ACTION-AT-A-DISTANCE AND COSMOLOGY

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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Certificate

This is to certify that this thesis entitled “Action-at-a-distance and Cosmology” submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by “Kaustubh Sudhir Deshpande” at “Inter-University Centre for Astronomy and Astrophysics (IUCAA) Pune”, under the supervision of “Prof. J. V. Narlikar” during the academic year 2012-2013.

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Abstract

Action-at-a-distance is an alternative approach to field theory in describing all the interactions. It is based on direct interactions between particles rather than through their coupling to a field having infinite degrees of freedom. First formulated in a relativistically invariant manner by Schwarzschild, Tetrode and Fokker, action-at-a-distance in electrodynamics was provided a paradigm by Wheeler and Feynman through their absorber theory of radiation. This theory, formulated in static and flat universe, uses advanced absorber response from the entire universe as the origin for radiation reaction but allows for both retarded and advanced interactions. In the first part of this work (section 3), we study extension of this formulation in cosmological models (using conformal invariance of electromagnetism). Self-consistency of advanced and retarded interactions is investigated in these models. Only those models can be considered to be viable which have only retarded interactions. It is found that steady-state and quasi-steady-state models satisfy this criterion while Friedman models don't. Thus the origin of arrow of time in electromagnetism can be attributed to the cosmological structure. In the second part (section 4), we study the formulation of action-at-a-distance electrodynamics in Riemannian space-times using Green's functions. Green's functions in curved space-times have a component which corresponds to propagation inside the light cone ("tail"), along with the usual propagation along the light cone. We evaluate the explicit expressions for Green's functions in de Sitter and Einstein-de Sitter space-times. This can be used in evaluating the "tail" in an electromagnetic signal.

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1 Introduction

Newton's law of gravitation and Coulomb's law for electrical charges, one of the very first laws of theoretical Physics, assumed instantaneous action-at-a-distance between particles. The gravitational and electrical effects due to masses and charges, respectively, were assumed to travel at infinite speed, in these laws. The experiments in electrodynamics, however, demonstrated that Coulomb's law was inadequate to explain their results. Gauss summarized this difficulty in a letter to Weber in 1845 [10] which says,

"...I would doubtless have published my researches long since were it not that at the time I gave them up I had failed to find what I regarded as the keystone, Nil actum reputans si quid superesset agendum: namely, the derivation of the additional forces - to be added to the interaction of electrical charges at rest, when they are both in motion - from an action which is propagated not instantaneously but in time as is the case with light..."

Thus Gauss suggested action-at-a-distance propagating at a finite speed i.e. speed of light. However, this did not get immediately formulated. Instead, Maxwell developed classical field theory of electrodynamics which had effects propagating at the speed of light. This was also found to be consistent with special relativity (which discarded instantaneous action-at-a-distance of Newton's and Coulomb's laws). It can be described by the following relativistically invariant action,

$$S = -\sum_a \int m_a da - \frac{1}{16\pi}\int F_{ik}F^{ik}d^4x - \sum_a \int e_a A_i da^i \tag{1.1}$$

where F_{ik} is the field, with infinite degrees of freedom, defined in terms of the 4-potential (A_i) as $F_{ik} = (A_{k;i} - A_{i;k})$. The particles $a, b, ...$ do not interact directly with each other but through their coupling with the field (described by the third term in S).

In early 19th century Schwarzschild, Tetrode and Fokker [18] developed a relativistically invariant action-at-a-distance theory. This partially found the answer to Gauss's problem. This theory can be described by the Fokker action which is given as follows.

$$S = -\sum_a \int m_a da - \sum_{a<b} \int e_a e_b \, \delta(s^2_{AB}) \, \eta_{ik} da^i db^k \tag{1.2}$$

The first term is the same inertial term as in field theoretic action. The second term represents electromagnetic interactions between two different particles a, b connected by a light cone[1], thus preserving relativistic invariance. This action with the following definitions of direct particle potentials $\left(A_i^{(b)}\right)$ and fields $\left(F_{ik}^{(b)}\right)$,

$$A_i^{(b)}(X) = e_b \int \delta(s^2_{XB})\eta_{ik} db^k, \quad F_{ik}^{(b)} = A_{k;i}^{(b)} - A_{i;k}^{(b)} \tag{1.3}$$

[1] implied by $s^2_{AB} = \eta_{ij}(a^i - b^i)(a^j - b^j)$ being zero

gives exactly Maxwell-like equations for $F_{ik}^{(b)}$ and Lorentz-like equations of motion for the particles [10]. This formulation hence resembles and seems to provide an alternative to the Maxwell's field theory. However, it faces the following problems.

1. The expressions for direct particle potential and field state that it is completely time-symmetric theory. A particle can interact with other particles connected to it by future as well as past null paths. i.e. Both retarded and advanced interactions are possible. This is contrary to the observations in the real world in which there are only retarded interactions.

2. In the absence of self-interactions, there is no obvious way of explaining radiation reaction in Fokker formulation.

3. This theory has to be generalized to curved space-time (and in quantum regime which we do not consider here).

The subsequent sections present answers to these problems, based on the work by Wheeler-Feynman and later by Hoyle-Narlikar, to develop a consistent action-at-a-distance theory of classical electrodynamics.

Notation and conventions

1. The signature of space-time is taken as $(+,-,-,-)$ i.e. flat space metric is $\eta_{ik} = diag(1,-1,-1,-1)$.

2. The Roman indices $i, j, k, ...$ run over all space-time indices (0, 1, 2, 3) while the Greek indices α, β, ... run over only the spatial indices (1, 2, 3).

3. The units are chosen with $c = 1$.

2 Wheeler-Feynman absorber theory in static, flat universe

In this section we describe the absorber theory of radiation proposed by Wheeler and Feynman (called as WF theory henceforth) for static and flat universe [18, 19]. Being action-at-a-distance theory, the main paradigm it sets is that of the role of the entire universe as absorber and advanced response from absorber as radiation reaction. This solves first two of the three problems stated in the previous section for the Fokker formulation of action-at-a-distance.

The asssumptions of the theory are as follows.

1. A point charge doesn't radiate electromagnetic radiation in charge-free space.

2. The fields acting on a given particle arise only from other particles. (i.e. no self-interaction)

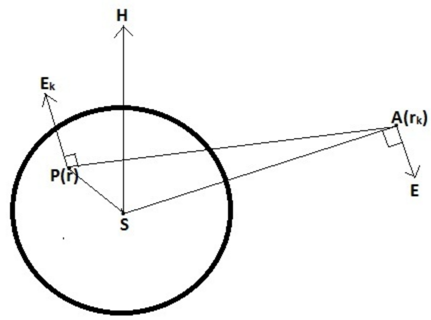

Figure 1: Absorber response for non-relativistic, high-density absorber medium

3. These fields are given by time symmetric solutions of Maxwell's equations i.e. half-retarded (F^{Ret}) plus half-advanced (F^{Adv}) Lienard-Wiechert solutions.

4. Universe is complete absorber i.e. all the radiation from a source charge gets absorbed.

The net field at any point is determined by the sum of the fields due to the source particle S (i.e. $F_S = \frac{1}{2}F_S^{Ret} + \frac{1}{2}F_S^{Adv}$) and the advanced response due to the absorber (F_A). Wheeler and Feynman demonstrate that if we assume the net field due to the source particle S at a typical absorber particle A to be full-retarded then the absorber response, in the neighbourhood of S, comes out to be $F_A = \frac{1}{2}F_S^{Ret} - \frac{1}{2}F_S^{Adv}$. Hence the net field at any point is $F_A + F_S = F_S^{Ret}$. This proves, in a self-consistent way[2], that the net interactions between any two particles are retarded in nature.

WF theory proves the above result for increasing level of generality; non-relativistic to relativistic motion of source particle, low to high density of absorber medium and finally the most general case i.e. just assuming complete absorption. We demonstrate here the result, in particular $F_A = \frac{1}{2}F_S^{Ret} - \frac{1}{2}F_S^{Adv}$, for non-relativistic motion and high-density absorber medium.

Consider a source particle S, with charge e and mass m, in such a medium. $A(\vec{r}_k)$ is a typical absorber particle with charge e_k and mass m_k. We will evaluate the field[3] at $P(\vec{r})$, a point in close vicinity of S. We consider P lying in a cavity of radius R, centered at S, with no absorber particles inside it (see Figure 1). A is considered to be much far from S as compared to the cavity size (i.e. $r_k >> R$).

Let $\vec{H}$ be the acceleration of the source particle. Consider Fourier components of $\vec{H}$ as $\vec{H}_\omega = \vec{H}_0 e^{-i\omega t}$. The full-retarded electric field ($\vec{E}$) due to S at A only for $\vec{H}_\omega$ is

[2] As stated in [8], "It is a feature of time symmetric theory that instead of considering the usual initial-value problem, one seeks for a self-cosistent solution of the equations. In a self-consistent solution, the fields of all charged particles are prescribed and it is shown that the physical description is the same as that obtained by considering all the particle-particle interactions of the theory."

[3] The "field" here represents direct particle field and is completely determined by the particle motion. It does not have any extra degrees of freedom.

given by,

$$E = H_0 e^{-i\omega t}\left(-\frac{e}{r_k}\right)\sin(\vec{H},\vec{r}_k)e^{i\omega r_k}\left(\frac{2}{1+n-ik}\right)e^{i\omega(n-ik-1)(r_k-R)} \tag{2.1}$$

The above calculation of E takes into account electric field in the far-field approximation, reflection at the cavity surface and phase lag due to transmission through a medium of refractive index $(n-ik)$[4]. The direction of $\vec{E}$ is as shown in Figure 1.

