# Implied Volatility in Markov Modulated GBM model

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by

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# Certificate

This is to certify that this dissertation entitled Implied Volatility in MMGBM model towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Sanjay NS at Indian Institute of Science Education and Research under the supervision of Dr. Anindya Goswami and Dr. Kedarnath Mukherjee, Professors, IISER Pune and NIBM Pune respectively, during the academic year 2018-2019.

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This thesis is dedicated to Math finance community

# Declaration

I hereby declare that the matter embodied in the report entitled Implied Volatility in Markov Modulated GBM model are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Anindya Goswami and Dr. Kedarnath Mukherjee and the same has not been submitted elsewhere for any other degree.

Sanjay NS

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# Abstract

Regime switching extension is one of the many generalizations of Black-Scholes model of asset price dynamics where one assumes that the volatility coefficient evolves as a hidden pure jump process. Under the assumption of Markov regime switching, we have considered the locally risk minimizing price of European call and put options. By pretending these prices or their noisy versions as traded prices, we have first computed the implied volatility (IV) of the underlying asset. Then by performing several numerical experiments we have investigated if the IV shows any dependence on the time to maturity (TTM) and strike price. We have restricted ourselves to the models where switching is not too fast. We have observed a clear dependence that matches with that in the well known empirically observed volatility smile qualitatively. Furthermore, we have experimentally validated that the at-the-money (ATM) IV is insensitive to the stock/strike price variations. However, ATM IV depends on TTM in a particular manner which is at par with the empirically observed stylized fact. Next an extensive simulation study on regime switching market shows that ATM-IV time series can identify the transitions of hidden Markov chain provided the coefficient values at different regimes are not too close to each other. Several plots have been presented to illustrate each numerical experiment. We also compare the simulated results with empirical data to examine the applicability of the proposed inference method.

# Contents

# Introduction

Pricing of the options is very important for traders in the stock market. But since the option prices are dependent upon underlying stock prices, it is not possible to model option prices directly without modelling stock prices. There are several different theoretical models of stock price dynamics. Black Scholes Merton (BSM) model is one of the basic model of stock price, used in option pricing. Some of the other models are Heston model, jump diffusion model, regime switching model,etc.

In the first chapter, we discuss about theoretical option pricing under MMGBM model. Since there is no closed form solution to MMGBM partial differential equation (PDE), we resort to numerical techniques to obtain option prices. The option prices are very crucial in obtaining Implied Volatility (IV) and a small change in option price might cause large change in IV. Hence precise numerical calculation of option prices is necessary. Since the domain of the MMGBM PDE is unbounded and the terminal data is also having unbounded support, one must truncate the domain before numerical computation. But this causes truncation errors and also computational complexity of solving PDE is higher than that of solving equivalent system of integral equations(IE). Hence we use the asymptotic behaviour of the solution for the boundary data of the equivalent IE. We discussed the stability and error propagation in the proposed numerical scheme.

In second chapter, we discuss some stylized facts about IV. In BSM model, the graph of IV vs strike price is a constant horizontal line. But that is not the case in real market. IV values for deep out of the money options are generally higher compared to IV for At-the-money(ATM) options. This is called volatility smile. With the help of several numerical experiments we showed that MMGBM model can capture volatility smile. Furthermore, we have experimentally validated that the ATM IV is insensitive to the stock/strike price variations. However, ATM IV depends on time to maturity in a particular manner which is at par with the empirically observed stylized fact.

It is important to note that in the regime switching model, the underlying Markov process is assumed to be observable which is not the case in real market. Given a real market data, challenge is to identify the market state at any given point of time. This is impossible if we know only stock price values.Because even if there seems to be an unusual change in the stock price, we can never be sure whether that is due to a discontinuous change in the volatility coefficient or a rare occasion of a very large increment in Brownian motion term. Hence in order to observe regime switching in markets, we use IV of ATM options. Since ATM IV is insensitive to stock/strike variations, the value of ATM IV changes only if the market regime changes. With the help of several numerical experiments, we showed that the ATM-IV time series can identify the transitions of hidden Markov chain provided the coefficient values at different regimes are not too close to each other.

Above results enable us to estimate the transition rate matrix of the hidden Markov chain. But that work will be done in the future.

# Preliminaries

**Definition 0.0.1.** *An option is a financial instrument which gives the buyer the right, but not the obligation, to buy/sell(call/put) certain number of shares of an asset at a fixed price* $K$ *(strike price) on/before(European/American) a fixed maturity time* $T$.

Throughout this text, mostly we will be dealing with call options only. If one encounters just 'option', it should be considered as call option.

**Definition 0.0.2.** *If current stock price* $s$ *is equal to Strike price , then such options are called At-the-Money(ATM) options.*

**Definition 0.0.3.** *If current stock price* $s$ *is lesser than Strike price of a call option, then such options are called Out-of-the-money(OTM) options. Similarly if current stock price is greater than the Strike price of a call option, then such options are called In-the-Money(ITM) options.*

## Continuous time markov chain

A stochastic process $\{X(t) : t \geq 0\}$ is called a Continuous-time Markov chain if $\forall t \geq 0, s \geq 0$, $i \in \chi, j \in \chi$,

$$\mathbb{P}(X(s+t) = j \mid X(s) = i, \{X(u) : 0 \leq u < s\}) = \mathbb{P}(X(s+t) = j \mid X(s) = i) = P_{ij}(t), \quad (1)$$

where $\chi = \{1, 2, \ldots k\}$ is the state space of the markov chain and $P_{ij}$ represents the probability that the Markov chain will be at state $j$ after $t$ unit of time , given that the current state is $i$. $P_{ij}(t)$ is called transition probability function. Furthermore, if

$$\mathbb{P}(X(s+t) = j \mid X(s) = i) = \mathbb{P}(X(t) = j \mid X(0) = i) = P_{ij}(t),$$

then $X_t$ is called time homogeneous Markov chain. We will be considering only the homogeneous Markov chains in this text.

A transition rate-matrix or Q-matrix ($\Lambda$) is a matrix describing the rate at which a continuous-time Markov chain moves between the states. The elements of the Q-matrix, $\lambda_{ij}$ are given by following equation.

$$\lambda_{ij} = \frac{dP_{ij}(t)}{dt} \mid_{t=0} \forall i, j \in \chi$$

This definition implies that the elements $\lambda_{ij}$ satisfy the following conditions.

1. $0 \leq -\lambda_{ii} < \infty$, $0 \leq \lambda_{ij}$ for $i \neq j$
2. $\sum_j \lambda_{ij} = 0 \forall i$

# Chapter 1

# Theoretical option price

## 1.1 Black Scholes Merton model

In 1973, Fischer Black, Myron Scholes and Robert C Merton introduced a mathematical model of underlying asset price dynamics to determine the price of a European call option on that underlying asset. The BSM model assumes that dynamics of stock price $\{S_t\}_{t\geq 0}$ follows geometric brownian motion(GBM), i.e,

$$dS_t = S_t(\mu dt + \sigma dW_t), S_0 > 0 \tag{1.1}$$

where $\mu$ is the drift parameter, $\sigma$ is the volatility parameter and $\{W_t\}_{t\geq 0}$ is the standard Wiener process. BSM model have certain assumptions laid on it. Namely

1. There is no arbitrage opportunity in the market.
2. Underlying asset doesn't pay dividend.
3. No transaction costs(Frictionless market)
4. It is possible to buy fractional amount of stocks also.
5. Short selling is also allowed.

### 1.1.1 BSM option pricing

Let $K, T, \sigma$ be as defined above and $r$ be continuously compounded rate of interest of an ideal bank. Let $C(t, s; K, T; r, \sigma)$ be the European call option price at time t and stock price $s$ where $K$ and $T$ are contract parameters and $r$ and $\sigma$ are market parameters. Often we simply denote it by $C(t, s)$. Assume $C$ is a smooth function of $t$ and $s$. To be more precise, it is twice differentiable with respect to $s$ and once differentiable with respect to $t$. Under the assumption of the BSM model, the call option price function $(t, s) \to C(t, s)$ satisfies the following linear parabolic partial differential equation

$$\frac{\partial C(t,s)}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 C(t,s)}{\partial s^2} + rs\frac{\partial C(t,s)}{\partial s} = rC(t,s) \tag{1.2}$$

on $(0, T) \times (0, \infty)$ with the following terminal and boundary conditions

$$C(T, s) = (s - K)^+$$
$$C(t, 0) = 0.$$

Interestingly the above PDE has an analytical solution. The value of the European call option price is given by

$$C(t, s; K, T; r, \sigma) = \Phi(d_1)s - K\exp(-r(T-t))\Phi(d_2) \tag{1.3}$$

where $d_1$ and $d_2$ are,

$$d_1 = \frac{1}{\sigma\sqrt{(T-t)}}\left(\ln(\frac{s}{K}) + (r + \frac{\sigma^2}{2})(T-t)\right)$$
$$d_2 = d_1 - \sigma\sqrt{(T-t)}$$

and $\Phi$ is the CDF of the standard normal distribution, $r$ is the continuously compounded rate of interest, and $(T-t)$ is time to maturity. The complete derivation of BSM PDE (1.2) and BSM option pricing formula (1.3) can be found in [3].

### 1.1.2 Drawbacks

The BSM model has several drawbacks. Not all of the assumptions are valid in real life. Following are some of the drawbacks.

1. It assumes that the growth rate, volatility coefficient and bank interest rate remain constant during the entire period of the option, which is not the case in actual market.

2. Since it assumes that stock price paths are continuous, it fails to capture opening gaps in the market.

3. The simple return of a stock is assumed to be normally distributed. However, the empirically that has increasingly heavier tail for shorter time steps.

## 1.2 Computation of Option Price in the Regime Switching GBM market model

The regime switching model of asset price dynamics assumes that the asset parameters have finitely many possible states. The collection of parameters stays at different states at different disjoint random duration. Furthermore, this market states are assumed to transit as a continuous time finite-state Markov chain. However, the asset price evolves as a geometric Brownian motion (GBM) during the inter-transition period. Thus the drift ($\mu$) and volatility ($\sigma$) parameters of the GBM are functions of a Markov chain $X := \{X_t\}_{t\geq 0}$. Let $\mathcal{X} := \{1, 2, \ldots, k\}$ be the state space of $X$. We further assume that $X$ is an irreducible Markov chain with a given rate matrix $\Lambda = (\lambda_{ij})_{k\times k}$ and $p_{ij} := \frac{\lambda_{ij}}{|\lambda_{ii}|}$ for $i \neq j$ is one step transition probabilities to state $j$ given that the Markov chain left state $i$. Thus $S_t$, the asset price at time $t$ satisfies

$$dS_t = \mu(X_t)S_t dt + \sigma(X_t)S_t dW_t, S_0 > 0. \tag{1.4}$$

where $W := \{W_t\}_{t\geq 0}$ is a standard Brownian motion on a given probability space $(\Omega, \mathcal{F}, P)$. This process $S := \{S_t\}_{t\geq 0}$ is called a Markov modulated geometric Brownian motion (MMGBM). Here we keep the ideal bank's rate as a deterministic constant $r$. That means in this model we are considering regime switching in risky asset only. Although this market is arbitrage free under admissible strategies, this is incomplete unlike GBM model [9]. The *fair* prices of a range of derivatives including call and put options have been extensively studied in the literature following [9]. For details, the readers may refer to [2], [4], [10], [15], [16], [17], [21], [22], [23], [25]. This list is merely indicative and not exhaustive.

