On Solutions of Nonlinear First-Order PDEs

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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Certificate

This is to certify that this thesis entitled "On Solutions of Nonlinear First-Order PDEs" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents the work by "Gaurav Prabhakar Sawant" at "TIFR Centre for Applicable Mathematics, Bengaluru", under the supervision of "Prof. K. T. Joseph" during the academic year 2012-2013.

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Acknowledgements

I sincerely thank Prof. Mythily Ramaswamy (Dean, TIFR-CAM) and Prof. K. T. Joseph (TIFR-CAM) for providing me with an opportunity to carry out my Thesis work at TIFR-CAM, Bengaluru, and thereby allowing me to use the library and computer facilities of the Institute, and Dr. Anisa Chorwadwala (IISER, Pune), for her kind suggestions regarding the subject area of the Thesis.

During my stay at TIFR-CAM, Bengaluru, I have been greatly profited from the discussions with, and comments by, many peers, in particular: A. Aggarwal, S. Arya, A. Bhattacharya, D. Bhattacharya, S. Bhattacharya, A. P. Chowdhury, I. Chowdhury, A.-J. Dasgupta, D. Ganguly, R. Gupta, R. Mishra, D. Ray, M. Sahoo, A. Sarkar, M. Sharma, M.-K. Singh, and P. Suchde. I am also grateful to Prof. Adimurthi, Prof. A. Apte, Prof. C. S. Aravinda, Prof. I. Biswas, Prof. D. Chakrabarti, Prof. G. D. V. Gowda, Prof. K. Sandeep, and Prof. S. Vadlamani for their valuable insightful suggestions from time to time.

I dedicate my thesis to my family for their unconditional and endless support and love throughout the years of my existence.

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March, 2013

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Abstract

We start with the linear PDEs: the Transport Equation, which arises in the physical phenomena where quantities like particles or energy are transferred inside a physical system via convection; the Laplace Equation, which accurately describes the behaviour of potentials, and thus has important applications in the fields of electromagnetics, astronomy and fluid mechanics; and the Heat Equation, which governs heat diffusion as well as other diffusive processes such as particle diffusion or the propagation of action potential in nerve cells. We derive explicit formulae for solutions of these PDEs and their nonhomogeneous counterparts using analytical techniques.

In the second chapter, we construct some exact solutions of some nonlinear first-order PDEs. Then we consider the Cauchy Problem by reducing the PDEs to the corresponding ODEs, and solve it locally.

In the third chapter, we consider the Hamilton-Jacobi Equations, and we use Calculus of Variations to obtain an explicit solution for the initial-value problem.

Next we consider scalar convex conservation laws, and introduce weak formulation of the initial-value problem. We derive an explicit formula for the solution and study its qualitative properties.

Finally we study the initial-boundary-value problem for scalar convex conservation laws. The boundary condition is prescribed as per Bardos, LeRoux and Nédélec [1], and explicit representation of the solution is obtained. The nature of solution for a special initial and boundary data is illustrated.

Contents

Chapter 1

Some Important Linear PDEs

We present the explicit solutions for the linear PDEs

1. The Transport Equation: $u_t + b \cdot \nabla u = 0$
2. Laplace's Equation: $\Delta u = 0$
3. Heat Equation: $u_t - \Delta u = 0$

and their nonhomogeneous counterparts. Unless otherwise stated, we denote a typical point in space by $x = (x_1, \ldots, x_n)$, and a typical time by t.

1.1 The Transport Equation

The Transport Equation with constant coefficients is the simplest of all PDEs:

$$u_t + b \cdot \nabla u = 0 \qquad \text{in } \mathbb{R}^n \times (0, \infty) \tag{1.1}$$

where $u = u(x,t)$ is the unknown function, $b = (b_1, \ldots, b_n)$ a fixed vector in $\mathbb{R}^n$, and ∇ is the spatial gradient operator: $\nabla u = (u_{x_1}, \ldots, u_{x_n})$.

We fix a point $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and define $z(s) := u(x+sb, t+s)$ for $s \in \mathbb{R}$. Then

$$\dot{z}(s) = \nabla u(x+sb, t+s) \cdot b + u_t(x+sb, t+s) = 0, \tag{1.2}$$

which makes $z(s)$ a constant. Further, for each point (x,t), u is constant on the line through (x,t) with the direction $(b,1) \in \mathbb{R}$. Hence the knowledge of the value of u at any point on each such line allows us to determine values of u everywhere in $\mathbb{R}^n \times (0,\infty)$.

We wish to compute u satisfying the initial-value problem

$$\begin{cases} u_t + b \cdot Du &= 0 \qquad \text{in } \mathbb{R}^n \times (0,\infty) \\ \qquad\quad u &= g \qquad \text{in } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{1.3}$$

with known $b \in \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$. The line through (x,t) with direction $(b,1)$ intersects the plane $\Gamma := \mathbb{R}^n \times \{t=0\}$ at the point $(x - tb, 0)$, giving

$$u(x,t) = g(x - tb) \qquad (x \in \mathbb{R}^n,\ t \geq 0) \tag{1.4}$$

as u is constant on this line and $u(x - tb, 0) = g(x - tb)$. Thus we have arrived at a (weak) solution of (1.3).

In case of the associated nonhomogeneous problem

$$
\begin{cases} u_t + b \cdot Du &= f \qquad \text{in } \mathbb{R}^n \times (0, \infty) \\ \qquad\qquad u &= g \qquad \text{in } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{1.5}
$$

we set $z(s) := u(x + sb, t + s)$ as before, and deduce

$$
\dot{z}(s) = \nabla u(x + sb, t + s) \cdot b + u_t(x + sb, t + s) = f(x + sb, t + s).
$$

Consequently

$$
\begin{aligned}
u(x,t) - g(x - tb) &= z(0) - z(-t) \\
&= \int_{-t}^{0} \dot{z}(s)\, ds \\
&= \int_{-t}^{0} f(x + sb, t + s)\, ds \\
&= \int_{0}^{t} f(x + (s - t)b, s)\, ds.
\end{aligned}
$$

Thus

$$
u(x,t) = g(x - tb) + \int_0^t f(x + (s - t)b, s)\, ds \qquad (x \in \mathbb{R}^n, t \geq 0) \tag{1.6}
$$

solves (1.5).

1.2 Laplace's Equation

Laplace's equation comes up in a wide variety of physical contexts. In a typical interpretation u denotes the density of some quantity in equilibrium, defined within a given open set $U \in \mathbb{R}^n$. If V is any smooth subregion within U, the net flux of u through ∂V is zero:

$$
\int_{\partial V} \mathbf{F} \cdot \boldsymbol{\nu}\, dS = 0,
$$

where $\mathbf{F}$ is the net flux density and $\boldsymbol{\nu}$ is the unit outer normal field. Application of Gauss-Green Theorem gives

$$
\int_V \nabla \cdot \mathbf{F}\, dx = \int_{\partial V} \mathbf{F} \cdot \boldsymbol{\nu}\, dS = 0.
$$

Since V is arbitrary, $\nabla \cdot \mathbf{F} = 0$. Physically the flux is proportional to the gradient ∇u and points in the opposite direction, suggesting $\mathbf{F} = -a \nabla u$ for some $a > 0$. It follows that

$$
\nabla \cdot (\nabla u) = \Delta u = 0, \tag{1.7}
$$

which is the Laplace Equation. The solutions of Laplace's Equation are sometimes called *harmonics*.

Since Laplace's equation is invariant under rotations, it is customary to seek a radial solution - that is, $u(x) = v(r)$ for $r = |x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$. For $i = 1, \dots, n$ we have

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}\left(\sum_{i=1}^n x_i^2\right)^{-\frac{1}{2}} 2x_i = \frac{x_i}{r} \qquad (x \neq 0)$$

We thus have

$$u_{x_i} = v'(r)\frac{x_i}{r}, \quad u_{x_i x_i} = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right),$$

and so

$$\Delta u = v''(r) + \frac{n-1}{r}v'(r).$$

Hence $\Delta u = 0$ if and only if

$$v'' + \frac{n-1}{r}v' = 0. \tag{1.8}$$

For $v' \neq 0$ we have

$$\log(v')' = \frac{v''}{v'} = \frac{1-n}{r}$$

and hence $v'(r) = \frac{a}{r^{n-1}}$ for some constant a. Consequently

$$v(r) = \begin{cases} b\log(r) + c & (n = 2) \\ \frac{b}{r^{n-2}} + c & (n \geq 3) \end{cases}$$

for some constants b and c. The Fundamental Solution of Laplace's Equation is then given by

$$\Phi(|x|) = \begin{cases} -\frac{1}{2\pi}\log(|x|) & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & (n \geq 3) \end{cases} \tag{1.9}$$

where $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$ and Φ is normalised: $\int_0^\infty \Phi(r)\,dr = 1$.

Remark For easier use, we write

$$-\Delta\Phi = \delta_0 \qquad \text{in } \mathbb{R}^n \tag{1.10}$$

with δ_0 being the Dirac measure on $\mathbb{R}^n$ that gives unit mass to the point 0.

Theorem 1.2.1. *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic in Ω. Then*

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\,dS = 0,$$

where ν is the outward unit normal to $\partial\Omega$.

Proof. Set $F := \nabla u$. Using the Gauss-Green theorem, we find

$$\int_\Omega \nabla \cdot F = \int_{\partial\Omega} F \cdot \nu\,dS.$$

Therefore, since u is harmonic, we get

$$\int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\,dS = 0,$$

as expected. □

Theorem 1.2.2 (Mean-Value Property for Laplace's Equation)**.** *If $u \in C^2(\mathbb{R}^n)$ is harmonic, then*

$$u(x) = \fint_{\partial B(x,r)} u \, dS = \fint_{B(x,r)} u \, dy \tag{1.11}$$

for each ball $B(x,r) \subset U$.

Proof. Setting

$$\phi(r) := \fint_{\partial B(x,r)} u(y) \, dS(y) = \fint_{\partial B(0,1)} u(x+rz) \, dS(z), \tag{1.12}$$

we obtain

$$\phi'(r) = \fint_{\partial B(0,1)} \nabla u(x+rz) \cdot z \, dS(z).$$

Application of Green's Formulae yields

$$\begin{aligned} \phi'(r) &= \fint_{\partial B(x,r)} \nabla u(y) \cdot \frac{y-x}{r} \, dS(y) \\ &= \fint \frac{\partial u}{\partial \nu} \, dS(y) \\ &= \frac{r}{n} \fint_{B(x,r)} \Delta u(y) \, dy = 0; \end{aligned}$$

hence ϕ is constant, and so

$$\phi(r) = \lim_{t \to 0} \phi(t) = \lim_{t \to 0} \fint_{\partial B(x,t)} u(y) \, dS(y) = u(x).$$

Moreover, a conversion to polar coordinates gives us

$$\begin{aligned} \int_{B(x,r)} u \, dy &= \int_0^r \left(\int_{\partial B(x,s)} u \, dS \right) ds \\ &= u(x) \int_0^r n\alpha(n) s^{n-1} \, ds \\ &= \alpha(n) r^n u(x), \end{aligned}$$

and hence

$$\fint_{B(x,r)} u \, dy = \frac{1}{\alpha(n) r^n} \int_{B(x,r)} u \, dy = u(x)$$

as expected. □

Theorem 1.2.3 (Converse to Mean-Value Property (Theorem 1.2.2))**.** *If $u \in C^2(\mathbb{R}^n)$ satisfies*

$$u(x) = \fint_{\partial B(x,r)} u \, dS$$

for each ball $B(x,r) \subset U$, then u is harmonic.

Proof. Suppose if possible that $\Delta u \not\equiv 0$. Then there exists some ball $B(x,r) \subset U$ in which $\Delta u > 0$, say. Then for the ϕ as in (1.12), we get

$$0 = \phi'(r) = \frac{r}{n} \fint_{B(x,r)} u(y)\, dy > 0,$$

a contradiction. Hence u must be harmonic. $\square$

Theorem 1.2.4 (Strong Maximum Principle)**.** *Suppose that $U \subset \mathbb{R}^n$ is bounded and open, and $u \in C^2(U) \cap C(\bar{U})$ is harmonic within U. Then $\max_{\bar{U}} u = \max_{\partial U} u$. Moreover, if U is connected and there exists a point $x_0 \in U$ with $u(x_0) = \max_{\bar{U}} u$, then u is constant within U.*

Proof. Suppose that there exists a point $x_0 \in U$ such that $u(x_0) = M := \max_{B(x_0,r)} u$. Then the mean-value property (1.11) gives

$$M = u(x_0) = \fint_{B(x,r)} u(y)\, dy \le M$$

for $0 < r < \operatorname{dist}(x_0, \partial U)$. The equality would hold only if $u \equiv M$ within $B(x_0, r)$, and hence $u(y) = M$ for all $y \in B(x_0, r)$. Hence the set $\{x \in U \,|\, u(x) = M\}$ is both open and relatively closed in U, and for a connected U, it equals U. Therefore u is constant within U, and thus $\max_{\bar{U}} u = \max_{\partial U} u$ as well. $\square$

Theorem 1.2.5 (Smoothness)**.** *If $u \in C(U)$ satisfies* (1.11) *for every ball $B(x,r) \subset U$, then $u \in C^\infty(U)$.*

Proof. Let $\eta(x)$ be the standard mollifier

$$\eta(x) = \begin{cases} C\ exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \ge 1 \end{cases}$$

with the constant C such that $\int_{-\infty}^{\infty} \eta(x)\, dx = 1$. We set

$$\eta_\epsilon := \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right),$$

and $u^\epsilon := \eta_\epsilon * u$ in $U_\epsilon = \{x \in U \,|\, \operatorname{dist}(x, \partial U) > \epsilon\}$. By this definition, $u^\epsilon \in C^\infty(U_\epsilon)$. If $x \in U_\epsilon$, we have

$$\begin{aligned} u^\epsilon(x) &= \int_U \eta_\epsilon(x-y) u(y)\, dy \\ &= \frac{1}{\epsilon^n} \int_{B(x,\epsilon)} \eta\left(\frac{|x-y|}{\epsilon}\right) u(y)\, dy \\ &= \frac{1}{\epsilon} \int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right) \left(\int_{\partial B(x,r)} u\, dS\right) dr \\ &= \frac{1}{\epsilon} u(x) \int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right) n\alpha(n) r^{n-1}\, dr \\ &= u(x) \int_{B(0,\epsilon)} \eta_\epsilon\, dy = u(x). \end{aligned}$$

Thus $u^\epsilon \equiv u$ in U_ϵ, and hence $u \in C^\infty(U_\epsilon)$ for every $\epsilon > 0$; in particular $u \in C^\infty(U)$. $\square$

Theorem 1.2.6 (Liouville's Theorem)**.** *A bounded harmonic $u : \mathbb{R}^n \to \mathbb{R}$ is constant.*

Proof. Take any two points $x, x' \in \mathbb{R}^n$, and take $r > 0$ large enough so that the balls $B(x,r)$ and $B(x',r)$ coincide except for a small proportion of their volumes. Now, since u is harmonic and bounded, the mean-value property (1.11) tells us that the averages of u over the two balls are arbitrarily close, and hence u must assume the same value inside $B(x,r) \cap B(x',r)$ for x and x'. Since the choice of these points is arbitrary, sending $r \to \infty$ confirms that u is indeed constant in $\mathbb{R}$. □

Remark 1: The proof of Theorem 1.2.6 is obtained from Nelson [7].

Remark 2: If u is a harmonic and $u \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, then $u \equiv 0$. To show this, we use the mean-value property and Hölder's inequality to get

$$\begin{aligned}
u(x) &= \frac{1}{\alpha(n)r^n} \int_{B(x,r)} u(y)\,dy \\
&\leq \frac{1}{\alpha(n)r^n} \|u\|_{L^p(B(x,r))} \|\text{Volume of } (B(x,r))\|_{L^q} \\
&= \frac{1}{\alpha(n)r^n} \|u\|_{L^p(B(x,r))} (\alpha(n)r^n)^{\frac{1}{q}} \\
&= (\alpha(n)r^n)^{-\frac{1}{p}} \|u\|_{L^p(B(x,r))} \\
&\to 0 \qquad \text{as } r \to 0.
\end{aligned}$$

This is valid for all balls $B(x,r)$, and hence $u \equiv 0$.

Theorem 1.2.7 (Harnack's Inequality)**.** *Suppose that V is an open connected set compactly contained in U. Then there exists a constant $C > 0$, depending only on V, such that*

$$\sup_V u \leq C \inf_V u$$

for all nonnegative harmonic functions u in U. In particular,

$$\frac{u(y)}{C} \leq u(x) \leq Cu(y)$$

for all $x, y \in V$.

Proof. Let $r = \frac{1}{4}\text{dist}(V, \partial U)$. For $x, y \in V$ with $|x - y| \leq r$, we have

$$\begin{aligned}
u(x) &= ⨍_{B(x,2r)} u\,dz \\
&\geq \frac{1}{\alpha(n)2^n r^n} \int_{B(y,r)} u\,dz \\
&= \frac{1}{2^n} ⨍_{B(y,r)} u\,dz \\
&= \frac{u(y)}{2^n}.
\end{aligned}$$

Switching the roles of x and y once, we obtain

$$2^n u(y) \geq u(x) \geq u(y).$$

Since V is connected and $\bar{V}$ is compact, there exists a finite cover $\{B_i\}_{i=1}^N$ of V, with each ball B_i having radius $\frac{r}{2}$ and $B_i \cap B_{i-1} \neq \emptyset$ for $i = 2, \ldots, N$. Then for all $x, y \in V$, we have

$$u(x) \geq \frac{1}{2^{n(N+1)}} u(y).$$

□

1.3 Poisson's Equation

Poisson's Equation, $-\Delta u = f$, is the nonhomogeneous analogue of Laplace's Equation.

Theorem 1.3.1 (Solution of Poisson's Equation)**.** *Define u by the convolution*

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy \\ &= \begin{cases} -\frac{1}{2\pi} \int_{\mathbb{R}^n} \log(|x-y|) f(y)\, dy & (n=2) \\ \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} f(y)\, dy & (n \geq 3). \end{cases} \end{aligned} \tag{1.13}$$

Then $u \in C^2(\mathbb{R}^n)$, and $-\Delta u = f$ in $\mathbb{R}^n$.

Proof. Since

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \Phi(y) f(x-y)\, dy,$$

we have

$$\frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \left[\frac{f(x+he_i-y) - f(x-y)}{h} \right] dy.$$

for $h \neq 0$ and an n-dimensional unit vector $e_i = (0, \ldots, \underbrace{1}_{i^{\text{th}} \text{ position}}, \ldots, 0)$. But as $h \to 0$,

$$\frac{f(x+he_i-y) - f(x-y)}{h} \to f_{x_i}(x-y)$$

uniformly on $\mathbb{R}^n$. Thus

$$u_{x_i}(x) = \int_{\mathbb{R}^n} \Phi(y) f_{x_i}(x-y)\, dy \qquad (i = 1, \ldots, n).$$

Likewise, we have

$$u_{x_i x_j}(x) = \int_{\mathbb{R}^n} \Phi(y) f_{x_i x_j}(x-y)\, dy \qquad (i, j = 1, \ldots, n).$$

Since the right hand side of the above equation is continuous in x, we have $u \in C^2(\mathbb{R}^n)$.

