d}{D}\right) D.$$

This completes the proof of the proposition. □

Proof of the theorem *(The large sieve)*: First let us set some notation. Let $d = p_1 p_2 ... p_t$ be a positive square free integer composed of primes dividing $P(z)$. By Chinese remainder theorem, for any $\underline{i} = (i_1, ..., i_t)$ with $1 \leq i_1 \leq w(p_1), ..., 1 \leq i_t \leq w(p_t)$, there exists a unique integer $w_{\underline{i},d}$ such that

$$w_{\underline{i},d} \equiv w_{i_j,p_j} (mod\ p_j) \forall 1 \leq j \leq t.$$

We denote by w_d the number of all posible $w_{\underline{i},d}$ appearing in this fashion. Now let $n \in S(A,P,z)$. This implies that

$$(n - w_{\underline{i},d}) = 1$$

for any d and $\underline{i}$ as above, and so by part 3 of the previous

proposition we obtain

$$\mu(d) = c_d(n - w_{\underline{i},d}) = \sum_{\substack{1\le a\le d\\(a,d)=1}} e\left(\frac{(n - w_{\underline{i},d})a}{d}\right).$$

We sum over all indices corresponding to d and over all integers $n \in S(A,P,z)$ and get

$$\mu(d)w(d)S(A,P,z) = \sum_{\substack{1\le a\le d\\(a,d)=1}} \sum_{w_{\underline{i},d}} e\left(\frac{w_{\underline{i},d}a}{d}\right) \sum_{n\in S(A,P,z)} e\left(\frac{na}{d}\right).$$

Squaring out the previous equation applying Cauchy Schwartz inequality gives

$$|\mu(d)w(d)S(A,P,z)|^2 \le \left(\sum_{\substack{1\le a\le d\\(a,d)=1}} \left|\sum_{w_{\underline{i},d}} e\left(\frac{-w_{\underline{i},d}a}{d}\right)\right|^2\right)\left(\sum_{\substack{1\le a\le d\\(a,d)=1}} \left|\sum_{n\in S(A,P,z)} e\left(\frac{na}{d}\right)\right|^2\right).$$

We write the first factor in the above expression as

$$\sum_{\substack{1\le a\le d\\(a,d)=1}} \sum_{w_{\underline{i},d}, w_{\underline{i}',d}} e\left(\frac{(w_{\underline{i}',d} - w_{\underline{i},d})a}{d}\right) = \sum_{w_{\underline{i}',d}, w_{\underline{i},d}} c_d(w_{\underline{i}',d} - w_{\underline{i},d}),$$

and, further, by using part 2 of the previous proposition, as

$$\sum_{w_{\underline{i}',d}, w_{\underline{i},d}, D|(d, w_{\underline{i},d} - w_{\underline{i}',d})} \mu\left(\frac{d}{D}\right) D = \sum_{D|d} \sum_{\substack{w_{\underline{i}',d}, w_{\underline{i},d}\\ w_{\underline{i}',d}\equiv w_{\underline{i},d}}} \mu\left(\frac{d}{D}\right) D$$

$$= \sum_{D|d} \mu(d/D) D w(d) w(d/D)$$

$$= dw(d) \sum_{E|d} \frac{\mu(E)w(E)}{E}$$

$$= dw(d) \prod_{p|D} \left(1 - \frac{w(p)}{p}\right)$$

$$= w(d) \prod_{p|D} (p - w(p)).$$

This gives us that

$$|\mu(d)w(d)S(A,P,z)|^2$$

$$\leq w(d) \prod_{p|D} (p - w(p)) \left(\sum_{\substack{1 \leq a \leq d \\ (a,d)=1}} \left| \sum_{n \in S(A,P,z)} e\left(\frac{na}{d}\right) \right| \right),$$

or, equivalently, that

$$\mu^2(d)S(A,P,z)^2 \prod_{p|d} \frac{w(p)}{p - w(p)} \leq \sum_{\substack{1 \leq a \leq d \\ (a,d)=1}} \left| sum_{n \in S(A,P,z)} e\left(\frac{na}{d}\right) \right|^2.$$

Now we sum the previous equation over $d \leq z$ and use the large sieve inequality with the sequence $(a_n)_{n \leq 1}$ choosen such that a_n is 1 if $n \in S(A,P,z)$ and 0 otherwise. We obtain

$$\sum_{d \leq z} \mu^2(d)S(A,P,z)^2 \prod_{p|d} \frac{w(p)}{p - w(p)} \leq (z^2 + 4\pi x)S(A,P,z),$$

which, after doin the obvious cancellations, completes the proof of the theorem.

