From Tate's Thesis to Automorphic Forms and Representations on $GL(2)$

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

Rahul Mistry

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

April, 2019

Supervisor: Dr. Chandrasheel Bhagwat

Certificate

This is to certify that this dissertation entitled From Tate's Thesis to Automorphic Forms and Representations on $GL(2)$towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Rahul Mistryat Indian Institute of Science Education and Research under the supervision of Dr. Chandrasheel Bhagwat, Assistant Professor, Department of Mathematics , during the academic year 2018-2019.

चं.रा. भागवत

Dr. Chandrasheel Bhagwat

Committee:

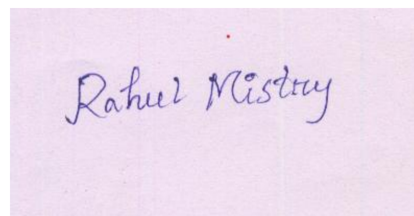
Dr. Chandrasheel Bhagwat

Dr. Baskar Balasubramanyam

This thesis is dedicated to My Mother and Dr. Chandrasheel Bhagwat

Declaration

I hereby declare that the matter embodied in the report entitled From Tate's Thesis to Automorphic Forms and Representations on $GL(2)$ are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Chandrasheel Bhagwat and the same has not been submitted elsewhere for any other degree.



Rahul Mistry

Acknowledgments

I would like to express my gratittude towards my guide Dr. Chandrasheel Bhagwat who has been contantly supportive of my mathematical ventures and have always tried to show me new ways of looking at things. I would also like to thank my TAC Dr. Baskar Balasubramanyam for the valuable inputs during our meetings. I have my mother to thank for the support she gave for my academic adventures and always supporitng me in every way in life. Finally, I thank the institute for all the memories and lessons it taught me during the 5 years.

Abstract

In this thesis we look at the celebrated Riemann-Zeta function and its generalizations and Tate's famous thesis which gave a way to arrive at the functional equations and meromorphic continuouations of such functions. We do this by consider the local fields and finally come to the global result suing a suitable topology to glue things together. The next level of generalization is realizing functions on the upper half plane as Automorphic Representations of a general linear group where the representations are not only one-dimensional because of the non-commutativity of the space.

Contents

Introduction

The Riemann-Zeta function, originally studied by Euler, admits a meromorphic continuation to all of the complex plane, even though initially it is defined for a certain half-plane, and the conjecture which is still unproven states that all non-trivial zeros of the function lie on the strip with real part $1/2$. To solve this, a step forward is to try and look at the generalizations and try to prove the general hypothesis. But classical methods are too cumbersome. On the advice of Emil Artin circa 1950, J. Tate made use of Fourier Analysis on adele groups to prove the analytic continuation and functional equation of the Dirichlet L-function, $L(s, \chi)$ (See, p.242, [RV99]).

The basic idea of Tate was to realize the local factors and global L-functions of χ, a Dirichlet character, as the greatest common divisor of a family of zeta integrals. The key is to take a nice topological ring R such as $\mathbb{Q}_p$, $\mathbb{R}$ or $\mathbb{A}_{\mathbb{Q}}$ and to consider integrals of the form:

$$Z(\chi, \phi) = \int \chi(x)\phi(x)dx$$

where χ is a character of $R^\times$ and ϕ a nice enough function on R. The functional equation reflects Fourier Duality between the pairs (χ, ϕ) and $(\chi^\vee, \hat{\phi})$, the dual character $\chi^\vee$ and the Fourier transform $\hat{\phi}$. The reason why this thesis is so remarkable is that, his methods can be easily adapted to derive the analytic continuation and functional equation of any type of L-functions.

The second part of the thesis, which is essentially chapter 6, talks about Modular forms, L-functions associated to it, Euler product of such functions. We also take a brief look at the Rankin-Selberg method for modular forms of $SL(2, \mathbb{Z})$. And we see how to realize a function on the upper-half plane, satisfying certain growth condition and under the action of congruence groups (more generally discontinuous groups) and how to pull it back to an adelic setting. Because Tate's celebrated thesis, can be thought of as the theory of Automorphic representation on $GL(1, \mathbb{A}_{\mathbb{Q}})$, $\mathbb{A}_{\mathbb{Q}}$ being the adele ring of $\mathbb{Q}$. Hence it is natural to construct a generalization to, say $GL(2, \mathbb{A}_{\mathbb{Q}})$. There are many reference for this, but we follow [DB97] as much as possible. Here we also needed some ideas about Basic Lie Theory (see [p.127, ch.2, DB97]). This part of the theory shall not be discussed in this thesis for sake of brevity and also to focus on the main goal, which is to realize the space of square-integrable functions, under the action of certain general linear groups, decompose into irreducible Hilbert subspaces and we also see when the Euler product should hold; in the case for $GL(1, \mathbb{A}_{\mathbb{Q}})$, the group being abelian, all representations were one-dimensional, hence Euler product always will exist if other conditions are suitable enough. So it is important to study the case for $GL(2)$ and further $GL(n)$, $n \geq 2$, which we have tried to do in this thesis.

Chapter 1

Preliminaries

1.1 Topological Groups

1.1.1 Definition and Examples

Definition 1.1.1. *A topological group G is a group with a topology satisfying the following additional properties:*

1. *Define map $f : G \times G \to G$, such that $f(g, h) = gh$. Then f is a continuous mapping where the domain has the product topology*

2. *The map $I : G \to G$, such that $I(g) = g^{-1}$ is a continuous mapping*

If the group G is finite, then we give it the discrete topology.

Examples:

1. $\mathbb{R}$ is a topological group w.r.t. addition.

2. $\mathbb{R}^*$, $\mathbb{R}^*_+$, $\mathbb{C}^*$ are topological groups with multiplication operation.

3. Let k be $\mathbb{R}$ or $\mathbb{C}$. Then $GL_n(k)$, the set of $n \times n$ matrices with non-zero determinant, forms a topological group w.r.t. multiplication, with the Euclidean topology given to it.
   This group has a subgroup, $SL_n(k)$, where all the elements have determinant 1. It is a closed subgroup of $GL_n(k)$.

Definition 1.1.2. *A locally compact topological group G, is a topological group that is both locally compact, i.e. every point $g \in G$, has a compact neighbourhood containing g, and G is also Hausdorff.*

For example, $\mathbb{R}$ or $\mathbb{C}$ with respect to addition, are locally compact topological groups. The set of p-adic numbers $\mathbb{Q}_p$ is also a locally compact group w.r.t. addition.
An interesting example is, the ring of Adeles, $\mathbb{A}_K$ for a number field K, is also a locally compact group w.r.t. addition operation.

1.1.2 Haar Measure

Let X be a set and Σ be a collection of subsets of X with the following properties:

1. $X \in \Sigma$.
2. if $A \in \Sigma$, then $A^c \in \Sigma$, where A^c is the complement of A in X.
3. Suppose $A_n \in \Sigma$, for all $n \geq 1$, then $\cup_{n=1}^{\infty} A_n \in \Sigma$.

X, together with such a collection Σ is called a *sigma*-algebra. If X is moreover a topological space, then we can take the smallest σ-algebra generated by the open sets of X, this is called the Borel σ-algebra of X.

Definition 1.1.3. *A positive measure μ on a space (X, Σ), is a mapping $\mu : \Sigma \to \mathbb{R}_+ \cup \{\infty\}$, such that it is countably additive, i.e.*

$$\mu(\cup_{n=1}^{\infty} A_n) = \Sigma_{n=1}^{\infty} \mu(A_n)$$

where $\{A_n\}_{n \geq 1}$ is a disjoint family of sets in Σ, the σ-algebra of X.
In particular, a positive measure defined on the Borel sets of a locally compact set X is called a Borel Measure.

Let X be a locally compact space and μ be a Borel measure and let S be a Borel subset of X. We say that a positive measure is outer-regular if

$$\mu(S) = \inf\{\mu(U) | S \subseteq U, U \text{ open in } X\}$$

We say that μ is inner-regular if

$$\mu(S) = \sup\{\mu(K) | K \subseteq S, K \text{ compact subset of } X\}$$

Definition 1.1.4. *A Radon measure on X is a Borel measure μ with the following properties:*

1. *μ is finite on compact subsets of X.*
2. *μ is outer-regular on all Borel sets.*
3. *μ is inner-regular on all open sets.*

Let G be a topological group and μ be a Borel measure on G. Then μ is called left translation invariant, if

$$\mu(gS) = \mu(S)$$

for all Borel subsets S of G and for all $g \in G$. Right translation invariance is defined in a similar manner.

Definition 1.1.5. *(Haar Measure) Let G be a locally compact group. A left (resc. right) Haar measure is a non-zero Radon measure that is left (resc. right) translation invariant. A Radon measure that is both left and right translation invariant is called a bi-variant Haar measure.*

We end this section with the following theorem:

Theorem 1.1.1. *Let G be a locally compact topological group. Then G admits a left Haar measure. This measure is unique upto multiplication by a scalar. (it is useful to note that a left Haar measure on G gives rise to a right Haar measure)*

Proof. See [p.12, ch.1, RV99]. $\square$

1.2 P-adic Numbers

This section we shall discuss about the p-adic numbers. We shall be referring to the first few sections of chapter 2 in [JK99].

1.2.1 Introduction

Definition 1.2.1. *(p-adic Integer) For a fixed prime number p, a p-adic integer is a formal expression of the form*

$$a_0 + a_1 p + a_2 p^2 + \cdots ,$$

where $0 \leq a_i < p$ for all $i = 0, 1, 2....$

The set of p-adic integers form a ring and it is denoted by $\mathbb{Z}_p$.

Proposition 1.2.1. *Every element in $\mathbb{Z}/p^n\mathbb{Z}$ can be uniquely expressed in the form*

$$a_0 + a_1 p \cdots + a_{n-1} p^{n-1} \pmod{p^n}$$

where $0 \leq a_i < p$ for all $i = 0, 1, 2, ..., n-1$.

Proof. See [p.101, ch.2, JK99]. □

1.2.2 Constructing the P-adic numbers

In analogy with the Laurent series from complex analysis, we extend the domain of p-adic integers by allowing formal series

$$\sum_{v=-m}^{\infty} a_v p^v = a_{-m} p^{-m} + \cdots + a_{-1} p^{-1} + a_0 + a_1 p \cdots$$

where $m \in \mathbb{Z}$ and $0 \leq a_v \leq p-1$. The set of such formal series form a field, this is denoted by $\mathbb{Q}_p$, the set of p-adic numbers.

Suppose $c = p^{-m} \frac{a}{b} \in \mathbb{Q}$, written by extracting the multiples of p from a and b. Here $m \in \mathbb{Z}$

and $(ab, p) = 1$. Suppose $\sum_{v=0}^{\infty}$ is the p-adic expression of $\frac{a}{b}$, then c is associated to the expression $\sum_{v=-m}^{\infty} a_v p^v \in \mathbb{Q}_p$. Hence we have a canonical mapping

$$\mathbb{Q} \to \mathbb{Q}_p$$

which takes $\mathbb{Z}$ into $\mathbb{Z}_p$ and is also injective, because if $a, b \in \mathbb{Z}$ have the same p-adic expressions, then $a - b$ is divisible by p^n for all $n > 0$, hence $a = b$.

Consider the following sequence of rings and ring homomorphisms:

$$\mathbb{Z}/p\mathbb{Z} \xleftarrow{\lambda_1} \mathbb{Z}/p^2\mathbb{Z} \xleftarrow{\lambda_2} \mathbb{Z}/p^3\mathbb{Z} \xleftarrow{\lambda_3} \mathbb{Z}/p^4\mathbb{Z} \cdots$$

here $\lambda_i : \mathbb{Z}/p^{i+1}\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$ is the cacnonical projection taking every $a \pmod{p^{i+1}}$ to $a \pmod{p^i}$. Consider the direct product

$$\prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z} = \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{Z}/p^n\mathbb{Z}\}$$

In this product we look at all tuples (x_n) such that $\lambda_n(x_{n+1}) = x_n$ for all $n \geq 1$. The set of all such that tuples is called the Projective limit of the sets $\mathbb{Z}/p^n\mathbb{Z}$,

$$\varprojlim_n \mathbb{Z}/p^n\mathbb{Z} = \{(x_n)_{n\in\mathbb{N}} \in \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z} : \lambda_n(x_{n+1}) = x_n \text{ for all } n \geq 1\}$$

Proposition 1.2.2. *Given any p-adic integer $c = \sum_{v=0}^{\infty}$, associating it with $\sum_{v=0}^{n-1} a_v p^v \in \mathbb{Z}/p^n\mathbb{Z}$ for every $n \geq 1$, yields a bijection*

$$\mathbb{Z}_p \cong \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}.$$

Every $c \in \mathbb{Q}_p$ can be written as $c = p^{-m}g$ where $g \in \mathbb{Z}_p$, here $-m$ is called the order of p of the element c, denoted $\mathrm{ord}_p(c)$. Using this representation addition, multiplication can be extended to $\mathbb{Q}_p$, which can be realized as the field of fractions of the integral domain $\mathbb{Z}_p$.

1.2.3 P-adic Absolute Value

Let $c \in \mathbb{Q}$, $c = p^m \frac{a}{b}$ such that $(ab, p) = 1$ and $m \geq 0$ an integer. Define a map:

$$v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$$

we put $v_p(0) = \infty$. It is easy to check:

1. $v_p(a) = \infty \iff a = 0$.
2. $v_p(ab) = v_p(a) + v_p(b)$.
3. $v_p(a + b) \geq \min\{v_p(a), v_p(b)\}$.

The map v_p is called the p-adic exponential valuation map. Using this exponential valuation, we can define the p-adic absolute value as follows:

$$|\cdot|_p : \mathbb{Q} \to \mathbb{R}, \ |a|_p = p^{-v_p(a)}.$$

The p-adic absolute value satisfies the following thing properties, because of the v_p satisfying the above three properties:

1. $|a|_p = 0 \iff a = 0$.
2. $|ab|_p = |a|_p \cdot |b|_p$.
3. $|a + b|_p \leq \max\{|a|_p, |b|_p\} \leq |a|_p + |b|_p$.

The fact that every integer can be written uniquely as product of prime powers (often called the Fundamental Theorem of Arithmetic) can be used to prove the following:

Proposition 1.2.3. *For every $a \in \mathbb{Q}^*$, $\prod_{p \leq \infty} |a|_p = 1$, where p runs though the sel of all primes and $|\cdot|_\infty$ is the usual absolute value induced from $\mathbb{R}$.*

Proof. See [p.108, ch.2, JK99]. □

Definition 1.2.2. *A Cauchy Sequence w.r.t. the absolute value $|\cdot|_p$ is a seuqnce $\{x_n\}$ such that given any $\epsilon > 0$, there exists a natural number N, such that for all $m, n > N$,*

$$|x_m - x_n|_p < \epsilon$$

holds.

