Representation theory of symmetric groups.

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This is to certify that this thesis entitled ”Representation theory of symmetric groups.” submitted towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune, represents work carried out by Venkata Raghu Tej Pantangi under the supervision of Amritanshu Prasad.

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Abstract

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This is an expository thesis exploring various results on representations of symmetric groups. Ordinary representation theory of symmetric groups (i.e representation theory over fields of characteristic zero) has been worked out by Frobenius, Schur and Young, around the beginning of the twentieth century. The modular representation theory of symmetric groups(i.e representation theory over field of positive characteristic) is still an active area of research. For example Calculation of Decomposition matrices for symmetric groups is still an important open problem. ”The Representation Theory of symmetric Groups” by G.D James [1] was the primary reference followed in the course of my Mater’s Thesis project.

Classifying of irreducible representations of symmetric groups over arbitrary fields and determining the corresponding decomposition matrices are the focus of this thesis.

Contents

Chapter 1

Introduction

Let G be a finite group, and k be a field. A representation of G over k is a group homomorphism $\rho : G \to GL(V)$, where V is any finite dimensional k- vector space and $GL(V)$ is the set of all invertible linear endomorphisms of V. It is convenient to denote a representation by the pair (ρ, V). Define kG to be the set of all maps $f : G \to k$. For $f, g \in kG$, we define their product

$$f.g(z) = \sum_{hk=z} f(h)g(k).$$

Clearly, this product along with point wise addition give a k- algebra structure to kG. Given a representation (ρ, V), V can be given a kG-module structure via the map ρ. Conversely given any finitely generated kG-module M, which is also a k-vector space, we can construct a representation $\rho : G \to GL(M)$ by the rule $\rho(g)(m) = 1_g.m$, where $1_g(g) = 1$ and $1_g(h) = 0$ for $h \neq g$. Therefore representations of G are same as finitely generated kG-modules. We say a representation (ρ, V) is irreducible if V, as a kG module is irreducible (i.e has no proper submodule). A representation (ρ, V) is called completely reducible if V can be written as a direct sum of irreducible kG modules.

Theorem. (Mashcke) If k is a field and G a finite group such that $char(k) \nmid |G|$, then every representation of G over k is completely reducible. Moreover if $char(k) \mid |G|$, then the representation corresponding to kG as a right module over itself is not completely reducible.

This is a well known result in representation theory that can be found in any text on representation theory of groups such as Curtis and Reiner [2]. So in the case when $char(k) \nmid |G|$, it is enough to find all the irreducible representations. An irreducible representation (ρ, V) of G over k is said to be absolutely irreducible if for any field extension K of k, the KG module $(V \otimes_{kG} K)$ is also irreducible. The field k is called a splitting field for G if all irreducible representations of G over k are absolutely irreducible. A conjugacy class of G is called p-regular if the order of elements in it is not divisible by p.

Theorem. (Brauer) Let k be a splitting field for a finite group G, of characteristic p . The number of inequivalent irreducible representations of G over k is the same as number of p-regular conjugacy classes in G.

An elegant proof of this result can be found in the article "Brauer Characters and Greens Theorem" (sporadic.stanford.edu/bump/brauer.ps) by Daniel Bump. An indecomposable module is a module which cannot be written as direct sum of its proper submodules. It is clear that classification of all indecomposable kG modules is enough to classify all representations of G over k. This still remains an open problem. However we have some results relating the so called "principal indecomposable modules" with the irreducible modules.

Theorem (Krull-Remak-Schmidt). Let $M \neq 0$ be a module which is both Noetherian and Artinian. Then E is a finite direct sum of indecomposable modules. Up to a permutation, the indecomposable components in such a direct sum are uniquely determined upto isomorphism.

The above result is quoted from Lang [3],Theorem 7.5 on p.441. It is easy to verify that all finitely generated kG-modules are both Artinian and Noetherian and thus the above results holds true for them. In particular kG, as a right kG- module can be written as a finite direct sum of indecomposable modules and upto permutation these indecomposable components are determined uniquely upto isomorphism. These indecomposable components are called principal indecomposable modules of kG.

Theorem (Brauer). There is a one-one correspondence between isomorphism classes of principal indecomposable kG modules and isomorphism classes of irreducible kG modules given by associating every indecomposable P to $P/Rad(P)$ ($RadP$ is the Jacobson Radical of P).

This is a special case of Theorem 3 on page 31 of Alperin [4]. Let $P_1, P_2 \ldots P_n$ be a set of representatives of isomorphism classes of principal indecomposable kG-modules. Then $U_1 := P_1/Rad(P_1), U_2 := P_2/Rad(P_2) \ldots U_n := P_n/Rad(P_n)$ is a complete set of representatives of isomorphism classes of irreducible kG-modules. Define c_{ij} to be the multiplicity of U_j as a composition factor of P_i in its Jordan-Holder composition series. The matrix $C := (c_{ij})$ is called the Cartan matrix of kG and the c_{ij}'s are called Cartan invariants.

Let K be a number field with O_K as its ring of integers. Let P be a prime lying over an integral prime p, then $k = O_K$ is a finite field of characteristic p. Since $char(K) = 0$, all KG modules are irreducible. Given a finite dimensional KG-module M, we have

Theorem. Let R_P denote the localization of O_K at P. There exists an $R_P[G]$ module M_P in M such that $KM_P = M$.

Now $\bar{M} := M_P/PR_PM_P$ is a finite dimensional kG module. Any module obtained by such a construction is called a kG-module associated to M. Let $M_1, M_2 \ldots M_m$ be a set of representatives of isomorphisms classes of irreducible KG modules and let $\bar{M}_1, \bar{M}_2 \ldots \bar{M}_m$ be the corresponding kG modules associated to them. Define d_{ij} to be the multiplicity of U_j as a composition factor of $\bar{M}_i$. The matrix $D := (d_{ij})$ is called a decomposition matrix of kG and the d_{ij}'s are called decomposition numbers.

Theorem. (Brauer and Nesbitt [5]) $D^T D = C$

Calculation of Decomposition and Cartan matrices for kG, when $char(k) \mid |G|$ is still an important open problem in representation theory.

This thesis explores various results on representation theory of symmetric groups over any arbitrary field. We follow a characteristic-free approach given in [1]. We begin with classification of irreducible representation of S_n and then focus on various results concerning decomposition numbers. **All the definitions and results concerning representations of symmetric group are essentially from the primary reference [1]. Most of the proofs also mimic those given in [1]. Unless otherwise mentioned, all the results and definitions are attributed to [1]**.

Chapter 2

Some Linear Algebra

Let G be a group, F a field and FG the group algebra generated by them. Let M be an FG module (a G representation). Let $<,>$ be a symmetric bilinear non-singular G-invariant form on M. Let U be a sub-module of M. As the form is G-invariant, we have $< u, vg >=< ug^{-1}, v >$ and thus $U^{\perp} = \{v \mid \ < u, v >= 0$ for all $u \in U\}$ is also a submodule. Let M^* be the dual space of M. Let V be a subspace of M and $V_0 = \{f \in M^* \mid f(V) = 0, \}$. Let $e_1, e_2 \dots e_k$ be a basis for V. Extend it a basis $e_1, e_2 \dots e_m$ of M. Let $f_1, f_2 \dots f_m$ be the basis of M^*, dual to $e_1, e_2 \dots e_m$. Observe that $f \in V_0$ if and only if $f(e_i) = 0$ for all $1 \leq i \leq k$. Therefore $f_{k+1}, f_{k+2} \dots f_m$ spans V_0. Thus we have $dim(V) + dim(V_0) = dim(M)$. Define

$$\theta : M \to M^* \text{ by } m \mapsto \theta_m \text{ where } \theta_m(x) =< m, x >$$

The form being non-singular makes θ, a linear isomorphism. Observe that $\theta(V^{\perp}) = V_0$. Therefore we have

$$dim(V) + dim(V^{\perp}) = dim(M). \tag{2.1}$$

The above equation implies that $V^{\perp\perp} = V$. Also given $0 \subset U \subset V \subset M$, we have $V^{\perp} \subset U^{\perp}$, and we may define

$$g : V \to (U^{\perp}/V^{\perp})^*, \text{ by } v \mapsto f_v, \text{ where}$$
$$f_v(x + V^{\perp}) =< v, x >$$

If $x + V^{\perp} = y + V^{\perp}$, we have $< v, x > - < v, y >=< v, x - y >= 0$. This shows that f_v is well defined. It is easy to see that g and f_v are linear. Now $Ker(g) = \{v \in V \mid$for all $x \in U^{\perp}, < v, x >= 0\} = V \cap U^{\perp\perp} = V \cap U = U$. Hence we have $V/U \cong (U^{\perp}/V^{\perp})^*$. Since $<,>$ is a G- invariant form, we have $V/U \cong (U^{\perp}/V^{\perp})^*$ as FG-modules. In particular we have $V \cong (M/V^{\perp})^*$.

Lemma 1. *For every FG sub-module V of M, $\frac{V}{V \cap V^{\perp}}$ is a self dual FG module.*

Proof. By second isomorphism theorem for modules, we have

$$\frac{V}{V \cap V^{\perp}} \cong \frac{V + V^{\perp}}{V^{\perp}}$$

Now by this and the the results proved prior to this lemma, we have

$$\frac{V + V^{\perp}}{V^{\perp}} \cong (\frac{V^{\perp}}{(V + V^{\perp})^{\perp}})^{*} \cong (\frac{V}{V \cap V^{\perp}})^{*}.$$

□

Definition 2. The gram matrix A of V defined with respect to a basis $\{e_1.e_2 \ldots e_k\}$ is the matrix whose (i, j) th entry is $< e_i, e_j >$.

Theorem 3. *The dimension of $\frac{V}{V \cap V^{\perp}}$ is equal to the rank of a gram matrix of V with respect to a given basis.*

Proof. Map V into V^* by the canonical map,f defined by the form $<,>$. Let f_v be the image of $v \in V$. Let $\{e_1, e_2 \ldots e_k\}$ be a given basis of V and let $\{\epsilon_1, \epsilon_2 \ldots \epsilon_k\}$ be its dual basis in V^*. It is easy to see that

$$f_{e_i} = \sum_{j=1}^{k} < e_i, e_j > \epsilon_j.$$

This implies that the gram matrix A with respect to $\{e_1, e_2 \ldots e_k\}$ is the same as the matrix of f with respect to the dual bases. Since $ker(f) = V \cap V^{\perp}$, we have $dim(\frac{V}{V \cap V^{\perp}}) = rank(A)$. □

Chapter 3

Specht Modules

3.1 Tableaux and Tabloids

Definition 4. A partition of n is a sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2 \ldots)$ such that $n = \sum_i \lambda_i$. In addition if $\lambda_i \geq \lambda_{i+1}$ for all i, λ is called a proper partition of n. If λ a proper-partition of n, we write $\lambda \vdash n$

Definition 5. If $\lambda = (\lambda_1, \lambda_2 \ldots \lambda_i \ldots)$ is a partition of n, then the diagram of λ, $[\lambda]$ is the pattern of n $\times$'s made up of r left aligned rows of $\times$'s with i-th row containing λ_i $\times$'s.

By (i, j)th node of $[\lambda]$, we mean the j-th $\times$ from the left in the i-th row. example: $\lambda = (4, 2^2, 1)$ then

$$[\lambda] = \begin{array}{llll} \times & \times & \times & \times \\ \times & \times & & \\ \times & \times & & \\ \times & & & \end{array}$$

We now define a partial order $\trianglerighteq$ by

Definition 6. If λ and μ are two partitions, we say $\lambda \trianglerighteq \mu$ if and only if

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i$$

for all $j \in \mathbb{N}$.

We now define a total order $\geq$ on partition by

Definition 7. If λ and μ are two partitions, we say $\lambda \geq \mu$ if and only if the least i for which $\lambda_i \neq \mu_i$ satisfies $\lambda_i > \mu_i$.

One can easily check that $\geq$ is a linear extension of $\trianglerighteq$ i.e. $\lambda \trianglerighteq \mu$ implies $\lambda \geq \mu$.

Give an partition λ, we define the conjugate partition of λ as $\lambda' = (\lambda'_1, \lambda'_2 \ldots)$, where $\lambda'_i = |\{i \,|\lambda_i \geq i\}|$.

Definition 8. A $\lambda-$tableau is one of the $n!$ array of integers obtained by replacing each node in $[\lambda]$ by one of the integers $1, 2 \ldots n$, without replacement.

example: The array $\begin{matrix} 1 & 2 \\ 3 & \end{matrix}$ is a $(2, 1)-$ tableau. The group S_n acts on $\lambda-$ tableau's in a natural way(by acting on each node individually). Many forms of the following lemma are going to be used to prove results about representations of S_n.

Lemma 9. *Let λ and μ be partitions of n, and t_1 and t_2 be λ and μ tableau respectively. If for all i, the integers in the ith row of t_2 belong to different columns of t_1, then $\lambda \trianglerighteq \mu$.*

Proof. No two of the μ_1 numbers in the first row t_2 are in the same column of t_1. So t_1 has at least μ_1 columns so we have $\lambda_1 \geq \mu_1$. Similarly no two of the μ_2 numbers in the second row of t_2 are in the same column of t_1. Therefore We must have $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$. Continuing in the way we get $\lambda \trianglerighteq \mu$. □

Definition 10. The Row stabilizer (Column-stabilizer) R_t (C_t) of a tableau t is the subgroup of S_n keeping the rows (columns) fixed set-wise.

It is simple to see that $R_{t\pi} = \pi^{-1} R_t \pi$ and $C_{t\pi} = \pi^{-1} C_t \pi$ for all $\pi \in S_n$. We now define an equivalence class on the set of $\lambda-$tableau's by $t_1 \sim t_2$ if and only if $t_1 \pi = t_2$ for some $\pi \in R_{t_1}$.

Definition 11. A tabloid $\{t\}$ is the equivalence class containing the tableau t.

Just as in the case of tableau, we represent a tabloid as an array of integers. The tabloid $\{t\}$, is represented as the diagram got by drawing lines between the rows of t. For example if $t = \begin{matrix} 1 & 2 \\ 3 & \end{matrix}$, then

$\{t\} = \begin{array}{|cc} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$ The group S_n acts on the set of $\lambda-$tabloids by $\{t\}\pi = \{t\pi\}$. This is well defined, since $t_1 \sim t_2$ implies for some $\sigma \in R_{t_1}$, $t_2 = t_1\sigma$. So we have $\pi^{-1}\sigma\pi \in \pi^{-1}R_{t_1}\pi = R_{t_1\pi}$, and therefore $\{t_1\pi\} = \{t_1\sigma\pi\} = \{t_2\pi\}$. Now we define a partial order $\trianglerighteq$ and a total order $\geq$ extending it on the set of $\lambda-$tabloids.

Definition 12. 1. Given a tableau t, let $m_{i,r}(t)$ be the total number of integers less than equal to i in the first r rows of t. We say $\{t_1\} \trianglerighteq \{t_2\}$ if and only if $m_{i,r}(t_1) \geq m_{i,r}(t_2)$ for all relevant i and r.

2. Write $\{t_1\} > \{t_2\}$ if and only if for some i

   - Whenever $j > i$, j and i are in the same row of $\{t_1\}$ and $\{t_2\}$
   - i is in a higher row of $\{t_2\}$ than $\{t_1\}$

Claim. *If $\{t_1\}$ and $\{t_2\}$ are $\lambda-$ tabloids, then $\{t_1\} \triangleleft \{t_2\}$ implies $\{t_1\} < \{t_2\}$.*

Proof. Assume $\{t_1\} \triangleleft \{t_2\}$ Consider the largest i such that $m_{i,r}(t_1) < m_{i,r}(t_2)$, for some r. If $j > i$, then $m_{j,r}(t_1) = m_{j,r}(t_r)$ for all r. This implies j is in the same row of both t_1 and t_2. It is clear by the choice of i, that i is in a higher row of t_1 than of t_2. Thus we have $\{t_1\} < \{t_2\}$. □

From the definition of $m_{i,r}(t)$, we get:

***Observation* 13.** *For $w < x$ with w and x being in the ath and bth row of t respectively, we have*

- $m_{i,r}(t(w,x)) - m_{i,r}(t) = 1$ *if $b \leq r < a$ and $w \leq i < x$*
- $m_{i,r}(t(w,x)) - m_{i,r}(t) = -1$ *if $a \leq r < b$ and $w \leq i < x$*

Here (w, x) is the transposition in S_n taking w to x and vice-versa. This gives us the following

Lemma 14. $\{t\} \triangleleft \{t(w,x)\}$ *if $w < x$ and w is lower than x in t.*

Lemma 15. *If $x-1$ is lower than x in t and t is a $\lambda-$tableau, then there is no $\lambda-$ tableau t_1 such that $\{t\} \lhd \{t_1\} \lhd \{t(x-1,x)\}$*

Proof. If s is any tableau with i in the rth row, the number

$$m_{i,j}(s) - m_{i-1,j}(s) = \text{the number of } i\text{'s in the first } j \text{ rows of } s.$$

Thus we have $m_{i,r}(s) - m_{i-1,r}(s) = 0$ if $r > j$ and $m_{i,r}(s) - m_{i-1,r}(s) = 1$ if $j \geq r$. Assume that the lemma is false. Let t_1 be the tableau such that $\{t\} \lhd \{t_1\} \lhd \{t(x-1,x)\}$ By Observation 13,

$$m_{i,r}(t) = m_{i,r}(t(x-1), s) \text{ if } i \neq x-1,$$

whence

$$m_{i,r}(t_1) = m_{i,r}(t) \text{ if } i \neq x-1$$

and

$$m_{i,r}(t) - m_{i-1,r}(t) = m_{i,r}(t_1) - m_{i-1,r}(t_1) \text{ if } i \neq x-1 \text{ or } x.$$

This implies that all numbers other than x and $x-1$ are in the same places in t and t_1. This requires either $\{t_1\} = \{t\}$ or $\{t_1\} = \{t(x-1,x)\}$. □

3.2 Specht Modules

Let $[n] = \{1,2,3,\ldots,n\}$. If $X \subset [n]$, S_X is the subgroup of S_n which fixes elements of $[n]$ outside X. Given a partition λ of n, the Young subgroup S_λ associated with it is the subgroup $(S_{\{1,2,\ldots\lambda_1\}} \times S_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \ldots \times S_{\{\lambda_i+1,\ldots,\lambda_i+\lambda_{i+1}\}} \times \ldots)$ of S_n. Let F be any arbitrary field and $\mu \vdash n$, then define M_F^μ to be the $F-$vector space spanned by the set of $\mu-$tabloids as a basis. Extending the natural action of S_n on the set of μ tabloids, makes M_F^μ an FS_n module. Since S_n acts transitively on the set of $\mu-$tabloids with the all the isotropy subgroups isomorphic (via conjugation) to S_μ, we have

Lemma 16. *M_F^μ is an FS_n- module that can be associated to the permutation representation obtained by the action of S_n on the right cosets of S_μ. M_F^μ is a cyclic module generated by any single $\mu-$tabloid, and $dim(M^\mu) = n!/(\mu_1!\mu_2!\ldots)$.*

If t is a tableau, we define the signed sum κ_t as $\sum_{\pi\in C_t} sgn(\pi)\pi$ and the polytabloid e_t as $\{t\}\kappa_t$.

Definition 17. The Specht module S_F^μ for the proper-partition μ is the submodule of M_F^μ spanned by the polytabloids.

Remark 18. Give a $\mu-$ tableau t, let $\rho_t = \sum_{\pi \in R_t} \pi$. The map $\theta : \rho_t\sigma \rightarrow \{t\}\sigma$ $(\sigma \in S_n)$, gives an FS_n module isomorphism from the ideal $\rho_t FS_n$ to M^μ_F. Restriction of θ to the ideal $\rho_t\kappa_t FS_n$ is an isomorphism onto the Specht module S^μ_F.