Now, for a fixed $\epsilon > 0$, we have

$$\begin{aligned} \Delta u(x) &= \int_{B(0,\epsilon)} \Phi(y) \Delta_x f(x-y)\, dy + \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi(y) \Delta_x f(x-y)\, dy \\ &=: I_\epsilon + J_\epsilon. \end{aligned} \tag{1.14}$$

For I_ϵ, we find the bound

$$|I_\epsilon| \le C\|\Delta f\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\epsilon)} |\Phi(y)|\,dy \le \begin{cases} C\epsilon^2|\log\epsilon| & (n=2) \\ C\epsilon^2 & (n=3). \end{cases} \tag{1.15}$$

We use integration by parts on J_ϵ to get

$$\begin{aligned} J_\epsilon &= \int_{\mathbb{R}^n\setminus B(0,\epsilon)} \Phi(y)\Delta_y f(x-y)\,dy \\ &= -\int_{\mathbb{R}^n\setminus B(0,\epsilon)} \nabla\Phi(y)\cdot\nabla_y f(x-y)\,dy \\ &\quad + \int_{\partial B(0,\epsilon)} \Phi(y)\frac{\partial f}{\partial\nu}(x-y)\,dS(y) \\ &=: K_\epsilon + L_\epsilon, \end{aligned} \tag{1.16}$$

where ν denotes the inward unit normal to $\partial B(0,\epsilon)$. Using (1.9), we obtain

$$|L_\epsilon| \le \|\nabla f\|_{L^\infty(\mathbb{R}^n)} \int_{\partial B(0,\epsilon)} |\Phi(y)|\,dS(y) \le \begin{cases} C\epsilon|\log\epsilon| & (n=2) \\ C\epsilon & (n=3). \end{cases} \tag{1.17}$$

Now, for $y \neq 0$, we have

$$\nabla\Phi(y) = \frac{-1}{n\alpha(n)}\frac{y}{|y|^n}$$

and

$$\nu = \frac{-y}{|y|} = \frac{-y}{\epsilon} \qquad \text{on } \partial B(0,\epsilon).$$

Consequently

$$\frac{\partial\Phi}{\partial\nu}(y) = \nu\cdot\nabla\Phi(y) = \frac{1}{n\alpha(n)\epsilon^{n-1}} \qquad \text{on } \partial B(0,\epsilon).$$

Then, performing integration by parts in K_ϵ and using the fact that Φ is harmonic away from the origin, we obtain

$$\begin{aligned} K_\epsilon &= \int_{\mathbb{R}^n\setminus B(0,\epsilon)} \Phi(y)\Delta_y f(x-y)\,dy - \int_{\partial B(0,\epsilon)} \frac{\partial\Phi}{\partial\nu}(y) f(x-y)\,dS(y) \\ &= \frac{-1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(0,\epsilon)} f(x-y)\,dS(y) \\ &= -⨍_{\partial B(x,\epsilon)} f(y)\,dS(y) \to -f(x) \qquad \text{as } \epsilon\to 0. \end{aligned} \tag{1.18}$$

Combining (1.14)-(1.18) and passing to the limit $\epsilon \to 0$, we get $\Delta u = -f$ as expected. □

Theorem 1.3.2 (Uniqueness)**.** *Suppose $U \subset \mathbb{R}^n$ is bounded and open and let $g \in C(\partial U)$ and $f \in C(U)$. Then there exists at most one solution to the boundary-value problem*

$$\begin{cases} \Delta u = -f & \text{in } U \\ \quad u = g & \text{on } \partial U. \end{cases} \tag{1.19}$$

Proof. Suppose if possible that u and $\tilde{u}$ solve (1.19). Then $w := u - \tilde{u}$ satisfies $\Delta w = 0$. An integration by parts shows that

$$0 = -\int_U w\Delta w\,dx = \int_U |\nabla w|^2\,dx.$$

Thus $\nabla w \equiv 0$ in U, and since $w = 0$ on ∂U, we have $w = u - \tilde{u} = 0$. □

For any function $u \in C^2(\bar{U})$ and any point $x \in U$, the following identity can be derived using Green's formulae:

$$u(x) = \int_{\partial U} \Phi(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y) - \int_U \Phi(y-x)\Delta u(y)\,dy. \tag{1.20}$$

For a fixed x we introduce a corrector function $\phi^x(y)$ that satisfies

$$\begin{cases} \Delta\phi^x = 0 & \text{in } U \\ \phi^x = \Phi(y-x) & \text{on } \partial U. \end{cases} \tag{1.21}$$

We define Green's Function for the region U as

$$G(x,y) := \Phi(y-x) - \phi^x(y) \qquad (x, y \in U, x \neq y)$$

This allows us to rephrase u as

$$u(x) = -\int_{\partial U} u(y)\frac{\partial G}{\partial \nu}(x,y)\,dS(y) - \int_U G(x,y)\Delta u(y)\,dy \qquad (x \in U), \tag{1.22}$$

where $\frac{\partial G}{\partial \nu}(x,y) = \nabla_y G(x,y)\cdot\nu(y)$ is the outer normal derivative of G with respect to the variable y. For a function $u \in C^2(\bar{U})$ that satisfies the boundary-value problem

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U, \end{cases} \tag{1.23}$$

we have a representation formula for Poisson's Equation:

$$u(x) = -\int_{\partial U} g(y)\frac{\partial G}{\partial \nu}(x,y)\,dS(y) + \int_U f(y)G(x,y)\,dy \qquad (x \in U) \tag{1.24}$$

1.4 Heat Equation

The heat equation, also known as the diffusion equation, describes the evolution in time of the density u of some quantity such as heat, chemical concentration, etc. If $V \subset U$ is any smooth subregion, the rate of change of the total quantity within V equals the negative of the net flux through ∂V:

$$\frac{d}{dt}\int_V u\,dx = -\int_{\partial V} \mathbf{F}\cdot\nu\,dS, \tag{1.25}$$

$\mathbf{F}$ being the flux density. Thus

$$u_t = -\nabla\cdot\mathbf{F} \tag{1.26}$$

Since $\mathbf{F}$ is proportional to the gradient of u and points opposite of u, we have $\mathbf{F} = -a\nabla u$ for a positive a. Scaling the system to $a = 1$ gives the homogeneous heat equation:

$$u_t - \Delta u = 0 \tag{1.27}$$

We consider u of the form

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) \qquad (x \in \mathbb{R}^n, t > 0), \tag{1.28}$$

where the constants α, β and the function $v : \mathbb{R}^n \to \mathbb{R}$ are to be found. To have u invariant under the scaling $u(x,t) \mapsto \lambda^\alpha u(\lambda^\beta x, \lambda t)$ for all $\lambda > 0$, $x \in \mathbb{R}^n$ and $t > 0$, we set $\lambda = t^{-1}$, write $v(y) := u(y, 1)$, and thereby compute

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot \nabla v(y) + t^{-(\alpha+2\beta)} \Delta v(y) = 0 \tag{1.29}$$

for $y := t^\beta x$. Setting $\beta = \frac{1}{2}$ and taking the common factors out from (1.29), we get

$$\alpha v + \frac{1}{2} y \cdot \nabla v + \Delta v = 0. \tag{1.30}$$

Further we assume v to be radial - that is, for some $w : \mathbb{R} \to \mathbb{R}$, we have $v(y) = w(|y|) =: w(r)$. Then

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0.$$

Now we set $\alpha = \frac{n}{2}$ and simplify the above to get

$$(r^{n-1} w')' + \frac{1}{2}(r^n w)' = 0,$$

and thus

$$r^{n-1} w' + \frac{1}{2} r^n w = a$$

for some constant a. Assuming w, w' to vanish for large r, we put $a = 0$ and obtain

$$w' = -\frac{1}{2} w,$$

which in turn gives

$$w = b \exp\left(-\frac{r^2}{4}\right)$$

for some constant b. Therefore, our guess for the solution of the heat equation (1.27) is

$$\Phi(x,t) = \begin{cases} b t^{\frac{-n}{2}} \exp\left(-\frac{|x|^2}{4t}\right) & (t > 0) \\ 0 & (t < 0) \end{cases} \tag{1.31}$$

for $x \in \mathbb{R}^n$. Further, we set

$$\begin{aligned} 1 &= \int_{\mathbb{R}^n} \Phi(x,t)\,dx \\ &= bt^{\frac{-n}{2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x|^2}{4t}\right) dx \\ &= 4^{\frac{n}{2}} b \int_{\mathbb{R}^n} \exp(|z|^2)\,dz \\ &= 4^{\frac{n}{2}} b \prod_{i=1}^{n} \int_{-\infty}^{\infty} \exp(z_i^2)\,dz_i \\ &= (4\pi)^{\frac{n}{2}} b, \end{aligned}$$

thereby getting $b = (4\pi)^{\frac{-n}{2}}$. With this b, the *fundamental solution* of the heat equation is

$$\Phi(x,t) = \begin{cases} (4\pi t)^{\frac{-n}{2}} \exp\left(-\frac{|x|^2}{4t}\right) & (t > 0) \\ 0 & (t < 0) \end{cases} \tag{1.32}$$

for $x \in \mathbb{R}^n$.

Theorem 1.4.1 (Solution of the Initial-Value Problem)**.** *Consider the initial-value problem*

$$u_t - \Delta u = 0 \qquad \text{in } \mathbb{R}^n \times (0,\infty) \tag{1.33}$$

$$u = g \qquad \text{on } \mathbb{R}^n \times \{t = 0\} \tag{1.34}$$

with $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. *Then*

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)\,dy \\ &= (4\pi t)^{\frac{-n}{2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right) g(y)\,dy \qquad (x \in \mathbb{R}^n, t > 0) \end{aligned} \tag{1.35}$$

satisfies (1.33) *and is* $C^\infty(\mathbb{R}^n \times (0,\infty))$. *Moreover, for each point* $x^0 \in \mathbb{R}^n$, *we have*

$$\lim_{\substack{(x,t)\to(x^0,0) \\ x\in\mathbb{R}^n, t>0}} u(x,t) = g(x^0). \tag{1.36}$$

Proof. The function $t^{\frac{-n}{2}} \exp\left(-\frac{|x|^2}{4t}\right)$ is infinitely differentiable, and has uniformly bounded derivatives of all orders on $\mathbb{R}^n \times [\delta,\infty)$ for every $\delta > 0$. Therefore $u \in C^\infty(\mathbb{R}^n \times (0.\infty))$. Moreover,

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \int_{\mathbb{R}^n} [(\Phi_t - \Delta_x \Phi)(x-y,t)]g(y)\,dy \\ &= 0 \qquad (x \in \mathbb{R}^n, t > 0). \end{aligned}$$

Now fix $x^0 \in \mathbb{R}^n$, and let $\epsilon > 0$. Choose $\delta > 0$ such that for $y \in \mathbb{R}^n$,

$$|g(y) - g(x^0)| < \epsilon \qquad \text{if} \quad |y - x^0| < \delta. \tag{1.37}$$

Thus for $|x - x^0| < \frac{\delta}{2}$, we have

$$\begin{aligned}
|u(x,t) - g(x^0)| &= \left|\int_{\mathbb{R}^n} \Phi(x-y,t)[g(y) - g(x^0)]\, dy\right| \\
&\leq \int_{B(x^0,\delta)} \Phi(x-y,t)|g(y) - g(x^0)|\, dy \\
&\quad + \int_{\mathbb{R}^n \setminus B(x^0,\delta)} \Phi(x-y,t)|g(y) - g(x^0)|\, dy \\
&=: I + J.
\end{aligned}$$

It follows from (1.37) that

$$I \leq \epsilon \int_{\mathbb{R}^n} \Phi(x-y,t)\, dy = \epsilon.$$

Furthermore, with $|x - x^0| \leq \frac{\delta}{2}$ and $|y - x^0| \geq \delta$, we also have

$$|y - x^0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x^0|,$$

and thus $|y - x| \geq \frac{1}{2}|y - x^0|$. As a consequence, we get

$$\begin{aligned}
J &\leq 2\|g\|_{L^\infty} \int_{\mathbb{R}^n \setminus B(x^0,\delta)} \Phi(x-y,t)\, dy \\
&\leq C t^{\frac{-n}{2}} \int_{\mathbb{R}^n \setminus B(x^0,\delta)} \exp\left(-\frac{|x-y|^2}{4t}\right) dy \\
&\leq C t^{\frac{-n}{2}} \int_{\mathbb{R}^n \setminus B(x^0,\delta)} \exp\left(-\frac{|y-x^0|^2}{16t}\right) dy \\
&= C t^{\frac{-n}{2}} \int_{\delta}^{\infty} \exp\left(-\frac{r^2}{16t}\right) r^{n-1}\, dr \to 0 \qquad \text{as } t \to 0^+.
\end{aligned}$$

Hence for $|x - x^0| < \frac{\delta}{2}$ and a small enough $t > 0$, $|u(x,t) - g(x^0)| < 2\epsilon$, from which (1.36) follows. □

Now consider the nonhomogeneous initial-value problem

$$u_t - \Delta u = f \qquad \text{in } \mathbb{R}^n \times (0,\infty) \tag{1.38}$$

$$u = 0 \qquad \text{on } \mathbb{R}^n \times \{t = 0\}. \tag{1.39}$$

For fixed s, the function

$$v = v(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy$$

is a solution of

$$\begin{cases} v_t(\cdot;s) - \Delta v(\cdot;s) = 0 & \text{in } \mathbb{R}^n \times (s,\infty) \\ u(\cdot;s) = f(\cdot,s) & \text{on } \mathbb{R}^n \times \{t = s\}. \end{cases} \tag{1.40}$$

Duhamel's Principle: A solution of (1.38)-(1.39) can be found in terms of solutions of (1.40) by integrating with respect to s:

$$\begin{aligned} u(x,t) &= \int_0^t v(x,t;s)\,ds \qquad (x\in\mathbb{R}^n, t>0) \\ &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dy\,ds \\ &= \int_0^t (4\pi(t-s))^{\frac{-n}{2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right) f(y,s)\,dy\,ds. \end{aligned} \tag{1.41}$$

Theorem 1.4.2 (Solution of Nonhomogeneous Problem)**.** *Suppose that* $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ *and* f *has compact support. Then* u *defined in* (1.41) *is* $C_1^2(\mathbb{R}^n \times (0,\infty))$ *and satisfies* (1.38)*. Further, for each point* $x^0 \in \mathbb{R}^n$*, we have*

$$\lim_{\substack{(x,t)\to(x^0,0) \\ x\in\mathbb{R}^n, t>0}} u(x,t) = 0. \tag{1.42}$$

Proof. We change variables to get

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,t-s)\,dy\,ds.$$

Since $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ has compact support and $\Phi(y,s)$ is smooth near $s=t>0$, we have

$$\begin{aligned} u_t(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_t(x-y,t-s)\,dy\,ds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\,dy \end{aligned}$$

and

$$u_{x_ix_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_{x_ix_j}(x-y,t-s)\,dy\,ds \qquad (i,j=1,\ldots,n).$$

Thus u_t, $\Delta_x u$, and similarly u, $\nabla_x u$ belong to $C(\mathbb{R}^n \times (0,\infty))$, that is, $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$.

We calculate

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \left[\left(\frac{\partial}{\partial t} - \Delta_x\right) f(x-y,t-s)\right] dy\,ds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\,dy \\ &= \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(y,s) \left[\left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y,t-s)\right] dy\,ds \\ &\quad + \int_\epsilon^t \int_{\mathbb{R}^n} \Phi(y,s) \left[\left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y,t-s)\right] dy\,ds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\,dy \\ &=: I_\epsilon + J_\epsilon + K. \end{aligned} \tag{1.43}$$

Now

$$|I_\epsilon| \leq (\|f_t\|_{L^\infty} + \|\Delta f\|_{L^\infty}) \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(y,s)\,dy\,ds \leq C\epsilon. \tag{1.44}$$

Also

$$\begin{aligned}
J_\epsilon &= \int_\epsilon^t \int_{\mathbb{R}^n} \left[\left(-\frac{\partial}{\partial s} - \Delta_y\right)\Phi(y,s)\right] f(x-y,t-s)\,dy\,ds \\
&\quad + \int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y,t-\epsilon)\,dy \\
&\quad - \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\,dy \\
&= \int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y,t-\epsilon)\,dy - K \qquad (1.45)
\end{aligned}$$

as Φ solves the heat equation. Combining (1.43)-(1.45), we obtain

$$\begin{aligned}
u_t(x,t) - \Delta u(x,t) &= \lim_{\epsilon\to 0} \int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y,t-\epsilon)\,dy \\
&= f(x,t)
\end{aligned}$$

with $x \in \mathbb{R}^n$, $t > 0$. The proof is complete with $\|u(\cdot,t)\|_{L^\infty} \leq t\|f\|_{L^\infty} \to 0$. □

Remark Combining Theorem 1.4.1 and Theorem 1.4.2, we find that

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\,dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dy\,ds,$$

under the hypotheses on f and g, is a solution of

$$\begin{aligned}
u_t - \Delta u = 0 &\qquad \text{in } \mathbb{R}^n \times (0,\infty) \\
u = g &\qquad \text{on } \mathbb{R}^n \times \{t = 0\}
\end{aligned}$$

Chapter 2

Nonlinear First-Order PDEs

We investigate general nonlinear first-order partial differential equations of the form

$$F(\nabla u, u, x) = 0,$$

where x belongs to an open subset U of $\mathbb{R}^n$; $F : \mathbb{R}^n \times \mathbb{R} \times \bar{U} \to \mathbb{R}$ is given, and $u : \bar{U} \to \mathbb{R}$ is the unknown. For $p \in \mathbb{R}^n$, $z \in \mathbb{R}$ and $x \in U$, we write

$$F = F(p, z, x) = F(p_1, \ldots, p_n, z, x_1, \ldots, x_n).$$

We also assume hereafter that F is smooth, and set

$$\begin{cases} \nabla_p F = (F_{p_1}, \ldots, F_{p_n}) \\ \nabla_z F = F_z \\ \nabla_x F = (F_{x_1}, \ldots, F_{x_n}) \end{cases}$$

In addition, we prescribe a boundary condition $u = g$ on a given $\Gamma \subset \partial U$ and $g : \Gamma \to \mathbb{R}$.

Note: We call the problem of solving a PDE with certain conditions on a hypersurface (in the above case, the subset $\Gamma \subset \partial U$) within the domain as the *Cauchy Problem*.

2.1 Complete Integrals

Suppose that $A \subset \mathbb{R}^n$ is open. Assume that for each parameter $a = (a_1, \ldots, a_n) \in A$ we have a C^2 solution $u = u(x; a)$ of the PDE

$$F(\nabla u, u, x) = 0 \tag{2.1}$$

We write

$$(\nabla_a u, \Delta_{xa} u) := \begin{pmatrix} u_{a_1} & u_{x_1 a_1} & \cdots & u_{x_n a_1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{a_n} & u_{x_1 a_n} & \cdots & u_{x_n a_n} \end{pmatrix}_{n \times (n+1)} \tag{2.2}$$

Then a *complete integral* in $U \times A$ is defined to be the C^2 function $u = u(x; a)$ that solves the PDE (2.1) and satisfies

$$\operatorname{rank}(\nabla_a u, \Delta_{xa} u) = n \qquad (x \in U, a \in A)$$

2.2 New Solutions from Envelopes

Let $u = u(x; a)$ be a C^1 function of $x \in U$, $a \in A$, where $U \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^m$ are open sets. Consider the equation

$$\nabla_a u(x; a) = 0 \qquad (x \in U,\ a \in A) \tag{2.3}$$

Suppose that we can solve (2.3) for the parameter a as a C^1 function of x: $a = \phi(x)$; thus

$$\nabla_a u(x; \phi(x)) = 0 \qquad (x \in U). \tag{2.4}$$

We then define the *envelope* of the functions $\{u(\cdot; a)\}_{a \in A}$ as

$$v(x) := u(x; \phi(x)) \tag{2.5}$$

This envelope v is sometimes called a *singular integral* of (2.1).

Theorem 2.2.1 (Construction of new solutions)**.** *Suppose that $u = u(\cdot, a)$ solves the PDE* (2.1) *for each $a \in A$ as above. If the envelope v defined by* (2.4) *and* (2.5) *exists as a C^1 function, then v solves* (2.1) *as well.*

Proof. We have $v(x) := u(x; \phi(x))$; and so for $i = 1, \ldots, n$

$$\begin{aligned} v_{x_i}(x) &= u_{x_i}(x; \phi(x)) + \sum_{j=1}^{n} u_{a_j}(x; \phi(x)) \phi^j_{x_i}(x) \\ &= u_{x_i}(x; \phi(x)) \qquad \text{(according to (2.4))} \end{aligned}$$

Hence for each $x \in U$,

$$F(\nabla v(x), v(x), x) = F(\nabla u(x; \phi(x)), u(x; \phi(x)), x) = 0.$$

□

2.3 Derivation of Characteristic ODE

Suppose u solves (2.1) with a boundary condition

$$u = g \qquad \text{on } \Gamma, \tag{2.6}$$

for given $\Gamma \subseteq \partial U$ and $g : \Gamma \to \mathbb{R}$. We fix a point $x \in U$ and try to find a curve inside U connecting x with a point $x^0 \in \Gamma$, along which we can calculate $u(x)$. Since we already know the value of u at x^0, we can compute u all along the curve, and so in particular at x.

