Large D Membrane Paradigm with Generic Four Derivative corrections to Gravity

Aditya Kar

Department of Physics

IISER Pune

Supervisor

Dr. Nabamita Banerjee

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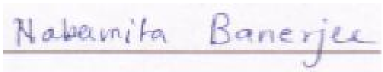
Certificate

This is to certify that this dissertation entitled Large D Membrane Paradigm with Generic Four Derivative corrections to Gravity should appear heretowards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents work carried out by Aditya Kar, under the supervision of Dr.Nabamita Banerjee, Assistant Professor, Department of Physics, during the academic year 2018-2019.



Dr.Nabamita Banerjee (Guide)
Committe:
Dr. Sachin Jain (TAC Member)

Declaration

I hereby declare that the matter embodied in the report entitled Large D Membrane Paradigm with Generic Four Derivative corrections to Gravity are the results of the work carried out by me at the Department of Physics,Indian Institute of Science Education and Research, Pune, under the supervision of Dr.Nabamita Banerjee and the same has not been submitted elsewhere for any other degree.

. Aditya Kar

. BS-MS 20141023

This Masters Thesis is largely based on the paper (**arXiv:1904.08273**) of the author with Dr. Taniya Mandal and Dr. Arunabha Saha, titled **"The large D membrane paradigm for general four derivative theory of gravity with cosmological constant"**

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Abstract

The aim of the Thesis is to find membrane equations dual to the dynamical black holes in large space-time dimensions D, in a theory of Einstein's gravity modified with the most general 4 derivative terms, both with or without a Cosmological constant. We treat the higher derivative term perturbatively and work to linear order in the parameter specifying the 4-derivative terms in the action. Finally we want to compute the stress tensor corresponding to this membrane.

Contents

Chapter 1

Introduction

The Einstein's Equations are a set of non-linear Partial Differential Equations with not many known exact solutions, other than the very symmetric cases of Schwarzschild black-hole, Reissner-Nordstrom black hole etc. Also it is notoriously difficult to construct dynamical solutions with horizon for general theory of relativity. And without matter there is no free parameter in Einstein equation which can be tuned to be small.

But recently, following the ideas proposed by Emparan, Suzuki and Tanabe (2013) a lot of work has been done by Shiraz Minwalla, Sayantani Bhattacharya, Arunabha Saha and others, where Einsteins Equation are solved perturbatively in large D dimensions , where the number of dimensions D acts as the perturbative parameter in the equation, and then solving for the metric in a power series in $\frac{1}{D}$. Since, the large D procedure was so useful in understanding many aspects of two derivative Einstein-Hilbert gravity, it becomes natural to generalise the procedure to higher-derivative gravity.

Recent progress has been made for exact Gauss-Bonnet gravity in flat background, by Arunabha Saha, where a very specific combination of the Gauss-Bonnet coefficients have been used.

As a part of the project, we would like to apply the procedure developed in the aforementioned works,for a generic combination of Gauss-Bonnet coefficients, with the expectation to get some non-triviality due to the generic contribution of each term in the action.

We will work out the membrane equations and the metric of the black hole to first non-trivial order in $\frac{1}{D}$ and linear order in the parameter specifying the 4 derivative term in the action. To be more precise we work with the action:

$$\mathcal{S} = \int d^D x \, \sqrt{-g}(R + \kappa(a_1 R^2 + a_2 R_{\mu\nu} R^{\mu\nu} + a_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma})) \quad ...(1)$$

where, the specific choice $a_1 = 1$,$a_2 = -4$,$a_3 = 1$ correspond to the exact Gauss-Bonnet action.‘

Chapter 2

Motivation

If we look at the expression of the Schwarzschild metric, which is an exact solution to the Einstein's Equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R \ g_{\mu\nu}$$

then we will see that there are two different length scales in Schwarzschild solution in general dimension D. Where, the Schwarzschild solution in D dimensions:

$$ds^2 = -f(r)d\tilde{t}^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{D-2} \text{ where } f(r) = 1 - \left(\frac{r_0}{r}\right)^{D-3}$$

Taking $D \to \infty$ naively :

- for the region $r > r_o$,where r_o is the horizon radius,the metric exponentially decays to flat Minkowski metric. over length scales of order $O(\frac{1}{D})$.
- when $r = r_o\big(1 + \frac{R}{D}\big)$ we have $f(R) = 1 - e^{-R}$ as $D \to \infty$ i.e. we have an exponential fall-off of the metric to flat space metric in this region.

Non-trivial physics is confined in the region of order $O(\frac{1}{D})$ about r_0 which is defined as the **Membrane Region** and this feature of Schwarzschild Metric is a key point for the construction of the Starting Ansatz.

The aim of the project will be to find a more generalised version of such exact solutions, like Schwarzschild solution, which has the presence of a region of order $O(\frac{1}{D})$, wherein lies all the nontrivial physics. However, the solutions will be approximate solution to whichever order, one is willing to perform in powers of $\frac{1}{D}$.

We will also find a set of equations, known as the Membrane equations, which are the constraint equations of General Relativity for our case. Solutions to the membrane equations are in one-one correspondence to solutions to the equations of motion.

Chapter 3

Methodology

3.1 Starting Ansatz

3.1.1 Solution to the Einstein Gauss-Bonnet Equations

In this case, we have to look at solutiuons to the equations of motion following from the action ... (1):

$$
\begin{aligned}
& R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \kappa \bigg(- \frac{g_{\mu\nu}}{2} (a_1 R^2 + a_2 R_{\gamma\delta} R^{\gamma\delta} + a_3 R_{\gamma\delta\alpha\beta} R^{\gamma\delta\alpha\beta}) + 2a_1 R R_{\mu\nu} \\
& -(2a_1 + a_2 + 2a_3) \nabla_\mu \nabla_\nu R + (2a_1 + \frac{1}{2} a_2) \Box R g_{\mu\nu} + (2a_2 + 4a_3) R_{\mu\gamma\nu\delta} R^{\gamma\delta} \\
& +(a_2 + 4a_3) \Box R_{\mu\nu} + 2a_3 R_{\mu\gamma\delta\alpha} R_{\nu}^{\gamma\delta\alpha} - 4a_3 R_{\mu\gamma} R_{\nu}^{\gamma} \bigg) = 0 \\
\implies \quad & E_{\mu\nu} = 0
\end{aligned}
$$

where $E_{\mu\nu}$ is the short-hand for the equations of motion.