We say that a $\mu-$tabloid $\{t\}$ is involved in an element $v \in M^\mu$ if it has a non-zero coefficient in the representation of v as the unique linear combination of $\mu-$tabloids. As $\kappa_{t\pi}\pi = \pi\kappa_t$, we have $e_{t\pi} = e_t\pi$ and therefore S^μ_F is a cyclic module generated by any one of the $\mu-$ polytabloids. Let $<,>$ be the unique bilinear form on M^μ_F for which the set of μ- tabloids is a orthonormal basis. Clearly this is a symmetric, S_n- invariant, non-singular bilinear form on M^μ_F irrespective of the field F. The following theorem by James is the first step in classifying irreducible modules of FS_n.

Theorem 19 (The submodule theorem)**.** *If U is a FS_n submodule of M^μ_F, then either $U \supset S^\mu_F$ or $U \subset S^{\mu\perp}_F$.*

In order to prove this theorem, we need the following lemma and its corollary.

Lemma 20. *Let λ and μ be partitions of n If t is a $\lambda-$tableau and t^* a $\mu-$ tableau such that $\{t^*\}\kappa_t \neq 0$. Then $\lambda \trianglerighteq \mu$, and if $\lambda = \mu$, then $\{t^*\}\kappa_t = \pm e_t$.*

Proof. Let a, b be two numbers in the same row of t^*. If a, b are in the same column of t as well, then the transposition $(a, b) \in C_t$. Let $\pi_1 \dots \pi_k$ be the coset representatives of $\{e,\ (a, b)\}$ as a subgroup of C_t. Without loss Of generality, we may assume $sgn(\pi_j) = 1$ for all $1 \leq j \leq k$. Thus we have $\kappa_t = (1 - (a, b))(\pi_1 + \dots \pi_k)$. Since a, b are in the same row of t^*, we get $\{t^*\}\kappa_t = \{t^*\}(1 - (a, b))(\pi_1 + \dots \pi_k) = 0$. This is contrary to our hypothesis that $\{t^*\}\kappa_t \neq 0$. So for all i, the numbers in ith row of t^* belong to different columns of t. Application of Lemma 9 implies $\lambda \trianglerighteq \mu$. Also if $\lambda = \mu$, then $\{t^*\}$ is involved in $\{t\}\kappa_t$, by construction. Therefore $\{t^*\} = \{t\}\pi$ for some $\pi \in C_t$. Hence $\{t^*\}\kappa_t = \pm\{t\}\kappa_t$. □

The following corollary follows from that fact that the set of $\mu-$tabloids form a basis of M^μ

Corollary 21. *If $u \in M^\mu_F$, and t is a $\mu-$tableau, then $u\kappa_t$ is a multiple of e_t*

Proof of the Submodule Theorem

Let t be any $\mu-$tableau. For $u, v \in M^{\mu}$, we have

$$< u\kappa_t,\ v > = \sum_{\pi \in C_t} < sgn(\pi)u\pi,\ v > \ \ (<,> \text{ is a bilinear form}) \tag{3.1}$$
$$= \sum_{\pi \in C_t} < u,\ sgn(\pi)v\pi^{-1} > \ \ (<,> \text{ is } S_n\text{-invariant}) \tag{3.2}$$
$$=< u,\ v\kappa_t > \tag{3.3}$$

Let U be any submodule of M^{μ}_F and let $u \in U$. By the above corollary, we have $u\kappa_t$ is a multiple of e_t. If $u\kappa_t \neq 0$, $e_t \in U$ and thus $S^{\mu}_F \subset U$. If for all $u \in U,\ u\kappa_t = 0$, by 3.1 we have

$$0 =< u\kappa_t, \{t\} >=< u, \kappa_t\{t\} >=< u, e_t >$$

i.e $S^{\mu\perp}_F \supset U$. Hence proved.

Define $D^{\mu}_F = S^{\mu}_F / S^{\mu}_F \cap S^{\mu\perp}_F$. By the results in the first section, we know D^{μ}_F is a self dual module.

Note: Unless otherwise mentioned, F is any arbitrary field and $S^{\mu}_F = S^{\mu}$ and $M^{\mu}_F = M^{\mu}$.

Theorem 22. *D^{μ}_F is zero or an absolutely irreducible FS_n module. Moreover if D^{μ}_F is non-zero, then $S^{\mu}_F \cap S^{\mu\perp}_F$ is the unique maximal ideal of S^{μ}_F, and D^{μ}_F is self dual.*

Proof. Let U be any submodule of S^{μ}_F, by submodule theorem, $U = S^{\mu}_F$ or $U \subset S^{\mu}_F \cap S^{\mu\perp}_F$. It is now clear that D^{μ}_F is either zero or irreducible. By Theorem 3, $dim(D^{\mu})_F$ is the rank of the Gram matrix with respect to any basis of S^{μ}. As polytabloids span S^{μ}_F, we can consider a basis inside the set of polytabloids. Since all the tabloids involved in a polytabloid have coefficient ± 1. This implies:

1. $dim_F(S^{\mu}_F) = dim_E(S^{\mu}_E)$ for any extension E of F.

2. The rank of the gram matrix of S^{μ}_F with respect to the polytabloid basis is same as its rank over the prime field k of F. Therefore the dimensions of D^{μ}_F as $F-$space and D^{μ}_k as k-vector space are the same.

One can establish an ES_n isomorphism between M^{μ}_E and $M^{\mu}_F \otimes_F E$, by sending $\{t\} \otimes 1$ to the a tabloid $\{t\}$ (t is any $\mu-$tableau.) By the rank nullity theorem and 1, this map upon restriction to $S^{\mu}_{F} \otimes_F E$ is an isomorphism onto S^{μ}_{E}. The same map sends $S^{\mu^{\perp}}_{F} \otimes_F E$ to $S^{\mu^{\perp}}_{E}$. Since $dim(S^{\mu}_{F} \cap S^{\mu\perp}_{F}) = dim(S^{\mu}_{F}) - dim(D^{\mu}_{F})$ and $dim(S^{\mu}_{E} \cap S^{\mu\perp}_{E}) = dim(S^{\mu}_{E}) - dim(D^{\mu}_{E})$, we have $dim(S^{\mu}_{F} \cap S^{\mu\perp}_{F}) = dim(S^{\mu}_{E} \cap S^{\mu\perp}_{E})$ and thus $S^{\mu}_{F} \cap S^{\mu\perp}_{F} \otimes_F E \simeq S^{\mu}_{E} \cap S^{\mu^{\perp}}_{E}$ under the map defined in this paragraph (by 1,2 in the previous paragraph). Thus we have $D^{\mu}_{E} \simeq D^{\mu}_{F} \otimes_F E$. This implies D^{μ}_{E} is non-zero and hence irreducible. Therefore D^{μ}_{F} is absolutely irreducible. □

3.3 Standard basis of specht modules

In this section, we find a basis of Specht modules consisting of polytabloids. Recall that given a set $X \subset \{1, 2 \ldots, n\}$, the group S_X is the subgroup of S_n which fixes all the elements outside X.

Definition 23 (Garnir Element)**.** Suppose that t is a given $\mu-$tableau. Let X and Y be subsets of the i^{th} and $(i+1)$st column of t for some i, and $\sigma_1, \sigma_2 \ldots \sigma_k$ be the coset representatives of the group $S_X \times S_Y$ in the group $S_{X \times Y}$. The element $G_{X,Y} = \sum_{i=1}^{k} sgn(\sigma_i)\sigma_i$ is called a Garnir element.

For all practical purposes, we take X to be the bottom most $|X|$ elements of the ith column and Y to be the topmost $|Y|$ elements of the $(i+1)$st column. Also we choose the coset representatives $\sigma_1, \sigma_2 \ldots \sigma_k$ in such a way that $t\sigma_1, t\sigma_2 \ldots t\sigma_k$ agree with t except on $X \cup Y$.

Theorem 24. *If $|X \cup Y| > \mu'_i$, then $e_t G_{X,Y} = 0$ for any base field.(μ'_i = length of the ith column.)*

Proof. Define $\overline{S_X S_Y} := \sum\{sgn(\sigma)\sigma \mid \sigma \in S_X \times S_Y\}$ and $\overline{S_{X \cup Y}} = \sum\{sgn(\sigma)\sigma \mid \sigma \in S_{X \cup Y}\}$. Now $|X \cup Y| > \mu'_i$, for all $\pi \in C_t$, a pair of numbers in $X \cup Y$ is in the same row of $t\pi$. Thus we have $\{t\pi\}\overline{S_{X \cup Y}} = 0$ and hence $\{t\}\kappa_t \overline{S_{X \cup Y}} = 0$. It is easy to verify that $\overline{S_X S_Y}$ is a factor of κ_t and $\overline{S_{X \cup Y}} = \overline{S_X S_Y} G_{X,Y}$. Therefore we have $0 = \{t\}\kappa_t \overline{S_{X \cup Y}} = |X|!|Y|! G_{X,Y}$. When the base field is $\mathbb{Q}$, we have $\{t\}\kappa_t G_{X,Y} = 0$. As the tabloid coefficients are integers, the result holds for any field. □

Definition 25. A tableau t is called a standard tableau if the numbers increase along the rows (left to right) and down the columns.

We shall prove that the set $\{e_t \mid t$ t is a standard $\mu-$tableau$\}$ is a basis for S^{μ} defined over any field.

Lemma 26. *If t has numbers increasing down the columns, then all the tabloids $\{t'\}$ involved in e_t satisfy $\{t'\} \trianglelefteq \{t\}$.*

Proof. If $\{t'\} \neq \{t\}$ is involved in e_t, $\{t'\} = \{t\}\pi$ for some $\pi \in C_t$. Since t is standard, in some column of t' there are integers $w < x$ such that w is lower than x. By Lemma14 we have $\{t'\} \triangleleft \{t'(w, x)\}$. If $t'(w, x)$ has its entries increasing down the columns, then obviously $\{t'(w, x)\} = \{t\}$. If this is not the case, repeat the process till we reach $\{t\}$. Hence the result. □

Theorem 27. *$S = \{e_t | t$ is a standard $\mu-$tableau$\}$ is a linearly independent set of S^μ.*

Proof. The above lemma shows that all the tabloids $\{t'\}$ involved in e_t, where t is a standard tableau satisfy $\{t'\} \trianglelefteq \{t\}$. In other words, if t is standard, $\{t\}$ is the last tabloid involved in e_t Let $\{e_{t_1}, e_{t_2} \dots e_{t_m}\}$ be the set S, where $\{t_1\} < \{t_2\} < \dots < \{t_m\}$. For every $1 < i \leq m$, e_{t_i} is not an element of span of $\{e_{t_1}, e_{t_2} \dots e_{t_{i-1}}\}$. This is because $\{t_i\} > t_j$ for all $1 \leq j \leq i - 1$. Thus by induction, the set S is a linearly independent set. □

From now on we refer to the elements of S as standard polytabloids. We now prove that the set S is in fact a basis for S^μ. For this, we shall apply Theorem 24. We define a new equivalence class on set of $\mu-$tableaux by $t_1 \sim t_2$ if and only if $t_1\pi = t_2$ for some $\pi \in C_{t_1}$. Notice that this is similar to the equivalence class used to define $\mu-$tabloids (we just replace R_t by C_t). Therefore on the equivalences classes $[t]$, we may define a total order in a way similar to the total order on tabloids. If t is not standard, by induction, we may assume that for all $[t'] < [t]$, e_t is a linear combination of standard polytabloids and prove that the results holds for t as well. Since for every $\pi \in C_t$, $e_t\pi = sgn(\pi)e_t$, we may assume without loss of generality that entries in t increase down the order. If t is not standard, there are two columns say jth and $j + 1$st with entries $a_1 < a_2 \dots a_r$ and $b_1 < b_2 < \dots < b_r$ respectively with $a_q > b_q$ for some q. Now consider the Garnir element $G_{X,Y}$ for the sets $X = \{a_1 \dots a_r\}$ and $Y = \{b_1 \dots b_q\}$. If $G_{X,Y} = \sum sgn(\sigma)\sigma$, by Theorem24, we have

$$0 = e_t \sum sgn(\sigma)\sigma = \sum sgn(\sigma)e_{t\sigma}.$$

Since $b_1 < \dots < b_q a_q < a_{q+1} \dots a_r$, we have $[t\sigma] < [t]$. Because $e_t = -\sum_{\sigma \neq 1} sgn(\sigma)e_{t\sigma}$, e_t is a linear combination of standard polytabloids. Therefore we have:

Theorem 28. *The set of standard $\mu-$polytabloids forms a basis for the specth module S^μ.*

Now we see some application of this standard basis.

Lemma 29. *If $v \in S^{\mu}_{\mathbb{Q}}$, and the coefficients of tabloids involved in v are all integers, then v is an integral linear combination of the standard polytabloids.*

Proof. We assume $v \neq 0$, otherwise, the lemma is vacuously true. Let $\{t\}$ be the last tabloid involved in v under the total order on tabloids. Since v can be written as a linear combination of standard polytabloids, by Lemma 26, the last tabloid involved in v is standard. Again Lemma 26 shows that the last tabloid involved in $v - ae_t$(here $a =< v, \{t\} >\in \mathbb{Z}$) is a standard tabloid $\{t'\}$ with $t' < t$. So by induction $v - ae_t$ is an integral linear combination of standard polytabloids. Therefore even v is an integral linear combination of standard polytabloids □

Corollary 30. *The matrices representing S_n over $\mathbb{Q}$ with respect to the basis of standard polytabloids of $S^{\mu}_{\mathbb{Q}}$ are integral matrices.*

Proof. We know that $e_t\pi = e_{t\pi}$. Now apply the lemma on $e_{t\pi}$. □

Corollary 31. *If $v \in S^{\mu}_{\mathbb{Q}}$ and the coefficients of tabloids involved in v are integers, then we may reduce all the integers modulo p and obtain an element of S^{μ}_{F}, where F is the Galois field of size p.*

Proof. We may consider $S^{\mu}_{\mathbb{Q}}$ and S^{μ}_{F} to be the $\mathbb{Q}$ and F span respectively of the set of $\mu-$polytabloids. By the above lemma, we have $v = \sum_i a_i e_i$ for some $a_i \in \mathbb{Z}$ and standard polytabloids e_i. Clearly the element $\bar{v} = \sum_i a_i(mod p)e_i$ is an element of S^{μ}_{F}. □

This corollary gives the following:

Theorem 32. *If F is the Galois field of size p, then S^{μ}_{F} is the p-modular representation of S_n obtained from $S^{\mu}_{\mathbb{Q}}$.*

***Observation* 33.** *Suppose that $\{t_1\} < \{t_2\} \ldots \{t_m\}$ are the standard $\mu-$tabloids. Now the only standard tabloid involved in e_{t_1} is t_1. If the coefficient of $\{t_1\}$ in e_{t_2} is a, then then only tabloid involved in $f_2 = e_{t_2} - a\{t_1\}$ is $\{t_2\}$. Continuing in this way, we get the basis $\{f_1, f_2 \ldots f_m\}$ with the property that each element involved a unique standard tabloid.*

The next lemma helps us to construct elements of $Hom_{FS_n}(M^{\lambda}_{F}, M^{\mu}_{F})$ from certain kind of elements of $Hom_{\mathbb{Q}S_n}(M^{\lambda}_{\mathbb{Q}}, M^{\mu}_{\mathbb{Q}})$ (here F is the Galois field of size p).

Lemma 34. *Suppose that $\theta \in Hom_{\mathbb{Q}S_n}(M^{\lambda}_{\mathbb{Q}}, M^{\mu}_{\mathbb{Q}})$ and that all tabloids involved in $\theta(\{t\})$ have integer coefficients ($\{t\} \in M^{\lambda}_{\mathbb{Q}}$). Then reducing all the integers modulo p, we obtain an element $\bar{\theta}$ of $Hom_{FS_n}(M^{\lambda}_{F}, M^{\mu}_{F})$($F$ is the field with p elements). Moreover if $ker(\theta) = S^{\lambda^{\perp}_{\mathbb{Q}}}$, then $ker(\bar{\theta}) \supset S^{\lambda\perp}_{F}$.*

Proof. By construction, $\bar{\theta} \in Hom_{FS_n}(M_F^\lambda, M_F^\mu)$.

Take a basis $f_1, f_2, \ldots f_k$ of $S_{\mathbb{Q}}^{\lambda^\perp}$ and extend by the standard basis of $S_{\mathbb{Q}}^{\lambda}$ to a basis $f_1, f_2 \ldots f_m$ of $M_{\mathbb{Q}}^\lambda$. Define $N = (n_{ij})$ to be the matrix with $n_{i,j} =< f_i, \{t_j\} >$. We may assume that N has integer entries. By row reducing the first k rows of N, we may assume that the first k rows of N (these correspond to the basis of $S_{\mathbb{Q}}^{\lambda^\perp}$) are linearly independent modulo p. Reducing entries of N modulo p, we obtain a set of m vectors, of which $m-k$ form the standard basis for S_F^λ and the other k are linearly independent and orthogonal to standard basis of S_F^λ. These k vectors form a basis for $S_F^{\lambda^\perp}$ because $dim(S_F^{\lambda^\perp}) = dim(M_F^\mu) - dim(S_F^\lambda) = k$. Thus we have constructed a basis of $S_{\mathbb{Q}}^{\lambda^\perp}$, whose vectors give a basis for $S_F^{\lambda^\perp}$, when coefficients of tabloids involved in them are reduced modulo p. Let B be this basis and $\bar{B}$ be the basis of $S_{\mathbb{F}}^{\lambda^\perp}$ obtained from B. Clearly if $\theta(B) = 0$, then $\bar{B} \subset ker(\bar{\theta})$ and hence the result. □

Chapter 4

Irreducible Representations of Symmetric Groups.

4.1 Classification of ordinary irreducible representations of S_n

In this section we show that for any field F of characteristic 0, $\{D^{\mu}_F \mid \mu \vdash n\}$ is the set of inequivalent irreducible representations. If $F = \mathbb{Q}$, $<,>$ is an inner product and therefore $S^{\mu}_{\mathbb{Q}} \cap S^{\mu\perp}_{\mathbb{Q}} = 0$. From the proof of Theorem 44, we may deduce that $S^{\mu}_{\mathbb{F}} \cap S^{\mu\perp}_{\mathbb{F}} = 0$ for any field F of characteristic zero ($\mathbb{Q}$ is the prime subfield of F). Therefore if $char(F) = 0$, the set $\{S^{\lambda}_F \mid \lambda \vdash n\}$ is a set of irreducible FS_n modules. The following lemma is useful in showing Specht modules corresponding to different proper-partitions are in fact inequivalent

Lemma 35. *Let θ be an element of $Hom_{FS_n}(M^{\lambda}, M^{\mu})$, such that $S^{\lambda} \nsubseteq ker(\theta)$, then $\lambda \trianglerighteq \mu$. Moreover if $\lambda = \mu$, then the restriction of θ to S^{λ} is a multiplication by a constant.*

Proof. Suppose that t is a $\lambda-$tableau. Since $e_t \notin ker(\theta)$, we have

$$0 \neq \theta(e_t) = \theta(\{t\}\kappa_t) = \theta(\{t\})\kappa_t = \text{a linear combination of } \mu\text{−tabloids}$$

So we have a $\mu-$tabloid $\{t*\}$ such that $\{t*\}\kappa_t \neq 0$ and thus by Lemma20 we get $\lambda \trianglerighteq \mu$. Schur's lemma gives us the proof of remainder of the lemma. □

We have seen that if $char(F) = 0$, $S^{\mu}_{\mathbb{F}} \cap S^{\mu\perp}_{\mathbb{F}} = 0$ and thus we get $M^{\mu}_F = S^{\mu}_F \oplus S^{\mu\perp}_F$. This implies that any non-zero element f of $Hom_{FS_n}(S^{\lambda}_F, M^{\mu}_F)$ can be extended to an element θ of

$Hom_{FS_n}(M^{\lambda}, M^{\mu})$, such that $ker(\theta) = S^{\lambda^{\perp}_F}$. This is because $ker(f) = 0$ as S^{μ}_F is irreducible. Application of the above lemma gives that $\lambda \trianglerighteq \mu$ if $Hom_{FS_n}(S^{\lambda}_F, M^{\mu}_F) \neq 0$. Thus we have:

Lemma 36. *$S^{\mu}_F \simeq S^{\nu}_F$ if and only if $\mu = \nu$.*

Proof. The discussion prior to the lemma proves that $\lambda \trianglerighteq \mu$ and $\mu \trianglerighteq \lambda$ and thus $\lambda = \mu$ □

Since the number of inequivalent ordinary irreducible representations of S^n is equal to number of proper-partitions of n, we have the following:

Theorem 37. *The Specht modules over $\mathbb{Q}$ are self dual and absolutely irreducible and give all the ordinary irreducible representations of S_n*

4.2 Classification of modular irreducible representations of S_n

In this section let F be any field of characteristic p, a prime. Since most of the results in this section depend only on $char(F)$, we denote M^{λ}_F, S^{λ}_F, D^{λ}_F by M^{λ}, S^{λ}, D^{λ} respectively. We have seen that D^{μ} is either irreducible or zero. $D^{\mu} \neq 0$ for certain kind of partitions called p-regular partitions.