It can be checked that $\mathbb{Q}$ is not complete w.r.t. $|\cdot|_p$ for all $p \leq \infty$, i.e. there exists non-convergent Cauchy sequences. One can *complete* the space by defining limits for all Cauchy sequences. The p-adic absolute value can be extended to all of $\mathbb{Q}_p$ by letting, for $x = \{x_n\} \in \mathbb{Q}_p$,

$$|x|_p := \lim_{n\to\infty} |x_n|_p \in \mathbb{R}.$$

So $\mathbb{Q}_p$ can be realized as the completion of $\mathbb{Q}$ w.r.t. $|\cdot|_p$.

Proposition 1.2.4. *Let p be a finite prime. The set*

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x| \leq 1\}$$

is a subring of $\mathbb{Q}_p$. It is also the completion of $\mathbb{Z}$ w.r.t. $|\cdot|_p$ in the field $\mathbb{Q}_p$.

Proof. See [p.112, ch.2, JK99] $\square$

The group of units of $\mathbb{Z}_p$, denoted $\mathbb{Z}_p^\times$ is the set $\{x \in \mathbb{Z}_p : |x|_p = 1\}$. Every element $x \in \mathbb{Q}_p^*$ admits a unique representation of the form $x = p^m \cdot u$, $m \in \mathbb{Z}$ and $u \in \mathbb{Z}_p^\times$.

Proposition 1.2.5. *The non-zero ideals of $\mathbb{Z}_p$ are the sets*

$$p^n\mathbb{Z}_p = \{x \in \mathbb{Z}_p : v_p(x) \geq n\}$$

and for $n \geq 0$

$$\mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}.$$

Proof. See [p.112, ch.2, JK99]. $\square$

Because of the isomorphism

$$\mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}$$

we can define a map

$$\mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$$

for all $n \geq 0$. Looking at elements of $\mathbb{Z}_p$ as formal power series in p, we take the image as the truncated polynomial with highest power of p being $n-1$. This map is surjective. All such maps for all $n \geq 0$, we can get a surjective homomorphism

$$\mathbb{Z}_p \to \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$$

Proposition 1.2.6. *The homomorphism*

$$\mathbb{Z}_p \to \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$$

is an isomorphism.

Proof. See [p.114, ch.2, JK99]. □

1.3 Valuations

The method used to obtained p-adic numbers from $\mathbb{Q}$ can be generalized arbitrary fields using the theory of (multiplicative)valuations.

Definition 1.3.1. *A Valuation on a field K is a function $|\cdot| : K \to \mathbb{R}$ with properties:*

1. $|x| \geq 0$ *and* $|x| = 0 \iff x = 0$.
2. $|xy| = |x| \cdot |y|$.
3. $|x + y| \leq |x| + |y|$.

For any two points $x, y \in K$, define distance between them

$$d(x, y) = |x - y|$$

makes K into a metric space, and a topological space.

Definition 1.3.2. *Two absolute values on K are called equivalent if they give the same topology on K.*

Proposition 1.3.1. *Two absolute values $|\cdot|_1$ and $|\cdot|_2$ are equivalent if there exists a real number $c > 0$, such that $|x|_1 = |x|_2^c$ for all $x \in K$.*

Proof. See [p.117, ch.2, JK99]. □

Theorem 1.3.2. *(Approximation Theorem) Let $|\cdot|_i$, for $i = 1, 2..., n$ be pairwise inequivalent absolute values on K. Let $a_1, a_2..., a_n$ ne given elements of K. Then for every $\epsilon > 0$, $\exists x \in K$ such that*

$$|x - a_i|_i < \epsilon \text{ for all } i = 1, 2..., n.$$

Proof. See [p.118, ch.2, JK99]. □

Definition 1.3.3. *A valuation $|\cdot|$ on K is non-archimedean, if $|n|$ is bounded for all $n \in \mathbb{N}$.*

We end this section here, but more about absolute values on Local fields of characteristic 0 shall be discussed in chapter 2 of this thesis.

Chapter 2

Structure of Arithmetic Fields

In this chapter we shall discuss the modular function on a field and explicitly find its form for an algebraic number field. The discussion on this chapter shall follow the treatment given in Chapter 4 of Fourier Analysis on Number Fields, Ramakrishnan, Valenza.

2.1 Module of an Automorphism

Let G be a locally compact additive group with Haar measure μ. Now if X is any Borel subset of G, then αX is again a Borel subset, because left multiplication by α is an automorphism. This implies, $\mu \circ \alpha$ is another Haar measure on G. We define the module of this automorphism, denoted $mod_G(\alpha)$, as

$$\mu(\alpha X) = mod_G(\alpha)\mu(X)$$

This map is multiplicative:

$$mod_G(\alpha\beta)\mu(X) = \mu(\alpha\beta X) = mod_G(\alpha)\mu(\beta X) = mod_G(\alpha)mod_G(\beta)\mu(X)$$

Let us take G to be a local field denoted k. Let V be a topological vector space over k. Then every $a \in k^*$ defines an automorphism of V via left multiplication, and we can extend mod_V by letting $mod_V(0) = 0$. In fact, $mod_k(a)$ can be thought of as the module of a acting on k itself.

Proposition 2.1.1. *Let k be a locally compact field with Haar measure μ. Then $mod_k : k \to \mathbb{R}_+$ is a continuous mapping.*

Proof. See [p.133, ch. 4, RV99]. $\square$

As a corollary we have, if k is a non-discrete local field, mod_k is unbounded, consequently k is not compact.

Using this modular map, we can define certain closed balls as follows:

Let k be a non-discrete local field. Let $m > 0$ be a positive integer. Consider

$$B_m = \{a \in k : mod_k(a) \leq m\}$$

We have the following important result:

Proposition 2.1.2. *B_m, as defined above, is compact.*

Proof. See [p.134, ch.4, RV99]. $\square$

Corollary 2.1.3. *For $a \in k$, $\lim_{n\to\infty} a^n = 0$ iff $mod_k(a) < 1$.*

This corollary can be used to show that the modular function is trivial on any discrete field l contained in k. The sets B_m, $m > 0$ constitute a local base at $0 \in k$ for the topology of k.

Theorem 2.1.4. *Let k be a locally compact, non-discrete field with Haar measure μ, then $\exists A \geq 1$, constant such that*

$$mod_k(a + b) \leq A \cdot sup\{mod_k(a), mod_k(b)\}, \forall a, b \in k.$$

Proof. See [p.136, ch.3, RV99]. $\square$

If mod_k satisfies the inequality in the previous theorem with $A = 1$, then we say k is Ultrametric, i.e.

$$mod_k(a + b) \leq \sup\{mod_k(a), mod_k(b)\}, \forall a, b \in k.$$

This is called the Ultrametric Inequality. Now the set of natural numbers $\mathbb{N}$ can be embedded in k by mapping n to $n \cdot (1_k)$, $1_k \in k$ multiplicative identity of k, then image of $\mathbb{N}$ is called the prime ring of k. Now $mod_k(n) \leq \sup\{mod_k(1_k)\} = 1$, hence in an ultrametric field, mod_k is bounded. The converse is stated in the next theorem:

Theorem 2.1.5. *If mod_k is bounded on the prime ring of k, then $mod_k \leq 1$ on the prime ring of k and k is ultrametric.*

Proof. See [p.139, ch.4, RV99]. □

2.2 Classification of Local Fields

Since we are interested in extensions over $\mathbb{Q}$, we shall consider char(k)=0. Before we go into the discussion, we shall state the main result of this chapter:

Theorem 2.2.1. *Let k be a locally compact, non-discrete field, such that char(k)=0, then, k is $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$.*

Before we start discussing the proof of the theorem, we need a proposition. Let V be a topological vector space over k, a non-discrete local field. Let W be a subspace of V of dimension n. Let $\{w_1, w_2, ..., w_n\}$ be a basis of W. Consider the map ϕ:

$$\phi : k^n \to W$$

$$\phi((a_i)_{i=1,..,n}) = \sum_{i=1}^{n} a_i w_i$$

Proposition 2.2.2. *Let k, V, W be defined as above, then:*

1. *Let U be any open neighborhood of 0 in V. Then $W \cap U \neq \{0\}$.*

2. *The mapping ϕ defined above is a homeomorphism.*

3. *W is closed and locally compact.*

4. *If V is locally compact, then dimension of V over k is finite. Moreover for all $a \in V$*

$$mod_V(a) = mod_k(a)^{dim(V)}.$$

Proof. See [p.141, ch.4, RV99]. $\square$

Proposition 2.2.3. *Let $F : \mathbb{N} \to \mathbb{R}_+$ be a function such that $F(mn) = F(m)F(n)$ for all $m, n \in \mathbb{N}$. Assume that $\exists A$, some constant such that $F(m+n) \leq A \cdot sup\{F(m), F(n)\}$ for all m, n. Then*

1. *$F(m) \leq 1$ for all m, or*
2. *$F(m) = m^\lambda$ for some positive constant λ for all m.*

Proof. See [p.138, ch.4, RV99]. $\square$

The Modulus function is defined on the prime ring of k as $mod_k(n) = mod_k(n \cdot 1_k)$. From the proposition above, we have two possibilities:

1. $mod_k(m) \leq 1$ for all m, which is equivalent to saying k is ultrametric, or
2. $\exists\lambda$, a positive constant such that $mod_k(m) = m^\lambda$ for all m.

2.3 Preliminary Analysis for Main Theorem

We shall assume char(k) $= 0$, k being the non-discrete local field in consideration. Now assume that mod_k is bounded on the prime ring of k, then $\{m \cdot 1_k : m \in \mathbb{N}\} \subset B_1$, where $B_1 = \{a \in k : mod_k(a) \leq 1\}$. Now since B_m is compact for all $m \in \mathbb{N}$, we have a limit point of the set $\{m \cdot 1_k : m \in \mathbb{N}\}$, i.e. if the limit point is a, then $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$, such that $\forall m \geq N$ $, mod_k(m \cdot 1_k - a) \leq \epsilon$. Now let $m_1, m_2 > N$, then $mod_k(m_1 \cdot 1_k - m_2 \cdot 1_k) = mod_k((m_1 \cdot 1_k - a) + (a - m_2 \cdot 1_k)) \leq \sup\{mod_k(m_1 \cdot 1_k - a), mod_k(a - m_2 \cdot 1_k)\} \leq \epsilon$ by the ultrametric inequality. So for large enough $m \in \mathbb{N}$, we can have $mod_k(n) < 1$, for example we can take $\epsilon = 1$, and $m_1 > N$ and $m_2 = m_1 + n$.

Since mod_k is multiplicative, the smallest integer $n \geq 1$ such that $,mod_k(n) < 1$, must be a prime number. Suppose $n = p_1p_2$, where p_1, p_2 are primes (the simplest case of a composite number), then $mod_k(n) = mod_k(p_1p_2) = mod_k(p_1) \cdot mod_k(p_2) < 1$, thus at least one of $mod_k(p_1)$ or $mod_k(p_2)$ is less than 1. But $n > p_1$ and $n > p_2$, contradicting minimality of n. Hence n must be a prime. We can construct a similar argument for any n. Now let p, a prime, be the smallest positive integer such that $mod_k(p) < 1$. Now $mod_k(mp) = mod_k(p + \cdots \text{m-times} + p) \leq mod_k(p) < 1$, by the ultrametric inequality. Hence $mod_k(mp) < 1$ for all $m \in \mathbb{N}$. So all multiples of p are strictly bounded above by 1.

Let r be a positive integer less than p. Then from minimality of p, $mod_k(r) \geq 1$, but $mod_k \leq 1$ on the prime ring, so $mod_k(r) = 1$. But $mod_k(r) = mod_k(r + mp - mp) \leq \sup\{mod_k(r + mp), mod_k(mp)\}$, by the ultrametric inequality, and since $mod_k(mp) < 1$, we have $1 = mod_k(r) \leq mod_k(r + mp)$, again since mod_k is bounded, $mod_k(r + mp) = 1$. Hence for all co-prime integers to p, mod_k is equal to 1.
From the above two paragraphs, we can conclude that p is the unique prime, such that mod_k is less than 1.

Now since $\text{char}(k) = 0$, $mod_k(p) \neq 0$ (in comparison, when we have a field with finite characteristic p, then $mod_k(p) = 0$; also by minimality of p, this is the unique prime for which $mod_k(p) < 1$). We choose a positive real number t, such that $mod_k(p) = p^{-t}$. Let $n = mp^s \in \mathbb{N}$ such that m and p are co-prime, then $mod_k(mp^s) = mod_k(p)^s = p^{-ts} = |n|_p^t$, where $|\cdot|_p$ is the p-adic absolute value on $\mathbb{Q}$.

Now if $mod_k(m) = |m|^\lambda$ for some positive constant λ, then this absolute value is equivalent to the usual absolute value on $\mathbb{R}$, denoted by $|\cdot|_\infty$. A summary of our discussion in this section is:
For all $n \in \mathbb{N}$, $mod_k(n) = |n|_v^t$
if $v = p$, a finite prime number, then $mod_k \leq 1$;
if $v = \infty$, then $mod_k(n) = |n|^\lambda$ holds for some positive constant λ.

2.4 Proof of Classification Theorem

We have k, a non-discrete locally compact field and $\text{char}(k) = 0$. Consider the map $\phi : \mathbb{Z} \to k$, such that $\phi(n) = n \cdot 1_k$. This can be extended to $\phi : \mathbb{Q} \to k$, by mapping respective inverses of non-zero elements of $\mathbb{Z}$. It can be seen that this map is infact a ring homomorphism. Now mod_k induces a map on $\mathbb{Q}$, taking all $x \in \mathbb{Q}$ to $|x|_v^t$. The topology generated by this absolute

value $|\cdot|_v$ is the same as that induced by the compact neighborhoods B_t, t positive real number, because distance between two points x, y is $|x-y|_v$. Thus the image of $\mathbb{Q}$ in k, is isomorphic to the completion of $\mathbb{Q}$ with respect to the metric $|\cdot|_v$. When $v = p$ a finite prime, then this completion is isomorphic to the p-adic numbers $\mathbb{Q}_p$. And if $v = \infty$, then the completion is $\mathbb{R}$.
Now is $v = p$, then k is a locally compact vector space over $\mathbb{Q}_p$, hence k is a finite extension of $\mathbb{Q}_p$. And if $v = \infty$, then k is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.