Let $\varphi(t, s, i; K, T; r, \sigma)$ be the locally risk minimizing price of European call option with strike price $K$ and maturity $T$ on a non dividend paying underlying stock (satisfying (1.4)) at time $t$ when the stock price is $s$, and the parameters are in $i^{th}$ state. Let us denote it simply by $\varphi(t, s, i)$ by surpressing the contract and model parameters. Under MMGBM model assumptions, the European call option price function $\varphi(t, s, i)$ satisfies the following system of parabolic partial differential equations (see [9], [8] for details),

$$\frac{\partial\varphi(t,s,i)}{\partial t} + \frac{1}{2}\sigma(i)^2 s^2 \frac{\partial^2\varphi(t,s,i)}{\partial s^2} + rs\frac{\partial\varphi(t,s,i)}{\partial s} + \lambda_i \sum_{j=1}^{k} p_{ij}\varphi(t,s,j) = r\varphi(t,s,i) \tag{1.5}$$

on $(0, T) \times (0, \infty) \times \mathcal{X}$ with the following terminal condition

$$\varphi(T, s, i) = (s - K)^+, \quad \forall s \in (0, \infty)$$

where $\varphi(t, s, i)$ is of at most linear growth in $s$. If $\Lambda$ is a null matrix, meaning Markov chain $X_t$ does not transit almost surely, the equation (1.5) exactly coincides with that of classical BSM model.

It is important to note that the domain of the PDE is unbounded and the terminal data is also having unbounded support. Hence one must truncate the domain before solving the option price equation numerically. Such truncation causes truncation error which depends on the values of the artificially imposed boundary data. As per our knowledge, the relevant error analysis is absent in the literature for the above mentioned problem. On the other hand in [13] it is shown that the computational complexity for solving the PDE problem on a large bounded domain is significantly higher than that for solving an equivalent system of integral equations on the same domain. In this section, we therefore obtain the asymptotic behaviour of the solution which should be used for the boundary data of the equivalent integral equation.

### 1.2.1 Integral equation

Let $C_i(t,s)$ be the theoretical BSM price function of the sme option with fixed interest rate $r$ and fixed $\sigma(i)$ for each $i \in \mathcal{X}$. The call option price function $\varphi(t,s,i)$ (as in (1.5)) satisfies the integral equation (1.6). The following two theorems are taken from [13] and [12].

**Theorem 1.** *(i)The Cauchy problem* (1.5) *has unique classical solution in the class of functions having at most linear growth.*
*(ii) The following integral equation has a unique solution in the class of functions belonging to* $C\Big([0,T]\times[0,\infty)\times\mathcal{X}\Big)\cap C^{1,2}\Big((0,T)\times(0,\infty)\times\mathcal{X}\Big)$ *having at most linear growth*

$$\varphi(t,s,i) = \exp(-\lambda_i(T-t))C_i(t,s) + \int_0^{T-t} \lambda_i \exp(-(\lambda_i + r)v) \tag{1.6}$$

$$\times \sum_j p_{ij} \int_0^{\infty} \varphi(t+v,x,j)\alpha(x;s,i,v)dxdv \tag{1.7}$$

*for all* $t \in [0,T]$*, where*

$$\alpha(x;s,i,v) = \frac{\exp(-\frac{1}{2}((\ln(\frac{x}{s}) - (r - \frac{\sigma(i)^2}{2})v)\frac{1}{\sqrt{v}\sigma(i)})^2)}{x\sigma(i)\sqrt{2\pi v}}$$

*is the pdf of Lognormal* $\Bigg(\ln s + \Big(r - \frac{\sigma^2(i)}{2}\Big)v, \sigma^2(i)v\Bigg)$ *distribution.*
*(iii) The solution* $\varphi(t,s,i)$ *of* (1.6) *solves* (1.5) *classically. This is the locally risk minimizing price of call option with strike price* $K$ *and maturity* $T$ *at time* $t$ *with* $S_t = s, X_t = i$.
*(iv)* $(s-K)^+ \le \varphi(t,s,i) \le s$ *for all* $t \in [0,T]$, $s \in (0,\infty)$

The proof of this theorem can be found in [13], [12] and [6].
The integral equation inherently includes all the boundary conditions because at $t = T$ the integral vanishes and hence

$$\varphi(T,s,i) = C_i(T,s) = (s-K)^+.$$

Again $\varphi(T,0,i) = 0$ and hence, the equation obtained from (1.6) by restricting $s = 0$ implies that

$$\varphi(t,0,i) = 0 \ \ \forall t \in [0,T].$$

For further discussion and proof of (iv) we refer to Proposition 4.1 in [12].
To obtain the European call option price under MMGBM model one needs to solve either (1.5) or (1.6). Since it is not possible to solve these analytically, we use numerical techniques to solve the integral equation.

### 1.2.2 Asymptotic behaviour of call price function

Consider the integral equation (1.6). We will derive an approximated integral equation using asymptotic behaviour of call option price function. The aim is to replace the infinite integral in equation (1.6) by a finite one and a correction term. The numerical scheme presented in [12] also replaces the infinite integral by a finite one but doesn't add correction term using the asymptotic behaviour of the solution. That approach relies on the fact that the infinite integral

$$\int_M^\infty \varphi(t+v,x,j)\alpha(x;s,i,v)dx$$

has diminishing effect for increasing value of $M$. However, to attain certain degree of accuracy in numerical solution, $M$ needs to be considerably large, which enhances the time complexity at the order of $M^2$. In view of this, the use of asymptotic behaviour of the solution $\varphi$ in approximating the above mentioned infinite integral would be particularly useful in reducing time complexity.

**Lemma 2.** *Let $\varphi$ be the solution of (1.6). Then for all $(t,s,i) \in [0,T] \times [0,\infty) \times \chi$,*

$$s - \varphi(t,s,i) = \exp(-\lambda_i(T-t))(s - C_i(t,s)) + \int_0^{T-t} \lambda_i \exp(-(\lambda_i + r)v) \times \sum_j p_{ij} \int_0^\infty (x - \varphi(t+v,x,j))\alpha(x;s,i,v)dxdv. \tag{1.8}$$

*Proof.* We first note that the mean of lognormal pdf $\alpha(\cdot;s,i,v)$ having the distribution $LN\left(\ln s + \left(r - \frac{\sigma^2(i)}{2}\right)v, \sigma^2(i)v\right)$, is equal to

$$\int_0^\infty x\alpha(x;s,i,v)dx = s^{rv}. \tag{1.9}$$

Hence,

$$\begin{aligned}
&\int_0^{T-t} \lambda_i \exp(-(\lambda_i + r)v) \times \sum_j p_{ij} \int_0^\infty x\alpha(x; s, i, v)dxdv \\
&= \int_0^{T-t} \lambda_i \exp(-(\lambda_i + r)v) s \exp(rv) \sum_j p_{ij} dv \\
&= s \int_0^{T-t} \lambda_i \exp(-\lambda_i v) dv \\
&= s(1 - \exp(-\lambda_i(T - t)).
\end{aligned}$$

Thus

$$s = s\exp(-\lambda_i(T - t)) + \int_0^{T-t} \lambda_i \exp(-(\lambda_i + r)v) \times \sum_j p_{ij} \int_0^\infty x\alpha(x; s, i, v)dxdv. \quad (1.10)$$

By subtracting the left and right side expressions of the (1.6) from those of the above equation, we get (1.8). □

A direct calculation involving the BSM European call option price formula gives the following limit expression

$$\lim_{s\to\infty} (s - C_i(t, s)) = K \exp(-r(T - t)) \quad \forall i \in \chi, \quad t \in [0, T].$$

This means that for a fixed strike and time to maturity, if stock price is too high, the difference between stock price and the BSM option price is close to the present value of strike price. Now we show that the MMGBM option price also has similar limiting behaviour. Define $g_i(t) = \lim_{s\to\infty}(s - \varphi(t, s, i))$. Therefore using equation (1.8),

$$\begin{aligned}
g_i(t) = &\lim_{s\to\infty} (s - \varphi(t, s, i)) \\
=&K \exp(-(\lambda_i + r)(T - t)) + \lim_{s\to\infty} \int_0^{T-t} \lambda_i \exp(-(\lambda_i + r)v) \\
&\times \sum_j p_{ij} \int_0^\infty (x - \varphi(t + v, x, j))\alpha(x; s, i, v)dxdv.
\end{aligned}$$

By using a substitution, $x = zs$ , the right side of the above expression is equal to

$$K\exp(-(\lambda_i + r)(T-t)) + \lim_{s\to\infty}\int_0^{T-t}\lambda_i\exp(-(\lambda_i + r)v)$$
$$\times\sum_j p_{ij}\int_0^\infty (zs - \varphi(t+v, zs, j))\overline{\alpha}(z; s, i, v)drdv$$

where $\overline{\alpha}(z; s, i, v)$ is the pdf of Lognormal $\left(\left(r - \frac{\sigma^2(i)}{2}\right)v, \sigma^2(i)v\right)$ distribution. Using Theorem 1 and dominated convergence theorem, we can pass the limit inside of the integration and conclude that $g$ satisfies the following integral equation,

$$g_i(t) = K\exp(-(\lambda_i + r)(T-t)) + \int_0^{T-t}\lambda_i\exp(-(\lambda_i + r)v)\times\sum_j p_{ij}g_j(t+v)dv. \quad (1.11)$$

Since $r > 0$, a direct application of Banach Fixed point theorem implies that (1.11) has a unique solution. Then a direct substitution shows that $g_i(t) = Ke^{-r(T-t)}$ is indeed the unique solution of (1.11). Thus

$$\lim_{s\to\infty}(s - \varphi(t, s, i)) = Ke^{-r(T-t)}.$$

Now we by splitting the domain of integration wrt $x$ variable into two parts, equation (1.6) can be rewritten as

$$\varphi(t, s, i) = \exp(-\lambda_i(T-t))C_i(t, s) + \int_0^{T-t}\lambda_i\exp(-(\lambda_i + r)v)\sum_j p_{ij}\left(\int_0^M(\varphi(t+v, x, j) - x)\right.$$
$$\left.\times\,\alpha(x; s, i, v)dx + \int_M^\infty(\varphi(t+v, x, j) - x)\alpha(x; s, i, v)dx + se^{rv}\right)dv$$

using (1.9). Again, since

$$\int_M^\infty \alpha(x; s, i, v)dx = 1 - F(M) \quad (1.12)$$

where $F$ is the CDF of the lognormal density $\alpha(\cdot, s, i, v)$,

$$\int_M^\infty \alpha(x; s, i, v)dx = 1 - \Phi\left(\frac{\ln M/S - (r - \frac{\sigma^2(i)}{2})v}{\sigma\sqrt{v}}\right) \quad (1.13)$$

where $\Phi$ is the cdf of the standard normal distribution. Hence using the asymptotic limit of solution function $\varphi$, on interval $[M, \infty)$, we obtain on the bounded domain $[0,T]\times[0,M]\times\mathcal{X}$ the following integral equation

$$\varphi_M(t,s,i) = \exp(-\lambda_i(T-t))C_i(t,s) + \int_0^{T-t} \lambda_i \exp(-(\lambda_i+r)v) \sum_j p_{ij} \Bigg( \int_0^M (\varphi_M(t+v,x,j)-x)$$

$$\times\, \alpha(x;s,i,v)dx - K\exp(-r(T-t-v)) \left(1 - \Phi\left(\frac{\ln M/S - (r - \frac{\sigma^2(i)}{2})v}{\sigma\sqrt{v}}\right)\right) + se^{rv}\Bigg) dv.$$

It is clear that for any given $(t,s,i) \in [0,T]\times[0,\infty)\times\mathcal{X}$, $\lim_{M\to\infty}\varphi_M(t,s,i) = \varphi(t,s,i)$, that is, $\varphi_M$ converges to $\varphi$ point-wise. However, the solution of above equation need not be non negative. Therefore we consider the following as the desired truncated problem

$$\varphi_M(t,s,i) = \max\Bigg(0, \exp(-\lambda_i(T-t))C_i(t,s) + \int_0^{T-t} \lambda_i \exp(-(\lambda_i+r)v) \sum_j p_{ij} \Bigg( \int_0^M (\varphi_M(t+v,x,j)-x)$$

$$\alpha(x;s,i,v)dx - K\exp(-r(T-t-v)) \left(1 - \Phi\left(\frac{\ln M/S - (r - \frac{\sigma^2(i)}{2})v}{\sigma\sqrt{v}}\right)\right) + se^{rv}\Bigg) dv\Bigg). \tag{1.14}$$

We aim to solve equation (1.14) to get the price of the European call option. A nemerical scheme is presented in the next section.

