Bound on Torsion Points on Elliptic Curves over Number Fields

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Certificate

This is to certify that this dissertation entitled Bound on Torsion Points on Elliptic Curves over Number Fields towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Nasit Darshan Prafulbhaiat Indian Institute of Science Education and Research under the supervision of Dr. Debargha Banerjee, Assistant Professor, Department of Mathematics , during the academic year 2018-2019.

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To

Mummy and Papa

Declaration

I hereby declare that the matter embodied in the report entitled Bound on Torsion Points on Elliptic Curves over Number Fields are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Debargha Banerjee and the same has not been submitted elsewhere for any other degree.

Nasit Darshan Prafulbhai

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Abstract

We propose a modified Rankin Selberg convolution, since the functional equation of Rankin-Selberg convolution for arbitrary cusp form doesn't respect critical line $s = 1/2$. We extend a result of Goldfeld and Hoffstein about the congruence of cusp forms in 'new' space under the assumption of the Riemann Hypothesis for modified Rankin-Selberg convolution.We prove Merel's conjecture which states that the Hecke operators act linearly independently on the winding cycle in the homology group $H_1(X_0(N), \mathbb{Z})$. We also provide an improvement on the bound of number of Hecke Operators which acts linearly independently on the space of cusp forms using estimates on Kloosterman Sums. It also gives linear independence of Poincare series.

Contents

Nomenclature

Γ	Congruence Subgroup
$\Gamma(N)$	$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\bmod N) \in SL_2(\mathbb{Z}) \right\}$
$\Gamma(s)$	$\int_0^\infty t^{s-1} e^{-s} ds$
$\Gamma_0(N)$	$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} (\bmod N) \in SL_2(\mathbb{Z}) \right\}$
$\Gamma_1(N)$	$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} (\bmod N) \in SL_2(\mathbb{Z}) \right\}$
Γ_∞	$\Gamma \cap SL_2(\mathbb{Z})_\infty$
$\mathbb{C}_{\mathscr{R}s>t}$	$\{s \in \mathbb{C}; \mathscr{R}s > t\}$
$\mathcal{H}$	$\{z \in \mathbb{C} : Im(z) > 0\}$
$\mathcal{H}^*$	$\mathcal{H} \cup \mathbb{Q} \cup \infty$
$\mathscr{R}s$	Real part of s
$C(X)$	$\{f : X \to \mathbb{C} : f \text{ is analytic}\}$
$f \ll_N g$	there exists constant C depends on N such that $f \leq Cg$
$J_0(N)$	$Jac(X_0(N))$
$J_1(N)$	$Jac(X_1(N))$
$L(f \times g; s)$	A modified Rankin Selberg L function for cusp form f and g

$q = e(\tau)$	$exp(2\pi\iota\tau)$
$SL_2(\mathbb{Z})$	$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$
$SL_2(\mathbb{Z})_\infty$	$\left\{ \pm \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} : m \in \mathbb{Z} \right\}$
T_n	Hecke Operator
$X_0(N)$	$\Gamma_0(N)\backslash\mathcal{H}^*$

Introduction

The Mordell-Weil theorem about elliptic curves E over $\mathbb{Q}$ says that for any number field K, $E(K)$ is finitely generated abelian group. So $E(K) \equiv \mathbb{Z}^r \oplus E(K)_{tor}$. The strong Uniform Boundness Conjecture (proved by Loic Merel in 1994, [11]) is $|E(K)_{tor}| \leq B(d)$, where $B(d)$ depends on $d = [K : Q]$ but neither E nor K. He explicitly proved bound $p \leq d^{3d^2}$ on largest prime divisor p of $|E(K)_{tor}|$. To prove Uniform Boundness conjecture, a key step is to show that for sufficiently large prime N, the Hecke operators $T_1, T_2, \cdots, T_D$ acts linearly independently on the winding cycle e from 0 to $\iota\infty$ when $\max\{800D^4, D^8\} < \frac{N}{(\log N)^4}$. Later in 1998, J.Vanderkam improved using analytic techniques that result is true when $D^2 \ll N$ (a chapter 3) Moreover, it is conjectured that result is true when $D \ll N$.

The aim of thesis is two-fold. Fold one is to understand the necessary background for the Hecke Operators and the proof given by Vanderkam. The second is to try to improve the bound on D.

Chapter 1 of thesis is an overview of the theory on Modular forms, the Hecke Operators and L function associated with cusp form. Chapter 1 is mostly consisting of the definitions. The first chapter also gives an isomorphism of uncompactified modular curves and enhanced elliptic curves. The Poincare series are defined in chapter 2. They are an important type of a basis of a set consisting of cusp forms. A reader can skip these chapters if he/she knows it properly.

How many number of Fourier coefficient $(a_1(f), a_2(f), \cdots)$ are required to define a modular form f of weight k at level N? Let $a_f(n) = a_g(n)$ for $n \leq A(N, k)$. When can we say $f \equiv g$ (f is congruent to g)? Sturm's theorem explicitly gives the form of $A(N, k)$ (see [12] and [15]). The fourth chapter is a review of that.

Goldfeld and Hoffstein have proved non-congruence of the newforms in terms of Fourier coefficients by using the Rankin Selberg L function for which the functional equation respects the critical line (i.e. $\Lambda(f \times g; s) = \Lambda(f \times g; 1-s)$). Since the Rankin Selberg L function for arbitrary 'new' cusp form does not have a functional equation which respects critical line, we propose a modification in the Rankin Selberg L function. As a result (Theorem 5.2.3), we get a new L functions (modified Rankin Selberg) which have Euler product structure when cusp forms are eigenforms and the functional equation which respects the critical line. We also discuss the non-vanishing region for the modified Rankin Selberg L functions in chapter 5. Assuming the Riemann Hypothesis for the modified Rankin Selberg Convolution (i.e. zeros of function lies on critical line) and using Goldfeld-Hoffstein's method, we get a better bound for $A(N,k)$ which is independent of k. This enable us to prove that the action of Hecke Operators $T_1, T_2, \cdots, T_D$ on the winding cycle e is linearly independent when $D \ll N$ (section 6.2)

We are also interested to find bound on Hecke operators which acts linearly independently on a set of cusp space. By using a estimation on Kloosterman Sums and similar trick (used for Vanderkam's theorem), we get that the action of Hecke Operators is linearly independently if $D \ll N^{1-\frac{1}{2k}}$. We also discuss the relation between linear independence of the Poincare series and linear independence of Hecke operators.

Chapter 1

Preliminaries

Chapter 1 emphasizes the theor of Modular Forms, a modular curve, a Complex Elliptic curve, the Hecke Operators, and the Jacobian of Modular curves. It also introduces L function associated with modular (or cusp) forms and approximated L function. The details can be found in [2],[14],[8] and [3]. At the end of chapter, We introduce thesis problem.

1.1 Modular Forms

Definition 1.1.1. *$SL_2(\mathbb{Z})$ is called as Modular Group.*

Define an action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$. For $\tau \in \mathcal{H}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau) = \frac{a\tau + b}{c\tau + d}$$

Definition 1.1.2. *Let k be an integer. $f : \mathcal{H} \to \mathbb{C}$ is called weakly modular of weight k if $\forall \alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$, $f(\alpha(\tau)) = (c\tau + d)^k f(\tau)$.*

Since $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in SL_2(\mathbb{Z})$, any weakly modular of weight k is periodic function of period 1 and $f(\frac{-1}{\tau}) = \tau^k f(\tau)$. Moreover, k is odd integer then $f \equiv 0$.

Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Then $\mathcal{H}$ is homeomorphic to punctured open unit disk $D' = D - 0$ with usual topology on both set under τ maps to $e(\tau) = e^{2\pi\iota\tau}$. So, any map f from $\mathcal{H}$ can be map g from D' as $g(q) = f(\frac{\log q}{2\pi\iota}))$. If f is weakly modular of weight k, we say f is holomorphic at ∞ if g extends analytically to $0 \in \mathbb{C}$. In this case f has fourier expansion $f(\tau) = \sum_{n\geq 0} a_n(f)q^n$ where $q = e(\tau)$.

Definition 1.1.3. *$f : \mathcal{H} \to \mathbb{C}$ is called modular form of weight k if f is weakly modular of weight k,f is holomorphic on $\mathcal{H}$ and f is holomorphic at ∞.*

$\mathcal{M}_k(Sl_2(Z)) = \{f : \mathcal{H} \to \mathbb{C} : f \text{ is modular form of weight } k\}$ is complex vector space because addition (pointwise) and scalar multiplication of modular form of weight k is also modular form of weight k. $\mathcal{M}(SL_2(\mathbb{Z}) = \bigoplus_{k\in\mathbb{Z}} \mathcal{M}_k(SL_2(\mathbb{Z}))$ is ring because product (pointwise) of modular form of weight m and n is modular form of weight $m + n$.

Definition 1.1.4. *Let f is modular form of weight k. f is called cusp form of weight k if fourier expansion of f vanishes at ∞*

$\mathcal{S}_k(Sl_2(Z)) = \{f : \mathcal{H} \to \mathbb{C} : f \text{ is cusp form of weight } k\}$ is complex vector subspace of $\mathcal{M}_k(SL_2(\mathbb{Z})$ and $\mathcal{S}(SL_2(\mathbb{Z}) = \bigoplus_{k\in\mathbb{Z}} \mathcal{S}_k(SL_2(\mathbb{Z}))$ is an ideal of $\mathcal{M}(SL_2(\mathbb{Z})$.

Definition 1.1.5. *$\Gamma(N)$ is called principle congruence subgroup of level N.*

Definition 1.1.6. Γ *(subgroup of $SL_2(\mathbb{Z})$) is congruence subgroup if $\exists N$ such that $\Gamma(N) \subset \Gamma$.*

$\frac{SL_2(\mathbb{Z})}{\Gamma(N)} \cong SL_2(\frac{\mathbb{Z}}{N\mathbb{Z}})$. And $|SL_2(\frac{\mathbb{Z}}{N\mathbb{Z}})| = N^3 \prod_{p|N}(1 - \frac{1}{p^2})$. So, any congruence subgroup has finite index in $SL_2(\mathbb{Z})$.

Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2^+(\mathbb{Q})$ and $\tau \in \mathcal{H}$ then factor of automorphy at γ is $j(\gamma, \tau) = (c\tau + d)$. $f : \mathcal{H} \to \mathbb{C}$. Define weight-k operator $[\gamma]_k$ such that $(f[\gamma]_k)(\tau) = (det(\gamma))^{k-1}j(\gamma, \tau)^{-k}f(\gamma(\tau))$. Since factor of automorphy is neither zero nor infinity, so f is meromorphic iff $f[\gamma]_k$.

Definition 1.1.7. *We say $f : \mathcal{H} \to \mathbb{C}$ is weakly modular form of weight k with respect to Γ if $f[\gamma] = f$ for all $\gamma \in \Gamma$.*

Definition 1.1.8. *Let Γ be a congruence subgroup and k be an integer. A function $f : \mathcal{H} \to \mathbb{C}$ is modular form of weight k with respect to Γ if f is holomorphic and weight k invariant under Γ and $f[\gamma]_k$ is holomorphic at ∞, $\forall \gamma \in SL_2(\mathbb{Z})$.*

Definition 1.1.9. *f is modular form of weight k with respect to Γ is called cusp form of weight k with respect to Γ if $f[\gamma]_k$ vanishes at infinity $\forall \gamma \in SL_2(\mathbb{Z})$.*

Similarly, we can define $\mathcal{M}_k(\Gamma)$,$\mathcal{S}_k(\Gamma)$, $\mathcal{M}(\Gamma) = \bigoplus_{k\in\mathbb{Z}} \mathcal{M}_k(\Gamma)$ and $\mathcal{S}(\Gamma) = \bigoplus_{k\in\mathbb{Z}} \mathcal{S}_k(\Gamma)$.

Example 1.1.1. *For $k > 2$ even, $G_k(\tau) = \sum_{(c,d)\neq(0,0)\in\mathbb{Z}\times\mathbb{Z}} \frac{1}{(c\tau+d)^k}$ is weight k modular form.*

Example 1.1.2. *$E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n$ is called Eisenstein Series of weight k. Where, $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and B_k is Bernoulli k-th number.*

Example 1.1.3. *Let $g_2(\tau) = 60G_4(\tau)$ and $g_3(\tau) = 140G_6(\tau)$ then discriminant function $\Delta(\tau) = (g_2(\tau))^3 - 27(g_3(\tau))^2$ is cusp form of weight 12.*

Example 1.1.4. *The Dedekind Eta function is $\eta(\tau) = q_{24} \prod_{n\geq 1} (1 - q^n)$ where $q_{24} = e^{2\pi\iota\tau/24}$. Then $\Delta(\tau) = (2\pi)^{12}\eta^{24}$.*

Definition 1.1.10. *A lattice $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z} \subset \mathbb{C}$ is a set, where $\{\omega_1, \omega_2\}$ is basis for $\mathbb{C}$ over $\mathbb{R}$.*

Definition 1.1.11. *A quotient of $\mathbb{C}$ by Λ (lattice) is known as complex torus.*

Proposition 1.1.1. *$\varphi : \frac{\mathbb{C}}{\Lambda} \to \frac{\mathbb{C}}{\Lambda'}$ is holomorphic map between complex torus. Then $\exists m\Lambda \subset \Lambda'$ such that $\varphi(z + \Lambda) = mz + b + \Lambda'$. And φ is invertible iff $m\Lambda = \Lambda'$.*

Corollary 1.1.2. *$\varphi : \frac{\mathbb{C}}{\Lambda} \to \frac{\mathbb{C}}{\Lambda'}$ is holomorphic map between complex torus.Then φ is homomorphism if and only if $b \in \Lambda'$ such that $\varphi(z + \Lambda) = mz + \Lambda'$.*

Definition 1.1.12. *A non-zero holomorphic homomorphism between complex torus is called an isogeny.*

Let $[N] : \frac{\mathbb{C}}{\Lambda} \to \frac{\mathbb{C}}{\Lambda}$ defined as $z + \Lambda \mapsto Nz + \Lambda$ is isogeny.Now kernel of map is isomorphic to $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$. We denote $\mathbb{C}/\Lambda$ by E, then kernel is denoted by $E[N]$.

Proposition 1.1.3. *If $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ then $E[N] = \langle \frac{\omega_1}{N} + \Lambda\rangle \times \langle \frac{\omega_2}{N} + \Lambda\rangle$.*

Let $\mu_N = \{z \in \mathbb{C} : z^N = 1\}$. Let $P, Q \in E[N]$, then $\begin{bmatrix} P \\ Q \end{bmatrix} = \gamma \begin{bmatrix} \frac{\omega_1}{N} + \Lambda \\ \frac{\omega_2}{N} + \Lambda \end{bmatrix}$ for $\gamma \in M_2(\mathbb{Z}/N\mathbb{Z})$. Then Weil pairing of P and Q is $e_N(P,Q) = e^{2\pi\iota \det \gamma/N}$.

Definition 1.1.13. *The Weierstrass $\mathscr{P}$- function for lattice Λ defined on $\mathbb{C}$, $z \notin \Lambda$, $\mathscr{P}(z) = \frac{1}{z^2} + \overline{\sum_{\omega\in\Lambda}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$ $z \notin \Lambda$, summation over non-zero element in lattice.*

Proposition 1.1.4. *For $k > 2$ is even, $G_k(\Lambda) = \overline{\sum_{\omega\in\Lambda}} \frac{1}{\omega^2}$. Let $\mathscr{P}$ is weierstrass function for lattice Λ. Then*

1. *Laurent expansion is $\mathscr{P}(z) = \frac{1}{z^2} + \sum_{n\geq 2 \text{ is even}} (n+1)G_{k+2}(\Lambda)z^n$. Radius of convergence is inf $\{|\omega| : \omega \in \Lambda\}$.*
2. *$(\mathscr{P}'(z))^2 = 4(\mathscr{P}(z))^3 - g_2(\Lambda)\mathscr{P}(z) - g_3(\Lambda)$, where $g_2(\Lambda) = 60G_4(\Lambda)$ and $g_3(\Lambda) = 140G_6(\Lambda)$.*
3. *Let $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ and $\omega_3 = \omega_1 + \omega_2$. Then $\mathscr{P}$ and $\mathscr{P}'$ satisfy $y^2 = 4(x - e_1)(x - e_2)(x - e_3)$ where $e_i = \mathscr{P}(\frac{\omega_i}{2})$ and e_i are distinct roots.*

Corollary 1.1.5. *$(\mathscr{P}, \mathscr{P}')$ is isomorphism between complex torus and elliptic curves.*

Corollary 1.1.6. *Discriminant function is non vanishing function on $\mathcal{H}$.*

Enhanced elliptic curves for $\Gamma_0(N)$ is pair of elliptic curve E and C is cyclic subgroup of $E[N]$. $(E, C) \sim (E', C')$ if there exists isomorphism between E and E' which maps C to C'. Then $S_0(N)$ is set of equivalence classes of enhanced elliptic curves of $\Gamma_0(N)$.

Enhanced elliptic curves for $\Gamma_1(N)$ is pair of elliptic curve E and $Q \in E$ of order N. $(E, Q) \sim (E', Q')$ if there exists isomorphism between E and E' which maps Q to Q'. Then $S_1(N)$ is set of equivalence classes of enhanced elliptic curves of $\Gamma_1(N)$.

An Enhanced Elliptic Curve for $\Gamma(N)$ is pair of elliptic curve E and (P, Q) is pair which generates $E[N]$ and Weil pairing $e_N(P.Q) = e^{2\pi\iota/N}.(E,(P,Q)) \sim (E',(P',Q'))$ if there exists isomorphism between E and E' which maps P to P' and Q to Q'. Then $S(N)$ is set of equivalence classes of enhanced elliptic curves of $\Gamma(N)$.

Let Γ be congruence subgroup.Then modular curve for Γ is $Y(\Gamma) = \{\Gamma\tau : \tau \in \mathcal{H}\} = \Gamma\backslash\mathcal{H}$.