Definition 38. A partition μ is called p-singular if for some i,

$$\mu_{i+1} = \mu_{i+2} = \ldots = \mu_{i+p} > 0.$$

Otherwise μ is called p-regular partition

A conjugacy class of S_n is called p-regular if the order of elements in it is not divisible by p.

Lemma 39. *The number of p-regular conjugacy classes of S_n is same as the number of p-regular partitions of n*

Proof. The number of p-regular conjugacy classes of S_n is same as the number of partitions μ of n where no non-zero part μ_i is divisible by p. We simplify the following ratio in two different ways to prove the lemma.

$$\frac{(1 - x^p)(1 - x^{2p})\ldots}{(1 - x)(1 - x^2)\ldots}$$

- Cancel equal factors $(1 - x^{mp})$ in the numerator and denominator. This leaves

$$\prod_{p\nmid i}(1 - x^i)^{-1} = \prod_{p\nmid i}(1 + x^i + x2i + x^{3i} + \ldots).$$

  Let the partition $(1^a, 2^b, 3^c \ldots)$ correspond to the multiplication of x^a from the 1st infinite sum, x^{2b} from the second infinite sum and so on. This correspondence shows that the co-efficient of x^n in the ratio is equal to number of partitions of μ of n such that no part μ_i is divisible by p.

- Now we rearrange the ratio to look as follows

$$\prod_{m=1}^{\infty} \frac{(1 - x^{mp})}{1 - x^m} = \prod_{m=1}^{\infty}(1 + x^m + x^{2m} \ldots x^{(p-1)m}).$$

  One can see that here, the coefficient of x^n is equal to number of p-regular partitions of n.

Comparing the coefficients of x^n in the above methods of simplification of the ratio gives us the lemma. □

Define g^μ to be $gcd(\{< e_t, e_{t*} > \ |e_t \, \& e_{t*}$ are polytabloids in$S^\mu_{\mathbb{Q}}\})$.

Lemma 40. *Suppose that μ is a partition with z_j parts equal to j. Then $\prod_{j=1}^{\infty}(z_j)!$ divides g^μ and g^μ divides $\prod_{j=1}^{\infty}(z_j!)^j$*

Note: Since all partitions have finitely many parts and $0! = 1$, there is no problem in taking the infinite products in the lemma.

Proof. Define an equivalence relation $\sim$ on the set of tabloids as follows: $\{t_1\} \sim \{t_2\}$ if and only if for all i and j, i and j belong to the same row of $\{t_2\}$ when they belong to the same row of $\{t_1\}$. In other words, we can go from $\{t_1\}$ to $\{t_2\}$ by "shuffling" rows. Clearly the size of each equivalence class is $\prod_{j=1}^{\infty}(z_j)!$. If $\{t_1\}$ is involved in a polytabloid e_t and we have $\{t_1\} \sim \{t_2\}$, then the definition of e_t shows that $\{t_2\}$ is involved in e_t. Moreover the sign of the coefficient of $\{t_2\}$ depends only on the sign of coefficient of $\{t_1\}$. This gives us that any two polytabloids have a multiple of $\prod_{j=1}^{\infty}(z_j)!$ tabloids in common and that $\prod_{j=1}^{\infty}(z_j)!$ divided the g^μ. Now let t be any tableau and $t*$ be the tableau obtained by reversing the order of numbers in the rows of t. Let $\pi \in C_t$ such that both i and $i\pi$ are in rows of t with the same lengths. Then $t\pi$ is involved in both e_t and e_{t*} with the same coefficient. Conversely any common tabloid of e_t and e_{t*} has this form. Therefore $< e_t, e_{t*} >= \prod_{j=1}^{\infty}(z_j!)^j$, and thus g^μ divides $\prod_{j=1}^{\infty}(z_j!)^j$ □

Corollary 41. *A prime p divides g^μ if and only if μ is p-singular.*

Proof. μ is p-singular if and only if p divides $z_j!$ for some j. By the above theorem, this happens if and only if p divides g^μ. □

Corollary 42. *If $t*$ is obtained by reversing the order of elements in each row of t, then $e_{t*}\kappa_t$ is a multiple of e_t, and this multiple is co-prime to p if and only if p is $\mu-$regular.*

Proof. Corollary of Lemma 20 shows that $e_{t*}\kappa_t$ is a multiple of e_t, $e_{t*}\kappa_t = he_t$ say. We have

$$h = h < e_t, \{t\} >=< he_t, \{t\} >=< e_{t*}\kappa_t, \{t\} >=< e_{t*}, e_t > .$$

In the proof of the above theorem, we have seen that $h = \prod_{j=1}^{\infty}(z_j!)^j$, which is co-prime to p if and only if μ is p-regular. □

Theorem 43. *D^μ is zero if and only if μ is p-singular.*

Proof. $S^\mu \subset S^{\mu^\perp}$ if and only if $< e_t, e_{t*} >= 0$ for all μ-tableau's t and $t*$. This is equivalent to p dividing g^μ. Lemma 40, implies the result. □

This proves that the set $\{D^\mu | \ \mu \ p\text{-}$ regular proper-partition of $n\}$ is a set of irreducible representations of S_n over the field F of characteristic p. We have shown that $\mathbb{Q}$ is a splitting field of S_n i.e all irreducible representations over $\mathbb{Q}$ are absolutely irreducible. We assume the following results from general representation theory:

1. If F is a splitting field of G and $char(F) = p \neq 0$, the number of irreducible representations of G over F is same as the number of p-regular conjugacy classes of G.

2. If $\mathbb{Q}$ is a splitting field of a group G, then every field F is a splitting field of F.

These results have been taken from Curtis and Riener [2](83.5 on page 591 and 83.7 on page 592). Once we prove that D^μ and D^ν are inequivalent representations for distinct p-singular partitions, we have:

Theorem 44. *Suppose that F is a field of characteristic $p \neq 0$. As μ varies over $\mu-$ regular proper-partitions of n, D^μ varies over a complete set of inequivalent FS_n modules. Moreover each D^μ is absolutely irreducible and self dual.*

We use the following lemma to prove that $D^\mu \ncong D^\nu$ for distinct p-regular partitions μ and ν.

Lemma 45. *Let λ and μ are partitions of n with λ being p-regular. Assume that U is an M^μ submodule such that there is a a non zero $\theta \in Hom_{FS_n}(S^\lambda, M^\mu/U)$. Then $\lambda \trianglerighteq \mu$ and if $\lambda = \mu$, we have $Im(\theta) \subset (S^\mu + U)/U$.*

Proof. Let t be a $\lambda-$tableau and $t*$ be the tableau obtained by reversing the order of elements in each row of t. By Corollary 41, we have $e_{t*}\kappa_t = he_t$ where $h \neq 0$(because λ is p-regular). As $\theta \neq 0$ and $h \neq 0$, we have $\theta(e_{t*}\kappa_t) = \theta(e_{t*})\kappa_t \notin U$. Now Lemma 20 implies that $\lambda \trianglerighteq \mu$. Now if $\mu = \lambda$, $\theta(e_t) = \theta(h^{-1}e_{t*})\kappa_t = me_t + U$ for some m in the prime field of F (again by Lemma 20). The lemma follows because S^λ is generated by e_t. □

Corollary 46. *Let λ and μ are partitions of n with λ being p-regular. Assume that U is an M^μ submodule such that there is a a non zero $\theta \in Hom_{FS_n}(D^\lambda, M^\mu/U)$. Then $\lambda \trianglerighteq \mu$ and $\lambda \triangleright \mu$ if $S^\mu \subset U$.*

Proof. We lift $\theta \neq 0$ onto S^λ as follows

$$S^\lambda \xrightarrow[Canonical]{} D^\lambda \xrightarrow[\theta]{} M^\mu/U$$

By the above lemma $\lambda \trianglerighteq \mu$. Now if $\lambda = \mu$, then $Im(\theta)$ is a non-zero submodule of $S^\mu + U/U$, so U does not contain S^μ. □

If $D^\lambda \cong D^\mu$ for two distinct p-regular partitions μ and λ, by the above corollary, we have $\lambda \trianglerighteq \mu$ and $\mu \trianglerighteq \lambda$. This completes the proof of Theorem 44.

Chapter 5

Semistandard homomorphisms

In this chapter we find a basis for $Hom_F(S^\lambda_F, M^\mu_F)$ except in the case $Char(F) = 2$ and λ is $2-$singular. We have already seen from Lemma 16, that M^μ is isomorphic to the permutation module associated with the action of S_n on S_n/S_μ. A tableau T of shape λ and type μ is one of the $n!/(\prod_i \mu_i!)$ objects obtained by replacing the nodes of $[\lambda]$ by μ_1 $1'$s, μ_2 $2'$s, so on. Denote the set of tableau of shape λ and type μ by $\mathcal{T}(\lambda,\mu)$. One can see that the set of $\lambda-$tableaux is same as $\mathcal{T}(\lambda,(1^n))$.

example: $\begin{matrix} 2 & 2 & 1 & 1 \\ 1 & 2 & & \end{matrix}$ is a tableau of shape $(4, 2)$ and type $(3, 3)$.

For this section, fix t to be a given $\lambda-$tableau. If $T \in \mathcal{T}(\lambda,\mu)$, define $T(i)$ to be the entry in T, which occurs in the same position as i occurs in t. We define the action of S_n on $\mathcal{T}(\lambda,\mu)$ by $(T\pi)(i) = T(\pi^{-1}(i))$. This is a transitive action on $\mathcal{T}(\lambda,\mu)$, with all the isotropy subgroups isomorphic to S^μ. Thus we may define M^μ_F to be the $F-$vector space spanned by $\mathcal{T}(\lambda,\mu)$. We make it a FS_nmodule of S_n action $\mathcal{T}(\lambda,\mu)$.

If $T_1, T_2 \in \mathcal{T}(\lambda,\mu)$, we say T_1 and T_2 are row equivalent (respectively, column equivalent) if $T_2 = T_1\pi$ for some $\pi \in R_t$ (respectively C_t).

Definition 47. If $T \in \mathcal{T}(\lambda,\mu)$, define $\theta_T \in Hom(M^\lambda, M^\mu)$ by the extending the relation

$$\theta_T(\{t\}) = \sum\{T_1 |\ T_1 \text{ is row equivalent to } T\}$$

to the unique S_n-invariant linear transformation.

Now we prove the simple

Theorem 48. *$C := \{\theta_T |\ T \in \mathcal{T}(\lambda,\mu)$ and the numbers are non-decreasing along each row of $T\}$*

Proof. The set $A = \{T \in \mathcal{T}(\lambda, \mu)$ and the numbers are non-decreasing along each row of $T\}$ is a set of representatives of the row equivalence classes of $\mathcal{T}(\lambda, \mu)$. The definition of θ_T and the fact that $\mathcal{T}(\lambda, \mu)$ is a basis of M^μ proves that C consists of linearly independent homomorphisms.

Suppose θ is a non-zero element of $Hom_{FS_n}(M^\lambda, M^\mu)$. If T and T' are row equivalent, then $T' = T\pi$ for some $\pi \in R_t$, and therefore

$$< \theta(\{t\}), T' >=< \theta(\{t\}), T\pi >=< \theta(\{t\}\pi^{-1}), T >=< \theta(\{t\}), T > .$$

Since A is a set of representatives of row equivalence classes, we have

$$\theta(\{t\}) = \sum_{T \in A} < \theta(\{t\}), T > \theta_T(\{T\}).$$

Since M^λ is a cyclic modules, we can say that θ is infact in the linear span of C. □

One can verify that

***Observation* 49.** $T\kappa_t = 0$ *if and only if some column of* T *contains two identical numbers.*

Define $\widehat{\theta^T}$ as the restriction of θ_T to S^λ. Now if T has two identical numbers in some column, then by the above observation, we have $\theta_T(S^\lambda) = 0$. To eliminate such cases, we consider the kind of tableau defined below.

Definition 50. A tableau $T \in \mathcal{T}(\lambda, \mu)$ is called semi-standard if the numbers in T are non-decreasing along the rows of T(left to right) and strictly increasing down the column. The set of semi-standard tableaux of type μ and shape λ is denoted by $\mathcal{T}_0(\lambda, \mu)$.

We defined column equivalence of two elements of $\mathcal{T}(\lambda, \mu)$. Denote the equivalence class containing T by $[T]$. We now define a partial order on the column equivalence classes.

Definition 51. Let $[T_1] \blacktriangleleft [T_2]$ if $[T_2]$ can be obtained form $[T_1]$ by interchanging w and x, where w belongs to a later column than x and $w < x$. We now define a partial order $\lhd$ by $T_1 \lhd T_2$ if and only if there is a chain of the form $[T_1] \blacktriangleleft [T_{i_1}] \blacktriangleleft [T_{i_2}] \blacktriangleleft \ldots \blacktriangleleft [T_{i_k}] \blacktriangleleft [T_2]$.

It is trivial to see that:

***Observation* 52.** *If* T *is semi-standard, and* T' *is row equivalent to* T, *then* $[T'] \lhd [T]$ *unless* $T' = T$.

Lemma 53. $\{\widehat{\theta}\}_T | T \in \mathcal{T}_0(\lambda,\mu)\}$ *is a linearly independent subset of* $Hom_{FS_n}(S^\lambda, M^\mu)$.

Proof. Consider any linear combination $\sum_{T\in\mathcal{T}_0(\lambda,\mu)} a_T\theta_T$, such that not all a_T's are zero. Let T_1 be such that $a_{T_1} \neq 0$ and $a_T = 0$ for all T such that $[T_1] \lhd [T_1]$. From the above observation,

$$\sum_{T\in\mathcal{T}_0(\lambda,\mu)} a_T\theta_T(\{t\}) = a_{T_1}T_1 + \text{a linear combination of tableaux } T_2 \text{ such that } [T_1] \ntrianglelefteq [T_2].$$

As $T_1 \in \mathcal{T}_0(\lambda,\mu)$, we have $T\kappa_t \neq 0$ and also by definition of $[T]$, we have $[T\kappa_t] = [T]$. This shows that

$$\sum_{T\in\mathcal{T}_0(\lambda,\mu)} a_T\theta_T(\{t\}\kappa_t) = \sum_{T\in\mathcal{T}_0(\lambda,\mu)} a_T\theta_T(\{t\})\kappa_t$$
$$= (a_{T_1}T_1 + \text{a linear combination of tableaux } T_2 \text{ such that } [T_1] \ntrianglelefteq [T_2])\kappa_t \neq 0.$$

Therefore $\sum_{T\in\mathcal{T}_0(\lambda,\mu)} a_T\widehat{\theta_T}$ is a non-zero element of $Hom_{FS_n}(S^\lambda, M^\mu)$. □

We shall prove that $\{\widehat{\theta}\}_T | T \in \mathcal{T}_0(\lambda,\mu)\}$ is a basis of $Hom_{FS_n}(S^\lambda, M^\mu)$ unless $char(F) = 2$ and λ is $2-$singular.

Lemma 54. *Suppose that* $\theta \in Hom_{FS_n}(S^\lambda, M^\mu)$ *is a non-zero homomorphism and* $\theta(e_t) = \theta(\{t\}\kappa_t) = \sum_{T\in\mathcal{T}(\lambda,\mu)}(S^\lambda, M^\mu)c_T T$ *for some* $c_T \in F$. *Unless* $char(F)$ *and* λ *is* $2-$*singular, we have*

1. $c_{T'} = 0$ *for all* T' *with repeated entries in some column*
2. $c_{T_1} \neq 0$ *for some* $T_1 \in \mathcal{T}_r(\lambda,\mu)$.

Proof. We prove both parts of the lemma separately

1. If T' has repeated entries in some column, there are i, j such that $i \neq j$ and i and j are in same column of t and $T'(i) = T'(j)$. Since $\kappa_t(i,j) = -\kappa_t$,

$$\sum c_T T(i,j) = \theta(\{t\}\kappa_t)(i,j) = \theta(\{t\})\kappa_t(i,j) = -\sum c_T T.$$

Because $T'(i,j) = T'$, $c_{T'} = 0$ (since $T(\lambda,\mu)$ is a basis of M^μ), unless $char(F) = 2$.

If $char(F) = 2$ and λ is $2-$regular, define π to be the permutation reversing the orders of elements in each row of t. By Corollary 42 , $\{t\}\kappa_t\pi\kappa_t = \{t\}\kappa_t$. Hence

$$\sum c_T T = \theta(\{t\}\kappa_t) = \theta(\{t\}\kappa_t)\pi\kappa_t = \sum C_t T\pi\kappa_t.$$

By Observation 49 , no tableau containing a column with a repeated entry has non-zero coefficient in $\sum c_T T\ \pi\kappa_t$, so $c_{T'} = 0$

If $\pi \in C_t$, we have $\kappa_t sgn(\pi)\pi = \kappa_t$. Therefore $\sum c_T T = \sum c_T sgn(\pi)T\pi$, and so if T_1 is column equivalent to T_2, then $c_{T_1} = \pm c_{T_2}$.

As $\theta \neq 0$, choose T_1 such that $c_{T_1} \neq 0$ and $c_{T*} = 0$ for all $[T_1] \lhd [T*]$. By part(1) of the lemma and the preceding paragraph, we may assume that numbers strictly increase along the column. If T_1 is semi-standard tableau, then the proof is complete. If T_1 is not semi-standard, there are two columns say jth with entries $a_1 < a_2 < \ldots < a_r$and $j+1$th with entries $b_1 < b_2 < \ldots < b_s$ such that $a_q < b_q$ for some q. Let $x_{i,j}$ denote the entry in the (i, j)th node of t. Let $G_{X,Y} = \sum sgn(\pi)\pi$ be the garnir element(cf 23) for the sets $X = \{x_{q,j}, \ldots x_{r,j}\}$ and $Y = \{x_{1,j+1}, \ldots x_{q,j+1}\}$. By Theorem 24 ,

$$\sum c_T T \sum sgn(\pi)\pi = \theta(\{t\}\kappa_t \sum sgn(\pi)\pi) = 0.$$

For any $T \in \mathcal{T}(\lambda,\mu)$, $T \sum sgn(\pi)\pi$ is a linear combination of tableaux agreeing on T except at $(1, j+1)$th, $(2, j+1)$th, …, $(q, j+1)$th; (q, j)th,…, (r, j)th nodes. All the tableaux involved in $T_1 \sum sgn(\pi)\pi$ have coefficients $\pm c_{T_1}$. But $\sum c_T T \sum sgn(\pi)\pi = 0$, therefore, there is a tableau $\overline{T}$ which agrees with T except on the nodes described above. Since $b_1 < b_2 < \ldots < b_q < a_q < \ldots < a_r$, we must have $[\overline{T}] \rhd [T_1]$. This contradiction to the choice of $[T_1]$. Therefore T is semi-standard. □

Using this we prove the main result of this section.

Theorem 55. *Unless $char(F) = 2$ and λ is $2-$singular, $B = \{\widehat{\theta}|\ T \in \mathcal{T}_0(\lambda,\mu)\}$ is a basis of $Hom_{FS_n}(S^\lambda, M^\mu)$.*

Proof. We have already seen that B is a linearly independent set. It is enough to show that B spans $Hom_{FS_n}(S^\lambda, M^\mu)$.

