Topology Of Complex Projective Varieties

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Certificate

This is to certify that this dissertation entitled Topology Of Complex Projective Varieties towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Vishwajeet S Bhoite at Indian Institute of Science Education and Research under the supervision of Prof. A J Parameswaran, Senior Professor, Department of Mathematics , during the academic year 2018-2019.

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This thesis is dedicated to my parents and my brother

Declaration

I hereby declare that the matter embodied in the report entitled Topology Of Complex Projective Varieties are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Prof. A J Parameswaran and the same has not been submitted elsewhere for any other degree.

Vishwajeet S Bhoite

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Abstract

In this project, I studied some of the interesting results about the topology of complex projective varieties. The project is based on the paper of Klaus Lamotke, titled "The Topology of Complex Projective Varieties After S. Lefschetz." Starting with Lefschetz Pencils, Dual Varieties this thesis covers deep results such as Lefschetz Hyperplane Section theorem, Weak Lefschetz theorem, and Hard Lefschetz Theorem. Along the way, it gives the proof of Lefschetz Hyperplane Section Theorem using Morse Theory, Picard-Lefschetz formula, and Monodromy theorem. Towards the end, we study topology in a neighborhood of a singular point on the complex hypersurfaces.

Contents

Introduction

The topology of complex surfaces was introduced by Émile Picard(1897) and extended to higher dimensions by Lefschetz (1924). During Lefschetz's time, the knowledge of topology was primitive, and Lefschetz's often appeals to geometric intuition where one would like to see more precise arguments. In [1], Klaus Lamotke presents Lefschetz's study more rigorously using the modern language of topology. Deligne and Katz (1973) have extended Picard–Lefschetz theory to varieties over more general fields. The main goal of this project is to study [1].

The required prerequisites in algebraic geometry can be found in the first two chapters of Shafarevich's book [5]. The main tool from differential topology is Ehresmann's fibration theorem. The prerequisites from algebraic topology the reader can refer to [2]. In chapter 1 we collect the prerequisites, most of the results are stated without proof. In chapter 2 we see Lefschetz's Pencil and dual Varieties. Lefschetz results and Weak Lefschetz Theorem are described in chapter 3, Equivalent statements of Hard Lefschetz Theorem are discussed in chapter 4. The Picard-Lefschetz formulas and the Monodromy Theorem are discussed in chapter 5 and chapter 6. The last chapter studies the topology in a neighborhood of a singular point of complex hypersurface.

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Chapter 1

Preliminary Definitions and Tools

The main aim of this chapter is to introduce preliminary definitions and tools which will be used through out the report. Most of the theorems presented in this chapter are without proof. In the following section we introduce some basic properties of algebraic sets, then we introduce complex manifolds and to the end of the chapter we state some tools from Algebraic topology which will be useful in later chapters.

1.1 Affine and Projective Varieties

Let k be an algebraically closed field. We define affine n-space over k to be the set $\mathbb{A}^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in k\}$. Let $A = k[x_1, x_2, \ldots, x_n]$ be the ring of polynomials in n-variables over k. Let $T \subset A$, by *zero set* of T we mean $Z(T) = \{P \in \mathbb{A}^n \mid f(P) = 0 \; for \; all \; f \in T\}$. We define *projective* N-space over k to be the set $\mathbb{P}_N = \mathbb{A}^{N+1} \setminus \{0\}/\sim$ where $(a_0, a_1, \ldots, a_n) \sim (\lambda a_0, \lambda a_1, \ldots, \lambda a_n)$ for all $\lambda \in k$, $\lambda \neq 0$. If T is any set of homogeneous polynomial in $S = k[x_0, \ldots, x_N]$ then the zero set of T is $Z(T) = \{P \in \mathbb{P}_N \mid f(P) = 0 \; for \; all \; f \in T\}$. The dual projective space $\check{\mathbb{P}}_N$ of $\mathbb{P}_N$ is the set of hyperplanes in $\mathbb{P}_N$. If $\mathbb{P}_N = \mathbb{P}(V)$ is a projective space associated to vector space V then $\check{\mathbb{P}}_N$ is the projective space associated to its dual vector space $\check{V}$.

Definition 1.1.1. A subset Y of $\mathbb{A}^n$ (*respectively* $\mathbb{P}_N$) is called *algebraic set*, if $Y = Z(T)$ for some $T \subset A$ (*respectively T is subset of homogeneous polynomials of S*). The set of all *algebraic sets* of $\mathbb{A}^n$ (*respectively* $\mathbb{P}_N$) form the closed sets of a topology on $\mathbb{A}^n$ (*respectively* $\mathbb{P}_N$), called the *Zariski topology.*

Definition 1.1.2. Let X be a topological space, a nonempty subset Y of X is called irreducible if it cannot be written as the union of two proper closed subsets of X.

We now define affine variety and projective variety.

Definition 1.1.3. An $affine\ variety$ (respectively *projective variety*) is an irreducible Zariski-closed subset of $\mathbb{A}^n$ (*respectively* $\mathbb{P}_N$) in the *Zariski topology*. An open subset of affine variety (respectively projective variety) is called $quasi-affine\ variety$ (respectively $quasi-projective\ variety$.)

To each subset Y of $\mathbb{A}^n$ (*respectively* $\mathbb{P}_N$) we assign an ideal (respectively homogeneous ideal) in A (*respectlively* S) called the ideal of Y given by $I(Y) = \{f \in A \mid f(P) = 0\ for\ all\ P \in Y\}$ (*respectively* $\{f \in S \mid f\ is\ homogeneous\ f(P) = 0\ for\ all\ P \in Y\}$). If Y is an affine (respectively projective) algebraic set then the ring $A(Y) = A/I(Y)$ (respectively $S(Y) = S/I(Y)$) is called the affine (respectively homogeneous) coordinate ring of Y.

Definition 1.1.4. If X is an algebraic set, we define the dimension of X to be $dim\ X = sup\{n \mid Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n,\ where\ each\ Z_i\ is\ irreducible\ Zariski-closed\ subset\ of\ X\}$.

We make a note of the fact that if Y is affine algebraic set the $dim\ (Y) = dimA(Y)$. And if Y is a projective variety with homogeneous coordinate ring $S(Y)$ then $dimY = dimS(Y) - 1$.

Having defined the objects in the category of varieties we now define the morphisms in the category.

Definition 1.1.5. Let Y be an quasi affine variety in $\mathbb{A}^n$. A function $f : Y \to k$ is regular at point $p \in Y$ is there is an open neighborhood U of p in Y such that $f = g/h$, where $g, h \in A$ such that h is nowhere zero on U.

Definition 1.1.6. Let Y be an quasi projective variety in $\mathbb{P}_N$. A function $f : Y \to k$ is regular at point $p \in Y$ if there is an open neighborhood U of p in Y such that $f = g/h$, where $g, h \in S$ are homogeneous polynomials of same degree such that h is nowhere zero on U.

A regular function is continuous, when k is identified with $\mathbb{A}^1$ with Zariski topology. We now define the category of varieties. By a variety we mean affine, quasi-affine, projective, or quasi-projective variety. We denote by $\mathcal{O}(Y)$, the ring of regular functions on Y. If $p \in Y$ we define the local ring of p in Y, $\mathcal{O}_p = \{(U, f) \mid U \textit{ is open neighborhood of } p \textit{ in } Y \textit{ and } f \in \mathcal{O}(U)\}/\sim$ where $(U, f) \sim (V, g)$ if $f = g$ on $U \cap V$.

Definition 1.1.7. If X and Y are two varieties, a morphism $\varphi : X \to Y$ is a continuous map such that for every open set V of Y if $f \in \mathcal{O}(V)$ then $f \circ \varphi \in \mathcal{O}(\varphi^{-1}(V))$.

Let $X \subset \mathbb{A}^N$ and $Y \subset \mathbb{A}^m$ be affine varieties. Then the product of X and Y in the category of varieties is $X \times Y \subset \mathbb{A}^{n+m}$ with the induced topology. The topology is not equal to product topology on general. We have $A(X \times Y) = A(X) \otimes_k A(Y)$ and $dim(X \times Y) = dim(X) + dim(Y)$. To define the product of projective spaces $\mathbb{P}_r$ and $\mathbb{P}_s$ we define a map $\psi : \mathbb{P}_r \times \mathbb{P}_s \to \mathbb{P}_N$ sending $(a_o, \ldots, a_i, \ldots, a_r) \times (b_0, \ldots, b_j, \ldots, b_s) \to (\ldots, a_i b_j, \ldots)$ where $N = rs + r + s$. ψ is an embedding and image of ψ is a subvariety of $\mathbb{P}_N$. If $X \subset \mathbb{P}_r$ and $\mathbb{P}_s$ are quasi-projective varieties then $X \times Y \subset \mathbb{P}_r \times \mathbb{P}_s$ is product of X and Y where we identify $X \times Y$ and $\mathbb{P}_r \times \mathbb{P}_s$ with image of ψ in $\mathbb{P}_N$. If X and Y are both projective then $X \times Y$ is projective.

We quote the following results without proof, the reader is referred to ch 1 of [5]

Theorem 1.1.1. *A subset $X \subset \mathbb{P}_n \times \mathbb{P}_m$ is Zariski-closed if and only if it is given by a system of equations $G_i(u_0, \ldots, u_n, v_0, \ldots, v_m) = 0$ $(i = 1, \ldots, t)$, homogeneous in each system of variables v_i and u_j separately. Every Zariski-closed subset of $\mathbb{P}_n \times \mathbb{A}^m$ is given by a system of equations $g_i(u_0, \ldots, u_n, y_1, \ldots, y_m) = 0 (i = 1, 2, ..t)$, homogeneous in each variables $u_0, \ldots, u_n$.*

Theorem 1.1.2. *The image of projective variety under a regular map is Zariski-closed.*

Theorem 1.1.3. *If $X \subset \mathbb{P}_N$ is a quasiprojective irreducible n-dimensional variety and Y is the zero set of m homogeneous polynomials on X and is not empty, then each of its components is of dimension atleast $n - m$.*

Theorem 1.1.4. *If $f : X \to Y$ is a regular mapping of irreducible varieties, $f(X) = Y$, $dimX = n$, $dimY = m$, then $m \leq n$ and*

1. *$dimf^{-1}(y) \geq n - m$ for every point $y \in Y$;*

2. there exists a non-empty open set $U \subset Y$ such that $dim f^{-1}(y) = n - m$ for $y \in U$.

Theorem 1.1.5. *If $f : X \to Y$ is a regular mapping of projective varieties, $f(X) = Y$, and if Y is irreducible, and all the fibres $f^{-1}(y)$ are irreducible and are of the same dimension, then X is irreducible.*

We now define singular and nonsingular points on a variety.

Definition 1.1.8. Let $Y \subset \mathbb{A}^n$ be an affine variety, and let $f_1, \ldots, f_t \in A = K[x_1, \ldots, x_n]$ be the set of generators for the ideal of Y. Y is *nonsingular at a point* $P \in Y$ if the rank of the martix $[(\partial f_i/\partial x_j)(P)]$ is $n - r$, where r is the dimension of Y. Y is *nonsingular* if it is nonsingular at everypoint.

A noetherian local ring R with maximal ideal m and residue field $k = A/m$, is regular local ring if $dim_k m/m^2 = dim A$.

Theorem 1.1.6. *Let $Y \subset \mathbb{A}^n$ be an affine variety. Let $P \in Y$ be a point. Then Y is nonsingular at P if and only if the local ring $\mathcal{O}_p$ is a regular local ring.*

Definition 1.1.9. Let Y be any variety. Y is nonsingular at a point $P \in Y$ if the local ring $\mathcal{O}_p$ is a regular local ring. Y is *nonsingular* if it is nonsingular at every point. Y is singular if it is not nonsingular.

Theorem 1.1.7. *Let Y be a variety. Then the set of singular points of Y is a proper Zariski-closed subset of Y.*

The tangent space Θ_x at a point $x \in X$ of a variety is defined as $(m_x/m_x^2)^*$, where m_x is the maximal ideal of the local ring $\mathcal{O}_x$ at x. Thus $x \in X$ is non-singular if $dim\Theta_x = dim X$.

