On Spectra of Graphs and Manifolds

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by

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This thesis is dedicated to my parents,

Anwar Unnisa Shahnaaz and Mir Zahid Ali.

For everything.

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Certificate

Certifed that the work incorporated in the thesis entitled *On Spectra of Graphs and Manifolds*, submitted by Ayesha Fatima was carried out by the candidate under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

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Abstract

One can define the notion of length spectrum for a simple regular periodic graph via counting the orbits of closed reduced cycles under an action of a discrete group of automorphisms [11]. We prove that this length spectrum satisfies an analogue of the 'Multiplicity one' property. We show that if all but finitely many primitive cycles in two simple regular periodic graphs have equal lengths, then all the cycles have equal lengths. This is a graph-theoretic analogue of a similar theorem in the context of geodesics on hyperbolic spaces [2]. We also prove, in the context of actions of finitely generated abelian groups on a graph, that if the adjacency operators [4] for two actions of such a group on a graph are similar, then corresponding periodic graphs are length isospectral.

We also consider the length-holonomy spectrum of compact hyperbolic spaces and using the analytic properties of the Selberg-Wakayama zeta function, we give some weak results in the direction of proving a strong multiplicity one property of the length-holonomy spectrum in the 3-dimensional case.

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Statement of Originality

The main results of this thesis which constitute original research are Theorems 2.2.1, 2.3.4, 3.7.2 and 3.7.3.
Lemma 2.3.3 and corollary 2.2.2 are original subsidary results, where the former is used in the proof of the main results while the latter follows from one of the main results.

Contents

Introduction

Notions analogous to the prime numbers have been explored in various contexts and many results have been proved in drawing out this analogy. For compact hyperbolic surfaces, this correspondence was first explored in the work of Selberg [24], wherein he proved a counterpart of the prime number theorem for primitive geodesics of compact Riemann surfaces. This was further explored in the works of Sarnak [23], Gangolli [8] and Wakayama [28] by developing the analytic theory of zeta functions for hyperbolic spaces which resemble the theory of L-functions for automorphic forms. In the case of finite graphs, the primitive cycles were studied and Hashimoto [15] proved that they satisfy a similar asymptotic distribution. The analogy between finite graphs and hyperbolic spaces was further developed by Sunada [27], Ihara [18] and Hashimoto [14, 15] by associating a zeta function to finite regular graphs.

One of the aim of this thesis is to explore the correspondence between compact locally symmetric Riemannian manifolds and simple regular periodic graphs. A result that we prove in this direction is an analogue of the classical strong multiplicity one theorem.

It is well known that the Fourier coefficients of cusp forms satisfy the classical strong multiplicity one theorem due to Atkin and Lehner: Let f and g are newforms for some Hecke congruence subgroup $\Gamma_0(N)$. Suppose that the eigenvalues of the Hecke operator at a prime p are equal for all but finitely many primes p. Then f and g are equal ([21, p.125]). In [2], Bhagwat and Rajan proved a multiplicity one type property for the length spectrum of even dimensional compact hyperbolic spaces.

We associate a notion of length spectrum to simple regular periodic graphs, which are basically countable connected graphs with an action of a subgroup of automorphisms which satisfy some properties.

A pair (X, Γ) consisting of a countable, infinite, connected graph X of bounded degree and a countable subgroup Γ of the automorphism group of X is called a *periodic graph* if Γ acts on $V(X)$ discretely and co-finitely. In this thesis, we also assume that the action of Γ is without inversions and with bounded co-volume.

We consider cycles in the graph X which are reduced (with no tail and backtracking). The action of the automorphism subgroup Γ gives rise to Γ-equivalence classes of reduced cycles. The set of the Γ-equivalence of reduced cycles is denoted by $[\mathcal{R}]_\Gamma$. The length spectrum is a function which counts the number of Γ-equivalence classes of reduced cycles of a given length.

The *length spectrum* of the periodic graph (X, Γ) is defined to be the function L_Γ on $\mathbb{N}$ given by

$$L_\Gamma(n) \;=\; \text{The number of } \xi \;\in\; [\mathcal{R}]_\Gamma \text{ such that } \ell(\xi) = n\ .$$

A closed path is called *primitive* if it is not obtained by going $n \;\geq\; 2$ times around some other closed path. The subset consisting of primitive reduced cycles is denoted by $\mathcal{P}$. The set of the Γ-equivalence of primitive reduced cycles is denoted by $[\mathcal{P}]_\Gamma$.

The *primitive length spectrum* of the periodic graph (X, Γ) is defined to be the function PL_Γ on $\mathbb{N}$ given by

$$PL_\Gamma(n) \;=\; \text{The number of } \xi \;\in\; [\mathcal{P}]_\Gamma \text{ such that } \ell(\xi) = n\ .$$

We show that the primitive length spectrum of simple $(q\,+\,1)$-regular graphs satisfy a strong multiplicity one property:

Theorem 0.0.1 (Multiplicity One property for Length Spectrum of Regular Periodic Graphs)**.** *Let* (X, Γ_1) *and* (X, Γ_2) *be two simple,* $q+1$ *regular periodic graphs. Further, assume that* Γ_1 *and* Γ_2 *act on* $V(X)$ *without inversions and with bounded co-volume and such that the stabilizer of any cycle with respect to either subgroup is trivial. Suppose* $PL_{\Gamma_1}(n) = PL_{\Gamma_2}(n)$ *for all but finitely many* $n \in \mathbb{N}$. *Then* $PL_{\Gamma_1}(n) = PL_{\Gamma_2}(n)$ *for all* $n \in \mathbb{N}$. *Furthermore, we can conclude that* $L_{\Gamma_1}(n) = L_{\Gamma_2}(n)$ *for all* $n \in \mathbb{N}$.

The proof of the theorem uses the analytic properties of the Ihara zeta function for periodic graphs, the construction of which was first given by Clair and Mokhtari-Sharghi in [5] and further studied by Guido, Isola and Lapidus in [10]. This zeta function is an analogue of the Riemann zeta function and satisfies many similar properties. The methods used in the proof are similar to the ones used in [2] and [22].

Another aspect of the length spectrum of compact hyperbolic manifolds of negative curvature that has been explored is its relation to the Laplace spectrum. In [9], Gangolli shows that the Laplace spectrum of the considered manifolds determines the length spectrum. This kind of relationship can be explored in the case of finite graphs, where the Laplace operator is self-adjoint and hence has a spectrum associated to it. The extension of this result to the case of periodic graphs is hindered by the fact that deformed Laplacian of such graphs may not be self-adjoint. We consider the case where the periodic graphs are with actions of finitely generated abelian groups and prove an analogue of the above result, albeit with a modified hypothesis. In the statement and the proof of this result, we crucially use a generalization of a construction given in [4], which expresses the adjacency operator of periodic graphs with $\mathbb{Z}$-action in terms of a finite matrix. This construction simplifies the determinant formula 1.4 by using the Fourier transform $\ell^2(\mathbb{Z}) = L^2(S^1)$.

Theorem 0.0.2. *Let* A_1 *and* A_2 *be the adjacency operators of the periodic* *graphs* $(X, \Gamma)_1$ *and* $(X, \Gamma)_2$ *respectively. Suppose* A_1 *and* A_2*, as* $n \times n$ *matrices, are conjugate. Then* $L_{\Gamma,1}(m) = L_{\Gamma,2}(m)$ *for all* $m \in \mathbb{N}$.

In the last part of the thesis, we consider the case of compact locally symmetric Riemannian spaces of the kind $X_\Gamma = \Gamma\backslash G/K$, where G is the connected component of identity in the isometry group $SO(n, 1)$ of the hyperbolic space $\mathbb{H}_n$, K is a maximal compact subgroup and Γ is a torsion-free uniform lattice in G. In this case, a parameter called holonomy class h_λ can be associated to the closed geodesics λ. The holonomy class of a closed geodesic is a conjugacy class in $SO(n - 1)$.

If we let $\mathcal{M}_{n-1}$ be the set of conjugacy classes in $SO(n - 1)$, the primitive length-holonomy spectrum of X_Γ is the function $\mathfrak{P}_\Gamma$ defined on $\mathbb{R} \times \mathcal{M}_{n-1}$ by, $\mathfrak{P}_\Gamma(a, [M]) = \#$ of primitive conjugacy classes $[\gamma]$ in Γ with $(l(\gamma), h_\gamma) = (a, [M])$.

It can be asked if this length-holonomy spectrum satisfies a strong multiplicity one property. The Selberg-Wakayama zeta function, which has been defined in [28], is given in terms of the length and holonomy of the primitive geodesics, and can thus be employed in answering this question. We explore this question in the case of $G = SO(3, 1)$ and arrive at weaker results. In the proofs of these results, we crucially use the fact that the holonomy class $b(p) := h_p$ of primitive closed geodesic p is a real number in $[0, 2\pi)$, which affords us a simplification of the Selberg-Wakayama zeta function.

If L_Γ is the length spectrum and PL_Γ is the primitive length spectrum of the space X_Γ, we prove the following result:

Theorem 0.0.3. *Let* $G = SO(3, 1)$*, and* Γ_1 *and* Γ_2 *be two uniform lattices in* G *such that* $\mathfrak{P}_{\Gamma_1}(a, b) = \mathfrak{P}_{\Gamma_2}(a, b)$ *for all but finitely many pairs* $(a, b) \in \mathbb{R} \times [0, 2\pi)$*. Then* $PL_{\Gamma_1}(l) = PL_{\Gamma_2}(l)$*, and hence* $L_{\Gamma_1}(l) = L_{\Gamma_2}(l)$*, for all* $l \in \mathbb{R}$.

Furthermore, if for any $p \in P_\Gamma$, we define $c(p) := \frac{b(p)}{a(p)}$, we can define a modified primitive length-holonomy spectrum as the function $\mathfrak{M}_\Gamma$ on $\mathbb{R}$ given by

$$\mathfrak{M}_\Gamma(c) \;=\; \#\,\text{of conjugacy classes } [p] \,\in\, P_\Gamma \text{ such that } c(p) \,=\, c.$$

We then prove the following result:

Theorem 0.0.4. *Let $G = SO(3,1)$, and Γ_1 and Γ_2 be two uniform lattices in G such that $\mathfrak{P}_{\Gamma_1}(a,\, b) = \mathfrak{P}_{\Gamma_2}(a,\, b)$ for all but finitely many pairs $(a,\, b) \in \mathbb{R} \times [0,\, 2\pi]$. Then $\mathfrak{M}_{\Gamma_1}(c) = \mathfrak{M}_{\Gamma_2}(c)$ for all $c \in \mathbb{R}$.*

Structure of the thesis

In the first chapter, we review the definitions from graph theory and define and state the properties of periodic graphs, the Ihara zeta function associated to them. In the second chapter, we define the length spectrum and state and proof the strong multiplicity one property for regular periodic graphs 2.2.1. We also state and prove a result 2.3.4 on the relation between the adjacency operator and the length spectrum in the case where the regular periodic graphs are with the action of finitely generated abelian groups. In the third chapter, we review the theory of compact locally symmetric Riemannian manifolds and the associated zeta functions, viz. the Selberg-Gangolli zeta and the Selberg-Wakayama zeta functions. In the case when $G = SO(3,\, 1)$, we state and prove weaker results 3.7.2, 3.7.3 in the direction of strong multiplicty one property of the length-holonomy spectrum.

Chapter 1

Periodic Graphs and their Zeta Functions

1.1 Preliminaries

In this section we give the basic definitions related to graph theory.

A *graph* X consists of a non-empty set $V(X)$ of elements called the vertices of X together with a set $E(X)$ of unordered pairs of vertices. An element $e = \{v_1, v_2\} \in E(X)$ is called an *edge* connecting the vertices v_1 and v_2; v_1 and v_2 are called adjacent vertices (written as $v_1 \sim v_2$). The vertices v_1 and v_2 are called the *end vertices* of the edge e. An edge $e = (v, v)$, with both the end vertices same, is called a *loop*. A graph is said to be *simple* if it has no loops. An edge e is said to be coming out of a vertex v if v is one of the end vertices of e. A graph X is said to be *countable* if both the vertex set $V(X)$ and the edge set $E(X)$ are countable.

The *degree* of a vertex $v \in V(X)$, denoted by $\deg(v)$, is the number of edges coming out of v, where we count the loops twice. For a non-zero positive integer q, a graph X is said to be *q-regular* if $\deg(v) = q$ for all $v \in V(X)$.

A graph is said to have *bounded degree*, if $d := \sup_{v \in V(X)} \deg(v) < \infty$. We call d the bounded degree of X.

A *path* of *length* m in X from $u \in V(X)$ to $v \in V(X)$ is a sequence of $m+1$ vertices $(u = v_0, \ldots, v = v_m)$ such that v_i is adjacent to v_{i+1} for $i = 0, \ldots, m$. A path can also be denoted by a sequence of edges $(e_1, \ldots, e_m)$ where e_i is an edge connecting v_{i-1} and v_i. A path is said to be *closed* if $u = v$. A graph is said to be *connected* if any two vertices can be joined by a path.

Definition 1.1.1 (Reduced Paths)**.** A path $(e_1, \ldots, e_m)$ is said to have *backtracking* if for any $i \in \{1, \ldots, m-1\}$, $e_i = e_{i+1}$ (traversed in opposite directions). A path with no backtracking is called *proper.*

A closed path is called *primitive* if it is not obtained by going $n \geq 2$ times around some other closed path.

A proper closed path $C = (e_1, \ldots, e_m)$ is said to have a *tail* if $\exists\, k \in \mathbb{N}$ such that $e_{m-j+1} = e_j$ (traversed in opposite directions) for some k consecutive values of j. Proper tail-less closed paths are called *reduced* closed paths. The set of reduced closed paths is denoted by $\mathcal{C}$.

A *cycle* is an equivalence class of closed paths, any of which can be obtained from another by a cyclic permutation of vertices. Simply put, a cycle is a closed path with no specified starting point. We denote the length of the cycle C, which is the number of edges in any closed path representing the cycle, by $\ell(C)$. This is also denoted by $|C|$ in literature. We use the terms interchangeably. The set of reduced cycles is denoted by $\mathcal{R}$. The subset consisting of primitive reduced cycles is denoted by $\mathcal{P}$. The primitive reduced cycles are also called *prime cycles.*

Definition 1.1.2 (Graph Automorphism)**.** *An automorphism* of a graph X is a permutation σ of the vertex set $V(X)$ such that (v_1, v_2) is an edge if and only if $(\sigma(v_1), \sigma(v_2))$ is an edge.