Suppose that $\theta \neq 0$ be an element of $Hom_{FS_n}(S^\lambda, M^\mu)$. By the previous lemma

$$\theta(\{t\}\kappa_t) = \sum c_T T, \text{where } c_{T_1} \neq 0 \text{ for some } T_1 \in \mathcal{T}_0(\lambda,\mu).$$

By Observation 52 , $(\theta - c_{T_1}\widehat{\theta}_{T_1})(\{t\}\kappa_t)$ is a linear combination of tableaux T_2 with the property $[T_1] \ntrianglelefteq [T_2]$. By induction, $(\theta - c_{T_1}\widehat{\theta}_{T_1})$ is a linear combination of elements of B, and therefore so is θ. Thus B is a basis. □

Definition 56. The elements of the set B as in the above are called semi-standard homomorphisms.

Remark 57. Given any total order $>$ on the set $\{1, 2, \ldots n\}$,we may define $\mathcal{T}_0(\lambda, \mu)$ to be the set of tableaux in $\mathcal{T}(\lambda, \mu)$ whose entries increase along the rows and strictly increase down the columns, with respect to $>$. One can see that all the previous results still hold.

Corollary 58. *Unless $char(F) = 2$ or λ is $2-$singular, every element of $Hom_{FS_n}(S^\lambda, M^\mu)$ can be extended to an element of $Hom_{FS_n}(M^\lambda, M^\mu)$.*

Proof. We can extend $\widehat{\theta_T}$ to θ_T. The corollary follows from the previous theorem. ☐

Theorem 55 and the above corollary can be false if $char(F) = 2$ and λ is $2-$singular. This is illustrated by the following example

Given $\{t\} = \begin{array}{c}\hline 1\\ \hline 2\\ \hline\end{array}$, then $e_t = \begin{array}{c}\hline 1\\ \hline 2\\ \hline\end{array} + \begin{array}{c}\hline 2\\ \hline 1\\ \hline\end{array}$. Now define $\theta \in Hom_{FS_n}(S^{(1^2)}, M^{(2)})$ by $\theta(e_t) = \begin{array}{cc}\hline 1 & 2\\ \hline\end{array}$. It is trivial to see that one cannot extend θ to an element of $Hom_{FS_n}(M^{(1^2)}, M^{(2)})$. Therefore Theorem 55 and the previous corollary need not hold when $char(F) = 2$ and λ is $2-$singular.

The following corollary gives us more information about $Hom_{FS_n}(S^\lambda, M^\mu)$.

Corollary 59. *Unless $char(F) = 2$ and λ is $2-$singular, $\lambda \ntrianglerighteq \mu$ implies $Hom_{FS_n}(S^\lambda, M^\mu) = 0$ and $Hom_{FS_n}(S^\lambda, M^\mu) \cong F$.*

Proof. By Theorem 55, $dim(Hom_{FS_n}(S^\lambda, M^\mu))$ is equal $|\mathcal{T}_0(\lambda, \mu)|$. Denote this number by $K_{\lambda\mu}$. Assume $K_{\lambda,\mu} \neq 0$, i.e there is a semi-standard tableau T of shape λ and type μ. The μ_1 $1'$s sit in the first row of T, thus $\mu_1 \leq \lambda_1$. The μ_1 1's and μ_2 $2'$s sit in the first two rows of T, thus $\mu_1 + \mu_2 \leq \lambda_1 + \lambda_2$. Continuing in this way, we get $\lambda \trianglerighteq \mu$. Clearly $K_{\lambda,\lambda} = 1$. Therefore $\lambda \ntrianglerighteq \mu$ implies $Hom_{FS_n}(S^\lambda, M^\mu) = 0$ and $dim(Hom_{FS_n}(S^\lambda, M^\mu)) = 1$ i.e $Hom_{FS_n}(S^\lambda, M^\mu) \cong F$. ☐

Corollary 60. *Unless $char(F) = 2$ and λ is $2-$singular, S^λ is indecomposable.*

Proof. If S^λ is a decomposable module, the projection map into one of the component gives a non-trivial element of $Hom_{FS_n}(S^\lambda, M^\lambda)$. This contradicts the previous corollary. ☐

Remark 61. In the later part of this thesis, we shall prove that $S^{(5,1^2)}$ is decomposable over any field of characteristic 2.

Chapter 6

Littlewood-Richardson Rule

We can now explicitly describe the composition factors of $M^{\mu}_{\mathbb{Q}}$. Since $\mathbb{Q}$ is a splitting field of S_n, the multiplicity of $S^{\lambda}_{\mathbb{Q}}$ is same as $dim(Hom_{\mathbb{Q}S_n}(S^{\lambda}_{\mathbb{Q}}, M^{\mu}_{\mathbb{Q}}))$. By Theorem 55, we know $dim(Hom_{\mathbb{Q}S_n}(S^{\lambda}_{\mathbb{Q}}, M^{\mu}_{\mathbb{Q}})) = K_{\lambda\mu}$. This implies the following theorem

Theorem 62 (Young's Rule)**.** *The multiplicity of $S^{\lambda}_{\mathbb{Q}}$ as a composition factor of $M^{\mu}_{\mathbb{Q}}$ is equal to the number of semi-standard tableau of shape λ and type μ.*

We have $M^{\mu}_{\mathbb{C}} = \oplus_{\lambda\vdash n} K_{\lambda\mu} S^{\lambda}_{\mathbb{C}}$. Again since $\mathbb{Q}$ is a splitting field, we also have $M^{\mu}_{\mathbb{Q}} = \oplus_{\lambda\vdash n} K_{\lambda\mu} S^{\lambda}_{\mathbb{Q}}$. The Young's Rule 62 is a special case of the "Littlewood-Richardson" rule which helps us calculate the composition factors of $Ind^{S_n}_{S_r \times S_{n-r}}(S^{\lambda}_{\mathbb{Q}} \otimes S^{\mu}_{\mathbb{Q}})$, where $\lambda \vdash r$, $\mu \vdash n-r$, and *Ind* denotes the induction process.

Before going into Littlewood-Richardson rule, we need to study combinatorial objects called sequences.

6.1 Sequences

Definition 63. Suppose that μ is a partition of a positive integer. A finite sequence of integers is called a sequence of type μ if for each i, i occurs μ_i times in the sequence.

Definition 64. Given a finite sequence of positive integers, we assign a quality(good or bad) to each term using the following rules:

1. All the $1'$s are good

2. An $i+1$ is good if and only if the number of good i's preceding it is strictly greater than the number of good $(i+1)'$s preceding it.

It follows from the definition that an $i+1$ is bad if and only if the number of previous i's is equal to the number of previous $i+1$'s and also if a sequence contains m $i-1'$s in succession, then the next m i's are good.

Definition 65. Let μ be any partition of n, let $\mu^{\#}$ be a proper-partition of some positive integer such that $\mu^{\#}_{i+1} \leq \mu^{\#}_{i} \leq \mu_i$, then $(\mu^{\#}, \mu)$ is called a pair of partitions for n.

Definition 66. Given any pair of partitions $(\mu^{\#}, \mu)$ for n, define $s(\mu^{\#}, \mu)$ to be the set of sequences of type μ in which for each i, the number of good i's is atleast $\mu^{\#}_{i}$.

The following observation is useful.

Observation 67. *If $\nu^{\#}_1 = \mu_1$ and $\nu^{\#}_i = \mu^{\#}_i$ for all $i > 1$, then $s(\nu^{\#}, \mu) = s(\mu^{\#}, \mu)$. This is true because all $1'$s are good.*

Given a pair of partitions for n, $(\mu^{\#}), \mu$, the diagram $[\mu^{\#}]$ is contained in $[\mu]$. We represent $(\mu^{\#}, \mu)$ by the diagram $[\mu^{\#}, \mu]$, obtained from $[\mu]$ by:

- drawing horizontal lines demarcating the rows of $[\mu]$
- drawing vertical lines to the right of the $\mu^{\#}_i$th element of the ith row of μ, for all i.

For example $((3,1),(4,2,1))$ is represented by

x	x	x	x
x	x		
x			

. In other words, the diagram enclosed by the vertical and horizontal lines in $[\mu^{\#}, \mu]$ is $[\mu^{\#}]$.

We now introduce operations R_c and A_c on a pair of partitions for n

Definition 68. Let $(\mu^{\#}, \mu)$ be a pair of partitions for n such that $\mu^{\#} \neq \mu$. Let $c > 1$ be an integer such that $\mu^{\#}_c < \mu_c$ and $\mu^{\#}_{c-1} = \mu_{c-1}$.

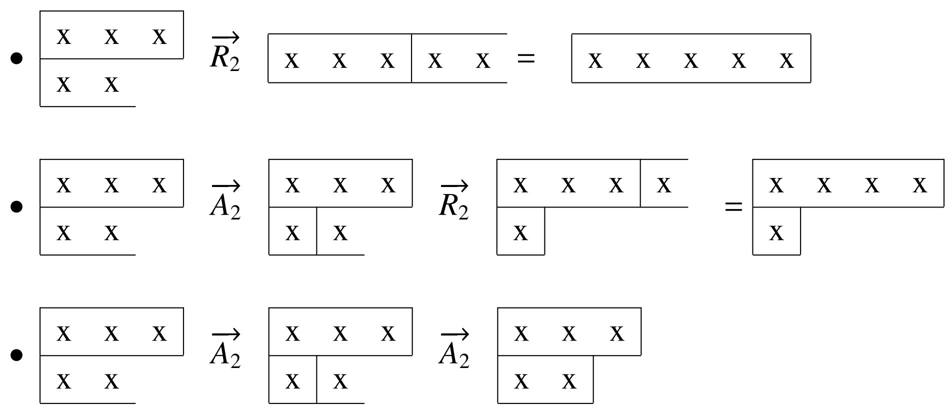
1. If $\mu^{\#}_{c-1} \gtrsim \mu^{\#}_c$, then $(\mu^{\#}A_c, \mu A_c)$ is the pair of partitions such that $\mu^{\#}_c A_c = \mu^{\#}_c + 1$, $\mu^{\#}_i A_c = \mu^{\#}_i$ for $i \neq c$ and $\mu A_c = \mu$. If $\mu^{\#}_{c-1} = \mu^{\#}$, then $(\mu^{\#}A_c, \mu A_c) = (0,0)$.

2. $\mu^{\#}R_c, \mu R_c$ is the pair of partitions such that $\mu^{\#}R_c = \mu^{\#}$ and $\mu_i R_c = \mu_i$ for $i \neq c, c-1$, $\mu_c R_c = \mu^{\#}_c$ and $\mu_{c-1}R_c = \mu_{c-1} + \mu_c - \mu^{\#}_c$.

Remark 69. • Since $s(\mu_{\#}, \mu) = s(\lambda^{\#}, \mu)$, where $\lambda_1^{\#} = \mu_1$ and $\lambda_i^{\#} = \mu_i^{\#}$ for $i > 1$, we may replace $(\mu_{\#}, \mu)$ by $(\lambda^{\#}, \mu)$ and take $c = 2$. Also we always enclose the first row by vertical of any $[\mu^{\#}, \mu]$

- The diagram $[\mu^{\#}]$ sits inside $[\mu^{\#}, \mu]$. The operation R_c merely "raises" nodes outside $[\mu^{\#}]$ in the cth row to the end of $c - 1$th row. The operation A_c "adds" one of the nodes of the c^{th}, outside $[\mu^{\#}]$ to the c^{th} row of $[\mu^{\#}]$.

The following example will help in understanding the essence of the above remark.



It is clear that any sequence of operations R_c, A_c on a given pair of partitions eventually leads to some pair of partitions of the form (λ, λ) for some proper-partition λ.

Lemma 70. *Given any pair of partitions $(\mu^{\#}, \mu)$, there is a pair $(0, \nu)$ and a sequence of operations R_c and A_c leading from $(0.\nu)$ to $(\mu^{\#}, \mu)$.*

Proof. Let $\mu_1^{\#}, \mu_2^{\#}, \ldots, \mu_r^{\#}$ be the non-zero parts of $\mu^{\#}$. Assign

$$\nu = (\mu_1^{\#}, \mu_2^{\#}, \ldots, \mu_r^{\#}, \mu_1 - \mu_1^{\#}, \ldots \mu_i - \mu_i^{\#}, \ldots).$$

We use A_c's to "enclose" the diagram $[\mu^{\#}]$, sitting in $[0, \nu]$, and then use R_c's to "raise" the nodes in the $r + 1$th, $r + 2$th ... to 1st, 2nd ... rows, thus transforming $(0, \nu)$ to $(\mu^{\#}, \mu)$ □

The following example captures the main idea in the proof of the previous lemma

Example: We obtain $((4, 3, 1), (4, 5, 2, 2))$ from $((0), (4, 3, 1, 2, 1, 2))$ by applying operations $A_2, A_2, A_2, A_3; R_3, R_5, R_6, R_4, R_5$, in that order.

The main theorem about sequences we shall apply is

Theorem 71. *Given any pair of partitions $(\mu^{\#}, \mu)$, the following is a bijection between the sets $s(\mu^{\#}, \mu) \setminus s(\mu^{\#} A_c, \mu A_c)$ and $s(\mu^{\#} R_c, \mu R_c)$:*

Given a sequence in $s(\mu^{\#}, \mu) \setminus s(\mu^{\#} A_c, \mu A_c)$, change all the bad $c's$ to $c - 1's$.

The proof of the previous result is purely combinatorial and can be found as Theorem 15.14(page no: 57) in ”The representation theory of symmetric group” [1].

6.2 Littlewood-Richardson Rule

If λ and ν are partitions of $n-r$ and r respectively($0 < r < n$), the Littlewood-Richardson rule gives us an algorithm to calculate the composition factors of $Ind^{S_n}_{S_{n-r}\times S_r}(S^{\lambda}_{\mathbb{Q}} \otimes S^{\mu}_{\mathbb{Q}})$. We define $[\lambda][\nu] = \sum_{\nu\vdash n} a_\nu[\nu]$, where a_ν is the multiplicity of $S^{\nu}_{\mathbb{Q}}$ as a composition factor of $Ind^{S_n}_{S_{n-r}\times S_r}(S^{\lambda}_{\mathbb{Q}} \otimes S^{\mu}_{\mathbb{Q}})$.

Remark 72. If μ is a partition of n, Young's Rule 62 implies: $[\mu_1][\mu_2]\ldots[\mu_i]\ldots = \sum_{\lambda\vdash n} K_{\lambda,\mu}[\lambda]$

Consider the additive group G, generated by $\{[\lambda] | \lambda$ is a partition of some intger$\}$. Given a pair of partitions $(\mu^{\#},\mu)$, we define a group endomorphisms $[\mu^{\#},\mu]^{\bullet}$ as follows:

Definition 73. $[\lambda]^{[\mu^{\#},\mu]^{\bullet}} = \sum a_\nu[\nu]([\lambda]^{[\mu^{\#},\mu]^{\bullet}}$ is the image of $[\lambda]$ under $[\mu^{\#},\mu]^{\bullet}$,) where $a_\nu = 0$ unless $\lambda_i \leq \nu_i$ for every i, and if $\lambda_i \leq \nu_i$ for every i, then a_ν is the number of ways if replacing the nodes of $[\lambda] \setminus [\nu]$ by integers such that :

1. The numbers are non-decreasing along rows
2. The numbers are strictly increasing along the columns
3. While reading from right to left in successive rows, we get a sequence in $s(\mu^{\#},\mu)$.

If $\mu^{\#} = \mu$, we denote $[\mu^{\#},\mu]^{\bullet}$ by $[\mu]^{\bullet}$.

Lemma 74. *If* $\mu = (\mu_1,\mu_2\ldots\mu_k,0,0\ldots0\ldots)$ *is a composition, then* $[0]^{[0,\mu]^{\bullet}} = [\mu_1][\mu_2]\ldots[\mu_k]$. *If* μ *is a partition,* $[0]^{[\mu]^{\bullet}} = [\mu]$.

Proof. The set $s(0,\mu)$ is the set of all sequences of type μ. Therefore if $[0]^{[0,\mu]^{\bullet}} = \sum a_\nu[\nu]$, by definition, a_ν is the number of tableaux of shape ν and type μ. i.e $a_\nu = K_{\nu,\mu}$. By the Remark 72 , we have $[0]^{[0,\mu]^{\bullet}} = [\mu_1][\mu_2]\ldots[\mu_k]$.

Let $[\nu]$ be any diagram appearing in $[0]^{[\mu]^{\bullet}}$. Then the nodes of ν can be replaced by $\mu_1$1's, $\mu_2$2's, and so on such that the conditions in the previous definition hold. Let i be the least number such that i appears in the jth row with $j > i$. No $i-1$'s appear higher than this by minimality of i. By condition 1 of the previous definition, $i-1$ cannot be to the right of i in the same row. Therefore i is not preceded by any $(i-1)$ when reading from right to left in successive rows, and hence i is bad in this sequence. This contradicts condition 3 of the previous definition. By condition 2 of the previous definition, no i can occur in the jth row with $j > i$. This implies $[\nu] = [\mu]$. It is easy to see that a_μ is 1. Therefore $[0]^{[0,\mu]^\bullet} = [\mu]$. □

The following lemma is central in proving the Littlewood-Richardson rule.

Lemma 75. $[\mu^\#, \mu]^\bullet = [\mu^\# A_c, \mu A_c]^\bullet + [\mu^\# R_c, \mu R_c]^\bullet$.

Proof. Suppose that μ is a composition of r, λ a partition of $n - r$ and ν, a partition of n, with $\lambda_i \leq \nu_i$ for all i. Let $[\lambda]^{[\mu^\#,\mu]^\bullet - [\mu^\# A_c, \mu A_c]^\bullet} = \sum a_\lambda[\lambda]$. If a_ν is non-zero,by definition of $[\mu^\#, \mu]^\bullet$ and that of $[\mu^\# A_c, \mu A_c]$, we can replace nodes of $[\nu] \setminus [\lambda]$ by $\mu_1$1's, $\mu_2$2's, and so on such we have a sequence in $s(\mu^\#, \mu) \setminus s(\mu^\# A_c, \mu A_c)$. Let A be the set of all objects we get by replacing nodes of $[\nu] \setminus [\lambda]$ by $\mu_1$1's, $\mu_2$2's, and so on such we have a sequence in $s(\mu^\#, \mu) \setminus s(\mu^\# A_c, \mu A_c)$ and B be the set of all objects we get by replacing nodes of $[\nu] \setminus [\lambda]$ by $\mu_1$1's, $\mu_2$2's, and so on such we have a sequence in $s(\mu^\# R_c, \mu R_c)$. If we can prove that an object Ξ of A satisfies

1. The numbers are non-decreasing along rows

2. The numbers are increasing along columns

if and only if the object $\Delta \in B$ obtained by changing all the bad c's in Ξ to $c - 1$, satisfies these conditions, the lemma will follow from Theorem 71 .

Suppose $\Xi \in A$ be an object satisfying the conditions (1) and (2) given in the first paragraph of this proof. A bad c cannot be to the right of a good c in the same row, because any c immediately after a bad c is bad. Therefore even after changing all the bad c's to $c-1$, the numbers remain non-decreasing along the row. Now assume $c-1$ occurs in place of $(i-1, j)$th node and a bad c occurs in place of (i, j)th node. Let m be maximal such that c occurs in the place of (i, j)th, $(i, j+1)$th, … $(i, j+m)$th nodes. The conditions in the first paragraph imply that $c - 1$ is in place of $(i-1, j)$th, $(i-1, j+1)$ … $(i, j+m)$th nodes. Since all the $c - 1$'s in any sequence of $s(\mu^\#, \mu) \setminus s(\mu^\# A_c, \mu A_c)$ are good, by definition of a good c, the c in the (i, j)th place is good. Therefore such a configuration is not possible. We can now conclude that the object $\Delta \in B$ obtained by changing the bad c's in Ξ satisfies the conditions given in the previous paragraph if Ξ satisfies them.