Usually such theories are sick in the sense that they have ghost modes in their spectrum about any spacetime. These are modes which can travel at speeds faster than light in that spacetime. But we treat the higher derivative terms perturbatively in the parameter κ. Thus,the terms with explicit 4 derivatives

acting on the metric will act as source terms and they do not change the order of the differential equation that needs to be solved to and the solution perturbatively in and hence we avert this issue.
Now, to construct our Starting Ansatz we need to generalise a Schwarzschild-like i.e spherically symmetric, static , stationary solution of $E_{\mu\nu}$,but perturbatively in κ.
Such a Schwarzschild-like solution in our case for generic Einstein-Gauss-Bonnet equations of motion, turns out to be of the form

$$ds^2 = -f(r)d\tilde{t}^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2$$

$$f(r) = 1 - \left(\frac{r_h}{r}\right)^{D-3}\left(1 + \frac{\tilde{\kappa}a_3}{r_h^2}\right) + \tilde{\kappa}a_3\frac{r_h^{2(D-3)}}{r^{2(D-2)}} + O\big(\tilde{\kappa}^2\big) \quad ...(2)$$

here, $\tilde{\kappa} = (D-3)(D-4)\kappa$

and, r_h is the horizon radius.

3.1.2 The Ansatz

Since we would like to find dynamical solutions to the gravity equation of motion perturbatively in $\frac{1}{D}$, we need to find an ansatz solution which solves for the equations of motion (eom) to leading order in $\frac{1}{D}$. Like in other previous works on the large D membrane black hole duality [1],[2] we will be using the key observation made [1],[2], [3] that the non-trivial dynamics associated with the black hole is contained within a thin **"Membrane Region"** of width $O(\frac{1}{D})$ about the horizon. Everywhere outside this thin region (outside the horizon) the spacetime exponentially goes to its asymptotic background. The algorithm solves for the gravity eom locally in a small region (of size $O(\frac{1}{D})$) about the horizon and then the global solution is obtained by 'smoothly' patching together all the local

solutions. The key ingredient of the algorithm is a starting ansatz solution with horizon on which we implement this procedure.
We choose our starting ansatz for our metric solution motivated by the form of the blackening factor in the metric...(2) as:

$$f(r) = 1 - \frac{1}{\psi^{D-3}}\left(1 + \tilde{\kappa} a_3 \frac{\mathcal{K}^2}{(D-2)^2}\right) + \tilde{\kappa} a_3 \frac{\mathcal{K}^2}{(D-2)^2} \frac{1}{\psi^{2(D-2)}}$$

here, the horizon is at $\psi = 1$, which in the Schwarzschild case was $\psi = \frac{r}{r_h}$ and $\mathcal{K}$ is the trace of the Extrinsic Curvature Tensor of $\psi = constant$ surfaces.

3.1.3 Metric Ansatz in Kerr-Schild co-ordinates

To come up with the form of the starting ansatz we have to see the full metric as a perturbation, added to a background metric.Hence, to do so we shall follow the strategy outlined in the references [1],[2], [3] and write the metric in Kerr-Schild co-ordinates.
The metric written in Kerr-Schild coordinates comes out to be:

$$ds^2 = ds^2_{flat} + \left(\frac{1}{\psi^{D-3}}\left(1 + \tilde{\kappa} a_3 \frac{\mathcal{K}^2}{(D-2)^2}\right) - \tilde{\kappa} a_3 \frac{\mathcal{K}^2}{(D-2)^2} \frac{1}{\psi^{2(D-2)}}\right)(O_\mu dx^\mu)^2 \quad ...(3)$$

where we have defined the unique one-form $O = n - u$ where n is the unit normal to $\psi = 1$ hypersurface embedded in flat background and u is a velocity vector field on the $\psi = 1$ hypersurface, such that,

$$u.u = -1, \; n.n = 1 \text{ and } n.u = 0.$$

Where "." is with respect to the Minkowski metric.
Note: The functions ψ, $\mathcal{K}$ and the one-forms n,u corrresponds to $\psi = \frac{r}{r_h}$, $\mathcal{K} = \frac{D-2}{r_h}$, $n = dr$ and $u = -dt$ for the Schwarzschild case.

3.2 Starting Point of Perturbation Theory

If we take the limit $D \to \infty$ naively , then the number of equations to be solved, which is $\frac{D(D+1)}{2}$, will also go to infinity.To circumvent this problem,we will impose a $SO(D-p-3)$ symmetry on the solution, so that only the metric is effectively dynamical only along the p+3 directions.Then, to take $D \to \infty$ we will keep the finite part,$p+3$ dimensional part of the D dimensions fixed and take the rest of the dimension to infinity,thus we have:

$$D = p + d + 3, \text{ where } d \to \infty \implies D \to \infty.$$

Now we promote the function ψ and the one form u in the metric ..(3) to be arbitrary functions of space-time preserving a $SO(D-p-3)$ symmetry having no D dependence in it.

The key-point is that the metric has a so-called fast direction along $d\psi$ along which the metric exponentially decays to flat space in the large D limit.Thus for $\psi >> 1$ equations of motion are trivially solved as flat space is a trivial solution to EGB equations.

The region $\psi << 1$ is causally disconnected from rest of the space-time and so we will not bother solving for the equations of motion in this region.

All of the non-trivial physics happens in the membrane region (the region of thickness $\frac{1}{D}$ about the horizon) and hence our aim will be to make our ansatz metric solve the equations of motion in this membrane region.

Chapter 4

Auxillary and Regularity conditions on the membrane

4.1 Auxilliary Condition

Even though the shape of the membrane and velocity vector on the membrane compeletely determines the mteric, but to arrive at a local expression of the metric and go for solving the Einstein equations we will need to define the function ψ and u away from the membrane hypersurface.One way to do so is to impose the conditions:

$$n.\nabla n = 0 \text{ and } n.\nabla u = 0.$$

Note, that this choice is not at all unique.
A different choice of the membrane data i.e ψ and u, such that $\psi \to \psi + \frac{1}{D}\delta\psi$ and $u \to u + \frac{1}{D}\delta u$ leaves the leading order ansatz invariant, but will only affect the explicit form of the sub-leading corrections. This ambiguity is fixed by requiring that $\psi = 1$ remains the membrane surface and u is the generator of the hypersurface.

Hence to make the ansatz meaningful, we need to consider a particular choice of ψ and u and define them on the membrane.

4.2 Regularity condition

We assume the following regularity conditions in our procedure for computing the metric.

- Solution is regular everywhere in the membrane region
- Solution is also analytic in D.
- Solution vanishes everywhere outside the membrane region, matching smoothly with flat background.

