Exploring the gravitational stability/instability for a class of spacetime geometries.

A thesis submitted towards partial fulfilment of
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by

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under the guidance of

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Certificate

This is to certify that this thesis entitled **"Exploring the gravitational stability/instability for a class of spacetime geometries"** submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by **Ashutosh Agnihotri** at **Indian Institute of Science Education and Research, Pune**, under the supervision of **Dr. Suneeta Vardarajan** during the academic year 2011-2012.

Student	Supervisor
Ashutosh Agnihotri	Suneeta Vardarajan

To my parents...

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Abstract

In this thesis we review some of the ideas that come up while investigating the classical stability of spacetime geometries. We restrict to the case in which there is no matter. We derive all the necessary material from the first principles and assume an elementary knowledge of relativity theory on the part of the reader. Later on we generalise these ideas to investigate the stability/instability of higher dimensional Schwarzchild spacetimes(Schwarzchild-Tangherlini) under Ricci flow [19]. This is a part of an ongoing work. We also outline our basic strategy and suggest how we plan to proceed on the problem.

Contents

Chapter 1

Introduction

Einstein's theory of general relativity provides classical description of physical spacetime structure and gravity. By solving Einstein's equation for spacetimes, we can extract information about the phenomena of interest. The non-linearity of the E.E. [1] makes it very difficult to obtain all solutions. In fact, very few exact solutions to E.E. are known.

Apart from finding exact solutions, one might ask the question, such as how a small inhomogeneity in some standard solution for e.g. Schwarzschild black hole evolves with time classically. If the deviation from a known exact solution is small, we can hope to find an approximate solution by considering

$$g_{ab} = g^0_{ab} + h_{ab} \tag{1.1}$$

where g^0_{ab} is the exact solution known and *linearising* the E.E. in h_{ab}. Classical stability analysis of black holes under gravitational perturbations has been an important subject of interest. A study of the standard Schwarzchild black holes was initiated by Regge and Wheeler [1] and their work was later polished by various other authors [2, 4, 3]. Recently, there has been an effort to extend this analysis to investigate the stability of higher dimensional black holes [5, 9].

The framework of Euclidean quantum gravity is another area where the *quantum* stability of black hole spacetimes becomes an important issue to

[1] Einstein Equation

investigate. To understand this better we give a very brief exposition of this approach. The Euclidean path integral for the theory is given by,

$$Z = \int D[g] \exp(-S[g]). \tag{1.2}$$

In the above expression $S[g]$ represents the Euclidean Einstein-Hilbert action and the path integral is performed over all metrics mod diffeomorphism degrees of freedom. A discussion about the full path integral is beyond the scope of this thesis. Instead, we present here the semi-classical approximation. In this approximation we look for the saddle-points of the action(Euclidean action) i.e. the classical solutions to Euclidean Einstein equation and perform Taylor series expansion about it. Thus near the classical solution, the metric can be expanded as

$$g_{MN} = \bar{g}_{MN} + \delta g_{MN}. \tag{1.3}$$

In the above equation $\bar{g}_{MN}$ denotes the classical (Euclidean) solution [2] and δg_{MN} represents off-shell perturbation. Further the action in the semi-classical approximation is then written as,

$$S[g] = S_0[\bar{g}] + S_2[\bar{g}, \delta g] + \dots \tag{1.4}$$

The term $S_2[\bar{g}, \delta g]$ is quadratic in the perturbations. It is this term that would be of concern to us in the present context. In transverse gauge, the part of $S_2[\bar{g}, \delta g]$ containing the traceless part of the perturbation is proportional to [10],

$$\int d^4x \sqrt{\bar{g}} h^{MN} \Delta_{\mathfrak{L}} h_{MN}. \tag{1.5}$$

Here $\Delta_{\mathfrak{L}}$ denotes the Euclidean Lichnerowicz operator 3.10and h_{MN} is a 2nd rank symmetric transverse-traceless tensor. To evaluate the path integral one can expand h_{MN} in eigenfunctions of the Lichnerowicz operator viz. solutions of the eigenvalue equation,

$$\Delta_{\mathfrak{L}} h_{MN} = \lambda h_{MN}. \tag{1.6}$$

[2] these are also referred to as gravitational instantons

Also appropriate boundary conditions are imposed so that h_{MN} are well-behaved. By well-behaved we mean that they vainish at infinity and have finite norm, and are regular everywhere else. Thus, if one finds a normalizable eigenmode of the Lichnerowicz laplacian with negative eigenvalue the effective action is decreased and this leads to the instability of the Euclidean solution in semi-classical approximation.

Such studies have been initiated in the past. And important mention is that carried by Gross, Perry and Yaffe (GPY) [11] to investigate the instabilities of the Euclidean Schwarzchild instanton using the path-integral formulation of relativity theory. GPY indeed found a normalizable tranverse-tracefree eigenmode of the Lichnerowicz laplacian with a negative eigenvalue. This established the instability of the Schwarzchild instanton in the semi-classical approximation.

Thus *computationally* speaking the problem of investigating the stability of gravitational instantons in the Euclidean quantum gravity approach using the semi-classical approximation boils down to studying the eigenmodes of the Lichnerowicz laplacian. Also it is to be noted that gauge freedom in the theory led us to restrict the perturbations to transverse-trace free class and thereof simplified the problem enormously. Even after gauge fixing the equations are intricately coupled and hence the problems still remains very difficult to solve. One of the solutions to this is to exploit the symmetry of spacetime and decompose the perturbation into appropriate parts and investigate the stability with respect to each of these.

1.0.1 Plan of the thesis

In this thesis we focus on various issues which would be useful to simplify the Lichnerowicz laplacian eigenvalue equation. We first discuss the gauge freedom in the problem and study how it can be used to impose transverse and traceless gauge conditions. We also study a couple of properties of the Lichnerowicz Laplacian with future application in mind.

We later extend these ideas to a problem we have been working on. This problem focusses on the investigation of stability/instability of higher dimensional Schwarzchild spacetimes(Schwarzchild-Tangherlini) under Ricci flow. In this connection we review some of the ideas of the formalism developed by previous authors while studying the classical stability of spacetime geometries [12, 14, 15, 16].

Chapter 2

Notation and Conventions

In this section we establish notation and convention.

2.1 Basic Formulae

The metric of the background is given by,

$$d\bar{s}^2 = \bar{g}_{MN} d\bar{x}^M d\bar{x}^N. \tag{2.1}$$

It is D dimensional. The formulae for various other quantities are given below. We have tried to follow the convention discussed in this section thoroughly throughout the thesis. In case a change from this convention in needed, a remark has been made in that particular section.

Christoffel symbols

$$\bar{\Gamma}^M{}_{NP} = \frac{1}{2}\bar{g}^{MQ}\left(-\partial_Q \bar{g}_{NP} + \partial_N \bar{g}_{QP} + \partial_P \bar{g}_{QN}\right) \tag{2.2}$$

Riemann tensor

$$\bar{R}^M{}_{NPQ} = \partial_P \bar{\Gamma}^M{}_{NQ} - \partial_Q \bar{\Gamma}^M{}_{NP} + \bar{\Gamma}^M{}_{SP}\bar{\Gamma}^S{}_{NQ} - \bar{\Gamma}^M{}_{SQ}\bar{\Gamma}^S{}_{NP} \tag{2.3}$$

$$\bar{R}_{MN} = \bar{R}^P{}_{MPN} \tag{2.4}$$

$$\bar{R} = \bar{R}^M{}_M \tag{2.5}$$

Einstein tensor

$$\bar{G}_{MN} = \bar{R}_{MN} - \frac{1}{2}\bar{g}_{MN}\bar{R} \tag{2.6}$$

2.2 Some important identities

Here we provide some important identities which would be used extensively in this thesis. Majority of them are concerned with the symmetry properties of Riemann tensor.