2.5 Ring of Integral Elements and Residue Field of a Local Field

Let k be a locally compact non-discrete field of characteristic 0. Consider the following sets: $A = \{x \in k | mod_k(x) \leq 1\}$, $A^\times = \{x \in k | mod_k(x) = 1\}$ and $P = \{x \in k | mod_k(x) < 1\}$. A is just $B_1 = \{x \in k | mod_k(x) \leq 1\}$, hence it is compact. If $a, b \in A$, then $mod_k(a+b) \leq \sup\{mod_k(a), mod_k(b) \leq 1\}$, and also $mod_k(ab) = mod_k(a) mod_k(b) \leq 1$, $1 \in A$, thus $a + b$ and ab are also elements of A. So A is a subring of k. Since $mod_k(a^{-1}) = mod_k(a)^{-1}$, $A^\times$ is a group w.r.t. multiplication. And finally P is an ideal of the ring A, because if $x, y \in P$, then $mod_k(x+y) \leq \sup\{mod_k(x), mod_k(y)\} < 1$, and for any $a \in A$ and $x \in P$, since $mod_k(a) \in A$, $mod_k(ax) = mod_k(a) mod_k(x) < 1$, so $ax \in P$. Hence P is an ideal.

A local ring, is an Integral Domain, such that it has a unique maximal ideal. A Discrete Valuation Ring (DVR) is a principal ideal domain which has a unique prime ideal; hence it is in particular a local ring. The following more concrete result holds:

Lemma 2.5.1. *A is a DVR, in particular a local ring. P is the unique prime ideal of A and $P = A\pi$ where π is the uniformizing parameter. Finally A/P is a finite field.*

Proof. See [p.145, ch.4, RV99]. □

If $A = \mathbb{Z}_p$, then A is a local ring, because using the absolute value map, we can define a local base for the identity 1 of A. The unique prime ideal is $p\mathbb{Z}_p$. And finally $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$. In general, $A/P \cong \mathbb{F}_{p^r}$. So $\text{card}(A/P) = q = p^r$. Next we move to discuss the roots of unity in a local field k.

2.6 Roots of unity in a Local Field

Let M be the set of roots of unity of order prime to p in k including 0, where p is the smallest prime for which $mod_k(p) < 1$. Then $M - \{0\}$ forms a group with multiplication as operation. We can define an injective homomorphism from M^* to $(A/P)^*$ which turns out to be an isomorphism. The proof can be found on p.149 of Ramakrishnan Valenza. Thus $M = M \cup \{0\}$ constitutes a complete coset representative for A/P and the polynomial $x^{q-1} - 1$ splits in k.

2.7 Global Fields

Definition 2.7.1. *A Global Field F is:*

1. *A finite extension of $\mathbb{Q}$ for characteristic 0.*
2. *A finitely generated function field in one variable over a finite field $\mathbb{F}_{p^r}$ for finite characteristic.*

A global field admits many absolute values and we analyze the global field by looking at the completions with respect to the different absolute value maps.

Definition 2.7.2. *Two absolute values $|\cdot|_1$ and $|\cdot|_2$ are called equivalent, i.e. they generate the same topology on a global field F, if for all $a \in F$, $|a|_1 = |a|_2^t$ for some positive real number t. A place of a global field is an equivalence class of non-trivial absolute values.*

Proposition 2.7.1. *Let $|\cdot|$ be an absolute value on a global field F. Then the following are equivalent:*

1. *$|\cdot|$ is ultrametric.*
2. *The image of $\mathbb{N}$ in F is bounded.*

In any case, we can say $|\cdot|$ is bounded by 1. We state the next proposition for a global field F of characteristic 0, but it is true for any arbitrary field.

Proposition 2.7.2. *Let F be a global field of characteristic 0 and let $|\cdot|$ be an absolute value om F. Then F can be embedded in a field that is complete with respect to an absolute value that is equivalent to $|\cdot|$.*

This can be seen by looking at the equivalence classes of Cauchy sequences in F, and completing it with respect to the relevant absolute value. The proof is a constructional one and is not needed in our discussion.

We close this chapter with the following important result:

Theorem 2.7.3. *Let $|\cdot|$ be an absolute value on $\mathbb{Q}$, a global field. Then either*

1. *$|\cdot|$ is equivalent to $|\cdot|_\infty$, the usual absolute value induced from $\mathbb{R}$, or*

2. *$|\cdot|$ is equivalent to a p-adic absolute value $|\cdot|$ for some prime p.*

Proof. See [p.158, ch.4, RV99]. $\square$

Chapter 3

Duality for Locally Compact Abelian Groups

3.1 Characters

Let G be a locally compact abelian topological group. A character of G is a continuous homomorphism

$$\chi : G \to \mathbb{C}^*$$

where $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in G$. If the image of χ is contained in $\mathbb{S}^1$, the unit circle on the complex plane, then χ is called a unitary character.

Denote by $\hat{G}$ the set of all unitary characters on G. If $\chi_1, \chi_2 \in \hat{G}$, then $\chi_1 \cdot \chi_2$ is again an element of $\hat{G}$, because of point-wise multiplication, $\chi_1 \cdot \chi_2(a) = \chi_1(a)\chi_2(a)$ for all $a \in G$. And $\chi^{-1}(a) = \chi(a^{-1})$. Hence $\hat{G}$ is a group. In fact, for a locally compact topological abelian group G, $\hat{G}$ is a locally compact abelian topological group. $\hat{G}$ is called the Dual group of G.

To see this, it is enough to specify a local base of neighbourhoods of the identity of $\hat{G}$, which is the trivial character. To do this we give $\hat{G}$ the compact-open topology as follows: Let $V \subseteq \mathbb{S}^1$ be an open neighbourhood of 1. Let $K \subseteq G$ be a compact subset. Define a set

$$W(K, V) = \{\chi \in \hat{G} | \chi(K) \subseteq V\}$$

Every such set $W(K, V)$, where K is compact in G and V is an open neighbourhood of 1 in $\mathbb{S}^1$, is an open set containing the trivial character. Hence this gives a topology on $\hat{G}$.

We define a subset of $\mathbb{S}^1$ before stating the next proposition. Define the exponential map $\phi : \mathbb{R} \to \mathbb{S}^1$, such that $\phi(x) = e^{2\pi i x}$. ϕ is a continuous homomorphism with kernel $\mathbb{Z}$. Now define for $0 < c \leq 1$, $N(c) = \phi((-c/3, c/3)) = \{e^{2\pi i t} | -c/3 < t < c/3\}$. In particular, $N(1) = \{e^{2\pi i t} | -1/3 < t < 1/3\}$.

Proposition 3.1.1. *Let G be an abelian topological group. The following statements hold:*

1. *A group homomorphism $\chi : G \to \mathbb{S}^1$ is continuous iff $\chi^{-1}(N(1))$ is a neighbourhood of the identity in G.*
2. *The family $\{W(K, N(1))\}_K$ as K ranges over compact subsets of G constitute a local base for the trivial character, giving $\hat{G}$ the compact-open topology.*
3. *If G is discrete then, $\hat{G}$ is compact.*
4. *if G is compact then, $\hat{G}$ is discrete.*
5. *If G is locally compact then, $\hat{G}$ is locally compact.*

Proof. See p.89 of Ramakrishnan, Valenza for a proof of 1, 2 and 5.

For 3, If G is discrete, $Hom(G, \mathbb{S}^1) = Hom_{\text{cont}}(G, \mathbb{S}^1)$. Suppose a sequence of characters $\{\chi_i\}(s)$ converges to $f(s)$ for every s. Then $\{\chi_i(s+t)\}$ converges to $f(s+t)$, but $\chi_i(s+t) = \chi_i(s)\chi_i(t)$ which converges to $f(s)f(t)$, i.e. $f \in \hat{G}$. So $Hom(G, \mathbb{S}^1)$ is closed in the space of all maps from $G \to \mathbb{S}^1$. Every element $g \in G$ can be mapped to any element in $\mathbb{S}^1$, hence the space of all maps from $G \to \mathbb{S}^1$ is isomorphic to $\prod_{g \in G}(\mathbb{S}^1)$, which is compact. Hence $Hom_{\text{cont}}(G, \mathbb{S}^1)$ is compact.

For part 4, Suppose G is compact. $f \in \hat{G}$ is continuous, so $f(G)$ is a compact subgroup of $\mathbb{S}^1$. Consider the basis $W(G, N(1)) = \{f \in \hat{G} | f(G) \subset N(1)\}$ where $N(1) = \{e^{2\pi i t} | t \in (-1/3, 1/3)\}$. But no non-trivial subgroup of $\mathbb{S}^1$ is contained in $N(1)$. Hence $W(G, N(1)) = \{\chi_o\}$, the trivial character. That is, the trivial character is open. Since $\hat{G}$ is a topological group, any other character is again open. Hence G is discrete. $\square$

3.2 The Fourier Inversion Theorem

In this section we shall state The Fourier Inversion Theorem without proof for a locally compact abelian group G with bi-variant Haar measure dx. Let the set $L^1(G)$ be the set of functions $f : G \to \mathbb{C}$, such that $||f||_1 = \int_G |f|dx < \infty$.

Definition 3.2.1. *For $f \in L^1(G)$, define the Fourier Transform of f, $\hat{f} : \hat{G} \to \mathbb{C}$ by*

$$\hat{f}(\chi) = \int_G f(x)\chi(x)dx$$

for $\chi \in \hat{G}$.

Since $|\chi(x)| = 1$ for all $x \in G$, we have $|\hat{f}(\chi)| \leq ||f||_1$ for all $\chi \in \hat{G}$ and $f \in L^1(G)$. We shall see explicitly what the Fourier Transform is for local fields in the next chapter.

Let $\mathscr{C}_c(G)$ be the set of all functions $f : G \to \mathbb{C}$, that are continuous and has compact support. For every p, $1 \leq p \leq \infty$, $\mathscr{C}_c(G)$ is contained in $L^p(G)$, where $L^p(G)$ is the set of all functions $f : G \to \mathbb{C}$, such that $||f||_p = \{\int_G |f|^p dx\}^{1/p} < \infty$ for finite p and it is replace by $||\cdot||_\infty$ for the remaining case, where dx is a Haar measure on G. The norm $||f||_p$ induces a topology on $L^p(G)$ and makes it a Banach Space. $\mathscr{C}_c(G)$ is dense in $L^p(G)$ for all p.

Definition 3.2.2. *A Haar measurable function $\phi : G \to \mathbb{C}$ in $L^\infty(G)$ is said to be of positive type if*

$$\int_G \int_G \phi(s^{-1}t)f(s)ds\overline{f(t)}dt \geq 0$$

for all $f \in \mathscr{C}_c(G)$.

Let $V(G)$ be the complex span of continuous functions of positive type. Let $V^1(G) = V(G) \cap L^1(G)$.

Theorem 3.2.1. *(Fourier Inversion) For all $f \in V^1(G)$, there exists a Haar Measure on $d\chi$ on the dual group $\hat{G}$ of G such that,*

$$f(x) = \int_{\hat{G}} \hat{f}(\chi)\chi(x)d\chi$$

The Fourier Transform viewed as a map, identifies $V^1(G)$ with $V^1(\hat{G})$.

See Sections 3.2 and section 3.3, Chapter 3 of Ramakrishnan, Valenza for proofs and more details.

3.3 Pontryagin Duality

In this brief section, we shall state the *cacnonical* isomorphism of topological groups between G a locally compact abelian topological group and $\hat{\hat{G}}$, which is the dual group of $\hat{G}$. This is called the Pontryagin Duality Theorem.

For a locally compact abelian group G, we construct $\hat{G}$ the set of continuous unitary characters on G. By repeating the same construction to $\hat{G}$, which itself is a locally compact abelian topological group, we denote the set of continuous unitary characters of $\hat{G}$ as $\hat{\hat{G}}$. Now if $\xi \in \hat{\hat{G}}$, then $\xi : \hat{G} \to \mathbb{S}^1$ is a continuous group homomorphism.

Let us define a map

$$\alpha : G \to \hat{\hat{G}}$$

where for all $g \in G$, $\alpha(g)$ is an element of $\hat{\hat{G}}$ and it is defined as $\alpha(g)(\chi) = \chi(g)$ for all $\chi \in \hat{G}$, this makes sense because $\chi(g) \in \mathbb{S}^1$. We can view $\alpha(g)$ as the evaluation of $g \in G$ on all the elements of $\hat{G}$. We shall end this section by stating the theorem for G.

Theorem 3.3.1. *(Pontryagin Duality) Let G be a locally compact abelian topological group and $\hat{\hat{G}}$ be the dual group of $\hat{G}$. The map $\alpha : G \to \hat{\hat{G}}$ defined above is an isomorphism of topological groups.*

Chapter 4

Adéles and Idéles

In this chapter we shall discuss the ring of Adéles and Idéles of a finite extension K of $\mathbb{Q}$. We shall be following chapter 5 of [RV99].

4.1 Restricted Direct Product

Let $J = \{v\}$ be a set of indices and J_∞ be a finite subset of J. Suppose we are given a locally compact group G_v for every $v \in J$ and for every $v \notin J_\infty$ we are given a compact open subgroup H_v of G_v.

Definition 4.1.1. *Restricted direct product of G_v w.r.t. H_v is,*

$$G = \prod_{v \in J}{}' G_v = \{(x_v) \prod_v \in G_v : x_v \in G_v \textit{ such that } x_v \in H_v \textit{ for all but finitely many } v\}$$

This is a subset of the direct product $\prod_v G_v$. We define topology on the restricted direct product, by specifying a neighbourhood base for the identity, say $\prod_v N_v$ where $N_v = H_v$ for all but finitely many v. This topology is not the same as the product topology. Let S be a finite subset of indices containing J_∞. Let $G_S = \prod_{v \in S} G_v \cdot \prod_{v \notin S} H_v$. Then the product topology on G_S is the same as the topology induced by the neighbourhood base of identity. Also for any such finite $S \supseteq J_\infty$, G_S is locally compact by Tychonoff's Theorem. It can be shown that G, the restricted direct product of G_v w.r.t. H_v is locally compact.

4.2 Characters

Let G be restricted direct product of G_v w.r.t. H_v for locally compact abelian groups G_v for all v. For $y \in G_v$, let y_v be the projection of y onto G_v, which can also be though of as the element $(1, 1, .., y_v, ...1, ...) \in G$.