Theorem 1.1.7. *Let $Y_0(N) = Y(\Gamma_0(N))$, $Y_1(N) = Y(\Gamma_1(N))$ and $Y(N) = Y(\Gamma(N))$. Then $Y_0(N) \cong S_0(N)$, $Y_1(N) \cong S_1(N)$ and $Y(N) \cong S(N)$.*

1.2 Modular Curves as Riemann Surfaces

Let Γ be congruence subgroup. Then $X(\Gamma) = \{\Gamma\tau : \tau \in \mathscr{H}^*\}$. Let $X_0(N) = X(\Gamma_0(N))$, $X_1(N) = X(\Gamma_1(N))$ and $X(N) = X(\Gamma(N))$

For topology on $X(\Gamma)$, $\phi : \mathcal{H} \to Y(\Gamma)$ is defined by $\tau \mapsto \Gamma\tau$. Define topology on $Y(\Gamma)$ by quotient topology where $\mathcal{H}$ has subspace topology of euclidean topology on $\mathbb{C}$. Define topology on $X(\Gamma)$ such that $Y(\Gamma)$ is dense subset.
Action of congruence subgroup Γ on $\mathcal{H}$ is properly discontinous. So, $Y(\Gamma)$ is hausdorff space. Let $\Gamma_\tau = \{\gamma \in \Gamma : \gamma(\tau) = \tau\}$ is known as isotropy subgroup.

Definition 1.2.1. *A $\tau \in \mathcal{H}$ is an elliptic point if Γ_τ is non-trivial. Corresponding point $\Gamma\tau \in Y(\Gamma)$ is also called elliptic.*

Corollary 1.2.1. *Throughout our discussion,For congruence subgroup Γ, for any $\tau \in \mathcal{H}$ we can find $\tau \in U \subset \mathbb{C}$ such that for any $\gamma \in \Gamma$, $\gamma(U) \cap U \neq \emptyset \Rightarrow \gamma \in \Gamma_\tau$. And U has no other elliptic points.*

Definition 1.2.2. *For $\tau \in \mathcal{H}$, period of τ is $h_\tau = |\frac{\{\pm I\}\Gamma_\tau}{\{\pm I\}}|$.*

Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ (where $\gamma \neq \pm I$) and $\tau \in \mathcal{H}$, then $\gamma(\tau) = \tau$ if and only if $c\tau^2 - (a-d)\tau - b = 0$ has solution. $\tau \in \mathcal{H}$ implies characteristic polynomial of matrix is either $x^2 + 1$ or $x^2 \pm x + 1$. Further $\gamma^3 = I$ or $\gamma^4 = I$ or $\gamma^6 = I$. So, h_τ is finite. Moreover, we can check, isotropy subgroup are cyclic. $h_\tau = h_{\gamma(\tau)}$ for any $\gamma \in \Gamma$. It is easy to show, imaginary part of elliptic point is strictly less than 2.

Now we can give local structre at any $\Gamma\tau \in Y(\Gamma)$. Let $h = h_\tau$ (it is well-defined). Let $\delta = \delta_\tau = \begin{bmatrix} 1 & \tau \\ 1 & -\overline{\tau} \end{bmatrix} \in SL_2(\mathbb{C})$. Then $\delta(\tau) = 0$ and $\delta(\overline{\tau}) = \infty$. Clearly, $h_{\delta(\tau)} = h$. Choose a nbd $\tau \in U$ as corollary 1.2.1., define $\psi : U \to \mathbb{C}$ by $\psi = \rho \circ \delta$ where $\rho(z) = z^h$. Now define $\varphi : \phi(U) \to \psi(U)$ such that $\varphi(\phi(z)) = \psi(z)$. This map is homeomorphism because fractional linear transformation is homeomorphism. It is trivial to check that map between $\varphi_1(\phi(U_1) \cap \phi(U_2))$ and $\varphi_2(\phi(U_1) \cap \phi(U_2))$ is analytic. So, it gives manifold structure on $Y(\Gamma)$.

Remarks 1.2.1. *Let $\mathcal{D} = \{z \in \mathbb{C} : |z| \geq 1, |Re(z)| \leq 1\}$. Then $f : \mathcal{D} \to Y(SL_2(\mathbb{Z}))$ defined by $f(z) = SL_2(\mathbb{Z})z$ is surjective map. $\mathcal{D}$ is called fundamental domain.*

Note that $Y(\Gamma)$ is connected but non-compact complex manifold of dimension 1. To make it compact, we required domain of ϕ map is compact because continous image of compact set is compact. So, we add $\mathbb{Q} \cup \{\infty\}$ in $\mathcal{H}$. Clearly, $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$ is compact.

Definition 1.2.3. *An element of set* $\Gamma \backslash (\mathbb{Q} \cup \{\infty\})$ *is called cusp for congruence subgroup* Γ.

Definition 1.2.4. *A* $s \in \mathbb{Q} \cup \{\infty\}$ *is called irregular cusps, if* $s = \delta(\infty)$ *then* $\delta^{-1}\Gamma_s\delta = \left\langle -\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \right\rangle$. *Otherwise, it is regular cusp.*

For topology on $\mathcal{H}^*$, define $U_r = \{z \in \mathcal{H} : \text{Im } (z) > r\} \cup \{infty\}$ are basic open set around infinity. For $\gamma \in SL_2(\mathbb{Z})$, $\gamma(\infty)$ is either rational number or infinity. Let $\gamma(\infty) = q \in \mathbb{Q}$, then set of $\gamma(U_r)$ form a base for open set aroud q. It is easy to observe that $|\Gamma \backslash (\mathbb{Q} \cup \{\infty\})|$ is finite. We extend ϕ in abuse of notation, $\phi : \mathcal{H}^* \to X(\Gamma)$. Give quotient topology on $X(\Gamma)$. Continous image of connected and compact is connected and compact respectively. So, $X(\Gamma)$ is connected and compact. To check hausdorffness, we have already checked for $\Gamma\tau_1$ and $\Gamma\tau_2$. For $\Gamma\tau$ and Γs, it is easy. When $x_1 = \Gamma s_1$ and $x_2 = \Gamma s_2$ where $x_1 \neq x_2$, let $s_1 = \gamma_1(\infty)$ and $s_2 = \gamma_2(\infty)$, $V_1 = \gamma_1(U_2)$ and $V_2 = \gamma_2(U_2)$. Let $t \in \phi(V_1) \cap \phi(V_2)$ implies $\exists \gamma \in \Gamma$ such that $\gamma(t_1) = t_2$, so t_1 is an elliptic point, but U_2 has no elliptic point, so $\gamma_2^{-1}\gamma\gamma_1 = SL_2(\mathbb{Z}_\infty)$ implies $x_1 = X_2$. So, $X(\Gamma)$ is an hausdorff space.

For manifold structure, we need only charts around cusps. Let $s \in \mathbb{Q} \cup \{\infty\}$, $\delta = \delta_s \in SL_2(\mathbb{Z})$ such that $\delta(s) = \infty$. Let $h = h_s = |SL_2(\mathbb{Z})_\infty / (\delta(\{\pm I\}\Gamma)\delta^{-1})_\infty|$ is called width of cusp s.Let $U = \delta^{-1}(U_2)$, define $\psi : U \to \mathbb{C}$ by $\psi = \rho \circ \delta$ where $\rho(z) = e^{2\pi\iota z/h}$ and $\psi(U) = V$. Then $\varphi : \phi(U) \to V$ such that $\varphi(\phi(z)) = \psi(z)$ defines homeomorphism. This gives manifold structrure around cusps.

Theorem 1.2.2. *For any congruence subgroup* Γ, $X(\Gamma)$ *is compact, connected and hausdorff space. Moreover,* $X(\Gamma)$ *is complex manifold of dimension 1. (So, it is Riemann Surface.)*

1.3 Dimension of $\mathcal{M}_k(\Gamma)$ and $\mathcal{S}_k(\Gamma)$

We start with genus of compact surfaces. There are saveral ways to define genus. Genus is number of handles. But mathematically, if manifold has Euler characteristic χ and genus $g = 1 - \frac{\chi}{2}$. And Euler characteristic $\chi = V - E + F$, where V is number of verices, E

is number of edges and F is number of faces via triangulation. Let $f : X \to Y$ be non-constant holomorphic map between compact Riemann surface, then f is surjective. $y \in Y$ and $x \in f^{-1}(y)$. (U, p) and (V, q) are local homeomorphism around x, y respectively. Then $qfp^{-1} : p(U) \to q(V)$ is holomorphic maps 0 to 0. Then order of 0, e_x is called ramification degree of x. Ramification degree is independent of local charts. $\sum_{x \in f^{-1}(y)} e_x = d$ is called degree of map f. Degree of map is independent of point in range space. Let g_X and g_Y is genus of X and Y, then by Riemann Hurwitz Formula, $2(g_X - 1) = 2d(g_Y - 1) + \sum_{x \in X}(e_x - 1)$.

Let $X(1) = SL_2(\mathbb{Z})\backslash\mathcal{H}^*$ and $X(\Gamma)$ are compact Riemann surfaces. $f : X(\Gamma) \to X(1)$ is surjective holomorphic map. And genus of $X(1)$ is zero. By Riemann Hurwitz Formula genus of $X(\Gamma)$ is $g = 1 + \frac{d}{12} - \frac{\epsilon_2}{4} - \frac{\epsilon_3}{3} - \frac{\epsilon_\infty}{2}$. Here ϵ_2, ϵ_3 and ϵ_∞ are number of elliptic points of order 2 and 3 and number of cusps.

Definition 1.3.1. *Let $f : \mathcal{H} \to \mathbb{C}$ and Γ is congruence subgroup. Then f is called and automorphic form of weight k respect to Γ if f is an meromorphic function, $f[\gamma] = f \forall \gamma \in \Gamma$ and $f[\gamma]$ has meromorphic continuation at ∞ for all $\gamma \in \Gamma$.*

So, we can define $\mathcal{A}_k(\Gamma)$ and $\mathcal{A}(\Gamma)$. Clearly, $\mathcal{S}_k(\Gamma) \subset \mathcal{M}_k(\Gamma) \subset \mathcal{A}_k(\Gamma)$.

Definition 1.3.2. *Let $V \subset \mathbb{C}$ is open. Then set of meromorphic differntials on V of degree n is $\Omega^n(V) = \{f(q)(dq)^n : f$ is meromorphic on $V\}$.*

By complex geometry, we can prove ...

Theorem 1.3.1. *$k \in \mathbb{N}$ is even. Γ is congruence subgroup.Then $\omega : \mathcal{A}_k(\Gamma) \to \Omega^{k/2}(X(\Gamma))$ is an isomorphism.*

Definition 1.3.3. *Let X is an compact Riemann surface. A divisor D on X is $D = \sum_{x \in X} n_x x$ where $n_x \in \mathbb{Z}$ and n_x is non-zero for finitely many $x \in X$. And degree of divisor D is $deg(D) = \sum_{x \in X} n_x$*

$f : X \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is meromorphic map then divisors of f is $div(f) = \sum_{x \in X} \nu_x(f)x$ where $\hat{\mathbb{C}}$ is Riemann sphere and $\nu_x(f)$ is ramification degree of x. Let $C(X) = \{f : X \to \mathbb{C} :$ f is meromorphic$\}$. Then, by complex analysis, $\forall f \in C(X)$, degree of f is 0.

Definition 1.3.4. *Let $D \in Div(X)$, then $L(D) = \{f \in C(X) : f = 0$ or $div(f) + D \geq 0\}$ $L(D)$ is complex vector space. $l(D) =$ dimension of $L(D)$.*

Definition 1.3.5. *A divisor $D \in Div(X)$ is principle divisor if $\exists \omega \neq 0 \in \Omega^1(X)$ such that $div(\omega) = D$.*

Theorem 1.3.2. ***Riemann-Roch Theorem*** *Let X is compact Riemann surface of genus g. $D \in Div(X)$ and $\omega \in \Omega^1(X)$. Then $l(D) = deg(D) - g + 1 + l(div(\omega) - D)$.*

Theorem 1.3.3. *Let Γ is congruence subgroup.Let g is genus of $X(\Gamma)$. $\epsilon_2, \epsilon_3, \epsilon_\infty^{reg}, \epsilon_\infty^{irr}$ are number of elliptic points of order 2,3 and regular and irregular cusps. Let $\epsilon_\infty = \epsilon_\infty^{reg} + \epsilon_\infty^{irr}$. Then*

k is even

$$dim(\mathcal{M}_k(\Gamma)) = \begin{cases} (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \epsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \epsilon_3 + \frac{k}{2}\epsilon_\infty & k \geq 2 \\ 1 & k = 0 \\ 0 & k \leq 0 \end{cases}$$

$$dim(\mathcal{S}_k(\Gamma)) = \begin{cases} (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \epsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \epsilon_3 + \frac{k-2}{2}\epsilon_\infty & k \geq 4 \\ g & k = 2 \\ 0 & k \leq 0 \end{cases}$$

k is odd

$$-I \in \Gamma \Rightarrow \mathcal{M}_k(\Gamma) = \mathcal{S}_k(\Gamma) = \{0\}$$

And

$$-I \notin \Gamma \Rightarrow \; dim(\mathcal{M}_k(\Gamma)) = \begin{cases} (k-1)(g-1) + \left\lfloor \frac{k}{3} \right\rfloor \epsilon_3 + \frac{k}{2}\epsilon_\infty^{reg} + \frac{k-1}{2}\epsilon_\infty^{irr} & k \geq 3 \\ 0 & k \leq 0 \\ \frac{1}{2}\epsilon_\infty^{reg} & k = 1, \epsilon_\infty^{reg} > 2g-2 \\ \geq \frac{1}{2}\epsilon_\infty^{reg} & k = 1, \epsilon_\infty^{reg} \leq 2g-2 \end{cases}$$

$$dim(\mathcal{S}_k(\Gamma)) = \begin{cases} (k-1)(g-1) + \left\lfloor \frac{k}{3} \right\rfloor \epsilon_3 + \frac{k-2}{2}\epsilon_\infty^{reg} + \frac{k-1}{2}\epsilon_\infty^{irr} & k \geq 3 \\ 0 & k \leq 0 \\ 0 & k = 1, \epsilon_\infty^{reg} > 2g-2 \\ \; dim(\mathcal{M}_k(\Gamma)) - \frac{1}{2}\epsilon_\infty^{reg} & k = 1, \epsilon_\infty^{reg} \leq 2g-2 \end{cases}$$

Proof. Let $f \in \mathcal{A}_k(\Gamma)$ Then $\mathcal{A}_k(\Gamma) = C(X(\Gamma))f$, $\mathcal{M}_k(\Gamma) = \{f_0 f : f_0 \in C(X(\Gamma)), div(f_0) + div(f) \geq 0\}$ And $\mathcal{S}_k(\Gamma) = \{f_0 f : f_0 \in C(X(\Gamma)), div(f_0) + div(f) > 0\}$.Let $\{x_i\}$ and $\{x_i'\}$ is

set of regular and irregular cusps, then
For k is even, $\mathcal{M}_k(\Gamma) \cong L(\lfloor div(f) \rfloor)$ and $\mathcal{S}_k(\Gamma) \cong L(\lfloor div(f) - \sum x_i - \sum x_i' \rfloor)$
For k is odd, $\mathcal{M}_k(\Gamma) \cong L(\lfloor div(f) \rfloor)$ and $\mathcal{S}_k(\Gamma) \cong L(\lfloor div(f) - \sum x_i - \frac{1}{2}\sum x_i' \rfloor)$
Now, applying Riemann-Roch theorem, we get our result. □

1.4 Hecke Operators

Let $\Gamma_1, \Gamma_2, \Gamma$ are congruence subgroup and $\alpha \in GL_2^+(\mathbb{Q})$ (i.e. determinant is positive). Then define $\Gamma_1\alpha\Gamma_2 = \{\gamma_1\alpha\gamma_2 : \gamma_i \in \Gamma_i, i = 1, 2\}$.

Lemma 1.4.1. *Continue to notation defined above, $\alpha^{-1}\Gamma\alpha \cap SL_2(\mathbb{Z})$ is also congruence subgroup. Let $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$. Then, $\Gamma_3\backslash\Gamma_2 \cong \Gamma_1\backslash\Gamma_1\alpha\Gamma_2$. Moreover, quotient is finite.*

Definition 1.4.1. *Let Γ_1, Γ_2 are congruence subgroup and $\alpha \in GL_2^+(\mathbb{Q})$. Let $\Gamma_1\alpha\Gamma_2 = \cup\Gamma_1\beta_j$ is disjoint union. Then double coset operator of weight k, $[\Gamma_1\alpha\Gamma_2]$ on $\mathcal{M}_k(\Gamma_1)$ is defined by $f[\Gamma_1\alpha\Gamma_2] = \sum f[\beta_j]_k$.*

Lemma 1.4.2. *$f \in \mathcal{M}_k(\Gamma_1)$ then $f[\Gamma_1\alpha\Gamma_2] \in \mathcal{M}_k(\Gamma_2)$. Similarly for cusp form also.*

Let $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$, then $\alpha^{-1}\Gamma_1(N)\alpha = \Gamma_1(N)$. Then diamond operator $\langle d \rangle$ for $gcd(d, N) = 1$ on $\mathcal{M}_k(\Gamma_1(N))$ is $\langle d \rangle f = f[\Gamma_1(N)\alpha\Gamma_1(N)]_k = f[\alpha]_k$.

Lemma 1.4.3. *Diamond operator does not depend on α. i.e. Let $\beta = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \Gamma_0(N)$ such that $w \equiv d (mod\ N)$ then $f[\alpha]_k = f[\beta]_k$*

Let $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ is multiplicative character, then $\mathcal{M}_k(N, \chi) = \{f \in \mathcal{M}_k(\Gamma_1(N)); \langle d \rangle f = \chi(d) f \forall d\}$. Then $\mathcal{M}_k(\Gamma_1(N)) = \oplus_\chi \mathcal{M}_k(N, \chi)$. If χ is identity, then $\mathcal{M}_k(N, 1) = \mathcal{M}_k(\Gamma_0(N))$.

Definition 1.4.2. *Let $\alpha = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$ where p is prime. Then T_p operator on $\mathcal{M}_k(\Gamma_1(N))$ defined by $T_p f = f[\Gamma_1(N)\alpha\Gamma_1(N)]$.*

Note that diamond operator on $\mathcal{M}_k(\Gamma_0(N))$ is identity operator for any $gcd(d, N) = 1$. Whenever T_p operator on $\mathcal{M}_k(\Gamma_0(N))$ is restricted operator.

Lemma 1.4.4. *Continue to notations, for $f \in \mathcal{M}_k(\Gamma_1(N))$,*

$$T_p f = \begin{cases} \sum_{j=0}^{p-1} f[\begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix}]_k & p|N \\ \sum_{j=0}^{p-1} f[\begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix}]_k + f[\begin{bmatrix} m & n \\ N & p \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}]_k & p \nmid N, mp - nN = 1 \end{cases}$$

We have defined diamond operator for all integers co-prime to N. $gcd(n, N) > 1$ then define $\langle n \rangle = 0$. And, $T_{p^r} = T_p T_{p^{r-1}} - p^{k-1} \langle p \rangle T_{p^{r-2}}$ for $r \geq 2$ defining T_1 is identity. And $T_{mn} = T_m T_n$ if $gcd(m, n) = 1$.