Suppose that this curve is parametrically described by the function $\mathbf{x}(s) = (x^1(s), \ldots, x^n(s))$ for some real parameter s. Assuming u to be a C^2 solution of (2.1), we define

$$z(s) := u(\mathbf{x}(s)) \tag{2.7}$$

We also set

$$\mathbf{p}(s) := \nabla u(\mathbf{x}(s)) \tag{2.8}$$

that is, $\mathbf{p}(s) = (p^1(s), \ldots, p^n(s))$, where

$$p^i(s) = u_{x_i}(\mathbf{x}(s)) \tag{2.9}$$

We differentiate (2.9) with respect to x_i to get:

$$\dot{p}^i(s) = \sum_{j=1}^{n} u_{x_i x_j}(\mathbf{x}(s))\dot{x}^j(s) \tag{2.10}$$

We also differentiate (2.1) with respect to x_i to get

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(\nabla u, u, x)u_{x_i x_j} + \frac{\partial F}{\partial z}(\nabla u, u, x)u_{x_i} + \frac{\partial F}{\partial x_i}(\nabla u, u, x) = 0 \tag{2.11}$$

Next we set

$$\dot{x}^j(s) = \frac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)) \qquad (j = 1, \ldots, n) \tag{2.12}$$

and evaluate (2.11) at $x = \mathbf{x}(s)$ to obtain the identity

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s))u_{x_i x_j}(\mathbf{x}(s)) + \frac{\partial F}{\partial z}(\mathbf{p}(s), z(s), \mathbf{x}(s))p^i(s) + \frac{\partial F}{\partial x_i}(\mathbf{p}(s), z(s), \mathbf{x}(s)) = 0$$

Substituting this and (2.12) into (2.10), we obtain, for $i = 1, \ldots, n$:

$$\dot{p}^i(s) = -\frac{\partial F}{\partial z}(\mathbf{p}(s), z(s), \mathbf{x}(s))p^i(s) - \frac{\partial F}{\partial x_i}(\mathbf{p}(s), z(s), \mathbf{x}(s)) \tag{2.13}$$

Finally, differentiating (2.7) gives

$$\dot{z}(s) = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(\mathbf{x}(s))\dot{x}^j(s) = \sum_{j=1}^{n} p^j(s)\frac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)), \tag{2.14}$$

using (2.9) and (2.12). We summarize the equations (2.12)-(2.14) as:

$$\begin{cases} (a) \quad \dot{\mathbf{p}}(s) = -\nabla_x F(\mathbf{p}(s), z(s), \mathbf{x}(s)) - \nabla_z F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \\ (b) \quad \dot{z}(s) = \nabla_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) \\ (c) \quad \dot{\mathbf{x}}(s) = \nabla_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \end{cases} \tag{2.15}$$

The system (2.15) of $2n + 1$ first-order ODEs comprises the *characteristic equations* of the nonlinear first-order PDE (2.1). The functions $\mathbf{p}(\cdot)$, $z(\cdot)$, $\mathbf{x}(\cdot)$ are called the *characteristics.*
We have thus proved

Theorem 2.3.1 (Structure of characteristic ODEs). *Let $u \in C^2(u)$ solve the nonlinear first-order PDE* (2.1) *in U. If $\boldsymbol{x}(\cdot)$ solves the ODE* (2.15)*(c), and $\boldsymbol{p}(\cdot) = \nabla u(\boldsymbol{x}(\cdot))$ and $z(\cdot) = u(\boldsymbol{x}(\cdot))$, then $\boldsymbol{p}(\cdot)$ and $z(\cdot)$ solve* (2.15)*(a) and* (2.15)*(b), respectively.*

Example 1: F is linear. Consider the PDE

$$F(\nabla u, u, x) = \mathbf{b}(x) \cdot \nabla u(x) + c(x)u(x) = 0 \qquad (x \in U). \tag{2.16}$$

Then $F(p, z, x) = \mathbf{b}(x) \cdot p + c(x)z$, and so

$$\nabla_p F = \mathbf{b}(x). \tag{2.17}$$

Equation (2.15)(c) then becomes

$$\mathbf{x}(s) = \mathbf{b}(\mathbf{x}(s)), \tag{2.18}$$

which is an ODE involving only $\mathbf{x}(\cdot)$. Furthermore, equation (2.15)(b) becomes

$$\dot{z}(s) = \mathbf{b}(\mathbf{x}(s)) \cdot \mathbf{p}(s). \tag{2.19}$$

Since $\mathbf{p}(s) = \nabla u(\mathbf{x}(\cdot))$, equation (2.16) simplifies equation (2.19) to yield

$$\dot{z}(s) = -c(\mathbf{x}(s))z(s) \tag{2.20}$$

Once $\mathbf{x}(s)$ is known by solving (2.18). In summary, equations (2.18) and (2.20) comprise the characteristic equations for the PDE (2.16).

Example 2: F is quasilinear. Consider the PDE

$$F(\nabla u, u, x) = \mathbf{b}(x, u(x)) \cdot \nabla u(x) + c(x, u(x)) = 0 \tag{2.21}$$

In this case $F(p, z, x) = \mathbf{b}(x, z) \cdot p + c(x, u(x))$; whence

$$\nabla_p F = \mathbf{b}(x, z)$$

Then we can write the characteristic equations of (2.21) as

$$\begin{cases} (a) & \dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s), z(s)) \\ (b) & \dot{z}(s) = -c(\mathbf{x}(s), z(s)) \end{cases} \tag{2.22}$$

For fully nonlinear F, we have to integrate equations (2.15) if possible.

2.4 Boundary conditions

We use the characteristic ODEs (2.15) to actually solve the boundary-value problem (2.1) and (2.6) locally near some $\Gamma \subset \partial U$. To simplify the calculations, we "flatten" a part of the boundary ∂U near a prescribed point $x^0 \in \partial U$. Then we can find smooth mappings $\mathbf{\Phi}$, $\mathbf{\Psi} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathbf{\Psi} = \mathbf{\Phi}^{-1}$ and $\mathbf{\Phi}$ straightens ∂U near x^0. A calculation with $V := \mathbf{\Phi}(U)$ and $v(y) := u(\mathbf{\Psi}(y))$ transforms our problem into the form

$$\begin{cases} G(\nabla v, v, y) & = 0 \quad \text{in } V \\ \qquad\qquad v & = h \quad \text{on } \Delta, \end{cases} \tag{2.23}$$

where $h(y) := g(\mathbf{\Psi}(y))$ and $\Delta := \mathbf{\Phi}(\Gamma)$. In effect, the boundary near x^0 is straightened out, as well as the form of the problem is preserved.

Assume that, near a given point $x^0 \in \Gamma$, the plane $\{x_n = 0\}$ coincides with Γ. A prescribed data $(p^0, z^0, x^0) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ that corresponds to

$$\mathbf{p}(0) = p^0, \quad z(0) = z^0, \quad \mathbf{x}(0) = x^0 \tag{2.24}$$

is said to be *admissible* if it satisfies the *compatibility conditions*

$$\begin{cases} z^0 &= g(x^0) \\ p_i^0 &= g_{x_i}(x^0) \qquad (i = 1, \dots, n-1) \\ F(p^0, z^0, x^0) &= 0. \end{cases} \tag{2.25}$$

Now we are in a position to solve the characteristic ODEs

$$\begin{cases} (a) \quad \dot{\mathbf{p}}(s) = -\nabla_x F(\mathbf{p}(s), z(s), \mathbf{x}(s)) - \nabla_z F(\mathbf{p}(s), z(s), \mathbf{x}(s))\mathbf{p}(s) \\ (b) \quad \dot{z}(s) = \nabla_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) \\ (c) \quad \dot{\mathbf{x}}(s) = \nabla_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \end{cases} \tag{2.26}$$

for a given point $y = (y_1, \cdot, y_{n-1}, 0) \in \Gamma$ near x^0, with the initial conditions

$$\mathbf{p}(0) = \mathbf{q}(y), \quad z(0) = g(y), \quad \mathbf{x}(0) = y. \tag{2.27}$$

The following lemma allows us to find the function $\mathbf{q}(\cdot) = (q^1(\cdot), \dots, q^n(\cdot))$ such that $\mathbf{q}(x^0) = p^0$ and the data $(\mathbf{q}(y), g(y), y)$ is admissible:

Lemma 2.4.1. *There exists a unique solution $\boldsymbol{q}(\cdot)$ giving the admissible data $(\boldsymbol{q}(y), g(y), y)$ for all $y \in \Gamma$ sufficiently close to x^0, provided the* noncharacteristic condition

$$F_{p_n}(p^0, z^0, x^0) \neq 0 \tag{2.28}$$

holds.

Remark If Γ is not flat near x^0, the noncharacteristic condition (2.28) can be modified to

$$\nabla_p F(p^0, z^0, x^0) \cdot \boldsymbol{\nu}(x^0) \neq 0 \tag{2.29}$$

for $\boldsymbol{\nu}(x^0)$ as the outward unit normal to ∂U at x^0.

2.5 Local Solution

Suppose that (p^0, z^0, x^0) is a noncharacteristic admissible data for the characteristic ODEs of (2.1) and (2.6), with the region Γ lying in the plane $\{x_n = 0\}$ near x^0. From Lemma 2.4.1, there is a function $\mathbf{q}(\cdot)$ so that $p^0 = \mathbf{q}(x^0)$ and $(\mathbf{q}(y), g(y), y)$ is admissible for all $y \in \Gamma$ near x^0. We write

$$\begin{cases} \mathbf{p}(s) = \mathbf{p}(y, s) = \mathbf{p}(y_1, \dots, y_{n-1}, s) \\ z(s) = z(y, s) = z(y_1, \dots, y_{n-1}, s) \\ \mathbf{x}(s) = \mathbf{x}(y, s) = \mathbf{x}(y_1, \dots, y_{n-1}, s) \end{cases}$$

and attempt to solve the characteristic ODEs (2.26), subject to initial conditions (2.27).

We employ the Inverse Function Theorem to obtain the following result:

Lemma 2.5.1 (Local Invertibility)**.** *For the noncharacteristic condition* $F_{p_n}(p^0, z^0, x^0) \neq 0$*, there exist an open interval* $I \subset \mathbb{R}$ *containing* 0*, a neighborhood* W *of* x^0 *in* $\Gamma \subset \mathbb{R}^{n-1}$*, and a neighborhood* V *of* x^0 *in* $\mathbb{R}^n$*, such that for each* $x \in V$ *there exist unique* $s \in I$*,* $y \in W$ *such that* $x = \boldsymbol{x}(y, s)$*, with the mappings* $x \mapsto s, y$ *being* C^2*.*

Using Lemma 2.5.1, we can then uniquely solve

$$\begin{cases} x = \mathbf{x}(y, s) \\ y = \mathbf{y}(x), \ \ s = s(x) \end{cases} \tag{2.30}$$

For $x \in V$, we define

$$\begin{cases} u(x) := z(\mathbf{y}(x), s(x)) \\ \mathbf{p}(x) := \mathbf{p}(\mathbf{y}(x), s(x)) \end{cases} \tag{2.31}$$

Finally, we use the local solutions of the characteristic ODEs to obtain a solution of the PDE:

Theorem 2.5.2 (Local Existence Theorem)**.** *The function* u *defined in* (2.31) *is* C^2 *and solves*

$$F(\nabla u(x), u(x), x) = 0 \qquad (x \in V)$$

with the boundary condition

$$u(x) = g(x) \qquad (x \in \Gamma \cap V).$$

Example 1: F is linear. For a linear homogeneous first-order PDE of the form

$$F(\nabla u, u, x) = \mathbf{b}(x) \cdot \nabla u(x) + c(x)u(x) = 0 \qquad (x \in U) \tag{2.32}$$

our noncharacteristic assumption (2.29) at $x^0 \in \Gamma$ becomes

$$\mathbf{b}(x^0) \cdot \boldsymbol{\nu}(x^0) \neq 0, \tag{2.33}$$

and thus does not involve z^0 or p^0. Furthermore if we specify the boundary condition

$$u = g \qquad \text{on } \Gamma, \tag{2.34}$$

we can find a unique admissible $\mathbf{q}(y)$ for $y \in \Gamma$ near x^0. Thus we can apply the Local Existence Theorem 2.5.2 to construct a unique solution of (2.32)-(2.34) in some neighborhood V containing x^0.

Example 2: F is quasilinear. For the quasilinear PDE

$$F(\nabla u, u, x) = \mathbf{b}(x, u) \cdot \nabla u + c(x, u) = 0 \tag{2.35}$$

the noncharacteristic assumption (2.29) at $x^0 \in \Gamma$ becomes $\mathbf{b}(x^0, z^0) \cdot \boldsymbol{\nu}(x^0) \neq 0$, where $z^0 = g(x^0)$. If we specify the boundary condition

$$u = g \qquad \text{on } \Gamma, \tag{2.36}$$

we can uniquely find the admissible $q(y)$ for $y \in \Gamma$ near x^0. Thus Theorem 2.5.2 yields the existence of a unique solution of (2.35)-(2.36) in some neighborhood V of x^0. We can compute this solution in V using the reduced characteristic equations (2.22), which do not explicitly involve $p(\cdot)$.

Remark In case of quasilinear F, the projected characteristics emanating from distinct points on Γ may intersect outside V, indicating that the local solution does not exist within all of U. In contrast, the projected characteristics in case of linear F never intersect, and thereby allow existence of a local solution within U.

Example 3: F is nonlinear. Consider the general Hamilton-Jacobi PDE

$$G(\nabla u, u_t, u, x, t) = u_t + H(\nabla u, x) = 0, \tag{2.37}$$

where $\nabla u = \nabla_x u = (u_{x_1}, \ldots, u_{x_n})$. We write $q = (p, p_{n+1})$, $y = (x, t)$ and get

$$G(q, z, y) = p_{n+1} + H(p, x);$$

so

$$\nabla_q G = (\nabla_p H(p, x), 1),\ \nabla_y G = (\nabla_x H(p, x), 0),\ \nabla_x G = 0.$$

Thus equation (2.15)(c) becomes

$$\begin{cases} \dot{x}^i(s) = \frac{\partial H}{\partial p_i}(\mathbf{p}(s), \mathbf{x}(s)) & (i = 1, \ldots, n) \\ \dot{x}^{n+1}(s) = 1. \end{cases} \tag{2.38}$$

In particular we can identify the parameter s with the time t.
Equation (2.15)(a) then becomes

$$\begin{cases} \dot{p}^i(s) = -\frac{\partial H}{\partial x_i}(\mathbf{p}(s), \mathbf{x}(s)) & (i = 1, \ldots, n) \\ \dot{p}^{n+1}(s) = 0, \end{cases}$$

while the equation (2.15)(b) becomes

$$\begin{aligned} \dot{z}(s) &= \nabla_p H(\mathbf{p}(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) + p^{n+1}(s) \\ &= \nabla_p H(\mathbf{p}(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) - H(\mathbf{p}(s), \mathbf{x}(s)) \end{aligned}$$

In summary, the characteristic equations for the Hamilton-Jacobi equation are:

$$\begin{cases} (a) & \dot{\mathbf{p}}(s) = -\nabla_x H(\mathbf{p}(s), \mathbf{x}(s)) \\ (b) & \dot{z}(s) = \nabla_p H(\mathbf{p}(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) - H(\mathbf{p}(s), \mathbf{x}(s)) \\ (c) & \dot{\mathbf{x}}(s) = \nabla_p H(\mathbf{p}(s), \mathbf{x}(s)) \end{cases} \tag{2.39}$$

for $\mathbf{p}(\cdot) = (p^1(\cdot), \ldots, p^n(\cdot))$, $z(\cdot)$, and $\mathbf{x}(\cdot) = (x^1(\cdot), \ldots, x^n(\cdot))$. The first and third of these equalities,

$$\begin{cases} \dot{\mathbf{p}} = -\nabla_x H(\mathbf{p}, \mathbf{x}) \\ \dot{\mathbf{x}} = \nabla_p H(\mathbf{p}, \mathbf{x}), \end{cases} \tag{2.40}$$

are called *Hamilton's Equations.* Once $\mathbf{x}(\cdot)$ and $\mathbf{p}(\cdot)$ are found by solving (2.40), the equation for $z(\cdot)$ is trivial.

In general, the form of the full characteristic equations can be quite complicated for fully nonlinear first-order PDE; however, in cases like the Hamilton-Jacobi Equation (2.37), a remarkable mathematical structure can be observed.

Chapter 3

Hamilton-Jacobi Equations

We study the initial-value problem for the Hamilton-Jacobi equation:

$$
\begin{cases} u_t + H(\nabla u) &= 0 \qquad \text{in } \mathbb{R}^n \times (0,\infty) \\ u &= g \qquad \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{3.1}
$$

Here $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is the unknown, $u = u(x,t)$, and $\nabla u = \nabla_x U = (u_{x_1}, \ldots, u_{x_n})$; the *Hamiltonian* $H : \mathbb{R}^n \to \mathbb{R}$ and the initial function $g : \mathbb{R}^n \to \mathbb{R}$ are given.

Two of the characteristic ODE for (3.1), the Hamilton's ODE

$$
\begin{cases} \dot{\mathbf{p}} &= -\nabla_x H(\mathbf{p}, \mathbf{x}) \\ \dot{\mathbf{x}} &= \nabla_p H(\mathbf{p}, \mathbf{x}), \end{cases} \tag{3.2}
$$

arise in the calculus of variations and in mechanics. Using calculus of variations we attempt to build a weak solution of the initial-value problem (3.1).

3.1 The Calculus of Variations

We call a given smooth function $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the *Lagrangian*. We write

$$
L = L(q,x) = L(q_1, \ldots, q_n, x_1, \ldots, x_n) \qquad (q, x \in \mathbb{R}^n)
$$

and

$$
\begin{cases} \nabla_q L = (L_{q_1}, \ldots, L_{q_n}) \\ \nabla_x L = (L_{x_1}, \ldots, L_{x_n}). \end{cases}
$$

We fix two points $x, y \in \mathbb{R}^n$ and a time $t > 0$, and introduce the *action functional*

$$
I[\mathbf{w}(\cdot)] = \int_0^t L(\dot{\mathbf{w}}(s), \mathbf{w}(s))\, ds \tag{3.3}
$$

defined for functions $\mathbf{w}(\cdot) = (w^1(\cdot), \ldots, w^n(\cdot)$ belonging to the *admissible class*

$$
\mathcal{A} = \{\mathbf{w}(\cdot) \in C^2([0,t]; \mathbb{R}^n \mid \mathbf{w}(0) = y, \mathbf{w}(t) = x\}
$$

Thus a C^2 curve $\mathbf{w}(\cdot)$ lies in $\mathcal{A}$ if it starts at a point y at time 0, and reaches the point x at time t. A basic problem in the *calculus of variations* is to find a curve $\mathbf{x}(\cdot) \in \mathcal{A}$ satisfying

$$I[\mathbf{x}(\cdot)] = \min_{\mathbf{w}(\cdot)\in\mathcal{A}} I[\mathbf{w}(\cdot)]. \tag{3.4}$$

That is, we look for the *minimizer* $\mathbf{x}(\cdot)$ of the functional $I[\cdot]$ among all the admissible $\mathbf{w}(\cdot) \in \mathcal{A}$.

Theorem 3.1.1 (Euler-Lagrange Equations)**.** *The function $\boldsymbol{x}(\cdot)$ defined above solves the system of* Euler-Lagrange Equations

$$-\frac{d}{ds}(\nabla_q L(\dot{\boldsymbol{x}}(s), \boldsymbol{x}(s))) + \nabla_x L(\dot{\boldsymbol{x}}(s), \boldsymbol{x}(s)) = 0 \qquad (0 \le s \le t) \tag{3.5}$$

Proof. Choose a smooth function $\mathbf{v} : [0, t] \to \mathbb{R}^n$, $\mathbf{v} = (v^1, \dots, v^n)$ that satisfies

$$\mathbf{v}(0) = \mathbf{v}(t) = 0. \tag{3.6}$$

For $\tau \in \mathbb{R}^n$ define

$$\mathbf{w}(\cdot) := \mathbf{x}(\cdot) + \tau\mathbf{v}(\cdot). \tag{3.7}$$

Then $\mathbf{w}(\cdot) \in \mathcal{A}$ and so

$$I[\mathbf{x}(\cdot)] \le I[\mathbf{w}(\cdot)].$$

Thus the real valued function

$$i(\tau) := I[\mathbf{x}(\cdot) + \tau\mathbf{v}(\cdot)]$$

has a minimum at $\tau = 0$, and consequently

$$\frac{di}{d\tau} = 0, \tag{3.8}$$

provided it exists.

We compute this derivative explicitly: We have

$$i(\tau) = \int_0^t L(\dot{\mathbf{x}}(s) + \tau\dot{\mathbf{v}}(s), \mathbf{x}(s) + \tau\mathbf{v}(s))\, ds,$$

and so

$$\frac{di}{d\tau} = \int_0^t \left(\sum_{j=1}^n L_{q_j}(\dot{\mathbf{x}} + \tau\dot{\mathbf{v}}, \mathbf{x} + \tau\mathbf{v})\dot{v}^j + L_{x_j}(\dot{\mathbf{x}} + \tau\dot{\mathbf{v}}, \mathbf{x} + \tau\mathbf{v})v^j \right) ds$$

We use (3.8) with $\tau = 0$ to get

$$0 = \frac{di}{d\tau}(0) = \int_0^t \left(\sum_{j=1}^n L_{q_j}(\dot{\mathbf{x}}, \mathbf{x})\dot{v}^j + L_{x_j}(\dot{\mathbf{x}}, \mathbf{x})v^j \right) ds$$

Now, (3.6) and an integration by parts yields

$$0 = \sum_{j=1}^n \int_0^t \left[-\frac{d}{ds}(L_{q_j}(\dot{\mathbf{x}}, \mathbf{x})) + L_{x_j}(\dot{\mathbf{x}}, \mathbf{x}) \right] v^j\, ds.$$

This identity is valid for all smooth functions $\mathbf{v} = (v^1, \dots, v^n)$ satisfying (3.6), and so

$$-\frac{d}{ds}(L_{q_j}(\dot{\mathbf{x}}, \mathbf{x})) + L_{x_j}(\dot{\mathbf{x}}, \mathbf{x}) = 0$$

for $0 \le s \le t$, $j = 1, \dots, n$. □

3.2 Hamilton's ODE

We assume that $\mathbf{x}(\cdot) \in C^2$ solves the Euler-Lagrange Equations (3.5). We set

$$\mathbf{p}(s) := \nabla_q L(\dot{\mathbf{x}}(s), \mathbf{x}(s)) \qquad (0 \leq s \leq t); \tag{3.9}$$

we call $\mathbf{p}(\cdot)$ the *generalized momentum* corresponding to the *position* $\mathbf{x}(\cdot)$ and *velocity* $\dot{\mathbf{x}}(\cdot)$. We make an important hypothesis:

$$\begin{cases} \text{Suppose for all } x, p \in \mathbb{R}^n \text{ that the equation } p = \nabla_q L(q, x) \\ \text{can be uniquely solved for } q \text{ as a smooth function of } p \text{ and } x, \\ q = \mathbf{q}(p, x). \end{cases} \tag{3.10}$$

Definition The *Hamiltonian* H associated with the Lagrangian L is

$$H(p, x) := p \cdot \mathbf{q}(p, x) - L(\mathbf{q}(p, x), x) \qquad (p, x \in \mathbb{R}^n)$$

where the function $\mathbf{q}(\cdot, \cdot)$ is defined implicitly by (3.10).