## 1.3 numerical scheme

To solve equation (1.14) we use step-by-step quadrature method. Let us take $\Delta t$ and $\Delta s$ as the time step and stock price step respectively. For $M_0$ and $N \in \mathbb{N}$, $n \in \{1, \dots N\}$, $l \in \{0,1,2,\dots N\}$, $m_0 \in \{0,1,\dots M_0\}$, $m \in \{0,1,\dots M_0\}$. We set

$$G(m, m_0, l, i) = \frac{\exp(-\frac{1}{2}((\ln(\frac{m_0}{m}) - (r - \frac{\sigma(i)^2}{2})l\Delta t)\frac{1}{\sqrt{l\Delta t}\sigma(i)})^2)}{m_0\sigma(i)\Delta s\sqrt{2\pi l\Delta t}},$$

$$\varphi_m^n(i) \approx \varphi(T - n\Delta t, m\Delta s, i),$$

$$\varphi_0^n(i) = 0.$$

For a function $\Psi$ over the interval $[0, n\Delta t]$, the quadrature rule is given by

$$\int_0^{n\Delta t} \Psi(v)dv = \Delta t \sum_{l=0}^{n} w_n(l)\Psi(l\Delta t) \tag{1.15}$$

where $w_n(l)$ are the weights corresponding to Simpson's rule for $n \geq 1$. Discretized equation is given by,

$$\varphi_m^n(i) = \max\Bigg(0,\ e^{-\lambda_i n\Delta t} C_i(T-n\Delta t, m\Delta s) + \Delta t w_n(0)\lambda_i \sum p_{ij}(\varphi_m^n(j) - m\Delta s) + \lambda_i \Delta t \sum_{l=1}^{n} w_n(l) e^{-l(\lambda_i+r)\Delta t}$$
$$\times \sum_j p_{ij}\Big(\sum_{m_0=1}^{M_0} \overline{w}(m_0)(\varphi_{m_0}^{n-l}(j) - m_0\Delta s)G(m, m_0, l, i)\Delta s + m\Delta s e^{rl\Delta t} - Ke^{-r(n-l)\Delta t}(1 - F(M_0\Delta s))\Big)\Bigg), \tag{1.16}$$

where $w_n$ and $\overline{w}(m_0)$ are weights. There are certain problems associated with Simpson's rule. For the Simpson's rule interval of the integration must be divided into even number of parts. We can choose $M_0$ to be even but we can't keep $n$ even always. Since $n$ runs from 0 to $N$, it takes odd values too. So we use combination of Simpson's and trapezoidal rule. If $n$ is even, we use Simpson's only. But if $n$ is odd, we use Simpson's upto and including $(n-1)^{th}$ nodes and trapezoidal rule for $(n-1)^{th}$ and $n^{th}$ nodes. To be more precise, for each $n \geq 1$ ,

$w_n(l) = W(n, l+1)$ where the weight matrix $W$ looks like $W = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & & & & \\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3} & & & \\ \frac{1}{3} & \frac{4}{3} & \frac{5}{6} & \frac{1}{2} & & \\ \frac{1}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{1}{3} & \\ \frac{1}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{5}{6} & \frac{1}{2} \\ & & & & & \\ & & & & & \end{bmatrix}$.

Here $n$ represents row number and $l$ represents column number.

### 1.3.1 stability analysis

**Lemma 3.** *Assume* $f(x) = \left(1 + \frac{a}{x}\right)^x$ *for* $x > 0$ *and* $a > 0$. *Then*

$$f(x) < e^a \quad \forall x > 0.$$

Proof: Note that $f(1) = (1+a) < e^a$ and $lim_{x\to\infty} f(x) = e^a$. Thus to prove the lemma it is sufficient to prove that $f'(x) > 0$ for all $x > 0$. To this end, we compute

$$\begin{aligned}
\ln f(x) &= x \ln\left(1 + \frac{a}{x}\right) \\
\frac{d}{dx} \ln f(x) &= \ln\left(1 + \frac{a}{x}\right) + \frac{x}{\left(1 + \frac{a}{x}\right)}\left(\frac{-a}{x^2}\right) \\
&= \ln\left(1 + \frac{a}{x}\right) - \frac{\frac{a}{x}}{1 + \frac{a}{x}} \\
&= \frac{1}{1 + \frac{a}{x}}\left[\left(1 + \frac{a}{x}\right)\ln\left(1 + \frac{a}{x}\right) - \frac{a}{x}\right].
\end{aligned}$$

Since the derivatives of $f$ and $\ln(f)$ have the same sign, we need to show

$$\frac{d}{dx} \ln f(x) > 0.$$

The above expression implies that the derivative is positive if and only if the function $h$ given by

$$h(y) := (1+y)\ln(1+y) - y$$

is positive for all $y > 0$. This is true because $h(0) = 0$ and

$$h'(y) = \frac{1+y}{1+y} + \ln(1+y) - 1 = \ln(1+y) > 0 \quad \forall y > 0.$$

Hence the proof. □

Let $\delta_n$ be a single isolated perturbation in computing $\varphi^n$ and $\epsilon_q$ be its effect in computing $\varphi^{n+q}$ . We are going to find out for which value of $\Delta t$, the numerical scheme will be stable under isolated perturbation. Let $a = \max_i \lambda_i$ , and $b = aF(M_0 \Delta s)$. Then $\epsilon_0 = \delta_n$. Then to calculate $\epsilon_1$ we refer to equation (1.16). Since $\varphi^{n-l}$ is multiplied by lognormal density and then integrated over $[0, M_0\Delta s]$, we obtain the following estimates

$$
\begin{aligned}
&\epsilon_1 \le \epsilon_0 F(M_0\Delta s)a\Delta t \le \epsilon_0 b\Delta t \\
&\epsilon_2 \le (\epsilon_0+\epsilon_1)b\Delta t \le \epsilon_0 b\Delta t(1+b\Delta t) \\
&\epsilon_3 \le (\epsilon_0+\epsilon_1+\epsilon_2)b\Delta t \le (\epsilon_0+\epsilon_0 b\Delta t+\epsilon_0 b\Delta t(1+b\Delta t))b\Delta t = \epsilon_0 b\Delta t(1+b\Delta t)^2 \\
&\vdots \\
&\epsilon_q \le \epsilon_0 b\Delta t(1+b\Delta t)^{q-1}.
\end{aligned}
$$

Let $\epsilon_T$ be total effect on $\varphi_m^N(i)$.

$$| \epsilon_T | \le \epsilon_{N-n} = \epsilon_0 b\Delta t(1+b\Delta t)^{N-n-1}.$$

As $\Delta t = \frac{T}{N}$, if

$$b\Delta t(1+\frac{bT}{N})^{N-n-1} \le 1, \tag{1.17}$$

then the effect would be less or equal to perturbation $\epsilon_0$ and the numerical scheme would be stable. On the other hand if we choose

$$\Delta t \le \frac{e^{-bT}}{b} \tag{1.18}$$

then from the Lemma 3,

$$\Delta t \le \frac{1}{b}\Big(1+\frac{bT}{N}\Big)^{-N}.$$

This implies that for any $n \ge 0$,

$$b\Delta t \le \Big(1+\frac{bT}{N}\Big)^{-N+n+1}.$$

Or in other words,

$$b\Delta t(1+\frac{bT}{N})^{N-n-1} \le 1.$$

This is exactly the sufficient condition (1.17) for stability of the scheme. Let $\delta_n$ be bounded by constant $\delta$ for all $n$. Then total effect $\epsilon$ of the perturbation in the value $\varphi_m^N(i)$ is given by

$$\sum_{n=1}^{N-1} \epsilon_{N-n} < (e^{bT}-1)\delta.$$

Hence we obtain the following theorem.

**Theorem 4.** *Let Let $a = \max_i \lambda_i$ , and $b = aF(M_0 \Delta s)$. For $\Delta t \leq \frac{e^{-bT}}{b}$ , the numerical scheme is strictly stable with respect to an isolated perturbation. Moreover, the scheme displays uniformly bounded error propagation.*

### 1.3.2 numerical example

We used following parameters during simulation.
Risk free interest rate $r = 0.05$. This is close to the usual bank rate in real world.
Time to maturity = 25 days $\approx$ 0.1 year
Rate matrix, drift parameter and volatility coefficients are

$\Lambda = \begin{bmatrix} -10 & \frac{20}{3} & \frac{10}{3} \\ 10 & -20 & 10 \\ \frac{10}{3} & \frac{20}{3} & -10 \end{bmatrix}$, $\mu = \begin{bmatrix} 0.08 & 0.09 & 0.1 \end{bmatrix}$ and $\sigma = \begin{bmatrix} 0.2 & 0.3 & 0.4 \end{bmatrix}$ respectively. We choose $N = 51$, $T = 0.1$, $K = 1$, $M = 1.5$, $M_0 = 400$. The contract parameters $T$ and $K$ are chosen keeping in mind the real market options. Usually difference between successive maturity date for Nifty50 Index options are about one month i.e, 25 trading days. If the grid points are less, errors in option prices will be high. Needless to say there is a trade off between error value and time complexity. Hence to balance both, these particular values of $N$ and $M_0$ are chosen. One can verify that $\Delta t = \frac{T}{N}$ is indeed satisfying condition (1.18). So numerical scheme is stable with respect to these parameter values.

The figure 1.1 shows the relation between option price and stock price when time to maturity is kept constant. We took time to maturity to be $T$. Horizontal axis represents stock price values and vertical axis represents option prices. There are 7 lines out of which 3 solid lines represent MMGBM regime option prices and other 3 dotted lines represent BSM option prices. The other line represents maturity value of options. We can observe that for lowest sigma regime, MMGBM prices are higher than the corresponding BSM prices. For other two regimes, BSM prices are slightly higher compared to MMGBM prices.

price under mmgbm.jpg

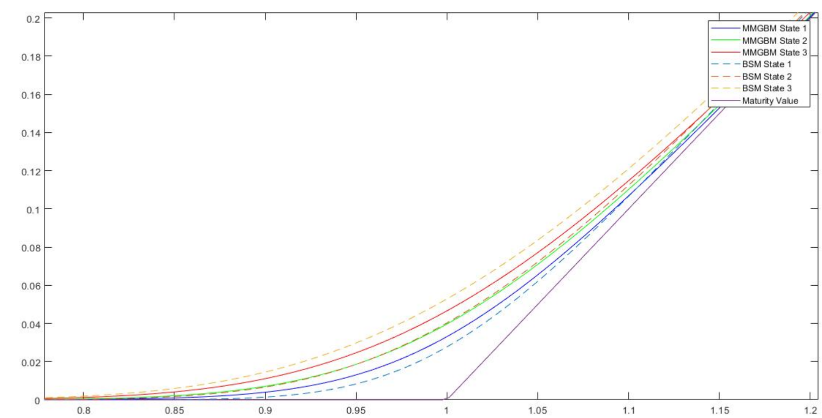


Figure 1.1: Vertical axis shows option price and horizontal axis shows stock price. Strike price is fixed at 1 and time to maturity 25 days.

# Chapter 2

# Implied Volatility in MMGBM model

### 2.0.1 Implied volatility

**Definition 2.0.1.** *The value of the volatility coefficient $\sigma$ of the asset price, for which theoretical BSM option price matches exactly with the observed market price of the same option is called the Implied volatility(IV) of the underlying asset.*

In other words, IV is the solution to the following implicit equation

$$C(t, s; K, T; r, IV) = M_p \tag{2.1}$$

where $C(t, s; K, T; r, IV)$ is the BSM call option price and $M_p$ is the market traded price of the option with same parameters.