Acceleration of A due to E is given by,

$$\vec{H}_k = \frac{e_k}{m_k}\vec{E}\,p(\omega) \tag{2.2}$$

where $p(\omega)$[5] is related to the refractive index by,

$$1-(n-ik)^2 = \frac{4\pi N e_k^2}{m_k\omega^2}p(\omega) \tag{2.3}$$

N is the number density[6] of absorber particles. Half-advanced field ($\vec{E}_k$) at P due to A (along the direction of $\vec{H}$) is given by,

$$E_k = -\frac{e_k}{2r_k}H_k\sin(\vec{H},\vec{r}_k)e^{-i\omega[r_k - r\cos(\vec{r},\vec{r}_k)]} \tag{2.4}$$

The net absorber response (E_A) is given by integrating E_k over the entire space i.e. $E_A = \int E_k N r_k^2 dr_k d\Omega$. This, after using all the above expressions, evaluates to the following,

$$E_A = \frac{1}{2}E^{Ret}(r) - \frac{1}{2}E^{Adv}(r) \tag{2.5}$$

where $E^{Ret}(r) = -\frac{e}{r}H_0\sin(\vec{H},\vec{r})e^{-i\omega(t-r)}$ and $E^{Adv}(r) = -\frac{e}{r}H_0\sin(\vec{H},\vec{r})e^{-i\omega(t+r)}$ are retarded and advanced fields due to S at P.

The above result for absorber response (after Fourier sum), when evaluated at S itself, gives the following well established result in electrodynamics for radiation reaction.

$$\vec{E}_A^{[S]} = \frac{2e}{3}\frac{d\vec{H}}{dt} \tag{2.6}$$

The fact that the final result is independent of the details of the absorber medium (e.g. N, $n-ik$) is related to the complete absorption. WF theory shows that the only assumption of complete absorption is sufficient to demonstrate the required result. This is explained below.

[4] Imaginary part of refractive index, k, denotes coefficient of absorption.

[5] $p(\omega)$ denotes frequency dependence of acceleration of absorber particles due to the incident electric field E. This is due to the interactions between the absorber particles, owing to their high-density.

[6] Here we assume uniform density of absorber particles.

Complete absorption implies a test particle placed outside the absorber medium experiences no radiation, i.e.

$$\sum_B \left(\frac{1}{2}F_B^{Ret} + \frac{1}{2}F_B^{Adv}\right)_{\text{[outside]}} = 0 \tag{2.7}$$

where the sum is over all the absorber particles. The individual sums, $\sum_B \left(\frac{1}{2}F_B^{Ret}\right)_{\text{[outside]}}$ and $\sum_B \left(\frac{1}{2}F_B^{Adv}\right)_{\text{[outside]}}$, represent net retarded and advanced waves. Since their sum is always zero, they have to be zero individually along with their difference.

$$\sum_B \left(\frac{1}{2}F_B^{Ret}\right)_{\text{[outside]}} = 0, \quad \sum_B \left(\frac{1}{2}F_B^{Adv}\right)_{\text{[outside]}} = 0 \tag{2.8}$$

$$\sum_B \left(\frac{1}{2}F_B^{Ret} - \frac{1}{2}F_B^{Adv}\right)_{\text{[outside]}} = 0 \tag{2.9}$$

Since $\sum_B \left(\frac{1}{2}F_B^{Ret} - \frac{1}{2}F_B^{Adv}\right)$ is a solution of Maxwell's equations for free space[7], owing to the property of having no extrema, it vanishes inside the absorber medium too. Hence,

$$\sum_B \left(\frac{1}{2}F_B^{Ret} - \frac{1}{2}F_B^{Adv}\right)_{\text{[everywhere]}} = 0 \tag{2.10}$$

The net field at a particle A (inside the absorber) is then given by,

$$\begin{aligned}\sum_{B\neq A} \left(\frac{1}{2}F_B^{Ret} + \frac{1}{2}F_B^{Adv}\right) &= \sum_{B\neq A} \left(F_B^{Ret}\right) + \left(\frac{1}{2}F_A^{Ret} - \frac{1}{2}F_A^{Adv}\right) - \sum_B \left(\frac{1}{2}F_B^{Ret} - \frac{1}{2}F_B^{Adv}\right) \\ &= \sum_{B\neq A} \left(F_B^{Ret}\right) + \left(\frac{1}{2}F_A^{Ret} - \frac{1}{2}F_A^{Adv}\right)\end{aligned}$$

where the first term is the sum of retarded fields due to all particles other than A while the second term (which only depends upon the motion of particle A) represents radiation reaction. Hence we get the desired expression of radiation reaction (or absorber response).

However, the net interactions can also be shown to be advanced (F^{Adv}), in the same self-consistent way[8]. As we will see in Section 4, both retarded and advanced solutions being self-consistent is the property of static and flat universe. It is not the same with other cosmological models.

[7]The sources for individual fields, F_B^{Ret} and F_B^{Adv}, are the same. Hence their difference satisfies Maxwell's equations for free space.

[8]Radiation reaction takes the form $\left(\frac{1}{2}F_A^{Adv} - \frac{1}{2}F_A^{Ret}\right)$ in that case.

3 Action-at-a-distance electrodynamics in cosmological models

As noted at the end of the section 2, WF theory shows that static and flat universe admits both self-consistent retarded and advanced net interactions.[9] Wheeler and Feynman in [18] argue that thermodynamics (or statistical mechanics) breaks this time symmetry. The initial conditions required for net advanced interactions have very low statistical probablity as compared to those for retarded interactions which are hence preferred over the former.

3.1 Hogarth

This argument of taking refuge to thermodynamics to determine electromagnetic time arrow was first criticized by Hogarth in 1962 [5]. Hogarth's argument was that Wheeler and Feynman did not consider time asymmetry due to cosmological structure. In an expanding universe, future and past absorbers (i.e. universe in perfect future or perfect past) have different properties (e.g. number density, refractive index). The retarded and advanced waves undergo redshift and blueshift, respectively. In order to have self-consistent retarded/ advanced solutions, future or past absorbers need to be perfect, respectively. This depends upon the specific cosmological model. Thus he concluded that electromagnetic arrow of time can be fixed by taking cosmological structure into account. In case of static and flat universe, both future and past absorbers are perfect and hence we have both self-consistent retarded and advanced solutions.

Hogarth analyzed various cosmological models for self-consistency of retarded or advanced solutions. Most of the expanding, singular big-bang models (e.g. Einstein-de Sitter model) have advanced while the steady-state model has retarded solution.

However, Hogarth's work had a few gaps or inconsistencies which were identified by Hoyle and Narlikar, as descibed in the next section.

3.2 Hoyle-Narlikar approach

Hogarth's work on WF theory in cosmological space-times had the following two gaps.

1. He used conformal invariance of electromagnetic action to apply the formulation of WF theory in flat space-time to cosmological space-times (which are conformally flat). However, action-at-a-distance electrodynamics has to be formulated in a general curved space-time, in order to establish the validity of these results.

[9] Retarded interactions follow the conventional notion of causality while advanced interactions are contrary to it.

2. The refractive index of the absorber medium was calculated from collisional damping by the absorber particles which has time asymmetry of thermodynamical origin.

This was first pointed out by Hoyle and Narlikar [8]. They filled both these gaps by developing generalized action-at-a-distance electrodynamics in Riemannian space-times (see(4)) and calculating refractive index from radiation reaction[10]. The results obtained by them were in agreement with those by Hogarth despite following different approach.

For a conformally flat metric $g_{ik} = \Omega^2(x^i)\eta_{ik}$, net electric field (retarded or advanced) at a distance r is given by[11]

$$E = -\frac{eH_0}{r} e^{-i\omega_0(t+\epsilon r)} \sin\theta \left(\frac{2}{1+n-ik}\right)_R .\eta \tag{3.1}$$

where $\eta = e^{-i\epsilon\int(n-ik-1)\omega dz}$, ω = proper frequency, ω_0 = frequency in the conformal coordinates, dz = proper distance, $\epsilon = \mp 1$ for retarded and advanced field, respectively.

For perfect absorption, $E \to 0$ i.e. $\eta \to 0$, as r approaches future or past boundary of the universe. This requires,

$$\epsilon \int k\omega dz \to \infty \tag{3.2}$$

Now, evaluating ω and dz in a proper frame (i.e. locally flat and comoving frame), the above reduces to, (see [15])

$$\epsilon \int k(r,t) \left(\frac{1+\epsilon V(r,t)}{1-\epsilon V(r,t)}\right) dr \to \infty \tag{3.3}$$

where $V(r,t) = \frac{dr}{dt}$.[12] Hence the necessary and sufficient conditions for self-consistency of net retarded and advanced interactions are the divergence of the following integrals.

$$\begin{aligned} I_R &= \int_{\text{future}} k(r,t) \left(\frac{1-V(r,t)}{1+V(r,t)}\right) dr \to -\infty \\ I_A &= \int_{\text{past}} k(r,t) \left(\frac{1+V(r,t)}{1-V(r,t)}\right) dr \to \infty \end{aligned} \tag{3.4}$$

The refractive index can be calculated from radiation reaction as follows. The motion of a charged absorber particle ($x(t)$) incident with electric field ε and frequency

[10] This ensures the conclusions about arrow of time are of purely electrodynamical origin.

[11] The result in (2.1) is valid here in conformal coordinates, due to conformal invariance in electromagnetism.

[12] Here we consider only such cosmological models in which the local motion is radial i.e. $\frac{d\theta}{dt} = \frac{d\phi}{dt} = 0$.

ω is determined by[13],

$$m\ddot{x} = e\varepsilon e^{-i\omega t} - \frac{2e^2\epsilon}{3}\dddot{x} \tag{3.5}$$

This gives the solution for displacement as,

$$x(t) = -\frac{e\varepsilon}{m\omega^2}\left(1 - \frac{2ie^2\epsilon}{3m}\omega\right)^{-1} e^{-i\omega t} \tag{3.6}$$

The refractive index of the medium polarized with charges described by $x(t)$ is given by,

$$(n - ik)^2 \varepsilon e^{-i\omega t} = \varepsilon e^{-i\omega t} + 4\pi N e x \tag{3.7}$$

Hence in the limit $\omega \to 0$[14],

$$1 - (n - ik)^2 = \frac{4\pi N e^2}{m\omega^2}\left[1 - \frac{2ie^2}{3m}\omega + O(\omega^2)\right] \tag{3.8}$$

The ω-dependence of k[15] in this asymptotic limit is then given by,

$$k \sim -\frac{N(\omega)}{\omega} \tag{3.9}$$

Although the classical theory is not valid in the limit $\omega \to \infty$[16], under some approximations [8], the refractive index is given by,

$$1 - (n - ik)^2 = \frac{4\pi N e^2}{m\omega^2}\left[1 + O\left(\frac{1}{\omega}\right)\right] \tag{3.10}$$

The ω-dependence of k[17] in this limit is then as follows.

$$k \sim \left[\frac{N(\omega)}{\omega^2}\right]^{1/2} \tag{3.11}$$

In the following sections we demonstrate the results for self-consistency of retarded/advanced interactions in various cosmological models by evaluating (3.4) using (3.9, 3.11).