Definition 1.1.10. Subvarieties $Y_1, \ldots, Y_r$ of a nonsingular variety X are transversal at a point $x \in \cap_{i=1}^r Y_i$ if

$$codim_{\Theta_{X,x}}(\cap_{i=1}^r \Theta_{Y_i,x}) = \sum_{i=1}^r codim_X(Y_i).$$

Remark. If subvarieties $Y_1, \ldots, Y_r$ of nonsingular variety X are transversal at x then $dim\Theta_{Y_i,x} = dim Y_i$, and $dim\Theta_{(\cap_{i=1}^r Y_i, x)} = dim \cap_{i=1}^r Y_i$, i.e. x is nonsingular point of each Y_i and $\cap_{i=1}^r Y_i$.

1.2 Some preliminaries about complex manifolds.

Definition 1.2.1. A function $f : \mathbb{C}^n \to \mathbb{C}$, is holomorphic at $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ if f has a power series expansion in some open neighborhood U of a given by

$$f(z) = \sum_{k_1,\ldots,k_n=0}^{\infty} a_{k_1 k_2 \ldots k_n}(z_1 - a_1)^{k_1} \ldots (z_n - a_n)^{k_n}.$$

In particular f is hohomorphic if it is holomorphic in each variable. A map $F = (F_1, \ldots, F_m) : V \to \mathbb{C}^m$, $V \subset \mathbb{C}^n$ is holomorphic if each F_i is holomorphic.

Definition 1.2.2. A function F is a *biholomorphic* on $W \subset \mathbb{C}^n$ if there exists an holomorphic inverse $G : F(W) \to W$.

Theorem 1.2.1. ***Inverse function theorem.*** *Let U, V be open sets of $\mathbb{C}^n$ and $f : U \to V$ a holomorphic function. Suppose that $z_0 \in U$ is such that $det\mathcal{J}_{\mathbb{C}}(f)(z_0) \neq 0$. Then there exists an open subset U' containing z_0 such that $f|_{U'} : U' \to f(U')$ is a biholomorphism.*

Proof. We give the sketch of proof. The real Jacobian of f has rank $2n$, since $det\mathcal{J}_{\mathbb{R}}(f) = |det\mathcal{J}_{\mathbb{C}}(f)|^2 \neq 0$. So by the real inverse function theorem, there is a local smooth inverse g. g is the required holomorphic inverse. □

Now let X be a topological manifold of dimension $2n$. A *local complex chart* (U, z) on X is an open subset $U \subset X$ and an homeomorphism $z : U \to V := z(U) \subset \mathbb{C}^n (\equiv \mathbb{R}^{2n})$. Two complex charts (U_a, z_a), (U_b, z_b) are compatible if the transition map $z_b \circ z_a^{-1} : z_a(U_a \cap U_b) \to z_b(U_a \cap U_b)$ is holomorphic. A *holomorphic atlas* of X is a collection $A = \{(U_a, Z_a)\}$ of local complex charts such that $\cup_a U_a = X$ and such that all transition maps are biholomorphic. A *complex analytic structure* on X is a maximal holomorphic atlas.

Definition 1.2.3. A *complex manifold* is a topological manifold together with a complex analytic structure.

Definition 1.2.4. A function $f : X \to \mathbb{C}$ on a complex manifold X is holomorphic if for all complex charts (U, φ) the map $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is holomorphic.

Definition 1.2.5. A map $f : X \to Y$ between complex manifolds is holomorphic if for all complex charts (U, φ) of X and (V, ψ) of Y, $\psi \circ \varphi^{-1}$ is holomorphic.

We now see that our main object of study is an smooth manifold which follows from the following proposition.

Proposition 1.1. Let $X \subset \mathbb{P}^N$ be a non-singular complex projective variety of dimension n, then X is a complex manifold of dimension n in the usual topology.

Proof. Consider any affine open cover $X = \cup U_i$. We show that each U_i is a complex manifold, which will follow from following lemma. $\square$

Lemma 1.2.2. *If Y is a nonsingular affine complex variety of dimension d in $\mathbb{A}^N$. Then Y is a complex manifold of dimension d and hence a smooth manifold of dimension $2d$.*

Proof. Let the ideal of X be generated by $f_1, f_2, \dots, f_t$. Let $p = (a, b) \in Y \subset \mathbb{A}^N$, where $a \in \mathbb{C}^d$ *and* $b \in \mathbb{C}^{N-d}$ since p is non singular rank of the martix $[(\partial f_i/\partial x_j)(p)]$ is $k = N - d$, where d is the dimension of Y. WLOG we assume that $det[(\partial f_i/\partial x_j)(p)]_{1\leq i,j\leq k} \neq 0$. Then by implicit function theorem there exists open neighborhood $U_1 \in \mathbb{C}^d$ containing a and $U_2 \in \mathbb{C}^k$ containing b and a holomorphic map $g : U_1 \to U_2$ such that for $(x_1, x_2) \in \mathbb{C}^d \times \mathbb{C}^k$ $f_i(x_1, x_2) = 0$ for $1 \leq i \leq k$ if and only if $x_2 = g(x_1)$. Now $p \in U_1 \times U_2 =: U$ Then take the chart around p to be $(U \cap X, \varphi)$ where $\varphi(x_1, x_2) = x_1$ be the projection on first d *coordinates.* $\square$

Consider a holomorphic function $f : U \to \mathbb{C}$, $U \subset \mathbb{C}^n$, $0 \in U$. The point 0 is critical if $Df(0) = 0$. It is non-degenerate if the Hessian matrix $D^2f(0) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right]$ is non-singular.

Theorem 1.2.3. ***Morse lemma***. *If 0 is a non-degenerate critical point of the function f, then there exists a holomorphic change of variables $x = \varphi(y),\ y = (y_1, .., y_n)$, $\varphi(0) = 0$ such that $f(\varphi(y)) = f(0) + y_1^2 + \dots + y_n^2$.*

1.3 Tools from algebraic topology.

We will be using singular homology with coefficients from a PID. In this section we will be stating the results from algebraic topology without proof,(see [2] for proofs)

Theorem 1.3.1. *Let $f : (X, A) \to (Y, B)$ be a continuous mapping between pairs of compact Euclidean neighborhood retracts (ENR), such that $f : X \setminus A \to Y \setminus B$ is a homeomorphism.*

Then f induces an isomorphism

$$f_* : H_*(X, A) \to H_*(Y, B)$$

of the relative singular homology.

We will be frequently using the homology groups of $\mathbb{CP}^n$, here we state them.

Theorem 1.3.2. *For coefficients in an PID, R, we have,*

$$H_p(\mathbb{CP}^n) = \begin{cases} R, & if\ p\ even,\ 0 \leq p \leq 2n \\ 0, & otherwise. \end{cases}$$

We now state the Universal coefficient theorem(UCT) for cohomology. Let G be a fixed abelian group. Consider a free resolution F of a abelian group H given by chain $0 \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \longrightarrow 0$, with $F_i = 0,\ i > 1$. Here F_0 is a free abelian group with elements of H as the generators and f_0 is the surjective map taking each generator to itself, and $F_1 = ker(f_0)$ and f_1 is inclusion. Take the dual cochain complex $Hom(F, G)$ and denote the n^{th} by $H^n(F, G)$. We define $Ext(H, G) := H^1(F, G)$.

Theorem 1.3.3. ***UCT*** *If a chain complex C of free abelian groups has homology groups $H_n(C)$ for each n, there is a split exact sequence*

$$0 \to Ext(H_{n-1}C, G) \to H^n(C, G) \xrightarrow{h} Hom(H_n(C), G) \to 0.$$

If H is a free abelian group we have $Ext(H, G) = 0$. So if $H_{n-1}(C)$ is a free abelian group then $H^n(C, G) \equiv Hom(H_n(C), G)$.

Instead of applying Hom to F we apply $\otimes$ to $0 \to F_1 \to F_0 \to 0$, to get the chain $F \otimes G$: $0 \to F_1 \otimes G \to F_0 \otimes G \to 0$. We denote $H_n(F \otimes G)$ by $Tor_n(F, G)$, and define $Tor(F, G) := Tor_1(F, G)$

Theorem 1.3.4. ***Künneth formula.*** *For a free chain complex C(i.e. each C_i is free) and* *an arbitrary chain complex D, there is a natural short exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \to H_n(C \otimes D) \to \bigoplus_{p+q=n-1} Tor(H_p(C), H_q(D) \to 0.$$

and this sequence splits.

Theorem 1.3.5. ***The topological Künneth formula.*** *If (X, A) and (Y, B) are CW pairs and R is a principal ideal domain, then there are natural short exact sequences*

$$0 \to \bigoplus_i (H_i(X, A, R) \otimes_R H_{n-i}(Y, B; R)) \to H_n(X \times Y, A \times Y \cup X \times B; R) \to$$
$$\bigoplus_i Tor_R(H_i(X, A; R), H_{n-i-1}(Y, B; R)) \to 0.$$

Theorem 1.3.6. ***Poincare Duality*** *If M is a compact $R-orientable$ $n-manifold$ with fundamental class $[M] \in H_n(M; R)$, then the map $D : H^k(M, R) \to H_{n-k}(M, R)$ defined by $D(\alpha) = [M] \cap \alpha$ is an isomorphism.*

Theorem 1.3.7. ***Ehresmann's fibration theorem*** *Let $f : E \to B$ be a proper submersion (i.e. the differential is a surjective linear map at all points of E). Then f is locally trivial fibration.*

Theorem 1.3.8. *Let M be a closed connected orientable n-manifold with boundary, let $\mu \in H_n(M)$ be the orientation. If coefficients from field are taken then the intersection form $\langle , \rangle : H^k(M; F) \times H^{n-k}(M; F) \to F$ is non-singular.*

We now state some preliminaries results which will be used in proving Picard-Lefschetz formula. Let $f : A \to B$ be a continuous mapping and $B^* \subset B$ a subspace such that f fibres $E = f^{-1}(B^*)$ locally trivially over B^*. Denote $F_y = f^{-1}(y)$ for $y \in B$ Let $w : I = [0, 1] \to B^*$ be a path from $a = w(0)$ to $b = w(1)$. The induced bundle w^*E over I is trivial, that is

$$W : F_a \times I \to E \subset A$$

with the following properties:

1. $f \circ W(x, t) = w(t)$ and $W(x, 0) = x$ for $x \in F_a$, $t \in I$.
2. let $t \in I$, $W_t : F_a \to F_{w(t)}$ given by $W_t(x) = W(x, t)$ is an homeomorphism

3. For any L with $F_a \cup F_b \subset L \subset A$ the lifting W is a mapping between pairs

$$W : F_a \times (I, \partial I) \to (A, L).$$

The path w determines W upto homotopy relative to ∂I and L and determines $W_1 : F_a \simeq F_b$ upto isotopy. Since the induced isomorphism in homology $(W_1)_*$ depends only on w, we denote

$$w_* = (W_1)_* : H_*(F_a) \simeq H_*(F_b).$$

Definition 1.3.1. If w is closed, W_1 is called a geometric monodromy and w_* is called algebraic monodromy along w.

Definition 1.3.2. Let $i \in H_1(I, \partial I)$ be a canonical generator. The map

$$\tau_w : H_q(F_a) \to H_{q+1}(F_a \times (i, \partial I)) \xrightarrow{W_*} oH_{q+1}(A, L)$$

$$\tau(x) = x \times i$$

is the extension along w.

We now list some properties of extensions

1. If $f^{-1}(image(w)) \subset L$, then we have $\tau_w = 0$.

2. Naturality. A commutative diagram

$$\begin{tikzcd} A \arrow[r, "(\Psi)"] \arrow[d, "f"] & A_1 \arrow[d, "f_1"] \\ B \arrow[r, "\varphi"] & B_1 \end{tikzcd} \tag{1.3.1}$$

with $\varphi(B^*) \subset B_1^*$ and $\varphi(L) \subset L_1$ induces a commutative diagram

$$\begin{tikzcd} H_q(F_a) \arrow[r, "(\Psi_a)_*"] \arrow[d, "\tau_w"] & H_q(F_1, \psi(a)) \arrow[d, "\tau_{\psi \circ w}"] \\ H_{q+1}(A, L) \arrow[r, "\Psi_*"] & H_{q+1}(A_1, L_1) \end{tikzcd} \tag{1.3.2}$$

3. $\partial_* : H_{q+1}(A, L) \to H_q(L)$ denotes the connecting homomorphism, then

$$(w_*(x)) - x = (-1)^q \partial_* \tau_w(x), \quad x \in H_q(F_a).$$

4. Composition. If w is a path from a to b and v is a path from b to c and if $F_a \cup F_b \cup F_c$ then $\tau_{v \circ w} = \tau_v \circ w_* + \tau_w$ and $(v\circ)_* = v_* \circ w_*$.