Chapter 5

Metric Correction and Gauge choice

5.1 Metric Correction

The starting ansatz mentioned in the previous section fails to solve the equations of motion beyond zeroth order in $\frac{1}{D}$ and hence to make it a solution, we have to add corrections in series in $\frac{1}{D}$.
Thus, considering the following most general decomposition of the full metric obeying $SO(D-p-3)$ symmetry, we have:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + e^{\phi(x)}d\Omega_d^2 \quad \mu,\nu = 0,1,...,p+2$$

$$d = D - p - 3$$

where we have:

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(0)}_{\mu\nu} + \sum_{k=1}^{\infty}\left(\frac{1}{D-3}\right)^k \left(\tilde{h}^{(k)}_{\mu\nu} + \tilde{\kappa}a_3\tilde{h}^{(k)}_{\mu\nu,\tilde{\kappa}}\right)$$

with

$$h_{\mu\nu}^{(0)} = \left(\frac{1}{\psi^{D-3}} \left(1 + \tilde{\kappa} a_3 \frac{\mathcal{K}^2}{(D-2)^2} \right) - \tilde{\kappa} a_3 \frac{\mathcal{K}^2}{(D-2)^2} \frac{1}{\psi^{2(D-2)}} \right) O_\mu O_\nu$$

and,

$$\phi = 2 \ln(S_0) + \sum_{k=1}^{\infty} \left(\frac{1}{D-3} \right)^k \left(\phi^{(k)} + \tilde{\kappa} a_3 \phi_{\tilde{\kappa}}^{(k)} \right)$$

Here, we are looking at the ansatz metric corrected to all orders in $\frac{1}{D}$. But we will be interested only in the correction to the first order i.e till $h_{\mu\nu}^{(1)}$.
Note that $h_{\mu\nu}^{(0)}$ is the leading ansatz that we had started with, and we have systematically added corrections as a power series in $\frac{1}{D}$ to make $h_{\mu\nu}^{(0)}$ a solution of the equation of motion. That is why, to make $h_{\mu\nu}^{(0)}$ a solution at first non-trivial order in $\frac{1}{D}$, we need to add the $h_{\mu\nu}^{(1)}$ piece to the metric.
Also, since $h_{\mu\nu}^{(0)}$ is an exact solution to the equations of motion itself on the $\psi = 1$ surface, we conclude that the corrections $h_{\mu\nu}^{(k)}$ must vanish when rrestricted to the $\psi = 1$ membrane hypersurface.

5.2 Gauge Fixing

We know that we have gauge freedom in our metric, which we have not used yet.Thus, to remove the gauge redundancy in our metric, we make the following convenient gauge choice as:

$$h_{\mu\nu}^{(n)} O^\mu = 0$$

where,

$$h_{\mu\nu}^{(n)} = \tilde{h}_{\mu\nu}^{(n)} + \tilde{\kappa} a_3 \tilde{h}_{\mu\nu,\tilde{\kappa}}^{(n)}$$

Chapter 6

Coordinate Rescaling

Since, the solution is non-trivial only in a region of thickness $\frac{1}{D}$ around the horizon, we can write the coordinate by expanding around an arbitrary point x_0 on the horizon by a thickness of $\frac{1}{D}$. Metric can also be written as expanding around the point x_0 .

6.1 Patch Coordinates

With the motivation that we had, that all the non-trivial physics is contained inside a region of order $\frac{1}{D}$ about the horizon, we define the so-called patch coordinates which are specialised to probe inside this membrane region, by changing the coordinates as follows:

$$x^\mu = x_0^\mu + \frac{1}{D-3}\alpha_a^\mu y^a$$

Or,

$$y^a = (D-3)(x^\mu - x_0^\mu)\alpha_\mu^a$$

We also define some rescaled quantities, to avoid spurious factors of D in our calculation while working with the patch co-ordinates.

The rescaled metric

$$G_{ab} = (D-3)^2 g_{ab} = \alpha_a^\mu \alpha_b^\nu g_{\mu\nu}$$
$$G^{ab} = \frac{1}{(D-3)^2} g^{ab} = \alpha_\mu^a \alpha_\nu^b g^{\mu\nu}$$

and,

$$\chi_a = (D-3)\partial_a \phi = \alpha_a^\mu \partial_\mu \phi$$

6.2 Choice of Patch Co-ordinates

Note that there are three special directions in our system , namely : 1)the unit timelike velocity vector u_μ, 2)the unit spacelike normal n_μ and 3)the radial direction of the d sphere. We will use a linear combination of these directions along with any arbitrary p directions as our coordinate for the $p+3$ dimensional coordinate system, which are namely, n,$O = n - u$ and $Z = dS - (n.dS)n$ for the 3 directions and Y^i for remaining p dimensions.Thus, the following scaled coordinates used are :

$$\begin{aligned}
R &= (D-3)(\psi - 1) \\
V &= (D-3)(x^\mu - x_0^\mu)O_\mu \\
z &= (D-3)(x^\mu - x_0^\mu)Z_\mu \\
y^i &= (D-3)(x^\mu - x_0^\mu)Y_\mu^i
\end{aligned}$$

Written in these rescaled coordinates the leading ansatz metric turns out to be of the form:

$$
\begin{aligned}
ds^2 =& 2\frac{S_0}{n_s}dVdR - (1-e^{-R})\left(1 - \tilde{\kappa}a_3\frac{n_s^2}{S_0^2}e^{-R}\right)dV^2 + \frac{dz^2}{1-n_s^2} + dy_i dy^i + O(\frac{1}{D}) \\
e^{\phi} =& S_0^2 + O(\frac{1}{D})
\end{aligned}
$$

6.3 Leading Order Correction

We find that the Leading Order Correction to our starting ansatz satisfying the proper gauge conditions, written out in our rescaled patch coordinates turn out to be of the form.

$$
\begin{aligned}
\tilde{h}^{(1)}_{ab} + \tilde{\kappa}a_3\tilde{h}^{(1)}_{ab,\kappa} \;=\;& (S_{VV}(R) + \tilde{\kappa}a_3 S_{VV,\tilde{\kappa}}(R))dV^2 + 2(S_{Vz}(R) + \tilde{\kappa}a_3 S_{Vz,\tilde{\kappa}}(R))dVdz \\
& + (S_{zz}(R) + \tilde{\kappa}a_3 S_{zz,\tilde{\kappa}}(R))dz^2 + (S_{tr}(R) + \tilde{\kappa}a_3 S_{tr,\tilde{\kappa}}(R))dy^i dy^i \\
& + 2(V_{Vi}(R) + \tilde{\kappa}a_3 V_{Vi,\tilde{\kappa}}(R))dVdy^i + 2(V_{zi}(R) + \tilde{\kappa}a_3 V_{zi,\tilde{\kappa}}(R))dzdy^i \\
& + (T_{ij}(R) + \tilde{\kappa}a_3 T_{ij,\tilde{\kappa}}(R))dy^i dy^j
\end{aligned}
$$

here, T_{ij} are trace-less in the sense that $T_{ii} = 0$.

All the terms are written according to their nature of transformation under the group $SO(D-p-3)$, as Scalars, Vectors, Tensors. Note further, that all these corrections should go to zero on the membrane, since on the membrane , our leading starting ansatz is already a solution, as any patch of the membrane resembles a patch of boosted Schwarzschild metric.