[13] $\epsilon = \mp 1$for retarded and advanced wave, respectively.

[14] This limit applies for the case of future infinity (i.e. for retarded waves), in an expanding universe.

[15] For $N(\omega) \to 0$ as $\omega \to 0$,$n \to 1$

[16] This limit applies for the case of past infinity i.e. for advanced waves, in an expanding universe.

[17] For $N(\omega) \to \infty$ as $\omega \to \infty$,$n \to 0$

3.2.1 de Sitter and Einstein-de Sitter model

de Sitter (dS) and Einstein-de Sitter (EdS) space-times are described by conformally flat metrics $g_{ik} = \Omega^2(t)\ \eta_{ik}$ with $\Omega^{dS}(t) = (-Ht)^{-1}$[18] and $\Omega^{EdS}(t) = Ct^2$[19]. These are cosmological solutions to Einstein's field equations for a universe with no spatial curvature (i.e. FRW metric with $k = 0$) with cosmological constant and matter domination, respectively.

In any FRW universe[20],

$$\omega \propto S(\tau)^{-1}, \quad N \propto S(\tau)^{-3} \quad \Rightarrow\ N \propto \omega^3 \tag{3.12}$$

Hence, using (3.9) and (3.11)

$$k \propto \begin{cases} -\omega^2 & \text{for retarded waves} \\ \omega^{1/2} & \text{for advanced waves} \end{cases} \tag{3.13}$$

For models with no spatial curvature (i.e. $k = 0$), $V(r,t) = \frac{dr}{dt} = 0$ and hence the integrals I_R and I_A in (3.4) become,

$$\begin{aligned} I_R &= \int_{\text{future}} k(r)dr \to -\infty \\ I_A &= \int_{\text{past}} k(r)dr \to \infty \end{aligned}$$

These integrals are evaluated below for de Sitter and Einstein-de Sitter space-times[21].

$$\begin{aligned} \omega_R(r) &= \omega_0 \left(\Omega\left(t_0 + r\right)\right)^{-1} \\ \omega_R^{dS}(r) &= \omega_0 \left(1 - Hr\right) \\ \omega_R^{EdS}(r) &= \omega_0 \left(1 + Dr\right)^{-2} \end{aligned}$$

$$I_R^{dS} \sim -\int_0^{H^{-1}} \left(1 - Hr\right)^2 dr, \quad I_R^{EdS} \sim -\int_0^{\infty} \frac{dr}{\left(1 + Dr\right)^4}$$

$$\begin{aligned} \omega_A(r) &= \omega_0 \left(\Omega\left(t_0 - r\right)\right)^{-1} \\ \omega_A^{dS}(r) &= \omega_0 \left(1 + Hr\right) \\ \omega_A^{EdS}(r) &= \omega_0 \left(1 - Dr\right)^{-2} \end{aligned}$$

[18] H =positive constant and $-\infty < t < 0$

[19] C =positive constant and $0 < t < \infty$

[20] $ds^2 = d\tau^2 - S(\tau)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right)$

[21] D is a positive constant ($D = \sqrt{C}$).

$$I_A^{dS} \sim \int_0^{\infty} \sqrt{(1+Hr)}dr, \quad I_A^{EdS} \sim \int_0^{D^{-1}} \frac{dr}{(1-Dr)}$$

We can see from the above expressions that I_R^{dS} and I_R^{EdS} converge while I_A^{dS} and I_A^{EdS} diverge to ∞. This implies that de Sitter and Einstein-de Sitter models have perfect past absorption (but imperfect future absorption) and hence self-consistent net advanced interactions (but not net retarded ones).

3.2.2 Steady-state model

Steady-state cosmological model (SS) [1, 6] is characterized by $N =$ constant and $\Omega(t) = (-Ht)^{-1}$[22]. The asymptotic form of k is then given by[23],

$$k \propto \begin{cases} -\omega^{-1} & \text{for retarded waves} \\ \omega^{-3} & \text{for advanced waves} \end{cases} \tag{3.14}$$

$$\begin{aligned} \omega_R^{SS}(r) &= \omega_0 (1 - Hr) \\ \omega_A^{SS}(r) &= \omega_0 (1 + Hr) \end{aligned}$$

$$I_R^{SS} \sim -\int_0^{H^{-1}} \frac{dr}{(1-Hr)}, \quad I_A^{SS} \sim \int_0^{\infty} \frac{dr}{(1+Hr)^3} \tag{3.15}$$

It can be easily verified that I_R^{SS} diverges to $-\infty$ while I_A^{SS} converges to a finite value. Hence steady-state model has self-consistent retarded but not advanced interactions.

3.2.3 Quasi-steady-state cosmology

Quasi-steady-state cosmology (QSSC), based on steady-state cosmology, is an alternative model to the standard cosmology. It was proposed by Hoyle, Burbidge and Narlikar in 1993 [7]. This model can be described by the scale factor $S(\tau)$ expanding exponentially over a large time scale (P) and oscillating over a much shorter time scale (Q). It can be approximated as [11, 17],

$$S(\tau) = e^{\frac{\tau}{P}} \left[1 + \eta \cos\left(\frac{2\pi\tau}{Q}\right) \right] \tag{3.16}$$

where $P >> Q$ and η is a constant with $0 < \eta < 1$. The number density (N) of particles oscillates between N_{min} and N_{max} during a cycle of time period Q but the

[22] $H =$ positive constant and $-\infty < t < 0$

[23] Asymptotic form of k in case of $N =$ constant is obtained from (3.8) and (3.10) in a different way than (3.9) and (3.11). See [8] for the same.

average density remains constant due to creation of matter at particular time epochs, periodically.

We look for the divergence of the integrals I_R^{QSSC} and I_A^{QSSC} to check the self-consistency of retarded and advenced interactions.

$$e^{\frac{\tau}{P}}\left(1-\eta\right)\leq S(\tau)\leq e^{\frac{\tau}{P}}\left(1+\eta\right)$$

$t=\int\frac{d\tau}{S(\tau)}$, conformal time, is obtained as follows.

$$\int d\tau\frac{e^{-\tau/P}}{(1+\eta)}\leq t\leq\int d\tau\frac{e^{-\tau/P}}{(1-\eta)}$$

$$-P\frac{e^{-\tau/P}}{(1-\eta)}\leq t\leq -P\frac{e^{-\tau/P}}{(1+\eta)}$$

$$\frac{(1-\eta)}{P}e^{\tau/P}\leq -\frac{1}{t}\leq\frac{(1+\eta)}{P}e^{\tau/P}$$

The conformal factor $\Omega(t)=S(\tau)$ follows the following inequalities.

$$e^{\tau/P}\left(1-\eta\right)\leq\Omega(t)\leq e^{\tau/P}\left(1+\eta\right)$$

$$-\frac{P}{t}\left(\frac{1-\eta}{1+\eta}\right)\leq\Omega(t)\leq -\frac{P}{t}\left(\frac{1+\eta}{1-\eta}\right)$$

The proper frequency for a retarded wave is given by, $\omega_R=\omega_0\left(\Omega\left(t_0+r\right)\right)^{-1}$, which then follows,

$$-\omega_0\frac{(t_0+r)}{P}\left(\frac{1-\eta}{1+\eta}\right)\leq\omega_R\leq -\omega_0\frac{(t_0+r)}{P}\left(\frac{1+\eta}{1-\eta}\right)$$

As the number density is almost constant (and oscillates between N_{min} and N_{max}), the absorption coefficient depends upon ω as $k\propto -\frac{\sqrt{N}}{\omega}$.

$$\frac{\sqrt{N_{max}}}{\omega_0}P\left(\frac{1+\eta}{1-\eta}\right)\frac{1}{(t_0+r)}\leq k\leq\frac{\sqrt{N_{min}}}{\omega_0}P\left(\frac{1-\eta}{1+\eta}\right)\frac{1}{(t_0+r)}$$

$$-\frac{C_2}{(C_0-r)}\leq k\leq -\frac{C_1}{(C_0-r)}$$

C_0, C_1 and C_2 are positive constants[24]. The integral $I_R^{QSSC}=\int_0^{C_0}kdr$ then satisfies,

$$-\int_0^{C_0}\frac{C_2}{(C_0-r)}dr\leq I_R^{QSSC}\leq -\int_0^{C_0}\frac{C_1}{(C_0-r)}dr$$

Since I_R^{QSSC} is sandwiched between two integrals both of which diverge to $-\infty$, $I_R^{QSSC}=-\infty$. Hence QSSC admits self-consistent retarded interactions. A similar analysis shows that it does not admit self-consistent advanced interactions.

[24] $C_0=-t_0$, $C_1=\frac{\sqrt{N_{min}}}{\omega_0}P\left(\frac{1-\eta}{1+\eta}\right)$, $C_2=\frac{\sqrt{N_{max}}}{\omega_0}P\left(\frac{1+\eta}{1-\eta}\right)$

3.2.4 FRW models with $k = \pm 1$

In this section we will check the self-consistency of retarded/advanced interactions in FRW models with $k = \pm 1$. The line element in these models is given by $ds^2 = d\tau^2 - S^2(\tau)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$. This line element can be written in conformally flat form by the following coordinate transformations.

$$R = \begin{cases} \sin^{-1} r & k = 1 \\ \sinh^{-1} r & k = -1 \end{cases}, \quad T = \int_{\tau_0}^{\tau} \frac{d\tau'}{S(\tau')} \tag{3.17}$$

$$\xi = \tfrac{1}{2}(T + R), \quad \eta = \tfrac{1}{2}(T - R)$$

$$X = \begin{cases} \tan\xi & k = 1 \\ \tanh\xi & k = -1 \end{cases}, \quad Y = \begin{cases} \tan\eta & k = 1 \\ \tanh\eta & k = -1 \end{cases}$$

$$\rho = \tfrac{1}{2}(X - Y), \quad t = \tfrac{1}{2}(X + Y)$$

This leads to the following form of line element,

$$ds^2 = \Lambda^2(t,\rho)(dt^2 - d\rho^2 - \rho^2 d\Omega^2)$$

where the conformal factor is given by[25],

$$\Lambda^2_{\pm}(t,\rho) = \frac{4\Sigma^2}{(1 \pm X^2)(1 \pm Y^2)} \tag{3.18}$$

where $\Sigma(T) = S(\tau)$.