$$\left[\bar{\nabla}_M, \bar{\nabla}_N\right] X_P = -\bar{R}^Q{}_{PMN} X_Q \tag{2.7}$$

$$\bar{R}_{KLMP} + \bar{R}_{KPLM} + \bar{R}_{KMPL} = 0 \tag{2.8}$$

$$\bar{\nabla}_K \bar{R}_{PQRS} + \bar{\nabla}_R \bar{R}_{KPQS} + \bar{\nabla}_Q \bar{R}_{RKPS} = 0 \tag{2.9}$$

$$\bar{\nabla}^P \bar{R}_{MNQP} = \bar{\nabla}_N \bar{R}_{MQ} - \bar{\nabla}_M \bar{R}_{NQ} \tag{2.10}$$

Chapter 3

Linearised Gravity

3.1 Linearised Einstein Equations

The vacuum Einstein field equation is given by

$$\bar{R}_{MN} - \frac{1}{2}\bar{g}_{MN}\bar{R} = 0. \tag{3.1}$$

Also, if a cosmological constant is included then the vacuum E.E. reads,

$$\bar{R}_{MN} - \frac{1}{2}\bar{g}_{MN}\bar{R} + \Lambda\bar{g}_{MN} = 0. \tag{3.2}$$

For most of the purpose in this thesis we set $\Lambda = 0$. And wherever $\Lambda \neq 0$, it has been explicitly mentioned in that particular context. On contracting over both the indices with Λ set to zero we have,

$$\bar{R} - \frac{D}{2}\bar{R} = 0. \tag{3.3}$$

If D $\neq$ 2 then $\bar{R} = 0$. On substituting this in eg.(3.1) vacuum Einstein equation becomes

$$\bar{R}_{MN}(\bar{g}) = 0. \tag{3.4}$$

Here the Ricci tensor $\bar{R}_{MN}$ has been calculated from the background metric $\bar{g}_{MN}$ and this is indicated in the above equation by putting $\bar{g}_{MN}$ in parantheses. For the perturbed spacetime the field equation is given by

$$\bar{R}_{MN}(\bar{g} + h) = 0. \tag{3.5}$$

In this case it is assumed that the perturbed spacetime geometry is also empty i.e. vacuum. In the linearised approximation we have

$$\bar{R}_{MN}(\bar{g}) + \delta\bar{R}_{MN}(h) = 0. \tag{3.6}$$

Since $\bar{R}_{MN}(\bar{g}) = 0$, the equations governing the perturbation is given by

$$\delta\bar{R}_{MN}(h) = 0. \tag{3.7}$$

3.2 Perturbation of the Ricci Tensors

We denote the perturbation of the metric tensor by $\delta\bar{g}_{MN} = h_{MN}$. Then the perturbation of the Ricci tensor and scalar are given as

$$\begin{aligned} 2\delta\bar{R}_{MN} = &-\bar{\nabla}^L\bar{\nabla}_L h_{MN} - \bar{\nabla}_M\bar{\nabla}_N h + \bar{\nabla}_M\bar{\nabla}_L h^L{}_N + \bar{\nabla}_N\bar{\nabla}_L h^L{}_M \\ &+ \bar{R}_{ML}h^L{}_N + \bar{R}_{NL}h^L{}_M - 2\bar{R}_{MLNS}h^{LS}, \end{aligned} \tag{3.8}$$

$$\delta\bar{R} = -\bar{R}^{MN}h_{MN} + \bar{\nabla}^M\bar{\nabla}^N h_{MN} - \bar{\nabla}^M\bar{\nabla}_M h. \tag{3.9}$$

3.3 Lichnerowicz Laplacian

We define Lichnerowicz laplacian by the following formula

$$(\Delta_{\mathfrak{L}}h)_{MN} = -\bar{\nabla}^P\bar{\nabla}_P h_{MN} + 2\bar{R}^P{}_{MNS}h^S{}_P + \bar{R}^P{}_M h_{NP} + \bar{R}^P{}_N h_{MP} \tag{3.10}$$

(3.10) Using this the perturbation of the Ricci tensor can be written as

$$2\delta\bar{R}_{MN} = (\Delta_{\mathfrak{L}}h)_{MN} - \bar{\nabla}_M\bar{\nabla}_N h + \bar{\nabla}_M\bar{\nabla}_P h^P{}_N + \bar{\nabla}_N\bar{\nabla}_P h^P{}_M \tag{3.11}$$

NOTE: At this point it is useful to notice that if we can somehow restrict our study to transverse and trace free perturbations i.e.

$$\bar{\nabla}^M h_{MN} = 0, \tag{3.12}$$

$$\bar{g}^{MN}h_{MN} = 0. \tag{3.13}$$

then the problem simplifies to studying the solutions of the following equation

$$(\Delta_{\mathfrak{L}}h)_{MN} = 0. \tag{3.14}$$

3.4 Gauge Invariance

Before we go further it is necessary that we study the concept of gauge invariance in linearised theory. To begin with we refer to the following equation,

$$g_{MN} = g^0_{MN} + h_{MN}. \tag{3.15}$$

This break-up is *not* unique. What this means is that there can be other coordinate systems too in which the complete metric can be written as a sum of unperturbed background metric i.e. g^0_{MN} and a *small* perturbation.

Consider an infinitesimal coordinate transformation of the form

$$x^K \to x'^K = x^K + \xi^K(x) \tag{3.16}$$

where ξ^K are arbitrary infinitesimal functions of x such that $|\partial_L \xi^Q| << 1$. Now the components of metric tensor transform under according to

$$g'_{MN}(x') = \frac{\partial x^P}{\partial x'^M}\frac{\partial x^Q}{\partial x'^N} g_{PQ}(x) \tag{3.17}$$

To first-order, the transformation matrix components are given by

$$\frac{\partial x^P}{\partial x'^M} = \delta^P{}_M - \partial_M \xi^P \tag{3.18}$$

We substitute this into the equation for transformation of metric components. To first-order this yields,

$$\begin{aligned} g'_{MN}(x') &= g_{MN}(x) - g_{MQ}\partial_N \xi^Q - g_{PN}\partial_M \xi^P \\ &= g_{MN} - \partial_M \xi_N - \partial_N \xi_M + \xi^Q \partial_N g_{MQ} + \xi^Q \partial_M g_{QN} \end{aligned} \tag{3.19}$$

Also we can Taylor expand the L.H.S. about x so that we evaluate both sides with the same numerical values of x. Then in first-order approximation we have,

$$g'_{MN} = g_{MN} - \partial_M \xi_N - \partial_N \xi_M + \xi^Q \partial_N g_{MQ} + \xi^Q \partial_M g_{QN} - \xi^Q \partial_Q g_{MN} \tag{3.20}$$

The last three terms on the R.H.S. can be clubbed together to yield the following simple formula,

$$g'_{MN} = g_{MN} - \nabla_M \xi_N - \nabla_N \xi_M \tag{3.21}$$

Now using the break-up of metric we have

$$g'_{MN} = g^0_{MN} + h_{MN} - \nabla_M \xi_N - \nabla_N \xi_M \tag{3.22}$$

We define

$$h'_{MN} = h_{MN} - \nabla_M \xi_N - \nabla_N \xi_M \tag{3.23}$$

Recalling that $|\partial_L \xi^Q| << 1$ we conclude that $|h'_{MN}| << 1$. Thus the transformed metric also qualifies as a valid candidate for linearised theory.

These are referred to as *gauge transformations*.