Theorem 3.2.1 (Derivation of Hamilton's ODE)**.** *The functions $\boldsymbol{x}(\cdot)$ and $\boldsymbol{p}(\cdot)$ defined in* (3.4) *and* (3.9) *satisfy Hamilton's equations* (3.2) *for* $0 \leq s \leq t$. *Furthermore, the mapping* $s \mapsto H(\boldsymbol{p}(s), \boldsymbol{x}(s))$ *is constant.*

Proof. From (3.9) and (3.10), we have $\dot{\mathbf{x}}(s) = \mathbf{q}(\mathbf{p}(s), \mathbf{x}(s))$. We write $\mathbf{q}(\cdot) = (q^1(\cdot), \dots, q^n(\cdot))$, and compute

$$\begin{aligned} \frac{\partial H}{\partial x_i}(p, x) &= \sum_{k=1}^n \left[p_k \frac{\partial q^k}{\partial x_i}(p, x) - \frac{\partial L}{\partial q_k}(q, x) \frac{\partial q^k}{\partial x_i}(p, x) \right] - \frac{\partial L}{\partial x_i}(q, x) \\ &= -\frac{\partial L}{\partial x_i}(q, x) \qquad \text{(following (3.10))} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial H}{\partial p_i}(p, x) &= \sum_{k=1}^n \left[p_k \frac{\partial q^k}{\partial p_i}(p, x) - \frac{\partial L}{\partial q_k}(q, x) \frac{\partial q^k}{\partial p_i}(p, x) \right] + q^i(p, x) \\ &= q^i(p, x) \qquad \text{(again from (3.10))} \end{aligned}$$

for $i = 1, \dots, n$. Thus

$$\frac{\partial H}{\partial p_i}(\mathbf{p}(s), \mathbf{x}(s)) = q^i(\mathbf{p}(s), \mathbf{x}(s)) = \dot{x}^i(s).$$

Similarly,

$$\begin{aligned} \frac{\partial H}{\partial x_i}(\mathbf{p}(s), \mathbf{x}(s)) &= -\frac{\partial L}{\partial x_i}(\mathbf{q}(\mathbf{p}(s), \mathbf{x}(s)), \mathbf{x}(s)) \\ &= -\frac{\partial L}{\partial x_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)) \\ &= -\frac{d}{ds}\left(\frac{\partial L}{\partial q_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)) \right) \qquad \text{(from (3.5))} \\ &= -\dot{p}^i(s). \end{aligned}$$

Finally,

$$\begin{aligned}\frac{d}{ds}H(\mathbf{p}(s),\mathbf{x}(s)) &= \sum_{i=1}^{n}\left(\frac{\partial H}{\partial p_i}\dot{p}^i + \frac{\partial H}{\partial x_i}\dot{x}^i\right)\\ &= \sum_{i=1}^{n}\left[\frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial x_i}\right) + \frac{\partial H}{\partial x_i}\left(\frac{\partial H}{\partial p_i}\right)\right] = 0.\end{aligned}$$

□

3.3 Legendre Transform

Suppose that the Lagrangian $L : \mathbb{R}^n \to \mathbb{R}$ is such that the mapping $q \mapsto L(q)$ is convex (hence continuous) and $\lim_{t\to 0} \frac{L(q)}{|q|} = +\infty$. We then define the *Legendre Transform* of L as

$$L^*(p) = \sup_{q\in\mathbb{R}^n} \{p \cdot q - L(q)\} \qquad (p \in \mathbb{R}^n) \tag{3.11}$$

The conditions on L imply that there exists some $q^* \in \mathbb{R}^n$ for which

$$L^*(p) = p \cdot q^* - L(q^*)$$

and the mapping $q \mapsto p\cdot q - L(q)$ has a maximum at $q = q^*$. But then $p = \nabla L(q^*)$, provided that L is differentiable at q^*. Hence the equation $p = \nabla L(q)$ is solvable for q in terms of p, $q^* = \mathbf{q}(p)$. Therefore

$$L^*(p) = p \cdot \mathbf{q}(p) - L(\mathbf{q}(p))$$

Since this is the definition of the Hamiltonian H associated with the Lagrangian L, we write

$$H = L^*. \tag{3.12}$$

Theorem 3.3.1 (Convex duality of Hamiltonian and Lagrangian)**.** *For a convex unbounded L and for H defined by* (3.11) *and* (3.12)*, the mapping $p \mapsto H(p)$ is convex. Furthermore,*

$$L = H^*. \tag{3.13}$$

Proof. For each fixed q, the function $p \mapsto p \cdot q - L(q)$ is linear, and consequently the mapping

$$p \mapsto H(p) = L^*(p) = \sup_{q\in\mathbb{R}^n} \{p \cdot q - L(q))\}$$

is convex. Indeed, if $0 \le \tau \le 1$, $p, \hat{p} \in \mathbb{R}^n$,

$$\begin{aligned}H(\tau p + (1-\tau)\hat{p}) &= \sup_q \{(\tau p + (1-\tau)\hat{p}) \cdot q - L(q)\}\\ &\le \tau \sup_q \{p \cdot q - L(q)\} + (1-\tau) \sup_q \{\hat{p} \cdot q - L(q))\}\\ &= \tau H(p) + (1-\tau)H(\hat{p}).\end{aligned}$$

Now fix some $\lambda > 0$,$p \ne 0$. Then

$$\begin{aligned}H(p) &= \sup_{q\in\mathbb{R}^n} \{p \cdot q - L(q)\}\\ &\ge \lambda|p| - L(\lambda\frac{p}{|p|}) \qquad (q = \lambda\frac{p}{|p|})\\ &\ge \lambda|p| - \max_{B(0,\lambda)} L.\end{aligned}$$

Thus $\liminf_{|p|\to\infty} \frac{H(p)}{|p|} \geq \lambda$ for all $\lambda > 0$. Now (3.12) suggests that

$$H(p) + L(q) \geq p \cdot q$$

for all $p, q \in \mathbb{R}^n$, and consequently

$$L(q) \geq \sup_{p \in \mathbb{R}^n} \{p \cdot q - H(p)\} = H^*(q).$$

On the other hand,

$$\begin{aligned} H^*(q) &= \sup_{p \in \mathbb{R}^n} \{p \cdot q - \sup_{r \in \mathbb{R}^n} \{p \cdot r - L(r)\}\} \\ &= \sup_{p \in \mathbb{R}^n} \inf_{r \in \mathbb{R}^n} \{p \cdot (q - r) + L(r)\}. \end{aligned} \tag{3.14}$$

Now since $q \mapsto L(q)$ is convex, there exists $s \in \mathbb{R}^n$ such that

$$L(r) \geq L(q) + s \cdot (r - q) \qquad (r \in \mathbb{R}^n).$$

Letting $p = s$ in (3.14), we find

$$H^*(q) \geq \inf_{r \in \mathbb{R}^n} \{s \cdot (q - r) + L(r)\} = L(q).$$

Therefore, H and L are Legendre transforms of each other. □

3.4 Hopf-Lax Formula

Theorem 3.4.1 (Solution of Hamilton-Jacobi Equation)**.** *Consider the initial-value problem*

$$\begin{cases} u_t + H(\nabla u) &= 0 \qquad a.e.\ in\ \mathbb{R}^n \times (0, \infty) \\ u &= g \qquad on\ \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{3.15}$$

Then the function u defined by

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\} \tag{3.16}$$

solves (3.15)*. Moreover, u is Lipschitz continuous, and differentiable a.e. in $\mathbb{R}^n \times (0, \infty)$.*

Proof. We break the proof down in the following sequence of lemmas:

Lemma 3.4.2 (Hopf-Lax Formula)**.** *Consider the minimization problem*

$$u(x,t) = \inf_{\boldsymbol{w}(\cdot) \in C^1} \left\{ \int_0^t L(\dot{\boldsymbol{w}}(s))ds + g(y) \mid \boldsymbol{w}(0) = y, \boldsymbol{w}(t) = x \right\}. \tag{3.17}$$

Then its solution $u = u(x,t)$ is given by (3.16)*.*

Proof of Lemma 3.4.2: Fix any $y \in \mathbb{R}^n$ and define $\mathbf{w}(s) := y + \frac{s}{t}(x-y)$ for $0 \le s \le t$. Then the definition of u in (3.17) implies

$$u(x,t) \le \int_0^t L(\dot{\mathbf{w}}(s))ds + g(y) = tL\left(\frac{x-y}{t}\right) + g(y),$$

and so

$$u(x,t) \le \inf_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}.$$

On the other hand, if $\mathbf{w}(\cdot) \in C^1$ satisfies $\mathbf{w}(t) = x$, employing Jensen's inequality we get

$$L\left(\frac{1}{t}\int_0^t \dot{\mathbf{w}}(s)ds\right) \le \frac{1}{t}\int_0^t L(\dot{\mathbf{w}}(s))ds.$$

If we set $\mathbf{w}(0) = y$, then

$$tL\left(\frac{x-y}{t}\right) + g(y) \le L(\dot{\mathbf{w}}(s))ds + g(y).$$

Consequently

$$u(x,t) \ge \inf_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}.$$

Moreover, since L is convex, this infimum is actually the minimum. Therefore, this u is indeed given by (3.17). □

Lemma 3.4.3 (Lipschitz Continuity)**.** *The function u defined in* (3.17) *is Lipschitz continuous in $\mathbb{R}^n \times [0,\infty)$, and $u = g$ on $\mathbb{R}^n \times \{t = 0\}$.*

Proof of Lemma 3.4.3: Fix $t > 0$, $x, \hat{x} \in \mathbb{R}^n$. Choose $y \in \mathbb{R}^n$ such that

$$tL\left(\frac{x-y}{t}\right) + g(y) = u(x,t). \tag{3.18}$$

Then

$$\begin{aligned} u(\hat{x},t) - u(x,t) &= \inf_x \left\{ tL\left(\frac{\hat{x}-z}{t}\right) + g(z) \right\} - tL\left(\frac{x-y}{t}\right) - g(y) \\ &\le g(\hat{x} - x + y) - g(y) \le \mathrm{Lip}(g)|\hat{x} - x|. \end{aligned}$$

Hence we have

$$|u(x,t) - u(\hat{x},t)| \le \mathrm{Lip}(g)|x - \hat{x}|. \tag{3.19}$$

Now select $x \in \mathbb{R}^n$,$t > 0$. Putting $y = x$ in (3.16) we get

$$u(x,t) \le tL(0) + g(x). \tag{3.20}$$

Furthermore,

$$\begin{aligned}
u(x,t) &= \min_{y\in\mathbb{R}^n}\left\{tL\left(\frac{x-y}{t}\right)+g(y)\right\}\\
&\geq g(x)+\min_{y\in\mathbb{R}^n}\left\{tL\left(\frac{x-y}{t}\right)-\operatorname{Lip}(g)|x-y|\right\}\\
&= g(x)-t\max_{z\in\mathbb{R}^n}\{\operatorname{Lip}(g)|z|-L(z)\} \qquad (z=\frac{x-y}{t})\\
&= g(x)-t\max_{w\in B(0,\operatorname{Lip}(g))}\max_{z\in\mathbb{R}^n}\{w\cdot z-L(z)\}\\
&= g(x)-t\max_{B(0,\operatorname{Lip}(g))}H.
\end{aligned}$$

Using (3.20), we conclude

$$|u(x,t)-g(x)|\leq Ct$$

for

$$C:=\max(|L(0)|,\max_{B(0,\operatorname{Lip}(g))}|H|). \tag{3.21}$$

Finally, select $x\in\mathbb{R}^n$, $0<\hat{t}<t$. Then $\operatorname{Lip}(u(\cdot,t)\leq\operatorname{Lip}(g)$ by (3.19). Consequently

$$|u(x,t)-u(x,\hat{t}|\leq C|t-\hat{t}|$$

for C defined in (3.21). □

Lemma 3.4.4 (An identity)**.** *For each $x\in\mathbb{R}^n$ and $0\leq s\leq t$, we have*

$$u(x,t)=\min_{y\in\mathbb{R}^n}\left\{(t-s)L\left(\frac{x-y}{t-s}\right)+u(y,s)\right\} \tag{3.22}$$

Proof of Lemma 3.4.4: Fix $y\in\mathbb{R}^n$, $0<s<t$ and choose $y\in\mathbb{R}^n$ so that

$$u(y,s)=sL\left(\frac{y-z}{s}\right)+g(z). \tag{3.23}$$

Since L is convex and $\frac{x-z}{t}=\left(1-\frac{s}{t}\right)\frac{x-y}{t-s}+\frac{s}{t}\frac{y-z}{s}$, we have

$$L\left(\frac{x-z}{t}\right)\leq\left(1-\frac{s}{t}\right)L\left(\frac{x-y}{t-s}\right)+\frac{s}{t}L\left(\frac{y-z}{s}\right).$$

Thus

$$\begin{aligned}
u(x,t) &\leq tL\left(\frac{x-z}{t}\right)+g(z)\\
&\leq (t-s)L\left(\frac{x-y}{t-s}\right)+sL\left(\frac{y-z}{s}\right)+g(z)\\
&= (t-s)L\left(\frac{x-y}{t-s}\right)+u(y,s).
\end{aligned}$$

This inequality is true for each $y \in \mathbb{R}^n$. Since $y \mapsto u(y,s)$ is continuous, we have

$$u(x,t) \leq \min_{y\in\mathbb{R}^n} \left\{ (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\} \tag{3.24}$$

Choose w such that

$$u(x,t) = tL\left(\frac{x-w}{t}\right) + g(w), \tag{3.25}$$

and set $y := \frac{s}{t}x + (1-\frac{s}{t})w$. Then $\frac{x-y}{t-s} = \frac{x-w}{t} = \frac{y-w}{s}$. Consequently

$$\begin{aligned}(t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s) &\leq (t-s)L\left(\frac{x-w}{t}\right) + sL\left(\frac{y-w}{s}\right) + g(w)\\ &= tL\left(\frac{x-w}{t}\right) + g(w)\\ &= u(x,t)\end{aligned}$$

by (3.25). Hence

$$\min_{y\in\mathbb{R}^n} \left\{ (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\} \leq u(x,t), \tag{3.26}$$

and hence u is indeed given by (3.16). □

Now, Rademacher's Theorem asserts that a Lipschitz function is differentiable almost everywhere (a.e.). Consequently, we know that u defined by (3.16) is differentiable a.e. for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, following Lemma 3.4.3.

Finally, we prove that u solves the Hamilton-Jacobi PDE whenever u is differentiable:

Theorem 3.4.5 (Solution of Hamilton-Jacobi Equation:)**.** *Suppose that $x \in \mathbb{R}^n$, $t > 0$, and that u defined by the Hopf-Lax Formula* (3.16) *is differentiable at a point $(x,t) \in \mathbb{R}^n \times (0,\infty)$. Then*

$$u_t(x,t) + H(\nabla u(x,t)) = 0.$$

Proof. We fix $q \in \mathbb{R}^n$, $h > 0$. From Lemma 3.4.4, we have

$$\begin{aligned}u(x+hq, t+h) &= \min_{y\in\mathbb{R}^n} \left\{ hL\left(\frac{x+hq-y}{h}\right) + u(y,t) \right\}\\ &\leq hL(q) + u(x,t).\end{aligned}$$

Hence

$$\frac{u(x+hq,t+h) - u(x,t)}{h} \leq L(q).$$

Allowing $h \to 0^+$, we compute

$$q \cdot \nabla u(x,t) + u_t(x,t) \leq L(q).$$

Since this inequality is valid for all $q \in \mathbb{R}^n$, we have

$$u_t(x,t) + H(\nabla u(x,t)) = u_t(x,t) + \max_{q\in\mathbb{R}^n}\{q \cdot \nabla u(x,t) - L(q)\} \leq 0. \tag{3.27}$$

Now choose z such that $u(x,t) = tL\left(\frac{x-z}{t}\right) + g(z)$. Fix $h > 0$ and set $s = t-h$, $y = \frac{s}{t}x + \left(1 - \frac{s}{t}z\right)$. Then $\frac{x-z}{t} = \frac{y-z}{s}$, and thus

$$\begin{aligned} u(x,t) - u(y,s) &\geq tL\left(\frac{x-z}{t}\right) + g(z) - \left[sL\left(\frac{y-z}{s}\right) + g(z)\right] \\ &= (t-s)L\left(\frac{x-z}{t}\right). \end{aligned}$$

That is,

$$\frac{u(x,t) - u((1-\frac{h}{t})x + \frac{h}{t}z, t-h)}{h} \geq L\left(\frac{x-z}{t}\right).$$

Allowing $h \to 0^+$ gives

$$\frac{x-z}{t} \cdot \nabla u(x,t) + u_t(x,t) \geq L\left(\frac{x-z}{t}\right).$$

Consequently

$$\begin{aligned} u_t(x,t) + H(\nabla u(x,t)) &= u_t(x,t) + \max_{q \in \mathbb{R}^n}\{q \cdot \nabla u(x,t) - L(q)\} \\ &\geq u_t(x,t) + \frac{x-z}{t} \cdot \nabla u(x,t) - L\left(\frac{x-z}{t}\right) \\ &= 0. \end{aligned}$$

This inequality and (3.27) complete the proof. □

Hence we have finally proved Theorem 3.4.1. □

3.5 Weak Solutions

Now we attempt to describe a weak solution for the Hamilton-Jacobi PDE (3.15).

Definition A function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be *semiconcave* if there exists a constant C such that

(3.28) $$g(x+z) - 2g(x) + g(x-z) \leq C|z|^2$$

holds for all $x, z \in \mathbb{R}^n$.

Definition A C^2 convex function $H : \mathbb{R}^n \to \mathbb{R}$ is called *uniformly convex* (with constant $\theta > 0$) if

(3.29) $$\sum_{i,j=1}^{n} H_{p_i p_j}(p)\xi_i \xi_j \geq \theta|\xi|^2 \qquad \forall\, p, \xi \in \mathbb{R}^n.$$

Definition A Lipschitz continuous function $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ that solves the initial-value problem (3.15) a.e. is said to be a *weak solution* if

(3.30) $$u(x+z,t) - 2u(x,t) + u(x-z,t) \leq C\left(1 + \frac{1}{t}\right)|z|^2$$

for some constant $C \geq 0$ and all $x, z \in \mathbb{R}^n$, $t > 0$.

Theorem 3.5.1 (Hopf-Lax formula as a weak solution). *Suppose that $H \in C^2$ is convex and unbounded, and g is Lipschitz continuous. If either g is semiconcave or H is uniformly convex, then u given by the Hopf-Lax Formula* (3.16) *is the unique weak solution of the initial-value problem* (3.15).

Proof. First of all, using the semiconcavity of g, we show that u is semiconcave in the spatial variable x.

Choose $y \in \mathbb{R}^n$ so that $u(x,t) = tL\left(\frac{x-y}{t}\right) + g(y)$. From the Hopf-Lax formula (3.16), we get

$$\begin{aligned} u(x+z,t) - 2u(x,t) + u(x-z,t) &\leq \left[tL\left(\frac{x-y}{t}\right) + g(y+z)\right] \\ &\quad - 2\left[tL\left(\frac{x-y}{t}\right) + g(y)\right] \\ &\quad + \left[tL\left(\frac{x-y}{t}\right) + g(y-z)\right] \\ &= g(y+z) - 2g(y) + g(y-z) \\ &\leq C|z|^2 \qquad \text{(from (3.28))} \end{aligned}$$

Lemma 3.5.2 (Uniform Convexity). *Suppose that H is uniformly convex (with constant θ) and u is defined by the Hopf-Lax Formula* (3.16). *Then*

$$u(x+z,t) - 2u(x,t) + u(x-z,t) \leq \frac{|z|^2}{\theta t}$$

for all $x, z \in \mathbb{R}^n$, $t > 0$.