Remark 4.2.1. *When using the large sieve theorem, we need lower bounds for the sum $L(z)$. One way of obtaining a lower bound is by considering the summation over primes $p \leq z$, and not over all integers d; that is,*

$$L(z) \geq \sum_{p<z} \frac{w(p)}{p - w(p)}.$$

In some situations the lower bound obtained in this manner is sufficient, as in the case of applications of the Euclidean algorithm.

4.3 Weighted sums of Dirichlet characters

We want to exploit the large sieve inequality in a slightly different direction than the one of previous section.

We want to recall for a positive integer $d \geq 2$, a **(Dirichlet) character** modulo d is a group homomorphism $\chi : (\mathbb{Z}/\mathbb{Z})^* \to \mathbb{C}^*$. One extends χ to all of $\mathbb{Z}$ by setting $\chi(n) = 0$ for any integer n not coprime to d. This is a periodic function, whose values are $\phi(d)-$th units of unity. The **trivial (or principal)** character modulo d, denoted χ_0, is defined by $\chi_0 = 1$ for all n coprime to

d. In this case that χ is non-trivial and has period strictly less than d, we say it is **imprimitive** character; otherwise, we say that it is **primitive**, of **conductor** d.

An important result about Dirichlet characters, which will be useful, is that for any non trivial character χ modulo d,

$$\sum_{M+1\leq n\leq M+N} \chi(n) \ll d^{1/2}\log d$$

for any M, N. This is known as Polya-Vinogradov inequality.

There is an important function, called the **Gauss sum**, which brings together the character χ modulo d and the exponential function $e\left(\frac{\cdot}{d}\right)$;

$$\tau(\chi) = \sum_{1\leq a\leq d} \chi(a) e\left(\frac{a}{d}\right).$$

One can show that if χ is a primitive character modulo d, then $|\tau(\chi)| = d^{1/2}$ and, for $(n,d) = 1$,

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{1\leq a\leq d} \overline{\chi}(a) e\left(\frac{na}{d}\right), \tag{4.3.1}$$

where $\overline{\chi}$ denotes the complex conjugate of χ. We will use these identities to obtain modified versions of large sieve inequality.

Theorem 4.3.1. *[4] (First modified large sieve inequality)*
Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers and let x, z be positive integers. Then

$$\sum_{d\leq z}\frac{d}{\phi(d)}\sum_{\chi}^{*}\left|\sum_{n\leq x}a_n\chi(n)\right|^2 \leq (z^2+4\pi x)\sum_{n\leq x}|a_n|^2,$$

where the summation $\sum_{\chi}^{*}$ is over primitive characters χ modulo d.

Proof. Let $d \leq z$ be fixed and let n be coprime to d. Let χ be a primitive character modulo d. We multiply the previous equation 4.3.1 by a_n , sum it over $n \leq x$ and then square it out. By using its preceding equation we obtain

$$\left|\sum_{n\leq x}a_n\chi(n)\right|^2 = \frac{1}{d}\left|\sum_{1\leq a\leq d}\overline{\chi}(a)\sum_{n\leq x}a_n e\left(\frac{na}{d}\right)\right|^2.$$

By applying Cauchy Schwartz inequality to the last sum we get that

$$\left|\sum_{n\leq x}\sum_{\substack{m\leq y\\ mn\leq u}}a_n b_m\chi(nm)\right| \ll \int_{-T}^{T}|A(t,\chi)B(t,\chi)|\min\left\{\frac{1}{|t|},2xy\right\}dt$$

$$+\frac{x^{3/2}y^{3/2}}{T}\left(\sum_{n\leq x}|a_n|^2\right)^{1/2}\left(\sum_{m\leq y}|b_m|^2\right)^{1/2}.$$