Lemma 4.2.1. *Let $\chi \in Hom_{cont.}(G, \mathbb{C}^*)$. Then χ is trivial on all but finitely many H_v. We have, for all $y \in G$, $\chi(y_v) = 1$ for all but finitely many v and*

$$\chi(y) = \prod_v \chi(y_v).$$

Proof. See [p.182, ch.5, RV99]. □

Lemma 4.2.2. *If for each v, $\chi_v \in Hom_{cont.}(G_v, \mathbb{C}^*)$ and $\chi_v|_{H_v} = 1$ for all but finitely many v, then $\chi = \prod_v \chi_v \in Hom_{cont.}(G, \mathbb{C}^*)$.*

Proof. See [p.183, ch.5, RV99]. □

For locally compact abelian group G_v, we can construct its dual, $\hat{G}_v$, the set of continuous homomorphisms of G_v with image in $\mathbb{S}^1$. Define $K(G_v, H_v) = \{\chi_v \in \hat{G}_v : \chi_v|_{H_v} = 1\}$ for all $v \notin J_\infty$. If $U \subseteq \mathbb{S}^1$ is a small enough neighbourhood of 1, then it contains no non-trivial subgroup. Then similar to the neighbourhoods defined for the compact-open topology, $W(H_v, U)$, the characters that map H_v into U, but the image is $\{1\}$, the trivial group, hence $W(H_v, U) = K(G_v, H_v)$, for a small enough U, hence $K(G_v, H_v)$ is open in $\hat{G}_v$. Let $\chi \in \hat{G}_v$. Consider the following diagram:

$$\begin{tikzcd}
G_v \arrow[r, "\chi"] \arrow[d] & \mathbb{S}^1 \\
G_v/H_v \arrow[ur, "\overline{\chi}"'] &
\end{tikzcd}$$

The above diagram is commutative. We can define a mapping from $K(G_v, H_v)$ to $(G_v/H_v)^\wedge$. This turns out to be an isomorphism of topological groups. H_v is open in G_v by assumption, hence G_v/H_v is discrete, so $(G_v/H_v)^\wedge$ is compact. So $K(G_v, H_v)$ is compact. Hence it makes sense to define a restricted directed product of $\hat{G}_v$ w.r.t. the subgroups $K(G_v, H_v)$.

Theorem 4.2.3. *Let G_v, H_v be defined as above. Then the restricted direct product of $\hat{G}_v$ w.r.t. $K(G_v, H_v)$ is topologically isomorphic to $\hat{G}$, i.e.*

$$\hat{G} \equiv \prod_v{}' \hat{G}_v.$$

Proof. See [p.184, ch.5, RV99] □

4.3 Haar Measure on Restricted Directed Products

Let $G = \prod'_{v \in J} G_v$ be the restricted direct product of locally compact abelian groups G_v w.r.t. compact subgroups $H_v \subseteq G_v$. Let dg_v be the (left) Haar Measure on G_v, normalized so that $\int_{H_v} dg_v = 1$ for almost all $v \notin J_\infty$.

Proposition 4.3.1. *There exists a unique Haar Measure on G such that for every finite set of indices S containing J_∞, the restriction of dg_S to*

$$G_S = \prod_{v \in S} G_v \times \prod_{v \notin S} H_v$$

is the product measure.

Proof. See [p.185, ch.5, RV99]. □

This proposition allows us to write

$$dg = \prod_v dg_v$$

for the (left) Haar Measure on G. The next proposition shows how to integrate with this Haar Measure.

Proposition 4.3.2. *Let G be the restricted direct product of locally compact groups as above with Haar measure dg.*

1. *Let f be an integrable function on G. Then*

$$\int_G f(g)dg = \lim_S \int_{G_S} f(g_S)dg_S.$$

If f is only assumed to be continuous this formal identity still holds provided we allow the integral to take infinite values.

2. *Let S_0 be a finite set of indices containing J_∞, and such that Vol$(H_v, dg_v) \neq 1$ for all $v \in S_0$. Suppose we are given a family of functions $f_v : G_v \to \mathbb{C}$ such that $f|_{H_v} = 1$ for all $v \notin S_0$. Let $g = (g_v) \in G$ and define*

$$f(g) = \prod_v f_v(g_v).$$

Then f is well-defined and continuous on G. If S is a finite set of indices containing S_0, we have

$$\int_{G_S} f(g_S)dg_S = \prod_{v \in S} (\int_{G_v} f_v(g_v)dg_v).$$

Moreover

$$\int_G f(g)dg = \prod_v (\int_{G_v} f_v(g_v)dg_v)$$

and $f \in L^1(G)$, provided the right-hand product is finite.

3. *Let $\{f_v\}$ and f be as in the previous part such that f_v is the characteristic function of H_v for all but finitely many v. Then f is integrable. Moreover if $\{G_v\}$ are abelian groups, then the Fourier Transform of f is integrable and is given by*

$$\hat{f}_v(g) = \prod_v \hat{f}_v(g_v).$$

Proof. See [p.187, ch.5, RV99]. □

Assume now that we are working with an abelian family $\{G_v\}$ of locally compact groups. Assume the respective measures dg_v are normalized so that $\mathrm{Vol}(H_v, dg_v) = 1$. Define for each v

$$d\chi_v = (dg_v)^\wedge$$

the dual measure of dg_v defined on the group $\hat{G}_v$, in the sense of the Fourier inversion theorem. By definition

$$\hat{f}_v(\chi_v) = \int_{G_v} f_v(g_v)\overline{\chi}_v(g_v)dg_v.$$

If f is the characteristic function of H_v, since characters are orthogonal, we get

$$\hat{f}_v(\chi_v) = \int_{H_v} \chi_v(g_v)dg_v = \begin{cases} \mathrm{Vol}(H_v) & \text{if } \chi|_{H_v} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

If H_v^* be the subgroup of $\hat{G}_v$ such that its elements are trivial on H_v, previously denoted $K(G_v, H_v)$, then $\hat{f}_v(\chi_v)$ is the characteristic function of H_v^*. Hence from Fourier Inversion theorem

$$\mathrm{Vol}(H_v)\mathrm{Vol}(H_v^*) = 1$$

but volume of H_v is 1 due to normalized Haar measure, so $\mathrm{Vol}(H_v^*) = 1$ which is calculated w.r.t. $d\chi_v$, hence we can define $d\chi = (dg)^\wedge$.

Proposition 4.3.3. *The measure $d\chi$ so defined is dual to dg. That is*

$$f(g) = \int_{\hat{G}} \hat{f}(\chi)\chi(g)d\chi$$

for all $f \in V^1(G)$.

($V^1(G) = V(G) \cap L^1(G)$ where $V(G)$ is the complex span of continuous functions on G.)

Proof. See [p.189, ch.5, RV99]. □

Chapter 5

Tate's Thesis

5.1 Introduction

We shall follow the discussion on Chapter 7, p.241 of Ramakrishnan, Valenza.
The well known Riemann Zeta function $\zeta(s)$, defined for $s \in \mathbb{C}$, $Re(s) > 1$ is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

A simple integral test shows that $\zeta(s)$ converges absolutely for $Re(s) > 1$. Euler had established that

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}}$$

for $Re(s) > 1$. Riemann was able to extend the domain of definition of $\zeta(s)$ to all of $\mathbb{C}$ by deriving a functional equation that related $\zeta(s)$ with $\zeta(1-s)$. Let $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ where $\Gamma(s) = \int_0^{\infty} e^x x^s \frac{dx}{x}$. Then

$$\Lambda(s) = \Lambda(1-s)$$

for all $s \in \mathbb{C}$.

A classical approach to proving analytic continuity and functional equation is to take the

Meliin Transform of a certain Theta function

$$\theta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z}$$

for all $z \in \mathbb{C}$. A generalization of $\zeta(s)$ is the following:

Let $L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ where χ is a Dirichlet character modulo $N \in \mathbb{N}$. If χ is the trivial character, i.e. $\chi(n) = 1$ for all $n \in \mathbb{N}$, then we get back the Riemann Zeta function. $L(s,\chi) = \prod_p \frac{1}{1-\chi(p)p^{-s}}$ similar to the Euler product of $\zeta(s)$. Similar functional equations and analytic continuation can be derived for $L(s,\chi)$ using a more general version of theta-function.

The idea of Tate's thesis is to representing the local factors as integrals, and then using Adélic topology to arrive at the global result. In our discussion we shall focus on fields with characteristic 0.

5.2 Characters and Schwartz-Bruhat Space of a local field F

We shall also try to follow the notation of chapter 7 of Ramakrishnan, Valenza.
Let F be a local field, char(F)=0. Let $|\cdot|$ be an absolute map on F. From the classification theorem, theorem 2.2.1., we know F is either $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$ for some prime $p \in \mathbb{Q}$. The possible absolute value maps are the following:

1. $F = \mathbb{R}$, then $|\cdot|$ is the usual absolute value.
2. $F = \mathbb{C}$, then $|z| = z\overline{z}$, i.e. square of the usual absolute value.
3. F a finite extension of $\mathbb{Q}_p$, then $|\cdot| = |\cdot|_p^{[F:\mathbb{Q}]}$, where $|\cdot|_p$ is the p-adic absolute value and $[F:\mathbb{Q}]$ is dimension of F over $\mathbb{Q}_p$ as a vector space.

Let dx be a Haar measure on F. Then F^* has Haar measure $d^*x = c\frac{dx}{|x|}$, it is a Haar-measure since it is translation invariant on F^* and $c > 0$ is a constant.

Let U_F be the group of units of F and $S_F = \{y \in \mathbb{R}_+^\times : y = |x| \text{ for some } x \in F^*\}$, then

$F^* = S_F \times U_F$. S_F is $\mathbb{R}_+^\times$ if F is Archimedean and for non-Archimedean F, $S_F = \{q^n | n \in \mathbb{Z}\}$, where q is the cardinality of the residue field of F. By $O_F = \{x \in F : |x| \leq 1\}$ we denote the ring of integers of F for non-Archimedean F and by $\mathfrak{p}$ we denote the unique prime ideal of O_F, it is the set $\{x \in F : |x| < 1\}$.

Now every continuous homomorphism $\chi : F^* \to \mathbb{C}^*$, it factors through the product $S_F \times U_F$. Let $X(F^*) = \mathrm{Hom}_{\mathrm{cont}}(F^*, \mathbb{C}^*)$, it is the set of continuous group homomorphisms from F^* to $\mathbb{C}^*$.

Definition 5.2.1. *An element of $\chi \in X(F^*)$ is called unramified if $\chi\big|_{U_F} \equiv 1$.*

Theorem 5.2.1. *For every unramified continuous character, there exists a complex number $s \in \mathbb{C}$ such that $\chi(x) = |x|^s$ for all $x \in F^*$.*

Proof. Define $V(F) = \{|x|_F : |\cdot|_F \text{is an absolute value on } F\}$. So, if $F = \mathbb{R}$, $V(\mathbb{R}) = \mathbb{R}_+^\times$.

$$\begin{tikzcd} F^\times \arrow[r, "\chi"] \arrow[d, "|\cdot|_F"'] & \mathbb{C}^\times \arrow[d] \\ V(F) \arrow[r, "\chi'"'] & \mathbb{C}^\times \end{tikzcd}$$

The above diagram is commutative because χ is unramified and χ' is a continuous character of $V(F) \to \mathbb{C}^\times$, i.e. $\chi(x) = \chi'(|x|_F)$, for all $x \in F^\times$. So it suffices to look at characters on $V(F)$.
$\mathbb{C}^\times \cong \mathbb{R}_+^\times \times \mathbb{S}^1$. So χ' factors though $\chi'_r : V(F) \to \mathbb{R}_+^\times$ and $\chi'_u : V(F) \to \mathbb{S}^1$. For F Archimedian, $V(F) = \mathbb{R}_+^\times$. From the previous paragraph, let $\chi : \mathbb{R}_+^\times \to \mathbb{R}_+^\times$ be the real part of the character. So $\chi(ab) = \chi(a)\chi(b)$ and $\chi(1) = 1$.For $t \in \mathbb{R}_+^\times$, Let $\chi'(t) = \log \chi(t)$. Then $\chi'(t_1 t_2) = \log(\chi(t_1)) + \log(\chi(t_2)) = \chi'(t_1) + \chi'(t_2)$. Now $\exists! x \in \mathbb{R}$, such that $t = e^x$, then let $z(x) = \chi'(t)$ such that $t = e^x$. Then $z(x_1 + x_2) = \chi'(e^{x_1 + x_2}) = \chi'(e^{x_1} e^{x_2}) = \chi'(t_1 t_2) = \chi'(t_1) + \chi'(t_2) = z(x_1) + z(x_2)$. And $z(0) = \chi'(e^0) = \log(\chi(1)) = \log(1) = 0$. So $z(x) = \sigma x$ for some $\sigma \in \mathbb{R}$. So $\chi'(t) = e^{\sigma x}$. So $\chi'(t) = \sigma \log(t)$ and finally, $\chi(t) = e^{\chi'(t)} = e^{\sigma \log(t)} = t^\sigma$ ■
For the unitary part of the character on $V(F)$, say $\xi : \mathbb{R}_+^\times \to \mathbb{S}^1$. But we have the following diagram:

$$\begin{tikzcd}
\mathbb{R}_+^\times \arrow[r, "\xi"] \arrow[d, "\exp(.)"', "\log(.)", leftrightarrow] & \mathbb{S}^1 \arrow[d, "\iota", leftrightarrow] \\
\mathbb{R} \arrow[r, "\xi'"'] & \mathbb{S}^1
\end{tikzcd}$$

The above diagram is commutative. Let $\xi'(x) = \xi(e^x) = \xi(t)$, $t \in \mathbb{R}_+^\times$, and $x \in \mathbb{R}$. Then $\xi'(x_1 + x_2) = \xi(e^{x_1+x_2}) = \xi(t_1)\xi(t_2) = \xi'(x_1)\xi'(x_2)$. Since $\xi'(x) \in \mathbb{S}^1$ for all $x \in \mathbb{R}$, $\log(\xi'(x)) \in i[0,1)]$. Let $z(x) = i^{-1}\log(\xi'(x))$ for all $x \in \mathbb{R}$. Then $z(x_1 + x_2) = i^{-1}\log(\xi'(x_1)) + i^{-1}\log(\xi'(x_2)) = z(x_1) + z(x_2)$ and $z(0) = \log(1) = 0$. So $z(x) = cx$ for some $c \in \mathbb{R}$.
So $z(x) = i^{-1}\log(\xi'(x)) = cx$, that is $\xi'(x) = e^{icx}$, so $\xi'(\log(t)) = e^{ic\log(t)} \implies \xi(t) = t^{ic}$.
Now taking the counterparts together, for F Archemedian, $\chi(x) = |x|_F^{\sigma+ic}$ for $\sigma, c \in \mathbb{R}$, i.e. $\chi(x) = |x|_F^s$, for some complex number $s \in \mathbb{C}$