Now, $V_{\pm}(t,\rho) = \frac{d\rho}{dt}$ is obtained in this case as follows.

$$V_{\pm}(t,\rho) = \frac{2\rho t}{\rho^2 + t^2 \pm 1} \tag{3.19}$$

The scale factor Σ is obtained by solving Friedman equation $\dot{S}^2 = \mp 1 + \frac{B}{S}$, for $k = \pm 1$ and $B =$ constant.

$$\Sigma_+ = \frac{2\,(X+Y)^2}{(1+X^2)\,(1+Y^2)} \tag{3.20}$$

$$\Sigma_- = \frac{1 + XY}{\sqrt{(1-X^2)\,(1-Y^2)}} - 1 \tag{3.21}$$

[25]The subscript $\pm$ denotes the quantities for $k = \pm 1$, respectively.

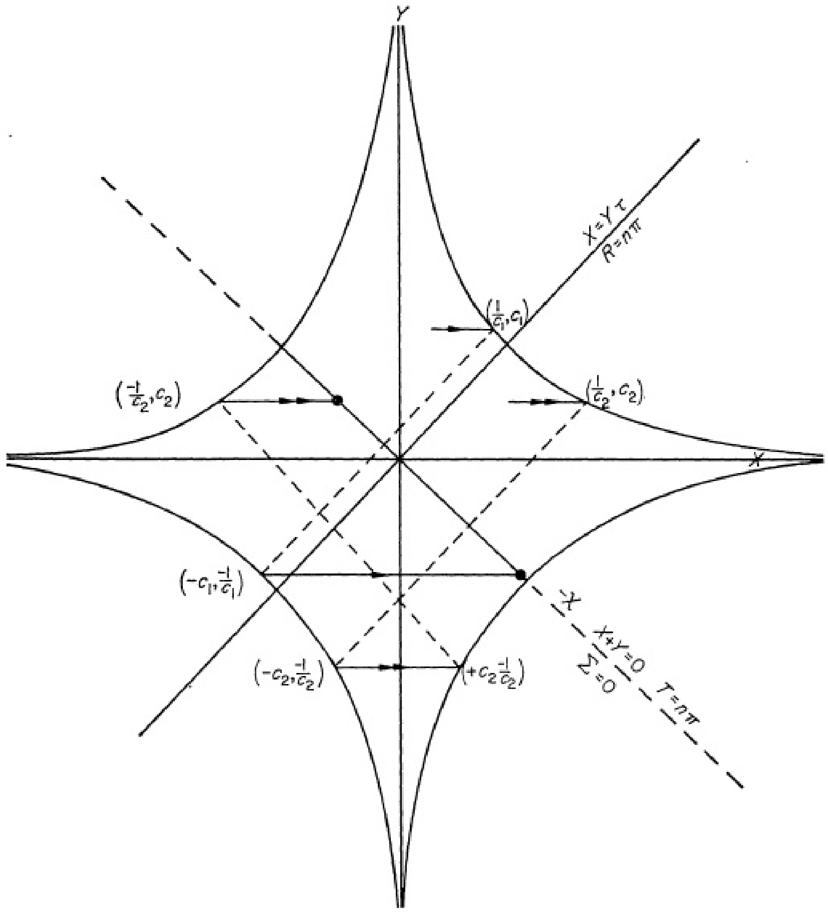

Figure 2: FRW universe with $k = +1$: $XY = \pm 1$ are bounds of the universe. Points (X, Y) and $(-\frac{1}{X}, -\frac{1}{Y})$ can be identified with each other. The line $X + Y = 0$ denotes the central singularity. Retarded null paths have Y = constant. (This figure is obtained from [15].)

Figure 2 shows various aspects of FRW universe with $k = 1$ in (X, Y) space. Retarded null paths have Y = constant while the advanced ones have X = constant.[26] Since the universe is closed, both retarded and advanced null paths finally reach the singularity. This can be observed in Figure 2. Hence, in the evaluation of both I_R and I_A, as $\omega \propto \Sigma^{-1}$, we need to take the absorption coefficient k in the asymptotic limit $\omega \to \infty$.

From (3.11) and (3.12), $k \propto \sqrt{\omega} \propto \Sigma^{-1/2}$ for both I_R and I_A.

$$I_R^+ = \int_{\text{future}} k \left(\frac{1-V}{1+V}\right) d\rho \propto -\int_{\text{future}} \frac{dX}{\sqrt{\Sigma_+}\,(1+X^2)} \tag{3.22}$$

[26] This is also valid for $k = -1$.

$$I_A^+ = \int_{\text{past}} k\left(\frac{1+V}{1-V}\right) d\rho \propto \int_{\text{past}} \frac{dY}{\sqrt{\Sigma_+}\,(1+Y^2)} \tag{3.23}$$

Using the expression for Σ_+ from (3.20), both the above integrals can be evaluated and found to be diverging to $-\infty$ and ∞, respectively [15]. This demonstrates that both retarded and advanced interactions are self-consistent in FRW model with $k = +1$.

In the case of $k = -1$, the asymptotic form of k is different as compared to $k = 1$[27].

$$k \propto \begin{cases} -\omega^2 & \text{retarded wave } (\omega \to 0) \\ \sqrt{\omega} & \text{advanced wave } (\omega \to \infty) \end{cases} \tag{3.24}$$

$$\begin{aligned} I_R^- &\propto -\int_{\text{future}} \frac{dX}{\Sigma_-^2\,(1-X^2)} \\ I_A^- &\propto \int_{\text{past}} \frac{dY}{\sqrt{\Sigma_-}\,(1-Y^2)} \end{aligned}$$

Using the expression of Σ_- from (3.21) and appropriate limits in the integrals, it can be seen that I_R^- converges to a finite value while I_A^- diverges to ∞ [15]. This shows that FRW model with $k = -1$ admits only self-consistent advanced solution.

3.3 A little about quantum aspects

We would like to mention here briefly that the action-at-a-distance theory presented and used in this entire work is restricted to classical physics. No quantum effects have been considered in any part. The calculations using asymptotic form of refractive index in the limit $\omega \to \infty$ (3.10) are not as soundly based as those in the limit $\omega \to 0$ due to the importance of quantum effects in the former case.

The quantum generalization of field theoretic approach to electrodynamics, quantum electrodynamics (QED), has been widely in agreement. However a completely quantum theory of action-at-a-distance is developed by Hoyle and Narlikar [10] explaining all the phenomena of QED.

4 Action-at-a-distance in Riemannian space-times: Non-local aspects

As noted in the section 3.2, action-at-a-distance electrodynamics has to be generalized to Riemannian space-times before applying it to cosmological space-times using conformal invariance. This requires curved space generalization of Fokker action

[27] It being an open universe, retarded waves travel to ∞ and hence the future boundary corresponds to $\omega \to 0$ limit.

(stated below) and the corresponding wave equation satisfied by the vector potential.

$$S = -\sum_a \int m_a da - \sum_{a<b} \int e_a e_b \delta(s_{AB}^2) \eta_{ik} da^i db^k \tag{4.1}$$

$$\Box A_i = 4\pi J_i \tag{4.2}$$

a and b denote different particles and J_i is the source current density. This generalization can only be done, keeping in mind non-localness of the problem, with the help of two-point functions such as parallel propagators and Green's functions. The generalized Fokker action can be written as,

$$S = -\sum_a \int m_a da - \sum_{a<b} \int e_a e_b \bar{G}_{i_A i_B} da^{i_A} db^{k_B} \tag{4.3}$$

where $\bar{G}_{i_A i_B}$ are Green's functions of the corresponding vector wave equation in curved space-time. We describe the mathematical formulation of parallel propagators and Green's functions [9] in the following two sections.

4.1 Parallel propagators

Let X^{i_A}be a 4-vector at a point A.[28] Suppose A can be connected with all other points B along a unique geodesic Γ_{AB}. If we parallel propagate X^{i_A}along Γ_{AB} to B, it transforms to X^{i_B}, which is a 4-vector at B. Since parallel propagation is a linear operation, we can write the following.

$$X^{i_B} = \bar{g}^{i_B}_{\ i_A} X^{i_A} \tag{4.4}$$

The functions $\bar{g}^{i_B}_{\ i_A}$are defined as parallel propagators, as they parallel propagate a vector from A to B. These are called as bivectors or two-point vectors as they transform as vectors at two points. The indices can be raised or lowered using the metric at the corresponding point thus giving the functions $\bar{g}^{i_B i_A}$, $\bar{g}_{i_B i_A}$, $\bar{g}_{i_B}^{\ \ i_A}$. These quantities can be used to parallel propagate a vector, with suitable contravariant or covariant indices, from A to B. It can also be shown that $\bar{g}^{i_B}_{\ i_A}$and $\bar{g}_{i_B}^{\ \ i_A}$are inverses of each other.

$$\bar{g}^{i_B}_{\ i_A} \bar{g}_{j_B}^{\ \ i_A} = \delta^{i_B}_{j_B}, \qquad \bar{g}^{i_B}_{\ j_A} \bar{g}_{i_B}^{\ \ i_A} = \delta^{i_A}_{j_A} \tag{4.5}$$

Also since there is symmetry with respect to A and B, we have the following symmetries for the parallel propagators (which can also be demonstrated by simple manipulations using properties listed above).

$$\bar{g}_{i_B i_A} = \bar{g}_{i_A i_B}, \qquad \bar{g}^{i_B i_A} = \bar{g}^{i_A i_B} \tag{4.6}$$

[28]Since we are dealing with non-local problem, it becomes essential to specify the location of a vector. $X^{i_A}$is a 4-vector only at A.