Now let $\Xi \in A$ and $\Delta \in B$ be the object obtained by changing all the bad c's in Ξ to $c-1$. Assume that Δ satisfies (1) and (2) in the first paragraph. Ξ will satisfy these condition unless a bad c lies is to the left of a $c-1$, in the same row or a bad c lies immediately above a good c, in the same column. Since all $c - 1's$ of a sequence in $s(\mu^\#, \mu) \setminus s(\mu^\# A_c, \mu A_c)$

are good and that the c following a good $c-1$ is good, no bad c lies to the left of a $c-1$. Suppose that a bad c occurs in $(i-1, j)$th place and a good c is in the (i, j)th place. Reading the sequence from right to left along successive rows, the number of $c-1$'s(since all $c-1$'s are good) to the left of the bad c in the $(i-1, j)$ position is atleast equal to the number of good c's in the ith row. Since the c in the (i, j)th place is good and Δ satisfies the conditions (1) and (2), every $c-1$ in the $i-1$th row to the left of $(i-1, j)$th place must have a good c immediately below it in the ith row. This provides a contradiction to the assumption that the c in the (i, j)th place is good. Thus Ξ satisfies (1) and (2) if Δ satisfies. This completes the proof of the lemma. □

Theorem 76 (**Littlewood-Richardson Rule**). $[\lambda][\mu] = [\lambda]^{[\mu]^\bullet}$.

Proof. Suppose that ν is a partition of n. By repeated application of A_c's and R_c's, we can go from $[0, \nu]$ to pairs of partitions of the form $[\pi, \pi]$. Then the previous lemma implies that we may write $[0, \nu]^\bullet = \sum_{\pi \vdash n} a_\pi [\pi]^\bullet \quad (a_\pi \in \mathbb{Z})$. It is clear that $a_\nu = 1$ and also that $a_\pi = 0$ unless $[\pi] \trianglerighteq [\nu]$. That is we may transform the set $\{[\pi] | \pi \vdash n\}$ to $\{[0, \nu]^\bullet | \nu \vdash n\}$ by an integral upper triangular matrix of determinant 1. Therefore we have integers b_α and c_β such that $[\lambda]^\bullet = \sum_{\alpha \vdash n} b_\alpha [0, \alpha]^\bullet$ and $[\mu]^\bullet = \sum_{\beta \vdash n} c_\beta [0, \beta]^\bullet$

Lemma 74 implies that $[\lambda]^{[\mu]^\bullet} = [0]^{[\lambda]^\bullet [\mu]^\bullet}$. We have

$$\begin{aligned}
[0]^{[\lambda]^\bullet [\mu]^\bullet} &= [0]^{\sum b_\alpha [0,\alpha]^\bullet \sum c_\beta [0,\beta]^\bullet} \\
&= \sum b_\alpha([\alpha_1][\alpha_2]\dots[\alpha_j]) \sum c_\beta([\beta_1]\dots[\beta_k]) \qquad \text{(by Lemma74)} \\
&= [0]^{\sum b_\alpha [0,\alpha]^\bullet} [0]^{\sum c_\beta [0,\beta]^\bullet} \qquad \text{(by Lemma74)} \\
&= [\lambda][\mu] \qquad \text{(by Lemma74).}
\end{aligned}$$

□

Chapter 7

Specht series for M^{μ}

In this chapter, we generalize Young's Rule 62 over arbitrary field. We find a filtration of submodules of M^{μ} with each factor isomorphic to a Specht Module. Such a filtration is called a Specht series. We have proved that if characteristic of the underlying field is 0, the Jordan-Holder composition series of M^{μ} is a Specht series. At the end of this chapter, we shall arrive at a very useful characterisation of Specht Modules.

Definition 77. Suppose that $(\mu^{\#}, \mu)$ be a pair of partitions for n and t a $\mu-$tableau. Let $e_t^{(\mu^{\#},\mu)} = \sum\{sgn(\pi)\{t\}\pi |\ \pi \in C_t$ and π fixes the numbers outside$[\mu^{\#}]\}$. $S^{(\mu^{\#},\mu)}$ is the submodule of M^{μ} spanned by $e_t^{(\mu^{\#},\mu)}$'s.

Example: If $\mu^{\#} = (3, 1)$ and $\mu = (3, 2, 1)$ and $t = \begin{array}{|ccc|} \hline 1 & 2 & 3 \\ \cline{2-3} 4 & \multicolumn{1}{|c}{5} & \\ \cline{1-1} \multicolumn{1}{c}{6} & & \end{array}$ (the numbers outside the box in t are the ones which are fixed), then

$$e_t^{\mu^{\#},\mu} = \begin{array}{ccc} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \cline{1-2} 6 & & \\ \cline{1-1} \end{array} \quad - \quad \begin{array}{ccc} \hline 4 & 2 & 3 \\ \hline 1 & 5 & \\ \cline{1-2} 6 & & \\ \cline{1-1} \end{array}$$

It is clear the $S^{(0,\,\mu)} = M^{\mu}$ and $S^{(\mu,\,\mu)} = S^{\mu}$ and that

***Observation* 78.** *If $\lambda_1 = \mu_1$ and for $\lambda_i = \mu_i^{\#}$ for $i > 1$, then $S^{(\lambda,\,\mu)} = S^{(\mu^{\#},\,\mu)}$.*

This observation hints a close relation between sequences and $S^{(\mu^{\#},\,\mu)}$.

Construction 79. Given a sequence of type μ, we construct a μ-tableau t as follows. If jth term of the sequence is a good i, put j as far left as possible in the ith row. If jth term of a sequence is a bad i, put j as far right as possible in the ith row. It is clear that different sequences in $s(0,\mu)$ correspond to tableaux belonging to different μ-tabloids. This construction gives a $1-1$ correspondence between $s(0,\mu)$ and the set of μ-tabloids. We may view $s(0,\mu)$ as a basis for M^{μ}.

Lemma 80. $\{e_t^{\mu^{\#},\,\mu} | t$ *corresponds to a sequence in* $s(\mu^{\#},\ \mu)$ *via the above construction*$\}$ *is a linearly independent subset of* $S^{(\mu^{\#},\,\mu)}$

Proof. Given a $\mu-$tableau t and a pair of partitions $(\mu^{\#},\ \mu)$, we say t is standard in $[\mu^{\#}]$ if the numbers in place of nodes in $[\mu^{\#}]$ sitting inside $[\mu]$ are increasing along the rows and down the columns. The Construction 79 takes sequences in $s(\mu^{\#},\ \mu)$ to $\mu-$tableaux t which are standard in $[\mu^{\#}]$. Using arguments similar to those in the proof of Lemma 26, we can say that $\{t\}$ is the last tabloid involved in $e_t^{\mu^{\#},\,\mu}$. Let $t_1 < t_2 < \ldots t_k$ be all the tableaux standard in $[\mu^{\#}]$. By inducing on the sets $A_i = \{e_{t_1}^{\mu^{\#},\,\mu}, e_{t_2}^{\mu^{\#},\,\mu} \ldots, e_{t_i}^{\mu^{\#},\,\mu}\}$, we get the result. □

In the course of this chapter, we will prove that the set in the above lemma is in fact a basis for $S^{(\mu^{\#},\,\mu)}$. In order to find a specht series for M^{μ}, we first prove that $S^{(\mu^{\#},\,\mu)} \diagup S^{\mu^{\#}A_c,\,\mu A_c} \simeq S^{\mu^{\#}R_c,\,\mu R_c}$.(Let t be a given $\mu-$tableau and $\pi_1, \pi_2 \ldots \pi_k$ be the co-set representatives of subgroup of C_t fixing elements outside $[\mu^{\#}]$ inside the subgroup of C_t fixing numbers outside $[\mu^{\#}A_c]$. One can verify that $e_t^{(\mu^{\#}A_c,\,\mu A_c)} = e_t^{\mu^{\#},\,\mu} \sum_{i=1}^{k} sgn(\pi_i)\pi$, and therefore $S^{\mu^{\#}A_c,\,\mu A_c} \subset S^{(\mu^{\#},\,\mu)}$.) We now wish to construct a FS_n-homomorphisms from $S^{(\mu^{\#},\,\mu)}$ to $S^{(\mu^{\#},\,\mu)}$. One homomorphisms defined in the next definition may do the job.

Definition 81. Suppose that $\mu = (\mu_1, \mu_2 \ldots)$ and $\nu = (\mu_1, \mu_2, \ldots \mu_i + \mu_{i+1} - \nu, \nu, \mu_{i+2} \ldots)$ for some positive integers i and ν such that ν is a partition. We define $\psi_{i,\nu} \in Hom_{FS_n}(M^{\mu}, M^{\nu})$ by

$$\psi_{i,\nu}(\{t\}) = \sum \{\{t_1\} |\ \{t_1\} \text{ agrees with } \{t\} \text{ in all except } i\text{th and } (i+1)\text{st rows,}$$
$$\text{and the } (i+1)\text{st row of } \{t_1\} \text{ is a subset of size } \nu \text{ of the } (i+1)\text{st row of } \{t\}\}.$$

Lemma 82. *$\psi_{c-1,\ \mu^\#_c}(S^{(\mu^\#,\ \mu)}) = S^{(\mu^\# R_c,\ \mu R_c)}$ and $S^{(\mu^\# A_c,\ \mu A_c)} \subset ker(\psi_{c-1,\ \mu^\#_c})$.*

Proof. Let t be any $\mu-$tableau, and let $\kappa_t = \sum\{sgn(\pi)\pi |\ \pi$ fixes numbers in t outside $[\mu^\#]\}$. Now move all but $\mu^\#_c$ numbers from cth row of t into the $c-1$th row. If the $\mu^\#_c$ numbers are the first $\mu^\#_c$ numbers, then we get a μR_c tableau say tR_c and
$\{tR_c\}\kappa_{t^\#} = e^{(\mu^\# R_c,\ \mu R_c)}_{tR_c}$.

If the $\mu^\#_c$ numbers are not the first $\mu^\#_c$ numbers, we still get a muR_c tableau say $\{t*\}$, but in this case, one of the numbers which has been moved up, say x lies inside $[\mu^\#]$. If y is the number above x in t, $1-(x,y)$ is a factor of $\kappa_{t^\#}$ and hence $\{t*\}\kappa_{t^\#} = 0$(because $\{t*\}(1-(x,y)) = 0)$.

$\psi_{c-1,\ \mu^\#_c}(e^{(\mu^\#,\ \mu)}_t) = \psi_{c-1,\ \mu^\#_c}(\{t\})\kappa_{t^\#}$(since $\psi_{c-1,\ \mu^\#_c}$ is FS_n-invariant). By definition $\psi_{c-1,\ \mu^\#_c}(\{t\})$ is the sum of all tabloids obtained my moving all but $\mu^\#_c$ from the cth row of $\{t\}$ into its $c-1$th row. The arguments in the previous paragraphs prove that
$\psi_{c-1,\ \mu^\#_c}(e^{(\mu^\#,\ \mu)}_t) = e^{(\mu^\# R_c,\ \mu R_c)}_{tR_c}$ and hence $\psi_{c-1,\ \mu^\#_c}(S^{(\mu^\#,\ \mu)}) = S^{(\mu^\# R_c,\ \mu R_c)}$.

If $(\mu^\# A_c,\ \mu A_c) = (0,0)$, by convention $\psi_{c-1,\mu^\#_c}(S^{(\mu^\# A_c,\ \mu A_c)}) = \psi_{c-1,\mu^\#_c}(0) = 0$. If not, let $\kappa_{t^\#,c} = \sum\{sgn(\pi)\pi |\ \pi$ fixes numbers in t outside $[\mu^\# A_c]\}$, then clearly $e^{(\mu^\# A_c,\ \mu A_c)}_t = \{t\}\kappa_{t^\#,c}$. Hence we have $\psi_{c-1,\mu^\#_c}(e^{(\mu^\# A_c,\ \mu A_c)}_t) = \psi_{c-1,\mu^\#_c}(\{t\})\kappa_{t^\#,c}$. Since $\mu^\#_c + 1 = \mu^\# A_c$, every tabloid $\{t_1\}$ involved in $\psi_{c-1,\mu^\#_c}(\{t\})$ has elements x and y in the same row such that $1-(x,y)$ is a factor of $\kappa_{t^\#,c}$ and therefore $\psi_{c-1,\mu^\#_c}(e^{(\mu^\# A_c,\ \mu A_c)}_t) = 0$. □

Theorem 83. *1. $\psi_{c-1,\ \mu^\#_c}(S^{(\mu^\#,\ \mu)}) = S^{(\mu^\# R_c,\ \mu R_c)}$ and*
$S^{(\mu^\#,\ \mu)} \cap \ker(\psi_{c-1,\ \mu^\#_c}) = S^{(\mu^\# A_c,\ \mu A_c)}$

2. $S^{(\mu^\#,\ \mu)} \big/ S^{\mu^\# A_c,\ \mu A_c} \simeq S^{\mu^\# R_c,\ \mu R_c}$

3. $dim(S^{(\mu^\#,\ \mu)}) = |s(\mu^\#,\ \mu)|$ and hence
$\{e^{\mu^\#,\ \mu}_t | t$ corresponds to a sequence in $s(\mu^\#,\ \mu)$ via Construction 79 $\}$ is a basis of $S^{(\mu^\#,\ \mu)}$.

4. $S^{(\mu^\#,\ \mu)}$ has a Specht series. The factors in this series are the Specht modules corresponding to the diagrams involved in $[0]^{[\mu^\#,\ \mu]^\bullet}$(c.f Definition 73)

Proof. By Lemma 70 , there is a pair of partitions $(0,\ \nu)$ from which we can reach $(\mu^\#,\ \mu)$ by a sequence of A_c's and R_c's. We have seen that $|s(0,\nu)| = dim(M^\nu)$, since $S^{(0,\ \nu)} = M^\nu$, we have $dim(0,\nu) = |s(0,\nu)|$.

Let $(\pi^\#, \pi)$ be a pair of partitions such that $|dim(S^{((\pi^\#,\pi))})| = |s((\pi^\#, \pi))|$. Now,

$$\begin{aligned}|s((\pi^\#, \pi))| &= dim(S^{(\pi^\#,\pi)})\\ &\geq dim(S^{(\pi^\# R_c, \pi R_c)}) + dim(S^{(\pi^\# A_c, \pi A_c)}) \quad \text{(Lemma82)}\\ &\geq |s(\pi^\# R_c, \pi R_c)| + |s(\pi^\# A_c, \pi A_c)| \quad \text{(Lemma80)}\\ &= |s(\pi^\#, \pi)| \quad \text{(Theorem71}\end{aligned}$$

) Therefore we must have $|s(\pi^\# R_c, \pi R_c)| = dim(S^{(\pi^\# R_c, \pi R_c)})$ and $|s(\pi^\# A_c, \pi A_c)| = dim(S^{(\pi^\# A_c, \pi A_c)})$. Since $dim(S^{(0,\nu)}) = |s(0, \nu)|$ and $(\mu^\#,\ \mu)$ result of application of a sequence of A_c's and R_c's, we have $|s(\mu^\#,\ \mu)| = dim(S^{(\mu^\#,\ \mu)})$. This result with Lemma80 and Lemma82 imply (1),(2) and (3).

If μ is a proper-partition of n, we have seen that $[0]^{[\mu]^\bullet} = [\mu]$(Lemma74). Since $S^{(\mu,\mu)} = S^\mu$, the specht series of $S^{(\mu,\mu)}$ is given by the diagrams involved in $[0]^{[\mu]^\bullet} = [\mu]$. Thus, we may now assume inductively that $S^{(\mu^\# A_c,\ \mu A_c)}$ and $S^{(\mu^\# R_c,\ \mu R_c)}$ have Specht series given by $[0]^{[\mu^\# A_c,\ \mu A_c]^\bullet}$ and $[0]^{[\mu^\# R_c,\ \mu R_c]^\bullet}$. By (1), and $[\mu^\#, \mu]^\bullet = [\mu^\# A_c, \mu A_c]^\bullet + [\mu^\# R_c, \mu R_c]^\bullet$(Lemma75), $S^{(\mu^\#,\ \mu)}$ has a Specht series given by $[0]^{[\mu^\#,\ \mu]^\bullet}$. □

The most important result of this chapter is the following corollary of Theorem 83 , which gives us a characterisation of Specht Modules.

Corollary 84. *If μ is a proper partition of n with k non-zero parts, then*

$$S^\mu = \bigcap_{i=2}^{k} \bigcap_{v=0}^{\mu_i - 1} ker(\psi_{i-1,v})$$

Chapter 8

Dimension of Specht Modules

In this chapter, we derive a formula to calculate the dimension of Specht module. To do so, we define ”hooks” of a partition. Hooks play a very important role in representation theory of S_n, especially in determining if a particular specht module is irreducible.

8.1 Hooks, Skew hooks and the Determinantal form

Definition 85. Let λ be a proper partition of n. Now consider the (i, j)th node of $[\lambda]$, then

1. the hook corresponding to (i, j) is the set $H_{i,j} = \{(i,l)|l > j\} \cup \{(k, j)|k \leq j\}$. The set $\{(i,l)|l > j\}$ is called the arm of $H_{i,j}$ and $\{(k, j)|k \leq j\}$, the leg. The leg length of a hook is $|\{(k, j)|k \leq j\}|$
2. the skew hook corresponding to (i, j) is the minimal set of nodes on the rim of $[\lambda]$, containing the last cell of ith row and the last cell of jth column such that they can be connected by drawing lines through them.
3. the hook length, h_{ij} of (i, j) is $\mu_i + \mu'_j + 1 - i - j$.
4. The hook graph of λ is the the object obtained by replacing each node of $[\lambda]$ with its corresponding hook length.

Examples:

- $H_{2,2}$ of the partition $(4,4,3)$ is the one enclosed by the horizontal and vertical lines in the following diagram $\begin{matrix} x & x & x & x \\ x & x & x & x \\ x & x & x & \end{matrix}$

- The $(2,2)$ skew hook of the partition $(4,4,3)$ is the one enclosed by the horizontal and vertical lines in the following diagram $\begin{matrix} x & x & x & x \\ x & x & x & x \\ x & x & x & \end{matrix}$

- The hook graph of $(4,4,3)$ is $\begin{matrix} 6 & 5 & 4 & 2 \\ 5 & 4 & 3 & 1 \\ 3 & 2 & 1 & \end{matrix}$

From Young's rule 62 , we know that if λ is a proper partition of n, $M^\lambda = \sum_{\mu \vdash n} K_{\mu,\lambda} S^\mu$. By the notation introduced in the chapter on Littlewood-Richardson rule, we may write $[\lambda_1][\lambda_2]\ldots = \sum K_{\mu,\lambda}[\mu]$, where $\lambda_1 \geq \lambda_2 \ldots$ are the parts of λ. If K is the matrix whose rows and columns indexed by proper partitions of n, with the (μ, λ)th entry being $K_{\mu,\lambda}$, then K is an upper triangular matrix with 1's in the diagonal, because $K_{\mu,\lambda} \neq 0$ if and only if $\mu \trianglerighteq \lambda$ and $K_{\mu,\mu} = 1$ for all $\mu, \lambda \vdash n$. Inverting the matrix K, we may write each $[\mu]$ as integral linear combinations of $([\lambda_1][\lambda_2]\ldots)$'s.

Theorem 86 (Determinental Form)**.** *If λ is a proper partition of n, then $[\lambda] = det(A)$, where A is a the matrix with its (i, j)th entry being the diagram $[\lambda_i - i + j]$, where $[m] = 0$ for all $m < 0$.(The size of the the matrix A is same as the number of non-zero parts of λ.) The matrix A is called the determinantal matrix of* $[\lambda]$

Remark 87. Note that here that $[0]$ is distinct from 0. $[0]$ acts like the multiplicative identity, this is because $[0][\lambda] = [0]^{[\lambda]^\bullet} = [\lambda]$ by the Littlewood-Richardson 76 and Lemma74. On the other hand, $0[\lambda] = 0$.

Proof. Let $\lambda_1, \lambda_2 \ldots \lambda_k$ be the non-zero parts of λ. By induction, we may assume that the theorem is true for partitions with less than k non-zero parts.

Observe that the last column of the matrix A are $[h_{1,1}], [h_{2,1}] \ldots h_{k,1}$, where $h_{i,j}$ is the hook length of the (i, j)th hook. This is because $h_{i,1} = \lambda_i + \lambda'_1 + 1 - i - 1 = \lambda_i - i + k$. Let s_i be the skew hook corresponding to $(i, 1)$th node of $[\lambda]$.