We will also be using relative version: Let $A' \subset A$ be a subspace, denote $E' = E \cap A'$ and $F'_y = F_y \cap A'$. Assume that f fibres the pair (E, E') locally trivially over B^* and F'_a is a strong deformation retract of A'. Then $W : (F_a, F'_a) \times (I, \partial I) = (F_a \times I, F_a \times \partial I \cup F'_a \times I) \to (A, L \cup A')$ and the relative extension is defined

$$\tau_w : H_q(F, F'_a) \to H_{q+1}((F_a, F'_a) \times (I, \partial I)) \xrightarrow{W_*} H_{q+1}(A, L \cup A')$$

$$\tau(x) = x \times i$$

The list of properties mentioned above are also true for the relative case.

Chapter 2

The modification of a projective variety with respect to a pencil of hyperplanes.

2.1 Pencil of hyperplanes and the Veronese embedding.

Let $\mathbb{P}_N$ denote $N-$*dimensional* complex projective space and $\check{\mathbb{P}}_N$ be its dual projective space. A line G in $\check{\mathbb{P}}_N$ is called a pencil of hyperplanes in $\mathbb{P}_N$. We use the notation $H_y \subset \mathbb{P}_N$ if $y \in \check{\mathbb{P}}_N$. Hence a pencil is denoted by $\{H_t\}_{t\in G}$. Let α and β represent two distinct points in G, then we have $G = \{a\alpha + b\beta \mid (a,b) \in \mathbb{P}_1\}$. If $p \in H_\alpha \cap H_\beta$, then it lies in every hyperplane of the pencil. We define the axis of the pencil to be $A = H_\alpha \cap H_\beta = \cap_{t\in G} H_t$. Thus a pencil in $\mathbb{P}_N$ consists of all hyperplanes which contain a fixed $(N-2)-$dimensional projective linear subspace A.

Let $X \subset \mathbb{P}_N$ be a closed, irreducible, nonsingular variety of dimension n. We intersect the variety by a pencil $\{H_t\}_{t\in G}$ of hyperplanes,

$$X_t = X \cap H_t, \quad t \in G$$

so that we have

$$X = \cup_{t \in G} X_t$$

is the union of hyperplane sections X_t. Let $X' = X \cap A$. We observe that $X \setminus X'$ can be looked at as the fibration over G with fibres $X_t \setminus X'$. We modify X along X' to get a new variety Y and a map $f : Y \to G$ such that the fibres $f^{-1}(t)$ are the whole of hyperplane sections X_t. We define the modification

$$Y = \{(x, t) \in X \times G \mid x \in H_t\}. \tag{2.1.1}$$

We observe that $Y = \Gamma \cap (X \times G)$, where $\Gamma = \{(x, y) \mid (x, y) \in \mathbb{P}_N \times \check{\mathbb{P}}_N \text{ and } x \in H_y\}$. Γ is called the universal hyperplane. To see Y is a variety it is enough to see Γ is a variety. Let $(z_0, z_1, \ldots, z_N)$ represent the coordinates of arbitrary point in $\mathbb{C}^{N+1}$ with respect to standard basis, and let $(w_0, w_1, \ldots, w_N)$ be the coordinates of the dual of $\mathbb{C}^{N+1}$ in the dual basis. Now, let $x = (a_0, a_1, \ldots, a_N)$ and $y = (b_0, b_1, \ldots, b_N)$, we have,

$$\begin{aligned}
(x, y) \in \Gamma \; iff \;\; & x \in H_y \\
iff \;\; & y(x) = 0 \\
iff \;\; & (\sum b_i e_i *)(\sum a_i e_i) = 0 \\
iff \;\; & \sum b_i a_i = 0 \\
iff \;\; & (x, y) \in Z(z_0 w_0 + z_1 w_1 + \ldots + z_n w_n = 0)
\end{aligned}$$

by 1.1.1 Γ is a variety.

We now have two projections

$$X \xleftarrow{p} Y \xrightarrow{f} G. \tag{2.1.2}$$

Let $Y' := p^{-1}(X') = X' \times G$. The complement is mapped isomorphically

$$p : Y \setminus Y' \simeq X \setminus X',$$

and each fibre of f is isomorphic to corresponding hyperplane section,

$$p : Y_t = f^{-1}(t) \simeq X_t, \quad t \in G.$$

2.2 Veronese Embedding

We now study *Veronese embedding* of projective spaces. Originally Lefschetz studied more general linear systems of hypersurfaces of X, and not just pencils $\{X_t\}_{t\in G}$ of hyperplane sections. Veronese embedding justifies that restricting to hyperplane sections does not diminish any generality.

Let $S = \{x_0^{i_1} \dots x_N^{i_N} \mid i_0 + i_1 + \dots + i_N = d\}$ be the set of monomials of degree d in $N+1$ variables. The number of elements in S is equal to $\binom{N+d}{d}$. Thus the set of all homogeneous polynomials of degree d is a vector space of dimension equal to $|S|$. Let $M = |S|-1$. Consider $\mathbb{P}_M$ whose homogeneous coordinates are represented by $v_{i_0\dots i_N}$ such that $i_0 + i_1 + \dots + i_N = d$ and $i_j \geq 0$. The Veronese embedding of degree d is defined to be $v_d : \mathbb{P}_N \to \mathbb{P}_M$ by $(x_0, \dots, x_N) \mapsto (\dots, v_{i_0 \dots, i_N}, \dots)$ where $v_{i_0\dots,i_N} = x_0^{i_1} \dots, x_N^{i_N}$.

We now see that $V_d(\mathbb{P}_N) = Z\{v_{i_0\dots,i_N} v_{j_0\dots,j_N} - v_{k_0\dots,k_N} v_{l_0\dots l_N} = 0 \mid i_0+j_0 = k_0+l_0, \dots, i_N + j_N = k_N + L_N\}$ Clearly any element of $V_d(\mathbb{P}_N)$ satisfies all the equation in the set. To see the converse we first note that for any point in the zero set of equations atleast one of the coordinate of the form $v_{0\dots d\dots 0}$ corresponding to the monomial u_i^d is nonzero. Let U_i be the set of points such that $v_{0\dots d \dots 0} \neq 0$. On U_i we define the inverse φ_i of v_d by $\varphi_i(z) = (z_{1\dots d-1\dots 0}, \dots, z_{0\dots d \dots 0}, \dots, z_{0\dots d-1\dots 1})$ these maps agree on the intersections because v_d is injective. Hence $v_d(\mathbb{P}_N)$ is defined by the equations above and v_d is an isomorphic embedding.

The importance of Veronese embedding is that if $F = \sum a_{i_0\dots i_n} x_0^{i_0} \dots x_N^{i_N}$ determines a degree d hypersurface $H \subset \mathbb{P}_N$, then $v_d(H) \subset v_d(\mathbb{P}_N) \subset \mathbb{P}_M$ is the intersection of $v_d(\mathbb{P}_N)$ with a corresponding hyperplane $H_F \subset \mathbb{P}_M$. Thus the Veronese embedding allows to reduce the study of problems concerning hypersurfaces to hyperplanes.

We have $v_d(F) = V(\mathbb{P}_N) \cap H_F$, the point $x \in F$ is nonsingular if H_F intersects $v_d(\mathbb{P}_N)$ at $v_d(x)$ transversally. If $X \subset \mathbb{P}_N$ is Zariski-closed and let $x \in X \cap F$ is non singular of both X and F, then F intersects X at x transversally, then H_F intersects $v_d(X)$ at $v(x)$ transversally.

2.3 Duality Theorem

We now study the dual variety of a projective variety X. We allow X to have singular points. We define the dual variety as $\check{X} \subset \check{\mathbb{P}}_N$ as the closure of set of all hyperplanes tangent to X. An hyperplane $H \subset \mathbb{P}_N$ is tangent to X if $T_xX \subset H$ for some nonsingular point $x \in X$. Thus $\check{X} = \overline{\{y \in \check{\mathbb{P}}_N \mid H_y \textit{ is tangent to } X\}}$. If X is non-singular the set of hyperplanes tangent to X is a closed set.

Theorem 2.3.1. *The dual of X is closed irreducible subvariety of atmost dimension $N-1$.*

Proof. Define $V'_X = \{(x, y) \in \mathbb{P}_N \times \check{\mathbb{P}}_N \mid x \in X_e \textit{ and } H_y \textit{ is tangent to } X \textit{ at } x\}$ where $X_e \subset X$ is non-empty open subset of nonsingular points of X. Let π_1 and π_2 be first and second projections respectively. A typical fibre of first projection is $\pi_1^{-1}(a) = \{(a, y) \mid H_y \textit{ is tangent to } X \textit{ at } a\}$. This fibre is isomorphic to $\check{\mathbb{P}}(T_a(\mathbb{P}_N)/T_aX)$ and hence has dimension $N - n - 1$. By 1.1.5 V'_X is irreducible and has dimension $N-1$. Thus the closure V_X of V'_X also has dimension $N-1$. The first projection maps V_X onto X since projection is closed map 1.1.2. We observe that $\check{X} = \pi_2(V_X)$, since projection is both continuous and closed hence $\pi_2(\overline{V'_X}) = \overline{\pi_2(V'_X)}$. The dimension of $\check{X}$ is atmost $N-1$ by 1.1.4. □

The set V_X as in the proof above is called the tangent hyperplane bundle of X.

Lemma 2.3.2. *Let $X \subset \mathbb{P}_N$ be a be a closed irreducible variety of dimension n, and let H be a hyperplane, $x \in X \cap H$ then X intersects H at x transversally if and only if $X_H = X \cap H$ is smooth at x.*

Proof. We have $T_x(X_H) = T_x(X) \cap T_x(H)$ and $T_x(X) \not\subset T_xH$ since the intersection is transversal. So we have $dim T_xX_H = n - 1$. Conversely assume X to be affine and let $I(Y) = (f_1, f_2, \dots, f_t)$ then the matrix $[(\partial f_i/\partial x_j)(P)]$ has rank $N - n$, now since X_H is smooth at x the jacobian of X_H has rank $N - n + 1$, which implies $gradH$ is linearly independent of span $((\partial f_i/\partial x_j)(P))$. We also have $T_xX = \cap T_xZ(f_i)$. If the intersection of X and H were not transversal then $\cap T_xZ(f_i) \subset T_xH$ which implies $gradH$ is in orthogonal complement of T_xX and hence is in the span of $(\partial f_i/\partial x_j)(P)$, a contradiction. □

We now study the duality theorem which gives relation between tangent hyperplane bundle of X and $\check{X}$. Towards that we make some construction. Define

$$W = \{(x, y) \in \mathbb{P}_N \times \check{\mathbb{P}}_N \mid x \in X \cap H_y\}. \tag{2.3.1}$$

We observe that $W = \Gamma \cap (X \times \check{\mathbb{P}}_N)$ where Γ is the universal hyperplane defined above and hence W is a variety. Let $p_1 : W \to X$ be first projection, a fibre of a point is $p_1^{-1}(x) = \{(x, y) \mid x \in H_y\}$. Thus all fibres are irreducible and have same dimension equal to $N-1$. Thus by 1.1.5 W is irreducible. By 1.1.4 there exist $x \in X$ such that $dim f^{-1}(x) = dimW - dimX$, hence W has $N + n - 1$ dimensions. We observe that $V_X \subset W$ and $\pi_1 = p_1 \mid V_X$. The open set of simple points is $W_e = p_1^{-1}(X_e)$. For a simple point $(c, b) \in W$ the second projection p_2 has maximal rank if and only if H_b intersects X at c transversally.

Since if p_2 has maximal rank at (c, b) then $p_2^{-1}(b)$ is smooth, which is isomorphic to X_b and hence by the lemma above H_b intersects X at c transversally. Conversely we have $T_c(H_b) = T_{(c,b)}(p_2^{-1}(p_2(c, b))) = kerTp_2(c, b)$ has dimension $n - 1$ which implies p_2 has maximal rank. As a result V'_X is the set of simple points of W which are critical with respect to p_2.