3.5 Gauge Invariance and Linearised vacuum Einstein equations

We recall that in vacuum the linearised Einstein equation is written as

$$\delta\bar{R}_{MN}(h) = 0.$$

In this section we primarily calculate how the linearised Ricci tensor behaves under the gauge transformation $h_{MN} \to h_{MN} + \nabla_M\xi_N + \nabla_N\xi_M$. To calculate this we first recall that variation of Ricci tensor (or the linearised Ricci tensor) is given by the formula

$$2\delta\bar{R}_{MN} = -\bar{\nabla}^P\bar{\nabla}_P h_{MN} + \bar{\nabla}^P\bar{\nabla}_M h_{PN} + \bar{\nabla}^P\bar{\nabla}_N h_{PM} - \bar{\nabla}_M\bar{\nabla}_N h.$$

The R.H.S. of the above equation has linear operators. So we just see what additional pieces we get by substituting $h_{MN} = \bar{\nabla}_M\xi_N + \bar{\nabla}_N\xi_M$. The detailed calculation is given below,

$$\begin{aligned}
2\delta\bar{R}_{MN} &= \bar{\nabla}^P[-\bar{\nabla}_P(\bar{\nabla}_M\xi_N + \bar{\nabla}_N\xi_M) + \bar{\nabla}_M(\bar{\nabla}_P\xi_N + \bar{\nabla}_N\xi_P) \\
&\quad + \bar{\nabla}_N(\bar{\nabla}_P\xi_M + \bar{\nabla}_M\xi_P)] - 2\bar{\nabla}_M\bar{\nabla}_N\bar{\nabla}^P\xi_P \\
&= \bar{\nabla}^P[[\bar{\nabla}_M, \bar{\nabla}_P]\xi_N + [\bar{\nabla}_N, \bar{\nabla}_P]\xi_M + (\bar{\nabla}_M\bar{\nabla}_N\xi_P + \bar{\nabla}_N\bar{\nabla}_M\xi_P)] \\
&\quad - 2\bar{\nabla}_M\bar{\nabla}_N\bar{\nabla}^P\xi_P
\end{aligned} \tag{3.24}$$

At this stage we recall the formula for the commutator of covariant derivatives acting of vector(or one-forms),

$$[\bar{\nabla}_M, \bar{\nabla}_N]\xi_P = -\bar{R}^K{}_{PMN}\xi_K. \tag{3.25}$$

Using this we have,

$$\begin{aligned}
2\delta\bar{R}_{MN} &= \bar{\nabla}^P[-\bar{R}^K{}_{NMP}\xi_K - \bar{R}^K{}_{MNP}\xi_K - \bar{R}^K{}_{PMN}\xi_K + 2\bar{\nabla}_N\bar{\nabla}_M\xi_P] \\
&\quad - 2\bar{\nabla}_M\bar{\nabla}_N\bar{\nabla}^P\xi_P \\
&= \bar{\nabla}^P[-\bar{R}_{KNMP}\xi^K - \bar{R}_{KMNP}\xi^K - \bar{R}_{KPMN}\xi^K + 2\bar{\nabla}_N\bar{\nabla}_M\xi_P] \\
&\quad - 2\bar{\nabla}_M\bar{\nabla}_N\bar{\nabla}^P\xi_P \\
&= -\bar{\nabla}^P[\xi^K(\bar{R}_{KNMP} + \bar{R}_{KMNP} + \bar{R}_{KPMN})] + 2\bar{\nabla}^P\bar{\nabla}_N\bar{\nabla}_M\xi_P \\
&\quad - 2\bar{\nabla}_M\bar{\nabla}_N\bar{\nabla}^P\xi_P
\end{aligned}$$

Here we use the following property of Riemann tensor,

$$\bar{R}_{KNMP} + \bar{R}_{KPNM} + \bar{R}_{KMPN} = 0. \tag{3.26}$$

Then we have,

$$\begin{aligned}
\delta\bar{R}_{MN} &= -\bar{\nabla}^P[\xi^K\bar{R}_{KNMP}] + \bar{\nabla}^P\bar{\nabla}_N\bar{\nabla}_M\xi_P - \bar{\nabla}_M\bar{\nabla}_N\bar{\nabla}^P\xi_P \\
&= -\bar{\nabla}^P\xi^K.\bar{R}_{KNMP} - \xi^K.\bar{\nabla}^P\bar{R}_{KNMP} + \bar{g}^{PQ}\bar{\nabla}_Q\bar{\nabla}_N\bar{\nabla}_M\xi_P - \bar{\nabla}_M\bar{\nabla}_N\bar{\nabla}^P\xi_P \\
&= -\bar{\nabla}^P\xi^K.\bar{R}_{KNMP} - \xi^K.\bar{\nabla}^P\bar{R}_{KNMP} + \bar{g}^{PQ}[\bar{\nabla}_N\bar{\nabla}_Q\bar{\nabla}_M\xi_P - \bar{R}^K{}_{MQN}\bar{\nabla}_K\xi_P \\
&\quad - \bar{R}^K{}_{PQN}\bar{\nabla}_M\xi_K] - \bar{\nabla}_M\bar{\nabla}_N\bar{\nabla}^P\xi_P \\
&= -\bar{\nabla}^P\xi^K.\bar{R}_{KNMP} - \xi^K.\bar{\nabla}^P\bar{R}_{KNMP} + \bar{g}^{PQ}[\bar{\nabla}_N(\bar{\nabla}_M\bar{\nabla}_Q\xi_P - \bar{R}^K{}_{PQM}\xi_K) \\
&\quad - \bar{R}^K{}_{MQN}\bar{\nabla}_K\xi_P - \bar{R}^K{}_{PQN}\bar{\nabla}_M\xi_K] - \bar{\nabla}_M\bar{\nabla}_N\bar{\nabla}^P\xi_P \\
&= -\bar{\nabla}^P\xi^K.\bar{R}_{KNMP} - \xi^K.\bar{\nabla}^P\bar{R}_{KNMP} - \bar{g}^{PQ}\bar{\nabla}_N(\bar{R}_{KPQM}\xi^K) \\
&\quad - \bar{R}_{KMQN}\bar{\nabla}^K\xi^Q - \bar{g}^{PQ}\bar{R}_{KPQN}\bar{\nabla}_M\xi^K \\
&= -\bar{\nabla}^P\xi^K.\bar{R}_{KNMP} - \xi^K.\bar{\nabla}^P\bar{R}_{KNMP} + \bar{\nabla}_N(\bar{R}_{KM}\xi^K) - \bar{R}_{PMKN}\bar{\nabla}^P\xi^K \\
&\quad + \bar{R}_{KN}\bar{\nabla}_M\xi^K
\end{aligned}$$

The first and fourth term on the R.H.S. cancel by symmetry properties of Riemann tensor(refer Chapter 2). To proceed further we recall that from Bianchi Identity we have the following relation,

$$\bar{\nabla}^P\bar{R}_{KNMP} = \bar{\nabla}_N\bar{R}_{KM} - \bar{\nabla}_K\bar{R}_{NM}. \tag{3.27}$$

Using this in the above calculation we have,

$$\begin{aligned}
\delta\bar{R}_{MN} &= -\xi^K[\bar{\nabla}_N\bar{R}_{KM} - \bar{\nabla}_K\bar{R}_{NM}] + \bar{R}_{KM}\bar{\nabla}_N\xi^K + \xi^K\bar{\nabla}_N\bar{R}_{KM} \\
&\quad + \bar{R}_{KN}\bar{\nabla}_M\xi^K \\
&= \xi^K\bar{\nabla}_K\bar{R}_{NM} + \bar{R}_{KM}\bar{\nabla}_N\xi^K + \bar{R}_{KN}\bar{\nabla}_M\xi^K
\end{aligned}$$

At this stage we notice that $\bar{R}_{MN}$ satisfies the vacuum Einstein equations i.e $\bar{R}_{MN} = 0$. Thus the R.H.S. of the above equation vanishes.

We conclude that the linearised vacuum Einstein equation is unaffected by the following gauge transformation,

$$h_{MN} \rightarrow h_{MN} + \bar{\nabla}_M\xi_N + \bar{\nabla}_N\xi_M.$$

3.6 Gauge Freedom

In the previous section we talked about invariance of the linearised Einstein equations under the gauge transformation $h_{MN} \to h_{MN} + \bar{\nabla}_M \xi_N + \bar{\nabla}_N \xi_M$. In this section we utilise this gauge freedom to simplify the linearised equations. We ask the following two questions,

- Can we restrict the study of perturbations such that they are transverse i.e. $\bar{\nabla}^M h_{MN} = 0$.
- Also can we restrict the perturbations to be traceless i.e. $\bar{g}^{MN} h_{MN} = 0$.