Definition 1.1.3. [10] A countable discrete subgroup Γ of automorphisms of X is said to act on $V(X)$

(1) *without inversions* if for any edge $e = \{v_1, v_2\}$, $\nexists \gamma \in \Gamma$ such that $\gamma(v_1) = v_2$ and $\gamma(v_2) = v_1$ (No edge is inverted),

(2) *discretely* if $\Gamma_v := \{\gamma \in \Gamma \mid \gamma(v) = v\}$ is finite,

(3) *with bounded co-volume* if $vol(X/\Gamma) := \sum_{v \in \mathcal{F}_0} \frac{1}{|\Gamma_v|} < \infty$, where $\mathcal{F}_0$ is a complete set of representatives of the equivalence classes in $V(X)/\Gamma$,

(4) *co-finitely* if $\mathcal{F}_0$ is finite.

The finite graph X/Γ is denoted by B. The set of vertices of this graph is $\mathcal{F}_0$ and the edges are the set $\{([v], [w]) | (v, w) \in E(X)\}$.

Definition 1.1.4 (Periodic Graph)**.** A pair (X, Γ) consisting of a countable, infinite, connected graph X of bounded degree and a countable subgroup Γ of the automorphism group of X is called a *periodic graph* if Γ acts on $V(X)$ discretely and co-finitely. In this thesis, as in [11], we also assume that the action of Γ is without inversions and with bounded co-volume.

Definition 1.1.5 (Γ-Equivalence Relation)**.** Two reduced cycles $C, D \in \mathcal{R}$ are said to be Γ-equivalent, and written as $C \sim_\Gamma D$, if there exists an isomorphism $\gamma \in \Gamma$ such that $D = \gamma(C)$. The set of Γ-equivalence classes of reduced cycles is denoted by $[\mathcal{R}]_\Gamma$. The set of Γ-equivalence classes of prime cycles is denoted by $[\mathcal{P}]_\Gamma$.

Since Γ is a subgroup of the automorphism group, the length of any two cycles in a Γ-equivalence class will be same. Therefore, for a Γ-equivalence class of reduced cycles $\xi \in [\mathcal{R}]_\Gamma$ we can define $\ell(\xi) := \ell(C)$ for any representative C in ξ.

Remark 1.1.6. Another notion of length, called effective length of a cycle C, has been defined in [10]. It is defined as the length of the prime cycle underlying C. We do not make use of this notion in this thesis.

Consider the action of Γ on the set of cycles. For any cycle C, the *stabilizer* of C in Γ is the subgroup $\Gamma_C := \{\gamma \in \Gamma \mid \gamma(C) = C\}$. Also, if C_1 and C_2 are Γ-equivalent, say $C_1, C_2 \in \xi$, then Γ_{C_1} and Γ_{C_2} are conjugates in Γ. Hence $|\Gamma_{C_1}| = |\Gamma_{C_2}|$. This enables us to define $S(\xi)$ as the cardinality of Γ_C for any C in ξ.

1.2 Zeta Functions of Graphs

The zeta function associated to finite graphs was defined by Yasutaka Ihara in his paper [18] on structure theorem of torsion-free discrete co-compact subgroups of $PGL(2, \mathbb{Q}_p)$. It was Serre who first suggested, in his book [25], the interpretation of Ihara's zeta function as a zeta function associated to certain $(p+1)$-regular, finite graphs.

Let X be a finite connected $(q+1)$-regular graph.

Definition 1.2.1 (Ihara zeta function)**.** The Ihara zeta function Z_X is defined by the Euler product

$$Z_X(u) := \prod_{C \in \mathcal{P}} (1 - u^{|C|})^{-1}, \quad \text{for} \quad |u| < \frac{1}{q}, \tag{1.1}$$

where $\mathcal{P}$ is the set of prime cycles of X.

We recall that the Riemann zeta function is given by

$$\zeta(s) = \prod_{p}(1 - p^{-s})^{-1} \quad \text{for} \quad \mathrm{Re}(s) > 1,$$

where p ranges over all rational primes. If we put $u := q^{-s}$, we observe that $u^{|C|} = (q^{|C|})^{-s}$ and $|u| < \frac{1}{q}$ if and only if $\mathrm{Re}(s) > 1$. This brings out the analogy between the Ihara and Riemann zeta functions.

The Ihara zeta function satisfies analogues of many of the properties of the Riemann zeta function, at least for the regular case. For example, the Ihara zeta function can be used to prove an analogue of the prime number theorem. This result gives an estimate on the number of prime cycles of a fixed length in a graph [26]. A version of Riemann hypothesis for $\zeta(s)$ has also been studied for the Ihara zeta function $Z_X(u)$ in the case of regular graph. The graphs satisfying this hypothesis have been completely classified (see [26]).

There have been generalizations of this zeta function in several directions. A brief survey of some of the generalizations and the consequent results can be found in [17].

In this thesis, we are concerned with the generalization of the Ihara zeta function to certain infinite graphs, namely periodic graphs of the form (X, Γ). This construction was first given by Clair and Mokhtari-Sharghi in [5] and further studied in [10].

The action of the automorphism subgroup Γ gives rise to Γ-equivalence classes of prime cycles. In constructing the Ihara zeta function for periodic graphs, the Euler product is taken over these classes. Furthermore, each term is normalized by an exponent related to the cardinality of the stabilizer of a representative of the equivalence class.

Let (X, Γ) be a periodic graph. The Ihara zeta function $Z_{X,\Gamma}$ of (X, Γ) is defined as follows.

Definition 1.2.2 (Ihara zeta function of Periodic Graphs).

$$Z_{X,\Gamma}(u) := \prod_{[C]_\Gamma \in [\mathcal{P}]_\Gamma} (1 - u^{\ell(C)})^{-\frac{1}{|\Gamma_C|}}, \tag{1.2}$$

for u sufficiently small so that the infinite product converges.

The following theorem gives the radius of convergence of the Ihara zeta function of a periodic graph (X, Γ) with bounded degree d. (See [10, Theorem 2.2 (i), p. 1345].

Theorem 1.2.3. *Let $Z_{X,\Gamma}$ be the zeta function of the periodic graph (X, Γ). Then, $Z_{X,\Gamma}(u)$ defines a holomorphic function in the open disc $|u| < \frac{1}{d-1}$.*

1.3 von Neumann Algebras

One of the important results of the finite Ihara zeta function is the determinant formula, which states that the inverse of the zeta function is a polynomial. More precisely, it shows that the inverse is the determinant of a matrix valued polynomial.

In order to obtain an analogous determinant formula for the infinite Ihara zeta function for periodic graphs, an analytic determinant was defined in [11]. The definition of this determinant uses the theory of von Neumann algebras, the relevant details of which we recall in this section. We refer the reader to the book 'Non-commutative Geometry'([6], Chapter 5, Section 1) by A. Connes for more details on this.

Definition 1.3.1 ($*$-algebra). A $*$-algebra is an algebra A equipped with a *conjugation* $*$, that is, a linear map $* : A \longrightarrow A$ such that

$$(a^*)^* = a \quad \text{and} \quad (ab)^* = b^* a^*$$

for all $a, b \in A$

Definition 1.3.2 (C^*-algebra)**.** A C^*-algebra is a Banach algebra A over $\mathbb{C}$ with a conjugate-linear involution $* : A \longrightarrow A$ such that

$$(ab)^* = b^* a^* \quad \text{and} \quad \|a^* a\| = \|a\|^2$$

for all $a, b \in A$

Let H be a complex Hilbert space and let $\mathfrak{L}(H)$ be the C^*-algebra of bounded operators from H into H. $\mathfrak{L}(H)$ is a Banach algebra equipped with the norm

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$$

and with the involution $T \longmapsto T^*$ defined by

$$\langle T^* x, y \rangle = \langle x, Ty \rangle \ \forall x, y \in H.$$

The *pre-dual* of $\mathfrak{L}(H)$, denoted by $\mathfrak{L}(H)_*$, is the subalgebra of trace class operators on H. The *ultraweak topology* on $\mathfrak{L}(H)$ is the weakest topology in which all elements of $\mathfrak{L}(H)_*$ are continuous, when considered as functions on $\mathfrak{L}(H)$.

Definition 1.3.3 (von Neumann Algebra)**.** A von Neumann algebra on H is a $*$-subalgebra of $\mathfrak{L}(H)$ containing the identity operator and which is closed under the ultraweak topology.

The commutant of a subset S of $\mathfrak{L}(H)$, denoted by S' is defined as

$$S' := \{T \in \mathcal{L}(H) ; TA = AT \, \forall \, A \in S\}.$$

The double commutant theorem of von Neumann states that a $*$-subalgebra M of $\mathfrak{L}(H)$ containing identity is a von Neumann algebra if and only if $M = (M')'$.

Consider now a periodic graph (X, Γ). Let $\mathcal{F} \subset V(X)$ be a set of representatives for the action of Γ on $V(X)$, i.e., the vertices of the finite graph

$B = X/\Gamma$. Consider the Hilbert space $\ell^2(\Gamma)$. Let the $*$-algebra of bounded operators on $\ell^2(\Gamma)$ be denoted by $\mathcal{N}$. We define a unitary representation λ of Γ on $\ell^2(\Gamma)$ by $(\lambda(\gamma)f)(x) := f(\gamma^{-1}x)$, for $\gamma \in \Gamma$, $f \in \ell^2(V(X))$, $x \in \Gamma$. Let $\mathcal{N}(\Gamma)$ be the $*$-subalgebra of $\mathcal{N}$ consisting of operators which commute with the action of Γ, i.e.,

$$\mathcal{N}(\Gamma) = \{T \in \mathcal{N}\,;\, T\lambda(\gamma) = \lambda(\gamma)T \;\; \forall\, \gamma \in \Gamma\}.$$

If $T \in (\mathcal{N}(\Gamma)^{'})^{'}$, then $TA = AT \;\; \forall\, A \in \mathcal{N}^{'}$. Clearly, $\lambda(\gamma) \in \mathcal{N}^{'} \;\; \forall\, \gamma \in \Gamma$ and hence $T \in \mathcal{N}$. The inclusion $\mathcal{N} \subset (\mathcal{N}(\Gamma)^{'})^{'}$ is obvious. Hence $(\mathcal{N}(\Gamma)^{'})^{'} = \mathcal{N}(\Gamma)$ and $\mathcal{N}(\Gamma)$ is a von Neumann algebra.

Definition 1.3.4 (von Neumann Trace)**.** The *von Neumann trace* of an element $T \in \mathcal{N}(\Gamma)$ is defined by

$$\mathrm{Tr}_\Gamma(T) = \langle T(\delta_e),\, \delta_e\rangle$$

where $\delta_e \in \ell^2(\Gamma)$, the Kronecker delta function, is a function which takes the value 1 on the identity element e of Γ and is zero elsewhere.

Furthermore, if A is a bounded operator on $\bigoplus_{i\in I} \ell^2(\Gamma)$ which commutes with the action of Γ, then von Neumann trace of A, $\mathrm{Tr}_\Gamma(A)$, can be written as $\sum_{i\in I} Tr_\Gamma(A_i)$, where A_i is the restriction of A to the i-th component of the direct sum. Choosing lifts of the elements in $\mathcal{F}$ to X, we can identify $\ell^2(V(X)) = \bigoplus_{v\in\mathcal{F}} \ell^2(\Gamma)$. Therefore, for $A \in \ell^2(V(X))$, the trace $Tr_\Gamma(A)$ is given by $\sum_{v\in\mathcal{F}}\langle Av,\, v\rangle$.

1.4 Determinant Formula and Functional Equation

The determinant formula was first proved by Ihara [18] for finite regular graphs. It was proved in full generality (for finite graphs) through the works of Sunada [27], Hashimoto [14, 15] and Bass [1].

We now introduce some notation, to be able to state the determinant formula for finite graphs. For a finite graph X, let $A = [A(v, w)]$, where $v, w \in V(X)$, be the matrix defined by $A(v, w)$ to be the number of edges which have v, w as the end vertices ($A(v, w)$ is set to be 0 when there are no edges between v and w). The matrix A is called the *adjacency matrix* of the graph X. Clearly, A is a symmetric matrix. Let Q be the diagonal matrix given by $Q(v, v) = \deg(v) - 1$. The matrix $\Delta = (Q + I) - A$ is called the *Laplacian* of the graph. A deformation of the Laplacian is defined by setting $\Delta(u) = I - Au + Qu^2$, $u \in \mathbb{C}$. The *Euler characteristic* of the finite graph X, denoted by $\chi(X)$, is defined as $\chi(X) := |V(X)| - |E(X)|$ If $d := \max_{v \in V(X)} \deg(v)$, we have

Theorem 1.4.1 (Determinant Formula for Finite Graph)**.**

$$\frac{1}{Z_X(u)} = (1 - u^2)^{-\chi(X)} det(\Delta(u)), \quad |u| < \frac{1}{d - 1} \tag{1.3}$$

A consequence of the the above theorem is that the finite Ihara zeta function can be meromorphically extended to the whole complex plane and the completions satisfy a functional equation.

An analogue of the determinant formula was proved by Clair and Mokhtari-Sharghi ([5]) for periodic graphs. A more direct proof of the same was given by Guido, Isola and Lapidus, in [11] for simple periodic graphs and in [10] for periodic graphs which are not necessarily simple.

In the periodic graph case, the determinant formula states that the Ihara zeta function is the reciprocal of a holomorphic function which, up to a multiplicative factor, is the determinant of a deformed Laplacian on the graph. The determinant used in this is an analytic determinant on a subset of $\mathcal{N}(\Gamma)$ and was first introduced in [5]. The construction of this determinant follows the construction of a positive valued determinant for finite factors (i.e., von Neumann algebras with finite centers) defined by Fuglede and Kadison in [7].

Definition 1.4.2. Let $(\mathcal{A}, \tau)$ be a von Neumann algebra endowed with a finite trace τ. Let $\mathcal{A}_0 = \{A \in \mathcal{A} : 0 \notin \text{conv}\sigma(A)\}$, where $\sigma(A)$ denotes the spectrum of A. For any $A \in \mathcal{A}$, define

$$\det_\tau(A) = \exp \circ \tau \circ \left(\frac{1}{2\pi i} \int_\Upsilon \log \lambda(\lambda - A)^{-1}\, d\lambda \right),$$

where Υ is the boundary of a connected, simply connected region Ω containing $\text{conv}\sigma(A)$ and log is a branch of the logarithm whose domain contains Ω.

The following proposition ([11, p. 10]) summarizes some of the properties of this determinant.