Theorem 2.3.3 (Duality Theorem)**.** *The tangent hyperplane bundle of X and $\check{X}$ coincide*

$$V_X = V_{\check{X}} \textit{ and hence } \check{\check{X}} = X.$$

Proof. Consider the set $U = \{(c, b) \mid c \in X_e,\ b \in \check{X},\ (c, b) \in V_X, \pi_2 = p_2 \mid V_X \textit{ has maximal rank } (= dim\check{X})\}$. U is open and nonempty subset of V_X. We prove $U \subset V_{\check{X}}$ and irreducibility of V_X will imply that $V_X \subset V_{\check{X}}$ and since $dimV_X = dimV_{\check{X}}$ we will get $V_X = V_{\check{X}}$. Let $(c, b) \in U$ then $\{c\} \times_c H \subset W$ where ${}_cH \subset \check{\mathbb{P}}_N$ is the hyperplane corresponding to c. Thus $T_{(c,b)}(\{c\} \times_c H) \subset T_{(c,b)}W$ which implies

$$T_{p_2}(T_{(c,b)}(\{c\} \times_c H)) \subset T_{p_2}(T_{(c,b)}W).$$

Now since p_2 maps $\{c\} \times_c H$ isomorphically onto ${}_cH$ we have $T_{p_2}(T_{(c,b)}\{c\} \times_c H) = T_b({}_cH)$. Since at (c, b), p_2 has rank less than N it must be $N - 1$. We have

$$T_{p_2}(T_{c,b}W) = T_b({}_cH).$$

We now observe that $T_{(c,b)}V_X \subset T_{(c,b)}W$ implies that

$$T\pi_2(T_{(c,b)}V_X) \subset T_{p_2}(T_{(c,b)}W) = T_b({}_cH)$$

and since $\pi_2 = p_2 \mid V_X$ has maximal rank $= dim\check{X}$ at (c,b), we have

$$T_b\check{X} = T\pi_2(T_{(c,b)}V_X) \subset T_b({}_cH).$$

Hence ${}_cH$ is tangent to $\check{X}$ at b and thus we get $(c,b) \in V_{\check{X}}$. □

We now see an application of Veronese embedding.

Proposition 2.1. All smooth hypersurfaces of $\mathbb{P}_N$ which have same degree d is diffeomorphic to one another.

Proof. Let $X \subset \mathbb{P}_N$ be smooth, consider $p_2 : W \setminus p_2^{-1}(\check{X}) \to \check{\mathbb{P}}_N \setminus \check{X}$ is a proper mapping which has a maximal rank $= N$ everywhere. Therefore $W \setminus p_2^{-1}(\check{X})$ is locally trivial fibre bundle over $\check{\mathbb{P}_N} \setminus \check{X}$ by Ehresmann's fibration theorem. Since $\check{\mathbb{P}} \setminus \check{X}$ is path connected all fibres (i.e. all transversal hyperplane sections X_y of X) are diffeomorphic to one another. If this is applied to the Veronese variety $X = v(\mathbb{P}_N)$ we get the desired result. □

2.4 Lefschetz Pencil.

In this section we define special type of pencils called *$Lefschetz$ pencils* and see their existence. We fix X to be irreducible, nonsingular projective variety of dimension n. We define class of X to be r if $\check{X}$ is a hypersurface of degree $r > 0$, and 0 if $dim\check{X} \leq N-2$.

Definition 2.4.1. A Lefschetz pencil on $X \subset \mathbb{P}_N$ is a pencil determined by a projective line $G \subset \check{\mathbb{P}}_N$ with the following properties

1. The axis A of the pencil intersects X transversally.
2. The modification Y of X along $X' = X \cap A$ is irreducible and non-singular.
3. The projection $f : Y \to G$ has $r =$ *class* X critical values and the same number of critical points each of which is non-degenerate.

Proposition 2.2. Let $b \in \check{\mathbb{P}}_N \setminus \check{X}$ (i.e. H_b intersects X transversally). Let E be the $(N-1)$-dimensional space of all projective lines in $\check{\mathbb{P}}_N$ through b. If $class\ X = 0$ the lines which do not meet $\check{X}$ form a non-empty open subset of E. If $class\ X = r > 0$ the lines which avoid the singular set of $\check{X}$ and intersect $\check{X}$ transversally form a non-empty open subset of E. For a line G in this set the intersection $G \cap \check{X}$ consists of $r = class\ X$ many points.

Proof. We consider the projection $p : \check{X} \to E$ with centre b, $p(y) = \textit{line through } b \textit{ and } y$. Now $p(\check{X})$ is a closed subset of E with $dim\ p(\check{X}) \leq dim\ \check{X}$, since image of projective variety under a morphism is closed. If $class\ X = 0$ the required open set is $E \setminus p(\check{X})$. If dimension $\check{X} = N - 1$ the subset $C \subset \check{X}$ consisting of singular points of $\check{X}$ and simple points y of $\check{X}$ where $p(y)$ is not transversal to $\check{X}$ is proper and closed, and since $\check{X}$ is irreducible $dim\ C \leq N - 2$. The required open set is $E \setminus p(C)$. □

We now prove the existence of Lefschetz pencil on X. By 2.2 there exists a projective line G which intersects $\check{X}$ transversally and avoids the singular set (for $class\ X = 0$ this means $G \cap \check{X} = \emptyset$.) We prove that the pencil $\{H_t\}_{t \in G}$ with axis A is a Lefschetz pencil in the following propositions.

Remark. If $class\ X > 0$ and $b \in G \cap \check{X} \subset \check{X}_e$, there is exactly one point $c \in X$ such that $(c, b) \in V = V_X = V_{\check{X}}$, because $V'_{\check{X}}$ is mapped isomorphically onto $\check{X}_e$ by π_2. We have

$$T_b({}_cH) = (Tp_2)(T_{(c,b)}W) = (T\pi_2)(T_{(c,b)}V) = T_b\check{X} \tag{2.4.1}$$

Proposition 2.3. The axis A intersects X transversally, and hence $X' = X \cap A$ and $Y' = p^{-1}(X')$ are non-singular and have dimension $n-2$ and $n-1$ respectively.

Proof. We observe that $Y \subset W$ where W is as defined above, and $Y = p_2^{-1}(G)$ is modification of X along X' and $f = p_2 \mid Y : Y \to G$. If $class\ X = 0$, $G \cap \check{X} = \emptyset$, and all hyperplanes of the pencil $\{H_t\}_{t \in G}$ intersect X transversally and hence A intersects X transversally since if $T_xX \subset T_xA$ then $T_xX \subset T_xH_t$ for all hyperplanes H_t a contradiction, and if $T_xX \cap T_xA$ is codimension one in T_xX say $T_xX = (T_xX \cap T_xA) + M$ then T_xX is contained in the hyperplane $T_xA + M$. Now let $class\ X > 0$, now if A did not intersect X transversally, we get a hyperplane H_b tangent to X at a point $c \in A$ i.e. $(c, b) \in V$. Now $c \in A \subset H_b$ dualizes to $b \in G \subset {}_cH$. Since G intersects $\check{X}$ transversally, ${}_cH$ also does, which means $(c, b) \notin V$ a contradiction.

Now since A intersects X transversally we have $N - dim\ X' = codim\ X + codim\ A$ and hence $dim\ X' = n-2$ which implies $dim\ Y' = n-1$. □

Lemma 2.4.1. *The projection $p_2 : W \to \check{\mathbb{P}}_N$ is transversal to G, i.e if $(c,b) \in W$ and $b \in G$ then $T_b\check{\mathbb{P}}_N = T_bG + (Tp_2)(T_{(c,b)}W)$.*

Proof. If p_2 has rank N at (c,b), $T_b\check{\mathbb{P}}_N = (Tp_2)(T_{(c,b)}W)$, else $(c,b) \in V$ and hence $T_b\check{X} = (Tp_2)(T_{(c,b)}W)$ by 2.4, and the lemma is proved since G intersects $\check{X}$ transversally. □

Proposition 2.4. The modification Y of X along X' is irreducible and non-singular.

Proof. Since p_2 is transversal to G, $Y = p_2^{-1}(G)$ is a submanifold of W of dimension n, and hence is nonsingular. Now since X is irreducible $X \setminus X'$ being open set of X is irreducible, which implies $Y \setminus Y'$ being isomorphic to $X \setminus X'$ is irreducible. $\overline{Y \setminus Y'}$ is irreducible component of Y, now any other irreducible component T of Y must be contained in Y' or else $T = T \cap \overline{Y \setminus Y'} \cup T \cap Y'$ would be reducible. Now since every irreducible component of Y has dimension n and cannot be contained in Y'. □

Proposition 2.5. The projection $f : Y \to G$ has $r = class\ X$ critical values, namely the points of $\check{X} \cap G$. There are same number of critical points.

Proof. Let $(c,b) \in Y$, we have $(Tf)(T_{(c,b)}Y) = (Tp_2)(T_{(c,b)}W) \cap T_bG$. Now if $b \in G \setminus \check{X}$ then $(c,b) \notin V$, and by 2.4 we have $(Tp_2)(T_{(c,b)}W) = T_b\check{\mathbb{P}}_N$ and hence f maximal rank 1 at (c,b). If $b \in G \cap \check{X}$ then $(c,b) \in V$ and again by 2.4 we have $(Tp_2)(T_{(c,b)}W) = T_b\check{X}$. And since $G \cap \check{X}$ transversally and hence (c,b) is critical point of f. By 2.4 there are no two critical points in same fibre of f. □

Proposition 2.6. Every critical point of $f : Y \to G$ is non-degenerate.

Proof. Let $(c,b)\ \epsilon\ V$ be a critical point of f. Choose projective coordinates of $(x_0 : \ldots : x_N)$ of $\mathbb{P}_N$ and dual coordinates $(y_0 : \ldots y_N)$ of $\check{\mathbb{P}}_N$ so that $b = (0 : \ldots : 0 : 1)$ and $c = (1 : 0 : \ldots, 0)$ and G is given by $y_1 = \ldots = y_{N-1} = 0$. The first projection fibres locally trivially with the explicit trivialization over $U = \{x\ \epsilon\ X \mid x_0 \neq 0\}$ given by

$$U \times \check{\mathbb{P}}_{N-1} \to p_1^{-1}(U),$$

$$(x,z) \to (x, (-\sum_{i=1}^{N} x_i z_i : x_0 z_1 : \ldots : x_0 z_N))$$

where $z = (z_1 : \ldots : z_N) \in \check{\mathbb{P}}_N$. Let $(t_1, t_2, \ldots, t_n)$ be holomorphic coordinates around c in X and $\zeta_1 = \frac{z_1}{z_N}, \ldots, \zeta_{N-1} = \frac{z_{N-1}}{z_N}$ be affine coordinates of $\mathbb{P}_{N-1}$ then w have holomorphic coordinates of W in a neighborhood of (c, b) is $(t_1, \ldots, t_n, \zeta_1, \ldots, \zeta_{N-1})$. So now we have $p_2 = (g(t,\zeta), \zeta_1, \ldots, \zeta_{N-1})$ and $f : Y \to G$ is given by $f(t) = g(t, 0)$. Now

$$Jac(p_2) = \begin{bmatrix} \frac{\partial g}{\partial t_1} & \cdots & \frac{\partial g}{\partial t_n} & * & * & * \\ 0 & \cdots & 0 & 1 & & \\ \cdots & \cdots & \cdots & & \ddots & \\ 0 & \cdots & 0 & & & 1 \end{bmatrix}$$

Now V is given by $\frac{\partial g}{\partial t_1} = \frac{\partial g}{\partial t_2} = \ldots = \frac{\partial g}{\partial t_n} = 0$. Therefore the Jacobian of the defining equations together with the Jacobian of p_2 must have rank $N + n - 1$ The big matrix

$$Jac(p_2) = \begin{bmatrix} \frac{\partial^2 g}{\partial t_1^2} & \cdots & \frac{\partial^2 g}{\partial t_1 \partial t_n} & * & * & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 g}{\partial t_n \partial t_1} & \cdots & \frac{\partial^2 g}{\partial t_n^2} & * & * & * \\ 0 & \ldots & 0 & * & * & * \\ 0 & \cdots & 0 & 1 & & \\ \cdots & \cdots & \cdots & & \ddots & \\ 0 & \cdots & 0 & & & 1 \end{bmatrix}$$

It has rank $N+n-1$ if and only if the rank of the hessian matrix of the second derivatives of f has maximal rank n.