Upon omitting the last column and the ith row of A, we get the determinantal matrix of $[\lambda \setminus s_i]$. Since the proper-partition corresponding to $[\lambda \setminus s_i]$ has less than k non-zero parts, induction hypothesis ensures that the result of expanding the determinant of A along its last column is
$[\lambda \setminus s_k][h_{k,1}] - [\lambda \setminus s_{k-1}][h_{k-1,1}] + \ldots \pm [\lambda \setminus s_1][h_{1,1}]$.

Now consider $[\lambda \setminus s_i][h_{i,1}]$. By Littlewood-Richardson 76, all the diagrams involved in $[\lambda \setminus s_i][h_{i,1}]$ are obtained adding $h_{i,1}$ nodes to $[\lambda \setminus s_i]$ such that no two added nodes are in the same column. $[\lambda \setminus s_i]$ definitely contains the last nodes of the 1st,2nd $\ldots$ $i-1$th rows of λ, and thus all the diagrams in $[\lambda \setminus s_i][h_{i,1}]$,

- contain the last nodes of 1st,2nd $\ldots i-1$th rows of λ and
- do not contain the last nodes of the $(i+1)$st,$(i+2)$nd $\ldots$ kth rows of $[\lambda]$.

Divide the diagrams in $[\lambda \setminus s_i][h_{i,1}]$ into 2 sets, according to whether or not the last node of ith row is in the diagram. It is clear that $[\lambda]$ is the only tableau involved in $[\lambda \setminus s_k][h_{k,1}] - [\lambda \setminus s_{k-1}][h_{k-1,1}] + \ldots \pm [\lambda \setminus s_1][h_{1,1}]$, which contains all the last nodes of all rows of $[\lambda]$. Observe that all other diagrams get cancelled in pairs and thus we have the result. □

Corollary 88. *$dim(S^\lambda) = n!det(\frac{1}{(\lambda_i - i + j)!})$, where $1/r! = 0$ if $r < 0$.*

Proof. The dimension of module M^μ corresponding to $[\mu_1][\mu_2]\ldots$ is $\left(\frac{n!}{\mu_1!\mu_2!\ldots}\right)$, and thus the corollary follows. □

Theorem 89 (Hook Lenght Formula)**.** *If λ is any proper partition with k non-zero parts and S^λ is the corresponding specht module over any dimension,*
$dim(S^\lambda) = n!\frac{\prod_{i<k} h_{i,1} - h_{k,1}}{\prod_i h_{i,1}!} = \frac{n!}{\text{product of hook lengths in } [\lambda]}$.

Proof. We proceed by induction on the number of non-zero parts of λ. By induction hypothesis, assume that the result is true for partitions with 2 parts. If λ has 3 parts, by the above corollary, we have

$$\frac{dim(S^{\lambda})}{n!} = det\begin{pmatrix} \frac{1}{(h_{1,1}-2)!} & \frac{1}{(h_{1,1}-1)!} & \frac{1}{h_{1,1}!} \\ \frac{1}{(h_{2,1}-2)!} & \frac{1}{(h_{2,1}-1)!} & \frac{1}{h_{2,1}!} \\ \frac{1}{(h_{3,1}-2)!} & \frac{1}{(h_{3,1}-1)!} & \frac{1}{h_{3,1}!} \end{pmatrix} \tag{8.1}$$

$$= \frac{1}{h_{1,1}!, h_{2,1}!h_{3,1}!} det\begin{pmatrix} h_{1,1}(h_{1,1}-1) & h_{1,1} & 1 \\ h_{2,1}(h_{2,1}-1) & h_{2,1} & 1 \\ h_{3,1}(h_{3,1}-1) & h_{3,1} & 1 \end{pmatrix} \tag{8.2}$$

$$= \frac{(h_{1,1}-h_{2,1})(h_{1,1}-h_{3,1})(h_{2,1}-h_{3,1})}{h_{1,1}!, h_{2,1}!h_{3,1}!} \tag{8.3}$$

This gives the first equality. To get the second equality, we induce on $n(\lambda$ is a partition of $n)$. Now,

$$\frac{1}{h_{1,1}!, h_{2,1}!h_{3,1}!} det\begin{pmatrix} h_{1,1}(h_{1,1}-1) & h_{1,1} & 1 \\ h_{2,1}(h_{2,1}-1) & h_{2,1} & 1 \\ h_{3,1}(h_{3,1}-1) & h_{3,1} & 1 \end{pmatrix} = \frac{1}{h_{1,1}h_{2,1}h_{3,1}} det\begin{pmatrix} \frac{1}{(h_{1,1}-3)!} & \frac{1}{(h_{1,1}-2)!} & \frac{1}{(h_{1,1}-1)!} \\ \frac{1}{(h_{2,1}-3)!} & \frac{1}{(h_{2,1}-2)!} & \frac{1}{(h_{2,1}-1)!} \\ \frac{1}{(h_{3,1}-3)!} & \frac{1}{(h_{3,1}-2)!} & \frac{1}{(h_{3,1}-1)!} \end{pmatrix}$$

By induction, the R.H.S of the above equation is same as

$$\frac{1}{h_{1,1}h_{2,1}h_{3,1}} \times \frac{1}{\prod(\text{hook lengths of the partition with parts}\lambda_1 - 1, \lambda_2 - 1, \lambda_3 - 1)}$$

,which is same as

$$\left(\frac{1}{\text{product of hook lengths in } [\lambda]}\right)$$

. This gives us the second equality. The induction step from partitions with 2 parts to 3 parts may be mimicked to get the induction step from partitions with k parts to $k+1$. Therefore the theorem is true by principles of mathematical induction. □

Chapter 9

Murnaghan-Nakayama Rule

Murnaghan-Nakayama rule is gives an algorithm for calculating the ordinary irreducible characters of S_n. The leg length of skew hook corresponding to a node is the same as the leg length of the corresponding hook(c.f Definition 85). Let χ^μ be the character corresponding to the irreducible module $S^\lambda_{\mathbb{Q}}$. By skew-r-hook, we mean a skew hook containing $r-$nodes.

Theorem 90 (Murnaghan-Nakayama Rule)**.** *Suppose that $\pi\rho \in S_n$, where ρ is an $r-$cycle and π is a permutation of the remaining $n-r$ numbers. Then*
$\chi^\mu(\pi\rho) = \sum_\nu\{(-1)^i\chi^\nu(\pi) | [\lambda] \setminus [\nu]$ *is a skew-r-hook of leg length i*$\}$.

Before going to the proof of this theorem, let us apply it to an example. Example: Suppose we want to find the value of $\chi^{(5,4,4)}$ on the class $(5,4,3,1)$. There are two skew-5-hooks,(the ones enclosed in the following diagrams)

$$\begin{array}{ccccc} x & x & x & x & x \\ x & x & x & x & \\ x & x & x & x & \end{array} \quad \text{and} \quad \begin{array}{ccccc} x & x & x & x & x \\ x & x & x & x & \\ x & x & x & x & \end{array}$$

. So only upon removing $[3,3,2]$ or $[5,3]$, we get a skew-5-hook and hence by the Murnaghan Nakayama Rule, we get
$\chi^{(5,4,4)}$ on $(5,4,3,1) = \chi^{(3,3,2)} - \chi^{(5,3)}$ on $(4,3,1)$. Upon repeated application of we have

$$\begin{aligned} \chi^{(5,4,4)} \text{ on } (5,4,3,1) &= \chi^{(3,3,2)} - \chi^{(5,3)} \text{ on } (4,3,1) \\ &= \chi^{(2,1,1)} - \chi^{(3,1)} + \chi^{(2,2)} \text{ on } (3,1) \\ &= \chi^{(2,2)} \text{ on } (3,1) \text{ (because there are no skew-3-hooks in } [2,1,1] \text{ or } [3,1]) \\ &= -\chi^{(1)} \text{ on } (1) \\ &= -1. \end{aligned}$$

It is evident that the only character table required in construction of character table of S_n using Murnaghan-Nakayam rule is that of S_1. A hook diagram is a diagram of the form $[x, 1^y]$.

Lemma 91. *Unless both* $[\alpha]$ *and* $[\beta]$ *are hook diagrams, no hook diagram is involved in* $[\alpha][\beta]$*. If* $[\alpha] = [a, 1^{(n-r-a)}]$ *and* $[\beta] = [b, 1^{(r-b)}]$ *then*
$[\alpha][\beta] = [a+b, 1^{n-a-b}] + [a+b-1, 1^{(n-a-b+1)}]+$ *some non-hook diagrams.*

Proof. If one of $[\alpha]$ or $[\beta]$ is not a hook diagram, then one of them contains a $(2,2)$ node. By Littlewood-Richardson 76 , $[\alpha][\beta] = [\alpha]^{[\beta]^\bullet} = [\beta]^{[\alpha]^\bullet}$ and therefore all tabloids involved in $[\alpha][\beta]$ contains a $(2,2)$ node whenever one of $[\alpha]$ or $[\beta]$ contains a $(2,2)$ node. By the definition of $[\alpha]^{[\beta]^\bullet}$(c.f Definition 73), any hook diagram $[\lambda]$ involved in $[\alpha][\beta]$ can accommodate a sequence of type β in $[\lambda] \setminus [\alpha]$ such that the numbers are non-decreasing along the rows and increasing along the rows. Therefore the first row of $[\lambda]$ must contain either $a+b$ or $a+b-1$ nodes and thus they the only hook representations involved in $[\alpha][\beta]$ are $[a+b, 1^{n-a-b}]$ and $[a+b-1, 1^{(n-a-b+1)}]$. The co-efficient of each of them is the number of ways in which they can accommodate a sequence of type μ such that when read from right to left along successive rows, it is in $s(\mu, mu)$. Clearly, this can be done in only one way for both $[a+b, 1^{n-a-b}]$ and $[a+b-1, 1^{(n-a-b+1)}]$. Hence the result. □

The following lemma is a special case of the Murnaghan Nakayama rule.

Lemma 92. *If* ρ *is an* $n-$*cycle and* ν *is a proper partition of* n*, then*

$$\chi^\nu(\rho) = \begin{cases} (-1)^{n-x} & \text{if } \nu = [x, 1^{n-x}] \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let α be a partition of r and β of $n-r$. Then the inner product $\left\langle \chi^{[\alpha][\beta]}, \chi^{(n)-(n-1,1)+\ldots\pm(1^n)} \right\rangle$ is zero. This is because, by the previous lemma, $[\alpha][\beta]$ contains no hook diagrams or has adjacent hook diagrams, each with co-efficient 1. The Frobenius reciprocity theorem implies that $\chi^{(n)-(n-1,1)+\ldots\pm(1^n)}$ is zero on all Young subgroups of the form $S_{(r,n-r)}$ with $0 < r < n$. This implies $\chi^{(n)-(n-1,1)+\ldots\pm(1^n)}$ is zero on all conjugacy classes of S_n, except perhaps on the conjugacy class containing the $n-$cycle ρ(let this conjugacy class be represented by (n)). Therefore the column vector which has $(-1)^{n-x}$ opposite $\chi^{(x,1^{n-x})}$ and zero opposite all other irreducible characters is orthogonal to all the columns of the character table of S_n, except the one associated with (n) .Since the character table is non-singular, this column vector is a multiple of the column associated with (n) of the character table. But since the entry opposite χ^{1^n} is 1, this is in fact the column associated with (n) as implicitly stated in the lemma. □

Lemma 93. *Suppose that λ is a partition of n and ν is a partition of $n - r$. Then*

1. *The multiplicity of $[\lambda]$ in $[\nu][x, 1^{(r-x)}]$ is zero unless $[\lambda] \setminus [\nu]$ is a union of skew hooks.*
2. *If $[\lambda] \setminus [\nu]$ is a union of m disjoint skew hooks having (in total) c columns,then multiplicity of $[\lambda]$ in $[\nu][x, 1^{(r-x)}]$ is the binomial coefficient $\binom{m-1}{c-x}$.*

Proof. By Littlewood-Richardson 76 , we know that $[\lambda]$ is involved in $[\nu][x, 1^{(r-x)}]$ if and only if $[\nu] \subset [\lambda]$ and it is possible to replace the nodes of $[\lambda] \setminus [\nu]$ by x 1's,one 2, one3 . . ., one $r - x$ in such a way that

1. Any column containing 1 has just one 1, which is at the top of the column.
2. For $i > 1$, $i + 1$ occurs in a later row than i; in particular, no two numbers greater than 1 appear in the same row.
3. The first non-empty row contains no number greater than 1.
4. Any row containing a number greater than 1 has that number at the end of the row.

This implies that $[\lambda] \setminus [\nu]$ does not contain the following diagram

x x

x x

, this is because neither of the left hand node(in the above diagram) can be replaced, either by a number greater than 1(by condition 4 given above); or by 1(by condition 1). Thus we conclude that $[\lambda] \setminus [\nu]$ is a union of skew hooks.

Now, suppose that $[\lambda] \setminus [\mu]$ is a disjoint union of m skew hooks, having c columns in total. Now replace nodes of $[\lambda] \setminus [\nu]$ with x 1's,one 2, one3 . . ., one $r - x$ as directed by the 4 conditions given above. Each column contains at most one 1(by 1) and also each column contains at least one 1, except may be the last column of the 2nd, 3rd . . . mth skew hook, by 2,3 and 4 (skew hooks are ordered from left to right). Therefore, $(c - m + 1)$ 1's are forced and the remaining $(x-c+m-1)$ 1's can be put in any of the $m-1$ spaces left at the top of the last columns in 2nd, 3rd . . . mth skew hooks. The condition 2 in the first paragraph ensures that the positions of numbers greater than 1 are determined once the positions of 1's are fixed. The multiplicity of $[\lambda]$ in $[\nu][x, 1^{(r-x)}]$ is therefore $\binom{m-1}{x-c+m-1} = \binom{m-1}{c-x}$, as claimed in the lemma. □

Proof of Murnaghan-Nakayama Rule 90

Let $a_{\nu\mu} = \left\langle \chi^{\lambda} \downarrow_{S_{n-r,r}}, \chi^{[\nu][\mu]} \right\rangle$, where μ is a partition of r and ν a partition of $n-r$. If ρ is an r cycle and π is a permutation of the remaining $n-r$ numbers, we have

$$\begin{aligned}\chi^{\lambda}(\rho\pi) &= \sum_{\nu,\mu} a_{\nu\mu}\chi^{\nu}(\pi)\chi^{\mu}(\rho)\\ &= \sum_{\nu}\chi^{\nu}(\pi)\sum_{x=1}^{r} a_{\nu,(x,1^{(r-x)})}(-1)^{(r-x)} \quad \text{(Lemma 92)}\end{aligned}$$

But by Frobenius Reciprocity, $a_{\nu,(x,1^{(r-x)})} = \left\langle \chi^{\lambda}, \chi^{[\nu][\mu]} \right\rangle$

$$= \binom{m-1}{c-x} \quad \text{(by the previous lemma.)}$$

Clearly $r \geq c \geq m$(c is the number of columns in $[\lambda] \setminus [\mu]$, r the number of nodes in $[\lambda] \setminus [\mu]$ and m the number of disjoint skew hooks into which $[\lambda] \setminus [\mu]$ can be split up), so

$$\begin{aligned}\sum_{x=1}^{r}\binom{m-1}{c-x}(-1)^{r-x} &= (-1)^{r-c}\left(\sum_{i=0}^{m-1}(-1)^{i}\binom{m-1}{i}\right)\\ &= \begin{cases}(-1)^{r-c} & \text{if } m=1\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

When $m = 1$, $[\lambda] \setminus [\mu]$ is a single skew$r-$hook of leg length $r-c$. This completes the proof.

The following result about modular characters of S_n, inspired by the Murnaghan-Nakayama rule is useful.

Theorem 94. *If ν is a partition of $n-r$, then the generalized character of S_n corresponding to*
$\sum\{(-1)^{i}[\lambda] \mid [\lambda] \setminus [\nu]$ *is a skew-r-hook of leg length* $i\}$ *is zero on all classes except those containing an $r-$cycle.*

Proof. From Lemma 93 , if $[\lambda]$ is involved in $[\nu]([r]-[r-1,1]+[r-2,1^2]\ldots\pm[1^r])$ $[\lambda]\setminus[\nu]$, then $[\lambda]\setminus[\nu]$ is a union of m disjoint skew hooks. Its coefficient will be $\sum_{x=1}^{r}\binom{m-1}{c-x}(-1)^{(r-x)}$($c$ is number of columns in $[\lambda] \setminus [\mu]$). This is $(-1)^{r-c}$ if $m = 1$ and 0 otherwise. Therefore we have
$[\nu]([r]-[r-1,1]+[r-2,1^2]\ldots\pm[1^r]) = \sum\{(-1)^{i}[\lambda] \mid [\lambda]\setminus[\nu]$ is a skew-r-hook of leg length $i\}$.

But, by definition, $Ind_{S_{(n-r,r)}}^{S_n}\chi^{[\nu]}\chi^{([r]-[r-1,1]+[r-2,1^2]\ldots\pm[1^r])}$ is zero on all of S_n, except perhaps on $S_{n-r,r}$. Lemma92 implies that it is zero on all of $S_{(n-r,r)}$ except on the class containing $\pi\rho$, where ρ is an r cycle and thus the result follows. $\square$

Chapter 10

Some Irreducible Specht Modules

Specht Modules are irreducible over any field of characteristic zero and since every field is a splitting field for S_n, a Specht module is irreducible over a field of characteristic p if and only if it is irreducible over the Galois field of size p. We will see some results which give information about irreducibility of Specht Modules. In this chapter, unless otherwise mentioned , S^μ is $S^\mu_{\mathbb{F}_p}$. Before proceeding further, we will state some combinatorial results about certain binomial coefficients without proof.

10.1 Combinatorial results

Definition 95. Suppose p is a prime and $n = n_0 + n_1 p + \ldots n_r p^r$ where, for each i, $0 \leq n_i < p$ and $n_r \neq 0$. Then

1. $\upsilon_p = max\{i | n_j = 0 \text{ for } j < i\}$
2. $\sigma_p = n_0 + n_1 \ldots n_r$
3. $l_p(n) = r + 1$

Lemma 96. $\upsilon_p(n!) = (n - \sigma_p(n))/(p-1)$

Lemma 97. *If* $a \geq b > 0$*,then* $\upsilon_p(\binom{a}{b}) < l_p(a) - l_p(b)$

Lemma 98. *If $a \geq b$*

$$a = a_0 + a_1 p + \ldots + a_r p^r \quad 0 \leq a_i < p$$
$$b = b_0 + b_1 p + \ldots + b_r p^r \quad 0 \leq b_i < p$$

such that $a_i \geq b_i$, then $\binom{a}{b} \equiv \prod_{i=0}^{r} \binom{a_i}{b_i}$ mod p. In particular, $p | \binom{a}{b}$ if and only if $a_i < b_i$ for some i

Corollary 99. *If $a \geq b \geq 1$, then all the binomial coefficients $\binom{a}{b}, \binom{a-1}{b-1} \ldots \binom{a-b+1}{1}$ are divisible by p if and only if $a - b \equiv -1$ mod $p^{l_p(b)}$.*

The proofs of these results may be found in Chapter 22 of [1].

10.2 Some irreducible Specht Modules

Unless otherwise mentioned $F = \mathbb{F}_1$ and $S^\mu = S^\mu_F$.

Lemma 100. *Suppose that $Hom_{FS_n}(S^\mu, S^\mu) \cong F$. Then S^μ is irreducible if and only if S^μ is self dual.*

Proof. If S^μ is irreducible, it is self dual because all irreducible modules of FS_n are self dual(Theorem 44). Let U be an irreducible sub-module of S^μ. If S^μ is self dual, there is a sub-module V of S^μ such that $S^\mu / V \cong U$. Since map

$$S^\mu \xrightarrow[cannonical]{} S^\mu / V \xrightarrow[iso]{} U$$

is a non-zero element of $Hom_{FS_n}(S^\mu, S^\mu) \cong F$, we must have $U = S^\mu$, so S^μ is irreducible. □

Recall that given a proper-partition μ, g^μ is the g.c.d of integers $< e_t, e_{t_1} >$, where e_t and e_{t_1} are polytabloids in $S^\mu_{\mathbb{Q}}$. Also recall that if t is a $\mu-$tableau, then $\kappa_t = \sum_{\pi \in C_t} sgn(\pi)\pi$ and $\rho_t = \sum_{\pi \in R_t} \pi$.