Chapter 7

Membrane Equation in Minkowski background

7.1 Membrane Equations

We know that not all of the equations of motion $E_{\mu\nu}$ are dynamical in nature. When a hypersurface is specified, the equations of motion can be naturally divided into two classes, the dynamical and the constraint equations of motion.We can solve the constraint equation in any one slice and then those constraint equations will be satisfied in all other slices after solving the dynamical equations of motion. The constraint equations of motion are defined as:

$$C_\mu = E_{\mu\nu} n^\nu.$$

In our case, the constraint equations are thus,

$$C_\mu = E_{\mu\nu} \partial^\nu \psi.$$

Solving the constraint equations: $C_\mu = 0$ at $R = 0$ will give us the so-called Membrane Equations of Motion:

$$\nabla.u = O\left(\frac{1}{D}\right)$$

$$\mathcal{P}^\mu_\alpha\left(\frac{\nabla^2 u_\mu}{\mathcal{K}}+u^\beta K_{\beta\mu}-u.\nabla u_\mu\left(1+\tilde{\kappa}a_3\frac{\mathcal{K}^2}{(D-3)^2}\right)-\frac{\nabla_\mu\mathcal{K}}{\mathcal{K}}\left(1-\tilde{\kappa}a_3\frac{\mathcal{K}^2}{(D-3)^2}\right)\right) = O\left(\frac{1}{D}\right)$$

Where,

$$\mathcal{P}^\mu_\alpha = \delta^\mu_\alpha - n^\mu n_\alpha + u^\mu u_\alpha.$$

The rest of the equations of motion, i.e the dynamical equations of motion are solved to find out the metric correction terms.

7.2 Inference

We expected a more nontrivial mixing of the co-efficients a_1,a_2 and a_3 in the membrane equations , but as is evident from the result, only the coefficient a_3 takes part in the membrane dynamics.This can be traced back to the fact that the α^0 solution satisfies $R_{\mu\nu} = 0$ and $R = 0$, which are the vacuum Einstein's Equation in Minkowski space-time.

Hence in the GB term of the action i.e $L_{GB} = \kappa(a_1R^2+a_2R_{\mu\nu}R^{\mu\nu}+a_3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$ the R^2 and $R_{\mu\nu}R^{\mu\nu}$ piece fails to contribute to the membrane equation, and only the $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ piece contributes, owing to the fact that $R_{\mu\nu\rho\sigma} \neq 0$ for solutions other than trivial flat space solution to Einstein Equation.

But we expect a more nontrivial mixing of coefficients to occur in AdS background since $R_{AB} = -(D-1)Lg_{AB}$ and $R = -D(D-1)L$. Thus, with this motivation we proceed to repeat the same procedure with AdS black brane for starters to find more non-trivial membrane equations.

Chapter 8

Ansatz in Ads space

8.1 Action in Presence of Cosmological Constant

The action now has a cosmological constant :

$$\mathcal{S} = \int d^D x \sqrt{-g}\,(R + (D-1)(D-2)l + \kappa(a_1 R^2 + a_2 R_{\mu\nu}R^{\mu\nu} + a_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})) \quad .... (2)$$

Where,

$$l = \frac{\Lambda}{(D-1)(D-2)} \quad \text{, is the cosmological constant}$$

8.2 Leading Ansatz from black Brane Solutions

The algorithm solves for the gravity eom locally in a small region (of size $O(1/D)$) about the horizon and then the global solution is obtained by 'smoothly' patching together all the local solutions. The key ingredient of the algorithm is a starting ansatz solution with horizon on which we implement this procedure.In this case, we use the black brane solution for convenience.

The static black brane solution to the Equations of Motion corresponding to the action (2) is:

$$ds^2 = -f(r)d\tilde{t}^2 + \frac{dr^2}{f(r)} + lr^2 d\Omega^2_{D-2}$$

where

$$f(r) = lr^2\Big(1 - \big(\frac{r_h}{r}\big)^{D-1}(1 + \kappa l(\mathcal{A} + \mathcal{B}) + \kappa l\mathcal{A} + \kappa l\mathcal{B}\big(\frac{r_h}{r}\big)^{2(D-)})\Big)$$

and

$$\begin{aligned} \mathcal{A} &= \frac{(D-4)}{(D-2)}\big((D-1)(Da_1 + a_2) + 2a_3\big) \\ \mathcal{B} &= (D-4)(D-3)a_3 \end{aligned}$$

Here, $r = r_h$ defines the horizon of our black brane solution.

8.3 Metric in Kerr-Schild Coordinates

The Black Brane Solution, written in Kerr-Schild coordinates take the following form:

$$\begin{aligned} ds^2 &= -lr^2(1+\tilde{\kappa}l\tilde{\mathcal{A}})dt^2 + \frac{dr^2}{lr^2(1+\tilde{\kappa}l\tilde{\mathcal{A}})} + lr^2 dx^2_{D-2} + \Big(\Psi^{-(D-3)}\big(1 + \tilde{\kappa}a_3\frac{\mathcal{K}^2}{(D-2)^2}\big) \\ &\quad -\tilde{\kappa}a_3\frac{\mathcal{K}^2}{(D-2)}\Psi^{-2(D-1)}\big)\Big)(O_\mu dx^\mu)^2 \end{aligned}$$

Here, the parameter $\tilde{\kappa} = (D-4)(D-3)\kappa$ is of order $O(1)$ and $\mathcal{K}$ is the trace of extrinsic curvature tensor of $r = r_h$ surface.
We also have the one form $O_\mu$defined as $O_\mu = n_\mu - u_\mu$, where $n_\mu dx^\mu = \frac{\sqrt{1-lr^2n_s^2}}{\sqrt{lr^2(1+\tilde{\kappa}a_1 l)}}dr + lr^2 n_s dS$ is the unit normal to the surface $r = r_h$. Here, the normalisation is carried out with respect to the asymptotic AdS spacetime.

Also, the velocity vector field u_μ is defined as $u_\mu dx^\mu = -\sqrt{lr^2(1+\tilde{\kappa}a_1 l)}dt$ such that,

$$n.n = 1 \quad n.u = 0 \quad u.u = -1,$$

here, all dot products are with respect to the asymptotic AdS spacetime given by:

$$ds^2 = -lr^2(1+\tilde{\kappa}l\tilde{\mathcal{A}})dt^2 + \frac{dr^2}{lr^2(1+\tilde{\kappa}l\tilde{\mathcal{A}})} + lr^2 dx^2_{D-2} \ .$$

Key point to note, is that the AdS length scale is modified by the presence of the 4-derivative terms, which warrants for greater care to be taken while arriving at the covariant form of the membrane equation.

8.4 Leading Ansatz in Rescaled Coordinates

Now our data, namely, 1) Shape of the membrane Ψ and 2) Velocity Vector field u_M can be taken to be arbitrary function and one-form respectively,respecting the $SO(D-p-3)$ symmetry that we had discussed in the previous case also.We can further uplift the data to the rest of the space-time following the auxilliary conditions, according the same Auxilliary Condition of minkowski background , only this time, all contractions are performed with respect to the AdS metric. Finally we will get the following form of the Leading Ansatz:

$$\begin{aligned} ds^2 &= 2\frac{D-3}{\mathcal{K}}dV dR - (1-e^{-R})\left(1-\kappa a_3\frac{\mathcal{K}^2}{(D-3)^2}e^{-R}\right)dV^2 \\ &+ \frac{lr_0^2}{1-lr_0^2 n_s^2}dz^2 + lr_0^2 dy_i dy^i + O(1/D) \end{aligned}$$

and

$$e^{\phi_0} = lr_0^2 S_0^2 + O(1/D)$$

where, $n_s = n.dS$, dot product being w.r.t. higher derivative corrected AdS background.