Proof. We use Taylor's Formula on (3.29) to see

$$H\left(\frac{p_1+p_2}{2}\right) \leq \frac{1}{2}H(p_1) + \frac{1}{2}H(p_2) - \frac{\theta}{2}\left|\frac{p_1-p_2}{2}\right|^2. \tag{3.31}$$

Since H is the Legendre Transform of the Lagrangian L, for given $p_1, p_2 \in \mathbb{R}^n$ there exist $q_1, q_2 \in \mathbb{R}^n$ such that

$$H(p_1) = p_1 \cdot q_1 - L(q_1)$$

and

$$H(p_2) = p_2 \cdot q_2 - L(q_2).$$

Once q_1, q_2 are known, we have

$$H\left(\frac{p_1+p_2}{2}\right) \geq \frac{p_1+p_2}{2} \cdot \frac{q_1+q_2}{2} - L\left(\frac{q_1+q_2}{2}\right).$$

We use the above three relations in (3.31) to get

$$\begin{aligned} \frac{p_1+p_2}{2} \cdot \frac{q_1+q_2}{2} - L\left(\frac{q_1+q_2}{2}\right) &\leq \frac{1}{2}[(p_1 \cdot q_1 - L(q_1)) + (p_2 \cdot q_2 - L(q_2))] \\ &\quad - \frac{\theta}{2}\left|\frac{p_1-p_2}{2}\right|^2. \end{aligned}$$

That is,

$$\begin{aligned}
\frac{1}{2}L(q_1)+\frac{1}{2}L(q_2) &\leq L\left(\frac{q_1+q_2}{2}\right)+\frac{1}{2}(p_1\cdot q_1+p_2\cdot q_2)\\
&\quad -\frac{1}{4}(p_1+p_2)\cdot(q_1+q_2)-\frac{\theta}{2}\left|\frac{p_1-p_2}{2}\right|^2\\
&\leq L\left(\frac{q_1+q_2}{2}\right)+\left[\frac{p_1-p_2}{2}\cdot\frac{q_1-q_2}{2}-\frac{\theta}{2}\left|\frac{p_1-p_2}{2}\right|^2\right] \qquad (3.32)
\end{aligned}$$

Now consider the particular Hamiltonian $H(\xi)=\frac{\theta}{2}|\xi|^2$. A simple calculation yields the corresponding Lagrangian as

$$L(\eta)=\sup_{\xi}\{\xi\cdot\eta-H(\xi)\}=\frac{1}{2\theta}|\eta|^2.$$

Setting $\xi=\frac{p_1-p_2}{2}$ and $\eta=\frac{q_1-q_2}{2}$ in (3.32), we obtain

$$\frac{1}{2}L(q_1)+\frac{1}{2}L(q_2)\leq L\left(\frac{q_1+q_2}{2}\right)+\frac{1}{2\theta}\left|\frac{q_1-q_2}{2}\right|^2. \qquad (3.33)$$

Now choose y so that $u(x,t)=tL\left(\frac{x-y}{t}\right)+g(y)$. We then calculate

$$\begin{aligned}
u(x+z,t)-2u(x,t)+u(x-z,t) &\leq \left[tL\left(\frac{x+z-y}{t}\right)+g(y)\right]-2\left[tL\left(\frac{x-y}{t}\right)+g(y)\right]\\
&\quad +\left[tL\left(\frac{x-z-y}{t}\right)+g(y)\right]\\
&= 2t\left[\frac{1}{2}L\left(\frac{x-z-y}{t}\right)+\frac{1}{2}L\left(\frac{x-z-y}{t}\right)-L\left(\frac{x-y}{t}\right)\right]\\
&\leq 2t\frac{1}{8\theta}\left|\frac{2z}{t}\right|^2 \qquad \text{(from (3.33))}\\
&\leq \frac{|z|^2}{\theta t},
\end{aligned}$$

as was to be proved. □

Lemma 3.5.3 (Uniqueness of weak solutions)**.** *Given a convex unbounded function $H\in C^2$ and a Lipschitz continuous function $g:\mathbb{R}^n\to\mathbb{R}$, there exists a unique weak solution of the initial-value problem* (3.15).

Proof. Suppose if possible that u and $\tilde{u}$ are two weak solutions of (3.15). Write $w:=u-\tilde{u}$.

At a point (y,s) where both u and $\tilde{u}$ are differentiable and solve (3.15), we have

$$\begin{aligned}
w_t(y,s) &= u_t(y,s)-\tilde{u}_t(y,s)\\
&= -H(\nabla u(y,s))+H(\nabla\tilde{u}(y,s))\\
&= -\int_0^1\frac{d}{dr}H(r\nabla u(y,s)+(1-r)\nabla\tilde{u}(y,s))dr\\
&= -\int_0^1\nabla H(r\nabla u(y,s)+(1-r)\nabla\tilde{u}(y,s))dr\cdot(\nabla u(y,s)-\nabla\tilde{u}(y,s))\\
&=: -\mathbf{b}(y,s)\cdot\nabla w(y,s).
\end{aligned}$$

Consequently,

$$w_t + \mathbf{b} \cdot \nabla w = 0 \qquad a.e. \tag{3.34}$$

For some smooth function $\phi : \mathbb{R} \to [0.\infty)$, set $v := \phi(w) \geq 0$. We multiply (3.34) by $\phi'(w)$ to get

$$v_t + \mathbf{b} \cdot \nabla v = 0 \qquad a.e. \tag{3.35}$$

Now choose $\epsilon > 0$ and define $u^\epsilon := \eta_\epsilon * u$, $\tilde{u}^\epsilon := \eta_\epsilon * \tilde{u}$, where

$$\eta(x,t) = \begin{cases} C\ exp\left(\frac{1}{|x|^2-1} + \frac{1}{|t|^2-1}\right) & if\ |x|, |t| < 1 \\ 0 & if\ |x|, |t| \geq 1 \end{cases} \tag{3.36}$$

is the standard mollifier in the variables x and t, and C is a constant such that $\int \eta(x,t)\, dx\, dt = 1$; and

$$\eta_\epsilon(x,t) = \frac{1}{\epsilon^{n+1}} \left(\frac{x}{\epsilon}, \frac{t}{\epsilon}\right).$$

Then

$$|\nabla u^\epsilon| \leq \mathrm{Lip}(u), \qquad |\nabla \tilde{u}^\epsilon| \leq \mathrm{Lip}(\tilde{u}), \tag{3.37}$$

and

$$\nabla u^\epsilon \to \nabla u, \qquad \nabla \tilde{u}^\epsilon \to \nabla \tilde{u} \tag{3.38}$$

a.e. as $\epsilon \to 0$. Furthermore, (3.30) implies that

$$\Delta u^\epsilon, \Delta \tilde{u}^\epsilon \leq C\left(1 + \frac{1}{s}\right) \tag{3.39}$$

for an appropriate constant C and all $\epsilon > 0$, $y \in \mathbb{R}^n$, $s > 2\epsilon$.

Now write

$$\mathbf{b}_\epsilon(y,s) := \int_0^1 \nabla H(r\nabla u^\epsilon(y,s) + (1-r)\nabla \tilde{u}^\epsilon(y,s))dr. \tag{3.40}$$

Then (3.35) becomes

$$v_t + \mathbf{b}_\epsilon \cdot \nabla v = (\mathbf{b}_\epsilon - \mathbf{b})\nabla v \qquad a.e.;$$

hence

$$v_t + \nabla \cdot (v\mathbf{b}_\epsilon) = (\nabla \cdot \mathbf{b}_\epsilon)v + (\mathbf{b}_\epsilon - \mathbf{b})\nabla v \qquad a.e. \tag{3.41}$$

Now

$$\begin{aligned} \nabla \cdot (\mathbf{b}_\epsilon) &= \int_0^1 \sum_{k,l=1}^n H_{p_k p_l}(r\nabla u^\epsilon + (1-r)\nabla \tilde{u}^\epsilon)(r u^\epsilon_{x_k x_l} + (1-r)\tilde{u}^\epsilon_{x_k x_l})dr \\ &\leq C\left(1 + \frac{1}{s}\right) \end{aligned} \tag{3.42}$$

for some constant C. Here we have used $\Delta H \geq 0$.

Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$, and set

$$R := max\{|\nabla H(p)| \ : \ |p| \leq max(\text{Lip}(u), \text{Lip}(\tilde{u}))\}. \tag{3.43}$$

Also define

$$\mathcal{C} := \{(x,t) \ : \ 0 \leq t \leq t_0, \ |x - x_0| \leq R(t_0 - t)\}.$$

Now, for

$$e(t) = \int_{B(x_0, R(t_0 - t))} v(x,t)dx,$$

we compute for a.e. $t > 0$:

$$\begin{aligned}
\dot{e}(t) &= \int_{B(x_0,R(t_0-t))} v_t dx - R \int_{\partial B(x_0,R(t_0-t))} vdS \\
&= \int_{B(x_0,R(t_0-t))} [-\nabla \cdot (v\mathbf{b}_\epsilon) + (\nabla \cdot \mathbf{b}_\epsilon)v + (\mathbf{b}_\epsilon - \mathbf{b}) \cdot \nabla v] dx \\
&\quad - R \int_{\partial B(x_0,R(t_0-t))} vdS \qquad \text{(following (3.41))} \\
&= - \int_{\partial B(x_0,R(t_0-t))} v(\mathbf{b}_\epsilon \cdot \nu + R) dS \\
&\quad + \int_{B(x_0,R(t_0-t))} [(\nabla \cdot \mathbf{b}_\epsilon)v + (\mathbf{b}_\epsilon - \mathbf{b}) \cdot \nabla v] dx \\
&\leq \int_{B(x_0,R(t_0-t))} [(\nabla \cdot \mathbf{b}_\epsilon)v + (\mathbf{b}_\epsilon - \mathbf{b}) \cdot \nabla v] dx \qquad \text{(from (3.37) and (3.40))} \\
&\leq C\left(1 + \frac{1}{t}\right) e(t) + \int_{B(x_0,R(t_0-t))} (\mathbf{b}_\epsilon - \mathbf{b}) \cdot \nabla v dx \qquad \text{(from (3.42))}
\end{aligned}$$

It follows from (3.37) and (3.38), and Dominated Convergence Theorem that

$$\dot{e}(t) \leq C\left(1 + \frac{1}{t}\right) e(t) \qquad \text{for a.e. } 0 < t < t_0. \tag{3.44}$$

Finally, fix $0 < \epsilon < r < t$ and choose the function ϕ such that

$$\phi(z) \begin{cases} = 0 & \text{if } |z| \leq \epsilon[\text{Lip}(u) + \text{Lip}(\tilde{u})] \\ > 0 & \text{otherwise.} \end{cases}$$

Since $u = \tilde{u}$ on $\mathbb{R}^n \times \{t = 0\}$,

$$v = \phi(w) = \phi(u - \tilde{u}) = 0 \qquad \text{at } \{t = \epsilon\}.$$

Thus $e(\epsilon) = 0$. Consequently Gronwall's inequality and (3.44) imply

$$e(r) \leq e(\epsilon) \ exp\left(\int_\epsilon^r C\left(1 + \frac{1}{s}\right) ds\right) = 0.$$

Hence

$$|u - \tilde{u}| \leq \epsilon[\text{Lip}(u) + \text{Lip}(\tilde{u})] \qquad \text{on } B(x_0, R(t_0 - r)).$$

Since this inequality if valid for all $\epsilon > 0$, $u \equiv \tilde{u}$ in $B(x_0, R(t_0 - r))$. Therefore, in particular, $u(x_0, t_0) = \tilde{u}(x_0, t_0)$. □

Thus the proof of Theorem 3.5.1 is complete. □

Chapter 4

Conservation Laws

4.1 Introduction to Conservation Laws

A *Conservation Law* is an equation of the form

$$u_t + \nabla \cdot (\mathbf{f}(u)) = 0. \tag{4.1}$$

It says that the rate of change of u contained in a domain $D \subset \mathbb{R}^n$ equals the flux of the vector field $\mathbf{f}$ into D:

$$\frac{d}{dt} \iiint_D u \, dx = \iint_{\partial D} \mathbf{f} \cdot \nu \, dS.$$

Many physical laws are conservation laws: the quantities u and $\mathbf{f}$ depend on the variables describing the state of a physical system, and their derivatives. When the effects of dissipations (eg. viscosity, heat conduction) are ignored, the conservation laws are of first-order, i.e., the quantities u and $\mathbf{f}$ are functions of the state variables but not of their derivatives. Here we shall try to develop a theory of the initial-value problem for scalar convex conservation laws in 1-dimensional space.

The initial value problem

$$u_t + (f(u))_x = 0 \tag{4.2}$$

consists of determining solutions u of (4.2) from the initial state

$$u(x, 0) = g(x) \tag{4.3}$$

for all future time.

Calculating the derivative in (4.2) we get a quasilinear equation

$$u_t + f'(u)u_x = 0. \tag{4.4}$$

If the initial data g is smooth and nondecreasing, then a global solution exists. Otherwise, the classical solutions of (4.4) always develop discontinuities after a finite time, and hence cannot be counted as regular solutions. In the following discussion we propose them as generalized ("weak") solutions.

Definition The function $u(x,t)$ is a weak solution of (4.2) with initial data g if u and $f(u)$ are integrable over every bounded set of the half-plane $t \geq 0$ and the relation

$$\int_0^\infty \int_{-\infty}^\infty [v_t u + v_x f(u)] \, dx \, dt + \int_{-\infty}^\infty v(x,0)g(x) \, dx = 0 \tag{4.5}$$

is satisfied for all smooth test functions $v : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ with compact support which vanish for large enough $|x| + t$, i.e., the vector field $(u, f(u))$ is divergence-free in the weak sense.

Weak solutions are in general not unique. To pick the physically feasible solution we need to impose additional conditions called *entropy conditions* on the solutions. Lax [6] proposed the condition

$$f'(u(x_-, t)) \geq f'(u(x_+, t)) \tag{4.6}$$

which says that the characteristics on either side of the discontinuity curve impinge on it.

Definition A weak solution satisfying (4.6) is called an entropy-weak solution.

4.2 An Explicit Formula

We now discuss convex conservation laws in one space dimension, i.e.,u and f denote scalar quantities. The conservation law (4.2) can now be written as a quasilinear equation

$$u_t + a(u)u_x = 0, \tag{4.7}$$

with $a = f'$. We require (4.7) to be nonlinear, which enforces the condition $f'' \neq 0$, meaning that f is either strictly convex or strictly concave. Here we assume that f is strictly convex, and

$$\lim_{|u| \to \infty} \frac{f(u)}{|u|} = \infty$$

Given such a function $f(u)$ defined for all u we have the Legendre Transform

$$f^*(s) = \sup_u \{us - f(u)\} \tag{4.8}$$

Let $u = G(s)$ be the value of u where the above supremum is achieved. Then we can easily show that

$$G(a(u)) = u \tag{4.9}$$

and

$$(f^*)'(s) = G(s) \tag{4.10}$$

Also it is easy to show that $G(s)$ and $f^*(s)$ are uniquely defined on the range of $a(u)$, that $f^*(s)$ is convex in that range, and that $f^*(s) \to \infty$ as s approaches the endpoints of the domain of dependence of f^*.

Using these auxiliary functions we assign a suitable function $u(x, t)$ to any bounded measurable initial function $g(x)$ in order to obtain a weak solution of the conservation law (4.2) with initial value $g(x)$. First we define $\Phi(y)$ as the integral of g:

$$\Phi(y) = \int_0^y g(\eta)\, d\eta. \tag{4.11}$$

Now the function

$$U(x, y; t) = \Phi(y) + tf^*\left(\frac{x - y}{t}\right) \tag{4.12}$$

is a continuous function of y for fixed x and t. It is easy to show that $U \to \infty$ as $y \to \pm\infty$. Therefore, it assumes a finite minimum in the interior.

Lemma 4.2.1. *Let y_1, y_2 be the values where U defined in* (4.12) *assumes its minimum for (x_1, t) and (x_2, t), respectively. If $x_1 < x_2$, then $y_1 \leq y_2$.*

Proof. Let f, and hence g, be convex. By definition of y_1, y_2 as minimum points, we have

$$\Phi(y_1) + tf^*\left(\frac{x_1 - y_1}{t}\right) \leq \Phi(y_2) + tf^*\left(\frac{x_1 - y_2}{t}\right)$$

and

$$\Phi(y_2) + tf^*\left(\frac{x_2 - y_2}{t}\right) \leq \Phi(y_1) + tf^*\left(\frac{x_2 - y_1}{t}\right).$$

Adding these two inequalities, we obtain

$$f^*\left(\frac{x_1 - y_1}{t}\right) + f^*\left(\frac{x_2 - y_2}{t}\right) \leq f^*\left(\frac{x_1 - y_2}{t}\right) + f^*\left(\frac{x_2 - y_1}{t}\right).$$

Since f^* is convex, it follows that $y_1 \leq y_2$. □

Lemma 4.2.2. *For a given t, barring the exception of countably many values of x, the function U assumes a minimum at a single point.*

Proof. For fixed x, t, denote the largest and the smallest values of y for which U attains a minimum by $y^+(x,t)$ and $y^-(x,t)$, respectively. By definition, $y^- \leq y^+$. On the other hand, Lemma 4.2.1 tells us that $y^+(x_1,t) \leq y^-(x_2,t)$ for $x_1 < x_2$. It follows that y^- and y^+ cannot differ except possibly at the points of discontinuity. Since y^- and y^+ are nondecreasing in x, the number of such discontinuities can only be countable. Thus, barring these countably many points, U attains minimum at a single point. □

We denote the minimum point by $y_0(x,t)$, and claim the following:

Theorem 4.2.3 (Unique Entropy-Weak Solution of Conservation Law)**.** *1. The function*

$$u(x,t) = G\left(\frac{x - y_0}{t}\right) \tag{4.13}$$

is the unique entropy-weak solution of (4.2)-(4.3)*.*

2. The x-integral of this solution $u(x,t)$ is equal to the value of the minimum of (4.12)*:*

$$u(x,t) = W_x(x,t)$$

where $W(x,t) = \min_y U(x,y;t)$

Proof. We first show that if g is smooth, then the function given by (4.13) matches with the smooth solution of the initial-value problem whenever the latter exists. For a continuous g, the function U in (4.12) has a continuous first derivative, and thus

$$g(y_0) - (f^*)'\left(\frac{x - y_0}{t}\right) = 0,$$

which in view of (4.10) and (4.12) is the same as

$$g(y_0) = G\left(\frac{x - y_0}{t}\right) = u(x,t). \tag{4.14}$$

Using (4.9) we get

$$\frac{x - y_0}{t} = a(g(y_0)).$$

(These equations show that the function $u(x,t)$ is constant along the straight line "characteristics", and that the slope of the characteristic line originating from $(y_0, 0)$ is $a(g(y_0))$.) Next, we write $u = \lim_{n\to\infty} u_N$, where

$$u_N(x,t) = \frac{\int_{-\infty}^{\infty} G\left(\frac{x-y}{t}\right) exp\left\{-N\left[\Phi(y) + tf^*\left(\frac{x-y}{t}\right)\right]\right\} dy}{\int_{-\infty}^{\infty} exp\left\{-N\left[\Phi(y) + tf^*\left(\frac{x-y}{t}\right)\right]\right\} dy}. \tag{4.15}$$

We denote

$$V_N = \int_{-\infty}^{\infty} \exp\left\{-N\left[\Phi(y) + tf^*\left(\frac{x-y}{t}\right)\right]\right\} dy. \tag{4.16}$$

From (4.10), we rewrite (4.15) as

$$u_N = \frac{1}{N}\frac{(V_N)_x}{V_N} = \left(\frac{1}{N}\log V_N\right)_x = (W_N)_x. \tag{4.17}$$

Likewise, we write $f(u) = \lim_{N\to\infty} f_N$, where

$$f_N(x,t) = \frac{\int_{-\infty}^{\infty} f\left[G\left(\frac{x-y}{t}\right)\right] exp\left\{-N\left[\Phi(y) + tf^*\left(\frac{x-y}{t}\right)\right]\right\} dy}{V_N}. \tag{4.18}$$

Using the definition of f^*, it is easy to show that

$$f(G(s)) = sG(s) - f^*(s).$$

Substituting this in (4.18), we get

$$f_N = \frac{1}{N}\frac{(-V_N)_t}{V_N} = (-W_N)_t. \tag{4.19}$$

From (4.17) and (4.19), we conclude that the vector fields (u_N, f_N) have zero divergence. Therefore, their limit $(u, f(u))$ is also divergence-free in the general sense. Indeed, $u(x,t)$ as prescribed in (4.13) is a weak solution of (4.2)-(4.3). From the relation

$$W_N = \frac{1}{N}\log V_N = \log(V_N)^{\frac{1}{N}}$$

and (4.16), we can determine

$$U(x,t) = \lim_{N\to\infty} W_N = \min_y \left\{\Phi(y) + tf^*\left(\frac{x-y}{t}\right)\right\}. \tag{4.20}$$

Integrating (4.17) with respect to x and letting $N \to \infty$, we get

$$U(x,t) = \int_{-\infty}^{x} u(\xi, t)\, d\xi.$$

Now write

$$\delta(t) = \max_x (y_0(x,t) - x)$$

Clearly,

$$U(x,t) = \min_{|x-y|<\delta(t)} \left\{ \Phi(y) + t f^* \left(\frac{x-y}{t} \right) \right\}. \tag{4.21}$$

Now $f^*(s)$ is strictly convex and unbounded, and $\Phi(y) \equiv \mathcal{O}(y)$ as g is bounded. It follows from (4.12) that $\delta(t) \to 0$ as $t \to 0$.