**Theorem 5.** *Implied Volatility exists for all the market traded options.*

Proof:
First we prove that $C(t, s; K, T; r, \cdot)$ is a bounded monotonic function and the range is an open interval given by $\big((s - K\exp\{-r(T-t)\})^+, s\big)$. Then we would show that $M_p$ must lie in the above range. We know that the function $C$ can be expressed using conditional expectation as below

$$C(t, s; K, T; r, \sigma) = E\Big[\exp\{-r(T-t)\}(S_T - K)^+ \mid S_t = s\Big] \tag{2.2}$$

where the process $\{S_t\}_{t\geq 0}$ satisfies $dS_t = S_t(rdt + \sigma dW_t)$. Thus for the degenerate case, i.e, for $\sigma = 0$,

$$C(t, s; K, T; r, 0) = (s - K\exp\{-r(T-t)\})^+.$$

.

From eq(1.3), as $\sigma \to \infty$ , $d_1 \to \infty$ and $d_2 \to -\infty$. This implies,

$$\lim_{\sigma\to\infty} C(t, s; K, T; r, \sigma) = s.$$

Therefore under continuity and strict monotonicity, $C(t, s)$ is bounded with respect to $\sigma$ for fixed $t$ and $s$. Furthermore for any fixed $t, s, K, T, r$, if the observed option value $M_p$ is in the range of $\big((s - K\exp\{-r(T-t)\})^+, s\big)$ , there is a unique $0 < IV < \infty$ such that $C(t, s; K, T; r, IV) = M_p$.

The continuity of $C(t, s)$ w.r.t $\sigma$ follows from direct observation of equation (1.3). The strict monotonicity follows from the positivity of the term $\frac{\partial C}{\partial \sigma}$, also known as Vega, which represents the change in the option price with respect to unit change in volatility coefficient. From the BSM formula, a direct calculation by employing partial derivative of option price with respect to $\sigma$ shows that the Vega is always positive. It is important to note that $M_p$, the market value of the option never goes beyond the above mentioned interval. We follow proof by contradiction method. To this end, first assume

$$0 < M_p \leq (s - K\exp\{-r(T-t)\})^+.$$

Then

$$0 < K\exp\{-r(T-t)\} \leq s - M_p.$$

It implies that

$$0 < K \leq (s - M_p)\exp\{r(T-t)\}.$$

Hence if a trader short sells one unit of stock to receive $s$ amount of money, purchases a call option at $M_p$ and keep the remaining amount in an ideal bank, s/he will get back $(s - M_p)\exp\{r(T-t)\}$ amount of money at the maturity. S/he can now exercise the option by spending $K$ amount of money to buy one unit of stock from the option writer. The stock s/he received can be returned to whom s/he borrowed from. Thus the trader is still left with non negative amount of money with no chance of loss, leading to an arbitrage. On the other hand if $M_p \geq S$, then a trader can write one unit of call option to receive $M_p$ amount of money and buy one unit of stock at price $s$ and still be left with some non negative amount of money. At the maturity if the option buyer excercises the option, the trader would give that one unit of stock to the buyer in exchange of $K$ amount of money. Thus at the end, irrespective of any circumstances, the trader has certain non negative profit with no chance of loss, leading to an arbitrage. Therefore by the no arbitrage principle, we can say that the observed market price of the option $M_p$ lies in the above mentioned open interval. Hence we can compute IV for all the market traded options. □

## 2.1 volatility smile in MMGBM model

BSM model implies that the IV, calculated from different Strike price options on a same underlying asset,with fixed time to maturity should be constant. But in reality this is not true. The plot of IV for a series of options with same time to maturity(TTM) but different strike price shows that IV values for deep out of the money($S < K$) options are generally higher compared to IV for ATM options. This is called volatility smile and BSM model fails to capture this stylized fact. Now we show that MMGBM model can explain this stylized fact. In the figure 2.1, horizontal axis represents strike values and vertical axis represents IV

call put.jpg

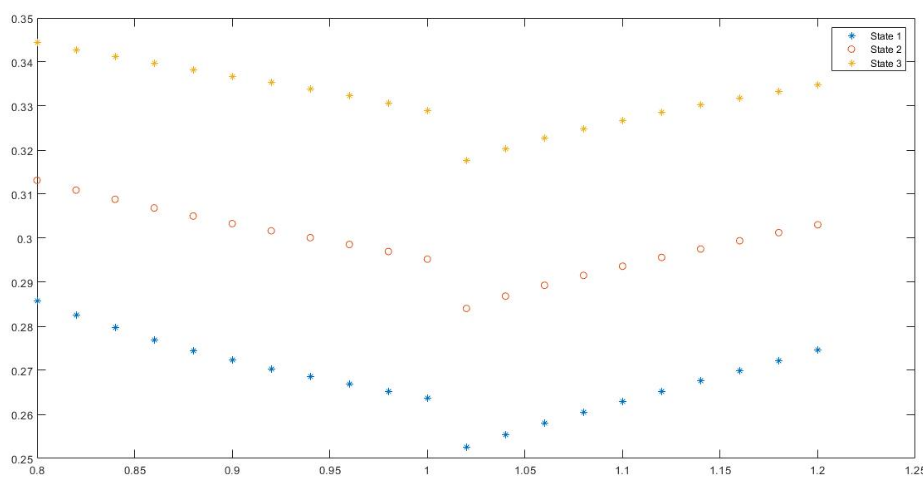


Figure 2.1: Horizontal axis represents strike values and vertical axis represents IV values.

values. Since we fix stock price as 1, strike prices above 1 correspond to out of the money call

options and strike prices below 1 correspond to out of the money put options. This curve is much more similar to what we observe in real market. Next step is to validate the result for different set of parameters. We have to establish that the same result holds true for a wide range of parameters. In order to do that, we choose extreme values of all parameters. Following are the values from which parameter values are chosen. We kept interest rate constant for all three regimes but we chose that constant from two values given below. All of the following extreme values of parameters are chosen based on real market values.
For all $i = 1, 2, 3$ :
Risk free interest rate $r \in \{0.01, 0.1\}$ (2 combinations)
volatility coefficient $\sigma(i) \in \{0.1, 0.5\}$ (8 combinations)
$\lambda(i) \in \{0.5, 3\}$ (8 combinations)
Transition probability matrix $P = \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix}$.
So total of 128 possible combination of parameters were obtained, out of which some cases where all three sigma are equal correspond to BSM cases. We are not considering those cases. For each set of parameters we varied stock price from 0.8 to 1.2 with step size 0.02. For all those 21 points we obtained option prices by solving integral equation. Then we obtain IV for all those points. Now to identify the pattern, we use polynomial fit approach. Since curve looks like parabola, we fit second degree polynomial to all the IV data points of each regime. If the leading coefficient in the quadratic regression equation is positive, then we can say that volatility smile pattern is observed. We obtained leading coefficient for each regime and for each set of parameters. It all came out to be positive. Hence we can say that MMGBM model captures volatility smile for a wide range of realistic parameter values. Figure 2.2 shows the smile coefficients (leading coefficient in quadratic fit) for each regime and for different parameter cases. We can observe that the coefficients are appearing in pairs from which we can infer that the result is not much sensitive to interest rate $r$ value.

When sigma was 0.5, smile coefficients are lesser than 1. But sigma 0.1 cases had smile coefficients higher than 1. Another observation is that the increase in lambda value increases the smile coefficient value.

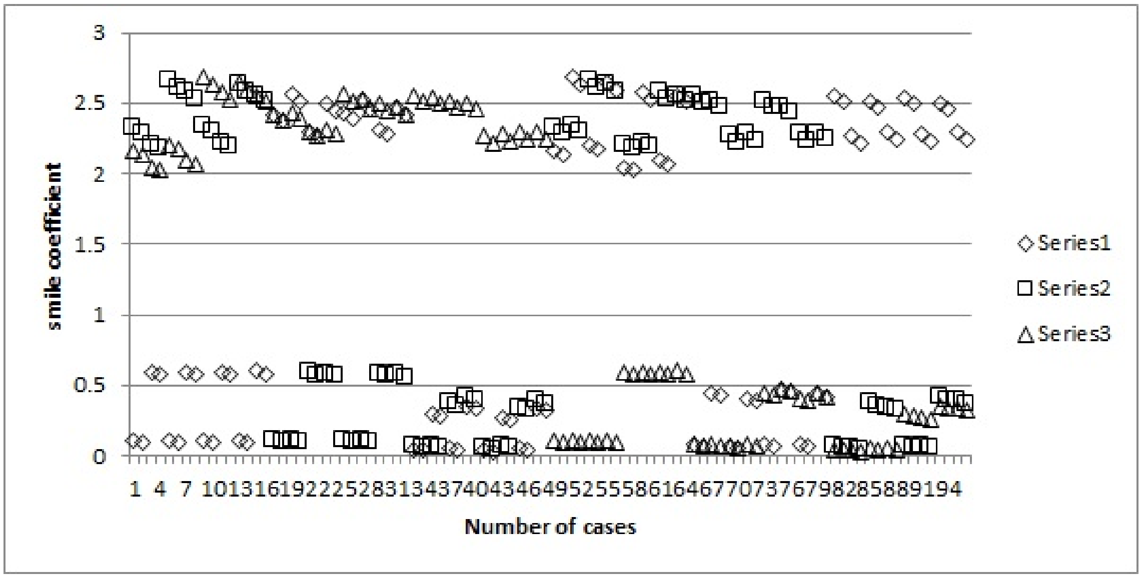


Figure 2.2:

## 2.2 IV for ATM options

In this section we study the behaviour of Implied Volatility of ATM options only. Let $\tau$ be time to maturity. Here we define new function $AV(K, \tau, i) : R_+ \times [0, T] \times \chi \to R_+$ which represents Implied volatility of At-the-Money options.

### 2.2.1 IV vs TIME to MATURITY

In this section we used 0.1, 0.2, 0.8 as $\sigma$ values for three regimes. We are using extreme values of sigma to enhance the effect. Horizontal axis is time to maturity and vertical axis is Implied volatility. As we can see in figure 2.3 IV decreases with time to maturity for lower volatility regimes. But for higher volatility regimes, IV increases with decrease in time to maturity. The same pattern is obtained for In the money and Out of the money options also.

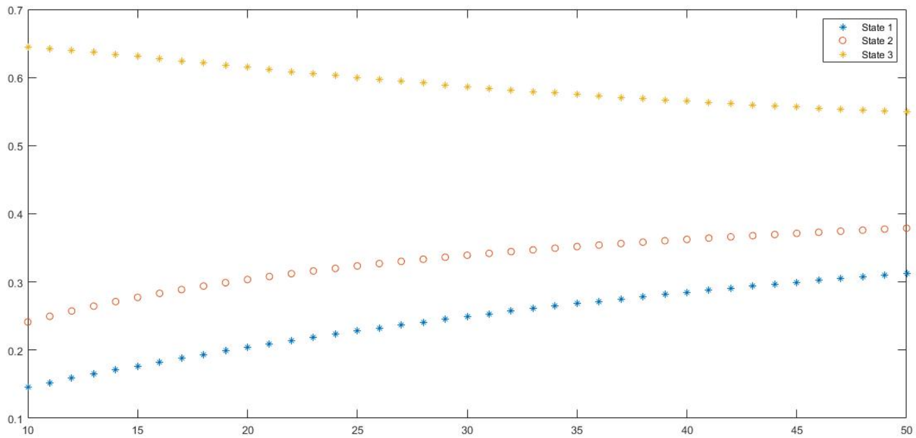


Figure 2.3: Horizontal axis represents time to maturity (in days unit) and vertical axis shows IV values. Stock/strike price is taken to be 1.