Proposition 1.4.3. *Let $(\mathcal{A}, \tau)$ be a von Neumann algebra endowed with a finite trace and $A \in \mathcal{A}_0$. Then*

1. $det_\tau(zA) = z^{\tau(I)} det_\tau(A)$*, for any* $z \in \mathbb{C} \setminus \{0\}$*,*

2. *if A is normal, and $A = UH$ is its polar decomposition,*

$$det_\tau(A) = det_\tau(U) det_\tau(H).$$

Consider the periodic graph (X, Γ). Let A be the adjacency operator of the graph X, i.e., A is an operator from $\ell^2(V(X))$ to itself defined by $(Af)(v) = \sum\limits_{w \sim v} f(w)$. Let Q be the operator from $\ell^2(V(X))$ to itself defined by $(Qf)(v) = (\deg(v) - 1)f(v)$ and for $u \in \mathbb{C}$, let

$$\Delta(u) := I - uA + u^2 Q.$$

We call $\Delta(u)$ the *deformed Laplacian* of (X, Γ). It is easy to check that $\Delta(u) \in \mathcal{N}(\Gamma)$. Recall that $d = \sup\limits_{v \in V(X)} \deg(v)$ and $B = X/\Gamma$ is the quotient graph. Let $\alpha := \frac{d + \sqrt{d^2 + 4d}}{2}$. Then it can be checked that, for $|u| < \frac{1}{\alpha}$, $0 \notin \sigma(\Delta(u))$.

Theorem 1.4.4 (Determinant Formula, [11])**.**

$$\frac{1}{Z_{X,\Gamma}(u)} = (1 - u^2)^{-\chi^2(X)} det_\Gamma(\Delta(u)) \quad \textit{for} \quad |u| < \frac{1}{\alpha}. \tag{1.4}$$

Here, $\chi^2(X)$ is the L^2-Euler characteristic of (X, Γ), defined as

$$\chi^2(X) := \sum_{v \in \mathcal{F}_0} \frac{1}{|\Gamma_v|} - \sum_{v \in \mathcal{F}_1} \frac{1}{|\Gamma_e|}$$

where $\mathcal{F}_0$ is the set of representatives of equivalence classes in $V(X)/\Gamma$ and $\mathcal{F}_1$ is the set of representatives of equivalence classes in $E(X)/\Gamma$. Also, $\det_\Gamma(\Delta(u)) = exp \circ Tr_\Gamma(Log(\Delta(u)))$, where Tr_Γ is the von Neumann trace, as defined in 1.3.

As in the case of finite graphs [26], the determinant formula allows the Ihara zeta function of periodic graphs to be extended to a larger domain. Furthermore, in the case of regular periodic graphs, there are several ways of completing the zeta function and each satisfies a functional equation.

Lemma 1.4.5 (Lemma 5.1, [10]). *Let (X, Γ) be a periodic graph such that X is a $(q + 1)$-regular graph. Then*

1. $\chi^2(X) = \chi(B) = |V(B)|(1 - q)/2 \in \mathbb{Z}$,
2. $Z_{X,\Gamma}(u) = (1 - u^2)^{\chi(B)} det_\Gamma(\Delta(u))^{-1}$ *for* $|u| < \frac{1}{q}$,
3. *a completion of $Z_{X,\Gamma}$ can be defined, which can be extended to a holomorphic function at least on the open set Ω_q (see figure 1.1).*

The set Ω_q is an open subset of $\mathbb{C}$ defined as below:

$$\Omega_q := \mathbb{R}^2 \setminus \left(\left\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = \frac{1}{q} \right\} \cup \left\{ (x, 0) \in \mathbb{R}^2 : \frac{1}{q} \le |x| \le 1 \right\} \right). \tag{1.5}$$

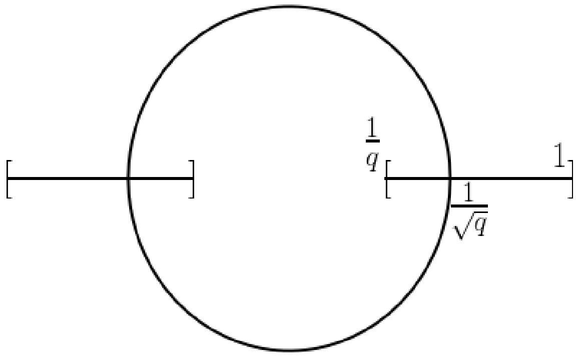

Figure 1.1: The set Ω_q

Theorem 1.4.6 (Functional equations; Theorem 5.2, [10])**.** *Let* (X, Γ) *be a periodic graph such that* X *is a* $(q+1)$*-regular graph. We can define the following completions of* $Z_{X,\Gamma}$ *which can be extended to a holomorphic function on the open set* Ω_q *(see figure 1.1):*

$$\xi_{X,\Gamma}(u) := (1 - u^2)^{-\chi(B)}(1 - u)^{|V(B)|}(1 - qu)^{|V(B)|}Z_{X,\Gamma}(u), \tag{1.6}$$

$$\Lambda_{X,\Gamma}(u) := (1 - u^2)^{-\chi(B)}(1 - u^2)^{|V(B)|/2}(1 - q^2u^2)^{|V(B)|/2}Z_{X,\Gamma}(u), \tag{1.7}$$

$$\Xi_{X,\Gamma}(u) := (1 - u^2)^{-\chi(B)}(1 + q\,u^2)^{|V(B)|}Z_{X,\Gamma}(u). \tag{1.8}$$

Furthermore, the above completions satisfy the following functional equations:

$$\xi_{X,\Gamma}(u) = \xi_{X,\Gamma}\left(\frac{1}{qu}\right), \tag{1.9}$$

$$\Lambda_{X,\Gamma}(u) = -\Lambda_{X,\Gamma}\left(\frac{1}{qu}\right), \tag{1.10}$$

$$\Xi_{X,\Gamma}(u) = \Xi_{X,\Gamma}\left(\frac{1}{qu}\right). \tag{1.11}$$

Chapter 2

Main Results on Spectra of Graphs

2.1 Length Spectrum Of Periodic Graphs

A notion of length spectrum has been defined and studied for compact Riemannian manifolds of negative curvature in [9]. Roughly speaking, the length spectrum of a manifold is a function which counts the number of closed geodesics of a given length. In this section, we define the analogous notions for finite graphs and for periodic graphs.

Definition 2.1.1 (Length Spectrum of a Finite Graph)**.** The *length spectrum* of the finite graph X is defined to be the function L_X on $\mathbb{N}$ given by

$$L_X(n) \; = \; \text{The number of } C \in \mathcal{R} \text{ such that } \ell(C) = n.$$

Definition 2.1.2 (Primitive Length Spectrum of a Finite Graph)**.** The *primitive length spectrum* of the finite graph X is defined to be the function PL_X

on $\mathbb{N}$ given by

$$PL_X(n) \;=\; \text{The number of } C \,\in\, \mathcal{P} \text{ such that } \ell(C) = n\ .$$

In the case of a periodic graph $(X,\,\Gamma)$, the above definitions must be modified by considering the Γ-equivalence classes of cycles.

Definition 2.1.3 (Length Spectrum of a Periodic Graph)**.** The *length spectrum* of the periodic graph $(X,\,\Gamma)$ is defined to be the function L_Γ on $\mathbb{N}$ given by

$$L_\Gamma(n) \;=\; \text{The number of } \xi \,\in\, [\mathcal{R}]_\Gamma \text{ such that } \ell(\xi) = n\ .$$

Definition 2.1.4 (Primitive Length Spectrum of a Periodic Graph)**.** The *primitive length spectrum* of the periodic graph $(X,\,\Gamma)$ is defined to be the function PL_Γ on $\mathbb{N}$ given by

$$PL_\Gamma(n) \;=\; \text{The number of } \xi \,\in\, [\mathcal{P}]_\Gamma \text{ such that } \ell(\xi) = n\ .$$

In the case of the length spectrum of compact hyperbolic spaces of even dimension, an analogue of the classical strong multiplicity was proved in [2]. We recall the classical strong multiplicity one theorem due to Atkin and Lehner: Let f and g are newforms for some Hecke congruence subgroup $\Gamma_0(N)$. Suppose that the eigenvalues of the Hecke operator at a prime p are equal for all but finitely many primes p. Then f and g are equal [21, p. 125].

In the next section we state and prove an analogue of this result for the length spectrum of regular periodic graphs.

2.2 Multiplicity One Property for the Primitive Length Spectrum of Regular Periodic Graphs

Theorem 2.2.1. *[Multiplicity One property for Primitive Length Spectrum of Regular Periodic Graphs] Let $(X,\, \Gamma_1)$ and $(X,\, \Gamma_2)$ be two simple, $q+1$ regular periodic graphs. Further, assume that Γ_1 and Γ_2 act on $V(X)$ without inversions and with bounded co-volume and such that the stabilizer of any cycle with respect to either subgroup is trivial. Suppose $PL_{\Gamma_1}(n) = PL_{\Gamma_2}(n)$ for all but finitely many $n \in \mathbb{N}$. Then $PL_{\Gamma_1}(n) = PL_{\Gamma_2}(n)$ for all $n \in \mathbb{N}$. Furthermore, we can conclude that $L_{\Gamma_1}(n) = L_{\Gamma_2}(n)$ for all $n \in \mathbb{N}$.*

Proof. Let (X, Γ_1) and (X, Γ_2) be as in section 2. Let Z_{Γ_1} and Z_{Γ_2} denote the Ihara zeta functions of $(X,\, \Gamma_1)$ and $(X,\, \Gamma_2)$, respectively. Let ξ_{Γ_1} and ξ_{Γ_2} be the extensions to Ω_q of Z_{Γ_1} and Z_{Γ_2} respectively, as defined above. Let B_i denote the finite graph X/Γ_i for $i = 1, 2$.

From the definition of the zeta function, we have in the region $|u| < \frac{1}{q}$,

$$\frac{\xi_{\Gamma_1}(u)}{\xi_{\Gamma_2}(u)} = \frac{(1 - u^2)^{-\chi(B_1)}\, [(1 - u)(1 - qu)]^{|V(B_1)|} \displaystyle\prod_{[C]_{\Gamma_1} \in [\mathcal{P}]_{\Gamma_1}} (1 - u^{\ell(C)})^{-1}}{(1 - u^2)^{-\chi(B_2)}\, [(1 - u)(1 - qu)]^{|V(B_2)|} \displaystyle\prod_{[C']_{\Gamma_2} \in [\mathcal{P}]_{\Gamma_2}} (1 - u^{\ell(C')})^{-1}}. \tag{2.1}$$

Under the hypothesis of Theorem 2.2.1, all but finitely many factors in the product terms of the numerator and the denominator of equation 2.1 cancel out. Therefore, there exist finite subsets S_1 and S_2 such that in $|u| < \frac{1}{q}$,

$$\frac{\xi_{\Gamma_1}(u)}{\xi_{\Gamma_2}(u)} = \frac{(1 - u^2)^{-\chi(B_1)}\, [(1 - u)(1 - qu)]^{|V(B_1)|} \displaystyle\prod_{i \in S_1} \left(1 - u^{\ell(C_i)}\right)^{-1}}{(1 - u^2)^{-\chi(B_2)}\, [(1 - u)(1 - qu)]^{|V(B_2)|} \displaystyle\prod_{j \in S_2} \left(1 - u^{\ell(C_j)}\right)^{-1}}.$$

Since the product terms in the above equation are over finite indexing sets and the other terms are all polynomials, the above expression defines a meromorphic function on the entire complex plane. Here we have crucially used the assumption that Γ_C is trivial for all cycles. On the other hand, $\dfrac{\xi_{\Gamma_1}}{\xi_{\Gamma_2}}$ is meromorphic on Ω_q and hence the two expressions must agree for all $u \in \Omega_q$. In particular,

$$\frac{\xi_{\Gamma_1}\left(\frac{1}{qu}\right)}{\xi_{\Gamma_2}\left(\frac{1}{qu}\right)} = \frac{\left(1 - \left(\dfrac{1}{qu}\right)^2\right)^{-\chi(B_1)} \left[\left(1 - \dfrac{1}{qu}\right)\left(1 - \dfrac{1}{u}\right)\right]^{|V(B_1)|}}{\left(1 - \left(\dfrac{1}{qu}\right)^2\right)^{-\chi(B_2)} \left[\left(1 - \dfrac{1}{qu}\right)\left(1 - \dfrac{1}{u}\right)\right]^{|V(B_2)|}} \times \frac{\prod_{i \in S_1}\left(1 - \left(\dfrac{1}{qu}\right)^{\ell(C_i)}\right)^{-1}}{\prod_{j \in S_2}\left(1 - \left(\dfrac{1}{qu}\right)^{\ell(C_j)}\right)^{-1}}$$

From the functional equation applied to the two zeta functions, we have for $u \in \Omega_q$,

$$\frac{\xi_{\Gamma_1}(u)}{\xi_{\Gamma_2}(u)} = \frac{\xi_{\Gamma_1}\left(\frac{1}{qu}\right)}{\xi_{\Gamma_2}\left(\frac{1}{qu}\right)}$$

Since $\chi(B_i) = |V(B_i)|(1 - q)/2$ for $i = 1, 2$, we can rearrange the terms to get

$$\begin{aligned} &N_1(u) \times \prod_{j \in S_2}\left(1 - u^{\ell(C_j)}\right) \prod_{i \in S_1}\left(1 - \left(\frac{1}{qu}\right)^{\ell(C_i)}\right) \\ &= N_2(u) \times \prod_{i \in S_1}\left(1 - u^{\ell(C_i)}\right) \prod_{j \in S_2}\left(1 - \left(\frac{1}{qu}\right)^{\ell(C_j)}\right), \end{aligned}$$

where N_1 and N_2 are as follows:

$$N_1(u) = \left(1 - u^2\right)^{(|V(B_1)| - |V(B_2)|) \times \frac{(q-1)}{2}} \left[(1 - u)(1 - qu)\right]^{|V(B_1)| - |V(B_2)|}$$

$$N_2(u) = \left(1 - \frac{1}{q^2u^2}\right)^{(|V(B_1)| - |V(B_2)|) \times \frac{(q-1)}{2}} \left[\left(1 - \frac{1}{u}\right)\left(1 - \frac{1}{qu}\right)\right]^{|V(B_1)| - |V(B_2)|}$$

The expressions $N_1(u)$ and $N_2(u)$ have no zeros in Ω_q. The zero of the first product term in the expression on the left-hand side lie on a circle of radius 1 centered at origin, while the zero of the second product lie on the circle of radius $\frac{1}{q}$ centered at origin. Similarly, the zero of the first product term in the expression on the right hand side lie on a circle of radius 1 centered at origin, while the zero of the second product lie on the circle of radius $\frac{1}{q}$ centered at origin. From the equality of the expressions, we conclude that the zeros of the expression on the left-hand side which lie on the unit circle coincide with the zeros of the expression on the right-hand side which lie on the unit circle. Hence we get equality of sets with multiplicity:

$$\left\{\frac{2\pi k}{\ell(C_i)} : k \in \mathbb{Z}; i \in S_1\right\} = \left\{\frac{2\pi k}{\ell(C_j)} : k \in \mathbb{Z}; j \in S_2\right\}$$

Thus we conclude that $PL_{\Gamma_1}(n) = PL_{\Gamma_2}(n)$ for all $n \in \mathbb{N}$.