Definition 1.4.3. *Let $\tau = x + y\iota \in \mathbb{C}$. Then hyperbolic measure μ, $d\mu(\tau) = \frac{1}{y^2} dxdy$.*

Lemma 1.4.5. *Let $\alpha \in GL_2(\mathbb{Q})$ and $\tau \in \mathbb{C}$. Then hyperbolic measure $d\mu(\alpha(\tau)) = d\mu(\tau)$.*

Let $\phi : \mathcal{H} \to \mathbb{C}$ such that $\phi(\gamma(\tau)) = \phi(\tau)$ for all $\gamma \in \Gamma$. Now we can define integration of ϕ on $X(\Gamma)$ by $\int_{X(\Gamma)} \phi(\tau) d\mu(\tau) = \sum \int_{\mathcal{D}} \phi(\alpha_i(\tau)) d\mu(\tau)$ where summation run over $\Gamma \backslash SL_2(\mathbb{Z})$.

Definition 1.4.4. *Volume of $X(\Gamma)$ is $V_\Gamma = \int_{X(\Gamma)} d\mu(\tau)$.*

Let $f, g \in \mathcal{M}_k(\Gamma)$, define $\phi(z) = f(z)\overline{g(z)}(Im(z))^k$. Clearly, $\phi(\gamma(z)) = \phi(z) \forall \gamma \in \Gamma$. Then $\phi(\alpha(z)) = C + \mathcal{O}(e^{-2\pi(Im(z))/h})(Im(z))^k$ where C depends only on constant term in fourier expansion of f and g. To make integration well-define, $\phi(\alpha(z)) \to 0$ as $Im(z) \to \infty$ for all $\alpha \in SL_2(\mathbb{Z})$. So, constant term must be 0. So, either f or g must be cusp form.

Definition 1.4.5. *Let Γ is congruence subgroup. Then* ***Petersson Inner Product****, $\langle , \rangle_\Gamma : \mathcal{S}_k(\Gamma) \times \mathcal{S}_k(\Gamma) \to \mathbb{C}$ defined by $\langle f, g \rangle = \frac{1}{V_\Gamma} \int_{X(\Gamma)} f(z)\overline{g(z)}(Im(z))^k d\mu(z)$ is well define inner product on $\mathcal{S}_k(\Gamma)$. So, $(\mathcal{S}_k(\Gamma), \langle , \rangle)$ is finite dimensional complex Hilbert Space.*

In Hilbert space $(H, \langle , \rangle)$, T is an operator then adjoint operator of T is T^* such that $\langle Tf, g \rangle = \langle f, T^* g \rangle$ for all $f, g \in H$. We want to find adjoint of double coset operator $[\Gamma \alpha \Gamma]$ for congruence subgroup Γ and $\alpha \in GL_2^+(\mathbb{Q})$.

Theorem 1.4.6. *Continue to notations, $\alpha' = det(\alpha)\alpha^{-1}$. Then,*

1. *$\alpha^{-1}\Gamma\alpha \subset SL_2(\mathbb{Z})$, $f \in \mathcal{S}_k(\Gamma)$ and $g \in \mathcal{S}_k(\alpha^{-1}\Gamma\alpha)$ then $\langle f[\alpha], g \rangle_{\alpha^{-1}\Gamma\alpha} = \langle f, g[\alpha'] \rangle_\Gamma$*
2. *$f, g \in \mathcal{S}_k(\Gamma)$ then $\langle f[\Gamma\alpha\Gamma], g \rangle = \langle f, g[\Gamma\alpha'\Gamma] \rangle$.*

Corollary 1.4.7. *For Hilbert Space $\mathcal{S}_k(\Gamma_1(N))$ with Petersson Inner Product, for $p \nmid N$, adjoint of Hecke operators is $\langle p \rangle^* = \langle p \rangle^{-1}$ and $T_p^* = \langle p \rangle^* T_p$.*

Since T_n and $\langle n \rangle$ for $gcd(n, N) = 1$ are commuting normal (i.e. commuting with its adjoint) operators, using spectral theorem of linear algebra on finite dimensional vector space, we can prove below result.

Theorem 1.4.8. *The Hilbert Space $\mathcal{S}_k(\Gamma_1(N))$ has an orthonormal basis of simultaneous eigenform (eigenvector) for Hecke operator $\{\langle n \rangle, T_n : gcd(n, N) = 1\}$.*

Proposition 1.4.9. *Let $\omega_N = [\begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix}]_k$ operator. Then for any Hecke Operator $T = \langle n \rangle$ or $T = T_n$, adjoint operator of T is $T^* = \omega_N T \omega_N^{-1}$*

Let $M|N$ then $\Gamma_1(N) \subset \Gamma_1(M)$. So, $\mathcal{S}_k(\Gamma_1(M)) \subset \mathcal{S}_k(\Gamma_1(N))$. Let $\alpha_d = \begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix}$. Then for any d dividing $\frac{N}{M}$,$[\alpha_d]_k$ operators (on $\mathcal{S}_k(\Gamma_1(M))$ maps to $\mathcal{S}_k(\Gamma_1(N))$) is injective.

Definition 1.4.6. *Let d be a divisor of $N \in \mathbb{N}$. Let $\iota_d : (\mathcal{S}_k(\Gamma_1(N/d)))^2 \to \mathcal{S}_k(\Gamma_1(N))$ such that $\iota_d(f, g) = f + d^{1-k} g[\alpha_d]_k$. Then subspace of oldforms in $\mathcal{S}_k(\Gamma_1(N))$ is*

$$\mathcal{S}_k(\Gamma_1(N))^{old} = \sum_{prime\ p|N} \iota_p((\mathcal{S}_k(\Gamma_1(N/p)))^2)$$

Since $\mathcal{S}_k(\Gamma_1(N))$ has Petersson Inner Product, we can define perpendicular notion. Then subspace of newform is

$$\mathcal{S}_k(\Gamma_1(N))^{new} = (\mathcal{S}_k(\Gamma_1(N))^{old})^{\perp}$$

Proposition 1.4.10. *$\mathcal{S}_k(\Gamma_1(N))^{old}$ and $\mathcal{S}_k(\Gamma_1(N))^{new}$ are invariant under $\langle n \rangle$ and T_n for all $n \in \mathbb{N}$. So, Both have simaltenous eigen form for Hecke operator.*

Let $f_p \in \mathcal{S}_k(\Gamma_1(N/p))$. If $f_p(z) = \sum\limits_{n \geq 1} a_n q^n$ then $\iota_p(0, f_p)(z) := \iota_p f_p = \sum\limits_{n \geq 1} a_n q^{np}$. Let $f = \sum\limits_{p|N} \iota_p f_p = \sum\limits_{n \geq 1} a_n(f) q^n$ then $a_n(f) = 0$ for all $gcd(n, N) = 1$.Converse of this statement is also true.

Theorem 1.4.11. ***Main Lemma*** *Let $f \in \mathcal{S}_k(\Gamma_1(N))$. Let f has Fourier expansion is $f(z) = \sum\limits_{n \geq 1} a_n(f) q^n$ such that $a_n(f) = 0$ if $gcd(n, N) = 1$. Then $\exists f_p \in \mathcal{S}_k(\Gamma_1(N/p))$ for all $p|N$ such that $f = \sum\limits_{p|N} \iota_p f_p$.*

Proof. Let define $\Gamma^1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}); \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} \pmod N \right\}$. Since $\alpha_N \Gamma_1(N) \alpha_N^{-1} = \Gamma^1(N)$, $\mathcal{S}_k(\Gamma_1(N))$ is isomorphic to $\mathcal{S}_k(\Gamma^1(N))$ via $f \mapsto N^{k-1} f[\alpha_N^{-1}]_k$. Since this map commutes with ι_d, It is enough to prove, given $f \in \mathcal{S}_k(\Gamma^1(N))$ with n^{th} fourier co-efficient in expansion is 0 if $gcd(n, N) = 1$ then $f = \sum_{p|N} f_p$ where $f_p \in \mathcal{S}_k(\Gamma^1(N/p))$. Let $\Gamma^0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}); \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \pmod N \right\}$. For $d|N$,$\Gamma_d = \Gamma_1(N) \cap \Gamma^0(N/d)$. Then $\Gamma_d \backslash \Gamma(N) = \left\{ \begin{bmatrix} 1 & b\frac{N}{d} \\ 0 & 1 \end{bmatrix} : 0 \le b < d \right\}$. Let $\pi_d : \mathcal{S}_k(\Gamma(N)) \to \mathcal{S}_k(\Gamma(N))$ defined by $\pi_d(f) = \frac{1}{d} \sum_{b=0}^{d-1} f[\begin{bmatrix} 1 & b\frac{N}{d} \\ 0 & 1 \end{bmatrix}]_k$ is a projection to $\mathcal{S}_k(\Gamma_d)$. Note under this map $\sum_{n \ge 1} a_n q_N^n \mapsto \sum_{d|n} a_n q_N^n$ and $\pi_r \pi_s = \pi_s \pi_r = \pi_{rs}$. Let $\pi = \prod_{p|N}(1 - \pi_p)$. Then Inclusion-Exclusion formula from combinatorics, $\pi : \sum_{n \ge 1} a_n q_N^n \mapsto \sum_{gcd(n,N)=1} a_n q_N^n$. So, our hypothesis says $f \in \ker(\pi) = \sum_{p|N} \ker(1 - \pi_p) = \sum_{p|N} \operatorname{im}(\pi_p)$ because π_p is projection operator. So our theorem is reduced to prove $\mathcal{S}_k(\Gamma^1(N)) \cap \sum_{p|N} \mathcal{S}_k(\Gamma_1(N) \cap \Gamma^0(N/p)) = \sum_{p|N} \mathcal{S}_k(\Gamma^1(N/p))$. Let $G = SL_2(\mathbb{Z}/N\mathbb{Z})$. Then $G = \prod_i SL_2(\mathbb{Z}/p_i^{e_i}\mathbb{Z})$, here product runs over prime factors of N. $\mathcal{S}_k(\Gamma(N))$ is representation of $SL_2(\mathbb{Z}/N\mathbb{Z})$. Let $H_i = \Gamma^1(p_i^{e_i})/\Gamma(p_i^{e_i})$ and $K_i = (\Gamma_1(p_i^{e_i}) \cap \Gamma_0(p_i^{e_i-1}))/\Gamma(p_i^{e_i})$ and $H = \prod_i H_i$. And fixed subspace of $\mathcal{S}_k(\Gamma(N))$ by subgroup H is denoted by $\mathcal{S}_k(\Gamma(N))^H$. Then we want to prove $\mathcal{S}_k(\Gamma(N))^H \cap \sum_i \mathcal{S}_k(\Gamma(N))^{K_i} = \sum_i \mathcal{S}_k(\Gamma(N))^{\langle H, K_i \rangle}$. This is a standard result of representation of finite groups. □

Definition 1.4.7. *A non-zero $f \in \mathcal{M}_k(\Gamma_1(N))$ is an eigenform for Hecke operator $\langle n \rangle$ and T_n for all $n \in \mathbb{Z}^+$ is an Hecke Eigenform. Let $f = \sum_{n=0}^{\infty} a_n(f) q^n$. If $a_1(f) = 1$ then eigenform is called Normalized Eigenform. $f \in \mathcal{S}_k(\Gamma_1(N))^{new}$ is newform if it is normalized eigenform.*

Theorem 1.4.12. *$\mathcal{B}_k(N) = \{f(n\tau) : f$ is new form of level M such that $nM|N\}$ is a basis of $\mathcal{S}_k(\Gamma_1(N))$.*

1.5 L Function of Modular Forms

Let $f = \sum_{n \ge 0} a_f(n) n^{\frac{k-1}{2}} q^n$ is modular form of weight k at level N. Then define L-function

$$L(f, s) = \sum_{n \ge 1} \frac{a_f(n)}{n^s}$$

Proposition 1.5.1. *$L(f,s)$ converges absolutely for $\mathscr{R}s > 1$.*

Let $g(s) = \int\limits_0^\infty f(\iota t) t^s \frac{dt}{t}$ is Mellin transform of f. Then $g(s) = (2\pi)^{-s}\Gamma(s)L(f, s - \frac{k-1}{2})$. Define $W_N : \mathcal{S}_k(\Gamma_1(N)) \to \mathcal{S}_k(\Gamma_1(N))$ by $W_N(f) = \frac{\iota^k}{(N\tau^2)^{k/2}} f(\frac{-1}{N\tau})$. Then W_N^2 is identity operator and W_N is self adjoint operator. So only possible eigenvalue of W_N is either +1 or -1. So eigenspace are

$$\mathcal{S}_k(\Gamma_1(N))^{\pm} = \{f \in \mathcal{S}_k(\Gamma_1(N)); W_N(f) = \pm f\}$$

Theorem 1.5.2. *Let $\Lambda_N(f,s) = N^{s/2+\frac{k-1}{4}} g(s + \frac{k-1}{2}) = \left(\frac{N}{4\pi^2}\right)^{\frac{s}{2}+\frac{k-1}{4}} \Gamma(s + \frac{k-1}{2}) L(f,s)$. Then $\Lambda_N(f,s)$ extends to an entire function such that $\Lambda_N(f,s) = \Lambda_N(W_N f, 1-s)$. Moreover, $L(f,s)$ has analytic continuation to whole plane. Moreover, if Let $f \in \mathcal{S}_k(\Gamma_1(N))^{\pm}$ then $\Lambda_N(f,s) = \pm\Lambda_N(f, 1-s)$*

For $\mathcal{S}_k(\Gamma_0(N))$, for all prime p with $p \nmid N$, $T_p^* = T_p$ by **Corollary 1.4.7**. Since commuting family of normal operators have simaltenous eigenvector.

Theorem 1.5.3. *$\mathcal{S}_k(\Gamma_0(N))$ has an orthonormal basis of simaltenous eigenform for $\{T_n;\ gcd(n,N) = 1\} \cup \{W_N\}$.*

Theorem 1.5.4. *$f \in \mathcal{S}_k(\Gamma_1(N), \chi)$ is newform if and only if*

$$L(f,s) = \prod_{p|N} \Big(1 - \frac{a_p}{p^s}\Big)^{-1} \prod_{p\nmid N} \Big(1 - \frac{a_p}{p^s} + \frac{\chi(p)}{p^{2s}}\Big)^{-1}$$

1.5.1 Approximated L function

Let f be an cusp form of weight k at level N and $W_N f = g$. Let $f = \sum\limits_{n\geq 1} a_f(n) n^{\frac{k-1}{2}} q^n$ and $g = \sum\limits_{n\geq 1} a_g(n) n^{\frac{k-1}{2}} q^n$.

Theorem 1.5.5. *Let $L(f,s)$ be an L function. Let $G(t)$ be an holomorphic even and bounded on vertical strip for $-4 < \mathscr{R}t < 4$ and $G(0) = 1$. Let $V_s(y) = \frac{1}{2\pi\iota} \int\limits_{\mathscr{R}t=3} y^{-t} G(t) \frac{\Gamma(s+t+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \frac{dt}{t}$. Then for $0 < \mathscr{R}s < 1$,*

$$L(f,s) = \sum_{n\geq 1} \frac{a_f(n)}{n^s} V_s\Big(\frac{2\pi n}{\sqrt{N}}\Big) + \Big(\frac{N}{4\pi^2}\Big)^{\frac{1}{2}-s} \frac{\Gamma(1-s+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \sum_{n\geq 1} \frac{a_g(n)}{n^{1-s}} V_{1-s}\Big(\frac{2\pi n}{\sqrt{N}}\Big)$$

Proof. : Let $I(f,s) = \frac{1}{2\pi\iota} \int\limits_{\mathscr{R}t=3} \Lambda_N(f, s+t) G(t) \frac{dt}{t}$. Now move the integration to $\mathscr{R}t = -3$ and using functional equation of $\Lambda_N(f,s)$ and since $\Lambda_N(f,s+t)G(t)$ is an holomorphic as function of t in vertical strip $-3 < \mathscr{R}t < 3$, we get $\Lambda_N(f,s) = I(f,s) + I(W_N f, 1-s)$.

$$
\begin{aligned}
I(f,s) &= \Big(\frac{N}{4\pi^2}\Big)^{\frac{s}{2}+\frac{k-1}{4}} \sum_{n\geq 1} \frac{a_f(n)}{n^s} \frac{1}{2\pi\iota} \int_{\mathscr{R}t=3} \Big(\frac{2\pi n}{\sqrt{N}}\Big)^{-t} G(t)\Gamma(s+t+\frac{k-1}{2})\frac{dt}{t} \\
&= \Big(\frac{N}{4\pi^2}\Big)^{\frac{s}{2}+\frac{k-1}{4}} \sum_{n\geq 1} \frac{a_f(n)}{n^s} \Gamma(s+\frac{k-1}{2}) V_s\Big(\frac{2\pi n}{\sqrt{N}}\Big) \\
I(W_N f, 1-s) &= \Big(\frac{N}{4\pi^2}\Big)^{\frac{1-s}{2}+\frac{k-1}{4}} \Gamma(1-s+\frac{k-1}{2}) \sum_{n\geq 1} \frac{a_g(n)}{n^{1-s}} V_{1-s}\Big(\frac{2\pi n}{\sqrt{N}}\Big) \text{ (similarly)}
\end{aligned}
$$

Now adding $I(f,s)$ and $I(W_N f, s)$ and dividing by $\Big(\frac{N}{4\pi^2}\Big)^{\frac{s}{2}+\frac{k-1}{4}} \Gamma(s+\frac{k-1}{2})$, we get our result. □

Corollary 1.5.6. *If $f \in \mathcal{S}_k(\Gamma_1(N))^{\pm}$ (i.e. $W_N f = \epsilon(f) f$). Then*

$$
L(f,s) = \sum_{n\geq 1} \frac{a_f(n)}{n^s} V_s\Big(\frac{2\pi n}{\sqrt{N}}\Big) + \epsilon(f)\Big(\frac{N}{4\pi^2}\Big)^{\frac{1}{2}-s} \frac{\Gamma(1-s+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \sum_{n\geq 1} \frac{a_f(n)}{n^{1-s}} V_{1-s}\Big(\frac{2\pi n}{\sqrt{N}}\Big)
$$

Corollary 1.5.7. *If $f \in \mathcal{S}_k(\Gamma_1(N))$ and $W_N f = g$ then, $L(f, \frac{1}{2}) = \sum\limits_{n\geq 1} \frac{a_f(n)+a_g(n)}{\sqrt{n}} V_{\frac{1}{2}}\Big(\frac{2\pi n}{\sqrt{N}}\Big)$*

Corollary 1.5.8. $L(f,\frac{1}{2}) = \begin{cases} 2\sum\limits_{n\geq 1} \frac{a_f(n)}{\sqrt{n}} V_{\frac{1}{2}}\Big(\frac{2\pi n}{\sqrt{N}}\Big) & f \in \mathcal{S}_k(\Gamma_1(N))^+ \\ 0 & f \in \mathcal{S}_k(\Gamma_1(N))^- \end{cases}$

1.6 Jacobian of Compact Riemann Surface

Let X is a compact Riemann Surface and $x_0 \in X$. Then $x \mapsto (\omega \mapsto \int_{x_0}^{x} \omega)$ defines(not properly) dual space on vector space $\Omega^1_{hol}(X)$. But integration from x_0 to x is not well-defined because there are two different paths differentiate by loops. Any loops on genus-g surface is $\mathbb{Z}-$ linear combination of longitudinal and latitudinal loops (Reference [3]). Let $A_1, A_2, \cdots, A_g$ and $B_1, B_2, \cdots, B_g$ are longitudinal and latitudinal loops on X. Then set of integration over loops is an abelian group of rank $2g$ which is also called 1^{st} homology group denoted by $H_1(X,\mathbb{Z}) = \mathbb{Z}\int_{A_1} \oplus \cdots \oplus \mathbb{Z}\int_{A_2} \oplus \mathbb{Z}\int_{B_1} \oplus \cdots \oplus \mathbb{Z}\int_{B_g} \cong \mathbb{Z}^{2g}$. We also state theorem of complex manifolds from [3] is $\Omega^1_{hol}(X)^{\wedge} = \mathbb{R}\int_{A_1} \oplus \cdots \oplus \mathbb{R}\int_{A_2} \oplus \mathbb{R}\int_{B_1} \oplus \cdots \oplus \mathbb{R}\int_{B_g}$.