8.5 Metric Correction

The Leading Ansatz does'nt solve the Einstein's Equations directly, as we had even in the previous section.So,to make our metric a solution to the Einstein's Equation, we add correction terms to our Leading Ansatz as a power seies in $\frac{1}{D-3}$.We again consider decomposition of the full $SO(D-p-3)$ metric in the following form, without loss of generality:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + e^{\phi(x)}d\Omega_d^2 \quad \mu, \nu = 0, 1, ..., p+2$$

$$d = D - p - 3$$

where we have:

$$g_{\mu\nu} = g_{\mu\nu}^{AdS} + \left(\tilde{h}_{\mu\nu}^{(0)} + \tilde{\kappa} a_3 \tilde{h}_{\mu\nu,\tilde{\kappa}}^{(0)}\right) + \sum_{k=1}^{\infty} \left(\frac{1}{D-3}\right)^k \left(\tilde{h}_{\mu\nu}^{(k)} + \tilde{\kappa} a_3 \tilde{h}_{\mu\nu,\tilde{\kappa}}^{(k)}\right)$$

and,

$$\phi = \phi^{(0)} + \sum_{k=1}^{\infty} \left(\frac{1}{D-3}\right)^k \left(\tilde{\phi}^{(k)} + \tilde{\kappa} a_3 \tilde{\phi}_{\tilde{\kappa}}^{(k)}\right)$$

Here, we are only interested in solving for the coefficient $h_{\mu\nu}^{(1)}$ and deriving the membrane equation to leading order.

8.6 Leading Order Correction

The first correction that we can add to our leading ansatz to make it a solution to the generic Einstein-Gauss Bonnet equations of motion,namely, $\tilde{h}_{\mu\nu}^{(1)}$ takes the same form as was assumed in the flat background section, with the same gauge conditions.

$$\begin{aligned}
\tilde{h}^{(1)}_{ab} + \tilde{\kappa} a_3 \tilde{h}^{(1)}_{ab,\kappa} &= (S_{VV}(R) + \tilde{\kappa} a_3 S_{VV,\tilde{\kappa}}(R))dV^2 + 2(S_{Vz}(R) + \tilde{\kappa} a_3 S_{Vz,\tilde{\kappa}}(R))dV dz \\
&\quad +(S_{zz}(R) + \tilde{\kappa} a_3 S_{zz,\tilde{\kappa}}(R))dz^2 + (S_{tr}(R) + \tilde{\kappa} a_3 S_{tr,\tilde{\kappa}}(R))dy^i dy^i \\
&\quad +2(V_{Vi}(R) + \tilde{\kappa} a_3 V_{Vi,\tilde{\kappa}}(R))dV dy^i + 2(V_{zi}(R) + \tilde{\kappa} a_3 V_{zi,\tilde{\kappa}}(R))dz dy^i \\
&\quad +(T_{ij}(R) + \tilde{\kappa} a_3 T_{ij,\tilde{\kappa}}(R))dy^i dy^j
\end{aligned}$$

Here T_{ij} are trace-less in the sense that $T_{ii} = 0$.

Chapter 9

Membrane Equation in Ads space

9.1 Membrane Equation

Finally, solving the constraint equation:

$$C_\mu = E_{\mu\nu} G^{\nu\delta} n_\delta$$

where, $E_{\mu\nu}$ are the equations of motion, $G_{\mu\nu}$ is the complete metric upto required order in $\frac{1}{D}$ and n_μ is the unit normal to the membrane hypersurface. We get similarly, th Membrane Equation for our Ads case as:

$$\nabla . u = O(\frac{1}{D})$$

$$\left(\frac{\nabla^2 u_\mu}{\mathcal{K}} + u^\nu K_{\mu\nu} - \frac{\nabla_\mu \mathcal{K}}{\mathcal{K}} \left(1 - \tilde{\kappa} a_3 \frac{\mathcal{K}^2}{(D-3)^2} \right) - (u.\nabla) u_M \left(1 - \tilde{\kappa} a_3 \frac{\mathcal{K}^2}{(D-3)^2} \right) \right) \mathcal{P}^\mu_\gamma = O(\frac{1}{D}) \quad ..... (4)$$

where, the notation is such that greek alphabetical indices correspond to the effective $p+3$ coordinates and the projection operator $\mathcal{P}_{\mu\nu} = g^{(AdS)}_{\mu\nu} + u_\mu u_\nu - n_\mu n_\nu$ is orthogonal to the direction u_μ and n_μ. Thus, we have the desired result for the Ads space membrane equation. Note, than even though only a_3 coefficient appears manifestly in our Ads Membrane Equation, all dot products are taken with respect to the Ads metric, whose length is modified by the presence of the 4-derivative terms, which ushers a nontrivial dependence on the other coefficients also,i.e, a_1 and a_2, which is otherwise hidden in the above geometrised form of the membrane equations.

Also, we have the dynamical equations of motion other than the constraint equations, which we will use to solve the metric correction terms in the R.H.S of the expression for $h^{(1)}_{\mu\nu}$, but the explicit expression of them are too complicated to be mentioned, so I will avoid writing them explicitly here.

The membrane equations pose a well-defined initial value problem for our metric and determines the shape function: ψ and the velocity vector: u_μ, which are $p+2$ many functions to be determined (one function ψ and $p+1$ independent components of the unit velocity vector field u_μ).On the other hand we have $p+2$ many equations (4).

The membrane equations can be interpreted as being equations of motion for some fictitious particles constrained on the dynamical membrane having energy density $u^0 = \gamma$, and velocity $\frac{u^i}{u^0}$.The first of the two equations (4) tells us a conservation law for these fictitious particles i.e they are allowed to move from one place to another on the membrane but cannot be created or destroyed.While the second equation is actually a Newton's equation for these particles where the R.H.S contains different type of force terms on the fictitious particles like shear viscosity, pressure force etc.

9.2 Quasi-Normal Modes

Now that we have the membrane equations at leading order, in presence of four derivative terms, we can compute the spectrum of linearised fluctutations about a static spherical membrane by solving the membrane equations.