Let $\eta(\delta)$ denote the oscillation of $\Phi(y)$ over an interval of length δ. Since $\Phi(y)$ is uniformly continuous, $\eta(\delta) \to 0$ as $\delta \to 0$. If m is a lower bound for g, from (4.21) we have the following lower bound for U:

$$U(x,t) \geq \Phi x - \eta(\delta) + mt.$$

On the other hand, for the particular case $y = x$, we already have

$$U(x,t) \leq \Phi(x) + t f^*(0).$$

These estimates show that $U(x,t) \to \Phi(x)$ uniformly as $t \to 0$, that is, $u(x,t) \to g(x)$ in a weak sense. The entropy condition follows from Lemma 4.2.1, and the uniqueness result follows from Quinn [8]. □

Remark 1: The uniqueness of entropy-weak solutions in more general setup has been shown by Kružkov [5].

Remark 2: We note that

$$u_0 \in L^\infty \implies u(x,t) \in BV_{Loc} \cap L^\infty,$$

that is, the solution map $u_0 \mapsto u(x,t)$ has a regularizing effect.

Now we show that a solution u depends continuously on g:

Theorem 4.2.4. *Let g_n be a sequence of functions which converges weakly to a limit g. Let u_n be the solution corresponding to the initial value g_n by* (4.13)*, and let u be the solution corresponding to g. Then $u_n \to u$ at all points of continuity of u.*

Proof. Let Φ_n and Φ denote the integrals of g_n and g. Let (x,t) be a point where the function U of (4.12) has a unique minimum y_0. Since $g_n \to g$ in a weak sense, $\Phi_n \to \Phi$ uniformly. Therefore the function

$$U_n(x,y;t) = \Phi_n(y) + t f^* \left(\frac{x-y}{t} \right) \tag{4.22}$$

achieves its minimum at a point y_n which tends to y_0 as $n \to \infty$. We write $u_n = (U_n)_x(x,t)$ and find

$$\begin{aligned} u_n(x,t) &= \frac{\partial}{\partial x}\left[\min_y U_n(x,y;t)\right] \\ &= \frac{\partial}{\partial x}\left[\Phi_n(y_n) + tf^*\left(\frac{x-y_n}{t}\right)\right] \\ &= f^*\left(\frac{x-y_n}{t}\right) = g_n(y_n), \end{aligned}$$

and hence

$$\begin{aligned} \lim_{n\to\infty} u_n(x,t) &= \lim_{n\to\infty} g_n(y_n) \\ &= g(y_0) = u(x,t) \qquad \text{(from (4.14))}, \end{aligned}$$

as was to be proved. □

4.3 Riemann Problem

Consider the initial-value problem for scalar conservation laws in one space dimension:

$$(4.23) \qquad \begin{cases} u_t + f(u)_x &= 0 \qquad in\ \mathbb{R}\times(0,\infty) \\ u(x,0) &= u_i(x). \end{cases}$$

When $u_i(x)$ is piecewise continuous with two pieces, i.e.

$$(4.24) \qquad u_i(x) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases},$$

the problem (4.23)-(4.24) is called *Riemann Problem* for the scalar conservation law (4.23), $u_l \neq u_r$ are the *initial states.*

We assume that $f \in C^2$ is uniformly convex, and we write $G = (f')^{-1}$.

Theorem 4.3.1 (Solution of Riemann Problem)**.** *1. If $u_l > u_r$, the entropy solution of the Riemann Problem* (4.23)-(4.24) *is*

$$(4.25) \qquad u(x,t) := \begin{cases} u_l & \text{if } \frac{x}{t} < \sigma \\ u_r & \text{if } \frac{x}{t} > \sigma \end{cases} \qquad (x\in\mathbb{R}, t>0),$$

where

$$(4.26) \qquad \sigma := \frac{F(u_l) - F(u_r)}{u_l - u_r}.$$

2. If $u_l < u_r$, the entropy-weak solution of the Riemann Problem (4.23)-(4.24) *is*

$$(4.27) \qquad u(x,t) := \begin{cases} u_l & \text{if } \frac{x}{t} < f'(u_l) \\ G\left(\frac{x}{t}\right) & \text{if } f'(u_l) < \frac{x}{t} < f'(u_r) \\ u_r & \text{if } \frac{x}{t} > f'(u_r) \end{cases} \qquad (x\in\mathbb{R}, t>0).$$

Proof. 1. First assume that $u_l > u_r$. Then u defined by (4.25)-(4.26) is an integral solution of (4.23). In particular, the Rankine-Hugoniot Condition holds as $\sigma = \frac{[[f(u)]]}{[[u]]}$. Furthermore,

$$f'(u_r) < \sigma = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \fint_{u_r}^{u_l} f'(r)\, dr < f'(u_l)$$

from the condition

$$f'(u_l) < \sigma < f'(u_r)$$

The entropy condition holds as well, since $u_l > u_r$.

2. Now consider the case $u_l < u_r$. Suppose that u is of the form

$$u(x,t) = w\left(\frac{x}{t}\right)$$

Then we have

$$\begin{aligned} u_t + (f(u))_x &= u_t + f'(u)u_x \\ &= -w'\left(\frac{x}{t}\right)\frac{x}{t^2} + f'(w)w'\left(\frac{x}{t}\right)\frac{1}{t} \\ &= w'\left(\frac{x}{t}\right)\frac{1}{t}\left[f'(w) - \frac{x}{t}\right] \end{aligned}$$

Assuming $w' \neq 0$, we get $f'\left(w\left(\frac{x}{t}\right)\right) = \frac{x}{t}$. Hence

$$u(x,t) = w\left(\frac{x}{t}\right) = G\left(\frac{x}{t}\right)$$

is a solution of the conservation law. Now $w\left(\frac{x}{t}\right) = u_l$ for $\frac{x}{t} = f'(u_l)$, and similarly $w\left(\frac{x}{t}\right) = u_r$ for $\frac{x}{t} = f'(u_r)$. As a consequence, the *rarefaction wave* u defined by (4.27) is continuous in $\mathbb{R} \times (0, \infty)$, and solves the PDE $u_t + f(u)_x = 0$ in each of its regions of definition. Thus u is an integral solution of (4.23)-(4.24). Furthermore, if G is Lipschitz continuous, we have

$$u(x+z,t) - u(x,t) = G\left(\frac{x+z}{t}\right) - G\left(\frac{x}{t}\right) \leq \frac{\text{Lip}(G)z}{t}$$

for $f'(u_l)t < x < x + z < f'(u_r)t$. This inequality implies that u also satisfies the entropy condition.

□

Remark Because of the regularizing effect, the solution (4.27) is continuous even though the initial data u_0 is discontinuous. In linear wave propagation, however, the singularities or discontinuities in the initial data propagate along the characteristics.

4.4 Asymptotic Behaviour

In this section, we study the behaviour of the entropy solution u of (4.23) as $t \to \infty$.

Theorem 4.4.1 (Behaviour in L^∞-norm)**.** *Consider the initial-value problem* (4.23)*. Assume that f is smooth and uniformly convex with $f(0) = 0$, and $g \in L^\infty$. Then there exists a constant C such that*

$$|u(x,t)| \leq \frac{C}{\sqrt{t}} \tag{4.28}$$

for all $x \in \mathbb{R}$, $t > 0$.

Proof. We put $\sigma = f'(0)$, thereby getting $G(0) = 0$, and thus

$$f^*(\sigma) = \sigma G(\sigma) - f(G(\sigma)) = 0, \qquad (f^*)'(\sigma) = 0. \tag{4.29}$$

Since f^* is uniformly convex, we have

$$\begin{cases} tf^*\left(\frac{x-y}{t}\right) & = tf^*\left(\frac{x-y-\sigma t}{t} + \sigma\right) \\ & \geq t\left[f^*(\sigma) + (f^*)'(\sigma)\left(\frac{x-y-\sigma t}{t} + \sigma\right) + \theta\left(\frac{x-y-\sigma t}{t}\right)^2\right] \\ & = \theta\frac{|x-y-\sigma t|^2}{t} \end{cases} \tag{4.30}$$

For some constant $\theta > 0$. If $M := \|g\|_{L^1}$ is the upper bound for $h = \int_0^x g\,dy$, we have

$$tf^*\left(\frac{x-y}{t}\right) + h(y) \geq \theta\frac{|x-y-\sigma t|^2}{t} - M.$$

On the other hand,

$$tf^*\left(\frac{x-(x-\sigma t)}{t}\right) + h(x-\sigma t) \leq M.$$

Thus at the minimizing point y_0 we have

$$\theta\frac{|x-y_0-\sigma t|^2}{t} \leq 2M;$$

and so

$$\left|\frac{x-y_0}{t} - \sigma\right| \leq \frac{C}{\sqrt{t}} \tag{4.31}$$

for some constant C. Further, since $G(\sigma) = 0$, we have

$$\begin{aligned} |u(x,t)| &= \left|G\left(\frac{x-y_0}{t}\right)\right| \\ &= \left|G\left(\frac{x-y_0}{t} - \sigma + \sigma\right) - G(\sigma)\right| \\ &\leq \operatorname{Lip}(G)\left|\frac{x-y_0}{t} - \sigma\right| \\ &\leq \frac{C}{\sqrt{t}} \end{aligned}$$

for any $x \in \mathbb{R}$ and $t > 0$, thereby proving (4.28). □

Although (4.28) asserts that $\|u\|_{L^\infty} \to 0$ as $t \to \infty$, $\|u\|_{L^1}$ need not vanish. So now we assume that g has compact support, and describe the L^1-evolution of u into a simple shape.

For prescribed constants p, q, d, σ with $p, q \geq 0$, $d > 0$, we define the corresponding *N-wave* with *speed* σ to be the function

$$N(x,t) := \begin{cases} \frac{1}{d}\left(\frac{x}{t} - \sigma\right) & \text{if } -\sqrt{pdt} < x - \sigma t < \sqrt{qdt} \\ 0 & \text{otherwise.} \end{cases} \tag{4.32}$$

Now for $\sigma = f'(0)$, we set $d := f''(0) > 0$, and also write

$$p := -2 \min_{y \in \mathbb{R}} \int_{-\infty}^{y} g\,dx, \qquad q := 2 \max_{y \in \mathbb{R}} \int_{y}^{\infty} g\,dx.$$

Further, we have $p, q \geq 0$ and $G'(\sigma) = \frac{1}{d}$.

Theorem 4.4.2 (Behaviour in L^1-norm)**.** *Suppose that u is a solution of the initial-value problem* (4.23)*, and consider the N-wave* (4.32) *with $p, q > 0$. Then there exists a constant C such that*

$$\int_{\infty}^{\infty} |u(x,t) - N(x,t)|\,dx \leq \frac{C}{\sqrt{t}} \tag{4.33}$$

for all $t > 0$.

Proof. We have

$$\begin{aligned} u(x,t) &= G\left(\frac{x - y(x,t)}{t}\right) \\ &= G\left(\frac{(x - \sigma t) - y(x,t)}{t} + \sigma\right) \\ &= G(\sigma) + G'(\sigma)\left(\frac{(x - \sigma t) - y(x,t)}{t}\right) \\ &\quad + \mathcal{O}\left(\left|\frac{(x - \sigma t) - y(x,t)}{t}\right|^2\right). \end{aligned}$$

Consequently

$$\left|u(x,t) - \frac{1}{d}\frac{(x - \sigma t) - y(x,t)}{t}\right| \leq \frac{C}{t}. \tag{4.34}$$

Since g has compact support, for some constant $R > 0$ we have $g \equiv 0$ on $\mathbb{R} \cap \{|x| \geq R\}$. Therefore

$$h(x) = \begin{cases} h_- & \text{if } x \leq -R \\ h_+ & \text{if } x \geq R \end{cases}$$

for constants $h_\pm$. Also

$$\min_{\mathbb{R}} h = -\frac{p}{2} + h_- = -\frac{q}{2} + h_+.$$

Set $\epsilon = \epsilon(t) := \frac{A}{\sqrt{t}}$ $(t > 0)$ for a constant A that we will select later. We claim that for a sufficiently large A,

$$u(x,t) = 0 \qquad \text{for } x - \sigma t < -R - \sqrt{pd(1+\epsilon)t}, \tag{4.35}$$

and

$$u(x,t) = 0 \qquad \text{for } x - \sigma t > R + \sqrt{qd(1+\epsilon)t}. \tag{4.36}$$

Since $(f^*)''(\sigma) = G'(\sigma) = \frac{1}{d}$ from (4.10), we deduce from (4.30)-(4.31) that

$$tf^*\left(\frac{x-y}{t}\right) = \frac{1}{d}\frac{|(x-\sigma t)-y|^2}{2t} + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) \qquad \text{as } t \to \infty. \tag{4.37}$$

For $x - \sigma t < -R - \sqrt{pd(1+\epsilon)t}$, we have $h(x-\sigma t) = h_-$, and so

$$tf^*\left(\frac{(x-(x-\sigma t))}{t}\right) + h(x-\sigma t) = tL(\sigma) + h_- = h_-.$$

Now if $y \leq -R$, then $f^* \geq 0$ implies

$$tf^*\left(\frac{x-y}{t}\right) + h(y) \geq h_-.$$

On the other hand, for $y \geq -R$, we estimate

$$\begin{aligned} tf^*\left(\frac{x-y}{t}\right) + h(y) &\geq \frac{1}{d}\frac{|(x-\sigma t)-y|^2}{2t} - \frac{p}{2} + h_- + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) \\ &\geq \frac{pd(1+\epsilon)t}{2dt} - \frac{p}{2} + h_- + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) \\ &= \frac{p}{2}\frac{A}{\sqrt{t}} + h_- + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) \\ &\geq h_-, \end{aligned}$$

provided A is large enough. Finally, we conclude that $y(x,t) = x - \sigma t$, and so $u(x,t) = G(\sigma) = 0$, proving (4.35). (4.36) can be proved following the same lines.

Now select z such that $h(z) = \min h = -\frac{p}{2} + h_-$ and $|z| \leq R$. Then (4.37) provides the estimate

$$\begin{aligned} tf^*\left(\frac{x-z}{t}\right) + h(z) &\leq \frac{1}{d}\frac{|(x-\sigma t)-z|^2}{2t} - \frac{p}{2} + h_- + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) \\ &\leq \frac{pd(1-\epsilon)t}{2dt} - \frac{p}{2} + h_- + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) \\ &= -\frac{p}{2}\frac{A}{\sqrt{t}} + h_- + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) \\ &< h_- \end{aligned}$$

for large enough A. Therefore

$$y(x,t) \geq -R \qquad \text{if } x - \sigma t = R - \sqrt{pd(1-\epsilon)t}. \tag{4.38}$$

Likewise, we can also show that

$$y(x,t) \leq -R \qquad \text{if } x - \sigma t = -R + \sqrt{qd(1-\epsilon)t}. \tag{4.39}$$

Since the mapping $x \mapsto y(x,t)$ is nondecreasing, for large enough t we have

$$\left| u(x,t) - \frac{1}{d}\left(\frac{x}{t} - \sigma\right) \right| \leq \frac{C}{t}$$

if

$$R - \sqrt{pd(1-\epsilon)t} < x - \sigma t < -R + \sqrt{qd(1-\epsilon)t}.$$

Finally, using the estimate $|u| = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right)$ obtained in Theorem 4.4.1, the fact that $|N| = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right)$ by definition, and the bound $\sqrt{(1 \pm \epsilon)t} - \sqrt{t} = \mathcal{O}(1)$, we get

$$\int_{\infty}^{\infty} |u(x,t) - N(x,t)|\, dx = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right)$$

as so desired. □

4.5 An Example with Burgers Equation

Burgers' Equation is a fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modeling of gas dynamics, nonlinear wave propagation as well as traffic flow. In one space dimension, it has the form

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad \text{in } \mathbb{R}^n \times (0,\infty). \tag{4.40}$$

Comparing with (4.2), we note

$$f(u) = \frac{u^2}{2}, \tag{4.41}$$

$$f^*(s) = \sup_u \{us - f(u)\} = \frac{s^2}{2} = f(s), \tag{4.42}$$

and

$$G(u) = (f')^{-1}(u) = u. \tag{4.43}$$

We apply the method of characteristics to (4.40) and obtain the characteristic ODE

$$\frac{dx}{dt} = u, \qquad \frac{du}{dt} = 0. \tag{4.44}$$

That is, u is constant along the characteristics that are straight lines in the x,t-plane. The solution of the ODE (4.44) is

$$x = ut + C_1, \qquad u = C_2, \tag{4.45}$$

with the constants C_1, C_2 depending on the prescribed initial or boundary data.

Example 1: Consider the Riemann problem on (4.40) with the initial condition

$$u_i(x) = \begin{cases} 0 & (x < a) \\ k & (a \le x \le b) \\ 0 & (x > b) \end{cases}, \tag{4.46}$$

where $k \in \mathbb{R}$ is a constant and $[a, b]$ is a prescribed x-interval. Applying Theorem 4.3.1, we obtain the solution as

1. For $k > 0$,

$$u(x,t) = \begin{cases} 0 & (x < a) \\ \frac{x-a}{t} & (0 \le \frac{x-a}{t} \le k) \\ k & (\frac{x-a}{t} > k) \cap (\frac{x-b}{t} < \frac{k}{2}) \\ 0 & (\frac{x-b}{t} > \frac{k}{2}) \end{cases}$$

   The shock line $\frac{x-b}{t} = \frac{k}{2}$ meets the rarefaction line $\frac{x-a}{t} = k$ at the point $(x,t) = (2b - a, \frac{2(b-a)}{k})$. From this point, a new shock curve is generated, which satisfies the initial-value problem

$$\frac{dx}{dt} = \frac{x}{2t}, \qquad (x_0, t_0) = (2b - a, \frac{2(b-a)}{k}),$$

   that is, its equation is

$$x^2 = \frac{k(2b-a)^2}{2(b-a)} t.$$

   Figure (4.1) shows the characteristic lines for (4.46) with $a = 0$, $b = 2$, $k = 1$.

2. For $k < 0$,

$$u(x,t) = \begin{cases} 0 & (\frac{x-a}{t} < \frac{k}{2}) \\ k & (\frac{x-b}{t} < k) \cap (\frac{x-a}{t} > \frac{k}{2}) \\ \frac{x-b}{t} & (k \le \frac{x-b}{t} \le 0) \\ 0 & (x > b) \end{cases}$$

   In this case, the shock line $\frac{x-a}{t} = \frac{k}{2}$ and the rarefaction line $\frac{x-b}{t} = k$ meet at the point $(x,t) = \left(2a - b, \frac{2(a-b)}{k}\right)$, and as before a new shock curve is generated from this point. It satisfies the ODE

$$\frac{dx}{dt} = \frac{x}{2t}, \qquad (x_0, t_0) = \left(2a - b, \frac{2(a-b)}{k}\right),$$

   that is, it has the equation

$$x^2 = \frac{k(2a-b)^2}{2(a-b)} t.$$

   Figure (4.2) shows the characteristic lines for (4.46) with $a = 0$, $b = 2$, $k = -1$.

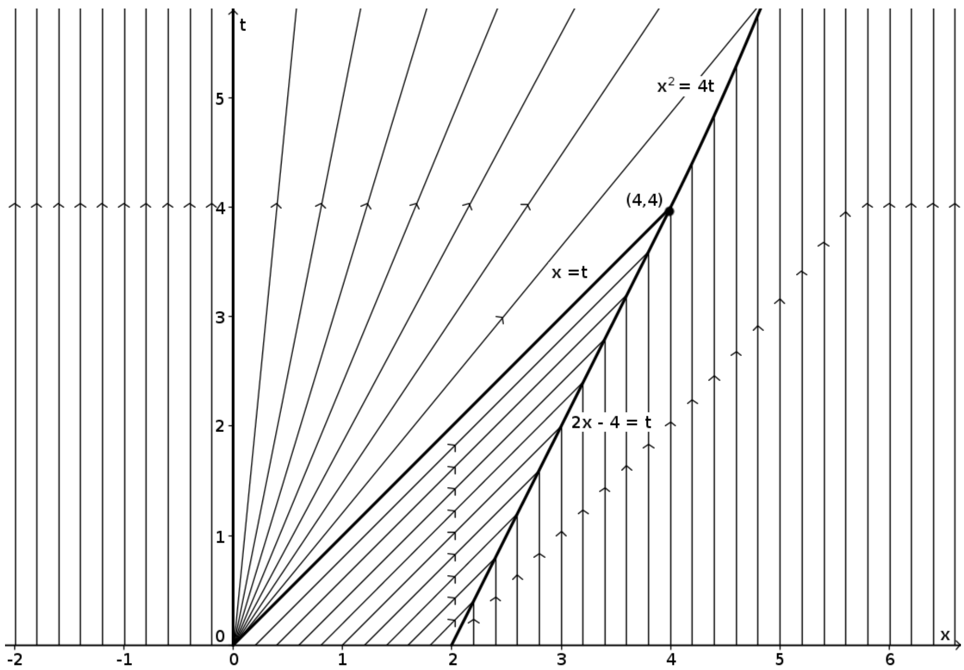

Figure 4.1: Characteristic lines for the Riemann Problem (4.40)-(4.46) with $a = 0$, $b = 2$, $k = 1$.