Our of the reasons for posing these questions stems from the variation formula of Ricci tensor. The formula is given by,

$$\begin{aligned} 2\delta \bar{R}_{MN} &= (\Delta_{\mathfrak{L}} h)_{MN} - \bar{\nabla}_M \bar{\nabla}_N h + \bar{\nabla}_M \bar{\nabla}_L h^L{}_N + \bar{\nabla}_N \bar{\nabla}_L h^L{}_M \\ &= (\Delta_{\mathfrak{L}} h)_{MN} + \bar{\nabla}_M \bar{\nabla}^L \left(h_{NL} - \frac{1}{2} \bar{g}_{NL} h \right) + \bar{\nabla}_N \bar{\nabla}^L \left(h_{ML} - \frac{1}{2} \bar{g}_{ML} h \right) \\ &= (\Delta_{\mathfrak{L}} h)_{MN} + \bar{\nabla}_M \bar{\nabla}^L \tilde{h}_{NL} + \bar{\nabla}_N \bar{\nabla}^L \tilde{h}_{ML} \end{aligned} \tag{3.28}$$

where we have defined $\tilde{h}_{MN}$ as

$$\tilde{h}_{MN} = h_{MN} - \frac{1}{2} \bar{g}_{MN} h. \tag{3.29}$$

Note that if we can make the last two terms on the R.H.S. of eq.(3.27) vanish then we are just left with the Lichnerowicz operator acting on the perturbation. We now show that this indeed can be done by exploiting the gauge freedom in the linear theory.

We perform a gauge transformation and go to a *primed* coordinate system in which the variation formula for Ricci tensor reads

$$2\delta \bar{R}'_{MN} = (\Delta_{\mathfrak{L}} h')_{MN} + \bar{\nabla}_M \bar{\nabla}^L \tilde{h}'_{NL} + \bar{\nabla}_N \bar{\nabla}^L \tilde{h}'_{ML} \tag{3.30}$$

In this system we assume that

$$\bar{\nabla}^M \tilde{h}'_{MN} = 0. \tag{3.31}$$

This condition is known as the *transverse gauge* condition. Recall that $\tilde{h}'_{NL} = h'_{NL} - \frac{1}{2}\bar{g}_{NL}h'$ and that $h'_{MN} = h_{MN} + \bar{\nabla}_M\xi_N + \bar{\nabla}_N\xi_M$. Substituting these in the transverse gauge condition we get

$$\bar{\nabla}^L\bar{\nabla}_L\xi_N + \bar{R}^K{}_N\xi_K = \bar{\nabla}^M(h_{MN} - \frac{1}{2}\bar{g}_{MN}h) \tag{3.32}$$

The above equation is a hyperbolic partial differential equation, which can be always solved locally. Thus, given *any* general perturbation h_{MN} we can solve the above equation for ξ_N, transform to $h'_{MN} = h_{MN} + \bar{\nabla}_M\xi_N + \bar{\nabla}_N\xi_M$ and hence impose the transverse gauge condition i.e. $\bar{\nabla}^M h'_{MN} = 0$.

Now the eq.(3.31) does not completely fix the gauge. We can add vectors η satisfying the homogeneous condition viz.

$$\bar{\nabla}^L\bar{\nabla}_L\eta_N + \bar{R}^K{}_N\eta_K = 0. \tag{3.33}$$

to ξ and still satisfy the R.H.S. of eq.(3.31).

If the background spacetime is vacuum,then this *residual* gauge freedom can be utilised to impose the tracefree condition too.[1]

It is important to note that this additional freedom to impose trace-free condition is possible because we considered vacuum i.e no *source* terms. The transversality condition can always be imposed.

[1]For a proof of this claim we refer the reader to Appendix **A** of the following paper[5]

3.7 Properties of Lichnerowicz Laplacian

In this section we prove two very important properties of the Lichnerowicz Laplacian. These are,

1. If h_{MN} is traceless i.e $\bar{g}^{MN}h_{MN} = 0$, then $(\Delta_{\mathfrak{L}}h)_{MN}$ is also traceless,
$$\bar{g}^{MN}(\Delta_{\mathfrak{L}}h)_{MN} = 0. \tag{3.34}$$

2. If the background metric $\bar{g}_{MN}$ is an Einstein metric and h_{MN} satisfies the transverse condition i.e $\bar{\nabla}^M h_{MN} = 0$, then
$$\bar{\nabla}^M(\Delta_{\mathfrak{L}}h)_{MN} = 0. \tag{3.35}$$

To prove the above two properties we recall the expression for the Lichnerowicz laplacian. It is given by,

$$(\Delta_{\mathfrak{L}}h)_{MN} = -\bar{\nabla}^L\bar{\nabla}_L h_{MN} + 2\bar{R}^L{}_{MNS}h^S{}_L + \bar{R}^L{}_M h_{NL} + \bar{R}^L{}_N h_{ML}$$

Proof 1:

$$\begin{aligned}
\bar{g}^{MN}(\Delta_{\mathfrak{L}}h)_{MN} &= -\bar{g}^{MN}\bar{\nabla}^L\bar{\nabla}_L h_{MN} + 2\bar{g}^{MN}\bar{R}^L{}_{MNS}h^S{}_L + \bar{g}^{MN}\bar{R}^L{}_M h_{NL} \\
&\quad + \bar{g}^{MN}\bar{R}^L{}_N h_{ML} \\
&= -\bar{\nabla}^L\bar{\nabla}_L(\bar{g}^{MN}h_{MN}) - 2\bar{g}^{MN}\bar{R}_{MLNS}h^{SL} + \bar{R}_{NL}h^{NL} \\
&\quad + \bar{R}_{LM}h^{ML} \\
&= 0 - 2\bar{R}_{LS}h^{SL} + 2\bar{R}_{LN}h^{NL} \\
&= 0.
\end{aligned}$$

Q.E.D.

Proof 2:

$$\begin{aligned}
\bar{\nabla}^M(\Delta_{\mathfrak{L}}h)_{MN} &= 2\bar{\nabla}^M(\bar{R}^L{}_{MNS}).h^S{}_L + 2\bar{R}^L{}_{MNS}\bar{\nabla}^M h^S{}_L + \bar{\nabla}^M\bar{R}^L{}_M.h_{NL} \\
&\quad + \bar{R}^L{}_M\bar{\nabla}^M h_{NL} + \bar{\nabla}^M\bar{R}^L{}_N.h_{ML} + \bar{R}^L{}_N\bar{\nabla}^M h_{ML} \\
&\quad - \bar{\nabla}^M\bar{\nabla}^L\bar{\nabla}_L h_{MN} \\
&= 2\bar{\nabla}^M(\bar{R}_{LMNS}).h^{SL} + 2\bar{R}_{LMNS}\bar{\nabla}^M h^{SL} + \bar{\nabla}^M\bar{R}^L{}_M.h_{NL} \\
&\quad + \bar{R}^L{}_M\bar{\nabla}^M h_{NL} + \bar{\nabla}^M\bar{R}^L{}_N.h_{ML} + 0 \\
&\quad - \bar{g}^{MP}\bar{g}^{LQ}\bar{\nabla}_P\bar{\nabla}_Q\bar{\nabla}_L h_{MN}
\end{aligned}$$

To simplify the equation we evaluate the last term on the R.H.S. separately. We then have,

$$
\begin{aligned}
\bar{\nabla}_P\bar{\nabla}_Q\bar{\nabla}_L h_{MN} &= \bar{\nabla}_Q\bar{\nabla}_P\bar{\nabla}_L h_{MN} - \bar{R}^K{}_{LPQ}\bar{\nabla}_K h_{MN} - \bar{R}^K{}_{MPQ}\bar{\nabla}_L h_{KN} \\
&\quad - \bar{R}^K{}_{NPQ}\bar{\nabla}_L h_{MK} \\
&= \bar{\nabla}_Q(\bar{\nabla}_L\bar{\nabla}_P h_{MN} - \bar{R}^K{}_{MPL}h_{KN} - \bar{R}^K{}_{NPL}h_{MK}) \\
&\quad - \bar{R}^K{}_{LPQ}\bar{\nabla}_K h_{MN} - \bar{R}^K{}_{MPQ}\bar{\nabla}_L h_{KN} - \bar{R}^K{}_{NPQ}\bar{\nabla}_L h_{MK} \\
&= \bar{\nabla}_Q\bar{\nabla}_L\bar{\nabla}_P h_{MN} - \bar{\nabla}_Q\bar{R}_{KMPL}.h^K{}_N - \bar{R}_{KMPL}\bar{\nabla}_Q h^K{}_N \\
&\quad - \bar{\nabla}_Q\bar{R}_{KNPL}.h_M^K - \bar{R}_{KNPL}\bar{\nabla}_Q h_M^K - \bar{R}_{KLPQ}\bar{\nabla}^K h_{MN} \\
&\quad - \bar{R}_{KMPQ}\bar{\nabla}_L h^K{}_N - \bar{R}_{KNPQ}\bar{\nabla}_L h_M^K
\end{aligned}
$$