$\square$

Chapter 3

Lefschetz's Theorems

In this chapter we will see Lefschetz results. The main results studied is Lefschetz's famous theorem on homology of hyperplane sections 3.2.1 and Weak Lefschetz Theorem 3.3.1

3.1 Main lemma.

We first prove two important lemmas 3.1.1 and 3.1.2.

Let $p : Y \to X$ be modification of X along X' as defined in 2.1.1. Since $Y' = X' \times G$ we have

$$H_q(Y') = H_q(X' \times G).$$

By Künneth formula 1.3.4 we have

$$H_q(X' \times G) \simeq H_q(X') \otimes H_0(G) \oplus H_{q-2}(X') \otimes H_2(G)$$

and by and 1.3.2 we get

$$H_q(X' \times G) \simeq H_q(X') \oplus H_{q-2}(X').$$

Using the inclusion $Y' \subset Y$ there is a cannonical homomorphism $\kappa : H_{q-2}(X') \to H_q(Y)$.

Lemma 3.1.1. *The sequence* $0 \to H_{q-2}(X') \xrightarrow{\kappa} H_q(Y) \xrightarrow{p_*} H_q(X) \to 0$ *is split exact se-*

quence.

Proof. We first show that the sequence splits by by producing a right inverse of p_*. Let $x \in H_q(X)$ then we have $x = u \cap [X]$ where $u \in H^{2n-q}(X)$ is Poincaré dual of x. Then we define the inverse $s : H_q(X) \to H_q(Y)$ by $s(x) = p^*(u) \cap [Y] \in H_q(Y)$ and $p(s(x)) = p_*(p^*(u) \cap [Y]) = u \cap p_*[Y] = u \cap [X] = x$. We use long exact sequences of (Y, Y') and (X, X'), by excision theorem 1.3.1 p'_* is an isomorphism, exactness follows by diagram chasing.

$$\begin{tikzcd}
H_{q+1}(Y) \arrow[r] \arrow[d, "p_*"] & H_{q+1}(Y, Y') \arrow[r, "\partial_*"] \arrow[d, "p'_*"] & H_q(X') \oplus H_{q-2}(X') \arrow[r] \arrow[d, "pr"] & H_q(Y) \arrow[r] \arrow[d, "p_*"] & H_q(Y.Y') \arrow[d, "p'_*"] \\
H_{q+1}(X) \arrow[r] & H_{q+1}(X, X') \arrow[r, "\partial_*"] & H_q(X') \arrow[r] & H_q(X) \arrow[r] & H_q(X.X')
\end{tikzcd}$$

□

Let $f : Y \to G$ be a holomorphic mapping between an n-dimensional compact complex manifold Y and a projective line G, so that f has r critical values and the same number of critical points each of which is non-degenerate. Let $x_1, \ldots, x_r$ be the critical points of f and $t_1, \ldots, t_r$ be the corresponding critical values. We decompose G into two closed hemisphere D_+ and D_- such that all the critical values are in the interior of D_+. We denote $G = D_+ \cup D_-$, $S^1 = D_+ \cap D_-$, $Y_+ = f^{-1}(D_+)$, $Y_- = f^{-1}(D_-)$, and $Y_0 = f^{-1}(S^1)$. Choose a point $b \in S^1$.

Lemma 3.1.2. ***Main lemma***

$$H_q(Y_+, Y_b) = \begin{cases} 0 & if \ q \neq n \\ free \ of \ rank \ r & otherwise. \end{cases}$$

Proof. We identify D_+ with closed unit in $\mathbb{C}$ by choosing suitable holomorphic coordinates that b corresponds to 1. We now choose small disks D_i with centre t_i for each critical value and radius ρ are chosen so that $D_i \cap D_j = \emptyset$ for $i \neq j$, and each D_i, $i = 1, 2, \ldots r$ is contained in D. See figure 3.1.2.

The lemma is proved in several steps

Step 1- Localization in the base space:

Let

$$T_i = f^{-1}(D_i) \ \ and \ \ F_i = f^{-1}(t_i + \rho).$$

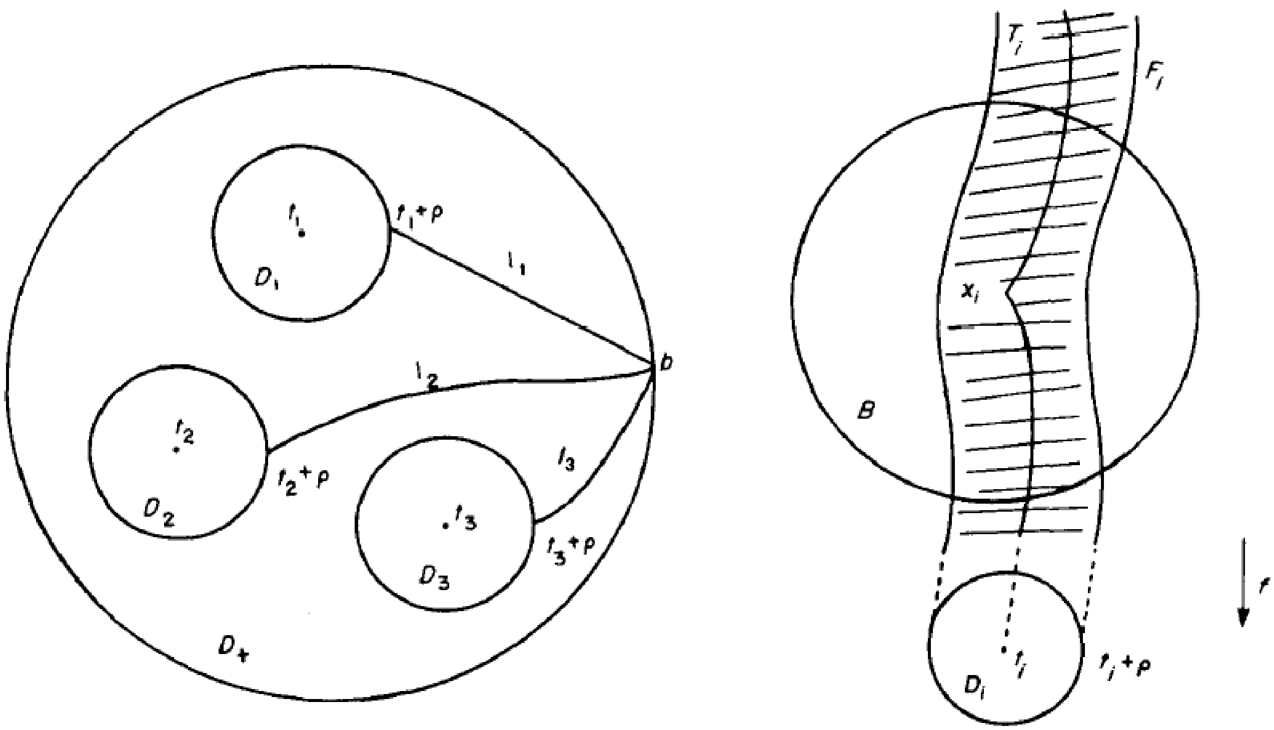

Fig. 1 Fig. 2

Figure 3.1: Isolating the critical points.

We first localize in the base space to reduce the investigation of (Y_+, Y_b) locally to (T_i, F_i) as follows: Let l_i be a smooth interval from b to $t_i + \rho$. We denote

$$l = \bigcup_{i=1}^{r} l_i \textit{ and } k = l \bigcup_{i=1}^{r} D_i.$$

We choose l_i to be disjoint from each other so that l can be contracted to b, and D_+ can be contracted to k.

claim 1. The fibre Y_b is strong deformation retract of $L = f^{-1}(l)$ and $K = f^{-1}(k)$ is a strong deformation retract Y_+. Hence we get

$$H_*(Y_+, Y_b) \simeq H_*(Y_+, L) \simeq H_*(K, L).$$

Proof. We use Ehresmann's fibration theorem 1.3.7. The map $f : Y_+ \setminus f^{-1}\{t_1, \dots, t_r\} \to D_+ \setminus \{t_1, \dots, t_r\}$ is a $\mathcal{C}^\infty$ locally trivial fibre bundle. Now by homotopy covering theorem, the contraction of l to b can be lifted so that L deformation retracts to Y_b. Similarly we lift the contraction $D_+ \setminus \{t_1, \dots, t_r\}$ to $l \cup \bigcup_{i=1}^{r}(D_i \setminus t_i)$ so that $L \cup \bigcup_{i=1}^{r}(T_i \setminus f^{-1}(t_i))$. Since t_i are the interior points of k the singular fibres can be filled in so that K is a deformation retract of Y_+. □

claim 2.

$$H_*(Y_+, Y_b) \simeq H_*(K, L) \simeq \bigoplus_{i=1}^{r} H_*(T_i, F_i).$$

Proof. We observe that $K \setminus L = f^{-1}(k) \setminus f^{-1}(l) = \bigcup_{i=1}^{r} T_i \setminus \bigcup_{i=1}^{r} F_i$. Hence the inclusion

$$(\bigcup_{i=1}^{r} T_i, \bigcup_{i=1}^{r} F_i) \hookrightarrow (K, L)$$

is an excision and by 1.3.1 the claim follows. □

Step 2- Localization in the total space:
Let

$$T = T_i \cap B \textit{ and } F = F_i \cap B.$$

We now localize in the total space and reduce the investigation of (T_i, F_i) to (T, F)

By morse lemma 1.2.3 we get a holomorphic coordinate chart $(U, \psi = (z_1, \ldots, z_n))$ centered at x_i such that

$$f(z) = t_i + z_1^2 + \ldots + z_n^2.$$

Let ϵ be small enough so that

$$B = \{z \in \mathbb{C}^n \mid ||z|| < \epsilon\}$$

is subset of $\psi(U)$. We denote $\psi^{-1}(B)$ again by B. We shrink radius ρ of D_i so that $\rho < \epsilon^2$ to get $D_i \subset f(B)$. see 3.1.2.

claim 3. $H_*(T, F) \simeq H_*(T_i, f_i)$.

Proof. Let $\partial B = \{z \in B \mid ||z|| = \epsilon\}$, and let $T' = T \cap \partial B$ and $F' = F \cap \partial B$. We consider

$$\begin{array}{ccc} (T, F) & \longrightarrow & (T_i, F_i) \\ \downarrow & & \downarrow \\ (T, T' \cup F) & \longrightarrow & (T_i, \overline{T_i \setminus B} \cup F_i) \end{array}$$

The bottom line is a excision because $T \setminus (T' \cup F) = T_i \setminus (\overline{T_i \setminus B} \cup F_i)$. Now $F_i \setminus \mathring{B}$ is strong deformation retract of $T_i \setminus \mathring{B}$ and F' is strong deformation retract of T' because f has maximal rank 2 of smooth manifolds on $T_i \setminus \mathring{B}$ and hence be Ehresmann's fibration theorem 1.3.7 we have $(T_i \setminus \mathring{B}, \partial T)$ diffeomorphic to $(F_i \setminus \mathring{B}, \partial F) \times D_i$, and D_i can be contracted to $t_i + \rho$. Thus the vertical lines have the same homology groups, completing the proof of the claim. $\square$

Step 3- Explicit calculation:

We will now use explicit coordinate description to calculate the homology groups of the pair (T, F). We have

$$T = \{z \in \mathbb{C}^n \mid |z_1|^2 + \ldots + |z_n|^2 \leq \epsilon^2 \textit{ and } |z_1^2 + \ldots + z_n^2| \leq \rho\} \tag{3.1.1}$$

$$F = \{z \in T \mid |z_1^2 + \ldots + z_n^2| = \rho\} \tag{3.1.2}$$

claim 4. F is diffeomorphic to unit sphere bundle

$$Q = \{(u,v) \in \mathbb{R}^n \times \mathbb{R}^n \mid ||u|| = 1, ||v|| \leq 1 \; and \; \langle u, v\rangle\}.$$

Proof. We decompose each coordinate $z_j = x_j + iy_j$ and let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ and therefore

$$F = \{(x,y) \mid ||x||^2 - ||y||^2\rho, \langle x, y\rangle = 0\}$$

now $\rho + 2||y||^2 \leq \;\epsilon\;^2$ implies that $||y|| \leq (\frac{\epsilon\;^2 - \rho}{2})^{1/2} = \sigma$

The diffeomorphism is given by

$$F \xrightarrow{\varphi} Q \tag{3.1.3}$$
$$(x,y) \mapsto \left(\frac{x}{||x||}, \frac{y}{\sigma}\right) \tag{3.1.4}$$

and the inverse is given by

$$Q \xrightarrow{\varphi^{-1}} F \tag{3.1.5}$$
$$(u,v) \mapsto (\sqrt{(\sigma^2||v||^2 + \rho)}u, \sigma v). \tag{3.1.6}$$

□

We get

$$H_{q-1}(F) = \begin{cases} 0 & if \; q \neq 1, \neq n \\ free \; of \; rank \; 1 & otherwise. \end{cases}$$

claim 5.

$$H_q(T,F) = \begin{cases} 0 & if \; q \neq n \\ free \; of \; rank \; 1 & otherwise. \end{cases}$$

Proof. By 3.1.1, T is linearly contracted to the origin by the contraction $H(z,t) = (1-t)z$. Hence the connecting homomorphism

$$\partial_* : H_q(T,F) \to H_{q-1}(F) \; for \; q \neq 0.$$

is an isomorphism. $\square$

$H_n(T, F)$ is generated by an orientation of real n-disk $\Delta = \{z \in T \mid all\ z_j\ real\}$

This completes the proof of the main lemma.