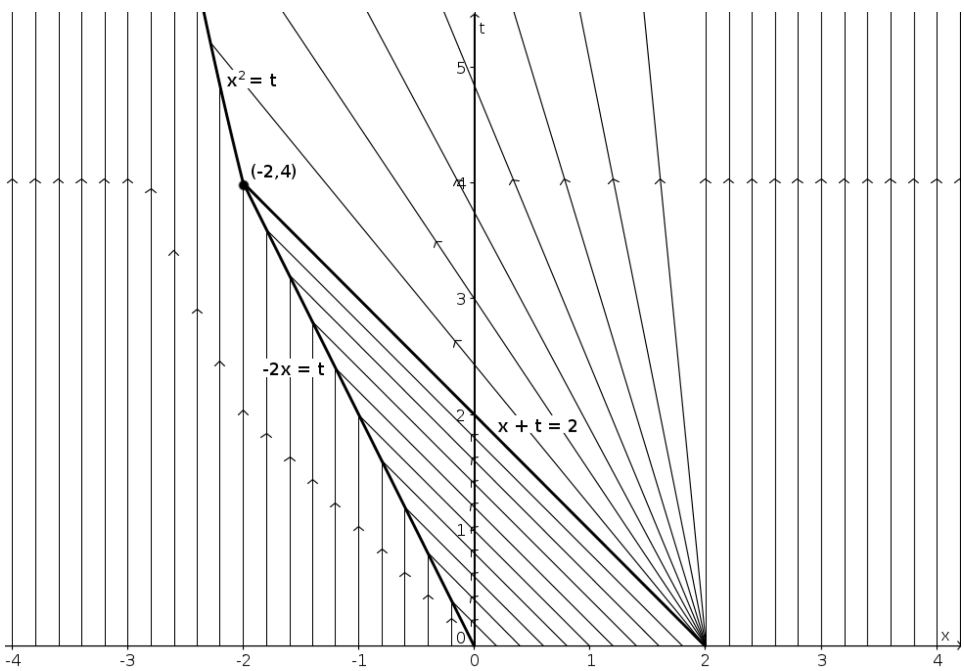

Figure 4.2: Characteristic lines for the Riemann Problem (4.40)-(4.46) with $a = 0$, $b = 2$, $k = -1$.

Example 2: Consider the Riemann problem on (4.40) once again, but with the initial condition

$$u_i(x) = \begin{cases} 1 & (x \le -1) \\ 0 & (-1 < x \le 0) \\ 2 & (0 < x \le 1) \\ 0 & (x > 1) \end{cases}. \tag{4.47}$$

Again, we apply Theorem 4.3.1 to obtain the solution

$$u(x,t) = \begin{cases} 1 & (\frac{x+1}{t} < \frac{1}{2}) \\ 0 & (\frac{x+1}{t} > \frac{1}{2}) \cap (x < 0) \\ \frac{x}{t} & (x > 0) \cap (\frac{x}{t} < 2) \\ 2 & (\frac{x}{t} > 2) \cap (\frac{x-1}{t} < 1) \\ 0 & (\frac{x-1}{t} > 1) \end{cases}.$$

In this example,

- The left shock line $2(x+1) = t$ meets the left rarefaction line $x = 0$ at the point $(x,t) = (0,2)$. The new shock curve generated from this point satisfies the ODE

  $$\frac{dx}{dt} = \frac{x}{2t} + \frac{1}{2}, \qquad (x_0, t_0) = (0, 2),$$

  and hence has the equation

  $$x = t - \sqrt{2t}. \tag{4.48}$$

- Also the right shock line $x - 1 = t$ meets the right rarefaction line $x = 2t$ at the point $(2,1)$. The shock curve generated here satisfies

  $$\frac{dx}{dt} = \frac{x}{2t}, \qquad (x_0, t_0) = (2, 1),$$

  and thus has the equation

  $$x^2 = 4t. \tag{4.49}$$

  (See Figure (4.3).)

- The two new shocks (4.48) and (4.49) meet again at the point $(x,t) = (2(2+\sqrt{2}), (2+\sqrt{2})^2)$. The third shock curve (See Figure (4.4)) generated from here satisfies

  $$\frac{dx}{dt} = \frac{x}{t}, \qquad (x_0, t_0) = (2(2+\sqrt{2}), (2+\sqrt{2})^2),$$

  and therefore is the straight line

  $$2x - 2 = t.$$

This example shows that, although a locally continuous solution through rarefaction may be generated initially, the shocks can annihilate its continuity at a certain time, and the discontinuity dominates after that.

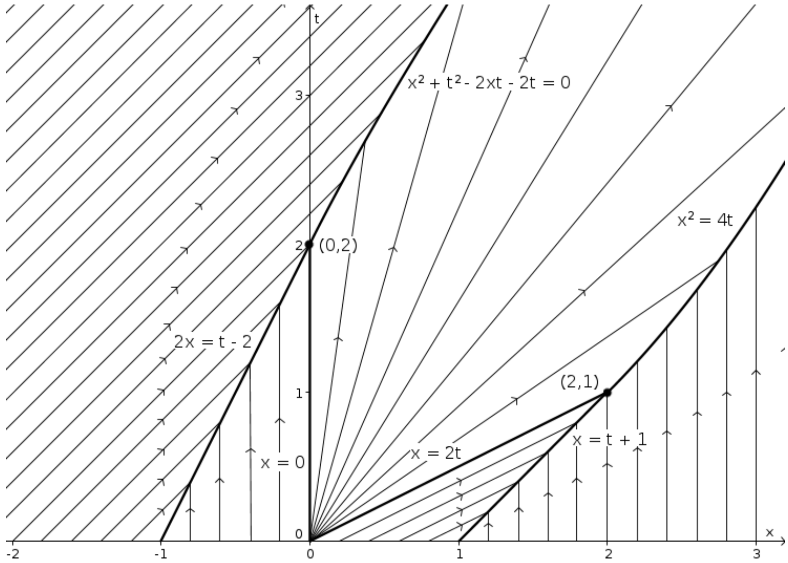

Figure 4.3: Characteristic lines for the Riemann Problem (4.40)-(4.47) near the origin.

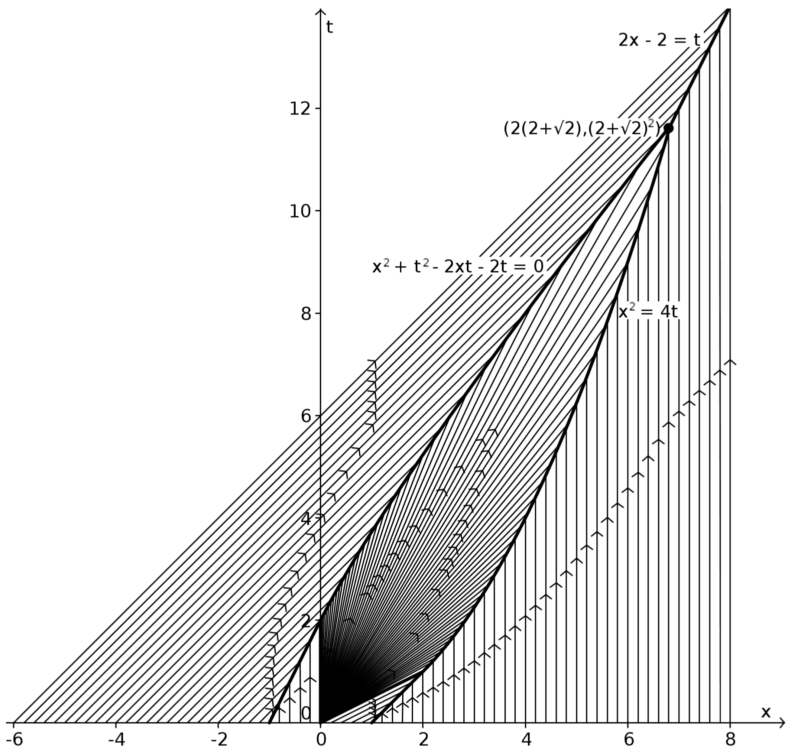

Figure 4.4: Characteristic lines for the Riemann Problem (4.40)-(4.47). Note that the two shocks initially generated as seen in Figure (4.3) meet again at the point $(x,t) = (2(2+\sqrt{2}), (2+\sqrt{2})^2)$; the new shock is generated along $2x - 2 = t$.

Chapter 5

Initial-Boundary-Value Problem

We consider the mixed initial-boundary problem for strictly convex conservation laws in one space dimension

$$u_t + (f(u))_x = 0 \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^+ \tag{5.1}$$

with the initial condition

$$u(x,0) = u_i(x). \tag{5.2}$$

As prescribed by Bardos, LeRoux and Nédélec [1], the boundary condition reads

$$\begin{cases} \text{either} & u(0,t) = \bar{u}_b(t) \\ \text{or} & f'(u(0,t)) \leq 0 \quad \text{and} \quad f(u(0,t)) \geq f(\bar{u}_b(t)) \end{cases} \tag{5.3}$$

for a given bounded function $u_b(t)$, where

$$\bar{u}_b(t) = \max\{u_b(t), \lambda\} \tag{5.4}$$

and λ is the unique point such that $f'(\lambda) = 0$ and $f(\lambda) = \inf_u f(u)$; the strict convexity of f allows this unique λ. We want the solution to satisfy the entropy condition

$$u(x_-, t) \geq u(x_+, t), \qquad x > 0, t > 0. \tag{5.5}$$

We also assume

$$\lim_{|u|\to\infty} \frac{f(u)}{|u|} = \infty. \tag{5.6}$$

Now we introduce some notations: For each fixed $x \geq 0$, $y \geq 0$, $t > 0$ and $\alpha \geq 0$, $C_\alpha(x,y,t)$ denotes the class of paths β in the quarter plane

$$D = \{(z,s) \in (\mathbb{R}^+ \cup \{0\}) \times (\mathbb{R}^+ \cup \{0\})\}.$$

Each path connects the point $(y,0)$ to (x,t) and is of the form $z = \beta(s)$, where β is a piecewise linear function with either one straight line (denoted by β_0) or three straight lines, where the absolute value of the slope of each straight line is at most α.

Let f^* be the Legendre Transform of f:

$$f^*(u) = \max_{\theta}[\theta u - f(\theta)].$$

Let $u_i(x) \in L^\infty(\mathbb{R}^+)$ and $u_b(t)$ be continuous and bounded on $(0, \infty)$, and let $\bar{u}_b$ be defined as in (5.4). Set $\alpha = \infty$. We define

$$\begin{aligned} J(\beta) &= -\int_{s:\beta(s)=0} f(\bar{u}_b(s))\, ds + \int_{s:\beta(s)\neq 0} f^*\left(\frac{d\beta}{ds}\right) ds \\ &= -\int_{s:\beta(s)=0} [f(\bar{u}_b(s)) - f(\lambda)]\, ds + \int_0^t f^*\left(\frac{d\beta}{ds}\right) ds \end{aligned} \tag{5.7}$$

as $f^*(0) = -\min f(u) = -f(\lambda)$, and

$$H(x,t,\beta) = \int_0^y u_i(z)\, dz + J(\beta). \tag{5.8}$$

Then we define

$$U(x,t) = \min_{\substack{\beta \in C_\alpha(x,y,t) \\ y \geq 0}} H(x,t,\beta). \tag{5.9}$$

for each fixed $x \geq 0$, $t > 0$. Further, let

$$A(x,y,t) = t f^*\left(\frac{x-y}{t}\right), \tag{5.10}$$

and with $\bar{C}_\alpha(x,y,t) = C_\alpha(x,y,t) \setminus \beta_0$, let

$$\begin{aligned} B(x,y,t) &= \min_{\beta \in \bar{C}_\alpha(x,y,t)} \{J(\beta)\} \\ &= \min_{\substack{0 \leq t_1 \leq t_2 \leq t \\ \frac{y}{t_1} \leq \alpha,\ \frac{x}{t-t_2} \leq \alpha}} \left[-\int_{t_1}^{t_2} f(\bar{u}_b(s))\, ds + t_1 f^*\left(\frac{y}{-t_1}\right) \right. \\ &\qquad \left. + (t-t_2) f^*\left(\frac{x}{t-t_2}\right) \right], \end{aligned} \tag{5.11}$$

and then define

$$Q(x,y,t) = \min[A(x,y,t), B(x,y,t)].$$

Since $B(x,y,t)$ is Lipschitz continuous for each fixed $y \geq 0$, $Q(x,y,t)$ is also Lipschitz continuous. Following Conway and Hopf [2], and Lax [6], we know that $U(x,t)$ is Lipschitz continuous as well, and $U_x(x,t)$ exists a.e.; we then denote

$$u(x,t) = U_x(x,t). \tag{5.12}$$

Finally we let

$$Q_1(x,y,t) = Q_x(x,y,t), \tag{5.13}$$

and

$$R(x,t) = \min_{y \geq 0}\left[Q(x,y,t) + \int_0^y u_i(z)\, dz\right]. \tag{5.14}$$

5.1 A Formula for the Solution

We introduce some more notations: for each fixed $x, y \geq 0$, $t > 0$, for $0 < t_1 < t_2 < t$ let $(t_2, t_1) = (t_2(x, y, t), t_1(x, y, t))$ denote a value for which (5.11) attains the minimum. Define

$$\begin{aligned} t_2^+(x, y, t) &= \max[t_2(x, y, t)], \\ t_2^-(x, y, t) &= \min[t_2(x, y, t)], \\ t_1^+(x, y, t) &= \max[t_1(x, y, t)], \\ t_1^-(x, y, t) &= \min[t_1(x, y, t)]. \end{aligned}$$

Also let $y_0(x, t)$ be a value of y that minimizes (5.14), and write

$$\begin{aligned} y_0^+(x, t) &= \max[y_0(x, t)], \\ y_0^-(x, t) &= \min[y_0(x, t)]. \end{aligned}$$

Following Jensen's inequality, we know that no two paths $\beta_1, \beta_2 \in C_\alpha(x, y, t)$ cross each other with different slopes inside D. Therefore, for $x_1 < x_2$,

$$y_0^-(x_2, t) \leq y_0^+(x_2, t) \leq y_0^-(x_1, t) \leq y_0^+(x_1, t), \tag{5.15}$$

and it follows from [4] that $y_0^\pm(x, t)$ are nondecreasing functions of x, $y_0^+(\cdot, t)$ is right continuous and $y_0^-(\cdot, t)$ is left continuous, $y_0^+(x, t) = y_0^-(x, t)$ a.e., and

$$\begin{aligned} y_0^+(x, t) &= y_0^+(x_+, t) &= y_0^-(x_+, t), \\ y_0^-(x, t) &= y_0^-(x_-, t) &= y_0^+(x_-, t). \end{aligned}$$

Lemma 5.1.1. *For fixed $t > 0$, Suppose that the minimum in* (5.9) *for $H(x, t, \beta)$ is attained for some $\bar{\beta} \in \bar{C}_\alpha(x, y_0(x, t), t)$. Let $x^* < x$ and β^* the path that attains the minimum in* (5.9) *for $H(x^*, t, \beta)$. Then*

$$\beta^* \in \bar{C}_\alpha(x^*, y_0(x^*, t), t).$$

Moreover,

$$t_1^\pm(x, y_0^\pm(x, t), t) = t_1^\pm(x^*, y_0^\pm(x^*, t), t)$$

and

$$y_0^+(x, t) = y_0^-(x, t) = y_0^+(x^*, t) = y_0^-(x^*, t).$$

Proof. Since any two paths β and β^* cannot cross, we have

$$\beta^* \in \bar{C}_\alpha(x, y_0(x^*, t), t).$$

For the same reason, it follows that

$$t_1^\pm(x, y_0^\pm(x, t), t) = t_1^\pm(x^*, y_0^\pm(x^*, t), t)$$

and

$$y_0^+(x, t) = y_0^-(x, t) = y_0^+(x^*, t) = y_0^-(x^*, t).$$

In particular, in $[0, x]$ $y_0^\pm(\cdot, t)$ is constant, and hence every point of $[0, x]$ is a point of continuity of $y_0^\pm(\cdot, t)$. □

Theorem 5.1.2. *Let $u(x, t)$ be defined by* (5.12)*. Then*

- *$u(x,t) = Q_1(x, y_0(x,t), t)$ solves* (5.1) *for a.e.* (x,t) *in the sense of distributions, where* $y_0(x,t)$ *is the unique minimizer of* (5.14) *and* Q_1 *is given by* (5.13).
- *For each $t > 0$ and $x > 0$, $u(x_{\pm}, t)$ exist and satisfy the entropy condition $u(x_-, t) \geq u(x_+, t)$.*
- *$u(x,t)$ satisfies the initial condition* (5.2).
- *$u(0_+, t)$ exists a.e. and satisfies the boundary condition* (5.3).

Proof. We write

$$V_N(x,t) = \int_0^\infty exp\left[-N\left(\int_0^y u_0(z)\,dz + Q(x,y,t)\right)\right] dy, \tag{5.16}$$

$$u_N(x,t) = \frac{\int_0^\infty Q_1(x,y,t) exp\left[-N\left(\int_0^y u_0(z)\,dz + Q(x,y,t)\right)\right] dy}{V_N(x,t)}, \tag{5.17}$$

$$f_N(x,t) = \frac{\int_0^\infty f(Q_1(x,y,t)) exp\left[-N\left(\int_0^y u_0(z)\,dz + Q(x,y,t)\right)\right] dy}{V_N(x,t)}, \tag{5.18}$$

and

$$U_N(x,t) = -\frac{1}{N}\log V_N, \tag{5.19}$$

according to Lax [6]. With $y_0(x,t)$ minimizing (5.14), we get from these definitions that

$$\lim_{N\to\infty} u_N(x,t) = Q_1(x, y_0(x,t), t), \tag{5.20}$$

$$\lim_{N\to\infty} f_N(x,t) = f[Q_1(x, y_0(x,t), t)], \tag{5.21}$$

and

$$\lim_{N\to\infty} U_N(x,t) = U(x,t). \tag{5.22}$$

Since

$$u_N(x,t) = -\frac{1}{N}\frac{(V_N)_x}{V_N} = (U_N)_x, \tag{5.23}$$

we get from (5.20), (5.22) and (5.23) that

$$U_x(x,t) = Q_1(x, y_0(x,t), t).$$

According to Joseph and Gowda [4], Q satisfies (5.1). Then

$$\begin{aligned}
(U_N)_t &= -\frac{1}{N}\frac{(V_N)_t}{V_N} \\
&= \frac{\int_0^\infty Q_t(x,y,t)exp\left[-N\left(\int_0^y u_0(z)\,dz + Q(x,y,t)\right)\right]\,dy}{V_N(x,t)} \\
&= \frac{\int_0^\infty -f(Q_1(x,y,t))exp\left[-N\left(\int_0^y u_0(z)\,dz + Q(x,y,t)\right)\right]\,dy}{V_N(x,t)} \\
&= -f_N \qquad (5.24)
\end{aligned}$$

That is,

$$(u_N)_t + (f_N)_x = 0.$$

Hence

$$\iint (u_N\phi_t + f_N\phi_x)\,dx\,dt = 0$$

for all test functions $\phi(x,t) \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$. Sending $N \to \infty$, we get

$$u_t + (f(u))_x = 0$$

as desired. The entropy condition follows from the nondecreasing nature of $(f^*)'$.

Now, any minimizer $\bar{\beta}$ for $H(x,t,\beta)$ cannot have any of its linear segments parallel to the x-axis (having so would make the slope of such a segment equal to ∞, and not less than $\alpha = \infty$). In fact, the slopes of each segment of $\bar{\beta}$ are uniformly bounded following the assumptions on $f(u)$, $u_i(x,t)$ and $u_b(x,t)$. Hence, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$u(x,t) = (f^*)'\left(\frac{x - y_0(x,t)}{t}\right)$$

for all $x \geq \epsilon$, $t \leq \delta$. Then, according to Lax [6], $\lim_{t\to 0} u(x,t) = u_i(x)$ a.e. $x \geq \epsilon$. Since ϵ is arbitrary, it follows that

$$\lim_{t\to 0} u(x,t) = u_i(x) \quad a.e.\ x \geq 0.$$

We have from Joseph and Gowda [4] that

$$u(x_\pm,t) = Q_1(x, y_0^\pm(x,t), t). \qquad (5.25)$$

Further, since $(f^*)'$ is nondecreasing, (5.5) is satisfied.