For F non-Archemedian, $V(F) = q^{\mathbb{Z}}$ for some $q \in \mathbb{N}$. Here $q = |O_F/P|$, where P is the unique maximal ideal of O_F generated by the *uniformizer*. So as we saw previously, unramified characters on $F^\times$ are the same as characters on $V(F)$ through the projection $|\cdot|_F$. So any $\xi : q^{\mathbb{Z}} \to \mathbb{C}^\times$ is determined by its value on q. Suppose $\chi(q) = q^s$ for some $s \in \mathbb{C}$, then $s = \log_q(\chi(q)) = \frac{\log(\chi(q))}{\log(q)}$, which is determined upto an integer multiple of $\frac{2\pi i}{\log(q)}$. Here we see that only the real part of s is determined uniquely. $\square$

Theorem 5.2.2. *Any element $\chi \in X(F^*)$ is of the for $\chi(x) = \omega(x)|x|^s$ for some $s \in \mathbb{C}$ and ω is an unramified character of F^* and $|cdot|$ is an absolute value on F.*

Proof. Consider the diagram:

$$\begin{tikzcd}
F^\times \arrow[r, "\chi"] \arrow[d] & \mathbb{C}^\times \\
q^{\mathbb{Z}} \times Q_F^\times \arrow[ur] &
\end{tikzcd}$$

Every quasi-character of $F^\times$ factors though the projection. For every $x \in F^\times$, $x = q^n \cdot u$ for some $u \in O_F^\times$. Let $\hat{x} = u$ and let $\chi|_{O_F^\times} = \chi'$. In particular, for $x \in O_F^\times$, $\chi(x) = \chi(\hat{x}) = \chi'(x)$. Now χ' being the restriction of a continuous map, is continuous. Since $O_F^\times$ is compact, $\chi'(O_F^\times)$ is a compact subgroup of $\mathbb{C}^\times$, hence it is contained in $\mathbb{S}^1$, so χ' is a unitary character on $O_F^\times$. And $x \to \hat{x}$ is a continuous homomorphism from $F^\times \to O_F^\times$, so $\chi'(\hat{\cdot}) : F^\times \to \mathbb{S}^1$ is a continuous homomorphism. Consider the character $z(x) := \chi(x)\chi'(\hat{x})^{-1}$. And for $x \in O_F^\times$, $z(x) = \chi(x)\chi'(\hat{x})^{-1} = \chi(x)\chi(x)^{-1} = 1$. So $z : F^\times \to \mathbb{C}^\times$ is an unramified character, hence $z(x) = |x|_F^s$ for some $s \in \mathbb{C}$. So $\chi(x) = \chi'(\hat{x})|x|_F^s$. $\square$

From theorems 4.2.1 and 4.2.2, we get the following classification for the elements of $X(F^*)$:

1. $F = \mathbb{R}$. $O_F^\times = \{+1, -1\}$. Any character $\chi : \{+1, -1\} \to \mathbb{S}^1$, $\chi(1) = \chi(1)^2 = 1$ and $\chi(-1)^2 = \chi(1) = 1$. So either $\chi(-1) = 1$ or $\chi(-1) = -1$. In the first case χ is trivial, and the second case, $\chi(x) = x/|x| = \mathrm{sgn}(x)$. So quasi-characters of $F^\times$ either looks like $|x|^s$ or $\mathrm{sgn}(x)|x|^s$ where $|\cdot|$ is the usual absolute value. $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{R}^\times, \mathbb{C}^\times)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{C}$.

2. $F = \mathbb{C}$. $O_F^\times = \mathbb{S}^1$. We know $\hat{\mathbb{S}^1} = \mathbb{Z}$. So quasi-characters look like $\chi_{s,n}(z) = (z/||z||^{1/2})^n ||z||^s$ where $||\cdot||$ is the square of the absolute value of $\mathbb{C}$, i.e. $\chi_{s,n} : r \cdot e^{i\theta} \to r^s \cdot e^{in\theta}$ for uniquely determined $n \in \mathbb{Z}$ and $Re(s)$. $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{C}^\times, \mathbb{C}^\times)$ is isomorphic to $\mathbb{Z} \times \mathbb{C}$.

3. F is non-Archemedian. Let $\chi : O_F^\times \to \mathbb{S}^1$. Since $\{1 + \mathfrak{p}^n\}_{n \geq 0}$ constitute a neighborhood basis of $1 \in O_F^\times$. So for large enough n, $\chi(1 + \mathfrak{p}^n) = \{1\}$ because there are no smaller subgroups of $\mathbb{S}^1$. Choose the smallest n, such that the above holds. Then $\mathfrak{p}^n$ is called the conductor of χ. For F non-Archemedian, we have seen that $Im(s)$ is not uniquely determined since log is a multivalued function.

Two characters $\chi_1, \chi_2 \in F$ are said to be equivalent, if $\chi_1\chi_2^{-1}$ is unramified. Let $s_1 = \log(\chi(q))/\log(q)$ and $s_2 = \log(\chi(q))/\log(q) + (2n\pi i)/\log(q)$ for some $n \in \mathbb{Z}$. Then for $x \in O_F^\times$, $\chi_1(x)\chi_2(x)^{-1} = \chi'(x)\chi'(x)^{-1}|x|^{s_1 - s_2 = 1} = 1$. And in particular, $|q|^{s_2 - s_1} = q^{2\pi i n/\log(q)} = q^{\log_q e^{(2i\pi n)}} = q^0 = 1$. So the space of quasi characters is $\{s \in \mathbb{C} | s \sim s' \text{ if } s - s' = m\frac{2\pi i}{\log(q)}, m \in \mathbb{Z}\}$.

The space of complex-valued functions which we shall be working with is called the Schwartz-Bruhat space of functions, denoted $\mathcal{S}(F)$, instead of an ad hoc space.

Definition 5.2.2. *(Schwartz-Bruhat Space)*

1. *For F Archimedean, a smooth functions such that all its derivatives decay faster than polynomials is called a Schwartz function.*

2. *A Schwartz-Bruhat function for F non-Archimedeanm, is the complex span of locally constant functions with compact support.*

It is interesting to note that the Fourier Transform is a bijection between spaces $\mathcal{S}(F)$ and $\mathcal{S}(\hat{F})$.

5.3 Local ζ-Function

First we shall discuss the notion of a self-dual measure on F. Let $\psi' : F \to \mathbb{S}^1$ be a continuous additive character of F. Define Fourier transform of $f \in \mathcal{S}(F)$ as

$$\hat{f}(\psi') = \int_F f(x)\psi'(x)dx$$

Now if this measure dx is equal to the Haar measure $d\chi$ used to define the Fourier Inversion formula in theorem 3.2.1, then we call dx a self-dual measure w.r.t. ψ', its dual being $d\chi$. Tate in his thesis had normalized the self-dual measure on F w.r.t. the *standard* additive continuous character (existence of such characters will be described later) such that the identity $f(x) = \hat{\hat{f}}(-x)$, following the treatment in Ramakrishnan, Valenza, chapter 7, we shall avoid this normalization of the Haar measure for the local case. For any $\chi \in X(F^*)$, we can define the shifted dual $\chi^\vee(x) = \chi(x)^{-1}|x|$ for all $x \in F$; explicitly for $\chi = \mu|\cdot|^s$, $\chi^\vee = \mu^{-1}|\cdot|^{1-s}$.

Let $f \in \mathcal{S}(F)$ and $\chi \in X(F^*)$ then, we can define an associated local zeta function:

$$Z(f,\chi) = \int_{F^*} f(x)\chi(x)d^*x$$

We immediately state the next theorem:

Theorem 5.3.1. *Let $f \in \mathcal{S}(F)$, $\chi = \mu|\cdot|^s$ be an element of $X(F^*)$ and $\sigma = Re(s)$ (called exponent of χ). Then the following are true:*

1. *$Z(f,\chi)$ is absolutely convergent for $\sigma > 0$.*

2. *If $\sigma \in (0,1)$, then we have the functional equation*

$$Z(\hat{f},\chi^\vee) = \gamma(\chi,\psi,dx)Z(f,\chi)$$

   for some $\gamma(\chi,\psi,dx)$ which is meromorphic as a function of $s \in \mathbb{C}$ and is independent of f.

3. *There exists a factor $\epsilon(\chi,\psi,dx) \in \mathbb{C}^*$ such that*

$$\gamma(\chi,\psi,dx) = \epsilon(\chi,\psi,dx)\frac{L(\chi^\vee)}{L(\chi)}$$

   .

Proof. See [RV99, p. 247, ch. 7] □

Lemma 5.3.2. *For all $\chi \in X(F^*)$ with exponent $\sigma \in (0,1)$, we have*

$$Z(f,\chi)Z(\hat{h},\chi^\vee) = Z(\hat{f},\chi^\vee)Z(h,\chi)$$

for all $f, h \in \mathcal{S}(F)$.

This implies that the ratio $Z(\hat{f},\chi^\vee)/Z(f,\chi)$ is independent of f.

Proof. See p.248, ch.7 of [RV99]. □

Below are some calculations regarding the proof of theorem 4.3.1, part 2 which follow the treatment given in [RV99] and include some omitted ones and show explicitly what we mean by a standard additive character of F.

Case 1: $F = \mathbb{R}$. Choose $f(x) = e^{-\pi x^2}$ in $S(\mathbb{R})$. Fix a unitary character $\psi(x) = e^{-2\pi i x}$. Define local zeta function w.r.t. $\chi(x) = |x|^s$,

$$Z(f,\chi) = \int_{\mathbb{R}^\times} f(x)\chi(x)d^*x = 2\int_0^\infty e^{-\pi x^2}|x|^{s-1}dx$$

Making substitution $u = \pi x^2$ and after some algebraic manipulations we get

$$Z(f,\chi) = \pi^{-s/2} \int_0^\infty e^{-u} u^{s/2-1} du = \pi^{-s/2}\Gamma(s/2)$$

Where $\Gamma(s)$ is the gamma function, $s \in \mathbb{C}$.
Now fourier transform of f is itself, i.e. $f = \hat{f}$. And the dual character of $|x|^s$ is itself as well. So $Z(\hat{f}, \chi^\vee) = Z(f,\chi)$.

Now for $\chi(x) = sgn(x)|x|^s$, choose $f_1(x) = xe^{-\pi x^2}$. Then

$$Z(f_1,\chi) = \int_{\mathbb{R}^\times} xe^{-\pi x^2} \frac{x}{|x|} |x|^{s-1} dx$$

After the same substitution as before we get

$$Z(f_1,\chi) = \pi^{-(\frac{s+1}{2})} \int_0^\infty e^{-u} u^{s-1/2} du = \pi^{-(\frac{s+1}{2})}\Gamma(\frac{s+1}{2})$$

Similar result holds for the local zeta factor of fourier transform of f and character χ as the previous result.

Proposition 5.3.3. *The fourier transform of $f(x) = e^{-\pi x^2}$ is itself.*

Proof.

$$\frac{d}{dy}\hat{f}(y) = \int_{-\infty}^\infty e^{-\pi x^2} \frac{d}{dy} e^{-2\pi ixy} dx = \int_{-\infty}^\infty e^{-\pi x^2}(-2\pi ix)e^{-2\pi ixy} dx$$
$$= i\int_{-\infty}^\infty (-2\pi x)e^{-\pi x^2} e^{-2\pi ixy} dx$$

Next we make substitution $u = \pi x^2$ when needed and integrate by parts to get

$$ie^{-2\pi ixy} \int -e^{-u} du - i \int (-2\pi iy) e^{-2\pi ixy} (\int -e^{-u} du) du$$

$$= 0 + i\int (2\pi iy) e^{2\pi ixy} e^{-\pi x^2} dx = (-2\pi y) f\hat{(y)}$$

So we get a differential equation, which we solve,

$$\int d(\hat{f}(y))/\hat{f}(y) = -2\pi y \implies \log(\hat{f}(y)) = -\pi y^2 + c$$

so we finally have, $\hat{f}(y) = Ce^{\pi y^2}$. Now $\hat{f}(0) = C$, we determine this next.

$$\hat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2} dx$$

Now

$$\hat{f}(0)^2 = \int_{-\infty}^{\infty} e^{-\pi x^2} dx \int_{-\infty}^{\infty} e^{-\pi y^2} dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)} dydx$$

Now changing to polar co-ordinates, $r^2 = x^2 + y^2$ and $dydx = rdrd\theta$,

$$\int_0^{\infty} \int_0^{2\pi} e^{-\pi r^2} rdrd\theta = 2\pi \int_0^{\infty} e^{-u} (2\pi)^{-1} du = 1$$

Hence $\hat{f}(0) = 1$, so $\hat{f}(y) = e^{-\pi y^2}$. $\square$

Now we return to our calculation of $Z(\hat{f}, \chi^\vee)$. For $\mathbb{R}$, we have $\chi^\vee(x) = |x|^{1-s}$ and $f_1(x) = xe^{-\pi x^2}$, then $\hat{f}_1(y) = iye^{-\pi y^2} = if_1(x)$. So $Z(\hat{f}_1, \chi^\vee) = Z(if_1, |\cdot|^{1-s}) = iZ(f_1, |\cdot|^{1-s}) = i\pi^{\frac{2-s}{2}} \Gamma(\frac{2-s}{2})$.

Case 2: $F = \mathbb{C}$ and $f_n(z) = (2\pi)^{-1} \bar{z}^n e^{-2\pi z\bar{z}}$ for $n \geq 0$ and $f_n(z) = (2\pi)^{-1} z^{-n} e^{-2\pi z\bar{z}}$ for $n < 0$. Quasi-characters look like $r \cdot e^{i\theta} \to r^s \cdot e^{in\theta}$.The measure on $\mathbb{C}^\times$ is taken to be twice the usual lebesgue measure. So

$$Z(f_n, \chi_{s,n}) = \int_{\mathbb{C}^\times} f_n(z) \chi_{s,n} 2d^*z$$

Changing to polar co-ordinates we get for $n \geq 0$

$$\int_0^{2\pi} \int_0^{\infty} (2\pi)^{-1} r^n e^{-in\theta} e^{-2\pi r^2} e^{in\theta} r^{2s} \frac{2}{r} drd\theta = 2 \int_0^{\infty} e^{-2\pi r^2} r^{2s+n-1} drd\theta$$

And similarly for $n < 0$

$$\int_0^{2\pi} \int_0^{\infty} (2\pi)^{-1} r^{-n} e^{-in\theta} e^{-2\pi r^2} e^{in\theta} r^{2s} \frac{2}{r} drd\theta = 2 \int_0^{\infty} e^{-2\pi r^2} r^{2s-n-1} dr$$

We shall write the following for all n

$$Z(f_n, \chi_{s,n}) = 2 \int_0^{\infty} e^{-2\pi r^2} r^{2s+|n|-1} dr$$

Make a change of variable $u = 2\pi r^2$ to get

$$2(2\pi)^{\frac{1-2s-|n|}{2}} \int_0^\infty e^{-u} u^{\frac{2s+|n|-1}{2}} \frac{1}{4\pi} \sqrt{2\pi} u^{-1/2} du = (2\pi)^{-(s+\frac{|n|}{2})} \int_0^\infty e^{-u} r^{s+\frac{|n|}{2}-1} du$$

$$= (2\pi)^{-(s+\frac{|n|}{2})} \Gamma(s + \frac{|n|}{2})$$

Now we calculate $Z(\hat{f}_n, \chi_{s,n}^\vee) = Z(\hat{f}_n, \chi_{1-s,-n})$. We need to find the fourier transform of f_n.