Thus the last term simplifies to,

$$
\begin{aligned}
\bar{g}^{MP}\bar{g}^{LQ}\bar{\nabla}_P\bar{\nabla}_Q\bar{\nabla}_L h_{MN} &= 0 + \bar{\nabla}^L\bar{R}_{KL}.h^K{}_N + \bar{R}_{KL}\bar{\nabla}^L h^K{}_N - \bar{\nabla}^L\bar{R}_{KNPL}.h^{PK} \\
&\quad - \bar{R}_{KNPL}\bar{\nabla}^L h^{PK} - \bar{R}_{KP}\bar{\nabla}^K h^P{}_N + \bar{R}_{KQ}\bar{\nabla}^Q h^K{}_N \\
&\quad - \bar{R}_{KNPQ}\bar{\nabla}^Q h^{PK}
\end{aligned}
$$

Plugging this expression and using the symmetry properties of the Riemann tensor, we have the following relation,

$$
\bar{\nabla}^M(\Delta_{\mathfrak{L}}h)_{MN} = (2\bar{\nabla}_S\bar{R}_{LN} - \bar{\nabla}_N\bar{R}_{SL})h^{SL}. \tag{3.36}
$$

Given that the background metric is Einstein i.e. $\bar{R}_{MN} = c\bar{g}_{MN}$, the R.H.S. of the above relation trivially goes to zero. Hence our claim is proved. [2]

Q.E.D.

[2]It is to be noted here that even if $\bar{g}_{MN}$ is not Einstein but R.H.S. goes to zero then too the transverse property of Lichnerowicz operator is preserved.

Chapter 4

Work in progress...

In this chapter we present the research problem that we have been working on. We utilise all the concepts that were reviewed in the previous chapters. We give a brief description of the strategy that we plan to follow. This work is in progress.

4.1 The Problem

We have been investigating the linear stability of higher dimensional Schwarzchild black holes(Schwarzchild Tangherlini) under Ricci flow[10] for a special class of static perturbations. The operator that governs the linearised flow of the perturbation is the Lichnerowicz Laplacian. As discussed in the paper[17] the stability/instability then depends on the spectrum of the Lichnerowicz Laplacian i.e the solutions to the eigenvalue equation

$$(\Delta_{\mathfrak{L}} h)_{MN} = \lambda h_{MN}.$$

Also it was remarked that solving the above equation was cumbersome as these were intricately coupled. One has to exploit the symmetry of the spacetime to decompose the perturbations and hence decouple the equations. In this section we study these issues and see how these can be implemented.

We study the formalism for the general case and then specialise to the higher dimensional Schwarzchild case.

4.2 Background Geometry

Consider an (m+n)-dimensional bulk spacetime M which is locally written as a product type

$$M^{m+n} = N^m \times K^n \tag{4.1}$$

We distinguish the tensors residing in these different manifolds, M, N^m and K^n by adopting the following index notation. We use upper case latin indices $K, L, M, N, \ldots$ to denote tensors on M, lower case latin indices $a, b, \ldots, h$ on N^m, and lower case latin indices $i, j, \ldots, p$ on K^n.

We also introduce the following coordinates $z^M = (y^a, x^i)$. Then our background spacetime metric is written as

$$\begin{aligned}\bar{g}_{MN} dz^M dz^N &= g_{ab}(y) dy^a dy^b + r^2(y)\gamma_{ij}(x) dx^i dx^j \\ &= g_{ab} dy^a dy^b + r^2 d\tilde{s}_n^2\end{aligned} \tag{4.2}$$

Further we assume that the metric γ_{ij} has a constant scalar curvature K on K^n. We call K^n as the *Base* manifold. Also the covariant derivatives, the connection coefficients and curvature tensors are denoted as

$$\bar{g}_{MN} dz^M dz^N \implies \bar{\nabla}_M, \bar{\Gamma}^M_{\ NL}, \bar{R}_{MNLS}, \tag{4.3}$$

$$g_{ab}(y) dy^a dy^b \implies D_a, \ \ {}^m\Gamma^a{}_{bc}(y), \ \ {}^m R_{abcd}(y), \tag{4.4}$$

$$\gamma_{ij}(x) dx^i dx^j \implies \hat{D}_i, \hat{\Gamma}^i{}_{jk}(x), \hat{R}_{ijkl} = K(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}). \tag{4.5}$$

4.2.1 Background Quantities

In this subsection we give the formulae for Christoffel symbols, Riemann tensors and other related quantities. These would be required in the calculation to be done further to calculate the expression for the Lichnerowicz Laplacian. The Christoffel symbols are given by:

$$\bar{\Gamma}^a{}_{bc} = \Gamma^a{}_{bc}; \bar{\Gamma}^a{}_{bj} = \bar{\Gamma}^a{}_{jb} = 0; \bar{\Gamma}^i{}_{ab} = 0;$$

$$\bar{\Gamma}^i{}_{ja} = \bar{\Gamma}^i{}_{aj} = \delta^i_j \frac{D_a r}{r};$$

$$\bar{\Gamma}^a{}_{ij} = -r\gamma_{ij}D^a r;$$

$$\bar{\Gamma}^i{}_{jk} = {}^{\Sigma}\Gamma^i{}_{jk}$$

NOTE: In the above calculation we have used the following conditions:

$$g_{ab} = g_{ab}(y); \gamma_{ij} = \gamma_{ij}(x); g_{ai} = 0.$$

The components of the Riemann tensor are:

$$\bar{R}^a{}_{bcd} = R^a{}_{bcd}; \bar{R}^a{}_{bcj} = -\bar{R}^a{}_{bjc} = 0;$$

$$\bar{R}^i{}_{ajb} = -\bar{R}^i{}_{abj} = -\delta^i_j \frac{D_b D_a r}{r};$$

$$\bar{R}^a{}_{ibj} = -\bar{R}^a{}_{ijb} = -r\gamma_{ji}D_b D^a r;$$

$$\bar{R}^a{}_{ibc} = 0; \bar{R}^i{}_{abc} = 0;$$

$$\bar{R}^i{}_{jka} = -\bar{R}^i{}_{jak} = 0;$$

$$\bar{R}^i{}_{ajk} = 0; \bar{R}^a{}_{ijk} = 0;$$

$$\hat{R}_{ijkl} = K(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk})$$

$$\bar{R}^i{}_{jkl} = [K - (D_a r)^2](\delta^i_k \gamma_{jl} - \delta^i_l \gamma_{jk})$$

It is important to note that the $\hat{}$ terms refer to the *Base* manifold and $\bar{}$ refers to the whole spacetime.

4.3 Solving the problem

One of the key strategies that is followed while investigating the (in)stability issues under Ricci flow is to split a perturbation in the following form [17],

$$h_{MN} = h^{TT}_{MN} + \frac{H}{D}\bar{g}_{MN} + \bar{\nabla}_M Y_N + \bar{\nabla}_N Y_M - 2\bar{g}_{MN}\frac{\bar{\nabla}^K Y_K}{D}, \tag{4.6}$$

where $\bar{h}^{TT}_{MN}$ is a transverse-tracefree part, H denotes the trace of h_{MN} and the last three parts make up traceless part of a divergence piece. We then study the flows of h^{TT}_{MN}, H and Y_M. In Ricci flow problems the trace of the perturbation cannot be *gauged* away, we have to study the flow of the trace too. The motivation for the above mentioned break-up of the perturbation variable stems from the fact that $(\Delta_{\mathfrak{L}} h^{TT})_{MN}$ gives us back a TT piece. Hence, the flow of h^{TT}_{MN} decouples from other flows and can be studies separately.