Remark. We now give an explicit retraction $R : T' \to F'$. Let $f(z) = t_i + re^{2\pi i\varphi}$, let $e^{-\pi i\varphi}z = z'$ so that $f(z') = r$. Let

$$R' : T' \to Q, \quad R'(z) = e^{\pi i\varphi}((x'/||x'||) + i(y'/||y'||)).$$

We define R to be composition of R' and the diffeomorphism of Q and F'.

$\square$

3.2 Lefschetz's Results

We now study many of the Lefschetz's results using techniques from algebraic topology, and the lemmas 3.1.1 and 3.1.2.
We observe that since f has no critical points in Y_- we get

$$Y_- = X_b \times D_-\ and\ Y_0 = X_b \times S^1$$

by 1.3.7 and hence $(Y_-, Y_0) = X_b \times (D_-, S^1)$. Now since $Y \setminus Y_+ = Y_- \setminus Y_0$ we have by 1.3.1

$$H_q(Y, Y_+) \simeq H_q(Y_-, Y_0) = H_q(X_b \times D_-, X_b \times S^1).$$

By 1.3.4

$$\bigoplus_i H_i(X_b) \otimes H_{q-i}(D_-, S^1) \simeq H_q(X_b \times D_-, X_b \times S^1).$$

$$H_p(D_-, S^1) = \begin{cases} R & if\ p = 2 \\ 0 & otherwise. \end{cases}$$

We get

$$H_q(Y, Y_+) \simeq H_{q-2}(X_b). \tag{3.2.1}$$

The homology sequence of the triple (Y, Y_+, Y_b)

$$\cdots \to H_{q+1}(Y_+, Y_b) \to H_{q+1}(Y, Y_b) \xrightarrow{L} H_{q+1}(Y, Y_+) \xrightarrow{\tau} H_q(Y_+, Y_b) \to \cdots$$

becomes

$$\cdots \to H_{q+1}(Y_+, Y_b) \to H_{q+1}(Y, Y_b) \xrightarrow{L} H_{q-1}(Y, Y_+) \xrightarrow{\tau} H_q(Y_+, Y_b) \to \cdots \tag{3.2.2}$$

Now by using 3.1.2 we get the isomorphism

$$L : H_{q+1}(Y, Y_b) \simeq H_{q-1}(X_b),\ \ q \neq n-1, n. \tag{3.2.3}$$

and a five term sequence

$$0 \to H_{n+1}(Y, Y_b) \xrightarrow{L} H_{n-1}(Y_b) \to H_n(Y_+, Y_b) \to H_n(Y, Y_b) \to H_{n-2}(X_b) \to 0 \tag{3.2.4}$$

Proposition 3.1. If $n > 1$, the generic hyperplane section X_b ($b \notin \check{X}$) is non singular and irreducible.

Proof. $b \notin \check{X}$, implies that H_b intersects X transversally and hence X_b is nonsingular. Now using 3.2.3

$$H_0(Y, Y_b) = H_{-2}(X_b) = 0, \quad H_1(Y, Y_b) = 0$$

Thus $H_0(Y_b) = H_0(Y)$ using the long exact sequence of the pair (Y, Y_b). Thus $H_0(Y_b) = 0$ since Y is connected because it is irreducible and nonsingular by proposition 2.4. Thus Y_b is connected and hence irreducible. □

Proposition 3.2. If $r =$ *class* X,

$$e(Y) = 2e(X_b) + (-1)^n r$$

and

$$e(X) = 2e(X_b) - e(X') + (-1)^n r.$$

Proof. From 3.1.1 we get

$$e(Y) = e(X) + e(X'),$$

and from 3.2.2 we have

$$e(Y) - e(Y_b) = e(Y, Y_b) = e(X_b) + (-1)^n r.$$

Thus

$$e(Y) = 2e(X_b) + (-1)^n r$$

and

$$e(X) = 2e(X_b) - e(X') + (-1)^n r.$$

□

We now present the Lefschetz's famous theorem on homology of hyperplane section.

Theorem 3.2.1. *The inclusions $X_b \hookrightarrow X$ induces isomorphisms $H_q(X_b) \to H_q(X)$ if $q < \frac{1}{2} dim X_b = n-1$, which is equivalent to $H_q(X, X_b) = 0$ for $q \leq n-1$.*

Proof. We use long exact sequence of the triple $(Y, Y_+ \cup Y', Y_b \cup Y')$. Using excision 1.3.1 we have

$$\begin{aligned} H_q(Y, Y_+ \cup Y') &= H_q(Y, Y_+ \cup (X' \times D_-)) \\ &\simeq H_q(Y_-, Y_0 \cup X' \times D_-) \\ \textit{using } 1.3.7 \qquad &= H_q(X_b \times D_-, X_b \times S^1 \cup X' \times D_-) \\ &\simeq H_{q-2}(X_b, X') \otimes H_2(D_-, S^1) \\ &\simeq H_{q-2}(X_b, X'). \end{aligned}$$

Now consider the inclusions

$$(Y_+, Y_b) \hookrightarrow (Y_+, Y_b \cup Y'_+) \hookrightarrow (Y_+ \cup Y', Y_b \cup Y').$$

The first inclusion induces an isomorphism in homology since $Y_b = X_b \times \{b\}$ is deformation retract of $Y_b \cup Y'_+ = X_b \times \{b\} \cup X' \times D_+$. The second inclusion induces isomorphism in homologies because of 1.3.1. Thus we have

$$H_*(Y_+ \cup Y', Y_b \cup Y') \simeq H_*(Y_+, Y_b) \tag{3.2.5}$$

The long exact sequence of $(Y, Y_+ \cup Y', Y_b \cup Y')$ is transformed into

$$\cdots \to H_{q+2}(Y_+, Y_b) \xrightarrow{p_*} H_{q+2}(X, X_b) \xrightarrow{L'} H_q(X_b, X') \xrightarrow{\tau'} H_{q+1}(Y_+, Y_b) \to \cdots . \tag{3.2.6}$$

Now by using 3.1.2 we get the isomorphism

$$L' : H_{q+1}(X, X_b) \simeq H_{q-1}(X_b, X'),\ q \neq n-1, n. \tag{3.2.7}$$

and a five term sequence

$$0 \to H_{n+1}(X, X_b) \xrightarrow{L'} H_{n-1}(X_b, X') \to H_n(Y_+, Y_b) \to H_n(X, X_b) \to H_{n-2}(X_b, X') \to 0 \tag{3.2.8}$$

We now use induction to complete the proof. The case $n = 1$ is obvious. We induct from $n-1$ to n. We observe that X' is a hyperplane in X_b, and hence apply the induction hypothesis to the pair (X_b, X') to get $H_q(X_b, X') = 0$ for $q \leq n-2$. the isomorphism 3.2.7 proves the theorem. □

Corollary 3.2.1.1. *$H^q(X, X_b) = 0$ for $q \leq n-1$, $n = dimX$, i.e. The inclusion $X_b \hookrightarrow X$ induces isomorphisms of cohomology groups in dimension strictly less than $n-1$ and a monomorphism of H^{n-1}. Also $H^n(X, X_b; R) \simeq Hom(H_n(X, X_b), R)$ where R is the coefficient ring.*

Proof. We apply universal coefficient theorem 1.3.3 to theorem 3.2.1, to get the result $H^q(X, X_b) = 0$ for $q \leq n-1$ and $H^n(X, X_b; R) \simeq Hom(H_n(X, X_b)R)$. □

The theorem 3.2.1 is generalised for hypersurfaces.

Corollary 3.2.1.2. *Let $X \subset \mathbb{P}_N$ be smooth irreducible $n-$dimensional variety. $F \subset \mathbb{P}_N$*

be hypesurface such that F intersects X transversally, then

$$H_q(X, X \cap F) = 0 \quad for\ q \leq n-1.$$

A subset $Y \subset \mathbb{P}_N$ if $Y = \bigcap_{i=1}^{r} F_i$ such that F_1 is smooth and F_k intersects $\bigcap_{i=1}^{k-1} F_i$ transversally and $\bigcap_{i=1}^{k} F_i$ are simple points of F_k. Y is $(N-r)$- dimensional variety. We use the corollary above to get

Corollary 3.2.1.3. *If $Y \subset \mathbb{P}_N$ is an $n-$dimensional smooth complete intersection, then $H_q(P_N, Y) = 0$ for $q \leq n$. Equivalently $Y \hookrightarrow \mathbb{P}_N$ induces isomorphism at homology groups in dimension strictly less than n and an epimorphism in dimension n.*

Proof. We consider the long exact sequence of the triple $(\mathbb{P}_N, \bigcap_{i=1}^{r-1} F_i, \bigcap_{i=1}^{r} F_i)$ and use induction on r. □

3.3 Weak Lefschetz Theorem

Let $\partial_* : H_n(Y_+, Y_b) \to H_{n-1}(Y_b) \simeq H_{n-1}(X_b)$, be connecting homomorphism. We define the module of "vanishing cycles" as

$$V = \partial_*(H_n(Y_+, Y_b)).$$

The long exact sequences of the pairs (Y_+, Y_b) and (X, X_b) form the following commutative diagram

$$\begin{array}{ccccccc} H_n(Y_+, Y_b) & \xrightarrow{\partial_*} & H_{n-1}(Y_b) & \longrightarrow & H_{n-1}(Y_+) & \longrightarrow & 0 \\ \downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 & & \\ H_n(X_+, X_b) & \xrightarrow{\partial_*} & H_{n-1}(X_b) & \xrightarrow{i_*} & H_{n-1}(X) & \longrightarrow & 0 \end{array} \tag{3.3.1}$$

All vertical arrows are induced by the restriction of $p : Y \to X$. p_1 is surjective because it occurs in the exact sequence 3.2.6 and $H_{n-2}(X_b, X') = 0$ according to 3.2.1 since X' is the hyperplane section of X_b. The middle one p_2 is an isomophism. Hence the Five lemma implies that p_3 is also an isomorphism. From the above commutative diagram we have

$$V = kernel(i_* : H_{n-1}(X_b) \to H_{n-1}(X)) = image(\partial_* : H_n(X, X_b) \to H_{n-1}(X_b)) \qquad (3.3.2)$$

and

$$rankV + rankH_{n-1}(X) = rankH_{n-1}(X_b) \qquad (3.3.3)$$

These observation have a cohomological counterpart

$$\begin{tikzcd} H^n(Y_+, Y_b) & H_{n-1}(Y_b) \arrow[l, "\delta*"'] & H_{n-1}(Y_+) \arrow[l] & 0 \arrow[l] \\ H_n(X_+, X_b) \arrow[u, "mono"] & H_{n-1}(X_b) \arrow[l, "\delta_*"] \arrow[u] & H_{n-1}(X) \arrow[l, "i^*"] \arrow[u] & 0 \arrow[l] \end{tikzcd} \qquad (3.3.4)$$

We define the module of "invariant cocycles".

$$\begin{aligned} I^* :=& kernel(\delta^* : H^{n-1}(Y_b) \to H^n(Y_+, Y_b)) && (3.3.5) \\ =& kernel(\delta^* : H^{n-1}(X_b)) \to H^n(X, X_b) && (3.3.6) \\ =& image(i^* : H^{n-1}(X) \to H^{n-1}(X_b)) && (3.3.7) \end{aligned}$$

The module I of invariant cycles is defined to be the Poincaré dual of I^*

$$I := \{u \cap [X_b] \mid u \in I^*\} \subset H_{n-1}(X_b). \qquad (3.3.8)$$

or equivalently

$$I = image(i_1 : H_{n+1}(X) \to H_{n-1}(X_b)) \qquad (3.3.9)$$

where $i_1 = Di^*D^{-1}$ and D is the duality map. Since i^* is injective, i_1 is also *injective* so that

$$rankI = rankH_{n+1}(X) = rankH_{n-1}(X) \qquad (3.3.10)$$

Theorem 3.3.1 (Weak Lefschetz Theorem)**.** *If coefficients in a field are taken rank* $I +$

$rank\ V = rank\ H_{n-1}(X_b)$.