We take the maximum over u and sum over χ and $d \leq z$ and obtain

$$\sum_{d\leq z}\frac{d}{\phi(d)}\sideset{}{^*}\sum_{\chi}\max_{u}\left|\sum_{n\leq x}\sum_{\substack{m\leq y\\ nm\leq u}}a_n b_m\chi(nm)\right| \ll \sum_{d\leq z}\frac{d}{\phi(d)}\sideset{}{^*}\sum_{\chi}\int_{-T}^{T}|A(t,\chi)B(t,\chi)|\min\left\{\frac{1}{|t|},2xy\right\}dt$$

$$+\frac{x^{3/2}y^{3/2}}{T}\left(\sum_{n\leq x}|a_n|^2\right)^{1/2}\left(\sum_{m\leq y}|b_m|^2\right)^{1/2}.$$

Now we use Cauchy Schwartz inequality, the previous corollary to obtain to obtain upper bounds for the first term on the right hand side.

$$\sum_{d\leq z}\frac{d}{\phi(d)}\sideset{}{^*}\sum_{\chi}\int_{-T}^{T}|A(t,\chi)B(t,\chi)|\min\left\{\frac{1}{|t|},2xy\right\}dt$$

$$\leq\left(\sum_{d\leq z}\frac{d}{\phi(d)}\left|\sideset{}{^*}\sum_{\chi}a_n\chi(n)\right|^2\right)^{1/2}\left(\sum_{d\leq z}\frac{d}{\phi(d)}\left|\sideset{}{^*}\sum_{\chi}b_m\chi(m)\right|^2\right)^{1/2}$$

$$\times\int_{-T}^{T}\min\left\{\frac{1}{|t|},2xy\right\}dt$$

$$\leq(z^2+4\pi x)^{1/2}(z^2+4\pi y)^{1/2}\left(\sum_{n\leq x}|a_n|^2\right)^{1/2}\left(\sum_{m\leq y}|b_m|^2\right)^{1/2}$$

$$\times(\log t+\log(2xy)).$$

For the second term on the right hand side we observe that there are $\phi(d)$ characters *modulo* d, hence

$$\frac{x^{3/2}}{y^{3/2}}T\sum_{d\leq z}\frac{d}{\phi(d)}\sideset{}{^*}\sum_{\chi}\left(\sum_{n\leq x}|a_n|^2\right)^{1/2}\left(\sum_{m\leq y}|b_n|^2\right)^{1/2}$$

$$\leq\frac{x^{3/2}y^{3/2}z^2}{T}\left(\sum_{n\leq x}|a_n|^2\right)^{1/2}\left(\sum_{m\leq y}|b_n|^2\right)^{1/2}.$$

Putting together the previous three equations and choosing $T = x^{3/2}y^{3/2}$ gives the desired inequality. □

4.4 The Bombieri-Vinogradov theorem

Define $\pi(x;d,a) = \sharp\{p\leq x : p\equiv a(mod\ d),\ p\ is\ prime\}$ and

$$lix = \int_2^x \frac{dt}{\log t}$$

In this section we study the error term that occurs in the asymptotic formula for $\pi(x;d,a)$ where a,d are co-prime integers and $d\leq x$. The following theorem is among the biggest breakthroughs of sieve theory.

Theorem 4.4.1. *[8](The Bombieri-Vinogradov theorem) For*

any $A > 0$ *there exists* $B = B(A) > 0$ *such that*

$$\sum_{d \leq \frac{x^{1/2}}{(\log x)^B}} \max_{y \leq x} \max_{(a,d)=1} \left| \pi(y; d, a) - \frac{liy}{\phi(d)} \right| \ll \frac{x}{(\log x)^A}.$$

4.5 The Titchmarsh divisor problem

Let a be a fixed integer. We consider the more complex question of determining the asymptotic behaviour of the function

$$\sum_{p \leq x} d(p + a),$$

where $d(.)$ is the divisor function. This is known in literature as the **Titchmarsh divisor problem.** This was first studied by Titchmarsh in 1930 and is related to the conjecture of Hardy and Littlewood formulating in 1922 and asserting that every sufficiently large integer can be written as the sum of a prime and two squares. Already in 1930 Titchmarsh showed that

$$\sum_{p \leq x} d(p + a) = O(x),$$

and that under a generalized Riemann hypothesis, an explicit asymptotic formula for $\sum_{p \leq x} d(p + a)$ also holds. Our goal in

this section is to describe a simple proof of Titchmarsh divisor problem.