### 2.2.2 IV vs stock/strike price

In this section we studied the variation of IV with respect to stock and strike prices. First we prove a theorem which is applicable not only to ATM but also to ITM and OTM options. Then we validate the result using numerical computation.

**Theorem 6.** *For a given market state, and fixed time to maturity, ATM IV does not depend on stock price. In other words, ATM IV is insensitive to stock price variations.*

Proof: Consider the regime switching GBM model as given in section 1.2. We know that the locally risk minimizing price $\varphi(t, s, i; K, T; r, \sigma)$ of a call option at time $t$ and $S_t = s$ with strike price $K$ and maturity time $T$ is given by

$$\varphi(t, s, i; K, T; r, \sigma) = E^* \Big( \frac{B_t}{B_T} (S_T - K)^+ \mid S_t = s, X_t = i \Big) \tag{2.3}$$

where $B_t = e^{rt}$ and $E^*$ is the expectation with respect to the risk neutral measure $P^*$. $S = \{S_u\}_{u \geq t}$ under $P^*$ solves

$$dS_u = rS_u du + \sigma(X_t) S_u dW_u^* \qquad S_t = s > 0 \tag{2.4}$$

where $W^*$ is brownian motion under $P^*$. Above equation has a closed form solution. We denote it by $S_u^{(t,s)}$. The solution is given by

$$S_u^{(t,s)} = s \exp\Big( \int_t^u (r - \frac{\sigma(X_{t'})^2}{2}) dt' + \int_t^u \sigma(X_{t'}) dW_{t'}^* \Big). \tag{2.5}$$

Hence

$$S_u^{(t,1)} = \exp\Big( \int_t^u (r - \frac{\sigma(X_{t'})^2}{2}) dt' + \int_t^u \sigma(X_{t'}) dW_{t'}^* \Big)$$

which implies,

$$S_u^{(t,s)} = s S_u^{(t,1)}.$$

Using the above equation, we rewrite equation (2.3).

$$\begin{aligned}
\varphi(t, s, i; K, T; r, \sigma) &= E^*\Big( \frac{B_t}{B_T} (S_T^{(t,s)} - K)^+ \mid S_t^{(t,s)} = s, X_t = i \Big) \\
&= E^*\Big( \frac{B_t}{B_T} s (S_T^{(t,1)} - \frac{K}{s})^+ \mid s S_t^{(t,1)} = s, X_t = i \Big) \\
&= s E^*\Big( \frac{B_t}{B_T} (S_T^{(t,1)} - \frac{K}{s})^+ \mid S_t^{(t,1)} = 1, X_t = i \Big) \\
&= s \varphi(t, 1, i; \frac{K}{s}, T; r, \sigma).
\end{aligned}$$

Since the BSM model is a special case of (1.4), the BSM call option price $C(t, s; K, T; r, \sigma)$ also satisfies

$$C(t, s; K, T; r, \sigma) = sC(t, 1; \frac{K}{s}, T; r, \sigma).$$

Fix a ratio $\frac{K}{s} = p > 0$. Then IV for a contract with strike price $sp$ and spot price $s$ at time $t$ is the solution of the following equation

$$\varphi(t, s, i; sp, T; r, \sigma) = C(t, s; sp, T; r, IV). \tag{2.6}$$

Using above equations,

$$\varphi(t, 1, i; p, T; r, \sigma) = C(t, 1; p, T; r, IV). \tag{2.7}$$

Since the equation (2.7) does not involve the spot price of stock, the IV obtained by using contracts with strike $sp$ does not depend on $s$. If $p < 1$, those contracts are ITM and if $p > 1$ those contracts are OTM contracts. For ATM options, $p = 1$. $\square$

### Error analysis using numerical experiments

We already showed that the ATM IV is constant given that the market regime and time to maturity are fixed. Although we have given theoretical proof, it is necessary to observe this feature using numerical experiments. The reason is, since there is no closed form solution to obtain MMGBM option prices, one always has to rely on numerical methods. Small errors in option prices might result in huge changes in IV. So unless our numerical scheme is robust, we don't observe the constancy of ATM IV. Hence we undertake some numerical experiments using the numerical scheme proposed in section 1.3. We will be dealing with ATM options only. However above theoretical result also applies to ITM and OTM options. Here we vary stock price from 0.8 to 1.2 with step size 0.025. Then we plot IV values for all stock price values and for all three regimes (Fig 2.4). The outcome is as expected and it reassures that the Implied volatility of ATM options is constant for any given regime. In the subsequent sections, we use this fact to observe regime switching in the markets.

Now we look at the errors in IV for different set of parameter combinations. Hence we used same set of parameter combinations as we have done in section 2.1. For each set of parameters we varied stock price from 0.8 to 1.2 with step size 0.025. For all those 17 points, option prices are obtained by solving integral equation. Then we calculate IV. For a given regime $i$, we calculated error by fixing time to maturity $\tau = T$,

$$error(i) = \left(\max_K AV(K,T,i) - \min_K AV(K,T,i)\right) / \min_K AV(K,T,i).$$

The highest error obtained for any regime and for any set of parameters was comparable to computational error and is of the order of $10^{-4}$. Hence we can say that ATM IV is indeed constant for a given regime for a wide range of market parameters and the proposed numerical scheme is good enough to reproduce the constancy of ATM IV.

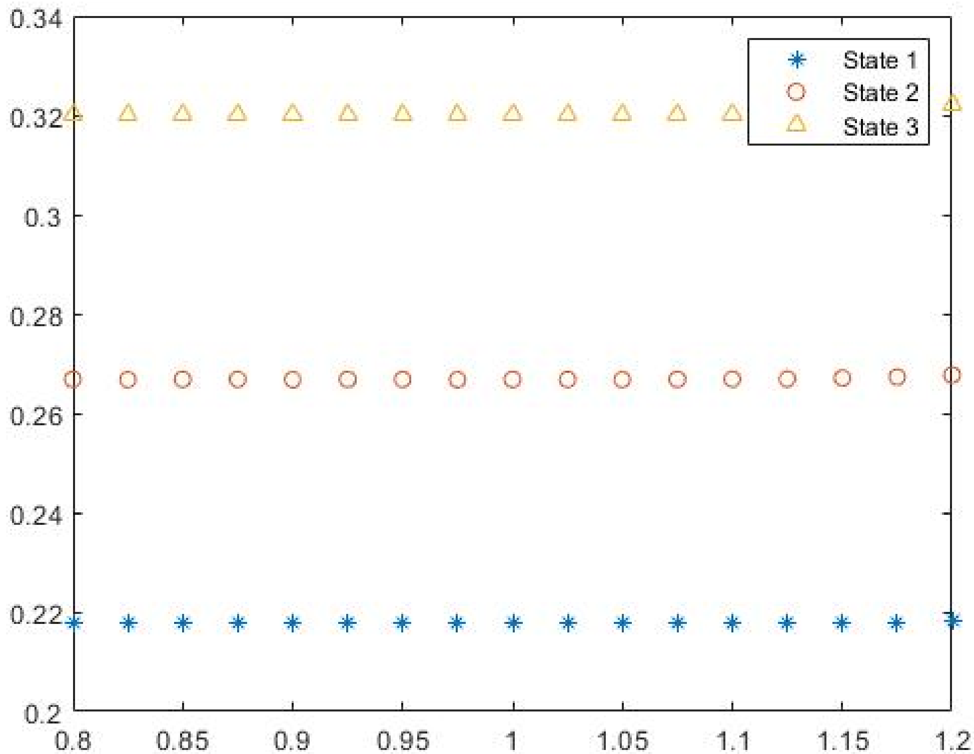


Figure 2.4: Vertical axis shows ATM IV values and Horizontal axis represents stock/strike values. Time to maturity 25 days and state 1, state 2 and state 3 corresponds to sigma values 0.2, 0.3 and 0.4 respectively.

# Chapter 3

# ATM IV time series

In the previous chapter, we have presented the Markov modulated GBM model and theoretical option pricing under the model assumptions. It is important to note that in the regime switching model, the underlying Markov process is assumed to be observable which is not the case in real market. Given a real market data, challenge is to identify the market state at any given point of time. This is impossible if we know only stock price values. Because even if there seems to be an unusual change in the stock price, we can never be sure whether that is due to a discontinuous change in the volatility coefficient or a rare occasion of a very large increment in Brownian motion term. So we can't say surely that there is a regime switching in the market just by looking at the spot prices. Hence we need more information of asset price dynamics than the dynamics itself reflects.

Interestingly the market traded price of options on the underlying asset do carry some additional(market perceived) information of the underlying asset dynamics. There are several types of options traded in the market on a daily basis. Each type of option corresponds to a variety of contracts depending on various contract parameters. However the Theorem 6 followed by the numerical experiments establishes that the IV obtained from ATM call option with a given time to maturity remains constant as long as the Markov volatility coefficient doesn't experience any transition. Hence our expectation is to observe the regime switching in IV time series plot for ATM options. An intuitive basis for this is the fact that whenever the sigma value of underlying asset changes, the change is reflected in the price of option, which in turn affects IV. Or in other words an increase or decrease in sigma

value corresponds to increase or decrease in ATM option prices. So whenever there is a transition in the sigma value of underlying asset it is expected that there will be a transition in ATM IV values. Moreover option prices are observable in the real market. Hence we can compute the IV values also. In the theoretical framework of MMGBM market, we ask the question whether it is possible to find out current market state by observing only IV of ATM options. One can argue that we can't compare the results of theoretical framework with real market because we assumed that the underlying Markov chain is observed in the theoretical MMGBM market but it is not observed in case of real market. So how can we say that real market follows MMGBM? The answer is market might have perceived the regime switching with the collective knowledge of all participants even though each participant is unaware of the current state of the market. Hence there is a possibility that the results of theoretical framework could be applicable in real market too.
We made few assumptions:

1. Market follows MMGBM and prices of options are the locally risk minimizing price [11]

2. Interest rate is assumed to remain constant.

3. In some of the next sections , we fixed a constant time to maturity. This means we made an assumption that every day there will be options with same time to maturity. In other words, we can say each day is an expiration date of a family of options.

4. We assume that options with all strike prices are available in the market.

5. The round off or truncation error in solving the price equation is negligible compared to the changes in the volatility coefficient.

Following is the step by step procedure to obtain IV time series for ATM options on a simulated MMGBM data of stock prices.

1. First we simulate a continuous time Markov chain according to given rate matrix. Here the number of states of Markov chain is three.

2. Then using the generated Markov chain, we simulate stock prices. The equation to simulate stock prices is as follows, which can be obtained by solving (1.1) using Ito's

calculus,

$$S_{t+dt} = S_t \exp\{(\mu(i) - \frac{\sigma(i)^2}{2})dt + \sigma(i)\sqrt{dt}Z_t\} \tag{3.1}$$

where $\mu(i)$ and $\sigma(i)$ corresponds to drift and volatility parameters for $i_{th}$ state and $Z_t$ are independent Standard normal random variables.

3. We fix each $s$ value as our strike price and obtain the option prices for different stock price and different time to maturity using discretized integral equation (1.16). Then use interpolation technique to get option price for the desired stock price. Since we are dealing with ATM options only, both strike and stock prices are same.

4. Using the above obtained option prices, we calculate IV at each time point. In other words, $IV_t$ is the solution of the following equation

$$C(t, s; s, \tau; r, IV_t, ) = \varphi(t, s, i; s, \tau; r, \sigma)$$

where $\tau$ is the fixed time to maturity, $r$ is the constant interest rate, $\sigma = \big(\sigma(1), \sigma(2), \sigma(3)\big)$ the volatility coefficients.

5. Next we plot $\{IV\}_t$. Then we compare this time series with the Markov chain realised in the MMGBM simulation. The purpose of this plot is to illustrate that the ATM IV time series do capture the instances of regime switching, the transition times of Markov chain.