Proof. Since $H_{n-1}(Y_+, Y_b) = 0$ by 3.1.2 we have $H^n(Y_+, Y_b) = Hom(H_n(Y_+, Y_b), R)$ by 1.3.3. So we get $I^* = \{u \in H^{n-1}(Y_b) \mid \langle u, x\rangle = 0\ for\ every\ x \in V\}$. Here $\langle -, -\rangle$ denotes the Kronecker pairing between cohomology and homology. By Poincaré duality the Kronecker pairing becomes the intersection form

$$H_{n-1}(X_b) \times H_{n-1}(X_b) \to R,$$

and thus

$$I = \{y \in H_{n-1}(X_b) \mid \langle y, x\rangle = 0\ for\ every\ x \in V\}. \tag{3.3.11}$$

Since the coefficients are taken from a field this form is non-degenerate by 1.3.8, and hence by 3.3.11 we get

$$rank\ I + rank\ V = rank\ H_{n-1}(X_b).$$

□

Chapter 4

The Hard Lefschetz Theorem

4.1 Hard Lefschetz Theorem

In this chapter we discuss several equivalent statements of "Hard Lefschetz Theorem" and consequences of it.

Let $[X_b] \in H_{2n-2}(X)$ be the fundamental class of the hyperplane section X_b and let $u \in H^2(X)$ be its Poincaré dual.

$$u \cap [X] = [X_b].$$

Theorem 4.1.1. *If field coefficients are chosen, the following statements are equivalent:*

1. $V \cap I = 0$
2. $V \oplus I = H_{n-1}(X_b)$
3. $i_* : H_{n-1}(X_b) \to H_{n-1}(X)$ *maps* I *isomorphically onto* $H_{n-1}(X)$.
4. $H_{n+1}(X) \simeq H_{n-1}(X),\ x \mapsto u \cap x.$ *is an isomorphism.*
5. *The restriction of the intersection form* $\langle -, - \rangle$ *from* $H_{n-1}(X_b)$ *to* V *remains non-degenerate.*
6. *The restriction of* $\langle -.- \rangle$ *to* I *remains non-degenerate.*

Proof.

$$1 \Leftrightarrow 2$$

by weak Lefschetz theorem 3.3.1.

$$2 \Rightarrow 3$$

since $i_* : H_{n-1}(X_b) \to H_{n-1}(X)$ is surjective (3.3.1) and $V = ker(i_*)$ $I \simeq H_{n-1}(X)$.

$$3 \Leftrightarrow 4$$

consider $H_{n+1}(X) \xrightarrow{i_1} H_{n-1}(X_b) \xrightarrow{i_*} H_{n-1}(X)$, now i_1 is injective and I is image of i_1 and $i_* \circ i_1(x) = \ x \mapsto u \cap x$, 4 follows from 3. Conversely if 4 is true $I = image(i_1)$ is mapped isomorphically onto $H_{n-1}(X)$ by i_*.

$$3 \Rightarrow 1$$

since $i_*(V) = 0$.

$$2 \Rightarrow 5 \ \ \textit{and} \ \ 2 \Rightarrow 6$$

by 2 and 3.3.11 $\langle -, - \rangle$ on $H_{n-1}(X_b)$ splits as direct sum of its restrictions to V and I,

$$\langle -, - \rangle = \langle -, - \rangle_V \oplus \langle -, - \rangle_I.$$

Now since $\langle -, - \rangle$ is non-degenerate, the direct summands must also be non-degenerate.

$$5 \Rightarrow 1$$

Let $z \in V \cap I$. Then $\langle z, v \rangle = \langle z, v \rangle_V = 0$ for every $v \in V$ now 5 implies $z = 0$.

$$6 \Rightarrow 1$$

Let $z \in V \cap I$. Then $\langle c, z \rangle = \langle c, z \rangle_I = 0$ for every $c \in I$ now 6 implies $z = 0$. □

We now state that the Hard Lefschetz Theorem is true without proof.

Theorem 4.1.2 (Hard Lefschetz Theorem)**.** *The statements* $1 - 6$ *in 4.1.1 are true if coefficients from field of characteristic zero are chosen.*

Let $X = X_0 \supset X_b = X_1 \supset X' = X_2 \supset X_3 \ldots \supset X_n \supset X_{n+1}$ where we obtain X_q is smooth generic hyperplane section of X_{q-1}. We denote the inclusions as

$$i_q : X_q \to X.$$

Let

$$I(X_q) \subset H_{n-q}(X_q)$$

be the module of invariant cycles for the pair

$$X_q \subset X_{q-1}$$

Using the hyperplane section theorem of Lefschetz we have the isomorphisms

$$i_* : H_k(X_q) \to H_k(X_j),\ j \leq q$$

for

$$n > k + q.$$

We then use duality to conclude that

$$i^* : H^k(X_j) \to H^k(X_q),\ j \leq q$$

for

$$n > k + q.$$

By 3 we deduce that

$$(i_q)_* : H_{n-q}(X_q) \to H_{n-q}(X) \textit{ maps } I(X_q) \textit{ isomorphically onto } H_{n-q}(X). \tag{4.1.1}$$

We now observe that

$$I_q^* = Image(i^* : H^{n-q}(X_{q-1}) \to H^{n-1}(X_q))$$

and, by Lefschetz hyperplane section theorem we have the isomorphisms

$$H^{n-q}(X_0) \xrightarrow{i^*} H^{n-q}(X_1) \to \ldots \xrightarrow{i^*} H^{n-q}(X_{q-1}).$$

Now using Poincaré duality we obtain

$$i_1 \text{ maps } H_{n+q}(X) \text{ isomorphically onto } I(X_q). \tag{4.1.2}$$

Iterating 6 we obtain

$$The\ restriction\ of\ the\ intersection\ form\ H_{n-q}(X)\ to\ I(X_q)\ remains\ non-degenerate. \tag{4.1.3}$$

The isomorphism $(i_q)_* : I(X_q) \to H_{n-q}(X)$ carries this form to a non-degenerate bilinear form on $H_{n-q}(X_q)$, and for odd $n-q$ this a skew-symmetric form, and thus the degeneracy assumption implies

$$dim H_{n-q}(X) = dim H_{n+q}(X) \in 2\mathbb{Z}.$$

and thus we get the following result.

Corollary 4.1.2.1. *The odd-dimensional Betti numbers of X are even.*

Remark. Consider $X = S^3 \times S^1$. Using Künneth formula we get $b_1(X) = 1$ and hence is not a complex projective variety by the corollary above.

The q-th power $u^q \in H^{2q}(X)$ is Poincaré dual to the fundamental class $[X_q] \in H_{2n-2q}(X)$ of X_q. Using 4.1.1 and 4.1.2 we obtain the following generalization of 4.

Corollary 4.1.2.2. $H_{n+q}(X) \simeq H_{n-q}(X),\ x \mapsto u^q \cap x.$ *is an isomorphism for* $q = 1, \dots, n$.

Definition 4.1.1. An element $c \in H_{n+q}(X),\ 0 \leq q \leq n$ is called primitive if

$$u^{q+1} \cap c = 0.$$

We denote $P_{n+q}(X) \subset H_{n+q}(X)$ as the subspace consisting of primitive elements.

Definition 4.1.2. An element $z \in H_{n-q}(X)$ is called effective if

$$u \cap z = 0.$$

We denote $E_{n-q}(X) \subset H_{n+q}(X)$ as the subspace consisting of effective elements.

Note that $c \in H_{n+q}(X)$ is primitive iff $u^q \cap c \in H_{n-q}(X)$ is effective.

Theorem 4.1.3 (Primitive Decomposition.). *Every element $c \in H_{n+q}(X)$ decomposes uniquely as $c = c_0 + u \cap c_1 + u^2 \cap c_2 + \ldots$ where $c_j \in H_{n+q+2j}(X)$ are primitive elements. and every element $z \in H_{n-q}(X)$ decomposes uniquely as $z = u^q \cap z_0 + u^{q+1} \cap z_1 + \ldots$ where $z_j \in H_{n+q+2j}(X)$ are primitive elements.*

Proof. We note that $u^q \cap c = z$ we get 4.1.2.2 as a consequence of the Primitive decomposition. Conversely, by induction starting with $q = n$. Clearly, a dimension count shows that $P_{2n}(X) = H_{2n}(X)$, $P_{2n-1}(X) = H_{2n-1}(X)$ and $c = c_0 + u \cap c_1 + u^2 \cap c_2 + \ldots$ is trivial for $q = n, n-1$ For the induction step it is sufficient to show that any element $c \in H_{n+q}(X)$ can be written uniquely as $c = c_0 + uc_1$, $c_1 \in H_{n+q+2}(X)$, $c_0 \in P_{n+q}(X)$. According to 4.1.2.2 we have an unique $z \in H_{n+q+2}(X)$ such that $u^{q+2} \cap z = u^{q+1} \cap c$ so that $c_0 = c - u \cap z \in P_{n+q}(X)$. To prove the uniqueness, assume

$$0 = c_0 + u \cap c_1, \; c_0 \in P_{n+q}(X).$$

Then $u^{q+1} \cap (c_0 + u \cap c_1)$ and hence $u^{q+2} \cap c_1 = 0$ therefore, $c_1 = 0$ which implies $c_0 = 0$ □

This theorem shows that the homology of X is completely determined by its primitive part. Moreover, the above proof implies

$$0 \leq dim P_{n+q} = b_{n+q} - b_{n+q+2} = b_{n-q} - b_{n-q-2}$$

and hence

$$1 = b_0 \leq b_2 \leq \ldots \leq b_{2\lfloor n/2 \rfloor}.$$

These inequalities introduce additional topological conditions of complex projective Varieties. For example, the sphere S^4 cannot be an complex projective variety because $b_2(S^4) = 0 < b_0(S^4) = 1$.

4.2 Homotopy Version of Lefschetz Theorem of Hyperplane Section.

In this section we give a stronger version of Lefschetz theorem on homology of hyperplane section (3.2.1). The proof presented here is as presented in [4]. Towards that we first give some definitions.

Definition 4.2.1. A critical point p of a smooth real valued function $f : M \to \mathbb{R}$ we define a symmetric bilinear form f_{**} on T_pM by $f_{**}(v, w) = \tilde{v}_p(\tilde{w}(f))$ where $\tilde{v}$ and $\tilde{w}$ are extensions of v and w to vector fields.

Definition 4.2.2. The index of a bilinear form on a vector space V, is defined to be the maximal dimension of the subspace of V on which H is negative definite.