Furthermore, we can check that, for $i = 1, 2$,

$$L_{\Gamma_i}(n) = \sum_{d|n} PL_{\Gamma_i}\left(\frac{n}{d}\right) \ \forall n \in \mathbb{N}.$$

Using this, we know that if $PL_{\Gamma_1}(n) = PL_{\Gamma_2}(n)$ for all $n \in \mathbb{N}$, then $L_{\Gamma_1}(n) = L_{\Gamma_2}(n)$ for all $n \in \mathbb{N}$. Hence $L_{\Gamma_1}(n) = L_{\Gamma_2}(n)$ for all $n \in \mathbb{N}$. $\square$

It follows from the above discussion that $N_1(u) = N_2(u)$ for all $u \in \Omega_q$. Without loss of generality, we can assume that $|V(B_1)| \geq |V(B_2)|$, which makes N_1 and N_2 polynomial expressions. Therefore, we can say that $N_1(u) = N_2(u)$ for all $u \in \mathbb{C}$. The zeros of $N_1(u)$ are $\{\pm 1, \frac{1}{q}\}$ while the zeros of $N_2(u)$ are $\{\pm\frac{1}{q}, 1\}$. Hence these two expressions have to be identically zero which can happen only when $|V(B_1)| = |V(B_2)|$. This proves the following corollary:

Corollary 2.2.2. *Suppose X, Γ_1, Γ_2 be as in Theorem 2.2.1. If $PL_{\Gamma_1}(n) = PL_{\Gamma_2}(n)$ for all but finitely many $n \in \mathbb{N}$ then $|V(X/\Gamma_1)| = |V(X/\Gamma_2)|$.*

2.3 Graphs With Action of Finitely Generated Abelian Groups

The relation between the length spectrum and the Laplace spectrum of some compact hyperbolic manifolds of negative curvature has been studied in [9]. For the manifolds considered, it is shown that the Laplace spectrum determines the length spectrum, a result we discuss in the next chapter. In the case of regular finite graphs, the adjacency operator is self-adjoint and hence we can consider the spectrum with respect to it. Two finite regular graphs X_1 and X_2 are said to be *isospectral* if the set of eigenvalues (with multiplicities) of the adjacency operators A_1 and A_2 of X_1 and X_2 are equal to each other.

We can use the determinant formula (1.3) to get the following lemma which shows that the length spectrum of two isospectral graphs is equal.

Lemma 2.3.1. *Suppose X_1 and X_2 are two finite $(q+1)$-regular graphs which are isospectral. Then $L_{X_1}(m) = L_{X_2}(m)$ for all $m \in \mathbb{N}$.*

Proof. Isospectrality of X_1 and X_2 implies that the corresponding adjacency matrices, A_1 and A_2, are similar. Let $\Delta_i(u) = I - A_i u + qu^2$ be the deformed Laplacian corresponding to X_i, for $i = 1, 2$. Then for any u in the disc $|u| < \frac{1}{q}$, $\Delta_1(u)$ is similar to $\Delta_2(u)$ and hence $\det(\Delta_1(u)) = \det(\Delta_2(u))$. If $Z_1(u)$ and $Z_2(u)$ are the Ihara zeta functions of X_1 and X_2 respectively, then using eq. 1.3 we get $Z_1(u) = Z_2(u)$ for all $|u| < \frac{1}{q}$.

Let $S_i = \{l \in \mathbb{N} \mid \ell(C) = l, C \in \mathcal{P}_i\}$, where $\mathcal{P}_i$ is the set of primitive reduced cycles in X_i for $i = 1, 2$. In other words, S_i is the multiset of the lengths that occur in the primitive length spectrum of X_i. Using this notation, the zeta function for the graph X_i, for $i = 1, 2$ can be written as

$$Z_i(u) = \prod_{l \in S_i} (1 - u^l)^{-1}.$$

Therefore, for $|u| < \frac{1}{q}$,

$$\prod_{l \in S_1} (1 - u^l)^{-1} = \prod_{m \in S_2} (1 - u^m)^{-1}$$
$$\prod_{l \in S_1} (1 + u^l + u^{2l} + \ldots) = \prod_{m \in S_2} (1 + u^m + u^{2m} + \ldots)$$

Let l_0 and m_0 be the smallest elements of S_1 and S_2 respectively. Suppose l_0 occurs with multiplicty k_1 and m_0 occurs with multiplicity k_2. Upon expansion, the expression of $Z_1(u)$ is of the form $Z_1(u) = (1 + k_1 u^{l_0} +$ higher powers of u) where as the expression of $Z_2(u)$ is of the form $Z_2(u) = (1 + k_2 u^{m_0} +$ higher powers of u). This implies that $l_0 = m_0$ and $k_1 = k_2$. Using the fact that the function $(1 - u^{l_0})$ has no poles in the disc $|u| < \frac{1}{q}$, we can inductively conclude that $S_1 = S_2$. Therefore $PL_{X_1}(m) = PL_{X_2}(m)$ and hence $L_{X_1}(m) = L_{X_2}(m)$ for all $m \in \mathbb{N}$. $\square$

In extending the above result to the case of periodic graphs we are impeded by the fact that the deformed Laplacian $\Delta(u)$ may not be self-adjoint. Furthermore, the determinant formula 1.4 is given in terms of the analytic determinant $\det_\Gamma$. In the rest of the section we state and proof an analogue of the above result, with a modified hypothesis, in the case of periodic graphs with actions of finitely generated abelian groups. We crucially use a generalization of a construction given in [4], which expresses the adjacency operator of periodic graphs with $\mathbb{Z}$-action in terms of a finite matrix.

Let (X, Γ) be a simple, $(q + 1)$-regular periodic graph such that Γ is a finitely generated abelian group. Therefore Γ can be written as $\mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_k}$, for some integers $r, n_1, n_2, \ldots, n_k$ such that $r \geq 0$, $n_i \geq 2$

for $i = 1, \ldots, k$, and $n_1 / n_2 / \ldots / n_k$. We also assume, as before, that the periodic graphs (X, Γ) are such that Γ acts on $V(X)$ without inversions and with bounded co-volume, and that Γ_C and Γ_v are trivial for every vertex v and every cycle C. The regularity assumption will give $\Delta(u) = I - uA + u^2 q$.

Let $\mathbb{Z}_{n_i} = \langle s_i \mid s_i^{n_i} = 1 \rangle$, for $i = 1, \ldots, k$, and let $\mathbb{Z}^r = \langle t_1 \rangle \times \langle t_2 \rangle \times \ldots \times \langle t_r \rangle$. Suppose that the number of orbits of action of Γ on X is n. Choosing the representatives of these orbits, we can identify

$$\ell^2(V(X)) = \bigoplus_n \ell^2(\Gamma).$$

Following [4], we describe below how the adjacency operator A can be written as a $n \times n$ matrix with entries in the ring

$$R = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}, s_1, s_2, \ldots, s_k] / \langle s_i^{n_i} - 1 \mid i = 1, \ldots, k \rangle.$$

Let $[v_1], [v_2], \ldots, [v_n]$ be the Γ-equivalence classes of $V(X)$. We look for the elements in the orbit $[v_j]$ which are adjacent to the representative vertex v_i of the orbit $[v_i]$. Any element in the orbit $[v_j]$ is of the form $\gamma \cdot v_j$ where $\gamma = (t_1^{u_1} t_2^{u_2} \ldots t_r^{u_r} s_1^{w_1} s_2^{w_2} \ldots s_k^{w_k}) \in \Gamma$. We write $A_{ij} = \sum \gamma$ where the sum is taken over all γ such that $v_i \sim \gamma \cdot v_j$.

Example 2.3.2. Consider the 4-regular graph $(Y, \mathbb{Z})$ with $V(Y) = \mathbb{Z} \cup \mathbb{Z}$ as shown in figure 2.1. Let $V(Y) = \{\ldots, v_{-1}, v_0, v_1, \ldots\} \cup \{\ldots, w_{-1}, w_0, w_1, \ldots\}$. The action of $\mathbb{Z} = \langle t \rangle$ we consider is given by $t \cdot v_i = v_{i+1}$ and $t \cdot w_i = w_{i+1}$. This action has two orbits, namely $[v_0]$ and $[w_0]$. Let A be the 2×2 matrix representing the adjacency operator of $(Y, \mathbb{Z})$. Then

$$A = \begin{pmatrix} t + t^{-1} & 1 + t \\ 1 + t^{-1} & t + t^{-1} \end{pmatrix}$$

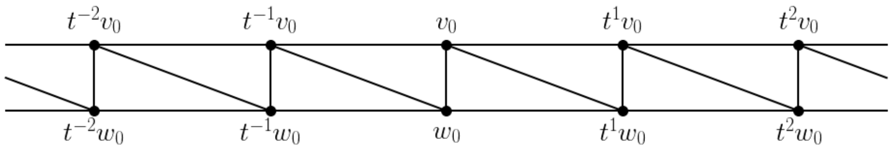

Figure 2.1: A 4-regular periodic graph

Under Fourier transform, $\ell^2(\mathbb{Z}) = L^2(S^1)$ where $S^1 = \{e^{i\theta} \mid \theta \in (-\pi, \pi]\}$ with normalized measure. Therefore we have

$$\ell^2(\Gamma) = \bigoplus_r L^2(S^1) \bigoplus \ell^2(\mathbb{Z}_{n_1}) \bigoplus \ell^2(\mathbb{Z}_{n_2}) \ldots \bigoplus \ell^2(\mathbb{Z}_{n_k}).$$

This shows that

$$\ell^2(V(X)) = \bigoplus_{nr} L^2(S^1) \bigoplus_n \ell^2(\mathbb{Z}_{n_1}) \bigoplus_n \ell^2(\mathbb{Z}_{n_2}) \ldots \bigoplus_n \ell^2(\mathbb{Z}_{n_k}).$$

Under the Fourier transform, multiplication by t_i becomes multiplication by the function $e^{i\theta_i}$ for $i = 1, 2, \ldots, r$. Hence $\Delta(u)$ is represented by a $n \times n$ matrix, whose entries are in terms of the variables $\theta_1, \theta_2, \ldots \theta_r, s_1, s_2, \ldots, s_k$. We denote this matrix by $M_{u,(X,\Gamma)}$. (The dependence on the above variables is suppressed for ease of notation.) To further simplify the notation, we denote $M_{u,(X,\Gamma)}$ with M_u when it is clear which periodic graph (X, Γ) is being referred. Therefore we have

$$
\begin{aligned}
\det_\Gamma(\Delta(u)) &= \exp \circ \mathrm{Tr}_\Gamma \circ \mathrm{Log}(\Delta(u)) \\
&= \exp \sum_{i=1}^{n} \mathrm{Tr}_\Gamma(\mathrm{Log}(\Delta(u)))_{ii} \\
&= \exp \sum_{i=1}^{n} \left(\sum_{\mathbb{Z}_{n_1}} \sum_{\mathbb{Z}_{n_2}} \dots \sum_{\mathbb{Z}_{n_k}} \left(\int_{S_1} \int_{S_1} \cdots \int_{S_1} (\mathrm{Log}(M_u))_{ii} \, d\theta_1 \dots d\theta_r \right) \right) \\
&= \exp \left(\sum_{\mathbb{Z}_{n_1}} \sum_{\mathbb{Z}_{n_2}} \dots \sum_{\mathbb{Z}_{n_k}} \left(\int_{S_1} \int_{S_1} \cdots \int_{S_1} \sum_{i=1}^{n} (\mathrm{Log}(M_u))_{ii} \, d\theta_1 \dots d\theta_r \right) \right) \\
&= \exp \left(\sum_{\mathbb{Z}_{n_1}} \sum_{\mathbb{Z}_{n_2}} \dots \sum_{\mathbb{Z}_{n_k}} \left(\int_{S_1} \int_{S_1} \cdots \int_{S_1} \mathrm{Tr}(\mathrm{Log}(M_u)) \, d\theta_1 \dots d\theta_r \right) \right) \\
&= \exp \left(\sum_{\mathbb{Z}_{n_1}} \sum_{\mathbb{Z}_{n_2}} \dots \sum_{\mathbb{Z}_{n_k}} \left(\int_{S_1} \int_{S_1} \cdots \int_{S_1} \log(\det(M_u)) \, d\theta_1 \dots d\theta_r \right) \right)
\end{aligned}
$$

Note that the matrix M_u in fact has entries which are polynomials in the variables $e^{\pm i\theta_1}, \dots, e^{\pm i\theta_r}, s_1, \dots, s_k$. Therefore we can say $M_u \in GL_n(R)$, after the change of variables $e^{i\theta_j} \longrightarrow t_j$, for $j = 1, \dots, r$.

Let $(X, \Gamma)_1$ and $(X, \Gamma)_2$ be two simple, regular periodic graphs. (The underlying infinite graph X of the two periodic graphs is same but has different Γ-actions.) Further, assume that both the actions of Γ on $V(X)$ are without inversions and with bounded co-volume. Let $M_{1,u}$ and $M_{2,u}$ denote $M_{u,(X,\Gamma)_1}$ and $M_{u,(X,\Gamma)_2}$ respectively. Let $\Delta_1(u)$ and $\Delta_2(u)$ be the deformed Laplacians, $L_{\Gamma,1}$ and $L_{\Gamma,2}$ be the length spectra, $PL_{\Gamma,1}$ and $PL_{\Gamma,2}$ be the primitive length spectra and Z_1 and Z_2 be the zeta functions of $(X,\Gamma)_1$ and $(X, \Gamma)_2$ respectively. We can conclude the following lemma by using the expression of $\det_{\Gamma_i}(\Delta_i(u))$ in terms of $M_{u,(X,\Gamma)_i}$.