We consider a coordinate system in which the background Ads metric takes the form:

$$ds^2 = -dt^2(1 + lr^2 + \kappa\mathcal{A}l^2r^2) + \frac{dr^2}{(1 + lr^2 + \kappa\mathcal{A}l^2r^2)} + r^2 d\Omega_{D-2}^2$$

Also for later convenience, let

$$\tilde{l} = l(1 + \kappa\mathcal{A}l)$$

We will be studying linearised perturbation about the static spherical surface:

$$r = 1$$

which is parametrised as:

$$r = 1 + \epsilon\delta r(t, \theta^a)$$

Using the above expression we can find out all the extrinsic curvature tensor components required for solving the membrane equation.

Similarly, we can perturb the velocity vector about a static configuration as,

$$u = -\sqrt{1 + \tilde{l}}dt + \epsilon\delta u_t(t, \theta^a)dt + \epsilon\delta u_a(t, \theta^a)d\theta^a$$

Note, that we will be working in linear order in ϵ.

Normalising the perturbed velocity vector w.r.t the induced metric till linear order in ϵ we will get:

$$\delta u_t(t,\theta^a) = -\frac{\tilde{l}\delta r}{\sqrt{1+\tilde{l}}}$$

The velocity perturbation δu_a can be parametrised as follows:

$$\delta u_a = \delta v_a + \nabla_a \Phi$$

where δv_a is the divergence-less part of the perturbation, i.e $\nabla^a \delta v_a = 0$. Finally using the scalar membrane equation till leading order in $\frac{1}{D}$, i.e:

$$\nabla . u = 0$$

one can show that:

$$\nabla_a \delta u^a = \nabla^2 \Phi = -\frac{(D-2)\partial_t \delta r}{\sqrt{1+\tilde{l}}}$$

Now that we have reduced all perturbations to the two key perturbations: δr and δv_a, we can mode-decompose the two linearised fluctuations as:

$$\delta r(t,\theta) = \sum_{j=0}^{\infty} \delta r_j Y_j(\theta) e^{-i\omega_j^s t}$$

and

$$\delta v_a(t,\theta) = \sum_{j=0}^{\infty} \delta v_j V_{a,j}(\theta) e^{-i\omega_j^v t} \ .$$

Then putting these perturbations in the membrane equation and solving for the frequency we find that the scalar and vector QNM angular frequency turns out to be:

$$\begin{aligned}\omega_j^s &= \pm\Bigg(\frac{1}{\sqrt{j(1+\tilde{l})-1}}\big((1-a_3\tilde{\kappa}(1+\tilde{l}))(j-1)\big)+\frac{a_1\tilde{\kappa}jl^2}{2\sqrt{j(1+\tilde{l})-1}}\Bigg)\\ &\quad +i(1-a_3\tilde{\kappa}(1+\tilde{l}))(1-j)\end{aligned}$$

and similarly for the vector mode we have:

$$\omega_j^v = i(1-a_3\tilde{\kappa}(1+\tilde{l}))(1-j)$$

Thus, we find that the Quasi-Normal Modes are decaying in large time scales and are thus stable solutions. Notice that the solutions are stable, with or without the higher derivative corrections. Thus, we have a well-behaved solution having no instability as such.

Chapter 10

Hypersurface

Summary

We have so far computed the membrane equations in both flat and Ads background. Now, we can take it further and ask whether there exists a Stress-Tensor corresponding to this membrane, whose conservation law leads to the membrane equation. Calculating the explicit form of this stress tensor for our Ads Membrane Equations will be the aim of the remaining of my project.
In this section I will outline the necessary concepts and procedure for computing this Stress-Tensor. Most of the contents can be found in more details in the references[1][2]

10.1 Hypersurfaces Basics

There are two equivalent, but distinct ways of representing hypersurfaces(codimension one sub-manifolds):

[1]”A Relativist's Toolkit”, by Eric Poisson
[2]”Lecture Notes on General Relativity”, by Matthias Blau

1. **Embeddings**:

If there is any explicit mapping from the hypersurface to the manifold.

$$\phi(x): \ \Sigma \rightarrow M$$

If the manifold M itself is of dimension n, then the image $\phi(\Sigma)$ is necessarily of dimension $n-1$.

The function ϕ has to satisfy two conditions:

a. Injectivity(one-one)

b. The n x $(n-1)$ Jacobian of the map ϕ should be of maximum rank i.e $n-1$.

For example, S^1 can be defined as a hypersurface of $\mathbb{R}^2$ through the explicit mapping:

$$\Sigma(S^1): \ x = a\cos t, y = a\sin t$$

2. **Embedded Hypersurface**:

We can use a function $S(x)$ as constraint, to define a hypersurface through a suitable condition like:

$$\Sigma = [\ x|S(x) = 0\ ]$$

For example,

$$\Sigma(S^1): \ x^2 + y^2 = a^2$$

10.2 Pullback and Pushforward

If we have an explicit mapping $x^\mu = x^\mu(y^a)$, from hypersurface to the manifold, then we can pushforward(or pullback) the tangent(or cotangent) space to(or from) the bigger manifold, through the linear map:

$$V^\mu = e^\mu_a v^a$$

where,

$$e^\mu_a = \frac{\partial x^\mu}{\partial y^a}$$

and similarly for the cotangent space,

$$v_a = V_\mu e^\mu_a \ .$$

Note that, the linear map e^μ_a is not a square matrix, and hence has no inverse.So we can only pushforward vectors and pullback covectors.Moreover,the above definition of pushforward implies that the basis vectors in the hypersurface tangent vector space, ∂_{y^a} gets pushed forward to e^μ_a for each $a = 1, 2, ..., (n-1)$. Furthermore, if we have a well-defined metric all over the bigger manifold,then we can also define a unique normal to the hypersurface, satisfying the condition:

$$g_{\mu\nu} n^\mu e^\nu_a = 0 \ ,$$

$$g_{\mu\nu} n^\mu n^\nu = \epsilon = \pm(\text{spacelike or timelike}) \ .$$

The n equations uniquely solves for the unit normal to the hypersurface.

10.3 Projection Operator

If instead of knowing a parametric relation, we have the function $S(x)$ as defined in the definition of embedded hypersurface, then we immediately have an expression for the un-normalised normal to the hypersurface as,

$$l_\mu = -\partial_\mu S$$

Then the unit normal will be:

$$n_\mu = \epsilon \frac{\partial_\mu S}{\sqrt{\partial_\nu S \partial^\nu S}}$$

further, we can define a projection operator on the hypersurface, which will project out the tangential components of any vector of the bigger manifold:

$$h_{\mu\nu} = g_{\mu\nu} - \epsilon n_\mu n_\nu$$

10.4 Induced Metric

We can define an induced metric on the hypersurface by performing a pullback on the metric defined in the full manifold as:

$$h_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b = h_{\mu\nu} e^\mu_a e^\nu_b$$

This induced metric appears in line elements restricted only along the hypersurface:

$$ds^2_\Sigma = h_{ab}\ dy^a dy^b$$

In literature, the induced metric is also known as **The First Fundamental Form**.