Definition 1.6.1. *The Jacobian of a compact Riemann surface X is $Jac(X) = \Omega^1_{hol}(X)^\wedge / H_1(X, \mathbb{Z})$.*

Let $f : X \to Y$ be analytic non-constant map between compact Riemann surface. Then we want to find map between $Jac(X)$ and $Jac(Y)$. Then $f^* : C(Y) \to C(X)$ defined by $f^*(g) = g \circ f$. So, $f^* : \Omega^1_{hol}(Y) \to \Omega^1_{hol}(X)$ such that $f^*(\omega) = \omega \circ f$. So we can define map duals. $F : \Omega^1_{hol}(X)^\wedge \to \Omega^1_{hol}(Y)^\wedge$ defined by $F(\int_{x_0}^{x}) = \int_{f(x_0)}^{f(x)}$. And let γ be a loop in X then $f(\gamma)$ is loop in Y. So, $F(\int_\gamma) = \int_{f(\gamma)}$. So, F maps $H(X, \mathbb{Z})$ to $H(Y, \mathbb{Z})$. So, $Jac(X) \to Jac(Y)$ by $[\varphi] \mapsto [\varphi \circ f^*]$ is well-defined.

Returning to our objects - modular curves, let Γ is congruence subgroup. Then $X(\Gamma)$ is compact Riemann surface. By using **Theorem 1.3.1**, $\omega^\wedge(\Omega^1_{hol}(X(\Gamma))^\wedge) = \mathcal{S}_2(\Gamma)^\wedge$. In abuse of notation, $H_1(X(\Gamma), \mathbb{Z}) := \omega^\wedge(H_1(X(\Gamma), \mathbb{Z}))$.

Definition 1.6.2. *Let Γ be congruence subgroup. Then Jacobian of modular curve $X(\Gamma)$ is $Jac(X(\Gamma)) := \mathcal{S}_2(\Gamma)^\wedge / H_1(X(\Gamma), \mathbb{Z})$*

Now let Γ_A and Γ_B are congruence subgroup such that $\alpha \Gamma_A \alpha^{-1} \subset \Gamma_B$ for some $\alpha \in GL_2^+(\mathbb{Q})$. Then $A = X(\Gamma_A)$ and $B = X(\Gamma_B)$ are compact Riemann surfaces. Let $f : A \to B$ is non-constant holomorphic map defined by $f(\Gamma_A \tau) = \Gamma_B \alpha(\tau)$. Since, below diagram commutes,

$$\begin{array}{ccc} \mathcal{S}_2(\Gamma_B) & \xrightarrow{[\alpha]_2} & \mathcal{S}_2(\Gamma_A) \\ {\scriptstyle \omega_B}\downarrow & & \downarrow{\scriptstyle \omega_A} \\ \Omega^1_{hol}(B) & \xrightarrow{f^*} & \Omega^1_{hol}(A) \end{array}$$

So, $F : \mathcal{S}_2(\Gamma_A)^\wedge \to \mathcal{S}_2(\Gamma_B)^\wedge$ is defined by $\varphi \mapsto \varphi \circ [\alpha]_2$. Since F maps $H_1(A, \mathbb{Z})$ to $H_1(B, \mathbb{Z})$. So, map $F^* : Jac(A) \to Jac(B)$ is well-defined.

Proposition 1.6.1. *Let T is Hecke operator either $T = T_n$ or $T = \langle n \rangle$. Then Hecke operator $T : J_1(N) \to J_1(N)$ defined by $[\varphi] \mapsto [\varphi \circ T]$ for all $\varphi \in \mathcal{S}_2(\Gamma_1(N))^\wedge$.*

Let $f \in \mathcal{M}_k(\Gamma)$ is normalized eigenform. If $f = \sum\limits_{n \geq 0} a_n(f) q^n$ then $a_n(f)$ are eigenvalues such that $T_n(f) = a_n(f) f$. Since eigenvalues of finite dimensional vector space are algebraic integers, $a_n(f)$ is algebraic integers.

Proposition 1.6.2. *$\mathcal{S}_2(\Gamma)$ has basis consists of simaltenous eigenform of Hecke operator such that all eigenvalues are rational integers.*

1.7 Introduction to Project

We are interested to find largest prime N occurs in torsion parts of any elliptic curves over field of degree d. Since $S_0(N) \cong Y_0(N)$ (by theorem 1.1.7), there must be relation between degree of extension of field and uncompactified modular curve $Y_0(N)$. This relation is proved by S.Kamienny in [9]. He has proved below result: .

Theorem 1.7.1. *Let d be a positive integer and $N > 60d - 24$ be a prime number. Suppose there is prime p satisfying,*
(1) $d < p - 1$
(2) $N > (1 + \sqrt{p^d})^2$
(3) there exists d weight 2 cusp forms $f_1, f_2, \cdots, f_d$ attached to eisenstein quotient J, that satisfy linear independence condition mod p.
Then there does not exist any elliptic curve with point of order N over any field of degree d.

L. Merel has proved below result in [11] which helps to prove 3rd criteria of theorem 1.7.1.

Theorem 1.7.2. *Let d be an positive integer and $N > 60d-24$ be a prime number. TFAE$\cdots$*
(1) 3rd criteria of theorem 1.7.1
(2) The action of the first d Hecke operator -$T_1, T_2, \cdots T_d$- on winding quotient e in $J_0(N)$, satisfy linear independence condition in characteristic 0.(Here, $(e, f) = \int_0^\infty f(\iota y)dy$)

Moreover, Merel has also proved that the action of Hecke operators -$T_1, T_2, \cdots, T_D$- on cycle (winding quotient), satisfy linear independence if $max(D^8, 400D^4) < \frac{N}{(\log N)^4}$. In 1998, Jeffrey Vanderkam has improved the result by using only analytic techniques. He proved,

Theorem 1.7.3. *Given $\delta > 0$, there exist constant c_δ such that $T_1, T_2, \cdots, T_D$ acts linearly independently on cycle e in $J_0(N)$ for all primes $N > c_\delta D^{2+\delta}$.*

Now onwards, we never talk about elliptic curves. So, **Theorem 1.7.1** and **Theorem 1.7.2** are assumed without proof.

Chapter 2

Poincare Series

Chapter 1 introduces a basis consisting of Hecke Eigenform for cusp space, while this chapter defines some different type of a basis for cusp space with some properties. For details, see [8] (-cha. 14) and, [7] (-cha. 3,4).

2.1 Poincare series as cusp forms

Let N be a prime number, $\Gamma = \Gamma_0(N)$ and $k \geq 2$ be a natural number. Then define **Poincare Series** for $m \geq 1$

$$P_m(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, \tau)^{-k} e(m\gamma\tau) \ ; \ e(a) = exp(2\pi\iota a)$$

Since P_m satisfy Cauchy-Riemann equation, P_m is holomorphic for $\tau \in \mathcal{H}$. It is easy to observe that series converges absolutely for $k > 2$. Let $\alpha \in \Gamma$,
$(P_m[\alpha]_k)(\tau) = j(\alpha,\tau)^{-k} P_m(\alpha\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, \alpha\tau)^{-k} j(\alpha,\tau) e(m\gamma\alpha\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma\alpha, \tau)^{-k} e(m\gamma\alpha\tau)$.
So, $(P_m[\alpha]_k)(\tau) = P_m(\tau), \forall \alpha \in \Gamma$. So, P_m is holomorphic and weight k invariant under Γ. To prove P_m is a modular form, it is sufficient to find fourier expansion.

Lemma 2.1.1. $\Gamma = \Gamma_0(N) = \Gamma_\infty \bigcup (\bigcup_{\substack{c>0 \\ c \equiv 0 (mod\ N)}} (\bigcup_{\substack{d (mod\ c) \\ gcd\ (c,d)=1}} \Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Gamma_\infty))$*, where* $ad - bc = 1$

Theorem 2.1.2. *$P_m(\tau) = \sum\limits_{n\geq 0} p(m,n)q^n$, where $q = e(\tau)$. Let $S(m,n,c)$ is a Kloosterman Sum and J_{k-1} is $(k-1)^{th}$ order J-Bessel function then,*

$$p(m,n) = \Big(\frac{n}{m}\Big)^{\frac{k-1}{2}}\Big\{\delta(m,n) + 2\pi\iota^{-k}\sum_{\substack{c>0\\ c\equiv 0(mod\ N)}}\frac{S(m,n,c)}{c}J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big)\Big\}$$

Proof. By help of **Lemma 2.1.1**, $P_m(\tau) = e(m\tau) + \sum\limits_{\substack{c>0\\ c\equiv 0(\text{mod } N)}}\sum\limits_{\substack{d(\text{mod } c)\\ \text{gcd }(c,d)=1}} I(c,d,\tau)$. Here,

$$\begin{aligned}
I(c,d,\tau) &= \sum_{n\in\mathbb{Z}}(c(\tau+n)+d)^{-k}e\Big(m\frac{a(\tau+n)+b}{c(\tau+n)+d}\Big)\\
&= \sum_{n\in\mathbb{Z}}(c(\tau+n)+d)^{-k}e\Big(\frac{ma}{c} - \frac{m}{c(c(\tau+n)+d)}\Big)\\
* \qquad &= \sum_{n\in\mathbb{Z}}\int\limits_{\mathbb{R}}(c(\tau+\nu)+d)^{-k}e\Big(\frac{ma}{c} - \frac{m}{c(c(\tau+\nu)+d)} - n\nu\Big)d\nu; \text{(Poisson Summation)}\\
&= \sum_{n\in\mathbb{Z}}e\Big(\frac{am+nd}{c}\Big)\Big\{\int_{-\infty+\iota y}^{\infty+\iota y}(c\nu)^{-k}e\Big(-n\nu - \frac{m}{c^2\nu}\Big)d\nu\Big\}e(n\tau); \text{(change of variable)}
\end{aligned} \tag{2.1}$$

By Cauchy Integral, inner integral is independent of y.Moreover, when $n \leq 0$ then inner integral is 0. Since $J_\nu(\tau) = \frac{\tau^\nu}{2^{\nu+1}\pi\iota}\int\limits_{\mathbb{R}} t^{-\nu-1}\exp\Big(t - \frac{\tau^2}{4t}\Big)dt$ (look at 8.412.2. in [6]). Since Kloosterman Sum is defined as $S(m,n,c) = \sum\limits_{\substack{d(\text{mod } c)\\ \text{gcd }(c,d)=1}} e\Big(\frac{am+nd}{c}\Big)$, we get our desired result. □

Since $\Gamma_0(N)$ have only 2 cusps- 0 and ∞. We need to check fourier expansion of Poincare series at 0. Let $\alpha = \begin{pmatrix} 0 & \frac{-1}{\sqrt{N}} \\ \sqrt{N} & 0 \end{pmatrix}$ then $\alpha(\infty) = 0$. Then similar calculation gives $(P_m[\alpha]_k)(\tau) = \sum\limits_{n\geq 0}\overline{p}(m,n)q^n$, where

$$\overline{p}(m,n) = \Big(\frac{n}{m}\Big)^{\frac{k-1}{2}}\Big\{2\pi\iota^{-k}\sum_{\substack{c\in\mathbb{N}\\ N\overline{N}\equiv 1(\text{mod } c)}}\frac{S(m\overline{N},n,c)}{c\sqrt{N}}J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c\sqrt{N}}\Big)\Big\}$$

So, above discription says that P_m has not non-zero constant term in fourier expansion around both cusps. So, $\{P_m : m \geq 1\}$ is set of cusp form of weight k for $\Gamma_0(N)$.

2.2 Petersson's Formula

Let $f \in \mathcal{S}_k(\Gamma_0(N))$. Let $f(\tau) = \sum\limits_{n\geq 1} a_f(n)q^n$. Then,

$$\begin{aligned}
V_{\Gamma_0(N)}\langle f, P_m\rangle &= \int_{X_0(N)} f(\tau) \sum_{\Gamma_\infty\backslash\Gamma_0(N)} \overline{j(\gamma,\tau)}^{-k}\overline{e(m\gamma\tau)}y^k d\mu(\tau) \\
&= \int_{X_0(N)} \sum_{\Gamma_\infty\backslash\Gamma_0(N)} f(\gamma\tau)\overline{e(m\gamma\tau)}(Im(\gamma\tau))^k d\mu(\tau) \\
&= \int_0^1 \int_0^{+\infty} f(\tau)e(-m\overline{\tau})y^{k-2}dxdy \text{ ;(translation of } \mathcal{D}) \qquad (2.2) \\
&= \sum_{n\geq 1} a_f(n) \int_0^1 e((n-m)x)dx \int_0^{+\infty} y^{k-2}\exp(-2\pi(m+n)y)dy \\
V_{\Gamma_0(N)}\langle f, P_m\rangle &= \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_f(m)
\end{aligned}$$

Since $\mathcal{S}_k(\Gamma_0(N))$ is finite dimensional vector space, linear span of $\{P_m; m \geq 1\}$ is closed subspace. Since orthogonal element to all Poincare series has all fourier coefficient 0 implies $\{P_m; m \geq 1\}$ is also basis of $\mathcal{S}_k(\Gamma_0(N))$.Moreover, Soumya Das and Satadal Ganguly has proved that $\{P_m; 1 \leq m \leq d\}$ is linearly independent in $\mathcal{S}_k(\Gamma_0(N))$, where d is dimension of $\mathcal{S}_k(\Gamma(1))$.

Since $V_{\Gamma_0(N)} \neq 0$, now onwards we write $\langle f, P_m\rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_f(m)$. Let $\mathscr{F}$ is set of normalized eigenform (basis) in $\mathcal{S}_k(\Gamma_0(N))$. Then $P_m = \sum\limits_{f\in\mathscr{F}} \langle f, P_m\rangle f$. By taking n^{th} fourier coefficient both side, we get Petersson's Formula. Similarly, $\langle W_N P_m, f\rangle = \epsilon(f)\langle P_m, f\rangle$ where $W_N f = \epsilon(f)f$. Since $W_N P_m = P_m[\alpha]_k$, we get below result.

Theorem 2.2.1. ***(Petersson's Formula)*** *If $\{f\}$ is a basis of $\mathcal{S}_k(\Gamma_0(N))$ then,*

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}} \sum_f a_f(m)a_f(n) = \delta(m,n) + 2\pi\iota^{-k} \sum_{c>0, N|c} \frac{S(m,n,c)}{c} J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big)$$

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}} \sum_f \epsilon(f)a_f(m)a_f(n) = 2\pi\iota^{-k} \sum_{\substack{c\in\mathbb{N} \\ N\overline{N}\equiv 1 (mod\ c)}} \frac{S(m\overline{N},n,c)}{c\sqrt{N}} J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c\sqrt{N}}\Big)$$

Lemma 2.2.2. *For any integer m, n and prime N,*

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}}\sum_{f} a_f(m)a_f(n) = \delta(m,n) + \mathcal{O}(N^{-k+1/2}(mn)^{\frac{k-1}{2}}\sqrt{gcd\ (m,n)}).$$

Proof. By using Weil Bound on Kloosterman sum ($S(m,n,c) \leq \sqrt{\text{gcd }(m,n,c)}\sqrt{c}\tau(c)$) where $\tau(n)$ is number of positive divisor of n and $J_{k-1}(x) \ll x^{k-1}$, we got $\cdots$

$$\sum_{c>0, N|c} \frac{S(m,n,c)}{c} J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big) \ll \sum_{t\geq 1} \frac{\sqrt{\text{gcd }(m,n,tN)}\sqrt{tN}\tau(tN)}{tN}\left(\frac{\sqrt{mn}}{tN}\right)^{k-1}$$
$$\ll N^{-k+\frac{1}{2}}(mn)^{\frac{k-1}{2}}\sqrt{\text{gcd }(m,n)}\sum_{t\geq 1}\frac{\tau(t)}{t^{k-1/2}}$$

Since $\tau(tN) \leq \tau(t)\tau(N)$ and summation over t is equal to $\zeta^2(k-1/2)$, lemma follows. □

Chapter 3

Proof of Vanderkam's Theorem

Linear independence is proved by classical way. Let $\sum\limits_{1\leq i\leq D} \alpha_i T_i e = 0$, then we want to prove $\alpha_i = 0$ for all $1 \leq i \leq D$ if hypothesis is satisfied. For reference, look at [16].