Lemma 101. *Let t be a $\mu-$tableau. Then*

1. *The gcd of coefficients of the tabloids involved in $\{t\}\kappa_t \rho_t$ is $g^{\mu'}$, where μ' is the conjugate partition of μ,*

2. $\{t\}\kappa_t \rho_t \kappa_t = \prod(\textit{hook lenghts in } [\mu])\ \{t\}\kappa_t$.

Proof. By definition $g^{\mu'} = gcd(\{< e_{t'}, e_{t'}\pi > \mid \pi \in S_n\})$, where t' is a μ'−tableau. Now,

$$\begin{aligned} sgn(\pi) < e_{t'}, e_{t'}\pi > &= sgn(\pi) < \{t'\},\ \{t'\}\kappa_{t'}\pi\kappa_{t'} > \\ &= \sum\{sgn(\pi)sgn(\sigma)sgn(\tau) \mid \sigma,\ \tau \in C_{t'},\ \sigma\pi\tau \in R_{t'}\} \\ &= \sum\{sgn(\omega) \mid \tau \in C_{t'},\ \omega\tau^{-1}\pi^{-1} \in C_{t'},\ \omega \in R_{t'}\} \\ &= \sum\{sgn(\omega) \mid \tau \in R_t,\ \omega\tau^{-1}\pi^{-1} \in R_t,\ \omega \in C_t\} \\ &=< \{t\}, \{t\}\kappa_t\rho_t\pi^{-1} > \\ &=< \{t\}\pi, \{t\}\kappa_t\rho_t > \end{aligned}$$

and hence the result 1 follows.

By Corollary21 , we have $\{t\}\kappa_t\rho_t\kappa_t = c\{t\}\kappa_t$ for some $c \in \mathbb{Q}$. By remark18 , $\rho_t\kappa_t\mathbb{Q}S_n \cong S^{\mu}_{Q}$ and hence we can say that $\rho_t\kappa_t\rho_t\kappa_t = c\rho_t\kappa_t$. By Maschke's theorem, let U be a right ideal of $\mathbb{Q}S_n$ such that $U \oplus \rho_t\kappa_t\mathbb{Q}S_n$. Multiplication on left by $\rho_t\kappa_t$ of $\mathbb{Q}S_n$ is a linear transformation. With respect the basis $\{\rho_t\kappa_t|tt$ is a standard tableau$\} \cup B$, where B is a basis of $U(\{\rho_t\kappa_t|tt$ is a standard tableau$\}$ is a basis of $\rho_t\kappa_t\mathbb{Q}S_n$ by Theorem 28), the matrix of this linear transformation is the block diagonal matrix $\left(\begin{array}{c|c} \mathrm{A} & \mathrm{B} \\ \hline \mathrm{C} & \mathrm{D} \end{array}\right)$, where A is a square diagonal matrix of size $dim(S^{\mu})$ whose diagonal entries are all equal to c, B is a zero matrix of size $dim(S^{\mu}) \times (n! - dim(S^{\mu})$, D a zero matrix of size $(n! - dim(S^{\mu}) \times (n! - dim(S^{\mu}))$ and C some matrix of size $(n! - dim(S^{\mu})) \times dim(S^{\mu})$. On the other hand its matrix with respect to the basis $\{\pi|\pi \in S_n\}$ has 1's along the diagonal since e (the identity permutation) has coefficient 1 in $\rho_t\kappa_t$. Since trace of both the matrices is same, we have $n! = cdim(S^{\mu})$. By the Hook length Formula 89 , we have $c = \prod$(hook lenghts in $[\mu]$) □

The first part of the lemma gives that $\frac{1}{g^{\mu'}}\{t\}\kappa_t\rho_t$, is an integer linear combination of the tabloids involved in it. This with Theorem 32 , shows that the map defined in the following definition is well defined.

Definition 102. Define $\theta \in Hom_{FS_n}(M^{\mu}, S^{\mu})$, given by
$\theta : \{t\} \mapsto (\frac{1}{g^{\mu'}}\ \{t\}\kappa_t\rho_t)_p$, where $(\frac{1}{g^{\mu'}}\ \{t\}\kappa_t\rho_t)_p$ is the element of S^{μ} obtained from the vector $\frac{1}{g^{\mu'}}\ \{t\}\kappa_t\rho_t \in S^{\mu}_{\mathbb{Q}}$, by reducing all the tabloids coefficients modulo p.

Theorem 103. *1. If $Im(\theta) \subset S^\mu$, equivalently if $ker(\theta) \supset S^{\mu^\perp}$, then S^μ is reducible.*

2. If $Im(\theta) = S^\mu$, equivalently if $ker(\theta) = S^\mu$ and if $Hom_{FS_n}(S^\mu, S^\mu) \cong F$, then S^μ is irreducible.

Proof. Suppose that $\phi \in Hom_{\mathbb{Q}S_n}(M^\mu_\mathbb{Q}, S^\mu_\mathbb{Q})$ is defined by $\phi(\{t\}) = \frac{1}{g^{\mu'}}\{t\}\kappa_t\rho_t$. By lemma 101 , ϕ sends $\{t\}\kappa_t$ to a non-zero multiple of itself. Since $dim(Im(\phi)) = dim(S^\mu_\mathbb{Q})$, we have $dim(ker(\phi)) = dim(S^{\mu^\perp})$. Submodule theorem 19 implies $ker(\phi) = S^{\mu^\perp}_\mathbb{Q}$. Theorem 34 tells us that $ker(\phi) = S^{\mu^\perp}_\mathbb{Q}$ implies $ker(\theta) \supseteq S^{\lambda^\perp}$. Therefore, $ker(\theta) \supset S^{\mu^\perp}$ if and only if $Im(\theta) \subset S^\mu$. The first part of the theorem follows now, because $Im(\theta)$ is a proper submodule of S^μ in this case.

If $ker(\theta) = S^{\mu^\perp}$, θ is an isomorphism between $M^\mu/S^{\mu^\perp}$ and S^μ. This implies S^μ is self dual. The second part of the theorem now follows from Lemma100 . □

Theorem 104. *Suppose that μ is a p-regular partition. Then S^μ is reducible if and only if p divides*

$$\frac{\{\prod \textit{hook lengths in } [\mu]\}}{g^{\mu'}}$$

.

Proof. The previous theorem and Corollary 59 imply that S^μ is reducible if and only if $ker(\theta) = S^{\mu\perp}$. When μ is p-regular, $S^\mu \cap S^{\mu^\perp}$ is the unique maximal ideal of S^μ and hence $M^\mu/S^{\mu^\perp}$ has the unique minimal ideal $\frac{S^\mu + S^{\mu^\perp}}{S^{\mu^\perp}}$. Therefore S^μ is reducible if and only if $ker(\theta) \supset S^\mu$.

But by Lemma 101 , we have

$$\begin{aligned} \theta(\{t\}\kappa_t) &= (\frac{1}{g_{\mu'}}\{t\}\rho_t\kappa_t)_p \\ &= (\frac{\{\prod \text{hook lengths in } [\mu]\}}{g^{\mu'}}\{t\}\kappa_t)_p \end{aligned}$$

. Since S^μ is a cyclic module, S^μ is reducible if and only if p divides the integer $\frac{\{\prod \text{hook lengths in } [\mu]\}}{g^{\mu'}}$. □

We need the following result relating S^{μ} and $S^{\mu'}$

Theorem 105. *$S^{\mu}_K \otimes S^{(1^n)}_K$ is isomorphic to the dual of $S^{\mu'}_F$, where μ' is the conjugate partition of μ. Where K is any field.*

Proof. We first prove the theorem for $K = \mathbb{Q}$. In this case , since $S^{\lambda}_{\mathbb{Q}}$ is self dual for any $\lambda \vdash n$, we need to prove $S^{\mu}_{\mathbb{Q}} \otimes S^{(1^n)}_{\mathbb{Q}} \cong S^{\mu'}_{\mathbb{Q}}$. Given t a $\mu-$tableau, let t' be the corresponding μ'tableau. For example if $t = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \end{matrix}$, then $t' = \begin{matrix} 1 & 4 \\ 2 & 5 \\ 3 & \end{matrix}$. Let $\rho_{t'} = \sum\{\pi |\ \pi \in R_{t'}\}$ and $\kappa_{t'} = \sum\{sgn(\pi)\pi |\ \pi \in C_{t'}\}$ as always. Let u be a generator for the cyclic module $S^{(1^n)}_{\mathbb{Q}}$, so that $u\pi = sgn(\pi)u$(since $dim(S^{(1^n)}_{\mathbb{Q}}) = 1$). Define $\theta : M^{\mu'}_{\mathbb{Q}} \to S^{(1^n)}_{\mathbb{Q}} \otimes S^{\mu}_{\mathbb{Q}}$ as the $\mathbb{Q}S_n$ mapping sending $\{t'\}$ to $(\{t\} \otimes u)\rho_{t'}$. Since $R_{t'} = C_t$ and $\pi u = sgn(\pi)u$, we have $(\{t\} \otimes u)\rho_{t'}\pi = sgn(\pi)\{t\pi\}\kappa_{t\pi} \otimes u$. Therefore θ sends $e_{t'} = \{t'\}\kappa_{t'}$ to $(\{t\} \otimes u)\kappa_{t'}\rho_{t'} = (\{t\}\kappa_t\rho_t \otimes u)$. Consider,

$$\begin{aligned} < \{t\}\kappa_t\rho_t,\ \{t\} > &=< \{t\}\kappa_t, \{t\}\rho_t > \\ &=< \{t\}\kappa_t, |R_t|\{t\} > \\ &= |R_t| \end{aligned}$$

Since $|R_t|$ is non-zero, $\theta(e_{t'}) \neq 0$ and hence $ker(\theta) \not\supseteq S^{\mu'}_{\mathbb{Q}}$. Therefore by sub modules theorem, $ker(\theta) \subseteq S^{\mu'^{\perp}}_{\mathbb{Q}}$. Now we have
$dim(S^{\mu}_{\mathbb{Q}}) = dim(Im(\theta)) = dim(M^{\mu'}_{\mathbb{Q}}/ker(\theta)) \geq dim(M^{\mu'}_{\mathbb{Q}}/S^{\mu'^{\perp}}_{\mathbb{Q}}) = dim(S^{\mu'}_{\mathbb{Q}})$. Similarly interchanging roles of μ and μ', we have $dim(S^{\mu'}_{\mathbb{Q}}) \geq S^{\mu'}_{\mathbb{Q}}$. Therefore $ker(\theta) = S^{\mu'^{\perp}}_{\mathbb{Q}}$. The proof for $K = \mathbb{Q}$ is now complete since $S^{\mu'}_{\mathbb{Q}} \cong M^{\mu'}_{\mathbb{Q}}/S^{\mu'^{\perp}}_{\mathbb{Q}} \cong S^{\mu}_{\mathbb{Q}} \otimes S^{1^n}_{\mathbb{Q}}$. The theorem holds for all fields of characteristic zero since $\mathbb{Q}$ is a splitting field.

If K is any field of characteristic p, then the theorem holds true for K if it is true over $F = \mathbb{F}_p$.By Theorem34 , we have a map $\bar{(}\theta)$ from $M^{\mu'}_F$ onto $S^{\mu}_F \times S^{1^n}_F$ with $ker(\bar{\theta}) \supseteq S^{\mu'^{\perp}}_F$. By dimensions, $ker(\bar{\theta}) = S^{\mu'^{\perp}}_F$. The theorem follows since $M^{\mu'}_F/S^{\mu'^{\perp}}_F$ is isomorphic to dual of $S^{\mu'}_F$. □

Recall that a hook partition is any partition of the form $(X, 1^y)$.

Theorem 106. *Suppose that μ is a hook partition. Then S^μ is irreducible if and only if one of the following holds*

(a) $\mu = (n)$ *or* (1^n)

(b) $p \nmid n$ *and* $\mu = (n-1, 1)$ *or* $(2, 1^{(n-2)})$

(c) $p \nmid n$ *and* $p \nmid 2$

Proof. $S^{(n)}$ and $S^{(1^n)}$ are one-dimensional and hence irreducible. We now assume $\mu = (x, 1^y)$, where $x > 1,\ y > 0$ and $x + y = n$. Let

$$t = \begin{array}{llll} 1 & \text{y+2} & \dots & \text{(y+x)} \\ 2 & & & \\ \vdots & & & \\ \text{(y+1)} & & & \end{array}$$

and $\bar{\kappa}_t = \sum\{sgn(\pi)\pi |\ \pi \in S_{\{2,3\dots y+1\}}\}$. Clearly
$\kappa_t = (1 - (1,2) - (1,3) - \dots - (1, y+1))\bar{\kappa}_t$. Considering κ_t and $\bar{\kappa}_t$ as elements of $\mathbb{Q}S_n$, we have
$\{t\}\kappa_t\rho_t\bar{\kappa}_t = \{t\}\kappa_t\bar{\kappa}_t\rho_t = y!\{t\}\kappa_t\rho_t$. Therefore we have

$$\begin{aligned} y!\{t\}\kappa_t\rho_t(1-(1,2)\dots-(1,y+1)) &= \{t\}\kappa_t\rho_t\kappa_t \\ &= \prod(\text{hook lengths in } [\mu])\ \{t\}\kappa_t \quad \text{by 101} \\ &= (x-1)!y!(x+y)\{t\}\kappa_t\ . \end{aligned}$$

By Lemma 40 , $g^{\mu'} = (x-1)!$ and thus
$\frac{1}{g^{\mu'}}\{t\}\kappa_t\rho_t(1-(1,2)\dots-(1-y+1)) = (x+y)\{t\}\kappa_t$. If θ is the homomorphism defined in Definition 102 ,
$\theta(\{t\}(1-(1,2)\dots-(1,y+1))) = (x+y)\{t\}\kappa_t$. By the virtue of definition of θ, we are back to working over $\mathbb{F}_p$. $\theta(\{t\}(1-(1,2)\dots-(1,y+1))) = (x+y)\{t\}\kappa_t$ shows that if $p \nmid (x+y) = n$, $Im(\theta) = S^\mu$. Since by Theorem 103 , this is equivalent to $ker(\theta) = S^{\mu^\perp}$, $S^{(x,1^y)}$ is self dual if $p \nmid n$. Corollary 59 implies $Hom_{FS_n}(S^\mu, S^\mu) \cong F$ if $p \neq 2$ or if $\mu = (n-1, 1)$. Thus by Lemma 100 , S^μ is irreducible in the case $p \nmid n$ and $p \neq 2$ or if $p \nmid n$ and$\mu = (n-1, 1)$. Since $(2, 1^{(n-2)})$ is the conjugate of $(n-1, 1)$, Theorem 105 implies S^μ is irreducible if $p \nmid n$ and $\mu = (2, 1^{(n-2)})$. Therefore S^μ is irreducible if

(a) $\mu = (n)$ or (1^n) or

(b) $p \nmid n$ and $\mu = (n-1, 1)$ or $(2, 1^{(n-2)})$ or

(c) $p \nmid n$ and $p \nmid 2$

Now if $p \mid n$, then $\{t\}(1 - (1,2) \ldots - (1, y+1)) \in ker(\theta)$. Let

$$t* = \begin{array}{lllll} (y+x) & (y+x-1) & \ldots & (y+2) & 1 \\ 2 & & & & \\ \vdots & & & & \\ (y+1) & & & & \end{array}$$

Since $x > 1$, $t*$ is a $(x, 1^y)$ tableau and hence we may define e_{t*}. It is clear that all the tabloids involved in e_{t*} contain 1 in the first row and hence $\{t*\}$ is the unique tabloid involved in both $\{t\}(1 - (1,2) \ldots - (1, y+1))$ and e_{t*}. So we have
$< \{t\}(1-(1,2) \ldots -(1, y+1)),\ e_{t*} >= 1$ and thus $\{t\}(1-(1,2) \ldots -(1, y+1)) \in ker(\theta) \setminus S^{\mu^{\perp}}$.
Therefore by Theorem103 , $S^{(x,1^y)}$ is reducible if $p \mid n$.

Finally we prove that $S^{(x,1^y)}$is reducible when $p = 2$ $x > 1$ and $y > 1$. By Theorem 105 , we may assume $x > y$. By Littlewood-Richardson Rule 76 , we have

$$[x][y] = [x+y] + [x+y-1, 1] + \ldots [x, y]$$
$$[x][1^y] = [x+1, 1^{y-1}] + [x, 1^y]$$

. If $p = 2$, then $S^{(y)} \cong S^{1^y}$ because $S^{(1^y)}$ is the sign representation of S_y and S^y , the trivial one. Therefore we have
$\chi^{(x+1,1^{y-1})} + \chi^{(x,1^y)} = \chi^{(x+y)} + \chi^{(x+y-1,1)} \ldots + \chi^{(x,y)}$ as 2−modular characters. We may now induce on y and prove that $\chi^{(x,1^y)} = \chi^{(x,y)} + \chi^{(x+2,y-2)} + \chi^{(x+4,y-4)} \ldots$. and so $\chi^{x,1^y}$ is a reducible 2−modular character. □

Remark 107. By Theorem 32 , one can see that the p-modular character corresponding to S^{μ} is same as the regular character corresponding to $S^{\mu}_{\mathbb{Q}}$.

Notice that the calculation of $g^{\mu'}$ was not difficult for hook partition μ. In general, it is not easy to calculate $g^{\mu'}$. However we shall classify all irreducible Specht modules corresponding to partitions of the type (x, y).

Lemma 108. *If* $\mu = (x, y)$*, then* $g^{\mu'} = y!gcd\{x!,\ (x-1)!1!,\ (x-2)!2!,\ \ldots, (x-y)!y!\}$

Proof. Let t_1 and t_2 be two μ'tableaux and
$X_{i,j} = \{k|k$ belongs to the ith column of t_1 and jth column of $t_2\}$ It is clear that the polytabloids e_{t_1} and e_{t_2} have the tabloid t_3 in common if and only if no two numbers from any one of the sets $X_{1,1} \cup X_{1,2},\ X_{2,1} \cup X_{2,2},\ X_{1,1} \cup X_{2,1},\ X_{1,2} \cup X_{2,2}$. Any row of $\{t_3\}$ must contain a number from $X_{1,2}$ and a number from $X_{2,1}$ or no numbers from $|X_{1,2} \cup X_{2,1}|$. Therefore $< e_{t_1}, e_{t_2} >= 0$ unless $|X_{1,2}| = |X_{2,1}|$.

If $|X_{1,2}| = |X_{2,1}|$, then the tabloid $\{t_3\}$ is common to e_{t_1} and e_{t_2} if and only if each of the first y rows of t_3 is occupied by only one element of $X_{2,1} \cup X_{2,2}$ and each row containing a number from $X_{2,1}$ contains a number from $X_{1,2}$. Thus e_{t_1} and e_{t_2} have $y!|X_{1,2}|!(x - |X_{1,2}|)!$ common tabloids.

Let $t_3 = t_1\pi_1$, where $\pi \in C_{t_1}$ be the tabloid representative of the tabloid $\{t_3\}$ common in e_{t_1} and e_{t_2}. Let $\sigma \in R_{t_3}$ be the permutation which interchanges each number in $X_{1,2}$ with a number $X_{2,1}$, leaving the other fixed. Clearly $sgn(\sigma) = (-1)^{|X_{1,2}|}$. Since $\{t_3\}$ is involved in e_{t_2}, we have $t_3\sigma = t_2\pi'$ for some $\pi' \in R_{t_2}$. Therefore $t_1\pi\sigma\pi'^{-1} = t_2$ and $sgn(\pi)sgn(\pi')$ depends only of t_1 and t_2 and not on t_3. Since $\{t_3\} = \{t_1\}\pi = \{t_2\}\pi'$,
$< e_{t_1}, e_{t_2} >= \pm y!(|X_{1,2}|!)(x - |X_{1,2}|!$.