10.5 Extrinsic Curvature

The Second Fundamental Form, or the Extrinsic Curvature tensor measures how the normal changes as one moves along the hypersurface. It describes how the hypersurface is embedded in the bigger space. The Extrinsic Curvature Tensor

K_{ab} takes the form:

$$K_{ab} = \nabla_\alpha n_\beta \ e^\alpha_a e^\beta_b$$

Or, equivalently,

$$K_{\alpha\beta} = h^\gamma_\alpha h^\delta_\beta \ \nabla_\gamma n_\delta = \frac{1}{2}\mathcal{L}_n g_{\alpha\beta}$$

Together, the first and the second fundamentral form are the initial value to be prescribed for solving the Einstein's Equations.
Also note that the trace of the extrinsic curvature tensor:$K = h^{\alpha\beta}K_{\alpha\beta} = \nabla^\alpha n_\alpha$ describes whether the normal appears to converge or diverge from the hypersurface, locally, depending on whether $K > 0$ or $K < 0$.

10.6 Gauss-Codazzi Equations

The Gauss-Codazzi Equations express the n-dimensional Riemann Tensor, RIcci Tensor, Ricci Scalar to the $(n-1)$- dimensional, hypersurface Intrinsic Curvature Tensor and Extrinsic Curvature Tensor.
The Gauss-Codazzi Equations leads to an expression of the Constraint Equation of General Relativity, in terms of the intrinsic and extrinsic curvature tensors.The equations are:

$$R_{\alpha\beta\gamma\delta}e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d = R_{abcd} + \epsilon(K_{ad}K_{bc} - K_{ac}K_{bd})$$

and

$$R_{\mu\alpha\beta\gamma}n^\mu e^\alpha_a e^\beta_b e^\gamma_c = K_{ab|c} - K_{ac|b}$$

where,

$$V_{a|b} = \partial_b V_a - \Gamma^e_{ab}V_e$$

$$\Gamma^e_{ab} = \frac{1}{2}h^{ed}(\partial_a h_{db} + \partial_b h_{da} - \partial_d h_{ab})$$

Here $K_{ac|b}$ is the covariant derivative of the extrinsic curvature with respect to the induced metric. Using the formula above, we can also find similar expressions for the Ricci Scalar and Ricci Tensor.Finally, we write down the constraint equation for the case of Einstein-Hilbert Gravity as:

$$-2\epsilon G_{\alpha\beta}n^{\alpha}n^{\beta} = R^{(3)} + \epsilon\big(K^{ab}K_{ab} - K^2\big)$$

and

$$G_{\alpha\beta}e^{\alpha}_{a}n^{\beta} = K^{b}_{a|b} - K_{,a}$$

while the remaining components $G_{\alpha\beta}e^{\alpha}_{a}e^{\beta}_{b}$ cannot be expressed solely in terms of h_{ab},K_{ab} and other related quantities.These two equations are important for initial value formulation of General Relativity. The membrane equations are basically the Gauss-Codazzi Equations rewritten in terms of ψ and u_{μ}.

10.7 Junction Condition

The Junction Conditions will play a key role in arriving at an expression for the Membrane Stress Tensor.

This section deals with the case where a manifold $\mathcal{M}$ is divided into two regions $\Sigma_{+}$and Σ_{-} which have a common boundary Σ. In Σ_{+} , we have the mertic $g^{+}_{\alpha\beta}$, with coordinates x^{+}_{α} and in the region Σ_{-}, we have the metric $g^{-}_{\alpha\beta}$ in coordinates x^{-}_{α}.The coordinates $x^{\pm}_{\alpha}$ need not be continuous across the hypersurface Σ,while, the intrinsic coordinates y^{a}, installed on the hypersurface are continuous across Σ.

If we define:

$$[A] = A(\Sigma_{+})|_{\Sigma} - A(\Sigma_{-})|_{\Sigma}$$

i.e the jump of A across the hypersurface.
Then we have the following identities holding true:

$$[e^{\alpha}_{a}] = 0 = [n^{\alpha}]$$

Now, to make sure that, we have defined a metric such that it solves the equations of motion everywhere in spacetime,when $g^{\pm}_{\alpha\beta}$ solves the equations in each half,one can prove that the first junction condition should hold:

$$\text{First Junction Condition}: [h_{ij}] = 0$$

This has to hold if we wish to smoothly join the two solutions across the hypersurface, such that the Einstein's Equations are satisfied everywhere.
Now the second junction condition holds as:

$$\text{Second Junction Condition}: [K_{ij}] = 0$$

A hypersurface is called *regular hypersurface*, when it also satisfies the second junction condition $[K_{ab}] = 0$.
Otherwise, when $[K_{\alpha\beta}] \neq 0$, the hypersurface is defined as *singular hypersurface*.
The presence of singular hypersurface corresponds to the presence of a Stress-Tensor localised completely on the hypersurface,which in analogy to electrostatics is the case of a charge distribution on a thin shell.

10.7.1 Surface Stress-Energy Tensor

After some more analysis, it can be figured out that for singular hypersurfaces,i.e $[K_{ab}] \neq 0$, we can define the Surface Stress-Energy Tensor to be as:

$$\begin{aligned} \mathcal{T}^{\alpha\beta}_{\Sigma} &= \delta(l) T^{ab} e^{\alpha}_{a} e^{\beta}_{b} \\ &= \delta(l) T^{\alpha\beta} \end{aligned}$$

where,

$$T_{ab} = -\frac{\epsilon}{8\pi}\big([K_{ab}] - [K]h_{ab}\big)$$

and l is the geodesic proper time or distance away from the hypersurface, l is > 0 in the Σ_{+} and < 0 in the Σ_{-} region. Thus, we see that if the second junction condition, $[K_{ab}] = 0$ is not satisfied, then the space-time is singular at Σ and the interpretation is that there is a surface stress-tensor $\mathcal{T}^{\alpha\beta}_{\Sigma}$ present at Σ.This is exactly, the stress-tensor that we are looking for.In the later sections we will see how to use this result to arrive at an explicit expression for the Membrane Stress-Tensor.

Chapter 11

Membrane Stress-Tensor

In this section we will discuss how to construct the Membrane Stress Tensor by matching interior ($\psi < 1$) and exterior ($\psi > 1$) solution, using the knowledge of the previous section and applying it for our case.Even though we do not have any interior solution of the metric, we can analytically continue the metric on the membrane to a solution for the interior such that it neither blows up or decays in length scales of order $\frac{1}{D}$.

11.1 Interior and Exterior Solution

In our case we have two regions, $\psi > 1$ and $\psi < 1$,that we will refer to as (out) and (in) respectively.

The Exterior Solution has an exponential falloff to the exterior of the membrane over length scales of order $O(\frac{1}{D})$, while it blows up in the interior region.

On the otherhand, the interior solution neither blows up nor decays in length scales of order $O(\frac{1}{D})$ away from the membrane.