Let $\mathscr{F}$ is basis of eigenform for $\mathcal{S}_2(\Gamma_0(N))$. Then $|\mathscr{F}| = g$, where g is genus of $X_0(N)$. Since $\mathcal{S}_2(\Gamma_0(1)) = \{0\}$, $\mathscr{F}$ is consists of newforms. Moreover, $(e, f) = \int\limits_0^\infty f(\iota y)dy = L(f, 1)$.

$$\begin{aligned} \sum \alpha_i(T_i e, f) = \sum \alpha_i(e, T_i f) = \sum \alpha_i a_f(i)(e, f) = \sum \alpha_i a_f(i) L(f, 1) = 0 \\ |\sum \alpha_i(T_i e, f)|^2 = \sum \alpha_i \overline{\alpha_j} a_f(i) a_f(j) |L(f, 1)|^2 = 0 \\ \sum_{f\in\mathscr{F}} \sum_{i,j} \alpha_i \overline{\alpha_j} a_f(i) a_f(j) |L(f, 1)|^2 = 0 \end{aligned} \tag{3.1}$$

Lemma 3.0.1. $|L(f, 1)|^2 = 2 \sum\limits_{l,m\geq 1} G_0(\frac{lm}{N}) \frac{a_f(l)a_f(m)}{\sqrt{lm}}$ *,where* $G_0(x) = \frac{1}{2\pi\iota} \int\limits_{Re(t)=3/4} \frac{\Gamma(1+t)^2}{(4\pi^2 x)^t} \frac{dt}{t}$.

Proof. By functional equation of newforms, $g(t) = \left(\frac{N}{4\pi^2}\right)^t \Gamma(1 + t)^2 L(f, 1 + t)^2$ is an even and entire function. By contour shift, we can prove $G_0(x) \leq \begin{cases} b & 0 < x < 1 \\ exp(-c\sqrt{x}) & x > 1 \end{cases}$. Consequently absolute convergence follows since $a_f(n) = \mathcal{O}(\sqrt{n})$. By change of variable to $t \to -t$ and functional equation, we get $R.H.S. = \frac{1}{2\pi\iota} \int_C g(t) \frac{dt}{t} = g(0) = |L(f, 1)|^2$.
Here, C is contour which has $Re(t) = \pm\frac{3}{4}$ as two side. Second equality follows by Cauchy

Residue Theorem. $\square$

After substituting **Lemma 3.0.1** in equation **3.1**,

$$\sum_{i,j}\alpha_i\overline{\alpha_j}\sum_{l,m}\frac{1}{\sqrt{lm}}G_0\Big(\frac{lm}{N}\Big)\sum_{f\in\mathscr{F}}a_f(i)a_f(l)a_f(j)a_f(m)=0$$

Since $f\in\mathscr{F}$ are newforms, $a_f(r)a_f(s)=\sum\limits_{d|\ \gcd(r,s)}a_f\Big(\frac{rs}{d^2}\Big)$. We make two pairs $i=Id_1$, $j=Jd_2$, $l=Ld_1$ and $m=Md_2$. clearly, $d_1,d_2\le D$. So,

$$\sum_{d_1,d_2\le D}\frac{1}{\sqrt{d_1d_2}}\sum_{Id_1,Jd_2\le D}\alpha_{Id_1}\overline{\alpha_{Jd_2}}\sum_{L,M}\frac{1}{\sqrt{LM}}G_0\Big(\frac{LMd_1d_2}{N}\Big)\sum_{f}a_f(IL)a_f(JM)=0 \qquad (3.2)$$

By **Theorem 2.2.1**, equation **3.2** divided in two below parts.

$$S_{main}=\sum_{d_1,d_2\le D}\frac{1}{\sqrt{d_1d_2}}\sum_{Id_1,Jd_2\le D}\alpha_{Id_1}\overline{\alpha_{Jd_2}}\sum_{IL=JM}\frac{1}{\sqrt{LM}}G_0\Big(\frac{LMd_1d_2}{N}\Big)$$

$$S_{off}=\sum_{d_1,d_2\le D}\frac{1}{\sqrt{d_1d_2}}\sum_{Id_1,Jd_2\le D}\alpha_{Id_1}\overline{\alpha_{Jd_2}}\sum_{L,M}\frac{1}{\sqrt{LM}}G_0\Big(\frac{LMd_1d_2}{N}\Big)\sum_{\substack{c>0\\ N|c}}\frac{S(IL,JM,c)}{c}J_1\Big(\frac{4\pi\sqrt{ILJM}}{c}\Big) \qquad (3.3)$$

3.1 A quadratic form

In this section, a lower bound for S_{main} is found.

Let $K=\gcd(I,J)$ and $A=K\frac{M}{I}=K\frac{L}{J}$. Let $I=I_1K$ and $J=J_1K$ Then,

$$S_{main}=\sum_{K\le D}\sum_{\gcd(I_1,J_1)=1}\sum_{d_1,d_2\le D}\frac{\alpha_{I_1d_1K}\overline{\alpha_{J_1d_2K}}}{\sqrt{I_1d_1J_1d_2}}\sum_{A}\frac{1}{A}G_0\Big(\frac{A^2I_1J_1d_1d_2}{N}\Big)$$

Since $\sum\limits_{\gcd(I_1,J_1)=1}\beta(I_1,J_1)=\sum\limits_{T}\mu(T)\sum\limits_{I_2,J_2}\beta(I_2T,J_2T)$. So above equations is,

$$S_{main}=\sum_{K}K\sum_{T}\mu(T)\sum I_2,J_2,d_1,d_2\frac{\alpha_{I_2d_1KT}\overline{J_2d_2KT}}{KT\sqrt{I_2d_1J_2d_2}}\sum_{A}\frac{1}{A}G_0\Big(\frac{A^2T^2I_2d_1J_2d_2}{N}\Big)$$

Again we use change of variable, let $U = I_2 d_1$, $V = J_2 d_2$ and $L = KT$. Let $x_c = \frac{\alpha_c}{\sqrt{c}}$. Then,

$$S_{main} = \sum_L L \sum_{U,V \le \frac{D}{L}} \tau(U)\tau(V) x_{UL} \overline{x_{VL}} \sum_{T|L} \frac{\mu(T)}{T} \sum_A \frac{1}{A} G_0\Big(\frac{A^2T^2UV}{N}\Big) \tag{3.4}$$

By contour integration, $\sum_A \frac{1}{A} G_0\Big(\frac{A^2}{X}\Big) = \frac{1}{2} \log (X) + c_0 + \mathcal{O}(1/X)$ for some constant c_o. Since $T^2UV < L^2UV < D^2 < N^{1-\delta}$, error term is small. Later we also show that error term can be ignored. We continue with only first two terms,

$$y_L = \sum_{UL<D} \tau(U) x_{UL}$$
$$y_L^* = \sum_{UL<D} \log (U) \tau(U) x_{UL}$$

In abuse of notation,

$$S_{main} = (\frac{1}{2} \log N + c_0) \sum_L \sum_{T|L} \frac{\mu(T)}{T} |y_L|^2 - \sum_L L|y_L|^2 \sum_{T|L} \frac{\mu(T) \log T}{T} - 2\mathscr{R} \sum_L \sum_{T|L} \frac{\mu(T)}{T} y_L y_L^*$$

Since $\sum_{T|L} \frac{\mu(T)}{T} = \frac{\phi(L)}{L}$ and $\sum_{T|L} \frac{\mu(T) \log T}{T} = -\frac{\phi(L)\psi(L)}{L}$, where $\psi(L) = \sum_{p|L} \frac{\log p}{p-1}$, we get

$$S_{main} = \sum_L \phi(L) |y_L|^2 (\frac{1}{2} \log N + c_0 + \psi(L)) - 2\mathscr{R} \sum_L \phi(L) y_L y_L^* \tag{3.5}$$

Let $\eta(n) = \sum_{d|n} \mu(d)\mu(n/d)$ (convolution of two Mobius function) then by Mobius inversion, $x_L = \sum_{UL<D} \eta(U) y_{UL}$. So,

$$\begin{aligned} y_L^* &= \sum_{UL<D} \tau(U) \log (U) \sum_{ULV<D} \eta(V) y_{UVL} \\ &= \sum_{WL<D} y_{WL} \sum_{UV=W} \eta(U)\tau(V) \frac{1}{2} \log V \end{aligned}$$

Let inner sum is $\Lambda(W)$, then $\Lambda(W) = \begin{cases} \log p & W = p^r, r \geq 1 \\ 0 & \text{else} \end{cases}$. Thus,

$$y_L^* = \sum_{UL<D} \Lambda(U) y_{UL}$$

By arithmetic and geometric mean inequality, $|2\mathscr{R} y_L \overline{y_{WL}}| \leq |2y_L\overline{y_{WL}}| \frac{|y_L|^2}{W} + W|y_{WL}|^2$. So second term in equation **3.5** is bounded by,

$$\sum_L \phi(L)|y_L|^2 \sum_{WL<D} \frac{\Lambda(W)}{W} + \sum_{WL<D} \phi(L) W \Lambda(W) |y_{WL}|^2$$

By Prime Number Theorem, $\sum\limits_{t<x} \frac{\Lambda(t)}{t} \leq \log x + c_1$ for some constant c_1. And for second term,

$$\sum_{M<D} \phi(M)|y_M|^2 \sum_{L|M} \frac{M\phi(L)}{L\phi(M)} \Lambda(M/L) = \sum_{L<D} \phi(L)|y_L|^2 (\log M + \psi(M))$$

So, second term in **3.5** is bounded by $\sum\limits_{L<D} \phi(L)|y_L|^2(\log D + c_1 + \psi(L))$. So,

$$S_{main} \geq \sum_L \phi(L)|y_L|^2 (\log \frac{\sqrt{N}}{D} + c_0 - c_1)$$

Since $D^2 < N^{1-\delta}$, S_{main} is bounded below by positive term for large enough N.

Now we deal with error term of summation of G_0. Error term has order of $N^{-\delta}$. So, extra term in equation **3.4** is be bounded by

$$N^{-\delta} \sum_L \sum_{U,V} \tau(U)\tau(V)|x_{UL}||x_{VL}|$$

By Mobius Inversion for x, above term is bounded by,

$$N^{-\delta} \sum_L \sum_{U,V} \tau(U)\tau(V) \sum_{W,X} |\eta(W)||\eta(X)||y_{UWL}||y_{XVL}|$$

Since $U, V, W, X < D$ for any $\epsilon > 0$, $|\tau(U)|, |\tau(V)|, |\eta(W)|, |\eta(X)| \ll D^\epsilon$,

$$N^{-\delta} D^\epsilon \sum_L \sum_{A,B} |y_{AL}||y_{BL}| = N^{-\delta} D^\epsilon \sum_{A,B} \tau(\gcd(A, B))|y_A||y_B| \ll N^{-\delta} D^\epsilon (\sum_L |y_L|)^2$$

By Cauchy's Inequality $((\sum_{1\leq t\leq n} x_t y_t)^2 \leq (\sum_t (x_t)^2)(\sum_t (yq_t)^2))$,

$$D^\epsilon N^{-\delta} (\sum_A |y_A|)^2 \ll D^\epsilon N^{-\delta} (\sum_A \phi(A)|y_A|^2)(\sum_A \frac{1}{\phi(A)}) \ll D^\epsilon N^{-\delta} \sum_A \phi(A)|y_A|^2$$

Finally we can say that S_{main} in equation **3.3**,

$$S_{main} \gg \log N \sum_L \phi(L)|y_L|^2 \tag{3.6}$$

3.2 The Kloosterman Sum and Bessel function

Using **Lemma 2.2.2** and $G_0(x) \ll e^{-r\sqrt{x}}$ for some constant $r > 0$, we can say in **equation 3.3** contribution from $G_0(x)$ is negligible when $x \leq (\log N)^3$. Similarly by using Weil's bound on Kloosterman Sum and $J_1(x) \ll x$, contribution of terms for $c > N^2$ is also negligible.

$$\sum_{Id_1, Jd_2 < D} |x_{Id_1}||x_{Jd_2}|\sqrt{IJ} \sum_{L,M} \frac{1}{\sqrt{LM}} G_0\Big(\frac{LMd_1d_2}{N}\Big) \sum_{\substack{c \geq N^2 \\ N|c}} \frac{S(IL, JM, c)}{c} J_1\Big(\frac{4\pi\sqrt{ILJM}}{c}\Big)$$
$$\ll N^{-1+\epsilon} \sum_{Id_1, Jd_2 < D} |x_{Id_1}||x_{Jd_2}|IJ$$

So, now we have to bound only below **equation 3.7**. Let taylor series expansion of

J-Bessel function, $J_1(x) = \sum\limits_{l\geq 0} b_l x^{2l+1}$ and integral representation of G_0 then

$$\int \frac{\Gamma(1+t)^2}{t} \sum_{l\geq 0} b_l N^t \sum_{Id_1, Jd_2 < D} x_{Id_1}\overline{x_{Jd_2}} \frac{(IJ)^{l+1}}{(d_1d_2)^t} \sum_{\substack{c<N^2 \\ N|c}} \frac{1}{c^{2l+2}} \times \sum_{\gcd\,(a,c)=1} \sum_{\substack{L,M \\ LM \ll \frac{N^{1+\epsilon}}{d_1d_2}}} (LM)^{l-t} e\Big(\frac{aIL + \overline{a}JM}{c}\Big) dt \quad (3.7)$$

Lemma 3.2.1. *Given integer a and c with $c \nmid a$. Let $b \in \mathbb{C}$ and $||a/c||$ denotes distance of a/c from nearest integer. Then $\forall \epsilon > 0$, $\sum\limits_{M}^{2M} m^b e\Big(\frac{am}{c}\Big) \ll M^b \; min \; \Big(M, \frac{c^{\epsilon}(1+|b|)}{||a/c||}\Big)$*

Proof. Second bound follows from absolute value while second bound follows from upper bound on partial sum and successive difference of m^{α} is bounded by $(1+|\alpha|)m^{-1\mathscr{R}\alpha}$. □

Lemma 3.2.2. *Let $0 < X < c$. Then $\forall \epsilon > 0$,*
$\sum\limits_{ab\equiv X(mod\ c)} min\Big(L, \frac{1}{||a/c||}\Big) min\Big(M, \frac{1}{||b//c||}\Big) \ll c^{\epsilon}(LM + c)$

Apply this lemma in **equation 3.7**,

$$S_{off} \leq \int \frac{\Gamma(1+t)^2}{t} \sum_{l\geq 0} b_l (l + |\mathscr{R}t|)^2 N^{\mathscr{R}t+\epsilon} \sum_{Id_1, Jd_2 < D} \frac{|x_{Id_1}||x_{Jd_2}|(IJ)^{l+1}}{(d_1d_2)^{\mathscr{R}t}} \Big(\frac{N}{d_1d_2}\Big)^{l-\mathscr{R}t} \times \sum_{\substack{c|N \\ c<N^2}} \frac{\Big(c + \frac{N}{d_1d_2}\Big)c^{\epsilon}}{c^{2l+2}} dt$$

Since summation over c is bounded by $N^{-2l-1+\epsilon}$ and using formula $|x_L| \ll \sum_{UL<D} |\eta(L)||y_{UL}|$, for all l, t inner sum (summation over I, J, d_1, d_2) is bounded by

$$N^{-\mathscr{R}t-l-1+\epsilon} D^{\epsilon} \Big(\sum_{I} |y_I| I^{l+1}\Big)^2 \ll N^{-\mathscr{R}t+\epsilon} D^{\epsilon} \Big(\frac{D^2}{N}\Big)^{l+1} \Big(\sum_{I} \phi(I)|y_I|^2\Big)$$

Since $D^2 \ll N$, rapid decaying of b_l and exponential decay of Gamma function on vertical line, S_{off} is negligible for lower bound (**equation 3.6**) for S_{main}. Since $S_{main} + S_{off} = 0$ we say $S_{main} = 0$. So, $y_L = 0$ for all $1 \leq L \leq D$. And consequently $\alpha_i = 0$ for all $1 \leq i < D$.

Chapter 4

Sturm's Theorem for Modular Forms

This chapter introduces congruence for Modular forms, i.e. number of Fourier coefficient required to define modular forms.
As per **Example 1.1.3**, $g_2 = 120\zeta(4)E_4$ is weight 4 modular form and Δ is weight 12 cusp form. By **Corollary 1.1.6**, $\Delta(\tau) \neq 0$ for all $\tau \in \mathcal{H}$. And Δ has simple zero at $\iota\infty$. Now define a function $j : \mathcal{H} \to \mathbb{C}$ by,

$$j(\tau) = 1728\frac{(g_2(\tau))^3}{\Delta(\tau)}$$

Then by modularity of g_2 and Δ, $j(\gamma(\tau)) = j(\tau)$ for all $\gamma \in SL_2(\mathbb{Z})$. Since Δ doesn't vanish on upper half, j is holomorphic on $\mathcal{H}$ and simple pole at $\iota\infty$. Moreover,j has Fourier expansion, $j(\tau) = \frac{1}{q}+744+196884q+\cdots$ where $q = e^{2\pi\iota\tau}$. By **Theorem 1.3.3**, $\mathcal{M}_0(\Gamma) = \mathbb{C}j$. Let $f \in \mathcal{M}_k(\Gamma_0(N))$. Let R be a subring of $\mathbb{C}$ which contains all fourier coefficient of f. Then we denote $f \in \mathcal{M}_k(\Gamma_0(N), R)$.

Definition 4.0.1. *Let F be a field over $\mathbb{Q}$ and $\mathcal{O}_F$ be a ring of integers and $\mathcal{P}$ be a prime ideal. Let $f \in \mathcal{M}_k(\Gamma_0(N), \mathcal{O}_F)$ with $f = \sum a_f(n)q^n$. Define $ord_{\mathcal{P}}(f) = min\{n : a_f(n) \notin \mathcal{P}\}$.(minimum of empty set is ∞)*

Theorem 4.0.1. ***Sturm's Theorem**: Let $f \in \mathcal{M}_k(\Gamma, \mathcal{O}_F)$. Let $\mathcal{P}$ be a prime ideal and $ord_{\mathcal{P}}(f) > \frac{k}{12}\left[SL_2(\mathbb{Z}) : \Gamma\right]$. Then $f \equiv 0(mod\ \mathcal{P})$.*

4.1 Proof for Full Modular Group

Proposition 4.1.1. *Continue to notation of **Theorem 4.0.1**, $\frac{f^{12}}{\Delta^k} \in \mathcal{P}\big[j\big]$, where $\mathcal{P}\big[j\big]$ is set of polynomials of j with coefficient in $\mathcal{P}$.(i.e. $ord_{\mathcal{P}}(f) = \infty$)*

Proof. Induction on k,
For $k = 1$, $f^{12} \in \mathcal{M}_{12}$. Since $\mathcal{M}_{12}(SL_2(\mathbb{Z})) = \mathbb{C}(E_4)^3 \oplus \mathbb{C}\Delta$, there exists $\alpha, \beta \in \mathcal{O}_F$ such that, $f^{12} = \alpha(E_4)^3 + \beta\Delta$. Then $\frac{f^{12}}{\Delta} = \alpha' j + \beta = \sum_{n\geq -1} a(n)q^n$.Since $ord_{\mathcal{P}}(f^{12}) > 1$, $\alpha, \beta \in \mathcal{P}$.
For general k, $12k = 4i+6j$. Moreover, $i = 3i_0$ and $j = 2j_0$. So, $f^{12} - E_4^i E_6^j$ is cusp of weight $12k$. So, $f^{12} = \alpha E_4^i E_6^j + \beta\Delta f_1$ for some $f_1 \in \mathcal{M}_{12(k-1)}$. Then, $\frac{f^{12}}{\Delta^k} = \alpha\left(\frac{E_4^3}{\Delta}\right)^{i_0}\left(\frac{E_6^2}{\Delta}\right)^{j_0} + \beta\frac{f_1}{\Delta^{k-1}}$. Since Δ is linear combination of E_4^3 and E_6^2, and applying induction hypothesis on f_1, we get our result.(Note that only constant term in fourier expansion of f^{12} is because of $E_4^i E_6^j$.) □

Proof of theorem 4.0.1 Since $ord_{\mathcal{P}}(f) > k/12$, $ord_{\mathcal{P}}(f^{12}) > k$. So, $\frac{f^{12}}{\Delta^k} \in \mathcal{P}\big[j\big]$. So, $f^{12} \in \Delta^k\mathcal{P}\big[j\big]$. Consequently, $ord_{\mathcal{P}}(f^{12}) = \infty$. And $ord_{\mathcal{P}}(f) = \infty$.