Proposition 5.3.4. *For all n, $\hat{f}_n(x) = i^{|n|} f_{-n}(x)$.*

Proof. Choose the character for fourier transform $\psi(x) = e^{-4\pi i < x, \bar{y} >}$, where $< x, \bar{y} >$ is the Hermitian inner-product of these two vectors over $\mathbb{R}$. Write $x = x_1 + ix_2$ and $y = y_1 + iy_2$. For $n = 0$, $f(x) = (2\pi)^{-1} e^{-2\pi|x|^2}$ and fourier transform of this is itself. Suppose $\hat{f}_m(y) = i^{|m|} f_{-m}(y)$ for some $m > 0$, i.e.

$$\hat{f}_m(y) = \int_{-\infty}^\infty \int_{-\infty}^\infty (x_1 - ix_2)^m e^{-2\pi(x_1^2+x_2^2)} e^{-4\pi i(x_1 y_1 - x_2 y_2)} dx_1 dx_2$$

$$= i^m (y_1 + iy_2)^{-m} e^{-2\pi(y_1^2+y_2^2)}$$

Write total derivative as $(\partial/\partial y_1 + i\partial/\partial y_2)$ and define $D = \frac{1}{4\pi i}(\partial/\partial y_1 + i\partial/\partial y_2)$. We know $D(h) = 0$ if and only if h is analytic. Now apply D to both sides of our induction hypothesis,

$$\int_{-\infty}^\infty \int_{-\infty}^\infty (x_1 - ix_2)^{m+1} e^{-2\pi(x_1^2+x_2^2)} e^{-4\pi i(x_1 y_1 - x_2 y_2)} dx_1 dx_2$$

$$= i^{m+1} (y_1 + iy_2)^{-(m+1)} e^{-2\pi(y_1^2+y_2^2)}$$

This in terms of notation means $\hat{f}_{m+1}(x) = i^{m+1} f_{-(m+1)}(x)$ and by induction we have proved our claim for $n \geq 0$. Now suppose $n < 0$. We observe that $f_n(-x) = (2\pi)^{-1}(-x)^{-n} e^{-2\pi|x|^2} = (-1)^n f_{-n}(x)$. From fourier inversion $\hat{\hat{f}}(x) = f(-x)$. So $\hat{\hat{f}}_{-n}(-x) = i^{-n} \hat{f}_n(-x) = i^{-n}(-1)^{-n} \hat{f}_n(x)$, this implies $\hat{f}_n(x) = i^{-n} f_{-n}(x)$, proving the proposition. $\square$

Now we can calculate $Z(\hat{f}_n, \chi_{s,n}^\vee) = Z(i^{|n|} f_n, \chi_{1-s,-n}) = i^{|n|} Z(f_n, \chi_{1-s,-n}) = i^{|n|}(2\pi)^{-(1-s+\frac{|n|}{2})}\Gamma(1-s+\frac{|n|}{2})$. The non-Archimedean case, i.e. when F is a finite extension of $\mathbb{Q}_p$ has been explicitly described at [p.253, ch.7, RV99].

5.4 Riemann-Roch and Poisson Summation

The discussion of this section closely follows section 7.2 of [RV99]. Let K be a Global field of characteristic 0, and $\mathbb{A}_K$ be its ring of adéles. Let $\mathcal{S}(K_v)$ be the Schawrtz-Bruhat space of functions of K_v, the local field for the place v of K.

Definition 5.4.1. *Define Adelic Schawrtz-Bruhat space of functions as*

$$\mathcal{S}(\mathbb{A}_K) := \otimes_v' \mathcal{S}(K_v) = \{\otimes_v f_v : f_v \in \mathcal{S}(K_v) \forall v \text{ and } f_v|_{O_v} = 1 \text{for almost all } v\}$$

For $\phi \in \mathcal{S}(\mathbb{A}_K)$, define

$$\widetilde{\phi}(x) = \sum_{\delta \in K} \phi(x+\delta)$$

The function $\widetilde{\phi}$ is invariant under translates by elements of K.

Definition 5.4.2. *Let $f : \mathbb{A}_K \to \mathbb{C}$ such that $\widetilde{f}$ and $\widetilde{\hat{f}}$ are both absolutely and uniformly convergent on compact subsets, then we say f is admissible.*

Lemma 5.4.1. *If $f \in \mathcal{S}(\mathbb{A}_K)$, then it is admissible.*

Proof. See [p.261, ch.7, RV99]. □

Theorem 5.4.2. *(Poisson Summation Formula) Let $f \in \mathcal{S}(\mathbb{A}_K)$, then $\widetilde{f} = \widetilde{\hat{f}}$, i.e.*

$$\sum_{\delta \in K} f(x+\delta) = \sum_{\delta \in K} \hat{f}(x+\delta)$$

for all $x \in \mathbb{A}_K$.

Proof. See [p.262, ch.7, RV99] □

Theorem 5.4.3. *Let $x \in \mathbb{A}_K^*$ and $f \in \mathcal{S}(\mathbb{A}_K)$, then*

$$\sum_{\delta \in K} f(\delta x) = \frac{1}{|x|} \sum_{\delta \in k} \hat{f}(\delta x^{-1}).$$

Proof. See [p.264, ch.7, RV99]. □

5.5 Global ζ-Function and Functional Equation

Let K be a Global field of characteristic 0. For each place v of K, let ψ_v be a continuous additive character of K_v and dx_v associated self-dual Haar measure. Explicitly, for every place v,

$\psi_v(x) = e^{-2\pi i \mathrm{tr}(x)}$ for v infinite

$\psi_v(x) = \psi_p(\mathrm{tr}(x))$ for v-finite, $v|p$

where the map ψ_p is the composition of maps $\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z} \to \mathbb{S}^1$. The first arrow takes a p-adic number $\sum_{k=-n}^{\infty} a_k p^k$ to $\sum_{k=-n}^{-1} a_k p^k$, this is called the p-adic fractional part. An example of such a character is $e^{-2\pi i \{x\}_p}$; $\{\cdot\}_p$ being the p-adic fractional part.

Let K/F be a finite extension of Global Fields. We define the trace map as follows:

$$\mathrm{tr} : \mathbb{A}_K \to \mathbb{A}_F$$

$$(x_v)_v \to \sum_{v|u} \mathrm{tr}_{K_v/F_u}(x_v)_u$$

as u ranges over places of F. So we have additive character

$$\psi_K(x) = \psi(\mathrm{tr}(x))$$

Let dx denote the Haar measure on $\mathbb{A}_K$ self-dual w.r.t. ψ_K. Moreover $\forall a \in \mathbb{A}_K^* = \mathbb{I}_{\mathbb{K}}$

$$d(ax) = |a|dx$$

Now we shall describe the Global Zeta function.

Let $\chi : \mathbb{I}_K/K^* \to \mathbb{C}^*$, i.e. a continuous group homomorphism of $\mathbb{I}_{\mathbb{K}}$ that is trivial on K^*. Then for $f \in \mathcal{S}(\mathbb{A}_k)$ and $\chi = \mu|\cdot|$ where $\mu : \mathbb{I}_K/K^* \to \mathbb{S}^1$ be a unitary character

$$Z(f, \chi) = \int_{\mathbb{I}_K} f(x)\chi(x)d^*x.$$

The Haar measure d^*x is induced by the product measure $\prod_v d^*x_v$ and we normalize d^*x_v by letting

$$d^*x_v = \frac{q_v}{q_v - 1}\frac{dx_v}{|x_v|}$$

where q_v is the cardinality of the residue field of K_v. This normalization gives the ring of integers of K_v, denoted O_v, volume 1. Define $\chi^\vee = \chi^{-1}|\cdot|$. Let $\mathbb{I}_K^1 = \{x \in \mathbb{I}_K : |x| = 1\}$, and $C_K^1 = \mathbb{I}_K^1/K^*$. C_K^1 is compact, hence it has finite Haar measure.

Theorem 5.5.1. *(Functional equation and Meromorphic Continuation) $Z(f,\chi)$ defined above, has the functional equation*

$$Z(f,\chi) = Z(\hat{f},\chi^\vee)$$

and it becomes a meromorphic function of s on the complex plane.
The extended $Z(f,\chi)$, is holomorphic everywhere, except when $\mu = |\cdot|^{-i\tau}$, in this case $Z(f,\chi)$ has poles at $s = i\tau$ and $s = 1+i\tau$ with corresponding values given by

$$-Vol(C_K^1 f(0)) \text{ and } VolC_K^1\hat{f}(0)$$

respectively.

Proof. See [p.272, ch.7, RV99]. $\square$

Chapter 6

Introduction to the Theory of Automorphic Forms and Representations

In this chapter, we follow the discussion given in first few sections of chapter 1, 2 and 3 of [DB97]. We try to generalize the method developed by Tate in his thesis to locally compact non-abelian groups where we shall need ideas from Representation Theory of Lie Groups (matrix groups) and we look at functions on the upper-half plane and then try to relate them to representations of a matrix group.

6.1 The Modular Group

We shall refer to section 1.2 of [DB97] for the discussion in this chapter.

Definition 6.1.1. *The Poincaré upper half plane is the set* $\mathcal{H} = \{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R}$ *and* $y > 0\}$.

Let $G = SL(2, \mathbb{R})$. Define action of $g \in G$ on $\mathcal{H}$ as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to g(z) = \frac{az + b}{cz + d}.$$

It is an easy check that $g_1(g_2(z)) = (g_1g_2)(z)$, i.e. it is a group action. But since the identity matrix I and $-I$, act trivially, we often pass to the group $PSL(2,\mathbb{R}) = G/\{\pm I\}$. We shall later see a more general action by the group $GL(2,\mathbb{R}^+)$.

More generally we allow $SL(2,\mathbb{C})$ (also $GL(2,\mathbb{C})$) to act on the Riemann Sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.

Consider the action of the set of upper triangular matrices in G on $\mathcal{H}$, suppose $x + iy \in \mathcal{H}$, then

$$\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} : i \to x + iy$$

Further the stabilizer of i under the action of G is $SO(2)$, because

$$\frac{ai+b}{ci+d} = i \implies a = d, b = -c$$

and since determinant is 1, we get $SO(2) = \{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) : a^2 + b^2 = 1\}$. The space of upper triangular matrices form a subgroup of G, say B. The upper-half plane $\mathcal{H}$ can be identified with the set of cosets of $G/SO(2)$, hence B acts transitively on $\mathcal{H}$ and we get

$$G = B \cdot SO(2)$$

this is called the Iwasawa Decomposition of $SL(2,\mathbb{R})$.

Definition 6.1.2. *We shall mainly be interested in the following type of subgroups of $SL(2,\mathbb{R})$:*

1. *Let $\Gamma(1) = SL(2,\mathbb{Z}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R}) : a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1)\}$. $\Gamma(1)$ is a discrete subgroup of $SL(2,\mathbb{R})$.*
2. *(Congruence subgroups) We denote by $\Gamma(N)$ for some $N \in \mathbb{N}$, the subgroups*

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) | a \equiv d \equiv 1 \pmod{N} \text{ and } b \equiv c \equiv 0 \pmod{N} \right\}.$$

Consider a map $\Gamma(1) \to SL(2,\mathbb{Z}/N\mathbb{Z})$, taking every matrix entry to its image in $\mathbb{Z}/N\mathbb{Z}$. Now $SL(2,\mathbb{Z}/N\mathbb{Z})$ is a finite group and hence $\Gamma(N)$ which is the kernel of this map, is normal in $\Gamma(1)$ and has finite index. A subgroup of $\Gamma(1)$ is called a congruence subgroup if it contains $\Gamma(N)$ for some N.

Proposition 6.1.1. *Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, \mathbb{R})^+$, then $Im(gz) = det(g)\frac{Im(z)}{|cz+d|^2}$ for all $z \in \mathcal{H}$.*

Proof. $\text{Im}(gz) = \text{Im}(\frac{az+b}{cz+d}) = \text{Im}(\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}) = \text{Im}(\frac{(adz+bc\bar{z})}{|cz+d|^2}) = \frac{(ad-bc)\text{Im}(z)}{|cz+d|^2}$.
In particular, if $g \in SL(2, \mathbb{R})$, then $\text{Im}(gz) = \frac{\text{Im}(z)}{|cz+d|^2}$. $\square$

Definition 6.1.3. *If Γ is a subgroup of G, then we say action of Γ of the upper-half plane is discontinuous if for any two compact subsets $K_1, K_2 \subset \mathcal{H}$, the set*

$$\{\gamma \in \Gamma | \gamma(K_1) \cap \gamma(K_2) \neq \emptyset\}$$

is finite.

Proposition 6.1.2. $\Gamma(1)$ *acts discontinuously on $\mathcal{H}$.*

Proof. See [p.19, ch.1, DB97]. $\square$

Definition 6.1.4. *Suppose Γ is a subgroup of $SL(2, \mathbb{R})$ acting discontinuously on $\mathcal{H}$. A fundamental domain for the action of Γ is an open set $F \subset \mathcal{H}$ such that*

1. *For every $z \in \mathcal{H}$, there exists $\gamma \in \Gamma$ such that $\gamma(z)$ lies in the closure of F in $\mathcal{H}$.*
2. *If for any two $z_1, z_2 \in F$, there exists $\gamma \in \Gamma$, such that $\gamma(z_1) = z_2$, then $z_1 = z_2$ and γ is the identity matrix.*

Proposition 6.1.3. *Consider the set*

$$F = \{z = x + iy \in \mathcal{H} | -1/2 < x < 1/2 \text{ and } |z| > 1\}$$

then F is a fundamental domain for $\Gamma(1)$.