Here we are looking at solutions linearised around a background metric as:

$$g_{\mu\nu} = g^b_{\mu\nu} + H_{\mu\nu}$$

Where,

$$\begin{aligned} H_{\mu\nu} &= F^{(out)}_{\mu\nu}[h_{ab}], \text{to the exterior} \\ & F^{(in)}_{\mu\nu}[h_{ab}], \text{to the interior} \end{aligned}$$

is the full corrected metric perturbation

And, h_{ab} is the induced metric on the hypersurface.The first junction condition requires the induced metric to be the same when approached from both the exterior and the interior.
Written more explicitly the two solutions are:

1. $\psi > 1$

$$H_{\mu\nu} = \left[\psi^{-(D-3)} \sum_{m=0}^{\infty} (\psi - 1)^m H^{(m)}_{\mu\nu}\right]$$

2. $\psi < 1$

$$H_{\mu\nu} = \left[\sum_{m=0}^{\infty} (\psi - 1)^m \tilde{H}^{(m)}_{\mu\nu}\right]$$

Where, $H^{(0)}_{\mu\nu} = \tilde{H}^{(0)}_{\mu\nu}$ is considered as the basic data for the solution, and the rest of the coefficients are expressed in terms of this basic data by solving for the linearised dynamical equations of motion.

11.2 Stress Tensor

After we have solved for the dynamical equations, and determined all the appropriate coefficients till some order in $\frac{1}{D}$, solving the constraint equation in any one slice of some foliation of space-time, automatically solves it for all other slices. Thus we get the membrane equation by demanding that it is satisified on $\psi = 1$ hypersurface.

But instead of doing so, we will compute a Stress-Tensor, and then the membrane equations will follow from the conservation of this stress tensor. As we had seen in the previous section,an expression for the Stress-Tensor for a singular hypersurface is :

$$\begin{aligned} \mathcal{T}^{\alpha\beta}_{\Sigma} &= \mid \partial\psi \mid \delta(\psi - 1) T^{ab} e^{\alpha}_{a} e^{\beta}_{b} \\ &= \mid \partial\psi \mid \delta(\psi - 1) T^{\alpha\beta} \end{aligned}$$

$$T_{\mu\nu} = -\frac{\epsilon}{8\pi}\left[T^{(out)}_{\mu\nu} - T^{(in)}_{\mu\nu}\right]$$

Where we have defined:

$$T^{(out)}_{\mu\nu} = K^{(out)}_{\mu\nu} - K^{(out)} h^{(out)}_{\mu\nu}$$

$$T^{(in)}_{\mu\nu} = K^{(in)}_{\mu\nu} - K^{(in)} h^{(out)}_{\mu\nu}$$

$K^{(out)}_{\mu\nu}$= Extrinsic Curvature Tensor with respect to the exterior solution,

$h^{(out)}_{\mu\nu}$=Projection operator on $\psi = 1$ hypersurface, with respect to the exterior solution and,

$K^{(out)} = h^{\mu\nu}_{(out)} K^{(out)}_{\mu\nu}$

Similar definition apply for the terms with superscript (in) terms,only with respect to the interior solution.

At the leading non-trivial order, the stress tensor takes the form of :

$$
\begin{aligned}
T^{(out)}_{\mu\nu} &= \left(\tilde{K}_{\mu\nu} - \tilde{K}\Pi_{\mu\nu}\right) + \frac{N}{2}\left(H^{(1)}_{\mu\nu} - H^{(1)}P_{\mu\nu}\right) - \frac{N}{2}(D-3)(H^{(0)}_{\mu\nu} - H^{(0)})P_{\mu\nu} \\
T^{(in)}_{\mu\nu} &= \left(\tilde{K}_{\mu\nu} - \tilde{K}\Pi_{\mu\nu}\right) + \frac{N}{2}\left(\tilde{H}^{(1)}_{\mu\nu} - H^{(1)}P_{\mu\nu}\right)
\end{aligned}
$$

Here,$\tilde{K}_{\mu\nu}$ and $\Pi_{\mu\nu}$ is the projection operator on the membrane with respect to the metric $\left[g^{(b)}_{\mu\nu} + H^{(0)}_{\mu\nu}\right]$.

$$
\begin{aligned}
H^{(n)} &= g^{\mu\nu}_{(b)} H^{(n)}_{\mu\nu} \\
\Pi^{\mu\nu} &= g^{\mu\nu}_{(b)} - n^{\mu}n^{\nu} + H^{\mu\nu}_{(0)} \\
\tilde{K}^{\mu\nu} &= K^{\mu\nu} - \frac{1}{2}[K^{\mu}_{\gamma}H^{\gamma\nu}_{(0)} + K^{\nu}_{\gamma}H^{\gamma\mu}_{(0)}]
\end{aligned}
$$

And, $K_{\mu\nu}$ and $P_{\mu\nu}$ are the extrinsic curvature tensor and projection operator on the membrane with respect to only the background metric $g^{(b)}_{\mu\nu}$.
We can express $H^{(1)}_{\mu\nu}$ in terms of $H^{(0)}_{\mu\nu}$ by solving the linearised dynamical equations of motion.Finally putting everything together we can show that the full Membrane Stress-Tensor turns out to be of the form:

$$
\begin{aligned}
T^{\mu\nu} &= -\frac{\epsilon}{8\pi}\left(T^{\mu\nu}_{(out)} - T^{\mu\nu}_{(in)}\right) \\
&= \frac{N}{2}(D-3)(H^{\mu\nu}_{(0)} - H_{(0)}P^{\mu\nu}) - \frac{N}{2}\left[H^{\mu\nu}_{(1)} - \tilde{H}^{\mu\nu}_{(1)} - (H^{(1)} - \tilde{H}^{(1)})P^{\mu\nu}\right]
\end{aligned}
$$

Thus solving for $H^{(1)}_{\mu\nu}$and $\tilde{H}^{(1)}_{\mu\nu}$ for our case, we can find an expression of the Stress-Tensor, whose conservation gives the membrane equations.
The rest of my project will be aimed at arriving at this Stress-Tensor for the case of our Higher Derivative corrected AdS Background Membrane.

Chapter 12

Conclusions

In this thesis, we have computed the membrane equations dual to a black brane solution in an asymptotically AdS background with a generic four-derivative correction to the Einstein-Hilbert Action, in large spacetime dimensions. Here, we have worked perturbatively up to first order corrections both in the expansion of $\frac{1}{D}$ and κ, the higher derivative parameter.

We have thus found that, in asymptotically AdS space,the presence of genric higher derivative terms modifies the background geometry on which the dual membrane is embedded.This causes the membrane equations to involve all the three coefficients of R^2,$R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$.

For the rest of my project I would be working out the explicit form of the stress tensor for our Ads Membrane Solution, whose conservation implies the membrane equations.

Another possible direction in the future can be to study large D membrane paradigm in this four derivative corrected theory till the second sub-leading order in an expansion in $\frac{1}{D}$ Also,it would be interesting to pursue this program in the presence of more higher derivative terms.

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