Lemma 2.3.3. *Suppose for a fixed u, the matrices $M_{1,u}$ and $M_{2,u}$ are conjugate in $GL_n(R)$. Then $det_{\Gamma_1}(\Delta_1(u)) = det_{\Gamma_2}(\Delta_2(u))$.*

Theorem 2.3.4. *Let A_1 and A_2 be the adjacency operators of the periodic graphs $(X,\Gamma)_1$ and $(X,\,\Gamma)_2$ respectively. Suppose A_1 and A_2, as $n \times n$ matrices, are conjugate. Then $L_{\Gamma,1}(m) = L_{\Gamma,2}(m)$ for all $m \in \mathbb{N}$.*

Proof. If A_1 and A_2 are similar, then for any complex number u in the disc $|u| < \frac{1}{q}$, $\Delta_1(u)$ is similar to $\Delta_2(u)$. From the lemma 2.3.3, we have $\det_{\Gamma_1}(\Delta_1(u)) = \det_{\Gamma_2}(\Delta_2(u))$ for all u such that $|u| < \frac{1}{q}$. From the determinant formula of the zeta function, we get $Z_1(u) = Z_2(u)$ for all $|u| < \frac{1}{q}$.

Let $S_i = \{l \in \mathbb{N} \mid \ell(C) = l, C \in [\mathcal{P}]_{\Gamma_i}\}$, for $i = 1, 2$. In other words, S_i is the multiset of the lengths that occur in the primitive length spectrum of $(X,\Gamma)_i$. Using this notation, the zeta function for the periodic graph $(X,\Gamma)_i$, for $i = 1, 2$ can be written as

$$Z_i(u) = \prod_{l \in S_i} (1 - u^l)^{-1}.$$

Therefore, for $|u| < \frac{1}{q}$,

$$\prod_{l \in S_1} (1 - u^l)^{-1} = \prod_{m \in S_2} (1 - u^m)^{-1}$$

$$\prod_{l \in S_1} (1 + u^l + u^{2l} + \ldots) = \prod_{m \in S_2} (1 + u^m + u^{2m} + \ldots)$$

Let l_0 and m_0 be the smallest elements of S_1 and S_2 respectively. Suppose l_0 occurs with multiplicity k_1 and m_0 occurs with multiplicity k_2. Upon expansion, the expression of $Z_1(u)$ is of the form $Z_1(u) = (1 + k_1 u^{l_0} +$ higher powers of $u)$ where as the expression of $Z_2(u)$ is of the form $Z_2(u) = (1 + k_2 u^{m_0} +$ higher powers of $u)$. This implies that $l_0 = m_0$ and $k_1 = k_2$. Using the fact that the function $(1 - u^{l_0})$ has no poles in the disc $|u| < \frac{1}{q}$, we can inductively conclude that $S_1 = S_2$. Therefore $PL_{\Gamma,1}(m) = PL_{\Gamma,2}(m)$ and hence $L_{\Gamma,1}(m) = L_{\Gamma,2}(m)$ for all $m \in \mathbb{N}$. $\square$

Chapter 3

Spectra of some Compact Locally Symmetric Riemannian Manifolds

3.1 Preliminaries

Selberg first constructed a generalization of the Riemannian zeta function, called the Selberg zeta function, for compact Riemann surface of genus bigger than 2 in [24]. Such a space is of the form $\Gamma\backslash\mathscr{H}$, where $\mathscr{H} = SL(2,\mathbb{R})/SO(2)$ is the upper half plane and Γ is a discrete subgroup of $SL(2,\mathbb{R})$. The Selberg zeta function is a complex valued function associated to the data (Γ, χ), where χ is the character of a finite dimensional unitary representation T of Γ. It uses the lengths of the primitive closed geodesics of the surface in place of primes.

Further generalizations of the Selberg zeta function were given by Wakayama [28] and Gangolli [9]. Gangolli constructed the zeta function $Z_\Gamma(s,\chi)$ for general compact locally symmetric space of negative curvature X_Γ, and showed how the location and the order of the zeros of $Z_\Gamma(s,\chi)$ gives information about the spectrum of the Laplace-Beltrami operator of X_Γ and about the topology of X_Γ. In this section, we give the relevant definitions needed to describe spaces of the type X_Γ. We refer to the Helgason's [16, *'Differential geometry and symmetric spaces'*] for a detailed discussion.

Let M be Riemannian manifold. Let p be a point in M and N_0 be the neighbourhood of 0 in T_pM (the tangent space of p) which is symmetric with respect to 0. Let N_p be $Exp_p(N_0)$, where Exp_p is the exponential map from T_pM to M. For any $q \in N_p$, we consider the geodesic $t \longrightarrow \gamma(t)$ within N_p such that $\gamma(0) = p$ and $\gamma(1) = q$. The mapping $q \longmapsto \gamma(-1)$ of N_p onto itself is called geodesic symmetry with respect to the point p.

Definition 3.1.1 (Riemannian Locally Symmetric Space)**.** A Riemannian manifold M is called a Riemannian locally symmetric space if for each $p \in M$ there exists a normal neighborhood of p on which the geodesic symmetry with respect to p is an isometry.

A map is said to be *involutive* if its square, but not itself, is equal to the identity map.

Definition 3.1.2 (Riemannian Globally Symmetric Space)**.** An analytic Riemannian manifold is called globally symmetric if each $p\, in\, M$ is the fixed point of a involutive isometry s_p on M.

It is know that ([16, Ch. IV, Lemma 3.1]) for each $p \in M$ there exists a normal neighborhood N_p of $p \in M$ such that s_p is the geodesic symmetry on N_p. It is also known that locally symmetric Riemannian manifolds, which are not globally symmetric, can be constructed as quotients of Riemannian globally symmetric spaces by discrete groups of isometries with no fixed points

For any Riemannian manifold M, let $I(M)$ be the group of isometries of M, endowed with the compact open topology. The identity component of $I(M)$ is denoted by $I_0(M)$. It is known that, when M is a globally symmetric space, $I(M)$ has a Lie group structure. Then the following theorem ([16, Ch. IV, Theorem 3.3]) gives the group theoretic description of locally and globally symmetric Riemannian spaces.

Theorem 3.1.3. *Let M be a Riemannian globally symmetric space and p_0 any point in M. If $G = I_0(M)$, and K is the subgroup of G which leaves p_0 fixed, then K is a compact subgroup of the connected group G and G/K is analytically diffeomorphic to M.*

Using the above theorem, we can represent any compact locally symmetric space of negative curvature X_Γ as $\Gamma\backslash G/K$, where G is a connected semisimple Lie group with finite centre, K is a maximal compact subgroup and Γ is a uniform lattice in G, i.e., a co-compact torsion free discrete subgroup of G.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to the involution θ determined by $\mathfrak{k}$. More specifically, $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigen-spaces of the involution map θ. Let $\mathfrak{a}_\mathfrak{p}$ be the maximal abelian subalgebra of $\mathfrak{p}$. The subalgebra $\mathfrak{a}_\mathfrak{p}$ can be extended to an algebra $\mathfrak{a}$ such that $\mathfrak{a} = \mathfrak{a}_\mathfrak{p} + \mathfrak{a}_\mathfrak{k}$, where $\mathfrak{a}_\mathfrak{k} = \mathfrak{a} \cap \mathfrak{k}$ and $\mathfrak{a}_\mathfrak{p} = \mathfrak{a} \cap \mathfrak{p}$. The *real rank* of G is defined to be the dimension of $\mathfrak{a}_\mathfrak{p}$, $\dim(\mathfrak{a}_\mathfrak{p})$. The rank of a locally symmetric Riemannian manifold is defined to be the rank of G. It is assumed all the locally symmetric Riemannian manifolds under consideration are of real rank 1, i.e., $\dim(\mathfrak{a}_\mathfrak{p}) = 1$.

Let $\mathfrak{g}^\mathbb{C}$ and $\mathfrak{a}^\mathbb{C}$ denote the complexifications of $\mathfrak{g}$ and $\mathfrak{a}$ and let $\Phi(\mathfrak{g}^\mathbb{C}, \mathfrak{a}^\mathbb{C})$ denote the set of roots of $(\mathfrak{g}^\mathbb{C}, \mathfrak{a}^\mathbb{C})$. An element of α in the dual space of $\mathfrak{a}^\mathbb{C}$ is called a *root* if

$$L_\alpha = \{g \in \mathfrak{g}^\mathbb{C} \,|hg - gh = \alpha(h)g \quad \forall\, h \in \mathfrak{a}^\mathbb{C}\}$$

is a non-zero subspace of $\mathfrak{g}^\mathbb{C}$.

Definition 3.1.4 (Ordered Vector Space)**.** Let V be a finite-dimensional vector space over R. V is said to be an *ordered vector space* if it is an ordered set and the ordering relation $>$ satisfies the conditions:

- $X > 0, Y > 0$ implies that $X + Y > 0$.
- If $X > 0$ and a is a positive real number, then $aX > 0$.

If $\{X_1, \ldots, X_n\}$ is a basis of V, then V can be turned into an ordered vector space by saying $X > 0$ if $X = \sum_{i=1}^{n} a_i X_i$ and the first non-zero number in the sequence $a_1, \ldots, a_n$ is > 0.

Suppose W a subspace of V. Let V^* and W^* denote their duals. Considered them turned into ordered vector spaces. The orderings are said to be *compatible*, if $\lambda \in V^*$ is positive whenever its restriction $\bar{\lambda}$ to W^* is positive.

We choose compatible ordering in the dual spaces of $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{a}_\mathfrak{p} + i\mathfrak{a}_\mathfrak{k}$, respectively. Since each root $\alpha \in \Phi$ is real valued on $\mathfrak{a}_\mathfrak{p} + i\mathfrak{a}_\mathfrak{k}$, this gives an ordering of Φ. Let Φ^+ be the set of positive roots under this order. Let

$$P_+ = \{\alpha \in \Phi^+ \mid \alpha \not\equiv 0 \,\text{on}\, \mathfrak{a}_\mathfrak{p}\},$$

$$P_- = \{\alpha \in \Phi^+ \mid \alpha \equiv 0 \,\text{on}\, \mathfrak{a}_\mathfrak{p}\}.$$

Let Σ be the set of restrictions of elements of P_+ to $\mathfrak{a}_\mathfrak{p}$. Since $\dim(\mathfrak{a}_\mathfrak{p}) = 1$, it is known that we can choose $\beta \in \Sigma$ such that 2β is the only other possible element in Σ. Let p be the number of roots in P_+ whose restriction to $\mathfrak{a}_\mathfrak{p}$ is β and let q be the number of roots in P_+ whose restriction is 2β. We choose $H_0 \in \mathfrak{a}_\mathfrak{p}$ such that $\beta(H_0) = 1$. Let $\rho = \frac{1}{2}\sum_{\alpha \in P_+} \alpha$. Also, we denote the number $\rho(H_0)$ by ρ_0.

If X_α is the root vector belonging to α, for any $\alpha \in \Phi^+$, then we put $\mathfrak{n}^\mathbb{C} = \sum_{\alpha \in P^+} \mathbb{C}X_\alpha$. If $\mathfrak{n} = \mathfrak{n}^\mathbb{C} \cap \mathfrak{g}$, then $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_\mathfrak{p} + \mathfrak{n}$ is the *Iwasawa decomposition* of $\mathfrak{g}$. Furthermore, if we let $A_\mathfrak{p} = \exp(\mathfrak{a}_\mathfrak{p})$ and $N = \exp(\mathfrak{n})$, then we get $G = KA_\mathfrak{p}N$, which is the *Iwasawa decomposition of* G. Also, we can write $G = KA_{\mathfrak{p}}K$, which is called the *Cartan decomposition* of G. The Weyl group of $(G, A_{\mathfrak{p}})$ is denoted by $\mathcal{W}$. The real dual of $\mathfrak{a}_{\mathfrak{p}}$ is denoted by Λ, while the complexification of Λ, $\Lambda + i\Lambda$, is denoted by $\Lambda^{\mathbb{C}}$.

3.2 Selberg-Gangolli Zeta Function

Let G be a connected semisimple Lie group with finite centre, K be a maximal compact subgroup and Γ be a uniform lattice in G, i.e., a co-compact torsion free discrete subgroup of G. These lattices need not be arthimetic. We fix a finite dimensional unitary representation T of Γ and let χ be its character. In [8], Gangolli defined a zeta function $Z_\Gamma(s, \chi)$ associated to the data (Γ, χ). In this section, we describe this zeta function, called the Selberg-Gangolli zeta function, and its properties.

Since $\Gamma\backslash G$ is compact, Γ contains no parabolic elements. Hence, any element $\gamma \in \Gamma \setminus \{e\}$ is semisimple and therefore lies in a Cartan subgroup of G. Upto conjugacy, there are only two Cartan subgroups of G, K and $MA_{\mathfrak{p}}$, where M is the centralizer of $A_{\mathfrak{p}}$ in K. Here K is compact and MA_p is non-compact. It is also assumed that the uniform lattice Γ is such that it has no elements of finite order. Since an element is elliptic if and only if it is of finite order, any $\gamma \in \Gamma$ is hyperbolic and is therefore conjugate to an element of $MA_{\mathfrak{p}}$. Furthermore, it can be shown that it can be chosen to be conjugate to an element of $MA_{\mathfrak{p}}^+$, where $A_{\mathfrak{p}}^+$ is the set $\{\exp(tH_0)\,;\, t \geq 0\}$. Let $h(\gamma) = m_\gamma(\gamma)h_{\mathfrak{p}}(\gamma)$ be an element of $MA_{\mathfrak{p}}^+$ conjugate to γ.

For any $h \in A_{\mathfrak{p}}$, let $u(h) = \beta(\log h)$. Then $u = u(h)$ can be considered as a parametrization on $A_{\mathfrak{p}}$ via which $A_{\mathfrak{p}}$ can be identified with $\mathbb{R}$. Under this parametrization, $A_{\mathfrak{p}}^+$ corresponds to the positive real axis. Define $u_\gamma := \beta(log\, h_{\mathfrak{p}}(\gamma))$. It is known that the value u_γ is independent of the choice of $h(\gamma)$.

Lemma 3.2.1. *For any negatively curved Riemannian manifold, and hence* *for X_Γ, there is a bijective correspondence between the set of closed geodesic classes in X_Γ and the set of conjugacy classes of* Γ.

Let C_Γ be a set of representatives of the Γ-conjugacy classes.

Definition 3.2.2. For $\gamma \in \Gamma$, the length $\ell(\gamma)$ of the conjugacy class $[\gamma]$ is defined to be the length of the unique closed geodesic in the corresponding free homotopy class in X_Γ.

Lemma 3.2.3 ([9])**.** *For any $\gamma \neq 1 \in \Gamma$, we have $\ell(\gamma) = u_\gamma$.*

The length spectrum of X_Γ is the function

$$L_\Gamma : \mathbb{R} \to \mathbb{N} \cup \{0\}$$

which to a real number l, assigns the number of Γ-conjugacy classes $[\gamma]$ such that $\ell(\gamma) = l$.