Proof. See [p.19, ch.1, DB97]. $\square$

Proposition 6.1.4. $SL(2, \mathbb{Z})$ *is generated by the two elements*

$$T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

Proof. See [p.20, ch.1, DB97]. $\square$

If we embed the upper-half plane in the Riemann Sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, then $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ is the topological boundary of $\mathcal{H}$. The point ∞ should be regarded no different from the other boundary points.

For example, $SL(2,\mathbb{R})$ acts transitively on $\mathbb{R} \cup \{\infty\}$.

This can be seen by consider the following mapping: Let $\mathcal{D}$ be the unit disc, then define a map $\mathcal{C} : \mathcal{H} \to \mathcal{D}$

$$C(z) = \frac{z-i}{z+i}$$

this map is called the Cayley Transform. $\mathcal{C}$ maps $\mathbb{R} \cup \{\infty\}$ onto the unit circle, hence they are considered equivalent to each other.

If Γ is a discontinuous group acting on $\mathcal{H}$, and $\Gamma\backslash\mathcal{H}$ be set of orbits of elements of $\mathcal{H}$ under the action of Γ. We give topology to the space $\Gamma\backslash\mathcal{H}$ as follows: a subset of $\Gamma\backslash\mathcal{H}$ is open if and only if its pre-image under the canonical projection $\mathcal{H} \to \Gamma\backslash\mathcal{H}$ is open in $\mathcal{H}$.

Let Γ be a congruence subgroup. A cusp of Γ is a point where the fundamental domain for Γ touches the boundary of $\mathcal{H}$. Let $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$. $SL(2,\mathbb{Q})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$. A subgroup of finite index can have only finitely many orbits on this set. An orbit of Γ in $\mathbb{P}^1(\mathbb{Q})$ is called a cusp of Γ. We have the more general notion of a cusp for a more general Γ

Definition 6.1.5. *Let Γ be a discontinuous group acting on $\mathcal{H}$ such that $\Gamma\backslash\mathcal{H}$ has finite volume, then by a cusp of Γ, we mean*

1. *a point $a \in \mathbb{P}^1(\mathbb{R})$ such that there exists $\gamma \in \Gamma$, $\gamma \neq I$ with $|tr(\gamma)| = 2$ and $\gamma(a) = a$, or*
2. *an orbit of such points under the action of Γ*

We shall end this section with a discussion on compactification of $\Gamma\backslash\mathcal{H}$ for a congruence group Γ. Consider the space $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$. Let $\mathcal{H}$ have to usual topology induced from the topology on the complex plane. We describe the topology in the neighbourhood of points $a \in \mathbb{Q} \cup \{\infty\}$ as follows:

1. for a point $a = \infty$, we take neighbourhoods of the form $\{\infty\} \cup \{z | \mathrm{Im}(z) \geq C$ for $C > 0, C \in \mathbb{R}\}$.
2. For $a \in \mathbb{Q}$, we take neighbourhoods $\{a\} \cup U$, where U is the interior of a circle touching a.

With this topology on $\mathcal{H}^*$, we give $\Gamma\backslash\mathcal{H}^*$ the quotient topology.

6.2 Modular Forms for $SL(2,\mathbb{Z})$

The discussion in the section follows the discussion of section 1.3, [p.26, ch.1, DB97].

Definition 6.2.1. *(Modular Form for $SL(2,\mathbb{Z})$) A modular form of weight k, where k is an even non-negative integer, is a holomorphic function f on $\mathcal{H}$, satisfying the identity:*

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$$

and f is holomorphic at the cusp ∞.

Since $\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right) \in \Gamma(1)$, so $f(z+1) = f(z)$ for all $z \in \mathcal{H}$. So f has the Fourier expansion:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z} = \sum_{n=-\infty}^{\infty} a_n q^n$$

where we have made the change in variable as $q = 2\pi i n z$. We have the following conditions:

1. If $a_n = 0$ for all $n \geq -N$, $N \in \mathbb{N}$, then f is meromorphic at ∞.
2. If $a_n = 0$ for all $n \geq -1$, then f is holomorphic at ∞.
3. If f is holomorphic at ∞ and $a_0 = 0$, the f is said to be cuspidal at ∞, and is called a a cusp form.

The modular forms of weight k for $\Gamma(1) = SL(2,\mathbb{Z}$ (resp. the cusp forms of weight k)form a space denoted $M_k(\Gamma(1))$ (resp. $S_k(\Gamma(1))$). It is useful to note that the space $M_k(\Gamma(1))$ is finite dimensional (see [p.26, ch.1, DB97]).

An automorphic function for Γ, a congruence subgroup is a function f such that $f(\frac{az+b}{cz+d}) = f(z)$, f is meromorphic on $\mathcal{H}$ and at ∞. It is a consequence of the maximum modulus principle that an automorphic function with no poles is constant. But an automorphic function maybe regarded as a Modular form of weight 0, hence we can say a modular form of wight 0 is constant.

Example 1. *Suppose $k \geq 4$ is an even integer. Define*

$$E_k(z) = \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} (mz+n)^{-k}.$$

$E_k(z)$ is a modular form of weight k (see [p.28, ch.1, DB97]).

6.3 L-Function associated to a Modular Form for $\Gamma(1)$

Suppose $f, g \in S_k(\Gamma(1))$, then it is easily checked that $f(z)\overline{g(z)}y^k$ stays invariant under action by $\Gamma(1)$.

Definition 6.3.1. *(Petersson Inner Product) Let $f, g \in S_k(\Gamma(1))$, then define*

$$\langle f, g\rangle = \int_{\Gamma(1)\backslash\mathcal{H}} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2}$$

It shall be shown later that $\frac{dxdy}{y^2}$ stays invariant under the action of $GL(2,\mathbb{Q})^\times$. Hence the inner-product and the integral is well-defined, because, $q^n = e^{-2\pi i n y}e^{2\pi i n x}$, thus as $y \to \infty$, a cusp form $f(z)$, having a fourier expansion $\sum_{n=1}^{\infty} a_n q^n$ decays rapidly as $q^n \to 0$. Hence the integral is rapidly convergent. It is easy to check that the Petersson Inner Product is Hermitian.

Definition 6.3.2. *Let $f(z) = \sum_{n=0}^{\infty} a_n q^n$ be an element of $M_k(\Gamma(1))$, then we can associate the following L-function to f:*

$$L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

$L(s, f)$ is convergent for sufficiently large s. The following proposition gives a sufficient estimate:

Proposition 6.3.1. *If f is cuspidal, its Fourier coefficients satisfy $a_n \leq Cn^{k/2}$ for some constant independent of n.*

Proof. See [p.32, ch.1, DB97]. □

We end this section with a result that establishes meromorphic continuation and functional equation for $L(s, f)$ defined above.

Proposition 6.3.2. *Let $L(s, f)$ be the L-function associated to a modular form. Define*

$$\Lambda(s, f) = (2\pi)^{-s}\Gamma(s)L(s, f)$$

then $\Lambda(s, f)$ extends to an analytic continuation in s if f is a cusp form.
If f is not cuspidal then $\Lambda(s, f)$ has poles at $s = 0$ and $s = k$. Moreover $\Lambda(s, f)$ satisfies

$$\Lambda(s, f) = (-1)^{k/2}\Lambda(k - s, f).$$

Proof. See [p.33, ch.1, DB97]. □

6.4 Hecke Operators

In this section we shall discuss a certain ring of operators acting on the space of Modular forms, introduced by Hecke (1937). The commutativity of this ring of operators result in Euler products of the associated L-functions introduced in the previous section. We shall closely follow the treatment in section 1.4 of [DB97, p.41].

Definition 6.4.1. *If f is a holomorphic function on $\mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, \mathbb{R})^+$, then define*

$$(f|\gamma)(z) = (det\ \gamma)^k/2(cz + d)^{-k} f(\frac{az + b}{cz + d}).$$

After an algebraic calculation, it can be verified that $(f|\gamma)\gamma' = f|(\gamma\gamma')$, i.e. it is a bona fide right action on holomorphic functions on $\mathcal{H}$. We would like to state the following group theoretic result before getting into Hecke Operators.

Proposition 6.4.1. *Let $\alpha \in GL(2, \mathbb{Q})^+$. Then for the double coset $\Gamma(1)\alpha\Gamma(1)$:*

$$\Gamma(1)\alpha\Gamma(1) = \bigcup_{i=1}^{N} \Gamma(1)\alpha_i,\ \ \alpha_i \in GL(2, \mathbb{Q})^+.$$

Moreover the number of right cosets is equal to $[\Gamma(1) : \alpha^{-1}\Gamma(1)\alpha \cap \Gamma(1)]$ which is finite.

Proof. See [p.42, ch.1, DB97]. □

Let $\alpha \in GL(2, \mathbb{Q}^+)$, let α_i be the right coset representatives of $\Gamma(1)\alpha\Gamma(1)$ as the previous propositions. Define a Hecke operator $T_\alpha = T(\alpha)$ on $M_k(\Gamma(1))$ as

$$f|T_\alpha = \sum f|\alpha_i$$

Now if $\gamma \in \Gamma(1)$, then

$$(f|T_\alpha)\gamma = \sum f|\alpha_i\gamma = \sum f|\gamma\alpha_i = \sum f|\alpha_i = \sum f|T_\alpha$$

So $f|T_\alpha$ is again a Modular form.

We can define a multiplication of two Hecke operators as follows:

$$T_\alpha \cdot T_\beta = \sum_{\sigma \in \Gamma(1)\backslash GL(2,\mathbb{Q})^+/\Gamma(1)} m(\alpha, \beta; \sigma) T_\sigma$$

where $m(\alpha, \beta; \sigma)$ is the cardinality of the set of indices (i, j) such that $\sigma \in \Gamma(1)\alpha_i\beta_j$. Using this definition of product it can checked that the product is associative, i.e. for all $\alpha, \beta, \gamma \in GL(2, \mathbb{Q})^+$, $(T_\alpha \cdot T_\beta)T_\gamma = T_\alpha(T_\beta \cdot T_\gamma)$. Thus the set of Hecke operators form an algebra, denoted $\mathcal{R}$.

Theorem 6.4.2. *The Hecke algebra $\mathcal{R}$ is commutative.*

Proof. See [p.45, ch.1, DB97] □

Since $\mathcal{R}$ is commutative, there is no distinction between a right action or left action of f.

Proposition 6.4.3. *The differential form $\frac{dxdy}{y^2}$ is invariant under the action of $GL(2, \mathbb{Q}^+)$.*

Now choose N large enough so that f, g are both modular forms for $\Gamma(N)$. We generalize the Petersson Inner product for all congruence cusp forms:

$$\langle f, g\rangle = \frac{1}{[\Gamma(1) : \Gamma(N)]} \int_{\Gamma(N):\Gamma(1)} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}.$$

Theorem 6.4.4. *The operator T_α is self-adjoint on $S_k(\Gamma(1))$, that is $\langle f|T_\alpha, g\rangle = \langle f, g|T_\alpha\rangle$.*

Proof. See [p.46, ch. 1, DB97]. □

The algebra $\mathcal{R}$ is a commutative family of self-adjoint operators on the finite-dimensional vector space $S_k(\Gamma(1))$. As a consequence, there exists a basis of the vector space consisting of eigenfunctions of all the Hecke operators. If f is such an eigenform, let $f = \sum A(n)q^n$, normalized so that $A(1) = 1$, we shall see that $L(s, f) = \sum A(n)n^{-s}$ has a Euler product.

Definition 6.4.2. *Let n be a positive integer. Let $T(n)$ be the sum of $T_{\alpha(d_1,d_2)}$ where d_1, d_2 are integers such that $d_1 d_2 = n, d_2 | d_1$ and $\alpha(d_1, d_2) = \begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix}$*

Suppose $f \in S_k(\Gamma(1))$ has Fourier expansion $\sum A(n)q^n$, and $T(n)f$ has Fourier expansion $\sum B(m)q^m$. Then we have the following relation among the coefficients:

$$B(m) = \sum_{ad=n, a|m} \left(\frac{a}{d}\right)^{k/2} dA\left(\frac{md}{a}\right).$$

Proposition 6.4.5. *Let f be a Hecke eigenform with eigenvalues $\lambda(n)$ normalized as $T(n)f = n^{1-k/2}\lambda(n)f$ and Fourier coefficients $A(n)$, then*

1. $A(1) \neq 0$.
2. *If $A(1) = 1$, then $\lambda(n) = A(n)$ for all n.*
3. *If $A(1) = 1$, then $A(mn) = A(m)A(n)$ for all m, n co-prime.*

Proof. See [p.48, ch.1, DB97]. □

We shall now see the Euler product for normalized Hecke eigenform.

Theorem 6.4.6. *Let f be a normalized Hecke eigenform, then*

$$L(s, f) = \sum A(n)n^{-s} = \prod_p (1 - A(p)p^{-s} + p^{k-1-2s})^{-1}.$$

Proof. See [p.49, ch.1, DB97]. □

6.5 Twisted L-function

In this section we shall slightly generalize the functional equation for $\Lambda(s,f)$ by considering a twisted version. We shall define a space $S_k(\Gamma_0(N),\psi)$, which is a subspace of $S_k(\Gamma_1(N),\psi)$, if $f \in S_k(\Gamma_0(N),\psi)$, it additionally satisfies $f|\gamma = \psi(d)$ where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Here $\Gamma_0(N)$ is the subgroup of $\Gamma(1)$ such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \pmod N$ and $\Gamma_1(N)$ has the further condition that $a \equiv d \equiv 1 \pmod N$.

Now for such an f, define $L(s,f,\chi) = \sum \chi(n)A(n)n^{-s}$, where χ is a primitive Dirichlet character modulo D and ψ is a Dirichlet modulo N and $(N,D)=1$. Now let $w_N = \left(\begin{smallmatrix} & -1 \\ N & \end{smallmatrix}\right)$ and let $g = f|w_N$. It can be seen that $g|\gamma = \overline{\psi(d)}g$, so $g \in S_k(\Gamma_0(N),\overline{\psi})$. So we again define $L(s,g,\overline{\chi}) = \sum \overline{\chi(n)B(n)}n^{-s}$.

We similarly define $\Lambda(s,f,\chi) = (2\pi)^{-s}\Gamma(s)L(s,f,\chi)$ and $\Lambda(s,g,\overline{\chi}) = (2\pi)^{-s}\Gamma(s)L(s,g,\overline{\chi})$.

Proposition 6.5.1. *There is a functional equation*

$$\Lambda(s,f,\chi) = i^k\chi(N)\psi(D)\frac{\tau(\chi)^2}{D}(D^2N)^{-s+k/2}\Lambda(k-s,g,\overline{\chi}).$$

where $\tau(\chi) = \sum_{n \pmod N} \chi(n)e^{2\pi i n/N}$ *is the Gauss Sum.*

Proof. See section.1.5 [p.59, ch.1, DB97]. □

The final theorem of this chapter due to Weil, shows that is sufficiently many twisted functional equations exists for f, then f is a modular form in $M_k(\Gamma_0(N),\psi)$.