By definition $g^{\mu'}$ is the gcd of integers $\pm y!(|X_{1,2}|!)(x - |X_{1,2}|!$ for all values of $|X_{1,2}|$. But since $0 \le |X_{1,2}| \le y$, we get the result. □

Definition 109. The p-power diagram $[\mu]^p$ for a partition μ is obtained by replacing each integer $h_{i,j}$ of the hook graph(c.f 85) for μ by $\upsilon_p(h_{i,j})$ (c.f Definition 95)

Now we are in a position to classify irreducible Specht modules corresponding to 2−part partitions.

Theorem 110. *Suppose that* $\mu = (x, y)$ *is p-regular proper partition. Then* S^μ *is reducible if and only if some column of* $[\mu]^p$ *contains two different columns.*

Proof. It is easily calculated that the hook lengths for $[\mu]$ are given by

$$\begin{aligned} h_{1,j} &= x - j + 2 \quad \text{for } 1 \le j \le y \\ h_{1,j} &= x - j + 1 \quad \text{for } y \le j \le x \\ h_{2,j} &= y - j + 1 \quad \text{for } 1 \le j \le y \end{aligned}$$

If there is a j such that $\upsilon_p(h_{1,j}) \ne \upsilon_p(h_{2,j})$, consider the largest j with this property. Then $j + p^r \le y + 1$ and

$\upsilon(h_{1,i}) = \upsilon(h_{2,1}) < r$ for $j+1 \leq i < j+p^r$. Clearly, $\{h_{1,i} | j \leq i < j+p^r\}$ is a set of p^r consecutive integers and hence $\upsilon_p(h_{1,j}) > r = \upsilon_p(h_{2,j})$. If $b = x-j+2$, since $\upsilon_p(b) > \upsilon_p(b-x+y-1)$ if and only if $\upsilon_p(b) > \upsilon_p(x-y+1)$, we have:

***Observation* 111.** *Some column of $[(x,y)]^p$ contains two different number if and only if there is an integer b such that $x-y+2 \leq b \leq x+1$ and $\upsilon_p(b) > \upsilon_p(x-y+1)$*

Now, $\prod$ hook lengths in $[x,y] = (y!(x+1)!)/(x-y+1)$ and $g^{\mu'} = y!gcd(\{x!,(x-1)!1!,\ldots(x-y)!y!\})$ by lemma 108 ,so by Theorem 104 , we have S^{μ} is reducible if and only if p divide
$\frac{x+1}{x-y+1}lcm(\{\binom{x}{x},\binom{x}{x-1}\ldots,\binom{x}{x-y}\})$. Because $(x+1)\binom{x}{b-1} = b\binom{x+1}{b}$, we have
$S^{(x,y)}$ is reducible if and only if there is an integer b such that $x-y+1 \leq b \leq x+1$ and $\upsilon_p(\frac{b}{x-y+1}\binom{x+1}{b})$. Comparing this result with 111 , we can conclude that $S^{(x+y)}$ is reducible if $[(x,y)]^p$ contains two different numbers.

On the other hand, assume no column of $[(x,y)]^p$ contains different numbers. Then for every b with $x-y+2 \leq b \leq x+1$, $\upsilon_p(b) \leq \upsilon_p(x-y+1)$. Now, if
$x-y+1 = a_r p^r + a_{r+1}p^{r+1}\ldots a_s p^s$ where $0 \leq a_i < p,\ a_r \neq 0 \neq a_s$, then
$x-y+1 < (a_{r+1}+1)p^{r+1} + a_{r+2}p^{r+2} + \ldots a_s p^s$
and $\upsilon_p((a_{r+1}+1)p^{r+1} + a_{r+2}p^{r+2} + \ldots a_s p^s) > \upsilon_p(x-y+1)$. Thus our assumption implies
$x+1 < (a_{r+1}+1)p^{r+1} + a_{r+2}p^{r+2} + \ldots a_s p^s$, and hence
$x+1 = c_0 + c_1 p + \ldots c_r p^r + a_{r+1}p^{r+1}\ldots a_s p^s\ (0 \leq c_i < p)$
and if $x-y+1 \leq b \leq x+1$,
$b = b_q p^q + \ldots b_r p^r + a_{r+1}p^{r+1}\ldots a_s p^s\ (0 \leq b_i < p,\ b_q \neq 0)$.
Therefore,
$x+1-b = c_0 + c_1 p \ldots + c_{q-1}p^{q-1} + d_q p^q \ldots + d_r p^r\ (0 \leq < p)$, where
$+d_q p^q \ldots + d_r p^r = c_q p^q \ldots + c_r p^r - b_q p^q \ldots - b_r p^r$.
By Lemma 96 ,we have

$$\begin{aligned}
\binom{\upsilon_p(x+1}{b)} &= (\sigma_p(b) + \sigma_p(x+1-p) - \sigma_p(x+1))/(p-1) \\
&= (b_q + \ldots + b_r + d_q + \ldots + d_r - c_q \ldots - c_r)/(p-1) \\
&= \upsilon_p(\binom{c_q p^q \ldots + c_r p^r}{b_q p^q + \ldots + b_r p^r}) \\
&\leq r-q \quad \text{by lemma 97} \\
&= \upsilon_p(x-y+1) - \upsilon_p(b).
\end{aligned}$$

Therefore, for $x - y + 1 \leq b \leq x + 1$, $\upsilon_p(\frac{b}{x-y+1}\binom{x+1}{b}) \leq 0$ and $S^{(x,y)}$ is irreducible. $\square$

Chapter 11

Decomposition matrix of S_n

Let $F = \mathbb{F}_p$ (field of size p) and $S^{\mu_1} \dots S^{\mu_d}$ be all the Specht modules over F ($\{\mu_1, \dots \mu_d\}$ is the set of all proper partitions of n). Let $\{D^{\lambda_1}, \dots D^{\lambda_c}\}$ be the set of all inequivalent irreducible modules over FS_n, where $\lambda_1 \dots \lambda_c$ are the p-regular partitions of n(c.f 44). Let $d_{i,j}$ be the multiplicity of D^{λ_j} in the composition series of S^{μ_i}. The $d \times c$ matrix $D = (d_{i,j})$ is called the decomposition matrix. Decomposition matrix is an invariant of representations of the group algebra FS_n. It is still an open problem to find the decomposition matrix of symmetric group over F. Through out this chapter, $F = \mathbb{F}_p$, $S^\mu = S^\mu_F$ and $D^\mu_F = D^\mu$. Let χ^μ be the p-modular character corresponding to S^μ and ϕ^λ, the p-modular irreducible character corresponding to D^λ(λ p-regular proper partition). By Remark 107 , χ^μ is same as the ordinary character corresponding to $S^\mu_{\mathbb{Q}}$. Therefore, D is the matrix of transformation between p-modular characters and ordinary irreducible characters.

Before going into results on decomposition matrix, we prove some elementary results about composition series of S^μ.

Theorem 112. *All the composition factors of M^μ are of the form $D^\lambda(= \dfrac{S^\lambda}{S^\lambda \cap S^{\lambda^\perp}})$ with $\lambda \rhd \mu$, except if μ is p-regular partition, when D^μ appear precisely once.*

Proof. By Corollary 46 , all the composition factors of M^μ/S^μ are of the form D^λ with $\lambda \rhd \mu$. Since $S^{\mu^\perp}$ is isomorphic to the dual of M^μ/S^μ, it has them same composition factors in the opposite order. We proved that $\dfrac{S^\lambda}{S^\lambda \cap S^{\lambda^\perp}}$) is non-zero if and only if λ is p-regular, in which case it is D^μ. Since $0 \subseteq S^\mu \cap S^{\mu^\perp} \subseteq S^\mu \subseteq M^\mu$ is a filtration of submodules for M^μ, the theorem is proved. $\square$

The following is an immediate and useful corollary of the above theorem.

Corollary 113. *If μ is p-regular, S^μ has a a unique top composition factor D^μ. If D is a composition factor of $S^\mu \cap S^{\mu^\perp}$, then $D \cong D^\lambda$ for some $\lambda \rhd \mu$. If μ is p-singular, all the composition factors of S^μ have the form D^λ with $\lambda \rhd \mu$.*

The following theorem gives us information about composition factors of hook representations when p is odd.

Theorem 114. *Suppose p is odd*

1. *If $p \nmid n$, all the hook representations of S_n over the field $\mathbb{F}_p = F$.*
2. *If $p \mid n$, then for all $(x, 1^y) \vdash n$ with $0 < x < n$ and $1 < y < n-1$, then for any $2 \leq x \leq n-1$ and $0 < y \leq n-2$, $S^{(x,1^y)}$ has two composition factors say D^{x+} and D^{x-} with $D^{x+} = D^{(x-1)-}$, where $D^{n-} = 0$, $D^{1+} = 0$, $D^{n+} = S^n$ and $D^{1-} = S^{1^n}$.*

Proof. We prove the result by induction on n. The result is vacuously true for $n = 1$ and we assume it is true for $n-1$. By Murnaghan-Nakayama rule, we have

$$Res^{S^n}_{S_{n-1}}(\chi^{x,1^y}) = \chi^{(x-1,1^y)} + \chi^{(x,1^{(y-1)})} \text{ if } x > 1, y > 0 \text{ and } x + y = n. \tag{11.1}$$

Case1: $p \nmid n$
By Theorem 106, all hook representations are irreducible in this case. We just need to prove they are inequivalent. This follows at once, because by induction they are inequivalent when restricted to S_{n-1}(c.f 11.1).

Case2: $p \mid n$ Clearly $p \nmid (n-1)$ and thus $\chi^{(x,1^{y-1})}$ and $\chi^{(x-1.1^y)}$ are two inequivalent irreducible p-modular characters of S_{n-1} by case1. Therefore by 11.1 , $Res^{S_n}_{S_{n-1}}(\chi^{(x,1^y)})$ has two modular constituents and thus $\chi^{(x,1^y)}$ has atmost two modular constituents. Since by Theorem 106 , $S^{(x,1^y)}$ is reducible of $p \mid n, \chi^{(x,1^y)}$ has precisely tow modular components say ϕ^+_x and ϕ^-_x, where $Res^{S_n}_{S_{n-1}}(\phi^+_x) = \chi^{(x-1,1^y)}$ and $Res^{S_n}_{S_{n-1}}(\phi^-_x) = \chi^{(x,1^{y-1})}$. Also Let $\phi^-_n = 0$ and $\phi^+_1 = 0$. If we show for every x, $\phi^-_{x-1} = \phi^+_x$, the theorem follows; no other equality can hold because there are different restrictions to S_{n-1}.

By Theorem 94 ,
$\chi^{(n)} - chi^{(n-1,1)} + \chi^{(n-1,1^2)} - \ldots \pm \chi^{1^n} = 0$
on all conjugacy classes of S_n except the (n). In particular, the above relation hold on all p-regular conjugacy classes. When written in terms of modular characters we have
$\chi^+_n - (\chi^-_{n-1} + \chi^+_{n-1}) + (\chi^-_{n-2} + \chi^+_{n-2}) \ldots \pm \chi^-_1 = 0.$

If some ϕ^-_{x-1} were not equal to ϕ^+_x, then ϕ^-_{x-1} appaears just once in the relation , contradicting the linear independence of modular characters of a group. $\square$

We shall use this result to find the decomposition matrix of S_5 and S_3 in the the case when $p = 3$

Theorem. When $p = 3$, the decomposition matrix of S_5 is :

	(5)	$(4,1)$	$(3,2)$	$(3,1^2)$	$(2^2,1)$
(5)	1	0	0	0	0
$(4,1)$	0	1	0	0	0
$(3,2)$	0	1	1	0	0
$(3,1^2)$	0	0	0	1	0
$(2^2,1)$	1	0	0	0	1
$(2,1^3)$	0	0	0	0	1
(1^5)	0	0	1	0	0

The partitions λ's indexing the rows correspond to the Specht modules S^λ's and the p-regular partitions μ's indexing the columns correspond to the Irreducible modules D^μ's.

Proof. The rows corresponding to $(5), (4,1)$ and $(3,1^2)$ come from Theorem 114 .

By taking $[\nu] = [2]$ and $r = 3$ in Theorem 94 , we have
$\chi^{(5)} - \chi^{(2^2,1)} + \chi^{(2,1^3)} = 0$ on all 3−regular conjugacy classes. By Theorem 114 , $\chi^{(5)}$ and $\chi^{(2,1^3)}$ are inequivalent and irreducible. Therefore $\chi^{2^2,1}$ has two components. Since $(2^2,1)$ is 3−regular partitions, one of the component must be $\phi^{(2^2,1)}$(cf Corollary 113) . Since $\chi^{(5)} = \phi^{(5)}$, we have $\chi^{(2,1^3)} = \phi^{(2^2,1)}$ and $\chi^{(2^2,1)} = \phi^{(5)} + \phi^{(2^2,1)}$ Again by 94 , we have the equation
$\chi^{(1^5)} - \chi^{(3,2)} + \chi^{(4,1)} = 0.$
Again here $\chi^{(1^5)}$ and $\chi^{(4,1)}$ are irreducible and inequivalent by Theorem 114 . Therefore $\chi^{(3,2)}$ has two modular constituents and by Corollary 113 , one of them has to be $\phi^{(3,2)}$. Since $\phi^{(4,1)} = \chi^{(4,1)}$, we have $\chi^{(3,2)} = \phi^{(3,2)} + \chi^{(4,1)}$ and $\chi^{(1^5)} = \phi^{(3,2)}$. This completes the result. $\square$

Theorem. When $p = 3$, the decomposition matrix of $S^{(3)}$ is of the form:

	(3)	(2,1)
(3)	1	0
$(2,1)$	1	1
(1^3)	0	1

Proof. The proof follows directly from 114 . □

***Observation* 115.** *1. If $\theta \in Hom_{FS_n(M^\lambda),S^\mu}$ such that $ker(\theta) \subseteq S^{\lambda^\perp}$, every composition factors of S^λ is a composition factor of S^μ. This is beacue $M/S^{\lambda^\perp}$ is isomorphic to dual of S^λ.*

2. If $\theta \in Hom_{FS_n}(S^\lambda, S^\mu)$, and if λ is p-regular, D^λ is a composition factor of S^μ. This is because when λ is p-regular , $S^\lambda \cap S^{\lambda^\perp}$ is the unique maximal ideal of S^λ.

A tableaux T of shape λ and type μ is called reverse semi-standard if the numbers are non-increasing along the rows and strictly decreasing down the columns The following theorem gives sufficient conditions for the premise of the above observation to hold.

Theorem 116. *Suppose that λ and μ are proper partitions of n and that T is a reverse semi-standard tableaux of shape λ and type μ. Let $N_{i,j}$ be the number of i's in the jth row of T.*

1. *If for all $i \geq 2$ and $j \geq 1$, $N_{i-1,j} \equiv -1 mod p^{a_{ij}}$ where $a_{ij} = l_p(N_{ij})$ (cf Definition 95), then $\theta_T \in Hom_{FS_n}(M^\lambda, M^\mu)$ as defined in Definition 47 is in fact an element of $Hom_{FS_n}(M^\lambda, S^\mu)$ such that* $\ker(\theta_T) \subseteq S^{\lambda^\perp}$
2. *If for all $i \geq 2$ and $j \geq 1$, $N_{i-1,j} \equiv -1 mod p^{b_{ij}}$, where*
   $b_{ij} = min(l_p(N_{i,j}), l_p(\sum_{m=1}^{i-1}(\lambda_{j+m-1} \sum_{s=j}^{\infty} N_{m,s})))$,
   then $\widehat{\theta_T}$, the restriction of θ_T to S^λ is a non-zero element of $Hom_{FS_n}(S^\lambda, S^\mu)$.

Proof. By Theorem 55 and Remark 57 , $\widehat{\theta_T}$ is a basis element for $Hom_{FS_n}(S^\lambda, M^\mu)$ and thus non-zero. Therefore $ker(\theta) \not\supseteq S^\lambda$ and thus by the Submodule Theorem , $ker(\theta) \subseteq S^{\lambda^\perp}$.

Let t be the lambda tableau used to define action of S_n on $\mathcal{T}(\lambda,\mu)$(in turn on M^μ), the set of tableaux of shape λ and type μ. By definition, $\theta_T(\{t\})$ is the sum of tableau of shape λ and type μ which are row equivalent to T.

Let $i \geq 2$ and $0 \leq v \leq \mu_i - 1$. We can choose $v_1, v_2 \ldots$ such that $0 \leq v_j \leq N_{i,j}$ for each j such that $\sum v_j = v$, because $\sum_{j=1}^{\infty} N_{i,j} = \mu_i$. Now, choose a tableau T_1 row equivalent to T and change all but v_j $i's$ in the jth row of T_1 to $i-1$'s. Let T_2 be the resulting tableau. By definition , each T_2 involved in $\psi_{i-1,v}(\theta_T(\{t\}))$ (c.f Definition 81) is obtained in this way from
$\prod_{j=1}^{\infty} \binom{N_{i-1,j}+N_{i,j}-v_j}{N_{i,j}-v_j}$
different tableaux row equivalent to T.

Since $\sum_{j=1}^{\infty} N_{i,j} = \mu_i > v = \sum_{j=1}^{\infty} v_j$, we may choose k such that $0 \leq v_k < N_{i,k}$. Now, if for all j, $N_{i-1,j} \equiv -1 mod p^{a_{i,j}}$, then by Corollary 99 ,
$\binom{N_{i-1,k}+N_{i,k}-v_k}{N_{i,k}-v_k}$ is divisible by p. Therefore if the hypothesis of part 1 of the theorem holds, by Corollary 84 , $\theta_T(M^{\lambda}) \subseteq S^{\mu}$.

Using similar arguments, under the hypothesis of part 2, it becomes evident that $\psi_{i-1,v}(\theta_T(\{t\}\kappa_t))$ does not involve T_2, unless
$N_{i,k} - v_k > \sum_{m=1}^{i-1}(\lambda_{k+m-1} - \sum_{s=k}^{\infty} N_{m,s})$.
For $m < i-1$, T_2 has $\sum_{s=k}^{\infty} N_{m,s}$ equal to m in rows $k, k+1, \ldots$, because T_2 is obtained from a tableau row equivalent to T. Similarly, T_2 has atleast $\sum_{s=k^{\infty} N_{i-1,s}} + N_{i,k} - v_k$ $i-1$'s in rows $k, k+1, \ldots$, since $N_{i,k} - v_k$ i's have been changed to $i-1$'s in the kth row. Therefore T_2 has atleast

$$N_{i,k} - v_k + \sum_{m=1}^{i-1}(\sum_{s=k}^{\infty} N_{m,s})$$

numbers less than equal to $i-1$ in rows $k, k+1, \ldots$. If this exceeds $\sum_{m=1}^{i-1} \lambda_{k+m-1}$, some column of T_2 must contain two identical numbers. In this case, T_2 is annihilated by κ_t. This shows that under the hypothesis of part 2, $\psi_{i-1,v}(\theta_T(\{t\}))\kappa_t = 0$ for $i \leq 2$ and $0 \leq v \leq \mu_i, 1$. Therefore by Corollary 84 , $\theta_T(\{t\}\kappa_t) \in S^{\mu}$, as we wished to prove. □

Now by Corollary 113 , composition factors of $S^{(n-m,m)}$ have the form $D^{(n-j,j)}$ with $j \neq m$.

Definition 117. Given two non-negative integers a and b, let

$$a = a_0 + a_1 p + \ldots a_r p^r \ (0 \leq a_i < p,\ a_r \neq 0)$$
$$b = b_0 + b_1 p + \ldots b_s p^s \ (0 \leq a_i < p,\ b_s \neq 0).$$

We say a contains b to base p if $s < r$ and for each i $b_i = 0$ or $b_i = a_i$

We define a function f_p on $\mathbb{N} \times \mathbb{N}$ by:

$$f_p(n,m) = \begin{cases} 1 \text{ if } n+1 \text{ contains } m \text{ to the base } p \\ 0 \text{ otherwise.} \end{cases}$$

The next result completely determines the decomposition numbers associated to $S^{(n-m,m)}$.

Theorem 118. *(James) The multiplicity of $D^{(n-j,j)}$ as a factor of $S^{(n-m,m)}$ is $f_p(n-2j, m-j)$.*

This theorem by James fills out many rows of the decomposition matrix in the case when n is small. The proof of this result may be found in [1] (pages:106-110).

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