From here on we focus on the TT part of the perturbation and would drop the superscript TT. Recalling the expression for the Lichnerowicz Laplacian,

$$(\Delta_{\mathfrak{L}} h)_{MN} = -\bar{\nabla}^L \bar{\nabla}_L h_{MN} + 2\bar{R}^L{}_{MNS} h^S{}_L + \bar{R}^L{}_M h_{NL} + \bar{R}^L{}_N h_{ML}$$

For our case we assume that the background metric satisfies the vacuum Einstein equation with cosmological constant set to zero i.e. $\bar{R}_{MN} = 0$. Then this further simplifies to,

$$(\Delta_{\mathfrak{L}} h)_{MN} = -\bar{\nabla}^L \bar{\nabla}_L h_{MN} + 2\bar{R}^L{}_{MNS} h^S{}_L \tag{4.7}$$

Our study of the Lichnerowicz Laplacian eigenvalue equation simplifies to studying the equation,

$$-\bar{\nabla}^L \bar{\nabla}_L h_{MN} + 2\bar{R}^L{}_{MNS} h^S{}_L = \lambda h_{MN}. \tag{4.8}$$

4.3.1 Classification of perturbations

The pertubations h_{MN} may be classified into tensor, vector and scalar depending on their tensorial behaviour w.r.t. coordinate transformations on the maximally symmetric $n-$ dimensional part of the whole spacetime. [1] Then tensors of rank at most 2 w.r.t. the maximally symmetric n- dimensional part can be decomposed into three types of components viz. tensor-type, vector-type and scalar-type components.

As discussed in the paper [9], the h_{ai} component of the perturbation can be decomposed in the following way,

$$h_{ai} = h^{(1)}_{ai} + \hat{D}_i h_a, \tag{4.9}$$

[1]This section basically reviews some of the major ideas formulated by previous authors, primarily Ishibashi, Sasaki and Kodama. Respective references are provided.

where $\hat{D}^i h^{(1)}_{ai} = 0$. Also h_{ij} can be decomposed as,

$$h_{ij} = h^{(2)}_{T\ ij} + \hat{D}_i h^{(1)}_{T\ j} + \hat{D}_j h^{(1)}_{T\ i} + h_L \gamma_{ij} + \left(\hat{D}_i \hat{D}_j - \frac{1}{n} \gamma_{ij} \hat{\Delta} \right) h^{(0)}_T, \qquad (4.10)$$

where $\hat{D}^j h^{(2)}_{T\ ij} = h^{(2)i}_{T\ i} = 0$ and $\hat{D}^i h^{(1)}_{T\ i} = 0$.

Therefore, the tensor part of the general perturbation h_{MN} consists of $h^{(2)}_{T\ ij}$, the vector part consists of $(h^{(1)}_{ai}, h^{(1)}_{T\ i})$ and the scalar part comprises of $(h_{ab}, h_a, h_L, h^{(0)}_T)$.

A linearised stability analysis of the higher dimensional Schwarzchild black hole under Ricci flow restricted to *tensor* perturbations i.e $h_{ab} = 0$, $h_{ai} = 0$ and $h_{ij} = h^{(2)}_{T\ ij}$ has been carried out previously [17]. To study the evolution of a general perturbation one has to study the *vector*[2] and *scalar*[3] perturbation on the *base* manifold.

[2]transforming as a rank 1 tensor under coordinate transformations on base manifold.

[3]transforming as a rank 0 tensor under coordinate transformations on base manifold.

4.4 Analysis of *Vector* perturbation

In our study of vector perturbation, we set all the parts barring the vector part to zero i.e. $h^{(2)}_{T\ ij}$, h_{ab}, h_a, h_L and $h^{(0)}_T$ are all set to zero. This is *not* a gauge choice. We are just restricting the type of perturbation we wish to study.

Thus for our case the perturbation is given as,

$$h_{ab} = 0, \tag{4.11}$$

$$h_{ai} = h^{(1)}_{ai}, \tag{4.12}$$

$$h_{ij} = \hat{D}_i h^{(1)}_{T\ j} + \hat{D}_j h^{(1)}_{T\ i}, \tag{4.13}$$

where $\hat{D}^i h^{(1)}_{T\ i} = 0$.

Further these components can be expanded in terms of harmonic tensors of the respective type [12]. Since the different harmonics decouple on the maximally symmetric space [14], we can study the Lichnerowicz eigenvalue equation corresponding to each type separately.

In the next section we give the perturbation equation for the general case and then apply it to the higher dimensional Schwarzchild case.[4]

4.5 Lichnerowicz operator revisited

For our case i.e. vacuum with cosmological constant set to zero, we saw that the eigenvalue equation for the Lichnerowicz Laplacian reduces to

$$(\Delta_{\mathfrak{L}} h)_{MN} = -\bar{\nabla}^L \bar{\nabla}_L h_{MN} + 2\bar{R}^L{}_{MNS} h^S{}_L = \lambda h_{MN}.$$

We now give the perturbation equations. Setting $M = a$ and $N = i$ in the above equation and utilising the Christoffel symbols and Riemann tensor

[4] For a general analysis of the stability under Ricci flow we need the expression for the Lichnerowicz laplacian acting on h_{ab}, h_{ai} and h_{ij}. The Lichnerowicz operator acting on h_{ab} is zero in the case for vector perturbation.

components previously we have

$$
\begin{aligned}
(\Delta_{\mathfrak{L}} h)_{ai} &= -\bar{\nabla}^c\bar{\nabla}_c h_{ai} - \bar{\nabla}^k\bar{\nabla}_k h_{ai} + 2\bar{R}^L{}_{aiS}h^S{}_L \\
&= -\frac{\hat{\Delta}h_{ai}}{r^2} - \frac{2}{r^3}D_a r\hat{D}^k h_{ki} + \frac{2}{r}D^b r\hat{D}_i h_{ab} - \frac{(n+2)}{r^2}D^b r D_a r h_{bi} \\
&\quad + \frac{n}{r}D^b r D_b h_{ai} - \frac{(n+1)}{r^2}(Dr)^2 h_{ai} - r\Box\left(\frac{h_{ai}}{r}\right) \\
&\quad - 2D_b D_a r h^b{}_i
\end{aligned}
\tag{4.14}
$$

Setting $m = i$ and $n = j$ we derive,

$$
\begin{aligned}
(\Delta_{\mathfrak{L}} h)_{ij} = -\Box h_{ij} - 2D^c\left(\frac{D_c r h_{ij}}{r}\right) - \frac{2}{r}D^c r D_c h_{ij} \\
4\frac{(Dr)^2}{r^2}h_{ij} - \frac{\hat{\Delta}h_{ij}}{r^2} - \frac{2}{r}D^a r(\hat{D}_i h_{aj} + \hat{D}_j h_{ia}) - \frac{n}{r}D^a r D_a h_{ij} \\
+ \frac{(2n+1)}{r^2}(Dr)^2 h_{ij} - 2D^a r D^b r h_{ab}\gamma_{ij} + 2r\gamma_{ij}D_d D^c r h^d{}_c \\
+ [K - (Dr)^2][\delta^k_j\gamma_{im} - \delta^k_m\gamma_{ij}]h^m{}_k
\end{aligned}
\tag{4.15}
$$

NOTE: In the above expression $\hat{\Delta}$ represents the laplacian operator on the maximally symmetric n-dimensional space and $\Box$ represents the operator $D^a D_a$ on the metric $g_{ab}dy^a dy^b$. Also, $(Dr)^2$ denotes $D^a r D_a r$.

4.6 Higher dimensional Schwarzchild case

We now specialise to the particular case of higher-dimensional Schwarzchild black hole(Schwarzchild-Tangherlini) case. Thus, K = 1 (remember that we considered background metric satisfying vacuum E.E.). The metric is given by

$$
ds^s = -f(r)dt^2 + g(r)dr^2 + r^2 d\tilde{s}^2_n, \tag{4.16}
$$

where $f(r) = \left(1 - \left(\frac{\alpha}{r}\right)^{n-1}\right)$, $f(r)g(r) = 1$ and $d\tilde{s}^2_n$ denotes the metric on the n- sphere S^n. In this case $\hat{\Delta}$ becomes the laplacian on S^n w.r.t. the metric γ_{ij}.