## 3.1 Parameters used and results obtained

For every stock price $s$ obtained from simulation, we fix it as strike price and solve (1.16) with a fixed particular time to maturity to get option prices. But we obtain option prices at grid points of $\Delta s$ step size. To obtain the option price for the desired stock price $s$ we need to do interpolation. Let

$$m_1 = \frac{s}{\Delta s}$$

Then the call option price for the desired $s$ is given by

$$\varphi(t, s, i) = \Big(m_1 - \lfloor m_1 \rfloor\Big)\varphi^n_{\lceil m_1 \rceil}(i) + \Big(\lceil m_1 \rceil - m_1\Big)\varphi^n_{\lfloor m_1 \rfloor}(i) \tag{3.2}$$

where $\lfloor \ \rfloor$ and $\lceil \ \rceil$ denote the integer floor and ceiling functions respectively.

We used same parameters as specified in section 1.3.2. But instead of taking 1.5 as $M$ value, we took 3. The reason is that while simulating stock price values, since we have taken $S_0$ to be 1, there are high chances that stock prices will go beyond 1.5. The plot of ATM IV time series showed us that ATM IV is constant for a given regime. Jumps in the ATM IV exactly matches with the jumps of volatility coefficient of the underlying asset. This shows that ATM IV plot captures the instances of regime switching, which can't be captured by stock price alone.

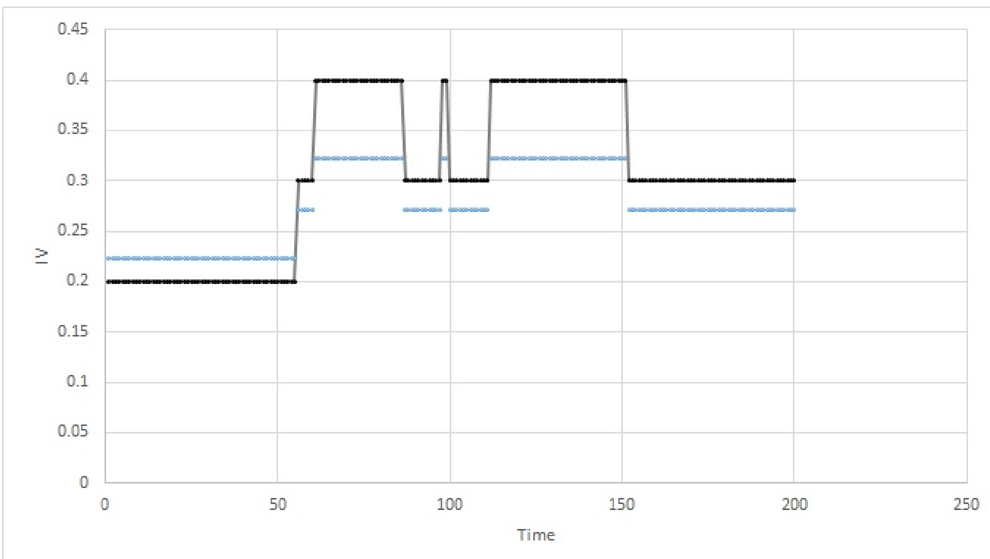


Figure 3.1: ATM IV time series. Time is in day units. TTM is kept constant.

- Line plot represent volatility parameter corresponding to the Markov states simulated. These are not observed in the actual market data. Horizontal lines represents the current market state.

- Point plots represent IV of At the Money options. These values can be computed from the observed market price of ATM options.

## 3.2 Real market data: some limitations

The above approach has limitations with real data. In actual market, not every day in a month is an expiration day of an ATM call option with fixed time to maturity on a fixed asset.Hence we can't obtain ATM IV time series data as stated above for all consecutive time points. For example, Nifty50 index has monthly expiry. So for a fixed maturity date, say 10 days, there will be only one data point in a month. That means 12 data points for a year. This is very low number of data. There is no straight forward way to circumvent this problem. Since time series of IV is not an uncorrelated time series,the surrogate data or resampling techniques do not help in increasing the number of data. But of course we can relax the condition of constant time to maturity and fix some interval of time to maturity and collect data of ATM options. One more problem with the real data is the absence of perfect ATM options. Not all strike price contracts are available in the real market whereas stock price takes any values. For example Nifty50 Index options have strike prices in the multiple of 100. So when Nifty50 is at 10230, there is no contract with strike price 10230. We choose nearest available i.e 10200 strike price as ATM option.

## 3.3 simulation study with varying time to maturity

In the above ATM IV simulated time series, we kept time to maturity as constant.But this is not much useful to deal with real data. So in this section we simulated the IV time series for ATM options with monotonically decreasing time to maturity, which is the case with real market. Here we fix the difference between two successive expiry dates as 20 trading days. Thus the Time to maturity(TTM) varies from 39 to 20 days. That means first trading day will have time to maturity 39 days and $20^{th}$ trading day will have TTM 20 days. Once again twenty first day will have 39 day TTM and so on. Same parameters as given in section 3.1 are used in simulation. Figure 3.2 is the plot obtained.

Point plots are IV points. These are observable in the real market. Line plot represents the regime switching. Horizontal lines give the sigma value of the corresponding state. Our initial expectation was to observe the regime switching in ATM IV time series with constant time to maturity. But the ATM IV plot with monotonically decreasing TTM showed that it is also capable of capturing the instance of regime switching.

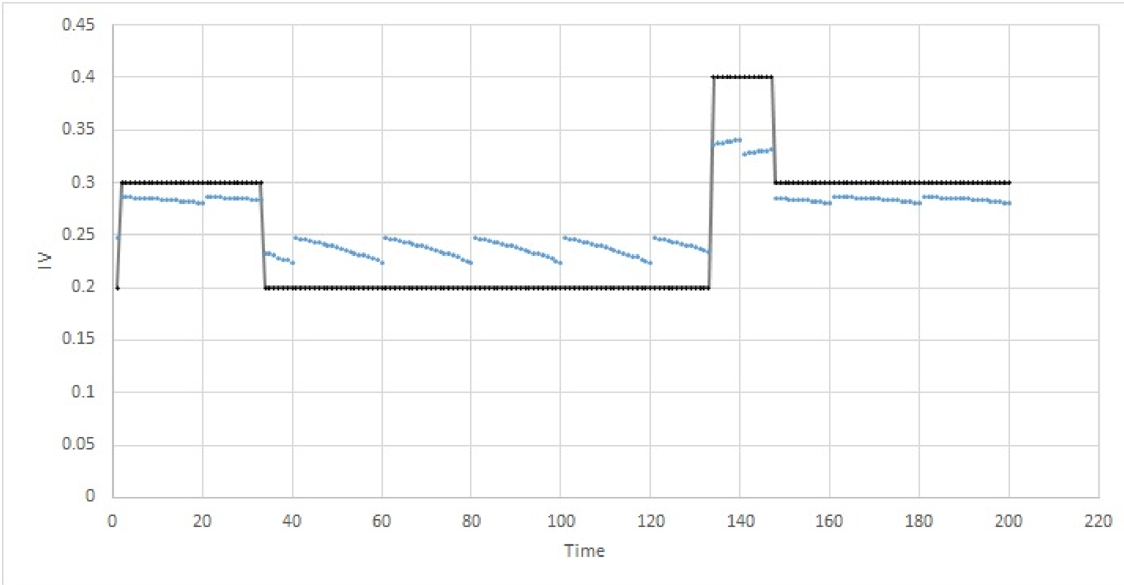


Figure 3.2: ATM IV time series. TTM varies from 40 to 20 days.

Vertical lines are drawn at every 20 days to indicate the expiry dates. We can notice that there is a small shift in IV after the expiry day even though the market is in the same state. For example,between 40 to 80 days market was in first state. But after the $60^{th}$ day, there is a small shift in the IV value. This small shift should not be confused with regime switching. Here market is in same regime and small jump in IV appears due to large change in time to maturity. We can also notice that in the first state, where sigma value is 0.2 , the IV decreases as the time to maturity decreases. But in state 3, which has sigma 0.4, the IV increases as the time to maturity decreases. In second state IV is almost constant throughout the expiry.

## 3.4 Simulation study with realistic ATM options

Now in this section, we simulate stock prices and compute IV values in same way as the realistic ATM options. We consider Nifty50 Index options as the basis for the study. Nifty50 Index options have strike prices in the multiples of 100. Its current price is around 10000 levels. But in simulation we normalize it to 1. If Nifty50 is at 10234, we consider 10200 strike price option as the ATM option. Hence in the simulation also we need to round off stock prices to two decimal points and consider those values as strike price of ATM option.

In this section, first we simulate stock prices using equation . Then we consider stock prices rounded off to two decimal places as strike prices. We calculate option prices and IV values as discussed in previous sections. Two cases are possible. One with constant time to maturity and other with varying time to maturity. We studied both the cases.

### 3.4.1 With constant TTM

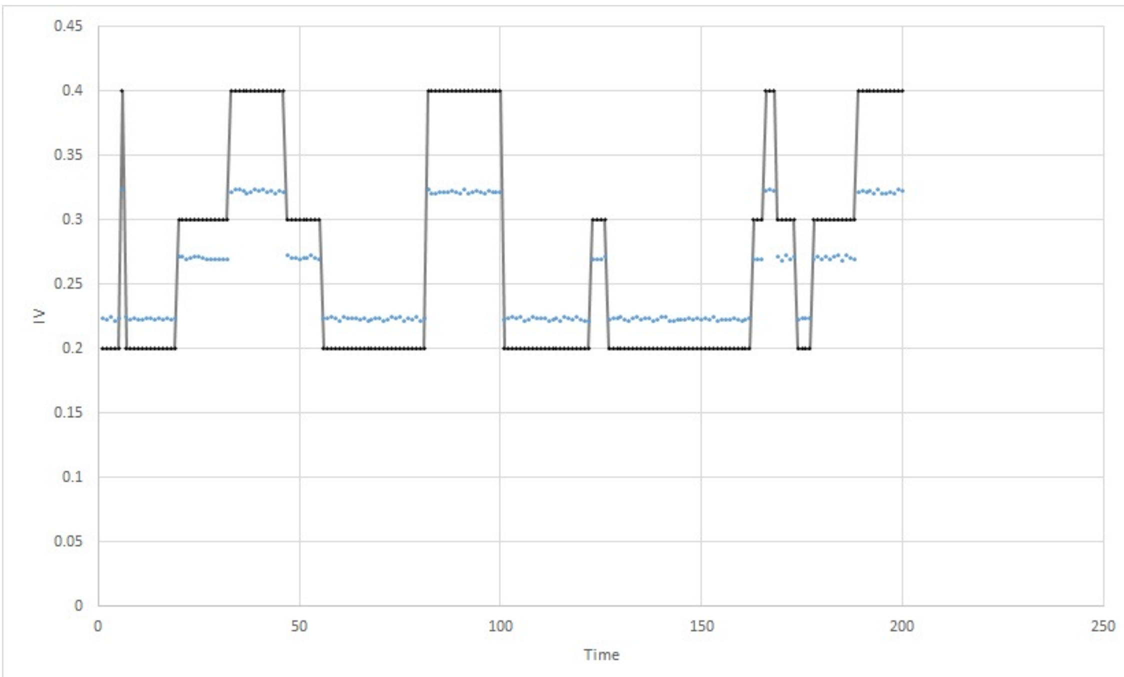


Figure 3.3: ATM IV time series for realistic ATM options. TTM is kept constant at 25 days

In the above figure we can clearly observe regime switching even though IV in a given state is not exactly constant as in fig. The small variations in IV within a given state is due to the fact that the options considered in this section are not exactly ATM options. Since we rounded off stock prices to get strike prices, the results we got in this section slightly vary from the result in section 3.1.