An conjugacy class $[\gamma] \neq 1$ of Γ is called *primitive* if $[\gamma] \neq [\delta^n]$ for any integer $n > 1$ and $\delta \in \Gamma$. Furthermore, it has been shown in [9] that any conjugacy class $[\gamma]$ can be written as $[\delta^n]$, for some primitive conjugacy class $[\delta]$ and integer $n \geq 1$. If $[\gamma] = [\delta^n]$, for some primitive $[\delta]$, then the number n is denoted by $j(\gamma)$. It is known that the primitive conjugacy classes correspond to primitive closed geodesics, which are closed geodesics which cannot be obtained by going around any other closed geodesic $n > 1$ times. Let P_Γ be the set of representatives of the primitive Γ-conjugacy classes.

The elements of P_+ are enumerated as $\alpha_1, \alpha_2, \ldots, \alpha_t$. Then, we define L as the semi lattice in $\mathfrak{a}_\mathfrak{p}^\mathbb{C}$ given by $\{\sum_{i=1}^t m_i\alpha_i\,;\, m_i \geq 0,\, m_i \in \mathbb{Z}\}$. For any $\lambda \in L$, the number of distinct ordered t-tuples $(m_1, m_2, \ldots, m_t)$ such that $\lambda = \sum_{i=1}^t m_i\alpha_i$. The character of the Cartan subgroup A which corresponds to λ is denoted by ξ_λ. Therefore, $\xi_\lambda(h) = \exp\lambda(\log(h))$.

For $s \in \mathbb{C}$, the zeta function $Z_\Gamma(s,\chi)(s)$ is defined by first defining its logarithmic derivative with respect to s, denoted by $\Psi_\Gamma(s, \chi)$, which is given as the following series convergent on $\mathrm{Re}\, s > 2\rho_0$ ([8]):

$$\Psi_\Gamma(s,\,\chi) \;=\; \kappa \sum_{\gamma\,\in\, C_\gamma\,\setminus\,\{1\}} \chi(\gamma)\, u_\gamma\, j(\gamma)^{-1}\, C(h(\gamma))\, \exp(\rho_0 \;-\; s)\, u_\gamma \tag{3.1}$$

Here $C(h(\gamma))$ is a positive function and κ is a positive integer, both of which depends only on the structure of G If $Z_\Gamma(s,\chi)$ is the Selberg-Gangolli zeta function, then

$$\frac{d}{ds}\,\log Z_\Gamma(s,\chi) \;=\; \Psi_\Gamma(s,\,\chi). \tag{3.2}$$

Furthermore, it has been shown in [9] that $Z_\Gamma(s,\chi)$ can be simplified to the following Euler product:

Definition 3.2.4 (Selberg-Gangolli Zeta Function)**.**

$$Z_\Gamma(s,\chi) \;=\; C \prod_{\delta\,\in\, PC_\Gamma}\;\prod_{\lambda\,\in\, L} (\det\,(I \;-\; T(\delta)\,\xi_\lambda(h(\delta))^{-1}\exp(-su_\delta)))^{m_{\lambda\kappa}}, \tag{3.3}$$

Here, $C \;\neq\; 0$ is a constant which depends only on G.

We state below some properties of this zeta function which have been proved in [9]:

Theorem 3.2.5.

1. *$Z_\Gamma(s,\chi)$ is holomorphic in the half plane $Re\, s \;>\; 2\rho_0$.*

2. *$Z_\Gamma(s,\chi)$ has a meromorphic continuation to the whole complex plane*

3. *$Z_\Gamma(s,\chi)$ satisfies the following functional equation*

$$Z_\Gamma(2\rho_0 -\; s,\chi) = Z_\Gamma(s,\chi)\exp\int_0^{s-\rho_0}\Phi(t)dt \tag{3.4}$$

where $\Phi(t) = \kappa\, vol(\Gamma\backslash G)\chi(1)c(it)^{-1}c(-it)^{-1}$. Here, $c(t)$ is the Harish-Chandra c-function, which is a function of one complex variable which depends only on $(G,\,K)$.

4. When $G = SO_0(2n+1, 1)$, Φ is a polynomial and so is $\int_0^{s-\rho_0} \Phi(t)dt$. This simplifies the functional equation.

Definition 3.2.6. An irreducible unitary representation π of G on a Hilbert space V is said to be spherical if there exists a non-zero vector $v \in V$ such that

$$\pi(k)v = v \quad \forall\, k \in K.$$

We denote the set of equivalence classes of the spherical representations of G by $\widehat{G}_s$. The *spherical spectrum* of a uniform lattice is the collection of spherical representations of G which occur in the decomposition of $L^2(\Gamma\backslash G)$.

Let T be a finite dimensional unitary representation of Γ with characteristic χ. We denote with U the unitary representation of G induced by T. Then U is a discrete direct sum of irreducible unitary representations occurring with finite multiplicity. Let $\{U_i, j \geq 0\}$ be the spherical representations among these and let $n_j(\chi)$ be their multiplicities. The spherical spectrum of a uniform lattice is precisely the set $\{U_i, j \geq 0\}$ with multiplicities when T is chosen to be the trivial representation of Γ.

We now state two theorems of Harish-Chandra [13] which will help us characterize the irreducible unitary spherical representations, which are also called representations of class 1.

Definition 3.2.7 (Positive definite function)**.** A complex valued continuous function ϕ on a topological group G is called positive definite if

$$\sum_{i,j=1}^{n} \phi(x_i^{-1}x_j)\alpha_i\bar{\alpha}_j \geq 0$$

for all finite sets $x_1,\, x_2,\, \ldots x_n$ in G and any set of complex numbers $\alpha_1,\, \alpha_2,\, \ldots \alpha_n$.

Theorem 3.2.8 (Harish-Chandra 1)**.** *There is a bijective correspondence between the set Ω of equivalence classes ω of representations of class 1 and the set β of all positive definite spherical functions φ on G satisfying $\varphi(e) = 1$.*

We write $G = KA_pN$, Iwasawa decomposition and define $H(x)$ as the unique element in $\mathfrak{a}_p$ such that $x = k(\exp H(x))n$, where $k \in K$ and $n \in N$.

Theorem 3.2.9. *[Harish-Chandra 2] Functions of the type*

$$\varphi_\nu(x) = \int_K e^{(i\nu-\rho)(H(xk))}dk, \qquad \nu \in \Lambda^{\mathbb{C}}$$

exhaust the class of spherical functions of G. Moreover, any two such functions φ_ν and φ_λ are identical if and only if $\nu = \lambda^s$ for some $s \in \mathfrak{W}$ (Weyl Group).

Using these two theorems, we get that each of the U_i is uniquely determined by a spherical function φ_{ν_j}, where the $\nu_j \in \Lambda^{\mathbb{C}}$ is uniquely determined up to action of the Weyl group.

Let $r_j^+(\chi) = \nu_j(H_0)$ and $r_j^-(\chi) = -\nu_j(H_0)$. Put $s_j^+ = \rho_0 + ir_j^+$ and $s_j^- = \rho_0 + ir_j^-$. Though these quantities depend on χ, where there is no danger of confusion the χ is not explicitly mentioned.

The following theorem from [8] shows that the information about the spherical spectrum of X_Γ is encoded in $Z_\Gamma(s, \chi)$.

Theorem 3.2.10 (Zeros and Poles of $Z_\Gamma(s, \chi)$)**.**

1. *The points s_j^+ and s_j^-, with $j \geq 1$, are zeroes of $Z_\Gamma(s,\chi)$ of order κn_j respectively. These are called the spherical zeroes.*

2. *The points $\rho_0 + ir_k$, $k \geq 0$ are either zeroes or poles of $Z_\Gamma(s,\chi)$ according to whether $-\chi(1)e_kE$ is positive or negative. These are called the topological zeros or poles. The order of the zero or pole $\rho_0 + ir_k$ is $|\chi(1)e_kE|$.*

3. *It also follows that the $\rho_0 + ir_k$ are either all poles or all zeroes, if they exist.*

4. *Also, since the elements r_k are purely imaginary, it follows that $\rho_0 + ir_k$ equals either $-k$ or $-2k$, $k \geq 0$, when the r_k exist.*

Here, E denotes the Euler-Poincaré characteristic of the manifold X_Γ.

When the dimension of X_Γ is odd, the Euler-Poincaré characteristic is zero, and hence the zeta function has only spectral zeros and no poles.

The following theorem from [9] shows the relation between the spherical spectrum and the length spectrum. Simply put, it shows that the spherical spectrum of the space, and hence the Laplace spectrum, determines the length spectrum. This theorem has been proved using the properties of the logarithmic derivative of $Z_\Gamma(s, \chi)$, showing its use as a tool to connect the geometric and the algebraic properties of the manifold X_Γ.

Theorem 3.2.11. *Let G be a connected semi-simple Lie group with finite centre and let Γ_1 and Γ_2 be two co-compact torsion free lattices in G. Assume that the spherical spectrum of the spaces $X_{\Gamma_i} = \Gamma_i\, G\backslash K$ for $i = 1, 2$ is same. Then,*

$$\{l_i \in \mathbb{R} \mid L_{\Gamma_1}(l_i) \neq 0\} = \{l_j \in \mathbb{R} \,/\, L_{\Gamma_2}(l_j) \neq 0\}.$$

In other words, the lengths that appear in the spectrum of X_{Γ_1} and X_{Γ_2} are the same.

3.3 Selberg-Wakayama Zeta Function

A further generalization of the Selberg zeta function was defined by Wakayama in [28]. This zeta function is also associated to a manifold X_Γ of the type $\Gamma\backslash G/K$, where G is a connected semisimple Lie group with finite centre, K is a maximal compact subgroup and Γ is a uniform lattice in G. Apart from an irreducible unitary representation of the lattice Γ, this zeta function also takes into consideration an irreducible unitary representation of K.

For any subgroup L of G, we denote the set of all equivalence classes of irreducible unitary representations of L by $\widehat{L}$. If $\pi \in \widehat{L}$ is a finite-dimensional representation, we put $\chi_\pi = \mathrm{tr}\pi$ and $d_\pi = dim\pi$.

We recall that M is the centralizer of $A_{\mathfrak{p}}$ in K. For $\tau \in \widehat{K}$, we put $\widehat{M_\tau} = \{\sigma \in \widehat{M} \mid [\sigma : \tau|_M] \neq 0]\}$. For ease of notation, we let $\tau_M := \tau|_M$ and $\alpha_\sigma = [\sigma : \tau|_M]$.

The Selberg-Wakayama zeta function $Z_\tau(s)$ of $X_\Gamma = \Gamma\backslash G/K$, associated with $\tau \in \widehat{K}$ is defined by the following Euler product:

Definition 3.3.1 (Selberg-Wakayama Zeta Function)**.**

$$Z_\tau(s) = \prod_{\sigma \in \widehat{M_\tau}} Z_\sigma(s)^{[\sigma:\tau|_M]} \tag{3.5}$$

where

$$Z_\sigma(s) = \prod_{p \in P_\Gamma} \prod_{\lambda \in L} (1 - \chi_\sigma(m_p)^{-1} \xi_\lambda(h(p))^{-1} e^{-su_p})^{m_\lambda}. \tag{3.6}$$

The Selberg-Wakayama zeta function satisfies many properties similar to the ones satisfied by the Selberg-Gangolli zeta function. The following theorem summarizes some of the results on Z_τ proved in [28] (Theorem 7.1-7.3, p.p. 287-291).

Theorem 3.3.2.

1. *Z_τ is holomorphic in the half plane $Re(s) > 2\rho_0$ and has analytic continuation as a meromorphic function on the whole complex plane.*

2. *Z_τ always have some zeros, which are called spectral zeros. The location and the order of the zeros contains information about the τ-spectrum $\widehat{G_\tau}$ in $L^2(\Gamma\backslash G)$ where*

$$\widehat{G_\tau} = \{\pi \in \widehat{G}; m_\Gamma(\pi) > 0,\, \tau \in \pi|_K\}.$$

Here m_Γ is the multiplicity of a unitary representation π of G in the right regular representation π_Γ of G on $L^2(\Gamma \backslash G)$.

3. *Apart from the spectral zeros, $\mathbb{Z}_\tau$ has certain series of zeros and poles which come from the Plancherel measure. These are called the topological zeros of Z_τ.*

4. *Z_τ satisfies the following functional equation:*

$$Z_\tau(2\rho_0 - s) = Z_\tau(s)\, exp\left(\int_0^{s-\rho_0} \Delta_\tau(t)\, dt \right). \tag{3.7}$$

Here Δ_τ is an expression which depends on G, K, χ and τ. In the case when dimension of X_Γ is odd, Δ_τ is a polynomial and the functional equation simplifies to $Z_\tau(2\rho_0 - s) = Z_\tau(s)$.

3.4 Length-holonomy Spectrum of $SO_0(n,1)$

In this section, we describe the holonomy class of a closed geodesic in $X_\Gamma = \Gamma \backslash G/K$ in the case where $G = SO_0(n, 1)$, the connected component of identity in the isometry group $SO(n, 1)$ of the hyperbolic space $\mathbb{H}_n$. We then give the definition of the length-holonomy spectrum of X_Γ.

Let $\lambda : [0, 1] \longrightarrow X$ be a closed geodesic in X_Γ. Let $\lambda(0) = \lambda(1) = P$ and $\lambda'(0) = v$.

Definition 3.4.1 (Parallel Transport of a vector along a geodesic)**.** Parallel transport of a vector v along a geodesic λ in X_Γ is a vector field Y_v along λ in X_Γ such that

1. $Y_v(t) \in T_{\lambda(t)} X_\Gamma$,

2. $Y_v(t)$ is parallel to v for all $t \in [0, 1]$,

3. $Y_v(0) = v$

Consider the subspace $W = \{w \in T_P X_\Gamma \ \ \langle v, w\rangle = 0\}$ of $T_P X_\Gamma$. Using the parallel transport define the map

$$T : W \longrightarrow W$$
$$w \longmapsto Y_w(1)$$

We now fix an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of W which extends to the basis $\{v_1, v_2, \ldots, v_n, v\}$ of $T_P X_\Gamma$. Then it can be checked that $T(v) = v$ and $\langle T(w_1), T(w_2)\rangle = \langle w_1, w_2\rangle$ for all $w_1, w_2 \in W$. Therefore the map T, which depends only on the geodesic λ, defines an element A of $SO(n-1)$ up to conjugacy.

Definition 3.4.2 (Holonomy Class)**.** The conjugacy class of A in $SO(n-1)$, as defined above, is called the holonomy class of the closed geodesic λ and is denoted by h_γ.

For any n, let the set of conjugacy classes of $SO(n)$ be denoted by $\mathcal{M}_n$.

When $G = SO_0(n, 1)$, it can be checked that M, the centralizer of $A_\mathfrak{p}$ in K is equal to $SO_0(n-1)$. Furthermore, it is known that the holonomy class of geodesic λ is precisely the conjugacy class $[m_\lambda]$.