4.2 Proof for Congruence Subgroup

Since Γ is congruence subgroup, there exists N such that $\Gamma(N) \subset \Gamma$. So, $f \in \mathcal{S}_k(\Gamma(N))$. Since $\mathcal{S}_k(\Gamma(N))$ has a basis of cusp forms whose Fourier coefficients are rational integer, for all $f \in \mathcal{S}_k(\Gamma(N), F)$, there exists $A_f \in F$ so that $A_f f \in \mathcal{S}_k(\Gamma(N), \mathcal{O}_F)$. Moreover, there exists basis whose Fourier coefficients are rational over $\mathbb{Q}(\zeta_N)$ (cyclotomic field with Nth root of unity).
Proof of Theorem 4.0.1: Consider $\frac{f^{12}}{\Delta^k}$ is a modular function at level N. Since $\Gamma(N)$ is normal in $SL_2(\mathbb{Z})$, for all $\gamma \in SL_2(\mathbb{Z})$, $f[\gamma]_k \in \mathcal{M}_k(\Gamma(N), F(\zeta_N))$. Moreover, there exists $A_\gamma \in F(\zeta_N)$ so $A_\gamma f[\gamma]_k \in \mathcal{M}_k(\Gamma(N), \mathcal{O}_{F(\zeta_N)})$.
Let K be a field extension of $F(\zeta_N)$ such that $\mathcal{P}\mathcal{O}_K$ be prinicipal and unramified ideal and let $\mathscr{P}$ be a prime ideal of $\mathcal{O}_K$ such that $\mathscr{P} | \mathcal{P}\mathcal{O}_K$ (existence of such field extension is Hilbert Class Field). If $ord_{\mathscr{P}}(f) < \infty$ then $ord_{\mathscr{P}}(A_\gamma f[\gamma]_k) < \infty$
Let $\Gamma(N)\backslash SL_2(\mathbb{Z}) = \{\gamma_1 = 1, \gamma_2, \cdots, \gamma_n\}$. Consider $\Phi = f\prod_{i=2}^{n} f[\gamma_i]_k$. It is easy to check

$\Phi \in \mathcal{M}_{kn}(SL_2(\mathbb{Z}))$. Since $\Gamma(N) \subset \Gamma$,

$$ord_{\mathscr{P}}(\Phi) \geq ord_{\mathscr{P}}(f) = ord_{\mathcal{P}}(f) > \frac{kn}{12} \geq \frac{k}{12}\big[SL_2(\mathbb{Z}) : \Gamma\big]$$

By result for full modular group, $ord_{\mathscr{P}}(\Phi) = \infty$. Since $ord_{\mathscr{P}}(\Phi) = ord_{\mathscr{P}}(f) + \sum_{i=2}^{n} ord_{\mathscr{P}}(A_\gamma f[\gamma]_k)$, we have $ord_{\mathscr{P}}(f) = ord_{\mathcal{P}}(f) = \infty$.

Chapter 5

Rankin Selberg Method at Level N

In this chapter, we propose a modification of Rankin Selberg Convolution of L function such that a modified Rankin Selberg L function for arbitrary cusp forms has a functional equation which respects critical line. We use techniques given in [13] and [4].

5.1 Non Holomorphic Eisenstein series at level N

Definition 5.1.1. *For* $(z,s) \in \mathcal{H} \times \mathbb{C}_{\mathscr{R}s>1}$,

$$F_N(z,s) = 1 + \sum_{\substack{m>0 \\ (mN,n)=1}} \frac{1}{|mNz+n|^{2s}}$$

Let $\zeta_N(s) = \sum\limits_{(n,N)=1} \frac{1}{n^s}$ then

$$2\zeta_N(2s)F_N(z,s) = \sum_{d|N} \frac{\mu(d)}{d^{2s}} G\Big(\frac{N}{d}z, s\Big)$$

Here $G(z,s) = \sum\limits_{(m,n)\neq(0,0)} \frac{1}{|mz+n|^{2s}}$ where summation runs over $\mathbb{Z}^2$. Then clearly $G(z,s)$ converges absolutely for $\mathscr{R}s > 1$ as a function of s. Let's also define quadratic form as

$Q_z(m,n) = \frac{1}{y}|mz+n|^2$. Then $disc(Q_z) = -4$ independent of z. Now define theta series by,

$$\theta_z(t) = \sum_{(m,n)} e^{-\pi Q_z(m,n)t}$$

Now take Mellin transform of θ_z then

$$\begin{aligned}
M(\theta_z)(s) &= \int_0^\infty \theta_z(x) x^s \frac{dx}{x} \\
&= \sum_{(m,n)\neq(0,0)} \int_0^\infty e^{-\pi Q_z(m,n)x} x^s \frac{dx}{x} + \int_0^\infty x^s \frac{dx}{x} \\
&= \Gamma(s)\left(\frac{y}{\pi}\right)^s G(z,s) + \int_0^\infty x^s \frac{dx}{x} \\
G^*(z,s) := \Gamma(s)\left(\frac{y}{\pi}\right)^s G(z,s) &= \int_0^\infty (\theta_z(x) - 1) x^s \frac{dx}{x}
\end{aligned}$$

By using Poisson summation for $e^{-\pi Q_z(m,n)t}$, we get $t\theta_z(t) = \theta_z(1/t)$. Since $\int_0^\infty = \int_0^1 + \int_1^\infty$ and using change of variable for second part and applying Poisson summation,

$$G^*(z,s) = \int_1^\infty (\theta_z(x) - 1)(x^s + x^{1-s}) \frac{dx}{x} + \frac{1}{s-1} - \frac{1}{s}$$

$$G^*(z,s) = G^*(z, 1-s)$$

So $G^*(z,s)$ has meromorphic continuation to a complex plane with only simple pole at $s = 0, 1$. Moreover, $\gamma \in SL_2(\mathbb{Z})$, $G^*(\gamma(z), s) = G^*(z,s)$ as a function of z.

5.2 Modified Rankin Selberg Function

Definition 5.2.1. *Let N is square-free integer. For $n = \prod_p p^{r_p} \in \mathbb{N}$, $\sigma_N(n) = \prod_{p|(n,N)} \left(\frac{p^{r_p+1}-1}{p-1}\right)$.*

Let $f, g \in \mathcal{S}_k(\Gamma_0(N))$ are Hecke Newforms. Let $f = \sum_{n\geq 1} a_n q^n$ and $g = \sum_{n\geq 1} b_n q^n$ then by **Theorem 1.5.4**

$$L(f,s) = \prod_{p|N}\left(1 - \frac{a_p}{p^s}\right)^{-1} \prod_{p\nmid N}\left(1 - \frac{a_p}{p^s} + \frac{p^{k-1}}{p^{2s}}\right)^{-1} = \prod_{p|N}\left(1 - \frac{a_p}{p^s}\right)^{-1} \prod_{p\nmid N}\left(1 - \frac{\alpha_p}{p^s}\right)^{-1}\left(1 - \frac{\beta_p}{p^s}\right)^{-1}$$

$$L(g,s)=\prod_{p|N}\Big(1-\frac{b_p}{p^s}\Big)^{-1}\prod_{p\nmid N}\Big(1-\frac{b_p}{p^s}+\frac{p^{k-1}}{p^{2s}}\Big)^{-1}=\prod_{p|N}\Big(1-\frac{b_p}{p^s}\Big)^{-1}\prod_{p\nmid N}\Big(1-\frac{\gamma_p}{p^s}\Big)^{-1}\Big(1-\frac{\delta_p}{p^s}\Big)^{-1}$$

So here for $p\nmid N$,$\alpha_p+\beta_p=a_p$, $\gamma_p+\delta_p=b_p$ and $\alpha_p\beta_p=p^{k-1}=\gamma_p\delta_p$. Now define

$$\mathcal{D}(f\times g,s)=\sum\frac{a_n\overline{b_n}}{n^s}$$

Since f,g are eigenforms,

$$\mathcal{D}(f\times g,s)=\prod_p\sum_{l\geq 0}\frac{a_{p^l}\overline{b_{p^l}}}{(p^l)^s}$$

Now trivial calculation gives below result,

$$\zeta_N(2s+2-2k)\mathcal{D}(f\times g,s)=\prod_{p|N}(1-\frac{a_p\overline{b_p}}{p^s})^{-1}\prod_{p\nmid N}(1-\frac{\alpha_p\overline{\gamma_p}}{p^s})^{-1}(1-\frac{\beta_p\overline{\gamma_p}}{p^s})^{-1}(1-\frac{\alpha_p\overline{\delta_p}}{p^s})^{-1}(1-\frac{\beta_p\overline{\delta_p}}{p^s})^{-1}$$

Now let's find $\langle y^sF_N(z,s)f(z),g(z)\rangle$,

$$\begin{aligned}\langle y^sF_N(z,s)f(z),g(z)\rangle&=\int_{X_0(N)}F_N(z,s)f(z)\overline{g(z)}y^{s+k}\frac{dxdy}{y^2}\\&=\int_0^\infty\int_0^1 y^{s+k}f(z)\overline{g(z)}\frac{dxdy}{y^2}\\&=\frac{1}{(4\pi)^{s+k-1}}\Gamma(s+k-1)\mathcal{D}(f\times g,s+k-1)\end{aligned}\tag{5.1}$$

First equality follows from Rankin's unfolding method. Now using result of **section 5.1**,

$$\frac{2}{(4\pi)^{s+k-1}}\Gamma(s+k-1)\zeta_N(2s)\mathcal{D}(f\times g,s+k-1)=\sum_{d|N}\frac{\mu(d)}{d^{2s}}\langle y^sG(\frac{N}{d}z,s)f(z),g(z)\rangle$$

$$\Phi(s)=\Big(\frac{N}{4\pi^2}\Big)^s\Gamma(s)\Gamma(s+k-1)\zeta_N(2s)\mathcal{D}(f\times g,s+k-1)=\frac{(4\pi)^{k-1}}{2}\sum_{d|N}\frac{\mu(d)}{d^s}\langle G^*(Nz/d,s)f,g\rangle$$

has meromorphic continuation to a complex plane since G^* has.

Let $p|N$, define $\beta=\begin{bmatrix}sp&-r\\N&p\end{bmatrix}\in\Gamma_0\Big(\frac{N}{p}\Big)\begin{bmatrix}p&0\\0&1\end{bmatrix}$. Then,

$$f[\beta]_k=\frac{-a_p}{p^{\frac{k}{2}-1}}f\text{ and }g[\beta]_k=\frac{-b_p}{p^{\frac{k}{2}-1}}g$$

Let $d|(N/p)$, $\begin{bmatrix} \frac{N}{pd} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} sp & -r \\ N & p \end{bmatrix} \begin{bmatrix} \frac{d}{N} & 0 \\ 0 & 1 \end{bmatrix} \in \Gamma(1)$. So,

$$G^*\Big(\frac{N\beta(z)}{pd}, s\Big) = G^*\Big(\frac{Nz}{d}, s\Big)$$

By change of variable in Petersson's Inner product ($z \mapsto \beta(z)$) and $\beta^{-1}\Gamma_0(N)\beta = \Gamma_0(N)$,

$$\langle G^*\Big(\frac{Nz}{pd}, s\Big) f(z), g(z)\rangle = \frac{a_p b_p}{p^{k-2}} \langle G^*\Big(\frac{Nz}{d}, s\Big) f(z), g(z)\rangle$$

So, $\Phi(s) = \frac{(4\pi)^{k-1}}{2}\Big(1 - \frac{a_p b_p}{p^{s+k-2}}\Big) \sum_{d|(N/p)} \frac{\mu(d)}{d^s} \langle G^*(Nz/d, s) f, g\rangle$. Now continuing this process for all $p|N$ and remember N is square free,

$$\Phi(s) = \frac{(4\pi)^{k-1}}{2} \prod_{p|N} \Big(1 - \frac{a_p b_p}{p^{s+k-2}}\Big) \langle G^*(Nz, s) f, g\rangle$$

$$\Lambda(f \times g, s) := \prod_{p|N} \Big(1 - \frac{a_p b_p}{p^{s+k-2}}\Big)^{-1} \Phi(s) = \frac{(4\pi)^{k-1}}{2} \langle G^*(Nz, s) f, g\rangle$$

So $\Lambda(f \times g, s)$ has meromorphic continuation to a complex plane with at most simple pole only at $s = 0, 1$. Moreover, $\Lambda(f \times g, s) = \Lambda(f \times g, 1-s)$. Now we have

$$\Lambda(f \times g, s) = \Big(\frac{N}{4\pi^2}\Big)^s \Gamma(s)\Gamma(s+k-1)\zeta_N(2s) \prod_{p|N} \Big(1 - \frac{a_p b_p}{p^{s+k-2}}\Big)^{-1} \mathcal{D}(f \times g, s+k-1)$$

Definition 5.2.2. *Let f, g are newform then modified Rankin Selberg L function,*

$$L(f \times g, s) = \zeta_N(2s) \prod_{p|N} \Big(1 - \frac{a_p b_p}{p^{s+k-2}}\Big)^{-1} \mathcal{D}(f \times g, s+k-1) = \prod_p L_p(s)$$

Then by product structure (begining of this section) we got for $\zeta_N(2s+2-2k)\mathcal{D}(f \times g, s)$,

$$\begin{aligned} p \nmid N \Rightarrow L_p(s-k+1) &= (1 - \frac{\alpha_p \overline{\gamma_p}}{p^s})^{-1}(1 - \frac{\beta_p \overline{\gamma_p}}{p^s})^{-1}(1 - \frac{\alpha_p \overline{\delta_p}}{p^s})^{-1}(1 - \frac{\beta_p \overline{\delta_p}}{p^s})^{-1} \\ &= \Big(1 - \frac{1}{p^{2s+2-2k}}\Big)^{-1} \sum_{l \geq 0} \frac{a_{p^l} \overline{b_{p^l}}}{(p^l)^s} \end{aligned} \tag{5.2}$$

$$
\begin{aligned}
p \mid N \Rightarrow L_p(s+1-k) &= (1-\frac{a_p\overline{b_p}}{p^s})^{-1}(1-\frac{a_p\overline{b_p}}{p^{s-1}})^{-1}\\
&= \sum_{l\geq 0}\frac{(a_p\overline{b_p})^l}{(p^l)^s}\sum_{l\geq 0}\frac{p^l(a_p\overline{b_p})^l}{(p^l)^s}\\
&= \sum_{l,m\geq 0}\frac{p^l(a_p\overline{b_p})^{l+m}}{(p^{l+m})^s}\\
&= \sum_{n\geq 0}\Big(\sum_{\substack{m+l=n\\ m,l\geq 0}}p^l\Big)\frac{(a_p\bar{b}_p)^n}{(p^n)^s} = \sum_{n\geq 0}\Big(\frac{p^{n+1}-1}{p-1}\Big)\frac{a_{p^n}\bar{b}_{p^n}}{(p^n)^s}\\
&= \sum_{n\geq 0}\sigma_N(p^n)\frac{a_{p^n}\overline{b_{p^n}}}{(p^n)^s}
\end{aligned} \tag{5.3}
$$

By combining above result,

$$L(f\times g,s) = \zeta_N(2s)\sum_{n\geq 1}\sigma_N(n)\frac{a_n\overline{b_n}}{n^{s+k-1}}$$

By Ramanujan-Petersson conjecture (proved by Deligne), $a_n \ll n^{\frac{k-1}{2}}$ and $b_n \ll n^{\frac{k-1}{2}}$ for $(n,N)=1$ and for $p|N$,$a_pb_p = \pm p^{k-2}$, we got below result.

Lemma 5.2.1. $r(n) = \sigma_N(n)\frac{a_n\overline{b_n}}{n^{k-1}} \ll 1$. *Here constant depends only on f,g.*

By using above lemma, $L(f\times g,s)$ (infinite series) converges absolutely for $\mathscr{R}s>1$.

Theorem 5.2.2. *Let $f,g\in\mathcal{S}_k(\Gamma_0(N))$ are Hecke Newforms, where N is square-free integer. Then*

$$L(f\times g,s) = \zeta_N(2s)\sum_{n\geq 1}\sigma_N(n)\frac{a_n\overline{b_n}}{n^{s+k-1}}$$

converges absolutely for $\mathscr{R}s>1$. Moreover, $L(f\times g,s)$ extends to meromorphic function on a complex plane with simple pole (at most) only at $s=0,1$. Let

$$\Lambda(f\times g,s) = \Big(\frac{N}{4\pi^2}\Big)^s\Gamma(s)\Gamma(s+k-1)L(f\times g,s)$$

then $\Lambda(f\times g,s)=\Lambda(f\times g,1-s)$, $Res_{s=1}(\Lambda(f\times g,s)) = \frac{(4\pi)^{k-1}}{2}\langle f,g\rangle = -Res_{s=0}(\Lambda(f\times g,s))$. So, $L(f\times g,s)$ and $\Lambda(f\times g,s)$ has poles at $s=0,1$ if and only if $\langle f,g\rangle\neq 0$.