### 3.4.2 With varying TTM

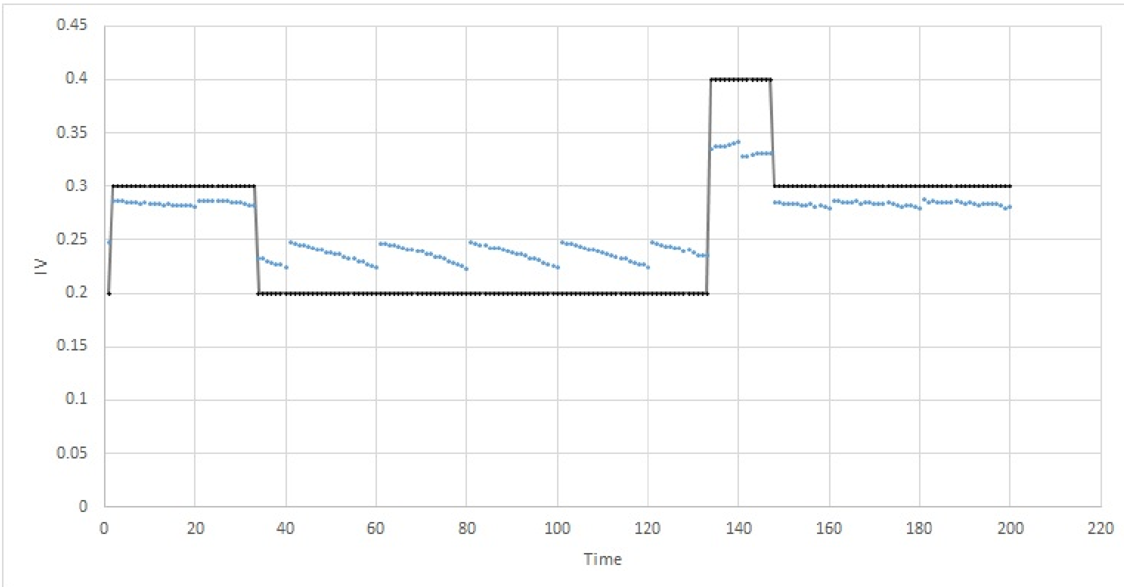


Figure 3.4: ATM IV time series for realistic ATM options. TTM varies from 40 to 20 days.

The results of this section are very similar to that of section 3.3. Regime switching is observable and the small variations in IV within a given regime is due to large change in time to maturity and rounded off strike price. Moreover variation of IV with respect to time to maturity is consistent with the results of the section 2.2.1. Similar feature has been observed in Nifty50 data also which will be discussed in the next section.

## 3.5 Empirical study

We collected data of NIFTY 50 Index options. We used the NSE website for this purpose and these are the following links :

1. https:// www.nseindia.com/ products/ content/ equities/ indices/ historical_index_data.htm
2. https:// www.nseindia.com/ products/ content/ derivatives/ equities/ historical_fo.htm
3. http:// www.option-price.com/ implied-volatility.php

First of all we use the first link and get the Nifty 50 index data for past 8 years. Then we look into the index value on a particular day . Then we decide the nearest tradable strike price and we choose the expiry in such a way that the time to maturity lies between the desired range. Then we use the second link and obtain the traded price, no of contracts information for the option with selected strike price. At the end we use the option price of that option to find out the implied volatility using the third link. Instead of manually doing it we used a python programme[1] to get the data automatically. Following figures show IV of NIFTY 50 ATM options for various fixed intervals of time to maturity from 2010 to 2018.

IV vs time T from 1 to 20.jpg

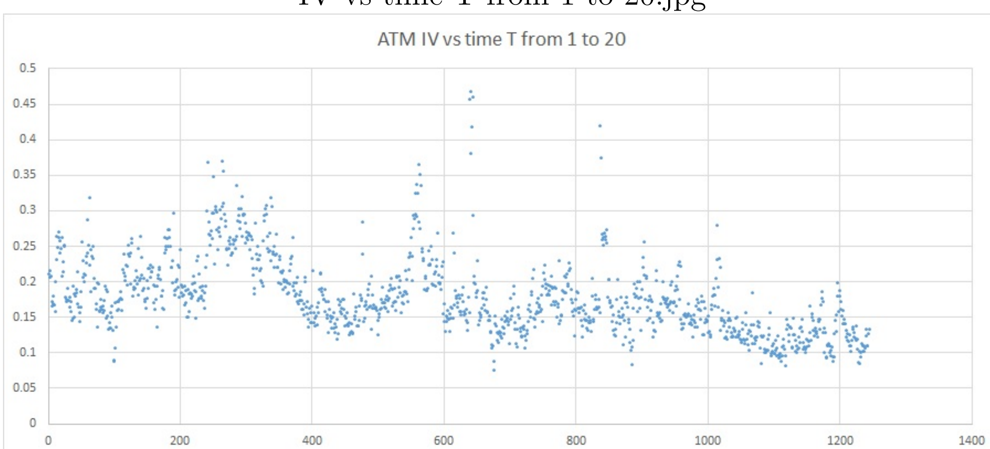


Figure 3.5: TTM from 20 to 1 days

Real IV data showed that the IV vs time plot has the same pattern irrespective of different time to maturity. Also the highest IV recorded was higher for options with time to maturity

[1]Thanks for Sweta Mahajan and Piyush for writing the code

IV vs time T from 20 to 40.jpg

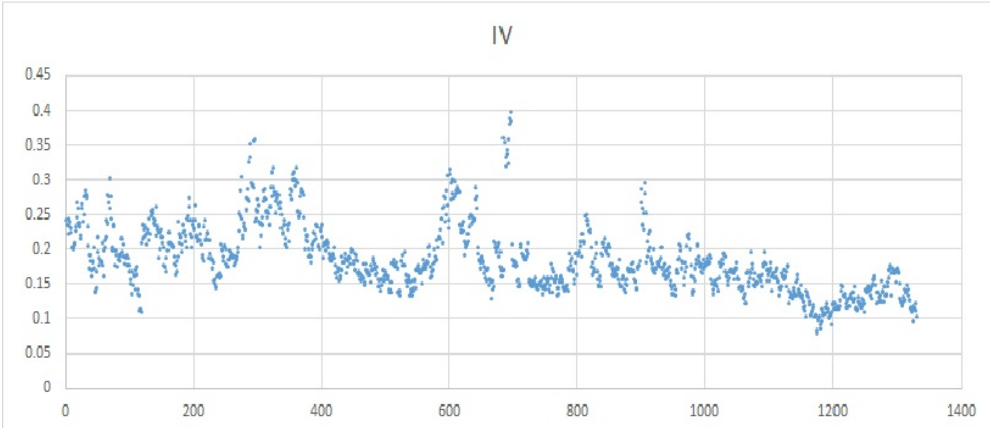


Figure 3.6: TTM from 40 to 20 days

IV vs time T from 40 to 60.jpg

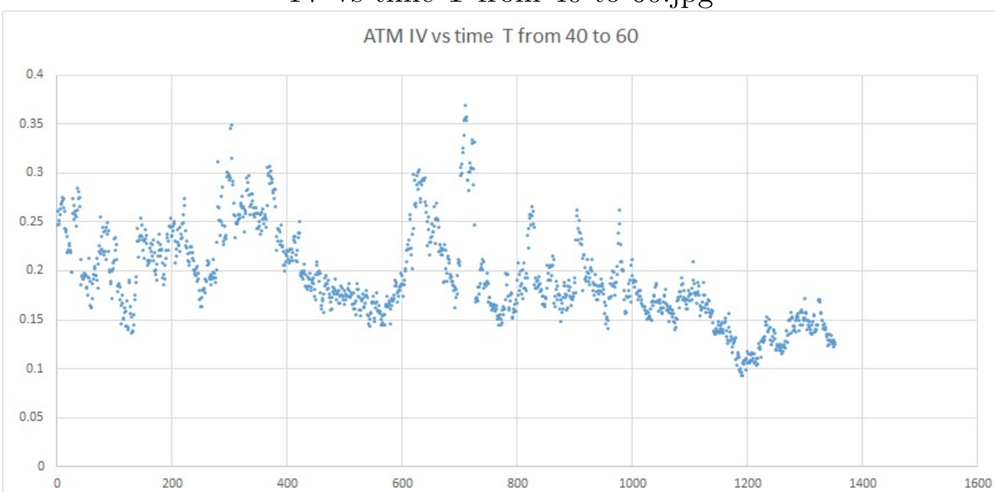


Figure 3.7: TTM from 60 to 40 days

1 to 20 days than the options in other two categories. By this we can say that IV is higher for near month expiry option contracts. On the other hand, we recall that the simulated IV (from MMGBM model) also showed similar feature which has been described in section 3.4.2.

In view of such close resemblance it is tempting to reproduce the IV time series obtained from Nifty50 ATM options (figure 3.6) using MMGBM model for some appropriate choice of parameters. However that leads to an estimation problem which is beyond the scope of present study. We have already plotted in section 3.4.2 and the figure 3.4 didn't match visually with that of figure 3.6. One may anticipate that the difference in patterns of actual and theoretical ATM time series is arising due to the mis-match in option prices. Actual market option price might be different from the theoretical MMGBM price which is causing these variations in ATM IV time series. In this connection it is natural to add some noise to option price and explain if that can bridge the gap between theoretical and empirical observations. But it would be messy and unwise thing to do because option price should lie in the range as discussed in Theorem 5 and the noisy version may exceed the boundary. Nevertheless one may add a multiplicative noise directly to ATM IV data. Hence we take up this experiment by considering the noise term to be iid Lognormal random variable. First we obtain the theoretical MMGBM ATM time series as in section 3.4.2 with the parameter values lambda and sigma chosen arbitrarily. Then noise is multiplied to resulting MMGBM ATM IV data points to make it visually and statistically comparable to Nifty50 ATM IV time series. By trial and error we found out the parameter value of lognormal distribution so that after multiplying such noise, spatial distribution of MMGBM IV data closely matches with that of Nifty50 IV data. Figure 3.8 shows how MMGBM IV time series looks after adding noise. This plot surely doesn't visually matching the empirical one. Or in other words, temporal distributions do not match. This, although not definitely, indicates that the MMGBM model may not explain the dynamics of the data under consideration. We have compared the empirical CDF of both the time series using empirical CDF vs theoretical CDF plot. In that, a straight line is obtained (fig 3.9), which implies both CDF are closely matching.