Definition 3.4.3 (Length-holonomy Spectrum)**.** The length-holonomy spectrum of X_Γ is defined to be the function $\mathfrak{L}_\Gamma$ defined on $\mathbb{R} \times \mathcal{M}_{n-1}$ by, $\mathfrak{L}_\Gamma(a, [M])$ = number of conjugacy classes $[\gamma]$ in Γ with $(l(\gamma), h_\gamma) = (a, [M])$.

3.5 Decompositions of $SO_0(3, 1)$

We now consider the case when $G = SO_0(3, 1)$ and give an explicit description of the Iwasawa decomposition.

The Lie algebra of $SO(3,1)$, denoted by $\mathfrak{so}(3,1)$ is given as

$$\{A \in M_4(\mathbb{R}) \mid A^T J + JA = 0\}$$

where

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.8}$$

It can be checked that

$$\mathfrak{so}(3,1) = \left\{ \begin{pmatrix} B & u \\ u^T & 0 \end{pmatrix} \middle| \, u \in \mathbb{R}^3 \text{ and } B^T = -B, \; B \in M_3(\mathbb{R}) \right\}$$

Thus any matrix of $\mathfrak{so}(3,1)$ can be uniquely written as

$$\begin{pmatrix} B & u \\ u^T & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & u \\ u^T & 0 \end{pmatrix}$$

where the first matrix is skew symmetric and the second matrix is symmetric and both belong to $\mathfrak{so}(3,1)$.

Define $\mathfrak{k}$ and $\mathfrak{p}$ as follows:

$$\mathfrak{k} = \left\{ \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \middle| \, B^T = -B \right\}$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & u \\ u^T & 0 \end{pmatrix} \middle| \, u \in \mathbb{R}^3 \right\}.$$

Then both $\mathfrak{k}$ and $\mathfrak{p}$ are subspaces of $\mathfrak{g} = \mathfrak{so}(3,1)$ and $\mathfrak{k}$ is a sub-algebra of $\mathfrak{g}$. In fact, $\mathfrak{k}$ is isomorphic to $\mathfrak{so}(3)$. Furthermore, we have $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. We have the decomposition $\mathfrak{so}(3,1) = \mathfrak{k} \oplus \mathfrak{p}$, which gives the Cartan Decomposition of $\mathfrak{so}(3,1)$. Also, if Θ is the map $\Theta(A) = -A^T$, then $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and the -1 eigenspaces of Θ respectively.

Let $\mathfrak{a}_\mathfrak{p}$ be the matrices of the form

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & \alpha & 0 \end{pmatrix} \tag{3.9}$$

Then $\mathfrak{a}_\mathfrak{p}$ is an abelian subalgebra of $\mathfrak{p}$. It can be checked that it is in fact the maximal abelian subalgebra in $\mathfrak{p}$. Consider the matrices in $\mathfrak{so}(3,1)$ of the form

$$\begin{pmatrix} 0 & 0 & -a & a \\ 0 & 0 & -b & b \\ a & b & 0 & 0 \\ a & b & 0 & 0 \end{pmatrix} \tag{3.10}$$

where a, $b \in \mathbb{R}$. Such matrices form an abelian subalgebra $\mathfrak{n}$ of $\mathfrak{so}(3,1)$. Furthermore, $\mathfrak{so}(3,1) = \mathfrak{k} \oplus \mathfrak{a}_\mathfrak{p} \oplus \mathfrak{n}$. This gives the Iwasawa Decomposition of $\mathfrak{so}(3,1)$.

Define $\mathfrak{m} = \left\{ \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in M_4(\mathbb{R}) \middle| B \in \mathfrak{so}(2) \right\}$. It can be checked that $\mathfrak{m}$ is a subalgebra of $\mathfrak{k}$ and is the centralizer of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{so}(3,1)$.

Let

$$\mathfrak{a}_\mathfrak{k} = \left\{ \begin{pmatrix} 0 & b & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| b \in \mathbb{R} \right\}. \tag{3.11}$$

Then $\mathfrak{a}_\mathfrak{k}$ is a maximal abelian subalgebra of $\mathfrak{k}$. Also, $\mathfrak{a}_\mathfrak{k}$ commutes with $\mathfrak{a}_\mathfrak{p}$. Therefore $\mathfrak{a} = \mathfrak{a}_\mathfrak{k} + \mathfrak{a}_\mathfrak{p}$ is a maximal abelian subalgebra of $\mathfrak{g}$ such that

$\mathfrak{a} \cap \mathfrak{p} = \mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{a} \cap \mathfrak{k} = \mathfrak{a}_{\mathfrak{k}}$. Then,

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & b & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & \alpha & 0 \end{pmatrix} \middle| \alpha, b \in \mathbb{R} \right\} \tag{3.12}$$

is a Cartan subalgebra of $\mathfrak{so}(3,1)$.

We can also check that

$$\exp \mathfrak{a} = A = \left\{ \begin{pmatrix} \cos b & \sin b & 0 & 0 \\ -\sin b & \cos b & 0 & 0 \\ 0 & 0 & \cosh \alpha & \sinh \alpha \\ 0 & 0 & \sinh \alpha & \cosh \alpha \end{pmatrix} \right\}, \tag{3.13}$$

$$\exp \mathfrak{a}_{\mathfrak{p}} = A_{\mathfrak{p}} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh \alpha & \sinh \alpha \\ 0 & 0 & \sinh \alpha & \cosh \alpha \end{pmatrix} \right\}, \tag{3.14}$$

$$\exp m = M = \left\{ \begin{pmatrix} \cos b & \sin b & 0 & 0 \\ -\sin b & \cos b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}. \tag{3.15}$$

Therefore, in this case, $M \cong SO(2)$ and hence the holonomy class of a closed geodesic λ is given by a parameter $b \in (0, 2\pi]$. The parameter α gives the length of the closed geodesic.

3.6 Root space decomposition of $\mathfrak{so}(3, 1, \mathbb{C})$

Let $\mathfrak{g}$ and $\mathfrak{a}$ be as above. Let $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{a}^{\mathbb{C}}$ denote the complexifications of $\mathfrak{g}$ and $\mathfrak{a}$ and let $\Phi(\mathfrak{g}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}})$ denote the set of roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}})$.

We now explicitly compute the roots of $\mathfrak{so}(3, 1, \mathbb{C})$. This is from Chapter IV, section 1 of [19].

An element of α in the dual space of $\mathfrak{a}^{\mathbb{C}}$ is called a root if

$$L_\alpha = \{g \in \mathfrak{g}^{\mathbb{C}} \,|hg - gh = \alpha(h)g \quad \forall\, h \in \mathfrak{a}^{\mathbb{C}}\}$$

is a non-zero subspace of $\mathfrak{g}^{\mathbb{C}}$.

Any $h \in \mathfrak{a}^{\mathbb{C}}$ can be written as

$$\begin{pmatrix} 0 & ih_1 & 0 & 0 \\ -h_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & ih_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.16}$$

for some complex numbers h_1 and h_2.

We know that $L_\alpha = \mathbb{C}E_\alpha$ where $E_\alpha \in \mathfrak{g}^{\mathbb{C}}$ is of the form

$$\begin{pmatrix} 0 & 0 & w & x \\ 0 & 0 & y & z \\ -w & -y & 0 & 0 \\ x & z & ih_2 & 0 \end{pmatrix} \tag{3.17}$$

for some $w, x, y, z \in \mathbb{C}$. Therefore, α is a root if there exists a non-zero solution for (w, x, y, z) such that $hE_\alpha - E_\alpha h = \alpha(h)E_\alpha$.

It can be shown that such a solution exists only when

$$\alpha(h) \in \{h_1 + ih_2, h_1 - ih_2, -h_1 - ih_2, -h_1 + ih_2\}$$

Denote these roots by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

We choose compatible ordering in the dual spaces of $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{a}_\mathfrak{p} + i\mathfrak{a}_\mathfrak{k}$, respectively. Any element of $\mathfrak{a}_\mathfrak{p}$ is of the form

$$x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & a & 0 \end{pmatrix} \tag{3.18}$$

where $a \in \mathbb{R}$. The vector space $\mathfrak{a}_\mathfrak{p}$ and hence it's dual $\mathfrak{a}_\mathfrak{p}^*$ are one dimensional over $\mathbb{R}$. Let $e_1 \in \mathfrak{a}_\mathfrak{p}^*$ such that $e_1(x) = a$ be a basis of $\mathfrak{a}_\mathfrak{p}^*$. Any element of $\mathfrak{a}_\mathfrak{p} + i\mathfrak{a}_\mathfrak{k}$ is of the form

$$y = \begin{pmatrix} 0 & ib & 0 & 0 \\ -ib & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & a & 0 \end{pmatrix} \tag{3.19}$$

where $a, b \in \mathbb{R}$. We can extend the basis e_1 of $\mathfrak{a}_\mathfrak{p}^*$ to a basis $\{e_1, e_2\}$ of $\mathfrak{a}_\mathfrak{p} + i\mathfrak{a}_\mathfrak{k}{}^*$ by letting $e_2(y) = b$. This choice of basis gives a compatible ordering of the dual spaces of $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{a}_\mathfrak{p} + i\mathfrak{a}_\mathfrak{k}$.

For $y \in \mathfrak{a}_\mathfrak{p} + i\mathfrak{a}_\mathfrak{k}$ as above,

$$\alpha_1(y) = b + a \tag{3.20}$$

$$= (e_1 + e_2)(y), \tag{3.21}$$

$$\alpha_2(y) = b - a \tag{3.22}$$

$$= (-e_1 + e_2)(y), \tag{3.23}$$

$$\alpha_3(y) = -b - a \tag{3.24}$$

$$= (-e_1 - e_2)(y) \quad \text{and} \tag{3.25}$$

$$\alpha_4(y) = -b + a \tag{3.26}$$

$$= (e_1 - e_2)(y). \tag{3.27}$$

Therefore α_1 and α_4 are positive roots according to the ordering fixed above.

Also, we know that for $G = SO_0(2n + 1, 1)$, $p = 2n$ and $q = 0$. Therefore for $G = SO_0(3, 1)$, $\rho_0 = 1$.

3.7 Towards Strong Multiplicity One Property for Length-Holonomy Spectrum of $SO_0(3,1)$

In this section we describe our efforts to prove the strong multiplicity one property for the length-holonomy spectrum of $SO_0(3,1)$. More precisely, we consider two uniform lattices Γ_1 and Γ_2 of $SO_0(3,1)$. The spaces associated with these lattices, $X_{\Gamma_i} = \Gamma_i \backslash G/K$ for $i = 1, 2$, have associated with them a primitive length-holonomy spectrum.

The primitive length-holonomy spectrum of X_Γ is defined to be the function $\mathfrak{P}_\Gamma$ defined on $\mathbb{R} \times [0, 2\pi)$ by,

$$\mathfrak{P}_\Gamma(a,\, b) = \# \text{ of conjugacy classes } [p] \in P_\Gamma \text{ such that } (a(p),\, b(p)) = (a, b).$$

We now define a modified length-holonomy spectrum of X_Γ. For any $p \in P_{\Gamma_i}$, we define $c(p) := \frac{b(p)}{a(p)}$. The *modified primitive length-holonomy spectrum* of X_Γ is defined to be the function $\mathfrak{M}_\Gamma$ on $\mathbb{R}$ given by,

$$\mathfrak{M}_\Gamma(c) = \# \text{ of conjugacy classes } [p] \in P_\Gamma \text{ such that } c(p) = c.$$

We assume that the primitive length-holonomy spectra of X_{Γ_1} and X_{Γ_2} is same for all but finitely many points in $\mathbb{R} \times [0,\, 2\pi)$, i.e., $\exists$ a finite subset $S \subset \mathbb{R} \times [0,\, 2\pi)$ such that $\mathfrak{P}_{\Gamma_1}(a,\, b) = \mathfrak{P}_{\Gamma_2}(a,\, b)$ $\forall \in (a,\, b) \notin S$. We want to check if this assumption implies that the primitive length-holonomy spectra are the same for all pairs (a, b). This is a work in progress and in this chapter we state and prove some partial results we have in this direction.

We first simplify the zeta function for the case $SO_0(3,1)$. Consider any $p \in P_\Gamma$, a primitive hyperbolic conjugacy class of Γ. Then p is conjugate to an element $h(p) = m_p \exp(l(p)H_0) \in MA_p^+$. Here

$$\log(h(p)) = \begin{pmatrix} 0 & b(p) & 0 & 0 \\ -b(p) & 0 & 0 & 0 \\ 0 & 0 & 0 & a(p) \\ 0 & 0 & a(p) & 0 \end{pmatrix} \in \mathfrak{a}, \tag{3.28}$$

$$m_p = \begin{pmatrix} \cos b(p) & \sin b(p) & 0 & 0 \\ -\sin b(p) & \cos b(p) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in M, \tag{3.29}$$

$$\exp(l(p)H_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh a(p) & \sinh a(p) \\ 0 & 0 & \sinh a(p) & \cosh a(p) \end{pmatrix} \in A_{\mathfrak{p}}^+ \tag{3.30}$$

for some $a(p)$, $b(p) \in \mathbb{R}$. (Here, $\ell(p) = a(p)$.)

If $\lambda \in L$ is of the form $m_1(e_1+e_2) + m_2(e_1-e_2)$, for m_1, $m_2 \in \mathbb{Z}^+ \cup \{0\}$, then

$$\lambda(\log h(p)) = (m_1 + m_2)a(p) + i(m_1 - m_2)b(p).$$

Therefore,

$$\xi(h(p)) = \exp\left((m_1 + m_2)a(p) + i(m_1 - m_2)b(p)\right).$$

Let τ be any irreducible representation of $K = SO(3)$. We know that any irreducible representation of $SO(3)$ is of odd dimension and hence dimension of τ is of the form $2m + 1$ for some positive integer m. Since

$M = SO(2)$ is an abelian group, any irreducible representation of $SO(2)$ is one dimensional. Thus the restriction of τ to $SO(2)$ is a direct sum of $2m+1$ one-dimensional representations, $\sigma_j : SO(2) \longrightarrow \mathbb{C}^{\times}$ given by $R(\theta) \longmapsto e^{ij\theta}$, for $j = -m, \ldots, m$. Here $R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Therefore $\chi_{\sigma_m}(m_p) = e^{im\theta}$. It can also be checked that, for $SO_0(3,1)$, $m_\lambda = 1$ for any $\lambda \in L$.