Theorem 4.5.1. *[7] Let a be a fixed integer. Then there exists a positive constant c such that*

$$\sum_{p\leq x} d(p+a) = cx + O\left(\frac{x\log\log x}{\log x}\right)$$

Proof. First let us observe that for any positive integer n,

$$d(n) = 2\sum_{\substack{d|n\\ d\leq\sqrt{n}}} 1 - \delta(n),$$

where

$$\delta(n) := \left\{ \begin{matrix} 1 \ if \ n \ is \ a \ square \\ 0 \ otherwise \end{matrix} \right\}.$$

Thus

$$\sum_{p\leq x} d(p+a) = 2\sum p \leq x \sum_{p\leq x}\left(\sum_{\substack{d|p+a\\ d\leq\sqrt{x}}} 1\right)\sum_{p\leq x}\delta(p+a)$$

$$= 2\sum_{d\leq\sqrt{x}} \pi(x;d,-a) + O(\sqrt{x}).$$

We recall that the Bombieri Vinogradov theorem allows us to control the error terms in the asymptomic formula for $\pi(x;d,-a)$ as long as $d \leq \sqrt{x}(\log x)^B$ for some positive constant B (to be

specified later). This suggests the right hand side of previous equation into two parts:

$$\sum_{d\leq\sqrt{x}}\pi(x;d,-a)=\sum_{d\leq\frac{\sqrt{x}}{(\log x)^B}}\pi(x;d,-a)+\sum_{\frac{\sqrt{x}}{(\log x)^B}\leq d\leq\sqrt{x}}\pi(x;d,-a).$$

For the first sum in previous equation we write

$$\sum_{d\leq\frac{\sqrt{x}}{(\log x)^B}}\pi(x;d,-a)=\sum_{d\leq\frac{\sqrt{x}}{(\log x)^B}}\left(\pi(x;d,-a)-\frac{lix}{\phi(d)}\right)+\sum_{d\leq\frac{sqrtx}{(\log x)^B}}\frac{lix}{\phi(d)}$$

and use Bombieri Vinogradov theorem to obtain the upper bound of

$$lix\sum_{d\leq\frac{\sqrt{x}}{(\log x)^B}}\frac{1}{\phi(d)}+O\left(\frac{x}{(\log x)^A}\right)$$

for any arbitrary $A>0$ and $B=B(A)$. Now we remark that there exists a positive constant c_0 such that for any x,

$$\sum_{d\leq x}\frac{1}{\phi(d)}=c_0\log x+O(1).$$

Therefore

$$\sum_{d\leq\frac{\sqrt{x}}{(\log x)^B}}\pi(x;d,-a)=\frac{c_0}{2}x+O\left(\frac{x\log\log x}{\log x}\right).$$

For the second sum we have the Brun Titchmarsh theorem and

again we obtain

$$\sum_{\frac{\sqrt{x}}{(\log x)^B} \le d \le \sqrt{x}} \pi(x; d, -a) \ll \frac{x}{\log x} \sum_{\frac{\sqrt{x}}{(\log x)^B} \le d \le \sqrt{x}} \frac{1}{\phi(d)} \ll \frac{x \log\log x}{\log x}.$$

The proof of the above theorem is complete by combining the previous four equations and setting $c = c_0/2$. □

Bibliography

[1] H. Davenport. Multiplicative number theory, volume 74 of Graduate Texts in Mathe- matics. Springer-Verlag, New York, third edition, 2000.

[2] P. Erdos and C. Pomerance, On the normal number of prime factors of n. Rocky Mountain J., 15 (1985), 34352.

[3] P. X. Gallagher, A larger sieve. Acta Arithm., 18 (1971), 7781.

[4] H. Halberstam and H. E. Richert, Sieve Methods, London Mathematical Society monographs, No 4, (London, New York: Academic Press, 1974).

[5] K. Ireland and M. Rosen, A Classical Introduction to Modern Number Theory.

[6] Yu-Ru Liu and M. Ram Murty, The Turan sieve and some of its applications, J. Ramanujan Math. Soc., 13:2 (1999), pp. 3549.

[7] E.C. Titchmarsh, A divisor problem. Rend. Circ. Mat. Palermo, 54 (1930), 41429.

[8] Alina Carmen Cojocaru and M Ram Murthy. An introduction to sieve methods and their applications, London Mathematical Society Student Texts 66. Cambridge University Press, New York 2005