4.2 Green's functions for wave equations

Green's functions provide the most general way of solving a differential equation. In order to understand how they work, consider the following general equation

$$\Theta_X F(x) = B(x) \tag{4.7}$$

where Θ_X is an operator while $F(x)$ and $B(x)$ are functions of x. Green's function $G(x, x')$ for Θ_X is defined by,

$$\Theta_X G(x, x') = \delta(x - x') \tag{4.8}$$

where $\delta(x - x')$ is Dirac delta function. The solution for $F(x)$ is then given by,

$$F(x) = \int_{-\infty}^{\infty} G(x, x')B(x')dx' \tag{4.9}$$

This can be easily verified by substituting the above solution in (4.7).

Now, the scalar wave equation in curved space time is given by,

$$\Box_X \phi(X) = \psi(X) \tag{4.10}$$

where $\Box\phi = \phi^{;i}_{;i}$ with $_{;i}$ denoting covariant derivative. The corresponding scalar Green's function $\bar{G}(X, X')$ and solution for $\phi(X)$ is then given by,

$$\begin{gathered} \Box_X \bar{G}(X, X') = \tfrac{1}{\sqrt{-\bar{g}(X,X')}}\delta_4(X, X') \\ \phi(X) = \int \bar{G}(X, X')\psi(X')\sqrt{-g(X')}d^4x' \end{gathered} \tag{4.11}$$

The vector wave equation is given by $F^{;k}_{ki} = 4\pi J_i$, where $F_{ki} = A_{i;k} - A_{k;i}$. After some manipulations and under Lorenz gauge (i.e. $A^{;i}_i = 0$),

$$4\pi J_i = (A_i)^{;k}_{;k} - (A_{k;i})^{;k} = \Box A_i - \left(A^{;k}_k\right)_{;i} + R_i^{\ k} A_k \tag{4.12}$$

this reduces to the following form.

$$\Box A_i + R_i^{\ k} A_k = 4\pi J_i \tag{4.13}$$

The equation satisfied by vector Green's function $\bar{G}_{i_X i_{X'}}(x, x')$ and solution for $A_{i_X}(x)$ is given by,

$$\begin{gathered} \Box_X \bar{G}_{i_X i_{X'}} + R^{\ k_X}_{i_X} \bar{G}_{k_X i_{X'}} = \tfrac{1}{\sqrt{-\bar{g}(X,X')}}\bar{g}_{i_X i_{X'}}\delta_4(X, X') \\ A_{i_X} = \int 4\pi \bar{G}_{i_X i_{X'}} J^{i_{X'}}\sqrt{-g(X')}d^4x' \end{gathered} \tag{4.14}$$

J_{i_X} produced by single particle with trajectory b^{i_B} and charge e_b is given by,

$$J_{i_X}(X) = e_b \int \bar{g}_{i_X i_B} \frac{\delta_4(X, B)}{\sqrt{-\bar{g}(X, B)}} db^{i_B} \tag{4.15}$$

This gives the following solution for A_{i_X} due to a single charge B.

$$A_{i_X} = 4\pi e_b \int \bar{G}_{i_X i_B} db^{i_B} \tag{4.16}$$

The electromagnetic part of generalized Fokker action can hence be written as,

$$S = -\sum_{a<b} \int e_a e_b \bar{G}_{i_A i_B} da^{i_A} db^{k_B} \tag{4.17}$$

which denotes the sum of interactions between two different particles.

Scalar and vector Green's functions are related by the following expression (which is proved in the Appendix A).

$$\left(\bar{G}_{i_X i_{X'}}\right)^{;i_X} = -\bar{G}_{;i_{X'}} \tag{4.18}$$

$\bar{G}(X, X')$ and $\bar{G}_{i_X i_{X'}}$are chosen to be the solutions of (4.11) and (4.14) which are symmetric with respect to X and X'.

$$\bar{G}(X, X') = \bar{G}(X', X), \qquad \bar{G}_{i_X i_{X'}} = \bar{G}_{i_{X'} i_X} \tag{4.19}$$

These symmetric Green's functions can be split into retarded and advanced parts. $G^{Ret}(X, X')$ and $G^{Adv}(X, X')$ are nonzero only when X is in the causal future and past of X', respectively.

$$\bar{G}(X, X') = \frac{1}{2} G^{Ret}(X, X') + \frac{1}{2} G^{Adv}(X, X') \tag{4.20}$$

Similar relations hold for vector Green's functions. G^{Ret} and G^{Adv} satisfy all the properties of $\bar{G}$ except the symmetry with respect to X and X'. The solutions obtained for scalar and vector wave equations using G^{Ret} or G^{Adv} give retarded and advanced solutions, respectively.

4.3 Evaluation of parallel propagators in conformally flat space-times

In this section, we evaluate parallel propagators for conformally flat space-times, which will be used further. Consider a metric of the form,

$$ds^2 = d\tau^2 - S^2(\tau)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) \tag{4.21}$$

Parallel propagation of a vector V^i along a path $x^i(\lambda)$ is given by,

$$0 = \frac{D}{d\lambda} V^i = \frac{dV^i}{d\lambda} + \Gamma^i_{jk}(\lambda) \frac{dx^j}{d\lambda} V^k \tag{4.22}$$

A null radial geodesic is given by, (for $\lambda = \tau$)

$$\begin{aligned}
\tau(\tau) &= \tau \\
r(\tau) &= r_1 + \int_{\tau 1}^{\tau} \frac{d\tau'}{S(\tau')} \\
\theta,\ \phi &= \text{constant}
\end{aligned}$$

Christofell symbols for the above metric are given by,

$$\Gamma^{\tau}_{rr} = S\dot{S}, \quad \Gamma^{\tau}_{\theta\theta} = r^2 S\dot{S}, \quad \Gamma^{\tau}_{\phi\phi} = r^2 \sin^2\theta S\dot{S}$$

$$\Gamma^{r}_{\theta\theta} = -r, \quad \Gamma^{r}_{\phi\phi} = -r\sin^2\theta, \quad \Gamma^{r}_{\tau r} = \tfrac{\dot{S}}{S}$$

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^{\theta}_{\tau\theta} = \tfrac{\dot{S}}{S}, \quad \Gamma^{\theta}_{r\theta} = \tfrac{1}{r}$$

$$\Gamma^{\phi}_{\theta\phi} = \cot\theta, \quad \Gamma^{\phi}_{\tau\phi} = \tfrac{\dot{S}}{S}, \quad \Gamma^{\phi}_{r\phi} = \tfrac{1}{r}$$

The differential equations satisfied by V^i are given by,

$$\tfrac{dV^\tau}{d\tau} = -\dot{S}V^r, \quad \tfrac{dV^r}{d\tau} = -\tfrac{\dot{S}}{S}V^r - \tfrac{\dot{S}}{S^2}V^t$$

$$\tfrac{dV^\theta}{d\tau} = -\left[\tfrac{\dot{S}}{S} + \tfrac{1}{rS}\right]V^\theta, \quad \tfrac{dV^\phi}{d\tau} = -\left[\tfrac{\dot{S}}{S} + \tfrac{1}{rS}\right]V^\phi$$

The parallel propagators, which are defined as $V^i = V^{j_1}\bar{g}^i_{j_1}$, can be obtained by solving the above equations.

$$\bar{g}_{\tau\tau} = \frac{1}{2}\left(\frac{S_1}{S} + \frac{S}{S_1}\right),\ \bar{g}_{\tau r} = \frac{S_1}{2}\left(\frac{S_1}{S} - \frac{S}{S_1}\right), \tag{4.23}$$

$$\bar{g}_{r\tau} = -\tfrac{S}{2}\left(\tfrac{S_1}{S} - \tfrac{S}{S_1}\right),\ \bar{g}_{rr} = -\tfrac{SS_1}{2}\left(\tfrac{S_1}{S} + \tfrac{S}{S_1}\right),$$

$$\bar{g}_{\theta\theta} = -rr_1SS_1,\ \bar{g}_{\phi\phi} = -rr_1SS_1\sin^2\theta$$

Transforming $\bar{g}_{ij}$ (which are bivectors) to conformal coordinates, i.e. for which the line element is $ds^2 = \Omega^2(t)(dt^2 - dx^2 - dy^2 - dz^2)$, gives

$$\begin{aligned}
\bar{g}_{tt} &= \frac{1}{2}(\Omega_1^2 + \Omega^2) \\
\bar{g}_{t\mu} &= \frac{1}{2}(\Omega_1^2 - \Omega^2)\frac{r_\mu}{r}, \quad \bar{g}_{\mu t} = \frac{-1}{2}(\Omega_1^2 - \Omega^2)\frac{r_\mu}{r} \\
\bar{g}_{\mu\nu} &= -\frac{r_\mu r_\nu}{2r^2}(\Omega_1 - \Omega)^2 + \Omega_1\Omega\eta_{\mu\nu}
\end{aligned}$$

e.g. For de Sitter space-time, i.e. $\Omega(t) = (-Ht)^{-1}$, the parallel propagators connecting $A(a, \vec{0})$ to $X(t, \vec{r})$ are given by,

$$\begin{aligned}
\bar{g}_{tt} &= \frac{1}{2H^2}\left(\frac{1}{t^2} + \frac{1}{a^2}\right), \\
\bar{g}_{t\mu} &= \frac{1}{2H^2}\left(\frac{t+a}{t^2+a^2}\right) r_\mu, \ \bar{g}_{\mu t} = \frac{-1}{2H^2}\left(\frac{t+a}{t^2+a^2}\right) r_\mu, \\
\bar{g}_{\mu\nu} &= -\frac{1}{2H^2}\frac{1}{t^2a^2} r_\mu r_\nu + \frac{1}{H^2 a t}\eta_{\mu\nu}
\end{aligned} \tag{4.24}$$

The above result for parallel propagators in de Sitter space-time is in agreement with [12].