Let $\mathcal{F} = \{h_1, h_2, \cdots h_T\} = \{h; h \in \mathcal{S}_k(\Gamma_0(N))$ is Hecke Newform$\}$ where $T = dim(\mathcal{S}_k(\Gamma_0(N)^{new})$. Let $h_i = \sum\limits_{n\geq 1} a_i(n)q^n$. Then for any $f, g \in \mathcal{S}_k(\Gamma_0(N))^{new}$, $f = \sum\limits_{1\leq i\leq T} \alpha_i h_i = \sum\limits_{n\geq 1} a_f(n)q^n$ and $g = \sum\limits_{1\leq i\leq T} \beta_i h_i = \sum\limits_{n\geq 1} a_g(n)q^n$ define

$$\begin{aligned}
L(f\times g, s) &= \sum_{1\leq i,j\leq T} \alpha_i\overline{\beta_j} L(h_i \times h_j, s) \\
&= \sum_{1\leq i,j\leq T} \alpha_i\overline{\beta_j}\zeta_N(2s)\sum_{n\geq 1}\sigma_N(n)\frac{a_i(n)\overline{a_j(n)}}{n^{s+k-1}} \\
&= \zeta_N(2s)\sum_{n\geq 1}\frac{\sigma_N(n)}{n^{s+k-1}}\Big(\sum_{1\leq i\leq T}\alpha_i a_i(n)\Big)\overline{\Big(\sum_{1\leq j\leq T}\beta_j a_j(n)\Big)} \\
L(f\times g, s) &= \zeta_N(2s)\sum_{n\geq 1}\sigma_N(n)\frac{a_f(n)\overline{a_g(n)}}{n^{s+k-1}}
\end{aligned} \tag{5.4}$$

Since $L(f\times g)$ is finite linear sum, it also converges absolutely for $\mathscr{R}s > 1$.

$$\begin{aligned}
\Lambda(f\times g, s) &= \sum_{1\leq i,j\leq T} \alpha_i\overline{\beta_j}\Lambda(h_i\times h_j, s) \\
&= \sum_{1\leq i,j\leq T} \alpha_i\overline{\beta_j}\frac{(4\pi)^{k-1}}{2}\langle G^*(Nz,s)h_i, h_j\rangle \\
\Lambda(f\times g, s) &= \frac{(4\pi)^{k-1}}{2}\langle G^*(Nz,s)f, g\rangle
\end{aligned} \tag{5.5}$$

So again by result of **section 5.1**, $\Lambda(f\times g, s) = \Lambda(f\times g, 1-s)$. Hence we have proved,

Theorem 5.2.3. *Let $f, g \in \mathcal{S}_k(\Gamma_0(N))^{new}$ are arbitrary cusp form, where N is square-free integer. Then*

$$L(f\times g, s) = \zeta_N(2s)\sum_{n\geq 1}\sigma_N(n)\frac{a_n\overline{b_n}}{n^{s+k-1}}$$

converges absolutely for $\mathscr{R}s > 1$. Moreover, $L(f\times g, s)$ extends to meromorphic function on a complex plane with simple pole (at most) only at $s = 0, 1$. Let

$$\Lambda(f\times g, s) = \Big(\frac{N}{4\pi^2}\Big)^s\Gamma(s)\Gamma(s+k-1)L(f\times g, s)$$

then $\Lambda(f\times g, s) = \Lambda(f\times g, 1-s)$, $Res_{s=1}(\Lambda(f\times g, s)) = \frac{(4\pi)^{k-1}}{2}\langle f, g\rangle = -Res_{s=0}(\Lambda(f\times g, s))$. So, $L(f\times g, s)$ and $\Lambda(f\times g, s)$ has poles at $s = 0, 1$ if and only if $\langle f, g\rangle \neq 0$.

Now we are interested to find logarithmic derivative of $L(f\times g,s)$. So let $A(s)=\sum_{n\geq 1}\frac{r(n)}{n^s}$. Then by **Lemma 5.2.1**, $A(s)$ converges absolutely for $\mathscr{R}s>1$ (Note that Lemma 5.2.1 is true for arbitrary cusp also). Moreover, $s=x+\iota y$ then $|A(s)|\ll\zeta(x)$ and $\forall\epsilon>0$ there exists constant c_ϵ such that

$$|\frac{A(s)}{c_\epsilon}-1|<1$$

Now take logarithm for $A(s)$ and applying Taylor series expansion for $\log(1+x)$,

$$\log\,A(s)=\log\,c_\epsilon-\sum_{l\geq 1}\frac{(-1)^l}{l}\Big(\frac{A(s)}{c_\epsilon}-1\Big)^l$$

For $(s,x)\in\mathbb{C}_{\mathscr{R}s>1+\epsilon}\times\mathbb{R}$ and $l\in\mathbb{N}$ define $F_l(s,x)=(-1)^l(l+1)\Big(\frac{A(s)}{c_\epsilon}-1\Big)^l\mathbb{1}_{(1+\frac{1}{l+1},\frac{1}{l})}(x)$,where $\mathbb{1}$ is simple function. Let $F(s,x)=\sum_{l\in\mathbb{N}}F_l(s,x)$. Then by (Theorem 25,[1]), we can take derivative of $\log\,A(s)$ with respect to s.

$$\frac{A(s)'}{A(s)}=\frac{1}{c_\epsilon}\Big(\sum_{n\geq 1}-\frac{r(n)\,\log\,n}{n^s}\Big)\left(\sum_{l\geq 0}\Big(1-\frac{A(s)}{c_\epsilon}\Big)^l\right).$$

Since $\log\,n<n^\epsilon$ for some $n>N_0$, $\frac{A(s)'}{A(s)}=\sum_{n\geq 1}\frac{a(n)}{n^s}$ converges absolutely for $\mathscr{R}s>1+\epsilon$ for all $\epsilon>0$. Now take logarithmic derivative of $L(f\times g,s)=\zeta_N(2s)A(s)$. So again for $\mathscr{R}s>1+\epsilon$,

$$\begin{aligned}
\log\,L(f\times g,s)&=-\sum_{p\nmid N}\log\,\Big(1-\frac{1}{p^{2s}}\Big)+\log\,A(s)\\
\frac{L'}{L}(f\times g,s)&=-\sum_{p\nmid N}\frac{2\log\,p}{p^{2s}\Big(1-\frac{1}{p^{2s}}\Big)}+(\log\,A(s))'\\
-\frac{L'}{L}(f\times g,s)&=\sum_{p\nmid N}2\log\,p\sum_{l\geq 1}p^{-2sl}-\sum_{n\geq 1}\frac{a(n)}{n^s}=\sum_{n\geq 1}\frac{\lambda(n)}{n^s}
\end{aligned}$$

Since both series converges absolutely for $\mathscr{R}s>1+\epsilon$ so sum also converges absolutely. Now fix $x_0\in\mathbb{R}>1+\epsilon$ for some $\epsilon>0$ then for any $s=x_0+\iota y$,

$$-\frac{L'}{L}(f\times g,s)\ll_{x_0}1 \tag{5.6}$$

5.3 Non-vanishing region for modified Rankin Selberg L function

Theorem 5.3.1. *Let f, g is cusp for of weight k (even) and N (square-free). Then modified Rankin Selberg $L(f \times g; s) \neq 0$ for any $s \in \Omega = \{s \in \mathbb{C}; \mathscr{R}s > 1\}$.*

Proof. Let's assume $L(f \times g; s_0) = 0$ for some $s_0 \in \Omega$ and order of vanishing is $\mathbf{Z} > 0$. Since $L(f \times g; s)$ is an analytic function in Ω, there exists neighborhood $U \subset \Omega$ around s_0 such that $L(f \times g; s) \neq 0$ for all $s \in U$ except s_0. Choose small neighborhood U such that $\mathscr{R}s > 1 + \epsilon$ for all $s \in U$. Let $\mathbf{C} \subset U$ is a smooth, closed and simple curve such that s_0 is an interior point for region closed by $\mathbf{C}$.

Recall from **Section 5.2**, $L(f \times g; s)$ has no poles in Ω and

$$\frac{L'}{L}(f \times g; s) = -\sum_{n \geq 1} \frac{\lambda(n)}{n^s}$$

converges absolutely for $\mathscr{R}s > 1 + \epsilon$. Then by Argument Principle,

$$2\pi\iota\mathbf{Z} = \int_{\mathbf{C}} \frac{L'}{L}(f \times g; s)ds = -\sum_{n \geq 1} \lambda(n) \int_{\mathbf{C}} n^{-s} ds = 0$$

The last equality follows since n^{-s} is an analytic function on U. So, $\mathbf{Z} = 0$, which is contradiction. □

Conjecture 1. *(**Riemann Hypothesis**) Let f, g be any cusp form of weight k(even) at level N (square-free). Then modified Rankin Selberg L function $M(f \times g, s) = 0$ then either s is a negative integer or $\mathscr{R}s = \frac{1}{2}$.*

Chapter 6

Congruence of Cusp forms

Let $f = \sum_{n\geq 1} a_f(n)q^n$ is cusp form of weight k for level N such that $a_f(n) = 0$ for all $1 \leq n \leq \frac{k}{12}[\Gamma(1) : \Gamma_0(N)]$ then recalling from **Cha. 4**, $f \equiv 0$. If we assume Riemann Hypothesis for modified Rankin-Selberg Convolution for $L(f \times f, s)$ then we can prove that $a_f(n) = 0$ for all $1 \leq n \leq A_N$,where $A_N \gg (\log N)^2(\log\log N)^4$ then $f \equiv 0$. We extended result of Goldfeld and Hoffstein; proven in (Theorem 2,[5]) using similar proof.

Let k be an even integer and N is square-free integer. Let $f, g \in \mathcal{S}_k(\Gamma_0(N))$ are non-zero cusp forms. Let $L(s) := L(f \times g, s)$ and $\Lambda(s) := s(1-s)\Lambda(f \times g, s) = s(1-s)D^sG(s)L(s)$,here $D = \frac{N}{4\pi^2}$ and $G(s) = \Gamma(s)\Gamma(s+k-1)$. Then by **Theorem 5.2.3**, $\Lambda(s)$ is an entire function for any f, g.

6.1 Extension of Goldfeld and Hoffstein's result

Lemma 6.1.1. *Let fix $\kappa > 1$. Let continue above notation with $D > \kappa$ and assuming Riemann hypothesis for $L(s)$ (i.e. all non-trivial zeros of $L(s)$ lies on critical strip $\mathscr{R}s = \frac{1}{2}$). Then for $x > 10\ log\ D$,*

$$\sum_{\substack{\gamma\in\mathbb{R} \\ \Lambda(\frac{1}{2}+\iota\gamma)=0}} \frac{sin^2(\gamma\ log\ x)}{\gamma^2} \ll_{G,\kappa} (log\ D)(log\ x)^2$$

Proof. By using Hadamard Product theorem (assume without proof) for an entire function, we can write $\Lambda(s)$ as product over zeros. Now applying functional equation $\Lambda(s) = \Lambda(1-s)$ and take logarithmic derivative,

$$\log\, D + \frac{G'}{G}(s) + \frac{L'}{L}(s) + \frac{1}{s} + \frac{1}{s-1} = \sum_{\gamma} \frac{1}{s - \frac{1}{2} - \iota\gamma}$$

Let $s = 2 + \iota y$ and $N(t) = \#\{\gamma \mid |\gamma - t| \leq 1\}$

$$\mathscr{R}\Big(\sum_{\gamma} \frac{1}{\frac{3}{2} + \iota(y-\gamma)}\Big) \geq \sum_{|\gamma - t| \leq 1} \frac{\frac{3}{2}}{\frac{9}{4} + 1} = \frac{6}{13} N(t)$$

Consequently,

$$N(t) \leq \frac{13}{6}\Big(\log\, D + \mathscr{R}\Big(\frac{G'}{G}(2+\iota t) + \frac{1}{2+\iota t} + \frac{1}{1+\iota t}\Big) + |\frac{L'}{L}(2+\iota t)|\Big)$$

As per **equation 5.6**, $|\frac{L'}{L}(2+\iota t)| \ll 1$ for all $t \in \mathbb{R}$ and upper bound on Digamma function ($\psi(s) = \frac{\Gamma'}{\Gamma}(s)$),

$$\begin{aligned} |t| \leq 1, D > e^3 \Rightarrow & N(t) \ll \log\, D \\ |t| \geq 1, \Rightarrow & N(t) \ll \log\, |t| + \log\, D \end{aligned} \tag{6.1}$$

Since |sin(t)/t| $\leq$ 1,

$$\begin{aligned} \sum_{\gamma} \frac{sin^2(\gamma \log\, x)}{\gamma^2} &= \sum_{|\gamma| \leq 1} \frac{sin^2(\gamma \log\, x)}{\gamma^2} + \sum_{|\gamma| > 1} \frac{sin^2(\gamma \log\, x)}{\gamma^2} \\ &= N(0)(\log\, x)^2 + 2\sum_{n \geq 0} \frac{N(2n+1)}{(2n+1)^2} \\ &\ll (\log\, D)(\log\, x)^2 + 2\sum_{n \geq 0} \frac{\log(2n+1) + \log\, D}{(2n+1)^2} \\ &\ll (\log\, D)(\log\, x)^2 + 2\zeta'(2) + (\log\, D)\zeta(2) \\ \sum_{\gamma} \frac{sin^2(\gamma \log\, x)}{\gamma^2} &\ll (\log\, D)(\log\, x)^2 \end{aligned} \tag{6.2}$$

Here last inequality follows because $x > 10 \log\, D$. $\square$

Theorem 6.1.2. *Let $f \neq g \in \mathcal{S}_k(\Gamma_0(N))^{new}$ are cusp forms such that $\langle f, g\rangle$=0, where N is square-free integer. Let $f = \sum\limits_{n\geq 1} a_n q^n$ and $g = \sum\limits_{n\geq 1} b_n q^n$. Let's assume Riemann Hypothesis (conjecture 1) for $L(f\times f, s)$ and $L(f\times g, s)$. Then for all $\kappa > 1$ there exists $C_\kappa > 0$ (depends only on κ, k) such that for all $N > \kappa$ there exists integer $n \leq C_\kappa (log\ N)^2 (log\ log\ N)^4$ for which $a_n \neq b_n$.*

Proof. Let consider integral

$$I = \frac{1}{2\pi\iota}\int_{(2)} \Big(\frac{x^{s-\frac{1}{2}} - x^{\frac{1}{2}-s}}{s-\frac{1}{2}}\Big)^2 \Big(-\frac{L'}{L}(s)\Big) ds$$

Here integration over imaginary line such that Real part is 2. Since (can be proved by contour shift integration)

$$\frac{1}{2\pi\iota}\int_{(2)} \frac{y^{s-\frac{1}{2}}}{(s-\frac{1}{2})^2} ds = \begin{cases} (\log\ y)^2 & y \geq 1 \\ 0 & y \leq 1 \end{cases}, \qquad -\frac{L'}{L}(s) = \sum_{n\geq 1}\frac{\lambda(n)}{n^s}$$

and recall from **equation 5.6** then for $x > 1$,

$$I = \sum_{n<x^2} \frac{\lambda(n)}{\sqrt{n}} \log\Big(\frac{x^2}{n}\Big)$$

To move line of integration from $\mathscr{R}s = 2$ to $\mathscr{R}s = 1/4$, we have to pick up residue at zeros of $\Lambda(s)$ on $\mathscr{R}s = 1/2$ and pole at $s = 1$.

$$Res_{s=\frac{1}{2}+\iota\gamma}\Big(\frac{x^{s-\frac{1}{2}} - x^{\frac{1}{2}-s}}{s-\frac{1}{2}}\Big)^2\Big(-\frac{L'}{L}(s)\Big) = -4\frac{sin^2(\gamma \log\ x)}{\gamma^2}$$

. So we get,

$$I = 4(x - 2 + \frac{1}{x}) - 4\sum_{\gamma}\frac{sin^2(\gamma\log\ x)}{\gamma^2} + \frac{1}{2\pi\iota}\int_{(1/4)}\Big(\frac{x^{s-\frac{1}{2}} - x^{\frac{1}{2}-s}}{s-\frac{1}{2}}\Big)^2\Big(-\frac{L'}{L}(s)\Big)ds$$

Since $\Lambda(s) = \Lambda(1-s)$,

$$-\frac{L'}{L}(s) = 2\ \log\ D + \frac{G'}{G}(s) + \frac{G'}{G}(1-s) + \frac{L'}{L}(1-s)$$

and use change of variable $s \mapsto 1-s$,

$$\begin{aligned} P &:= \frac{1}{2\pi\iota}\int_{(1/4)} \Big(\frac{x^{s-\frac{1}{2}} - x^{\frac{1}{2}-s}}{s-\frac{1}{2}}\Big)^2 \Big(-\frac{L'}{L}(s)\Big) ds \\ &= \frac{1}{2\pi\iota}\int_{(3/4)} \Big(\frac{x^{s-\frac{1}{2}} - x^{\frac{1}{2}-s}}{s-\frac{1}{2}}\Big)^2 \Big(2 \log D + \frac{G'}{G}(s) + \frac{G'}{G}(1-s) + \frac{L'}{L}(s)\Big) ds \\ &= J - \frac{1}{2\pi\iota}\int_{(3/4)} \Big(\frac{x^{s-\frac{1}{2}} - x^{\frac{1}{2}-s}}{s-\frac{1}{2}}\Big)^2 \Big(-\frac{L'}{L}(s)\Big) ds \\ &= J + 4(x-2+\frac{1}{x}) - I \end{aligned} \tag{6.3}$$

Last equality follows by again picking up residue at $s = 1$ and transfer integral to $\mathscr{R}s = 2$. Here $J = \frac{1}{2\pi\iota} \int\limits_{(3/4)} \Big(\frac{x^{s-\frac{1}{2}} - x^{\frac{1}{2}-s}}{s-\frac{1}{2}}\Big)^2 \Big(2 \log D + \frac{G'}{G}(s) + \frac{G'}{G}(1-s)\Big)$

$$2I = 2\sum_{n<x^2} \frac{\lambda(n)}{\sqrt{n}} \log\Big(\frac{x^2}{n}\Big) = 8\Big(x-2+\frac{1}{x}\Big) - 4\sum_{\gamma} \frac{sin^2(\gamma \log x)}{\gamma^2} + J \tag{6.4}$$

$L_1(s) = L(f \times f, s)$ and $L_2(s) = L(f \times g, s)$. By using **Theorem 5.2.3**, L_1 has meromorphic continuation to a complex plane with simple pole at $s = 0, 1$ while L_2 has analytic continuation to a whole complex plane. Now by giving appropriate index, we got

$$2I_1 = 2\sum_{n<x^2} \frac{\lambda_1(n)}{\sqrt{n}} \log\Big(\frac{x^2}{n}\Big) = 8\Big(x-2+\frac{1}{x}\Big) - 4\sum_{\gamma} \frac{sin^2(\gamma \log x)}{\gamma^2} + J$$

$$2I_2 = 2\sum_{n<x^2} \frac{\lambda_2(n)}{\sqrt{n}} \log\Big(\frac{x^2}{n}\Big) = -4\sum_{\gamma'} \frac{sin^2(\gamma' \log x)}{\gamma'^2} + J$$