adding noise.jpg

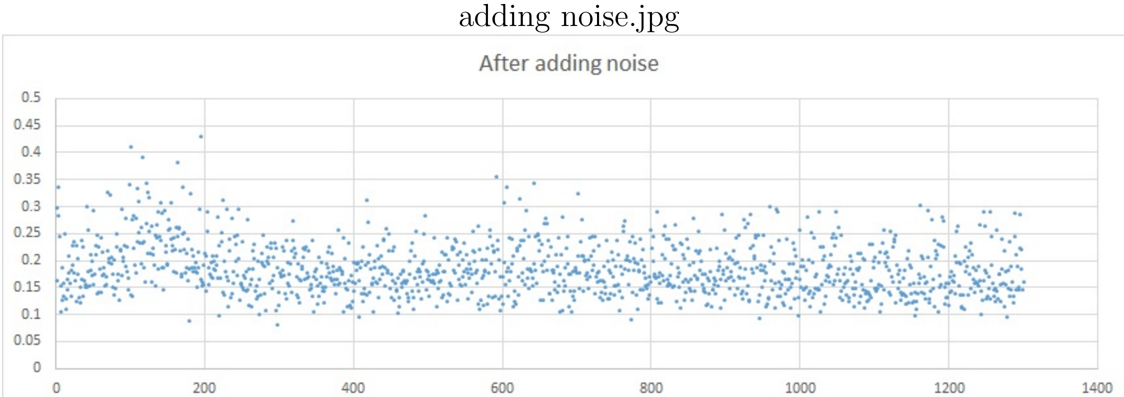


Figure 3.8: Simulated ATM IV data set after adding noise. TTM varies from 40 to 20 days

vs mmgbm ecdf.jpg

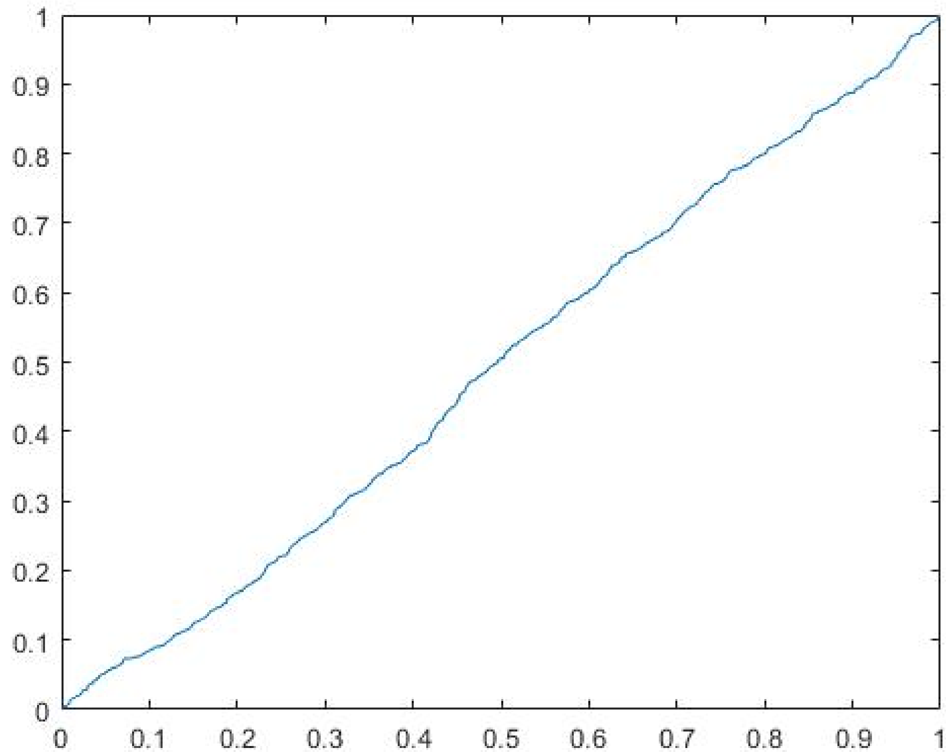


Figure 3.9: Plot of Nifty data CDF vs MMGBM data CDF. From 0 to 0.4 we divide the interval into 100 parts and calculated CDF of both NIFTY data and MMGBM IV data and then plotted those CDF against each other.

# Conclusion

The proposed numerical scheme to solve for MMGBM option prices is fairly accurate and stable under certain condition on $\Delta t$.

We have given the proof of existence of Implied Volatility for all market traded options (Theorem 5). We have observed a clear dependence between IV and strike price under MMGBM model that matches with that of the well known empirically observed volatility smile qualitatively. Dependence between ATM IV and time to maturity is at par with the empirically observed stylized fact. Both the above observations are supported by several numerical experiments.

ATM IV is insensitive to stock/strike variations as long as market state is unchanged. ATM-IV time series can identify the transitions of hidden Markov chain provided the coefficient values at different regimes are not too close to each other. We have proved it theoretically (Theorem 6) and also shown it computationally.

All the above results enable us to estimate the transition rate matrix of the Markov chain, which can be done in future.

# Appendix

## Matlab code (fig 1.1)

```
d = 1.0/(sqrt(2*pi));
T = 0.1; % MATURITY
N=51;
dt=T/(N-1);
u=zeros(41,400,3);
M=400;
Max=3;
dx= Max/M;
P= [0,2/3,1/3;0.5,0,0.5;1/3,2/3,0];
lambda=[10,20,10];
mu= [0.08 0.09 0.1];
R=[0.05,0.05,0.05];
SIG=[0.2,0.3,0.4];
st=1;
w=zeros(N);
w(1,1)=1/2;
for i=2:N
    w(i,1)=1/3;
    if mod(i,2)==0
        w(i,i)=4/3;
        w(i-1,i)=1/2;
    else
        w(i,i)=5/6;
        w(i-1,i)=1/3;
    end
    for j=i+1:N
        if mod(i,2)==0
            w(j,i)=4/3;
        else
            w(j,i)=2/3;
        end
    end
end
for m0=2:M
                        if mod(m0,2)==0
                            q(m0)=2/3;
                        else
                            q(m0)=4/3;
                        end
    end
    q(1)=1/3;
    q(M+1)=1/3;


tic
```

```
    for k=1:3
        for i=1:N
            for j=1:M
                t=(i-1)*dt;
                s=j*dx;

eta(i,j,k)=s*normcdf((log(s/st)+(R(k)+0.5*SIG(k)^2)*t)/(SIG(k)*sqrt(t)))-
st*exp(-R(k)*t)*normcdf((log(s/st)+(R(k)-0.5*SIG(k)^2)*t)/(SIG(k)*sqrt(t)));
            end
        end
    end
    %at maturity price
    for j=1:M
        for i=1:3
            u(1,j,i)=max(0.0,dx*(j)-st);
            eta(1,j,i)=u(1,j,i);
        end
    end
    %Construction of Matrix G( m,m0,l,i)
    for i=1:3
        for l=2:N
            for m=1:M
                for m0=1:M

                    G(m,m0,l,i)=exp(-0.5*(( log(m0/m)-(R(i)-
0.5*SIG(i)^2)*((l-1)*dt))/(SIG(i)*sqrt((l-1)*dt) ))^2)*d/(m0*SIG(i)*sqrt((l-
1)*dt));
                end

            end
        end
    end



     for n=2:N

        for m=1:M
            for i=1:3
                vint=0;
                for l=2:n
                    xint=0;

                    for m0=1:M
                        psum=0;
                        for j=1:3
                           psum=psum+P(i,j)*((u(n-l+1,m0,j)-m0*dx));
                        end
                        xint=xint+psum*G(m,m0,l,i)*q(m0+1);
                    end
                    vint=vint+w(n-1,l)*exp(-R(i)*(l-1)*dt)*lambda(i)*exp(-
lambda(i)*(l-1)*dt)*(xint+m*dx*exp(R(i)*(l-1)*dt)-st*exp(-R(i)*(n-l)*dt)*(1-
logncdf(Max,log(m*dx)+(R(i)-0.5*SIG(i).^2)*(l-1)*dt,SIG(i)*sqrt((l-1)*dt))));
                end
```

```
                B(i)=dt*vint+eta(n,m,i)*exp(-lambda(i)*n*dt)-
0.5*lambda(i)*dx*dt;
            end
            u(n,m,:)=(eye(3)-w(n-1,1)*dt*diag(lambda)*P)\B';
            u(n,m,:)=max(0.0,u(n,m,:));
        end
    end
toc



mm=dx:dx:Max;
mmm=1:1:length(mm);
plot(mm,u(N,mmm,1),'b')
hold on
plot(mm,u(N,mmm,2),'g')
hold on
plot(mm,u(N,mmm,3),'r')
plot(mm,eta(N,mmm,1),'--')
hold on
plot(mm,eta(N,mmm,2),'--')
hold on
plot(mm,eta(N,mmm,3),'--')
plot(mm,u(1,mmm,3))
legend('MMGBM State 1','MMGBM State 2','MMGBM State 3','BSM State 1','BSM
State 2','BSM State 3','Maturity Value')
```

# Bibliography


[1] Arendt W., Batty C., Hieber, M. and Neubrander, F. (2001). *Vector-valued Laplace Transforms and Cauchy Problems*. Birkhauser.

[2] Basak G. K., Ghosh M. K. & Goswami A. (2011). Risk Minimizing Option Pricing for a Class of Exotic Options in a Markov-Modulated Market. *Stochastic Analysis and Applications*, 29(2), 259-281.

[3] Black F. and Scholes M. (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy, 81*(3), 637-654.

[4] Bo, Lijun, Yongjin Wang, and Xuewei Yang. Markov-modulated jump-diffusions for currency option pricing. Insurance: Mathematics and Economics 46.3 (2010), 461-469.

[5] Buffington J. and Elliott R. J. (2002). American options with regime switching *Intl. J. Theor. Appl. Finance, 5*, 497-514.

[6] Das, M. K.; Goswami, A. and Patankar, T.(2018). Pricing derivatives in a regime switching market with time inhomogeneous volatility. *Stoch. Anal. Appl., 36*(4), 700-725.

[7] Heath David, Platen Eckhard and Schweizer M. (2001). A comparison of two quadratic approaches to hedging in incomplete markets. *Mathematical Finance, 11*, 385-413.

[8] Deshpande A. and Ghosh M. K. (2008). Risk Minimizing Option Pricing in a Regime Switching Market. *Stoch. Ann. App., 26*(2), 313-324.

[9] DiMasi G. B., Kabanov M. Yu. and Runggaldier W. J. (1994). Mean-Variance hedging of options on stocks with Markov volatitlity. *Theory Probab. Appl. 39,* 173-181.

[10] Fan, K., Shen, Y., Siu, T. K., & Wang, R. Pricing foreign equity options with regime-switching. Economic Modelling, 37(2014), 296-305.

[11] Föllmer H. and Schweizer M. (1991). Hedging of contingent claims under incomplete information. *Applied Stochastic Analysis: Stochastic Monographs, 5*, 389-414.

[12] Goswami, A., Patel, J., and Shevgaonkar, P.(2016). A system of non-local parabolic PDE and application to option pricing. Stochastic Analysis and Applications 34(5), 893-905.

[13] Goswami A. and Saini, R. K. (2014). Volterra equation for pricing and hedging in a regime switching market.*Cogent Economics & Finance*, 2(1), pp 939769.

[14] Guo X. (2002). Information and option pricing. *Quantitative Finance 1*, 38-44.

[15] Jobert A. and Rogers L. C. G. (2006). Option pricing with Markov-modulated dynamics. *SIAM J. Control Optim., 44*, 2063-2078.

[16] Li, Jinzhi, and Shixia Ma. Pricing options with credit risk in Markovian regime-switching markets. Journal of Applied Mathematics 2013(2013), 621371, 9 pages.

[17] Lian, Yu-Min, Jun-Home Chen, and Szu-Lang Liao. Option pricing on foreign exchange in a Markov-modulated, incomplete-market economy. Finance Research Letters 16 (2016), 208-219.

[18] Mamon R. S. and Rodrigo M. R. (2005), Explicit solutions to European options in a regime switching economy. *Operations Research Letters, 33*, 581-586.

[19] Schweizer M. (1991). Option hedging for semi-martingales. *Stochastic Processes and their Applications*, 339-363.

[20] Schweizer M. (2001). A guided tour through quadratic hedging approaches, E. Jouini, J. Cvitanić, M. Musiela (eds.). *Option Pricing Interest Rates and Risk Management.* Cambridge University Press, 538-574.

[21] Siu, Tak Kuen, Hailiang Yang, and John W. Lau. Pricing currency options under two-factor Markov-modulated stochastic volatility models. Insurance: Mathematics and Economics 43.3 (2008), 295-302.

[22] Su, Xiaonan, Wensheng Wang, and Kyo-Shin Hwang. Risk-minimizing option pricing under a Markov-modulated jump-diffusion model with stochastic volatility. Statistics & Probability Letters 82.10 (2012), 1777-1785.

[23] Swishchuk, Anatoliy, Maksym Tertychnyi, and Robert Elliott. Pricing currency derivatives with Markov-modulated Lévy dynamics. Insurance: Mathematics and Economics 57 (2014), 67-76.

[24] Tsoi A. H., Yang H. and Yeung S. N. (2000). European option pricing when the risk free interest rate follows a jump process. *Comm. Statist. Stoch. Models, 16*, 143-166.

[25] Wang, W., Jin, Z., Qian, L., and Su, X. Local risk minimization for vulnerable European contingent claims on nontradable assets under regime switching models. Stochastic Analysis and Applications 34.4 (2016), 662-678.