4.3.1 Equivalence between cosmological and Doppler shifts

This subsection demonstrates a result using the non-local properties of parallel propagators. Cosmological redshift is the shift in wavelength of a wave transmitted from space-time point B to A. This effect, due to the curvature of space-time, is given by the following expression (for FRW metrics with $k = 0, \pm 1$),

$$\frac{\lambda_A}{\lambda_B} \equiv 1 + z = \frac{S_A}{S_B} \tag{4.25}$$

where "z" is positive for S being increasing function of time. i.e. There is redshift for waves going forward in time in an expanding universe.

Doppler effect gives shift in the wavelength due to relative velocity between the two frames at A and B. This is given by,

$$\frac{\lambda_A}{\lambda_B} \equiv 1 + z = \sqrt{\frac{1+v}{1-v}} \tag{4.26}$$

where v = relative spatial velocity between A and B.

Here we demonstrate that cosmological redshift is equivalent to the Doppler redshift observed due to the apparent relative velocity between two space-time points [13]. We demonstrate this result for FRW metric with $k = 0$. The apparent relative velocity can be obtained by parallel propagating 4-velocity vector between the two points.

Consider an observer "1", at rest at B, emitting waves which are being observed by another observer "0", also at rest at A. 4-velocity of "1" at B is given by $(v_1^\tau, v_1^r, v_1^\theta, v_1^\phi) = (1, 0, 0, 0)$. Let v_0^i be the 4-velocity obtained by parallel propagating v_1^i from B to A. v_0^i is the 4-velocity that "0" will assign to "1". Using parallel propagators obtained in the previous section (see(4.23)), we get the following.

$$v_0^\tau = \frac{1}{2}\left(\frac{S_1}{S_0} + \frac{S_0}{S_1}\right), \ v_0^r = \frac{1}{2S_0}\left(\frac{S_1}{S_0} + \frac{S_0}{S_1}\right), \ v_0^\theta = v_0^\phi = 0 \tag{4.27}$$

Hence the magnitude of relative velocity (v) and Doppler shift (z), assigned by "0" to "1", are given by,

$$v = \frac{dl}{dt} = g_{rr}\frac{v_0^r}{v_0^\tau} = \frac{S_0^2 - S_1^2}{S_0^2 + S_1^2} \tag{4.28}$$

$$1 + z = \sqrt{\frac{1+v}{1-v}} = \frac{S_0}{S_1} \tag{4.29}$$

This demonstrates that cosmological redshift is equivalent to Doppler redshift due to apparent relative velocity between A and B.

4.4 Evaluation of Green's functions for wave equation

The scalar and vector wave equations in flat space-time are given by,

$$\Box_X \phi(X) = \psi(X), \quad \Box A_i = 4\pi J_i \tag{4.30}$$

The corresponding scalar and vector Green's functions are given by the following. (See Appendix B.)

$$\bar{G}(X, X') = \frac{1}{4\pi}\delta(s_{XX'}^2), \quad \bar{G}_{ij} = \eta_{ij}\bar{G} \tag{4.31}$$

$$;\ s_{XX'}^2 = \eta_{ij}(x^i - x'^i)(x^j - x'^j)$$

Hence the vector potential and Fokker action (electromagnetic part) in flat space-time are given by the following.

$$A_i = e_b \int \delta(s_{XB}^2)\eta_{ik}db^k, \quad S = \sum_{a<b}\int e_a e_b \delta(s_{AB}^2)\eta_{ik}da^i db^k \tag{4.32}$$

4.4.1 Green's functions in conformally flat space-times

Green's functions in curved space-times have the following general structure as shown by DeWitt and Brehme in [3],

$$\begin{aligned}
\bar{G}(X, A) &= \frac{1}{4\pi}\left[\Delta^{1/2}\delta(s_{XA}^2) - \frac{1}{2}v\theta(s_{XA}^2)\right] \\
\bar{G}_{i_X i_A} &= \frac{1}{4\pi}\left[\Delta^{1/2}\bar{g}_{i_X i_A}\delta(s_{XA}^2) - \frac{1}{2}v_{i_X i_A}\theta(s_{XA}^2)\right]
\end{aligned} \tag{4.33}$$

where $\Delta = det\left\|\left(\frac{1}{2}s_{XA}^2\right)_{;i_X i_A}\right\| . [\bar{g}(X, A)]^{-1}$. $v,\ v_{i_X i_A}$ are functions of x^i.

These functions have non-zero coefficients not just for $\delta(s_{XA}^2)$, as is the case for flat space-time, but also for $\theta(s_{XA}^2)$. This results in the propagation of electromagnetic potential, given by (4.14) or (4.16), not just along null cone but also inside the null cone. This gives rise to the so called "tail" in the transmission of electromagnetic signals in curved space-times.

In this section we first describe a general scheme to evaluate Green's functions for conformally flat space-times (i.e. with metric $g_{ik} = \Omega^2(t)\eta_{ik}$) (as followed in [12]). Then we calculate explicitly the Green's functions for two cosmological space-times, de Sitter and Einstein-de Sitter space-times.

The retarded electromagnetic 4-potential produced by a charge A with a trajectory $a^{i_A}(\lambda)$ is given by (see(4.16)),

$$A_{i_X}(x) = 4\pi e \int G^{Ret}_{i_X i_A} da^{i_A} \tag{4.34}$$

Charge creation can be incorporated in direct interparticle action by defining direct particle fields suitably as below.

$$A^{i_X}_{;i_X} = 4\pi e \int \left(G^{Ret\ i_X}_{i_A}\right)_{;i_X} da^{i_A} = 4\pi e \int_{A_0} -G^{Ret}_{;i_A} da^{i_A} = 4\pi e G^{Ret}(X, A_0) \tag{4.35}$$

The above demonstration of violation of Lorenz gauge condition involves the use of (4.18) and particle creation at A_0. The Maxwell-like equations satisfied by this potential are given by,

$$F^{ik}_{;k} - \left(A^j_{;j}\right)^{;i} = -4\pi J^i \tag{4.36}$$

Using the form of the metric, $g_{ik} = \Omega^2(t)\ \eta_{ik}$, we can write the above equation in a better form. Consider the underlined indices $(\underline{i}, \underline{j})$ and $\hat{A}, \hat{F}$ denoting the corresponding vectors/tensors in η_{ik}. A_i can be decomposed as,

$$A_i = \hat{A}_{\underline{i}} + \phi_{;\underline{i}} \tag{4.37}$$

where ϕ is a scalar. Hence the following relations hold.

$$F_{ik} = \hat{F}_{\underline{ik}}, \quad F^{ik}_{;k} = \frac{1}{\Omega^4}\hat{F}^{\underline{ik}}_{;\underline{k}} \tag{4.38}$$

$$A^i_{;i} = \frac{1}{\Omega^2}\hat{A}^{\underline{i}}_{;\underline{i}} + \frac{1}{\Omega^2}\left[\underline{\Box}\phi - \frac{2\Omega_{;\underline{i}}}{\Omega}\left(\hat{A}^{\underline{i}} + \phi^{;\underline{i}}\right)\right] \tag{4.39}$$

ϕ can be chosen such that $A^i_{;i} = \frac{1}{\Omega^2}\hat{A}^{\underline{i}}_{;\underline{i}}$ thus satisfying the following equation.

$$\underline{\Box}\phi - \frac{2\Omega_{;\underline{i}}}{\Omega}\left(\hat{A}^{\underline{i}} + \phi^{;\underline{i}}\right) = 0 \tag{4.40}$$

This can be rewritten as $\Box\phi = \frac{2\Omega_{;\underline{i}}}{\Omega^3}\hat{A}^{\underline{i}}$ and hence is the scalar wave equation. The vector wave equation, as stated below, is obtained by using all the above relations.

$$\underline{\Box}\hat{A}^{\underline{i}} - \frac{2\Omega^{;\underline{i}}}{\Omega}\hat{A}^{\underline{j}}_{;\underline{j}} = 4\pi J^{\underline{i}} \tag{4.41}$$

Considering the charge created at $A_0(a,\vec{0})$ and our coordinate frame $(t,\vec{r})$ being its rest frame, the 4-current density $J^{\underline{i}}$ is given by,

$$J^{\underline{i}} = e\delta_3(r)\theta(t-a)(1,\vec{0}) \tag{4.42}$$

i.e. it has only temporal component (charge density) and zero space component (3-current density). Hence the solution for $\hat{A}^{\underline{i}}$ has the form $\hat{A}^{\underline{i}} = (\chi(t,r),\vec{0})$.[29] Substituting this form of $\hat{A}^{\underline{i}}$ in (4.41), we get

$$\frac{\partial^2\chi}{\partial t^2} - \frac{2\dot{\Omega}}{\Omega}\frac{\partial\chi}{\partial t} - \frac{\partial^2\chi}{\partial r^2} - \frac{2}{r}\frac{\partial\chi}{\partial r} = 4\pi e\delta_3(r)\theta(t-a) \tag{4.43}$$

The above differential equation can be solved, once the form of $\Omega(t)$ is specified, to obtain $\chi(t,r)$. Then $G^{Ret}(X,A_0)$ can be obtained using (4.35) as follows.

$$G^{Ret}(X,A_0) = \frac{1}{4\pi e\Omega^2(t)}\frac{\partial\chi}{\partial t}(t,r) \tag{4.44}$$

$\phi(t,r)$, which satisfies the scalar wave equation (4.40), can be obtained using G^{Ret} as follows.

$$\phi(t,r) = \int G^{Ret}[t,\vec{r};t',\vec{r'}]\,\frac{2\dot{\Omega}(t')}{\Omega^3(t')}\,\chi(t',r')\,\sqrt{-g(t',r')}\,d^3r'dt' \tag{4.45}$$

The complete solution for $A^i(X)$ is then obtained from (4.37) as the following.

$$A^i(t,r) = \left(\left(\chi(t,r) + \frac{\partial\phi}{\partial t}\right), \frac{\partial\phi}{\partial r}, 0, 0\right) \tag{4.46}$$

The retarded potential given by (4.34) for the J^i given by (4.42) is,

$$A_{i_X}(x) = 4\pi e\int_a^{t_f} G^{Ret}_{i_X 0_A}\,da \tag{4.47}$$

which gives the following expression for $G^{Ret}_{i_X 0_A}$.

$$G^{Ret}_{i_X 0_A} = -\frac{1}{4\pi e}\frac{\partial A_{i_X}}{\partial a} \tag{4.48}$$

Purely spacelike components, $G^{Ret}_{\mu_X\nu_A}$, can be obtained using $\left(G^{Ret}_{i_X\mu_A}\right)^{;i_X} = -G^{Ret}_{;\mu_A}$ and substituting the results for G^{Ret} and $G^{Ret}_{i_X 0_A}$ obtained above. Thus we get scalar and vector Green's functions from (4.44) and (4.48).