Since J does not depends on f, g, we give same index. And L_2 has no pole so $8(x-2+x^{-1})$ is removed. Let's assume that $\lambda_1(n) = \lambda_2(n)$ for all $n < x^2$. By subtracting I_2 from I_1,

$$8(x-2+x^{-1}) - 4\sum_{\gamma} \frac{sin^2(\gamma \log x)}{\gamma^2} + 4\sum_{\gamma'} \frac{sin^2(\gamma' \log x)}{\gamma'^2} = 0$$

By using **Lemma 6.1.1**,

$$\begin{aligned} x - 2 + \frac{1}{x} &\leq \frac{1}{2}\sum_{\gamma}\frac{sin^2(\gamma \log\, x)}{\gamma^2} \\ &\ll (\log\, D)(\log\, x)^2 \\ x &\ll (\log\, D)(\log\log\, D)^2 \end{aligned} \tag{6.5}$$

So, there is a contradiction if $x \gg (\log\, D)(\log\log\, D)^2$. □

Corollary 6.1.3. *Let $f \in S_k(\Gamma_0(N))^{new}$ be arbitrary cusp form such that $a_f(n) = 0$ for all $n \leq A_N$,where $A_N \gg (log\ N)^2(log\ log\ N)^4$. Then $f \equiv 0$.*

Proof. Let's assume $f \not\equiv 0$. Then by **Theorem 5.2.3**, $L(f \times f; s)$ has meromorphic continuation with simple pole at $s = 0$ and $s = 1$. Then, for $n \leq A_N$, n^{th} coefficient of $\frac{L'}{L}(s)$ is

$$\lambda(n) = \begin{cases} 2\ \log\, p & n = p^{2l}; p \nmid N \\ 0 & else \end{cases}$$

Now let $x = \sqrt{A_N}$ then $2\sum_{n<x^2}\frac{\lambda(n)}{\sqrt{n}}\log\left(\frac{x^2}{n}\right)$ is bounded by $\mathcal{O}(\log\, x)$. On the other hand by **equation 6.4**, we get,

$$x - 2 + x^{-1} = \frac{1}{2}\sum_{\substack{\gamma\in\mathbb{R} \\ \Lambda(f\times f;\frac{1}{2}+\iota\gamma)=0}}\frac{sin^2(\gamma\ \log\, x)}{\gamma^2} + \mathcal{O}(\log\, x) + \mathcal{O}((\log\, N)(\log\, x))$$

Again by **Lemma 6.1.1**, $x - 2 + x^{-1} \ll (\log\, N)(\log\, x)^2$. Consequently, $x \ll (\log\, N)(\log\log\, N)^2$ which is contradiction. □

6.2 Improvement of Vanderkam's Theorem

Now we want to check linear independence of Hecke operators in the homology of $H_1(X_0(N), \mathbb{Z})$. Let's recall from **Cha. 3**, e be the winding cycle then e acts on $S_2(\Gamma_0(N))$ by

$$(e, f) = \frac{1}{2\pi}L(f, \frac{1}{2}).$$

Theorem 6.2.1. *Let $\mathcal{F} = \{f_1, f_2, \cdots, f_L\}$ are Hecke Newform for $S_2(\Gamma_0(N))$ such that $L(f_i, \frac{1}{2}) \neq 0$, where N is prime. Then $\{T_1e, T_2e, \cdots, T_Le\}$ acts linearly independently on $\mathcal{S}_2(\Gamma_0(N))$.*

Proof. Let $\sum_{1\le i\le L} \alpha_i T_i e = 0$. (we want to show that $\alpha_i = 0$.) Since $T_i's$ are linear operator it is sufficient to check on basis. So, we check on basis consisting of Hecke Newforms, $\mathcal{F}^* = \{f_1, f_2, \cdots, f_g\}$ where g is genus of $X_0(N)$ or $dim(S_k(\Gamma_0(N)))$. $\mathcal{F} \subset \mathcal{F}^*$ and remaining elements have $L(f_i, \frac{1}{2}) = 0$. Let $f_i = \sum_{n\ge 1} a_i(n)q^n$. So for $f_j \in \mathcal{F}^*$,

$$\sum_{1\le i\le L} \alpha_i (T_i e, f_j) = \sum_{1\le i\le L} \alpha_i (e, T_i f_j) = \sum_{1\le i\le L} \alpha_i a_j(i) \frac{1}{2\pi} L(f_j, \frac{1}{2}) = 0$$

So for $1 \le j \le L$,

$$\sum_{1\le i\le L} \alpha_i a_j(i) = 0$$

Now let A is $L \times L$ square matrix such that $[A]_{ij} = a_i(j)$ and $\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_L]^T$ is column vector then $A\alpha = 0$. By linear algebra, $\Big\{A\alpha = 0 \Rightarrow \alpha = 0\Big\} \Leftrightarrow \Big\{A^T\alpha = 0 \Rightarrow \alpha = 0\Big\}$ So now we have

$$\sum_{1\le i\le L} \alpha_i a_i(j) = 0 \text{ for } 1 \le j \le L$$

Now consider $f = \sum_{1\le i\le L} \alpha_i f_i = \sum_{n\ge 1} a_f(n) q^n$ then $a_f(n) = 0$ for all $1 \le n \le L$. By (Theorem 2, [10]), we can say that $L \ge \frac{1}{6} g = \frac{1}{72} N + \mathcal{O}(1)$. So for large enough N, $L \gg (\log N)^2 (\log\log N)^4$. Then by **Theorem 6.1.3**, we have $f \equiv 0$. So $\alpha_i = 0$ for all $1 \le i \le L$. Hence, theorem is proved. □

Chapter 7

Linear Independence of Hecke Operator

This chapter is based on work of supervisor and Satadal Ganguly. They prove below theorem by using analytic techniques. By using this result, we prove linear independence of Poincare series (see corollary 7.0.2).

Theorem 7.0.1. *The Hecke Operators* $T_1, T_2, \cdots, T_D$ *acts linearly independenty on* $\mathcal{S}_k(\Gamma_0(N))$ *when* $D \ll N^{1-\frac{1}{2k}}$ *and* N *is prime.*

Corollary 7.0.2. *The Poincare series* $P_1, P_2, \cdots, P_D$ *are linearly independent elements in* $\mathcal{S}_k(\Gamma_0(N))$ *when* $D \ll N^{1-\frac{1}{2k}}$ *and* N *is prime.*

7.1 Proof for Hecke operators

Let $\sum\limits_{1\leq d\leq D} c_d T_d = 0$ for some compplex number c_d. We wish to prove $c_d = 0$. Since there is basis consisting of Hecke eigenform, we need to check only on this basis. Let $B_k(N) = \{f_1, f_2, \cdots, f_g\}$ is basis where g is dimension of $\mathcal{S}_k(\Gamma_0(N))$ then $T_n f_i = a_i(n)$. So we have,

$$\sum_{1\leq d\leq D} c_d a_i(d) = 0 \text{ for } 1 \leq i \leq g \Leftrightarrow \sum_{1\leq i\leq g} |\sum_{1\leq d\leq D} c_d a_i(d)|^2 = 0 \tag{7.1}$$

Now opening square and using Petersson's formula (Theorem 2.2.1),

$$\sum_{m \le D} |c_m|^2 + 2\pi i^{-k} \sum_{1 \le m,n \le D} c_m \overline{c_n} \sum_{c \ge 1} \frac{S(m,n;Nc)}{Nc} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{Nc}\right) = 0 \tag{7.2}$$

Let's recall trick which we use to prove Vanderkam's Theorem (Theorem 1.7.3), and apply here,

$$\begin{aligned} S_{main} &= \sum_{1 \le m \le D} |c_m|^2 \\ S_{off} &= \sum_{1 \le m,n \le D} c_m \overline{c_n} \sum_{c \ge 1} \frac{S(m,n;Nc)}{Nc} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{Nc}\right) \end{aligned} \tag{7.3}$$

Let's denote inner sum by $A(m,n)$. Then break $A(m,n) = A^+(m,n,c_0) + A^-(m,n,c_o)$ into two parts where A^+ denotes summation over $c > c_0$ and later denotes summation over $1 \le c \le c_0$.

We wish to study $A^+(m,n,c_0)$ first. We require below lemma for that.

Lemma 7.1.1. *We have the estimates, for any $\delta > 0$*

$$\sum_{c > c_0} \frac{(a,c)}{c^{1+\delta}} \ll \tau(a)^2 c_0^{-\delta}.$$

Proof. We write the sum over c as a double sum

$$\sum_c = \sum_{d|a} \sum_{(a,c)=d}$$

and make a change of variables $c = dc_1$ with $(a, c_1) = 1$. Capturing the coprimality condition using the Möbius function and making another change of variables we obtain

$$\sum_c = \sum_{d|a} d \sum_{r|a} \mu(r) \sum_{c_2} \frac{1}{(rdc_2)^{1+\delta}}.$$

The results follow from this upon trivial estimation. □

We also need the bound $J_{k-1}(x) \ll x^{k-1}$ and the Weil bound $|S(m,n,c)| \le \sqrt{c(m,n,c)}\tau(c)$, where $\tau(m)$ denotes the total number of divisors of a postive integer m. Recall that $\tau(m) = O_\varepsilon(m^\varepsilon)$ for any $\varepsilon > 0$. By the above bounds and the lemma, we obtain the estimate

$$\begin{aligned} A^+(m,n,c_0) &\ll \sum_{c>c_0} \frac{\sqrt{Nc(m,n,Nc)}(Nc)^\varepsilon}{Nc}\left(\frac{\sqrt{mn}}{Nc}\right)^{k-1} \\ &\ll \frac{(mn)^{(k-1)/2}}{N^{k-1/2-\varepsilon}} \sum_{c>c_0} \frac{\sqrt{((m,n),c)}}{c^{k-1/2-\varepsilon}} \\ &\ll \frac{(mn)^{(k-1)/2}}{N^{k-1/2} c_0^{k-3/2}} (NDc_0)^\varepsilon \end{aligned}$$

last inequality follows since $m, n \le D$. Suppose S^+_{off} denotes the sum of those terms in S_{off} for which $c > c_0$. We estimate this as follows:

$$\begin{aligned} S^+_{off} &\ll N^{-k+1/2} c_0^{-k+3/2} (NDc_0)^\varepsilon \sum_{m\le D}\sum_{n\le D} |m^{(k-1)/2} c_m n^{(k-1)/2} \overline{c_n}| \\ &\ll N^{-k+1/2} c_0^{-k+3/2} D^k ||c||^2 (NDc_0)^\varepsilon, \end{aligned} \tag{7.4}$$

where we have used Cauchy's inequality to estimate the double sum.

Now we estimate the other part of the sum, i.e., the sum S^-_{off} which is the sum over those c with $c \le c_0$. First we recall a standard integral representation of the J-Bessel function (see [6]). For $x > 0$, one can write

$$J_{k-1}(x) = \frac{1}{2\pi i}\int_{(\alpha)} \frac{\Gamma(-s)}{\Gamma(2+s)}\left(\frac{x}{2}\right)^{k-1+2s} ds$$

for any $\alpha \in (-1/2, 0)$. Writing the Bessel function as above we insert the sum over m and n inside the integral and write

$$S^-_{off} = \frac{1}{2\pi i}\int_{(\alpha)} \frac{\Gamma(-s)(2\pi)^{k-1+2s}}{\Gamma(2+s)} \sum_{c\le c_0} \frac{1}{(Nc)^{k+2s}} \sum_{m\le D}\sum_{n\le D} c_m m^{(k-1)/2+s} \overline{c_n} n^{(k-1)/2+s} S(m,n;Nc)$$

Shifting the contour to the right to the line $\Re s = 1/2$ and picking up the residue from the pole at $s = 0$ (coming from $\Gamma(-s)$), we get

$$S^-_{off} = \mathrm{Res}_{s=0} + \frac{1}{2\pi i}\int_{(1/2)} \frac{\Gamma(-s)(2\pi)^{k-1+2s}}{\Gamma(2+s)} \sum_{c\le c_0} \frac{1}{(Nc)^{k+2s}} \sum_{m\le D}\sum_{n\le D} c_m m^{(k-1)/2+s} \overline{c_n} n^{(k-1)/2+s} S(m,n;Nc)$$

and

$$\mathrm{Res}_{s=0} = \frac{-(2\pi)^{k-1}}{\Gamma(2)} \sum_{c \le c_0} \frac{1}{(Nc)^k} \sum_{m \le D} \sum_{n \le D} c_m m^{(k-1)/2} \overline{c_n} n^{(k-1)/2} S(m,n;Nc)$$

Now we estimate the double sum over m and n, in the expression for the residue as well as in the integral, using the Weil bound first. By the Weil bound and again using Lemma 7.1.1, we have that

$$\mathrm{Res}_{s=0} \ll \frac{D^k ||\mathbf{c}||^2}{N^{k-1/2}} (NDc_0)^{\varepsilon}.$$

Similarly,

$$\int_{(1/2)} \ll \frac{D^{k+1} ||\mathbf{c}||^2}{N^{k+1/2}} (NDc_0)^{\varepsilon} \int_{-\infty}^{\infty} \left| \frac{\Gamma(-1/2+it)(2\pi)^{k-1+2(1/2+it)}}{\Gamma(2+1/2+it)} \right| dt$$

and

$$\int_{-\infty}^{\infty} \left| \frac{\Gamma(-1/2+it)(2\pi)^{k-1+2(1/2+it)}}{\Gamma(2+1/2+it)} \right| \ll \int_{-\infty}^{\infty} |1+it|^{-3} dt \ll 1$$

by the Stirling approximation for $\Gamma(s)$. Thus

$$S_{off}^{-} \ll ||\mathbf{c}||^2 \left(\frac{D^{k+1}}{N^{k+1/2}} + \frac{D^k}{N^{k-1/2}} \right) (NDc_0)^{\varepsilon}$$

Now, we choose $c_0 = N$. Suppose $D \ll N^{1-1/2k}$. Then from Eq. (7.4), we see that S_{OD}^{+} is very small compared to $||\mathbf{c}||^2$; for example,

$$S_{off}^{+} \ll N^{-k} ||\mathbf{c}||^2$$

and the above two bounds yield

$$S_{off}^{-} \ll N^{-1/2k+2\varepsilon} ||\mathbf{c}||^2.$$

Hence

$$S_{off} = S_{off}^{+} + S_{off}^{-} \ll N^{-1/2k+2\varepsilon} ||\mathbf{c}||^2.$$

Since $\varepsilon > 0$ is arbitrary, by choosing $\varepsilon < 1/4k$ and the relation

$$S_{main} + 2\pi i^{-k} S_{off} = 0$$

we get $S_{main} = 0$. Now from equation 7.3, $c_d = 0$ for all $1 \le d \le D$.

7.2 Proof for Poincare Series

Let's recall Poincare series (chapter 2) defined as

$$P_m(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,z)^{-k} e(m\gamma z) = \sum_{n\geq 1} p(m,n)q^n$$

;where

$$p(m,n) = \left(\frac{n}{m}\right)^{(k-1)/2} \left\{ \delta(m,n) + 2\pi\iota^{-k} \sum_{c\geq 1} \frac{S(m,n;Nc)}{Nc} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{Nc}\right) \right\}$$

Then $\{P_1, P_2, \cdots\}$ consists another basis for $S_k(\Gamma_0(N))$. i.e. only element which is orthogonal to set is 0. Now we prove corollary 7.0.2.

Proof. Let $D \ll_k N^{1-\frac{1}{2k}}$,

$$\sum_{1\leq d\leq D} c_d P_d = 0$$

Now for any cusp form f, $\langle f, P_n\rangle = ca_f(n)$,where $c \neq 0$ is constant depends on k, n. Apply this result to any eigenform,

$$0 = \langle f, \sum_d c_d P_d\rangle = \sum_d c_d \alpha_d a_f(d)$$

So, we get 7.1 for all $f \in B_k(N)$. So, we get

$$\sum_{1\leq d\leq D} c_d P_d = 0 \Leftrightarrow \sum_{1\leq d\leq D} c_d \alpha_d T_d = 0$$

Now by theorem 7.0.1, $\alpha_d c_d = 0 \Rightarrow c_d = 0$.

□

Bibliography

[1] E.P.van der Ban. Tools from lebesgue integration, 2005.

[2] Fred Diamond and Jerry Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[3] H. M. Farkas and I. Kra. *Riemann surfaces*, volume 71 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1992.

[4] Paul Garrett. Basic rankin selberg, July 2010.

[5] Dorian Goldfeld and Jeffrey Hoffstein. On the number of Fourier coefficients that determine a modular form. In *A tribute to Emil Grosswald: number theory and related analysis*, volume 143 of *Contemp. Math.*, pages 385–393. Amer. Math. Soc., Providence, RI, 1993.

[6] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Academic Press, Inc., San Diego, CA, sixth edition, 2000. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger.

[7] Henryk Iwaniec. *Topics in classical automorphic forms*, volume 17 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997.

[8] Henryk Iwaniec and Emmanuel Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[9] S. Kamienny. Torsion points on elliptic curves over fields of higher degree. *Internat. Math. Res. Notices*, 6:129–133, 1992.

[10] E. Kowalski and P. Michel. A lower bound for the rank of $J_0(q)$. *Acta Arith.*, 94(4):303–343, 2000.

[11] Loïc Merel. Bornes pour la torsion des courbes elliptiques sur les corps de nombres. *Invent. Math.*, 124(1-3):437–449, 1996.

[12] M. Ram Murty. Congruences between modular forms. In *Analytic number theory (Kyoto, 1996)*, volume 247 of *London Math. Soc. Lecture Note Ser.*, pages 309–320. Cambridge Univ. Press, Cambridge, 1997.

[13] A. P. Ogg. On a convolution of L-series. *Invent. Math.*, 7:297–312, 1969.

[14] Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*, volume 11 of *Publications of the Mathematical Society of Japan.* Princeton University Press, Princeton, NJ, 1994. Reprint of the 1971 original, Kanô Memorial Lectures, 1.

[15] William Stein. *Modular forms, a computational approach*, volume 79 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2007. With an appendix by Paul E. Gunnells.

[16] Jeffrey VanderKam. Linear independence of Hecke operators in the homology of $X_0(N)$. *J. London Math. Soc. (2)*, 61(2):349–358, 2000.