To obatin the equations in this case we restrict a in our equation for vector perturbation to $t \equiv 0$ and $r \equiv 1$. Thus we have the following two equations for the two different values that a takes,

$$
\begin{aligned}
(\Delta_{\mathfrak{L}} h)_{0i} &= -\bar{\nabla}^c\bar{\nabla}_c h_{0i} - \bar{\nabla}^k\bar{\nabla}_k h_{0i} + 2\bar{R}^L{}_{0iS}h^S{}_L \\
&= -\frac{\hat{\Delta}h_{0i}}{r^2} - \frac{2}{r^3}D_0 r\hat{D}^k h_{ki} + \frac{2}{r}D^b r\hat{D}_i h_{0b} - \frac{(n+2)}{r^2}D^b r D_0 r h_{bi} \\
&\quad + \frac{n}{r}D^b r D_b h_{0i} - \frac{(n+1)}{r^2}(Dr)^2 h_{0i} - r\Box\left(\frac{h_{0i}}{r}\right) \\
&\quad - 2D_b D_0 r h^b{}_i \\
&= -\frac{\hat{\Delta}h_{0i}}{r^2} + \frac{2}{r}D^b r\hat{D}_i h_{0b} + \frac{n}{r}D^b r D_b h_{0i} \\
&\quad - \frac{(n+1)}{r^2}(Dr)^2 h_{0i} - r\Box\left(\frac{h_{0i}}{r}\right),
\end{aligned}
\tag{4.17}
$$

$$
\begin{aligned}
(\Delta_{\mathfrak{L}} h)_{1i} &= -\bar{\nabla}^c\bar{\nabla}_c h_{1i} - \bar{\nabla}^k\bar{\nabla}_k h_{1i} + 2\bar{R}^L{}_{1iS}h^S{}_L \\
&= -\frac{\hat{\Delta}h_{1i}}{r^2} - \frac{2}{r^3}D_1 r\hat{D}^k h_{ki} + \frac{2}{r}D^b r\hat{D}_i h_{1b} - \frac{(n+2)}{r^2}D^b r D_1 r h_{bi} \\
&\quad + \frac{n}{r}D^b r D_b h_{1i} - \frac{(n+1)}{r^2}(Dr)^2 h_{1i} - r\Box\left(\frac{h_{1i}}{r}\right) \\
&\quad - 2D_b D_1 r h^b{}_i \\
&= -\frac{\hat{\Delta}h_{1i}}{r^2} - \frac{2}{r^3}\hat{D}^k h_{ki} + \frac{2}{r}D^b r\hat{D}_i h_{1b} \\
&\quad - \frac{(n+2)}{r^2}D^b r h_{bi} + \frac{n}{r}D^b r D_b h_{1i} - \frac{(n+1)}{r^2}(Dr)^2 h_{1i} \\
&\quad - r\Box\left(\frac{h_{1i}}{r}\right)
\end{aligned}
\tag{4.18}
$$

The above equations can be further simplified by using the following formulae for Christoffel symbols. These are given by

$$\bar{\Gamma}^0{}_{00} = 0; \bar{\Gamma}^0{}_{01} = \frac{f'}{2f}; \bar{\Gamma}^1{}_{00} = \frac{f'}{2g}; \bar{\Gamma}^0{}_{11} = 0; \bar{\Gamma}^1{}_{01} = 0; \bar{\Gamma}^1{}_{11} = \frac{g'}{2g};$$

$$\bar{\Gamma}^0{}_{ij} = 0; \bar{\Gamma}^i{}_{0j} = 0; \bar{\Gamma}^0{}_{0i} = 0; \bar{\Gamma}^i{}_{00} = 0; \bar{\Gamma}^1{}_{ij} = -\frac{r}{g}\gamma_{ij};$$

$$\bar{\Gamma}^i{}_{1j} = \frac{1}{r}\delta^i_j; \bar{\Gamma}^1{}_{1i} = 0; \bar{\Gamma}^0{}_{00} = 0; \bar{\Gamma}^i{}_{11} = 0; \bar{\Gamma}^0{}_{1i} = 0; \bar{\Gamma}^i{}_{10} = 0;$$

$$\bar{\Gamma}^1{}_{i0} = 0; \bar{\Gamma}^i{}_{jk} = \hat{\Gamma}^0{}_{00} = 0$$

Also note that since we are restricting to static perturbations, we have $\partial_0 h_{MN} = 0$. Also $fg = 1$. Using this information we have the following expressions,

$$\begin{aligned}
(\Delta_{\mathfrak{L}} h)_{0i} = & -\frac{\hat{\Delta} h_{0i}}{r^2} + 2\frac{f}{r}\hat{D}_i h_{01} + \frac{nf}{r}\partial_1 h_{0i} - (n+1)\frac{f}{r^2} h_{0i} \\
& - rf\partial_1^2\left(\frac{h_{0i}}{r}\right) + \left\{\frac{3}{2}rf' - \frac{f'}{2}\right\}\partial_1\left(\frac{h_{0i}}{r}\right) \\
+ & \left\{\frac{f}{2}\partial_1\left(\frac{f'}{f}\right) - \frac{f'^2}{2f}\right\} h_{0i},
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
(\Delta_{\mathfrak{L}} h)_{1i} = & -\frac{\hat{\Delta} h_{1i}}{r^2} - \frac{2}{r^3}\hat{D}^k h_{ki} + \frac{2f}{r}\hat{D}_i h_{11} - \frac{(n+2)}{r^2} f h_{1i} + \frac{nf}{r}\partial_1 h_{1i} \\
& - \frac{(n+1)}{r^2} f h_{1i} - rf\partial_1^2\left(\frac{h_{1i}}{r}\right) - \frac{3}{2}rf'\partial_1\left(\frac{h_{1i}}{r}\right) \\
& - \left\{\frac{f}{2}\partial_1\left(\frac{f'}{f}\right) + \frac{f'^2}{2f}\right\} h_{1i}
\end{aligned} \tag{4.20}$$

Similarly we can write the equation for $(\Delta_{\mathfrak{L}} h)_{ij}$ in the Schwarzchild-Tangherlini spacetime.

4.7 Vector Harmonics

In the previous section we remarked that vector harmonics may be used to expand the vector part of the perturbation h_{MN}. In this section we give a brief review of the vector harmonics and see how these can be used to expand the vector perturbation. Also, recall that in vector perturbation case $h_{ab} = 0$, $h_{ai} = h^{(1)}_{ai}$ and $h_{ij} = \hat{D}_i h^{(1)}_{T\ j} + \hat{D}_j h^{(1)}_{T\ i}$.

The vector harmonics are defined as solutions to the eigenvalue equation,

$$\hat{\Delta}\mathbb{V}_i = -k^2\mathbb{V}_i. \tag{4.21}$$

These are divergence-free i.e. $\hat{D}_i \mathbb{V}^i = 0$. To expand vector-type perturbations of a rank 2 tensor, we define the vector-type harmonic tensor as,

$$\mathbb{V}_{ij} = -\frac{1}{2k}\left(\hat{D}_i\mathbb{V}_j + \hat{D}_j\mathbb{V}_i\right). \tag{4.22}$$

The vector-type harmonic tensors satisfy the following properties,

$$\left[\hat{\Delta} + k^2 - (n+1)K\right] \mathbb{V}_{ij} = 0, \tag{4.23}$$

$$\mathbb{V}^i_i = 0, \hat{D}_i \mathbb{V}^i_j = \frac{k^2 - (n-1)K}{2k} \mathbb{V}_j. \tag{4.24}$$

The eigenvalue k^2 is non-negative and $k^2 = 0$ occurs only for $K = 0$ i.e flat space for which the harmonic vectors become constant vectors. In our study we do not consider this trivial case and focus on non-constant harmonic vectors. An important point needs to be clarified here. For vector-type tensor harmonics to be defined $k^2 - (n-1)K > 0$. But $k^2 > 0$ and $K > 0$ does not guarantee the positivity of $k^2 - (n-1)K$. If $k^2 - (n-1)K < 0$ then vector-type harmonics should vanish i.e. $\hat{D}_i \mathbb{V}_j + \hat{D}_j \mathbb{V}_i = 0$. This is precisely the definition of Killing vector.