4.4.2 de Sitter space-time

The parallel propagators for de Sitter space-time are calculated in (4.24). The procedure described in the previous section gives the scalar and vector Green's functions in this space-time, as stated below [12].[30]

[29] The source charge being at rest at the origin, the potential is of electrostatic origin with radial symmetry.

[30] All Green's functions are retarded here. The vector Green's functions have first index corresponding to $X(t,\vec{r})$ while the second to $A_0(a,\vec{0})$.

$$
\begin{aligned}
G[t,\vec{r};a,\vec{0}] &= \frac{H^2}{4\pi}\left[at\frac{\delta(t-a-r)}{r}+\theta(t-a-r)\right] \\
G_{00} &= \frac{1}{4\pi}\left[\frac{t^2+a^2}{2at}\frac{\delta(t-a-r)}{r}-\frac{8at}{3}\left(\frac{3}{D^2}-\frac{4r^2}{D^3}\right)\theta(t-a-r)\right] \\
G_{0\mu} &= \frac{1}{4\pi}\left[\frac{(a+t)r_\mu}{2at}\frac{\delta(t-a-r)}{r}-\frac{8r_\mu t}{3}\left\{\frac{1}{D^2}+\frac{4a(a+t)}{D^3}\right\}\theta(t-a-r)\right] \\
G_{\mu 0} &= \frac{1}{4\pi}\left[-\frac{(a+t)r_\mu}{2at}\frac{\delta(t-a-r)}{r}+\frac{8r_\mu a}{3}\left\{\frac{1}{D^2}+\frac{4t(a+t)}{D^3}\right\}\theta(t-a-r)\right] \qquad (4.49)\\
G_{\mu\nu} &= \frac{1}{4\pi}\left[-\left(\frac{r_\mu r_\nu}{2at}-\eta_{\mu\nu}\right)\frac{\delta(t-a-r)}{r}-\left\{\frac{r_\mu r_\nu}{(a+t)^2}-\eta_{\mu\nu}\right\}f(t,r)\theta(t-a-r)\right]
\end{aligned}
$$

$; D = (t+a)^2 - r^2$

$f(t,r) = \frac{4(a+t)^2}{3D^2}\left[\ln\left(\frac{4at}{D}\right)+\frac{2(a^2+t^2+r^2)}{D}+\frac{12at(a+t)^2}{D^2}-\frac{3(a^2+t^2)}{4at}-1\right]$

4.4.3 $\Omega(t) = Ct^n$

In this section we evaluate scalar Green's functions for conformally flat space-times with $\Omega(t) = Ct^n$ (for integral value of n) by following a different approach [2, 4]. We first define a reduced symmetric Green's function $\bar{g}(X, X')$ by,

$$
\bar{G}(X,X') = \frac{\bar{g}(X,X')}{\Omega(t)\Omega(t')} \qquad (4.50)
$$

Then the equation (4.11) satisfied by $G(X, X')$ gives,

$$
\left(\frac{\partial^2}{\partial t^2}-\nabla^2-\frac{\ddot{\Omega}}{\Omega}\right)\bar{g}(X,X') = \delta(t-t')\delta_3(x-x') \qquad (4.51)
$$

where ∇^2 is Laplacian operator and $\frac{\ddot{\Omega}}{\Omega} = \frac{n(n-1)}{t^2}$ for $\Omega = Ct^n$. Expanding $\bar{g}(X, X')$ in Fourier modes in space,

$$
\bar{g}(X,X') = \frac{1}{(2\pi)^3}\int \tilde{g}(t,t';\vec{k})e^{i\vec{k}.(\vec{x}-\vec{x}')}d^3k \qquad (4.52)
$$

Putting this expression in the equation above and integrating over space gives,

$$
\left(\frac{\partial^2}{\partial t^2}+k^2-\frac{n(n-1)}{t^2}\right)\tilde{g}(t,t';\vec{k}) = \delta(t-t') \qquad (4.53)
$$

Let the retarded reduced Green's function in Fourier space be $\tilde{g}^{Ret}(t,t';\vec{k}) = \theta(t-t')\hat{g}(t,t';k)$. Substituting this in the above equation gives,

$$
\left(\frac{\partial^2}{\partial t^2}+k^2-\frac{n(n-1)}{t^2}\right)\hat{g}(t,t';k) = 0 \qquad (4.54)
$$

along with the boundary conditions,

$$\hat{g}(t=t';k)=0, \quad \frac{d\hat{g}}{dt}(t=t';k)=1 \tag{4.55}$$

The general solution of the above differential equation for $\hat{g}$ (for an integral value of n) is given by,

$$\hat{g}_l(t,t';k)=AktJ_l(k\,|t|)+BktN_l(k\,|t|) \tag{4.56}$$

where J_l and N_l are Bessel and Neumann functions with $n=l+1$ or $n=-l$. The arbitrary constants A, B can be fixed by the boundary conditions stated above.

This problem can also be solved more easily using $\hat{L}_l=\left(-\frac{\partial}{\partial t}+\frac{l+1}{t}\right)$ as ladder operator [2], which gives the solution for higher values of l by the relation $\hat{L}_l\hat{g}_l=\hat{g}_{l+1}$. This is explained in detail in Appendix C. Thus starting from $\hat{g}_0$, the solution for flat space-time, Green's functions for all the higher-l can be computed by successively operating $\hat{L}_l$.

$l=1$ corresponds to both de Sitter ($n=-1$) and Einstein-de Sitter ($n=2$) space-times. The reduced Green's function for these two space-times, as calculated below, is the same.

The general solution for $\hat{g}_0$ (i.e. for no specific boundary conditions) is given by $\hat{g}_0=A(t')\frac{\sin(k\triangle t)}{4\pi k}+B(t')\frac{\cos(k\triangle t)}{4\pi k}$, with $A(t')$ and $B(t')$ arbitrary functions of t'.

$$\begin{aligned}\hat{g}_1 &= \hat{L}_0\hat{g}_0\\ &= \frac{C(t')}{4\pi k}\left[\frac{\sin(k\triangle t)}{t}-\cos(k\triangle t)\right]+\frac{D(t')}{4\pi k}\left[\frac{\cos(k\triangle t)}{t}+\sin(k\triangle t)\right]\end{aligned}$$

Imposing the boundary conditions fixes $C(t')$ and $D(t')$ and gives the following solution for $\hat{g}_1$.

$$\hat{g}_1(t,t';k)=\frac{1}{4\pi k}\left[\left(1+\frac{1}{k^2tt'}\right)\sin(k\triangle t)-\frac{\triangle t}{ktt'}\cos(k\triangle t)\right] \tag{4.57}$$

Evaluating reduced Green's function in position space from (4.52) using the above form of Fourier components gives,

$$g^{Ret}(X,X')=\frac{1}{4\pi}\left[\frac{\delta(t-t'-|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|}+\frac{\theta(t-t'-|\vec{r}-\vec{r}'|)}{tt'}\right] \tag{4.58}$$

The full scalar Green's functions for both de Sitter and Einstein-de Sitter space-times is then obtained from the above common reduced Green's function (see (4.18)).

$$\begin{aligned}G^{Ret}_{dS}[t,\vec{r};a,\vec{0}] &= \frac{H^2}{4\pi}\left[at\frac{\delta(t-a-r)}{r}+\theta(t-a-r)\right]\\ G^{Ret}_{EdS}[t,\vec{r};a,\vec{0}] &= \frac{1}{4\pi C^2}\left[\frac{1}{a^2t^2}\frac{\delta(t-a-r)}{r}+\frac{1}{a^3t^3}\theta(t-a-r)\right]\end{aligned}$$

The scalar Green's function for de Sitter space-time is in agreement with that computed in the previous section (4.49). G^{Ret} can be calculated for other integral values of n by the procedure described above.

The vector Green's functions can be calculated from scalar Green's functions by the procedure described in the section 4.4.1 . However, it becomes extremely difficult in the case of Einstein-de Sitter space-time to get the explicit expressions for the same.

5 Summary and Discussion

This section summarizes the work presented in the previous sections. We studied action-at-a-distance theory, in particular its formulation in electrodynamics by Wheeler and Feynman (WF theory), its application to cosmological models (using conformal invariance) and the generalization to any Riemannian space-time. The first part (i.e. section 3) deals with the investigation of existence of self-consistent retarded and advanced solutions in various cosmological models while the second part (i.e. section 4) has evaluation of Green's functions for wave equation in curved space-times.

5.1 Self consistent retarded/advanced solutions in cosmological models

Model	Future absorber	Past absorber	Net interaction
Static and flat	Perfect	Perfect	Ambiguous
de Sitter	Imperfect	Perfect	Advanced
Einstein-de Sitter	Imperfect	Perfect	Advanced
FRW $k = 1$	Perfect	Perfect	Ambiguous
FRW $k = -1$	Imperfect	Perfect	Advanced
Steady-state	Perfect	Imperfect	Retarded
Quasi-steady-state	Perfect	Imperfect	Retarded

Table 1: Self-consistency of retarded/advanced solutions

The above table summarizes the results of section 3. This analysis provides a tool for limiting viable cosmological models. Since the net interactions are found to be retarded in our universe, only those cosmological models are viable which have self-consistent retarded solution. From the above table, we can see that only steady-state and quasi-steady-state models satisfy this condition while other Friedman models fail. Although this can not be the sufficient criterion to decide for the correct cosmological model, it certainly is a necessary criterion, in action-at-a-distance formulation.