Finally, we have two candidates for $u(0_+,t)$ for a.e. $t > 0$: if

$$u(0_+,t) = \lim_{x\to 0}(f^*)'\left(\frac{x}{t - t_2^+(x, y_0^+(x,t), t)}\right), \qquad (5.26)$$

then

$$\frac{\partial J}{\partial t_2} = 0$$

gives us

$$f\left((f*)'\left(\frac{x}{t - t_2^+}\right)\right) = f(\bar{u}_b(t_2^+)).$$

Passing the above to the limit $x \to 0$, (5.26) gives us

$$f(u(0_+,t)) = f(\bar{u}_b(t)). \tag{5.27}$$

In the case $f'(u(0_+,t)) \geq 0$, we have, in addition,

$$u(0_+,t) = u_b(t). \tag{5.28}$$

On the other hand, if

$$u(0_+,t) = \lim_{x\to 0} (f^*)' \left(\frac{x - y_0^+(x,t)}{t} \right) \quad a.e., \tag{5.29}$$

we have

$$u(0_+,t) = (f^*)' \left(\frac{-y_0^+(x,t)}{T} \right),$$

and

$$f'(u(0_+,t)) = f' \left((f^*)' \left(\frac{-y_0^+(x,t)}{t} \right) \right) = \frac{-y_0^+(x,t)}{t} \leq 0.$$

Since the path joining $(0,t)$ to $(y_0^+(x,t),t)$ minimizes $H(x,t,\beta)$ in (5.9), we have

$$\begin{aligned} tf^* \left(\frac{-y_0^+(0,t)}{t} \right) + \int_0^{y_0^+(0,t)} u_i(z)\,dz \quad \leq \quad & -\int_{t-\Delta t}^{t} f(\bar{u}_b(s))\,ds \\ & + (t-\Delta t) f^* \left(\frac{-y_0^+(0,t)}{t-\Delta t} \right) \\ & + \int_0^{y_0^+(0,t)} u_i(z)\,dz, \end{aligned}$$

that is,

$$\frac{tf^*\left(\frac{-y_0^+(0,t)}{t}\right) - (t-\Delta t) f^*\left(\frac{-y_0^+(0,t)}{t-\Delta t}\right)}{\Delta t} \leq -\frac{1}{\Delta t} - \int_{t-\Delta t}^{t} f(\bar{u}_b(s))\,ds.$$

Passing to the limit $\Delta t \to 0$, we find

$$f^* \left(\frac{-y_0^+(0,t)}{t} \right) - \left(\frac{-y_0^+(0,t)}{t} \right) (f^*)' \left(\frac{-y_0^+(0,t)}{t} \right) \leq -f(\bar{u}_b(t)),$$

that is,

$$f \left[(f^*)' \left(\frac{-y_0^+(0,t)}{t} \right) \right] \leq -f(\bar{u}_b(t)),$$

and hence

$$f(u(0_+,t)) \geq f(\bar{u}_b(t)). \tag{5.30}$$

The boundary condition (5.3) is met by (5.27), (5.28) and (5.30), and the proof is complete. □

5.2 Riemann Initial-Boundary-Value Problem for Burgers' Equation

Consider the one-dimensional Burgers' Equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^+ \tag{5.31}$$

subject to the initial condition

$$u(x,0) = k_i \tag{5.32}$$

and the boundary condition

$$u(0,t) = k_b \tag{5.33}$$

prescribed in a weak sense (5.3), with $k_i, k_b \in \mathbb{R}$ being constants. We apply the analysis done in the previous section to obtain the solution of (5.31)-(5.32)-(5.33) in various cases:

Case 1: $k_i, k_b \geq 0$, $k_i > k_b$

In this case, a rarefaction wave is generated with the bounding curves $x = k_i t$ and $x = k_b t$ (See Figure (5.1)). The explicit solution is

$$u(x,t) = \begin{cases} k_b & (0 \leq \frac{x}{t} \leq k_b) \\ \frac{x}{t} & (k_b \leq \frac{x}{t} \leq k_i) \\ k_i & (\frac{x}{t} \geq k_i) \end{cases} .$$

This solution is continuous, though the initial and boundary conditions are different.

Case 2: $k_b \geq 0$, $k_i + k_b \geq 0$

In this case, a shock line is generated along $x = \frac{k_i + k_b}{2} t$ (See Figure (5.2)). Here the explicit solution is

$$u(x,t) = \begin{cases} k_b & (0 \leq \frac{x}{t} < \frac{k_i + k_b}{2}) \\ k_i & (\frac{x}{t} > \frac{k_i + k_b}{2}) \end{cases} ,$$

and is thus discontinuous.

Case 3: $k_i \geq 0$, $k_b \leq 0$

Here, since our domain is restricted to $\mathbb{R}^+ \times \mathbb{R}^+$, the left arm of the rarefaction wave does not appear in the solution unless $k_b = 0$ (See Figure (5.3)). The solution is continuous, and is given by

$$u(x,t) = \begin{cases} \frac{x}{t} & (0 \leq \frac{x}{t} \leq k_i) \\ k_i & (\frac{x}{t} \geq k_i) \end{cases} .$$

Case 4: $k_i \leq 0$, $k_i + k_b \leq 0$

Due to the restriction of the domain $\mathbb{R}^+ \times \mathbb{R}^+$ (See Figure (5.4)), the solution generated in this case is simply the initial state:

$$u(x,t) = k_i.$$

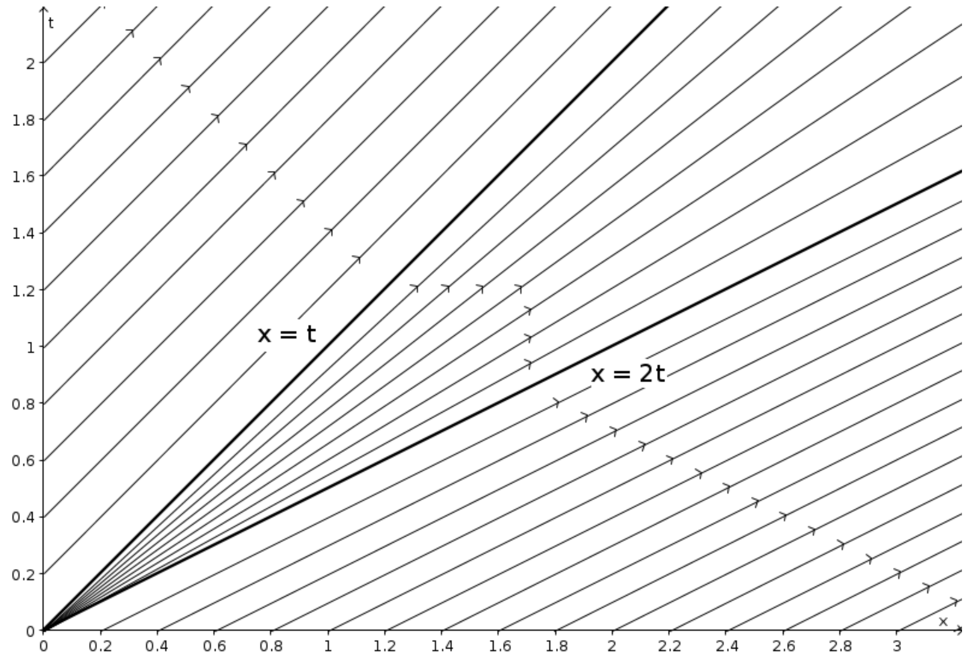

Figure 5.1: Characteristic lines for the problem (5.31)-(5.33) with $k_i = 2$ and $k_b = 1$. The rarefaction is spanned between the lines $x = t$ and $x = 2t$.

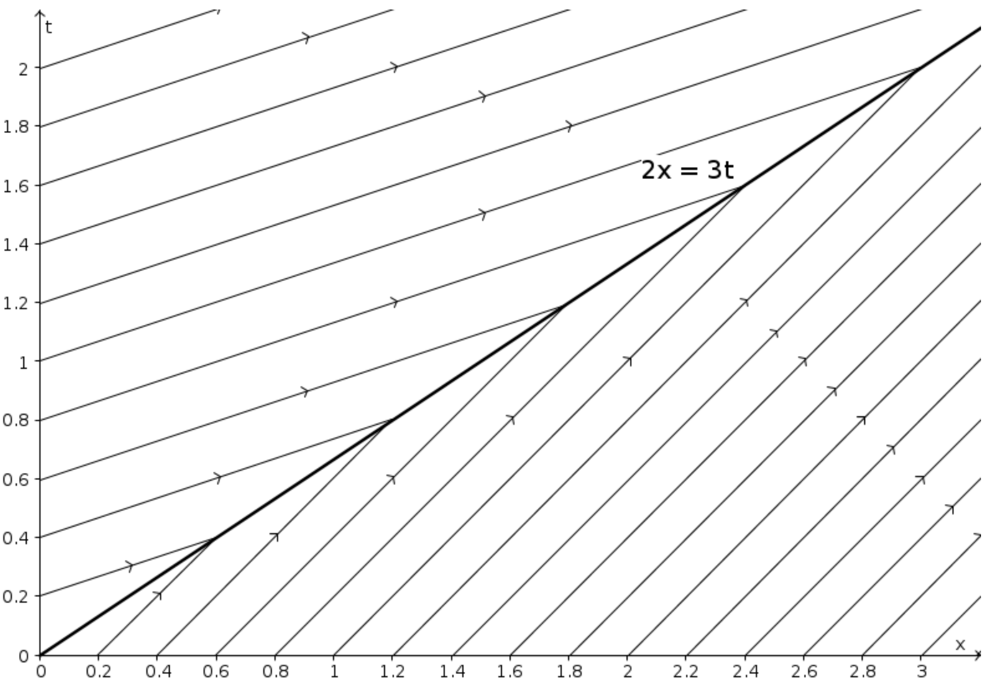

Figure 5.2: Characteristic lines for the problem (5.31)-(5.33) with $k_i = 1$ and $k_b = 2$. The shock is generated along the line $2x = 3t$.

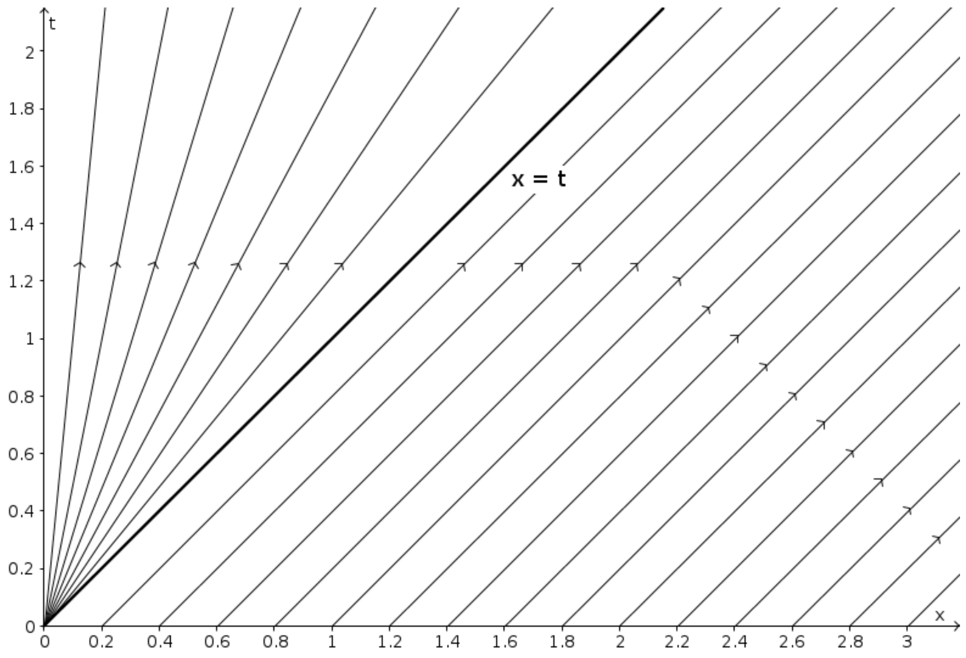

Figure 5.3: Characteristic lines for the problem (5.31)-(5.33) with $k_i = 1$ and $k_b = -2$.

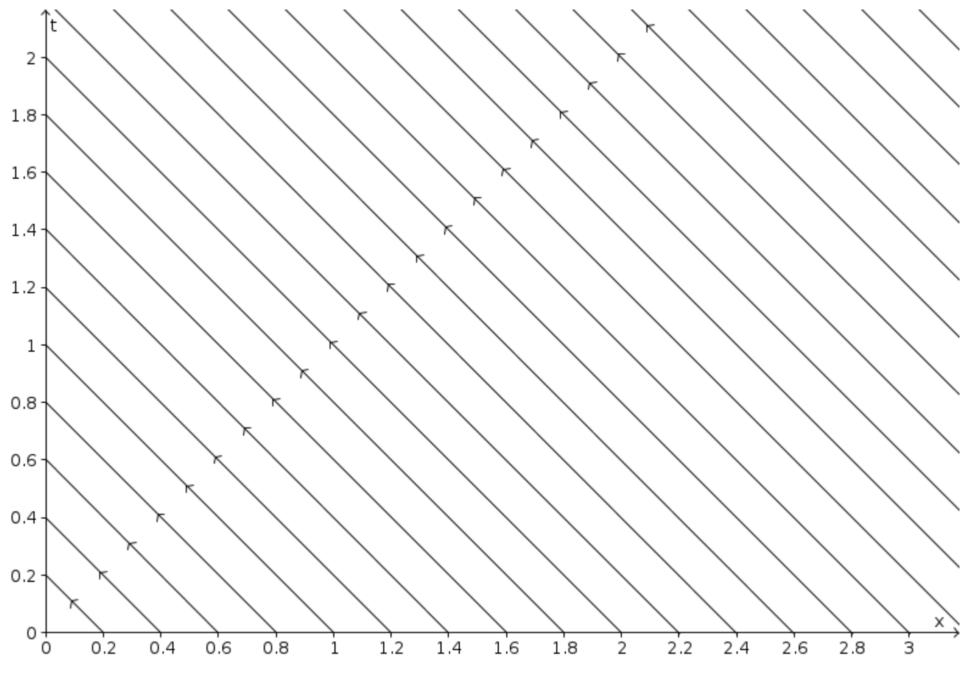

Figure 5.4: Characteristic lines for the problem (5.31)-(5.33) with $k_i = -1$ and $k_b = \frac{1}{2}$.

Appendix A

Convolution and Smoothing

Suppose that $U \subset \mathbb{R}^n$ is open. For fixed $\epsilon > 0$, define

$$U_\epsilon := \{x \in U \,|\, \operatorname{dist}(x, \partial U) > \epsilon\}.$$

- The standard mollifier $\eta \in C^\infty(\mathbb{R}^n)$ is given by

  $$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & (|x| < 1) \\ 0 & (|x| \geq 1) \end{cases},$$

  where the constant C is such that

  $$\int_{\mathbb{R}^n} \eta \, dx = 1.$$

- For the fixed $\epsilon > 0$, the functions

  $$\eta_\epsilon(x) := \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right)$$

  are C^∞ and satisfy

  $$\int_{\mathbb{R}^n} \eta_\epsilon \, dx = 1, \qquad \operatorname{supp}(\eta_\epsilon) \subset B(0, \epsilon).$$

- If $f : U \to \mathbb{R}$ is locally integrable, we define the mollification of f in U_ϵ by $f^\epsilon := \eta_\epsilon * f$, that is,

  $$f^\epsilon(x) = \int_U \eta_\epsilon(x-y) f(y) \, dy = \int_{B(0,\epsilon)} \eta_\epsilon(y) f(x-y) \, dy$$

  for $x \in U_\epsilon$.

- The mollification f^ϵ satisfies the following properties:

  1. $f^\epsilon \in C^\infty(U_\epsilon)$.
  2. $f^\epsilon \to f$ as $\epsilon \to 0$
  3. If $f \in C(U)$, then $f^\epsilon \to f$ uniformly on any compact subset of U.
  4. If $1 \leq p < \infty$ and $f \in L^p_{\text{loc}}(U)$, then $f^\epsilon \to f$ in $L^p_{\text{loc}}(U)$.

Appendix B

Some Useful Theorems

B.1 Gauss-Green Theorem

Assume that U is a bounded open subset of $\mathbb{R}^n$, and that ∂U is C^1. Suppose that $u, v \in C^1(\bar{U})$. Then

$$\int_U u_{x_i}\, dx = \int_{\partial U} u\nu^i dS \qquad (i = 1, \ldots, n),$$

where ν is the outward unit normal to ∂U.

The integration by parts formula follows by putting uv in place of u:

$$\int_U u_{x_i} v\, dx = -\int_U u v_{x_i}\, dx + \int_{\partial U} uv\nu^i\, dS \qquad (i = 1, \ldots, n).$$

B.2 Green's Formulae

For $u, v \in C^2(\bar{U})$,

1. $$\int_U \Delta u\, dx = \int_{\partial U} uv\nu^i\, dS \qquad (i = 1, \ldots, n).$$

2. $$\int_U \nabla v \cdot \nabla u\, dx = -\int_U u\Delta v dx + \int_{\partial U} \frac{\partial v}{\partial \nu}\, dS.$$

3. $$\int_U (u\Delta v - v\Delta u)\, dx = \int_{\partial U} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right)\, dS.$$

B.3 Inverse Function Theorem

Let $U \subset \mathbb{R}^n$ be an open set. Assume that $\mathbf{f} \in C^1(U; \mathbb{R}^n)$, $\mathbf{f} = (f^1, \ldots, f^n)$ and

$$J\mathbf{f}(x_0) = \left| \frac{\partial(f^1, \ldots, f^n)}{\partial(x_1, \ldots, x_n)}(x_0) \right| \neq 0.$$

Then there exist an open set $V \subset U$, with $x_0 \in V$, and an open set $W \subset \mathbb{R}^n$, with $\mathbf{f}(x_0) \in W$, such that

- The mapping $\mathbf{f} : V \to W$ is one-one and onto.
- The inverse function $\mathbf{f}^{-1} : W \to V$ is C^1.
- If $\mathbf{f} \in C^k$, then $\mathbf{f}^{-1} \in C^k$ $(k = 2, \dots)$.

B.4 Dominated Convergence Theorem

Let $\{f_n\}$ be a sequence of real valued measurable functions on a measure space (S, Σ, μ). Suppose that the sequence converges pointwise to a function f and is dominated by some integrable function g in the sense that

$$|f_n(x)| \leq g(x)$$

for all n in the index set of the sequence and all points $x \in S$. Then f is integrable and

$$\lim_{n\to\infty} \int_S |f_n - f| \, d\mu = 0,$$

which also implies

$$\lim_{n\to\infty} \int_S f_n \, d\mu = \int_S f \, d\mu.$$

B.5 Rademacher's Theorem

If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ is Lipschitz continuous, then f is differentiable a.e. in U; that is, the points in U at which f is not differentiable form a set of Lebesgue measure zero.

Appendix C

Some Useful Inequalities

C.1 Hölder's Inequality

Let (S, Σ, μ) be a measure space and let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, for all measurable real or complex valued functions f and g on S,

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

C.2 Jensen's Inequality

Let (Ω, A, μ) be a measure space. If g is a real valued function that is μ-integrable, and if φ is a convex function on $\mathbb{R}$, then:

$$\varphi\left(\int_a^b f(x)\,dx\right) \leq \frac{1}{b-a}\int_a^b \varphi((b-a)f(x))\,dx,$$

where $a, b \in \mathbb{R}$, and $f : [a, b] \to \mathbb{R}$ is a non-negative real valued function that is Lebesgue integrable.

C.3 Gronwall's Inequality

C.3.1 Differential Form

Let I denote an interval of the real line of the form $[a, \infty)$ or $[a, b]$ or $[a, b)$ with $a < b$. Let β and u be real valued continuous functions defined on I. If u is differentiable in the interior I° of I and satisfies the differential inequality

$$u'(t) \leq \beta(t)u(t), \qquad t \in I^\circ,$$

then u is bounded by the solution of the corresponding differential equation $y'(t) = \beta(t)\,y(t)$:

$$u(t) \leq u(a)\exp\left(\int_a^t \beta(s)\,ds\right)$$

for all $t \in I$.

C.3.2 Integral Form

Let I denote an interval of the real line of the form $[a, \infty)$ or $[a, b]$ or $[a, b)$ with $a < b$. Let α, β and u be real valued continuous functions defined on I. Assume that β and u are continuous and that the negative part of α is integrable on every closed and bounded subinterval of I.

1. If β is nonnegative and if u satisfies the integral inequality

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)\, ds \qquad \forall\, t \in I,$$

   then

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(r)\, dr\right) ds, \qquad t \in I.$$

2. If, in addition, the function α is nondecreasing, then

$$u(t) \leq \alpha(t) \exp\left(\int_a^t \beta(s)\, ds\right), \qquad t \in I.$$

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