The vector perturbation[5] of the metric is then expanded as below,

$$h_{ab} = 0, h_{ai} = r f_a \mathbb{V}_i, h_{ij} = 2r^2 H_T \mathbb{V}_{ij}. \tag{4.25}$$

Such decompositions have been already utilised to explore the classical stability of a class of spacetimes [14]. Using these techniques one can decouple the equations and hence study the Lichnerowicz Laplacian eigenvalue equation in detail. One of the key features to be noted is that the symmetry of the *base* manifold enabled us to carry out the decomposition which other wise could not have been possible.

[5]Here a point needs to be clarified. $h_{ai} = h^{(1)}_{ai}$ and $h_{ij} = \hat{D}_i h^{(1)}_{T\ j} + \hat{D}_i h^{(1)}_{T\ j}$. Thus, $h^{(1)}_{T\ i} = -r^2 \frac{H_T}{k} \mathbb{V}_i$

Chapter 5

Summary and Discussion

This thesis primarily reviews the basics formalism to carry out perturbation analysis for a class of spacetime geometries. We already discussed in the introduction how this formalism may be useful to explore the instabilities in Euclidean path integral approach to quantum gravity. The ideas were then applied to study the (in)stablity of higher dimensional Schwarzchild spacetimes under Ricci flow. We briefly summarise the main points of the thesis in this section:

1. The gauge-freedom of the theory enables us to impose the transverse-traceless conditions. Although *transverse* conditions may always be imposed in general but *traceless* condition can only be imposed in *vacuo*.

2. In TT gauge, the stability of the class of spacetimes we have considered depends on the spectrum of the Lichnerowicz operator.

   In the case of Ricci flow problem we focussed on, the strategy is to break the perturbation variable into a transverse-traceless piece, a trace and a traceless piece of a divergence. The TT part decouples from others because the Lichnerowicz operator is also TT. We then study this TT part.

3. To simplify the Lichnerowicz operator eigenvalue equation we need to impose certain symmetry conditions on the *base* manifold. In our case

we assumed that that base manifold is maximally symmetric and has a constant curvature.

4. The symmetry of the base manifold enables one to decompose the perturbation into its tensor, vector and scalar part. These in turn can be expanded in terms of *respective* harmonics on the base manifold. The Lichnerowicz eigenvalue equation can then be studies for each type of perturbation separately.

5.1 Future Directions

We are presently analysing the flow of vector perturbations using the strategy suggested. Also, we plan to analyse the scalar perturbations moving forward on the same track. We believe that systematic analysis as suggested would help us better understand the unstable modes(if any) of the higher dimensional Schwarzchild spacetimes(Schwarzchild-Tangherlini) under Ricci flow.

We hope to generalise the ideas discussed in this thesis to study certain problems which will help us understand better the stabilities/instabilities of classical spacetimes in quantum gravity.

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Appendix A

Variational Formulae

In this section we review(and derive) some of the important formulae related to linear perturbation theory in arbitrary background.

A.0.1 Variation of Christoffel symbols

Let g(s) be a one-parameter family of metrics such that $\frac{\partial}{\partial s} g_{ij} = h_{ij}$, then we have

$$\frac{\partial}{\partial s}\Gamma^k{}_{ij} = \frac{1}{2} g^{kl} \left(-\nabla_l h_{ij} + \nabla_i h_{jl} + \nabla_j h_{il}\right) \tag{A.1}$$

To prove the above formula we follow the usual trick of computing at an arbitrary point p in the manifold in normal coordinates. Then we have $\partial_k g_{ij}(p) = 0$. Hence, $\Gamma^k{}_{ij}(p) = 0$. Also in such coordinates all the covariant derivatives can be replaced by partial derivatives. Thus at p we have,

$$\begin{aligned} \frac{\partial}{\partial s}\Gamma^k{}_{ij} &= \frac{1}{2} g^{kl} \left(-\partial_l \frac{\partial}{\partial s} g_{ij} + \partial_i \frac{\partial}{\partial s} g_{jl} + \partial_j \frac{\partial}{\partial s} g_{il}\right) && \text{(A.2)} \\ &= \frac{1}{2} g^{kl} \left(-\partial_l h_{ij} + \partial_i h_{jl} + \partial_j h_{il}\right) && \text{(A.3)} \end{aligned}$$

Now since at p partial derivatives and covariant derivatives can be used interchangeably hence we have

$$\frac{\partial}{\partial s}\Gamma^k{}_{ij} = \frac{1}{2}g^{kl}\left(-\nabla_l h_{ij} + \nabla_i h_{jl} + \nabla_j h_{il}\right) \tag{A.4}$$

Also since both sides of the above equation are components of tensors hence its validity is guaranteed in all coordinates systems.

A.0.2 Variation of Riemann and Ricci

Recalling the definition of Riemann and Ricci tensors we have

$$R^i{}_{jkl} = \partial_k \Gamma^i{}_{lj} - \partial_l \Gamma^i{}_{kj} + \Gamma^i{}_{km}\Gamma^m{}_{lj} - \Gamma^i{}_{lm}\Gamma^m{}_{kj} \tag{A.5}$$

$$R_{jl} = R^k{}_{jkl} = \partial_k \Gamma^k{}_{lj} - \partial_l \Gamma^k{}_{kj} + \Gamma^k{}_{km}\Gamma^m{}_{lj} - \Gamma^k{}_{lm}\Gamma^m{}_{kj} \tag{A.6}$$

By same logic as in case of Christoffel symbols, the variation of Ricci tensor is given by

$$\frac{\partial}{\partial s}R_{jl} = \nabla_k\left(\frac{\partial}{\partial s}\Gamma^k{}_{lj}\right) - \nabla_l\left(\frac{\partial}{\partial s}\Gamma^k{}_{kj}\right) \tag{A.7}$$

On substituting from equation we have

$$\begin{aligned}\frac{\partial}{\partial s}R_{jl} &= \nabla_k\left\{\frac{g^{ki}}{2}\left(-\nabla_i h_{lj} + \nabla_l h_{ij} + \nabla_j h_{il}\right)\right\} \\ &\quad - \nabla_l\left\{\frac{g^{ki}}{2}\left(-\nabla_i h_{kj} + \nabla_k h_{ij} + \nabla_j h_{ik}\right)\right\} \\ &= \nabla^i\left\{\frac{-\nabla_i h_{lj} + \nabla_l h_{ij} + \nabla_j h_{il}}{2}\right\} - \frac{\nabla_l \nabla_j h}{2}\end{aligned} \tag{A.8}$$

Now recall the following relation

$$[\nabla_a, \nabla_b]V_c = -R^m{}_{cab}V_m \tag{A.9}$$

Using this we have the variation of Ricci tensor as

$$2\frac{\partial}{\partial s}R_{jl} = -\nabla^i\nabla_i h_{lj} - \nabla_j\nabla_l h + R_{ml}h^m{}_j + R_{mj}h^m{}_l - 2R_{mjkl}h^{mk} + \nabla_l\nabla^i h_{ij} + \nabla_j\nabla^i h_{il} \tag{A.10}$$

To obtain the formula for the variation of Ricci scalar we recall that Ricci scalar is given as

$$R = g^{ij}R_{ij} \tag{A.11}$$

On differenting w.r.t. s we obtain

$$\frac{\partial}{\partial s}R = R_{ij}\frac{\partial}{\partial s}g^{ij} + g^{ij}\frac{\partial}{\partial s}R_{ij} \tag{A.12}$$

On plugging in the expression for variation of Ricci tensor calculated before we have

$$\frac{\partial}{\partial s}R = -h_{ij}R^{ij} + \nabla^i\nabla^j h_{ij} - \nabla^k\nabla_k h \tag{A.13}$$