Theorem 6.5.2. *(Weil) Suppose ψ is a Dirichlet character modulo N a positive integer. Suppose $A(n)$, $B(n)$ satisfy $|A(n)|$, $|B(n)| = O(n^K)$ for sufficiently large real number K. If N,D are relatively prime and χ is a primitive Dirichlet character modulo D, let $L_1(s,\chi) = \sum \chi(n)A(n)n[-s]$, $L_2(s,\overline{\chi}) = \sum \overline{\chi(n)}B(n)n^{-s}$, define $\Lambda_1(s,\chi) = (2\pi)^{-s}\Gamma(s)L_1(s,\chi)$ and $\Lambda_2(s,\overline{\chi}) = (2\pi)^{-s}L_2(s,\overline{\chi})$.*
Let S be a finite set of primes, including the ones diving N. Assume that whenever the conductor D of χ is either 1 or a prime, $\Lambda_1(s\chi)$ and $\Lambda_2(s,\overline{\chi})$ originally defined re(s) sufficiently large, have analytic continuation to all s, are bounded on all vertical strip $\sigma_1 \le$ re$(s) \le \sigma_2$, and satisfy functional equation:

$$\Lambda_1(s,\chi) = i^k\chi(N)\psi(D)\frac{\tau(\chi)^2}{D}(D^2N)^{-s+k/2}\Lambda_2(k-s,\overline{\chi}).$$

Then $f(z) = \sum A(n)q^n$ *is a modular form in* $M_k(\Gamma_0(N), \psi)$.

Proof. See [p.61, ch.1, DB97]. □

6.6 The Rankin-Selberg Method

We follow the reference [p.65, ch.1, DB97], section. 1.6. The Rankin-Selberg method is a powerful way of proving functional equations of sufficiently many L-functions attached to an automorphic form, which proves the existence of the automorphic form. For the sake of brevity, we shall outline the basic ideas and state the main results. Define automorphic form for $SL(2, \mathbb{Z})$

$$E(z, s) = \pi^{-s}\Gamma(s)\frac{1}{2}\sum \frac{y^s}{|mz+n|^{2s}}$$

It is convergent for $\mathrm{re}(s) > 1$ and $E(\gamma z, s) = E(z, s)$ for all $\gamma \in \Gamma(1)$.

Let ϕ be an automorphic function for $\Gamma(1)$, i.e. $\phi(\gamma z) = \phi(z)$ for all $\gamma \in \Gamma(1)$. Suppose $\phi(x + iy) = O(y^{-N})$ for $N > 0$ as $y \to \infty$. Because $\phi(z+1)\phi(z)$, it has a Fourier expansion $\sum_{-\infty}^{\infty} \phi_n(y)e^{2\pi inx}$, where $\phi_n(y) = \int_0^1 \phi(x+iy)e^{-2\pi inx}dx$. Let ϕ_0 be the constant term. Define

$$M(s, \phi_0) = \int_0^{\infty} \phi_0(y)y^s\frac{dy}{y}$$

the Mellin transform of ϕ_0. Let $\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s)M(s-1, \phi_0)$.

Proposition 6.6.1. *With the above hypotheses, we have*

$$\Lambda(s) = \int_{\Gamma(1)\backslash\mathcal{H}} E(z, s)\phi(z)\frac{dxdy}{y^2}.$$

Lambda(s) *so defined has meromorphic continuation to all* s *wit hat most simple poles at* $s = 0$ *and* $s = 1$*. The residue of* $\Lambda(s)$ *at* $s = 1$ *is* $\frac{1}{2}\int_{\Gamma(1)\backslash\mathcal{H}} \phi(z)\frac{dxdy}{y^2}$.

Proof. See [p.70, ch.1, DB97]. □

We end this section with an important application of this method. If $f(z) = \sum A(n)q^n$ and $g(z) = \sum B(n)q^n$ are modular forms, then $\sum A(n)B(n)n^{-s}$ has an analytic continuation to all s. We need to define the following:

$L(s, f \times g) = \zeta(2s-2k+2)\sum_{n=1}^{\infty} A(n)B(n)n^{-s}$ and $\Lambda(s, f \times g) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+1)L(s, f \times g)$.

Theorem 6.6.2. *With notation as above and $\Lambda(s, f \times g)$ originally defined for re(s) sufficiently large, it has a meromorphic continuation to all s. It is holomorphic everywhere except possible poles at $s = k$ and $s = k-1$. We get functional equation*

$$\Lambda(s, f \times g) = \Lambda(2k-1-s, f \times g).$$

The residue at $s = k$ is $\frac{1}{2}\pi^{1-k}\langle f, g\rangle$, the Petersson Inner product defined over $\Gamma(1)\backslash\mathcal{H}$.

Proof. See [p.72, ch.1, DB97]. Rough outline:

1. Let $\phi(z) = f(z)\overline{g(z)}y^k$, this is an automorphic form
2. At least one of f or g needs to be a cusp form
3. f, g should be Hecke eigenforms
4. Find $\phi_0(y)$
5. Define $M(s, \phi_0)$ the Mellin transform
6. Define $\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s)M(s-1, \phi_0)$ and derive the functional equation.

□

6.7 Classical Automorphic Forms and Representations

This section follows section 3.2 of [p.278, DB97].

Let $G = GL(2, \mathbb{R})^+$, it acts on $C^\infty(G)$ by right-translation, $\phi(g)F(x) = F(xg)$. Let Γ be a discrete subgroup of G such that $\Gamma\backslash G$ has finite volume. There is an action of the Lie Algebra of G which are 2 matrices with Lie Bracket $[X, Y] = XY - YX$. The action is

$$X(F(g)) = \frac{d}{dt}F(ge^{tX})|_{t=0}$$

for $F \in C^\infty(G)$. Let $K = SO(2)$. We say such an F is K-finite, if $\rho(k)F$ for all $k \in K$ forms a finite dimensional vector space. Let $C(\Gamma\backslash G, \chi, \omega)$ be space of continuous functions $F : G \to \mathbb{C}$ such that

1. $F(\gamma g) = \chi(\gamma)F(g)$ for all $\gamma \in \Gamma$, $g \in G$.
2. $F(zg) = \omega(z)F(g)$ where $z \in Z(\mathbb{R})$, which is the center of G, consisting of scalar matrices, $g \in G$.

Let $A(\Gamma, \chi, \omega)$ be the set of functions that are K-finite and Z-finite, where Z is the center of the universa enveloping algebra containing the Lie algebra of G, finiteness defined in a similar manner to K-finiteness, also they satisfy $|F(g)| < C||g||^N$ for some constant C, N where $||g||$ is the length of the vector $(g, \det(g)^{-1})$ calculated in $\mathbb{R}^5$. A cusp form of $A(\Gamma, \chi, \omega)$, also satisfies the following, if $a = \infty$, then if either $\chi(\tau_r) \neq 1$ where $\tau_r \in \Gamma$ is the generator of the stabilier of ∞ inside Γ, although it is not necessary for it to be a generator, or $\int_0^r F((\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix})g)dx = 0$. If $a \neq \infty$, then we use conjugation to similarly define cuspidality.

Definition 6.7.1. *Let $g \in G$. Define a slash operator on f a function on the upper-half plane:*

$$f|_z kg = \big(\frac{c\bar{z} + d}{|cz + d|}\big)^k f\big(\frac{az + b}{cz + d}\big)$$

for $g = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in G$.

A Maass form is not holomorphic, but rather is an eigenform of the non-Eucidean Laplacian operator

$$\Delta_k = -y^2\big(\delta^2/\delta x^2 + \delta^2/\delta y^2\big) + iky\delta/\delta x$$

this operator commutes with the action given in the definition above.

The space $L^2(\Gamma\backslash G, \chi, \omega)$ is the space of square integrable functions satisfying the conditions 1 and 2 above, and $L^2_0(\Gamma\backslash G, \chi, \omega)$ is the subspace of cusp forms, cuspidality defined in the same sense as $A(\Gamma, \chi, \omega)$, but the sense of equality being almost everywhere.

Let $\phi \in C_c^\infty(G)$. If $f \in L^2(\Gamma\backslash G, \chi, \omega)$, we define

$$(\rho(\phi)f)g = \int_G f(gh)\phi(h)dh.$$

Definition 6.7.2. *Siegel sets are nicely shaped substitutes for the fundamental domains.*

1. $\mathcal{F}_{c,d}$ *is the set of* $z = x + iy \in \mathcal{H}$ *such that* $0 \leq x \leq d$ *and* $y \geq c$.

2. $\mathcal{F}_d^{\infty}$ *is the set of points on the upper half plane with* $0 \leq x \leq d$ *and no condition on* y.

Proposition 6.7.1. *(Gelfand, Garev and Piatetski-Shapiro) Let* $\phi \in C_c^{\infty}(G)$.

1. *There exists a constants depending on* ϕ*, such that for all* $f \in L_0^2(\Gamma\backslash G, \chi, \omega)$*, we have* $sup_{g\in G}|\rho(\phi)f(g)| \leq C||f||_2$*, where* $||\cdot||_2$ *is the* L^2 *norm.*

2. *The restriction of the operator* $\rho(\phi)$ *to* $L_0^2(\Gamma\backslash G, \chi, \omega)$ *is a compact operator.*

Theorem 6.7.2. *The space* $L_0^2(\Gamma\backslash G, \chi, \omega)$ *decomposes into direct sum of Hilbert subspaces which are invariant under the right regular representation* ρ*. Let* H *be such a subspace, then the* K*-finite vectors in* H *are dense and every* K*-finite vector is automatically an element of* $C^{\infty}(\Gamma\backslash G, \chi, \omega)$*. The* K*-finite vectors form an irreducible admissible* $(\mathfrak{g}, K)$ *module contained in* $A_0(\Gamma\backslash G, \chi, \omega)$*.(*$\mathfrak{g}$ *is the Lie algebra associated to* G*).*

Proof. See [p.289, ch.3, DB97]. □

Theorem 6.7.3. 1. *Let* (π, V) *be an irreducible admissible unitary representation of* $GL(2, \mathbb{R})$*. Then the multiplicity of* π *in the decomposition of* $L_0^2(\Gamma\backslash G, \chi, \omega)$ *is finite.*

2. *Let* $\lambda \in \mathbb{C}$ *and let* σ *be a character of* K*. Then* $A_0(\Gamma, \chi, \omega, \lambda, \rho)$ *is finite dimensional.*

Proof. See [p.290, ch.3, DB97]. □

6.8 Automorphic Representations of $GL(n)$

The reference for the discussion in this section is [p.291, ch.3, DB97], section.3.3. We shall also follow the notation of [DB97] for less confusion.

Let A be the adele ring of a number field F. A_f denote the ring of finite adeles (only containing the non-archimedean local factors). An Automorphic representation of $GL(n, A)$ is rather a representation of $GL(n, A_f)$, which is simultaneously a $(\mathfrak{g}_{\infty}, K_{\infty})$ module, where $\mathfrak{g}_{\infty} = \prod_{v \text{ Archimedean}} \mathfrak{gl}(n, F_v)$ and K_{∞} is similarly defined.

The group $GL(n, A)$ is the restricted direct product of the groups $GL(n, F_v)$ w.r.t. the maximal compact subgroups $GL(n, O_v)$ for non-archimedean places and for the archimedean places we again take the maximal compact subgroups. $GL(n, A)$ is unimodular, i.e. the left and right Haar measures coincide. Note that we assume $n = 2$ and $A = \mathbb{Q}$ for the proofs but state them for the most general case possible.

Theorem 6.8.1. *Let F be an algebraic number field.*

1. *$SL(n, F_\infty)SL(n, F)$ is dense in $SL(n, A)$.*
2. *Let K_0 be an open compact subgroup of $GL(n, A_f)$. Assume that the image of K_0 in $A_f^\times$ under the determinant map is $\prod_{v \notin S_\infty} O_v^\times$. Then the cardinality of*

$$GL(n, F)GL(n, F_\infty)\backslash GL(n, A)/K_0$$

is equal to the class number of F (which is the cardinality of the class group of F, order of the group generated by fractional ideals, called the class group of F and S_∞ is the set containing all the non-archimedean places.)

Proof. See [p.294, ch.3, DB97]. □

Proposition 6.8.2. *Suppose that A is the adele ring of $\mathbb{Q}$. The inclusion $SL(2, \mathbb{R}) \to GL(2, A)$ induces a homomorphism*

$$\Gamma_0(N)\backslash SL(2, \mathbb{R}) \cong Z(A)GL(2, \mathbb{Q})\backslash GL(2, A)/K_0(N)$$

($Z(A)$ is the center of $GL(n, A)$).

Proof. See [p.294, ch.3, DB97]. □

In light of the previous theorem, if we have a function f defined on the upper-half plane satisfying certain properties, then one can pull it back via this isomorphism and approach it from the adelic point of view. Hence we have the following result:

Proposition 6.8.3. *(Gelfand, Garev and Piatetski-Shapiro) Let $\phi \in C_c^\infty(GL(n, A))$.*

1. *There exists constant $C > 0$, depending on ϕ such that*

$$sup_{g \in GL(2,A)} |\rho(\phi)f(g)| \leq C||f||_2.$$

for all $f \in L_0^2(GL(2,F)\backslash GL(2,A),\omega)$.

2. *The operator* $\rho(\phi)$ *is compact on* $L_0^2(GL(n.F)\backslash GL(n,A),\omega)$.

Proof. See [p.297, ch.3, DB97]. □

We end this section and the thesis with the following theorem:

Theorem 6.8.4. *The space* $L_0^2(G(n,F)\backslash GL(n,A),\omega)$ *decomposes into a Hilbert space direct sum of irreducible invariant subspaces.*

Proof. We assume again $n = 2$, $F = \mathbb{Q}$. The proof is also similar to theorem 6.7.2. See [p.299, ch.3, BD97] for the theorem. □

Bibliography

[RV99] Ramakrishnan, Dinakar; Valenza, Robert J., Fourier Analysis on Number Fields, Springer-Verlag, New York, Inc., 1999, ISBN: 978-81-8128-515-7.

[JK99] Neukirch, Jurgen, Algebraic Number Theory, Springer-Verlag, Berlin, 1999, ISBN: 3-540-65399-6.

[DB97] Bump, Daniel, Automorphic Forms and Representations, Cambridge University Press, Cambridge, 1997, ISBN: 0-521-65818-7.