With these simplifications, the Selberg-Gangolli-Wakayama zeta function for $SO_0(3,1)$ is as follows:

$$Z_\tau(s) = \prod_{k=-m}^{m} \prod_{p \in P_\Gamma} \prod_{\lambda \in L} (1 - e^{-(ikb(p) + (m_1+m_2)a(p) - (m_1-m_2)b(p) + sa(p))}). \quad (3.31)$$

Let Z_{Γ_1} and Z_{Γ_2} be the Selberg-Wakayama zeta functions of the spaces X_{Γ_1} and X_{Γ_2}, respectively. We consider ratio of the corresponding zeta functions,

$$\frac{Z_{\Gamma_1}(s)}{Z_{\Gamma_2}(s)} = \frac{\prod\limits_{k=-m}^{m} \prod\limits_{p \in P_{\Gamma_1}} \prod\limits_{m_1, m_2 \in \mathbb{N}} (1 - e^{-(ikb(p) + (m_1+m_2)a(p) - (m_1-m_2)b(p) + sa(p))})}{\prod\limits_{k=-m}^{m} \prod\limits_{q \in P_{\Gamma_2}} \prod\limits_{l_1, l_2 \in \mathbb{N}} (1 - e^{-(ikb(q) + (l_1+l_2)a(q) - (l_1-l_2)b(q) + sa(q))})}. \quad (3.32)$$

Under our assumption, there exist finite indexing sets S_1 and S_2 such that the above fraction becomes,

$$\frac{Z_{\Gamma_1}(s)}{Z_{\Gamma_2}(s)} = \frac{\prod\limits_{k=-m}^{m} \prod\limits_{p \in S_1} \prod\limits_{m_1, m_2 \in \mathbb{N}} (1 - e^{-(ikb(p) + (m_1+m_2)a(p) - (m_1-m_2)b(p) + sa(p))})}{\prod\limits_{k=-m}^{m} \prod\limits_{q \in S_2} \prod\limits_{l_1, l_2 \in \mathbb{N}} (1 - e^{-(ikb(q) + (l_1+l_2)a(q) - (l_1-l_2)b(q) + sa(q))})}. \quad (3.33)$$

It can be checked that both the numerator and the denominator of the above ratio is holomorphic, which gives the following lemma:

Lemma 3.7.1. *The ratio* $\frac{Z_{\Gamma_1}(s)}{Z_{\Gamma_2}(s)}$ *(3.33) is meromorphic.*

Let $T(s) = \frac{Z_{\Gamma_1}(s)}{Z_{\Gamma_2}(s)}$. We have proved that $T(s)$ is a meromorphic function. We also know that, for $i = 1, 2$, Z_{Γ_i} and hence $T(s)$ admits a meromorphic continuation to $\mathbb{C}$. Therefore the expressions must match $\forall\, s \in \mathbb{C}$, i.e.,

$$\frac{Z_{\Gamma_1}(1 - s)}{Z_{\Gamma_2}(1 - s)} = \frac{\prod_{k=-1}^{1} \prod_{p \in S_1} \prod_{m_1, m_2 \in \mathbb{N}} \left(1 - e^{-(ikb(p) + (m_1+m_2)a(p) - (m_1-m_2)b(p) + (1-s)a(p))}\right)}{\prod_{k=-1}^{1} \prod_{q \in S_2} \prod_{l_1, l_2 \in \mathbb{N}} \left(1 - e^{-(ikb(q) + (l_1+l_2)a(q) - (l_1-l_2)b(q) + (1-s)a(q))}\right)}. \tag{3.34}$$

For $G = SO_0(n, 1)$, when n is odd and hence for $G = SO_0(3, 1)$, it is know that $Z_\Gamma(s) = Z_\Gamma(1 - s)$ [12]. This implies that $T(s) = T(1 - s)$. Any zero of $T(s)$ is either a zero of the numerator in the expression for T or pole of the denominator. But we have proved that the denominator of the expression is analytic on $\mathbb{C}$. Hence the zeros of $T(s)$ are precisely the zeros of the numerator of the expression of $T(s)$. Let A denote the set of zeros of $T(s)$. It can be deduced that that the elements of A are precisely the complex numbers of the form $s = s_1 + s_2$ where

$$s_1 = -m_1 - m_2 \tag{3.35}$$

$$s_2 = \frac{-b(p)(m_1 - m_2 + k) - 2n\pi}{a(p)}. \tag{3.36}$$

Here $m_1, m_2 \in \mathbb{Z}^+$, $k = -1, 0, 1$, $p \in S_1$ and $n \in \mathbb{Z}$.

We know that $T(s) = T(1-s)$. Therefore we get that $s \in A \Rightarrow 1 - s \in A$. But this is not true. Therefore we get that $A = \emptyset$, i.e., $T(s)$ has no zeros. By a similar we can conclude that $T(s)$ has no poles. Therefore the zeros of the numerator in the expression for $T(s)$ cancel out with the zeros of the denominator.

More precisely, for any fixed $m_1, m_2 \in \mathbb{Z}^+$, $k_1 \in \{-1, 0, 1\}$, $n_1 \in \mathbb{Z}$ and $p \in S_1$, there exist $l_1, l_2 \in \mathbb{Z}^+$, $k_2 \in \{-1, 0, 1\}$, $n_2 \in \mathbb{Z}$ and $q \in S_2$

such that

$$m_1 + m_2 = l_1 + l_2 \tag{3.37}$$

$$\frac{-b(p)(m_1 - m_2 + k_1) - 2n_1\pi}{a(p)} = \frac{-b(q)(m_1 - m_2 + k_2) - 2n_2\pi}{a(q)} \tag{3.38}$$

Let A_{Γ_1} be the multiset of zeros of the expression in the numerator of $T(s)$ and let A_{Γ_2} be the multiset of zeros of the expression in the denominator of $T(s)$. In terms of the new notation, the equations 3.37 and 3.38 state that $A_{\Gamma_1} = A_{\Gamma_2}$.

Consider now the elements of A_{Γ_1} which lie on the line $Re(s) = 0$. We have $(m_1, m_2) = (0, 0)$ for these elements. Also, each such zero appears with the same multiplicity in A_{Γ_2} and will have $(l_1, l_2) = (0, 0)$.

Therefore we have the following equality of the multisets,

$$\left\{\frac{-b(p)k_1 - 2n_1\pi}{a(p)} \mid p \in S_1,\, k_1 = -m, \ldots, m,\, n_1 \in \mathbb{Z}\right\} =$$
$$\left\{\frac{-b(q)k_2 - 2n_2\pi}{a(q)} \mid q \in S_2,\, k_2 = -m, \ldots, m,\, n_2 \in \mathbb{Z}\right\} \tag{3.39}$$

Now consider the case when τ is the trivial representation of $SO(3)$. For this τ, we have $m = 0$ and 3.39 becomes the following equality of multisets:

$$\left\{\frac{2n_1\pi}{a(p)} \mid p \in S_1,\, n_1 \in \mathbb{Z}\right\} =$$
$$\left\{\frac{2n_2\pi}{a(q)} \mid q \in S_2,\, n_2 \in \mathbb{Z}\right\} \tag{3.40}$$

Let s_0 the element in the set on the left hand side which is strictly positive and closest to 0. For $n \leq 0$, $\frac{2n\pi}{a(p)} \leq 0$. For $n > 1$, $\frac{2n\pi}{a(p)} \frac{2\pi}{a(p)}$. Therefore, $s_0 = \frac{2\pi}{a(p)}$ for some $p \in S_1$. Let $s_0^{'}$ be the element in the set on the right hand side which is strictly positive and closest to 0. By similar argument, it can be concluded that $s_0^{'} = \frac{2\pi}{a(q)}$ for some $q \in S_2$. Now, the equality of the sets in equation 3.40 implies that for the $p \in S_1$ corresponding to s_0, there exists a $q \in S_2$ (corresponding to $s_0^{'}$) such that $\frac{2\pi}{a(p)} = \frac{2\pi}{a(q)}$.

Suppose the zero $\frac{2\pi}{a(p)}$ has multiplicity > 1. It has the same multiplicity in the set on right hand side of eq. 3.40. Therefore, there exist a $0 < n^{'} \in \mathbb{Z}$ and $q^{'} \in S_2$ such that $\frac{2\pi}{a(p)} = \frac{2n^{'}\pi}{a(q^{'})}$. If $n^{'} > 1$, then

$$\frac{2\pi}{a(q^{'})} < \frac{2n^{'}\pi}{a(q^{'})} = \frac{2\pi}{a(q)}.$$

This is a contradiction. Hence $n^{'} = 1$ and

$$\frac{2\pi}{a(q^{'})} = \frac{2\pi}{a(q^{'})} = \frac{2\pi}{a(q)}.$$

Therefore the zeros $\frac{2\pi}{a(p)}$ and $\frac{2\pi}{a(q)}$ appear with the same multiplicity in the sets on the left hand side and the set on the right hand side of eq. 3.40, respectively.

We can now remove the points of the form $\frac{2n\pi}{a(p)}$ and points of the form $\frac{2n\pi}{a(q)}$ from the sets on the left-hand side and right-hand side of eq. 3.40 respectively and repeat the argument to get the following result:

Theorem 3.7.2. *Let $G = SO_0(3,1)$ and Γ_1 and Γ_2 be two uniform lattices in G such that $\mathfrak{P}_{\Gamma_1}(a, b) = \mathfrak{P}_{\Gamma_2}(a, b)$ for all but finitely many pairs $(a, b) \in \mathbb{R} \times [0, 2\pi]$. Then $PL_{\Gamma_1}(l) = PL_{\Gamma_2}(l)$, and hence $L_{\Gamma_1}(l) = L_{\Gamma_2}(l)$, for all $l \in \mathbb{R}$.*

Let $A_{\Gamma_1}^{(0,0)}$ be the multiset on the left hand side of the above equation and let $A_{\Gamma_2}^{(0,0)}$ be the multiset on the right hand side. Consider now the smallest point y in $A_{\Gamma_1}^{(0,0)}$ such that $y > 0$. This point is of the form $\frac{2n_1\pi}{a(p)}$ for some $n_1 \in \mathbb{Z}$ and $p \in S_1$. Recall that $a(p) > 0$ for all $p \in P_{\Gamma_i}$ $(i = 1, 2)$. If $n_1 \leq 0$ then $y \leq 0$. Also $\frac{2n_1\pi}{a(p)} > \frac{2\pi}{a(p)}$ for $n_1 > 1$. Therefore we conclude that $n_1 = 1$ and $y = \frac{2\pi}{a(p)}$ for some $p \in S_1$. Similarly, we can argue that the smallest point in $A_{\Gamma_2}^{(0,0)}$ is of the form $\frac{2\pi}{a(q)}$ for some $q \in S_2$. The equality of multisets above implies that $\frac{2\pi}{a(p)} = \frac{2\pi}{a(q)}$ and they occur with same multiplicities. By removing all the points of the form $\frac{2n_1\pi}{a(p)}$ $\left(\text{resp. } \frac{2n_2\pi}{a(q)}\right)$ from $A_{\Gamma_1}^{(0,0)}$ (resp. $A_{\Gamma_2}^{(0,0)}$), we can repeat the argument above to conclude the below equality of sets with multiplicity:

$$\Big\{a(p) \mid p \in S_1\Big\} = \Big\{a(q) \mid q \in S_2\Big\}. \tag{3.41}$$

Therefore we get the equality of the primitive length spectrum with multiplicity:

$$\Big\{a(p) \mid p \in P_{\Gamma_1}\Big\} = \Big\{a(q) \mid q \in P_{\Gamma_2}\Big\}. \tag{3.42}$$

Now we look again the case where τ is the 3-dimensional representation of $SO(3)$ and hence $m = 1$. Using the equality of sets in equation 3.42, we can remove the elements of the form $\frac{2n_1\pi}{a(p)}$ ($n_1 \in \mathbb{Z}$ and $p \in S_1$) from the set on the left hand side of equation 3.39 and of the form $\frac{2n_2\pi}{a(q)}$ ($n_2 \in \mathbb{Z}$ and $q \in S_2$) from the set on the right hand side to get the below equality of multisets:

$$\left\{\frac{-b(p)k_1 - 2n_1\pi}{a(p)} \mid p \in S_1,\, k_1 = \pm 1,\, n_1 \in \mathbb{Z}\right\} =$$
$$\left\{\frac{-b(q)k_2 - 2n_2\pi}{a(q)} \mid q \in S_2,\, k_2 = \pm 1,\, n_2 \in \mathbb{Z}\right\} \tag{3.43}$$

Consider the point in the set on the left hand side which is positive and closest to 0. It is of the form

$$s_2 = \frac{-b(p)k - 2n\pi}{a(p)}$$

for some $n \in \mathbb{Z}$, $p \in S_1$, and $k = \pm 1$. We know that $b(p^{-1}) = 2\pi - b(p)$ and $a(p^{-1}) = a(p)$. Therefore

$$\begin{aligned}\frac{-kb(p^{-1} - 2n\pi)}{a(p^{-1})} &= \frac{-k(2\pi - b(p)) - 2n\pi}{a(p)} \\ &= \frac{b(p)k - 2(n + k)\pi}{a(p)}.\end{aligned}$$

Replacing p with p^{-1} if needed, we can assume that $k = -1$. Therefore

$$s_2 = \frac{b(p) - 2n\pi}{a(p)}.$$

If $n > 0$, $b(p) - 2n\pi < 0$. But since $s_2 > 0$, this is not possible and hence $n \leq 0$. For ease of notation, we put $m = -n$, which gives

$$s_2 = \frac{b(p) + 2m\pi}{a(p)}.$$

But

$$\frac{b(p) + 2m\pi}{a(p)} \geq \frac{b(p)}{a(p)}.$$

Hence the point in the set on the left hand side which is positive and closest to 0 is of the form $s_2 = \frac{b(p)}{a(p)}$ for some $p \in S_1$. Using similar arguments, one can conclude that the point in the set on the left hand side which is positive and closest to 0 is of the form $\frac{b(q)}{a(q)}$ for some $q \in S_2$. From the equality of the sets 3.43, we get that $\frac{b(p)}{a(p)} = \frac{b(q)}{a(q)}$. Using arguments similar to the ones used in the proof of Theorem 3.7.2, we get the following theorem:

Theorem 3.7.3. *Let $G = SO_0(3,1)$ and Γ_1 and Γ_2 be two uniform lattices in G such that $\mathfrak{P}_{\Gamma_1}(a, b) = \mathfrak{P}_{\Gamma_2}(a, b)$ for all but finitely many pairs $(a, b) \in \mathbb{R} \times [0, 2\pi]$. Then $\mathfrak{M}_{\Gamma_1}(c) = \mathfrak{M}_{\Gamma_2}(c)$ for all $c \in \mathbb{R}$.*

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