Also, this has interesting implications on the origin of arrow of time. The choice of the direction of time is *ad hoc* in field theory i.e. the retarded solution is chosen arbitrarily over the advanced one. However in action-at-a-distance formulation, origin of time asymmetry can be attributed to the large scale structure of the universe. The universe has such a cosmological stucture that it provides the correct absorber response to produce net retarded interactions, thus fixing the arrow of time. Steady-state and quasi-steady-state models are suitable candidates for such a cosmological structure.

5.2 Green's functions for wave equations

In section 4.4, we evaluated Green's functions for wave equation in conformally flat space-times, in particular for those having conformal factor as $\Omega(t) = Ct^n$. We evaluated explicit expressions for de Sitter, Einstein-de Sitter space-times and also provided a general procedure for all other integral values of n. This shows that the Green's functions have the form (4.33) which has non-zero coefficients for the heaviside theta function ($\theta(s^2_{XA})$) too. Hence the vector potential (A_{i_X}), for a source localized in space and time, has the following general form (using (4.14)),

$$A_{i_X}(x^i) = \Lambda^{(\delta)}_{i_X}\delta(s^2_{XA}) + \Lambda^{(\theta)}_{i_X}\theta(s^2_{XA}) \tag{5.1}$$

The non-zero functions of x^i, $\Lambda^{(\delta)}_{i_X}$ and $\Lambda^{(\theta)}_{i_X}$, show that the propagation of wave is not restricted to null cone but also inside the null cone. The propagation inside the light cone can also be perceived as a wave travelling with speed less than that of light. This is referred to as the "tail" in the electromagnetic signals.

The coefficients $\Lambda^{(\delta)}_{i_X}$ and $\Lambda^{(\theta)}_{i_X}$ can be calculated for specific space-times and also compared (for orders of magnitude with respect to the spatial distance from the source). This will give the extent to which the tail affects a signal. The detectable effects due to such a tail in pulsed radiation from a pulsar are presented by P. E. Roe [16].

Appendix

A Relation between scalar and vector Green's functions

The scalar and vector Green's functions satisfy the following equations.

$$\begin{array}{c}\Box_x \bar{G}(x,x') = \frac{1}{\sqrt{-\bar{g}(x,x')}}\delta_4(x,x') \\ \Box_x \bar{G}_{i_X i_{X'}} + R^{k_X}_{i_X}\bar{G}_{k_X i_{X'}} = \frac{1}{\sqrt{-\bar{g}(x,x')}}\bar{g}_{i_X i_{X'}}\delta_4(x,x')\end{array} \tag{A.1}$$

Now, differentiating both sides of the above equations with appropriate indices,

$$\begin{aligned}\left(\Box_x \bar{G}(x,x')\right)_{;i_{X'}} &= \left(\frac{1}{\sqrt{-\bar{g}(x,x')}}\delta_4(x,x')\right)_{;i_{X'}} \\ \left(\Box_x \bar{G}_{i_X i_{X'}} + R^{k_X}_{i_X}\bar{G}_{k_X i_{X'}}\right)^{;i_X} &= \left(\frac{1}{\sqrt{-\bar{g}(x,x')}}\bar{g}_{i_X i_{X'}}\delta_4(x,x')\right)^{;i_X}\end{aligned}$$

Evaluating right sides of the above equations by integrating over test function f(x),

$$\begin{aligned}\int f(x)\left(\frac{1}{\sqrt{-\bar{g}(x,x')}}\delta_4(x,x')\right)_{;i_{X'}}\sqrt{-g}d^4x &= \int f(x)\left(\delta_4(x,x')\right)_{;i_{X'}} d^4x \\ &= \left(\int f(x)\left(\delta_4(x,x')\right) d^4x\right)_{;i_{X'}} \\ &= f(x')_{;i_{X'}}\end{aligned}$$

$$\begin{aligned}&\int f(x)\left(\frac{1}{\sqrt{-\bar{g}(x,x')}}\bar{g}_{i_X i_{X'}}\delta_4(x,x')\right)^{;i_X}\sqrt{-g}d^4x \\ =\ &\int_V \left[f(x)\frac{1}{\sqrt{-\bar{g}(x,x')}}\bar{g}_{i_X i_{X'}}\delta_4(x,x')\right]^{;i_X}\sqrt{-g}d^4x \\ =\ &-\int f(x)^{;i_X}\frac{1}{\sqrt{-\bar{g}(x,x')}}\bar{g}_{i_X i_{X'}}\delta_4(x,x')\sqrt{-g}d^4x \\ =\ &\left[\oint_{\partial V} f(x)\frac{1}{\sqrt{-\bar{g}(x,x')}}\bar{g}_{i_X i_{X'}}\delta_4(x,x')d\Sigma^{i_X} - f(x')_{;i_{X'}}\right] \\ =\ &-f(x')_{;i_{X'}}\end{aligned}$$

The surface integral term vanishes as the integral is over the boundary at infinity and integrand involves $\delta_4(x,x')$. Thus we get,

$$\left(\Box_x \bar{G}_{i_X i_{X'}} + R^{k_X}_{i_X} \bar{G}_{k_X i_{X'}}\right)^{;i_X} = -\left(\Box_x \bar{G}(x,x')\right)_{;i_{X'}} \tag{A.2}$$

The above equation reduces to the following.

$$\Box_x(-\bar{G}(x,x')_{;i_{X'}}) = \Box_x(\bar{G}_{i_X i_{X'}}{}^{;i_X}) \tag{A.3}$$

This establishes the required relation,

$$\left(\bar{G}_{i_X i_{X'}}\right)^{;i_X} = -\bar{G}_{;i_{X'}} \tag{A.4}$$

B Green's functions in flat space-time

The scalar and vector wave equations in flat space-time are given by,

$$\Box_X \phi(X) = \psi(X), \quad \Box A_i = 4\pi J_i \tag{B.1}$$

and hence scalar and vector Green's functions satisfy,

$$\Box_X \bar{G}(X,X') = \delta_4(X,X'), \quad \bar{G}_{ij} = \eta_{ij}\delta_4(X,X') \tag{B.2}$$

From the above equations, we get $\bar{G}_{ij} = \eta_{ij}\bar{G}$.
Now, solving for $\bar{G}$,

$$\begin{aligned} \Box_X \bar{G}(X,X') &= \delta_4(X,X') \\ \Box_Z \bar{G}(Z) &= \delta_4(Z) \end{aligned}$$

where $z^i = x^i - x'^i$. Taking Fourier transform of $\bar{G}(Z)$ and $\delta_4(Z)$ and equating the integrands,

$$\begin{aligned} \Box_Z \bar{G}(Z) &= \Box_Z \left[\frac{1}{(2\pi)^4}\int d^4k\, \tilde{G}(k)\, e^{-ik^j z_j}\right] \\ \delta_4(Z) &= \frac{1}{(2\pi)^4}\int d^4k\, e^{-ik^j k_j} \end{aligned}$$

we get, $\tilde{G}(k) = -\frac{1}{k^j k_j}$.

$$\begin{aligned} \bar{G}(Z) &= -\frac{1}{(2\pi)^4}\int d^4k \frac{1}{k^j k_j} e^{-ik^j z_j} \\ &= -\frac{1}{(2\pi)^4}\int d^3k e^{-i\vec{k}.\vec{z}} \int_{-\infty}^{\infty} dk_0 \frac{e^{-ik_0 z_0}}{k_0^2 - k^2} \end{aligned}$$

Evaluation of the k_0 integral above using contour integral, gives G^{Ret} and G^{Adv} for $z_0 > 0$ and $z_0 < 0$, respectively.

$$G^{Ret}(X, X') = \frac{\theta(z_0)}{2\pi}\delta(s^2_{XX'}),\ G^{Adv}(X, X') = \frac{\theta(-z_0)}{2\pi}\delta(s^2_{XX'}) \tag{B.3}$$

$$\bar{G}(X, X') = \frac{1}{4\pi}\delta(s^2_{XX'}) \tag{B.4}$$

; $s^2_{XX'} = \eta_{ij}(x^i - x'^i)(x^j - x'^j)$

C Ladder operator for getting scalar Green's function

The Fourier space scalar Green's function for conformally flat space-time with $\Omega(t) = Ct^n$ satisfies the following equation, (see 4.54)

$$\left(\frac{\partial^2}{\partial t^2} + k^2 - \frac{l(l+1)}{t^2}\right)\hat{g}_l(t, t'; k) = 0 \tag{C.1}$$

where $n = l + 1$ or $n = -l$. Define operators $\hat{L}^1_l = \left(-\frac{\partial}{\partial t} + \frac{l+1}{t}\right)$ and $\hat{L}^2_l = \left(\frac{\partial}{\partial t} + \frac{l+1}{t}\right)$. The above equation can then be written as,

$$\hat{L}^2_l\hat{L}^1_l\hat{g}_l = k^2\hat{g}_l \tag{C.2}$$

Now operating both sides of the above equation by $\hat{L}^1_l$ gives,

$$\begin{aligned} k^2\hat{L}^1_l\hat{g}_l &= \hat{L}^1_l\hat{L}^2_l\hat{L}^1_l\hat{g}_l \\ k^2\left(\hat{L}^1_l\hat{g}_l\right) &= \left(\hat{L}^2_l\hat{L}^1_l + \left[\hat{L}^1_l, \hat{L}^2_l\right]\right)\left(\hat{L}^1_l\hat{g}_l\right) \end{aligned}$$

It can be easily checked that the commutator of the two operators is given by $\left[\hat{L}^1_l, \hat{L}^2_l\right] = \frac{2(l+1)}{t^2}$ and also $\hat{L}^2_l\hat{L}^1_l + \frac{2(l+1)}{t^2} = \hat{L}^2_{l+1}\hat{L}^1_{l+1}$. This gives,

$$\hat{L}^2_{l+1}\hat{L}^1_{l+1}\left(\hat{L}^1_l\hat{g}_l\right) = k^2\left(\hat{L}^1_l\hat{g}_l\right) \tag{C.3}$$

and hence we obtain,

$$\hat{L}^1_l\hat{g}_l = \hat{g}_{l+1} \tag{C.4}$$

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