We now state an important result in Morse theory without proof the reader is referred to Theorem 3.5 in [4] for the proof

Theorem 4.2.1. *Let f be a differentiable function on a manifold M with no degenerate critical points, and if $M^a = \{x \,|f(x) \leq a\}$ is compact, then M has a homotopy type of a CW-complex, with one cell of dimension r for each critical point of index r.*

Theorem 4.2.2. *(Lefschetz) Let X, X_b be as in theorem 3.2.1, then we have $\pi_r(X, X_b) = 0$ for $r < k$.*

Proof. We use the fact that some small neighborhood U of X_b can be deformed into X_b within X. Consider the function $f : X \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & if\ x \in X_b \\ 1/||p - x||^2 & otherwise. \end{cases}$$

The critical points of f have index $\geq k$ and these points are non-degenerate critical points with $\epsilon \leq f \leq \infty$. Hence X has the homotopy type of $f^{-1}[0, \epsilon]$ with finitely many cells of dimension $\geq k$ attached. Choose ϵ small enough such that $f^{-1}[0, \epsilon] \subset U$. Every map from $(I^r, \dot{I}^r)$ into (X, X_b) is deformed into a map

$$(I^r, \dot{I}^r) \to (f^{-1}[0, \epsilon], X_b) \subset (U, X_b),$$

since $r < n$ and thus is deformed into X_b. $\square$

Chapter 5

The Picard-Lefschetz formulas

Let $f : Y \to G$ be as in chapter 3. Let $G^* = G \setminus \{t_1, \dots t_r\}$ where $t_1, \dots t_r$ are singular values of $f : Y \to G$, we similarly remove the corresponding singular fibres from Y, to get $Y^* = Y \setminus f^{-1}\{t_1, \dots, t_r\}$. Now by Ehresmann's fibration theorem $f : Y^* \to G^*$ is a locally trivial fibre bundle with typical fibre $Y_b \simeq X_b$

Definition 5.0.1. The fundamental group $\pi_1(G^*, b)$ acts on the homology of Y_b. This action is called *monodromy* of $f : Y \to G$.

Definition 5.0.2. An elementary path encircling a singular value t_v is

$$w_v = l_v^{-1}.\omega_v.l_v \tag{5.0.1}$$

where if t is the local coordinate of G in a neighborhood of t_v, we choose $\rho > 0$ small enough so that the disk D_v with centre t_v and radius ρ does not meet any $t_u \neq t_v$, l_v is any path in G^* from b to $t_v + \rho$ and $\omega_v(s) = t_v + \rho e^{2\pi i s}$, $0 \leq s \leq 1$.

See figure 5

We have $\pi_1(G^*, b) = \langle [w_1], \dots, [w_r] \mid [w_1] \cdot [w_2] \cdots [w_r] = 1 \rangle$

We look at the action of elementary paths w_i on $H_q(Y_b)$. Consider the following sequence of homomorphisms induced by inclusions

$$H_n(T, F) \xrightarrow{\simeq} H_n(T_i, F_i) \hookrightarrow H_n(Y_+, L) \xrightarrow{\simeq} H_n(Y_+, Y_b). \tag{5.0.2}$$

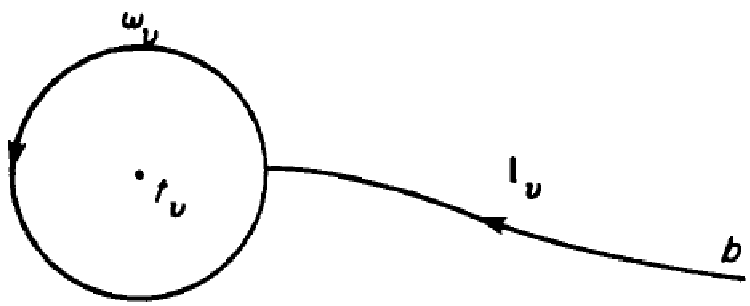

Figure 5.1: An elementary path around t_v.

3.1.2 implies that an orientation of the disk Δ determines a generator $[\Delta]$ of $H_n(T, F)$. The injective map 5.0.2 transforms $[\Delta]$ into an element $\Delta_i \in H_n(Y_+, Y_b)$. The generators $\Delta_1, \ldots, \Delta_r$ of $H_n(Y_+, Y_b)$ are transformed by the connecting homomorphism into $\delta_1, \ldots \delta_r \in H_{n-1}(Y_b)$, $i = 1, \ldots r$.

δ_i is called the vanishing cycle and Δ_i is called the corresponding thimble, the geometric boundary $\partial\Delta = S^{n-1} \subset F \subset F_i$ is an embedded $(n-1) - sphere$ in F_i. Since $f^{-1}(l_i)$ is trivially fibred there is an embedding
$j : F_i \times l_i \to Y$, $j(F_i \times l_i) = f^{-1}(l_i)$, $j(y, t_i + \rho) = y$ and $f \circ j(y, r) = r$ for $y \in F_i$ and $r \in l_i$
Then the thimble

$$C_i = \Delta \cup j(S^{n-1} \times l_i)$$

represents Δ_i.

The boundary of C_i is an embedded $(n-1)$-sphere in Y_b, which represents δ_i. As the sphere ∂C_i is moved along the thimble from Y_b following l_i into $F_i = Y_{t_i+\rho}$ and furthur into the singular fibre Y_{t_i} it vanishes at the critical point x_i.

Theorem 5.0.1. *The normal bundle of the vanishing cycle ∂C_i in Y_b is isomorphic to the tangent bundle of $(n-1)$-sphere, The self-intersection number is*

$$\langle \delta_i, \delta_i = \begin{cases} 0 & if\ n\ even \\ (-1)^{(n-1)/2}.2 & if\ n\ odd. \end{cases}$$

Proof. F is a tubular neighborhood of S^{n-1} in F_i, and S^{n-1} lies in F as the zero section Q_0 lies in the tangent bundle Q of the $n-1$ sphere. The self-intersection number of Q_0 in Q is known to be 0 or 2 depending on whether n is odd or even. This number is calculated with respect to the orientation of Q (first orientation of Q_0 and then the corresponding orientation of a fibre.) The orientation induced by complex structure of F on Q differs from the usual by the factor $(-1)^{(n-1)(n-2)/2}$. Thus the self intersection number of S^{n-1} in F_i is $(-1)^{(n-1)(n-2)/2}$ The orientation preserving diffeomorphism $h_i : Y_b \simeq F_i$, $h_i(y) = j(y,b)$, $y \in F_i$ maps S^{n-1} onto ∂C_i. $\square$

Let $f : T \to D = \{t \in C \mid |t| \leq \rho\}$ (see 3.1.1) be $f(z) = z_1^2 + \ldots z_n^2$, with $D^* = D \setminus 0$, typical fibre F. We have the relative extension (see 1.3.2) $\tau_\omega : H_{n-1}(F, F') \to H_n(T, F)$ along the path $\omega : I \to D \setminus 0$, given by $\omega(t) = \rho e^{2\pi i t}$.

Lemma 5.0.2. *Suppose $s = \partial_*[\Delta] = [S^{n-1}] \in H_{n-1}(F)$, and $c \in H_{n-1}(F, F')$ so that $\langle c, s\rangle = 1$. Then $\tau_\omega(c) = -(-1)^{n(n-1)/2}[\Delta]$.*

Proof. Since $H_n(T,F) = \langle[\Delta]\rangle$ we have $\tau_\omega(c) = \gamma[\Delta]$ with $\gamma \in \mathbb{Z}$, we prove $\gamma = -(-1)^{n(n-1)/2}$

Consider the following commutative diagram,

$$\begin{tikzcd}
H_n(F \times I, \partial(F \times I)) \arrow[r, "W_*"] \arrow[d, "\partial_*"', "\simeq"] & H_n(T, T' \cup F) \arrow[r, "\simeq", "inc^*"'] \arrow[d, "\simeq"] & H_n(T,F) \arrow[d, "\partial_*"] & \\
H_{n-1}(\partial(F \times I)) \arrow[r, "W^*"] \arrow[d, "\simeq"] & H_{n-1}(T' \cup F) \arrow[r, "R_*", "\simeq"'] & H_{n-1}(F) \arrow[r, "(Re)_*", "\simeq"'] & H_{n-1}(S^{n-1}) \arrow[d, "\simeq"] \\
H_{n-1}(\partial(Q \times I)) \arrow[rrr, "g^*"] & & & H_{n-1}(Q_0) \\
H_{n-1}(\partial(C \times I)) \arrow[rrr, "g^*"] \arrow[u, "\simeq", "inc^*"'] & & & H_{n-1}(Q_0) \arrow[u, "="]
\end{tikzcd}$$

We have

1. $W : F \times I \to T$ given by $W(x,t) = e^{\pi i t}.z$
2. $R : T' \cup F \to F$, $R|F = id_F$ and $R|_T'$ is the retraction given in 3.1.2
3. $Re = real\ part$
4. Q is as in the proof of 3.1.2,$Q_0 = \{(u,0) \in \mathbb{R}^n \times \mathbb{R}^n \mid ||u|| = 1\}$ and $Q' = \{(u,v) \in Q \mid ||v|| = 1\}$
5. $g : \partial(Q \times I) = Q' \times I \cup Q \times \partial I \to Q_0$, where $g(u + iv, t) = Re(e^{i\pi t}(u + iv))$

6. $C = \{e_1 + iv| \ v \in \mathbb{R}^n, v \perp e_1\}$ where $e_1 = (1, 0 \dots 0) \in \mathbb{R}^n$.

Start with $c \times i \in H_n(F, \partial(F \times I))$ the upper line of the diagram gives us $\tau_\omega(c) = \gamma \cdot [\Delta]$ The isomorphisms of the right boundary convert this element into $\gamma \cdot [Q_0]$. Here Q_0 is orientated as the unit sphere of the canonically orientated $\mathbb{R}^n$. The isomorphisms of the left boundary applied to $c \times i$ followed by g_* yield $\gamma \cdot [Q_0]$.

claim 6. The orientation which $c \in H_{n-1}(F, F')$ determines differs by a factor $(-1)^{n(n-1)/2}$ from the orientation of the coordinate system $(v_2, \dots, v_n)$ on C.

Consider a neighborhood of e_1 in F. The coordinate system of F is $(v_2, \dots, v_n, u_2 \dots u_n)$, where $(u_2, \dots u_n)$ is the positively oriented coordinate system of Q_0. The orientation of $(v_2, \dots, v_n)$ differs from the orientation of c by the factor as the canonical orientation of F differs the orientation of $(v_2, \dots, v_n, u_2 \dots u_n)$ since $\langle c, s\rangle = 1$. The canonical orientation of F is given by the complex coordinate system $(u_2 + iv_2, \dots, u_n + iv_n)$ which yields a positively orientated real system $(u_2, v_2, \dots u_n, v_n)$. The orientation due to $(u_2, v_2, \dots u_n, v_n)$ differs from the one due to $(v_2, \dots v_n, u_2, \dots u_n)$ by the sign of the corresponding permutation by $(-1)^{1+2+3+\dots+n-1} = (-1)^{n(n-1)/2}$.

claim 7. The degree of $g : \partial(C \times I) \to Q_0$ is -1.

The only inverse image of the point $-e_2 \in Q_0$ is the point $(e_1 + ie_2, 1/2) \in \partial C \times I \subset \partial(C \times I)$. Therefore the local mapping degree of g at $(e_1 + ie_2, 1/2)$ is γ. The orientation of C is given by $(v_2, \dots, v_n)$ followed by the canonical orientation of I determines an orientation of $C \times I$ and hence of $\partial(C \times I)$. $(v_3, \dots v_n, t)$ is positively orientated coordinate system of $\partial(C \times I)$ in a neighborhood of $(e_1 + ie_2, 1/2)$. The positively orientated coordinate system $(u_1, u_3, \dots, u_n)$ is chosen in the neighborhood of $-e_2$ in Q_0. In these coordinates $g(v_3, \dots, v_n, t) = (cos\pi t, -sin\pi t \cdot v_3, \dots, -sin\pi t \cdot v_n)$. The Jacobian of g at $(e_1 + ie_2, 1/2)$ is negative, and hence the degree of g is -1.

Combining claim 1 and claim 2 we get $\gamma = -(-1)^{n(n-1)/2}$. □

Theorem 5.0.3. *Let $f : T_v \to D_v$ be as in 3.1.2, $D_v^* = D_v \setminus T_v$ and $t_v + \rho$ be the base point. The absolute extension along the path ω_v is*

$$\tau_{\omega_v} : H_{n-1}(F_v) \to H_n(T_v, F_v), \quad \tau_{\omega_v}(x) = -(-1)^{n(n-1)/2}\langle x, s\rangle \cdot [\Delta].$$

Proof. $H_n(T_v, F_v)$ is freely generated by $[\Delta]$, and $s = \partial_*[\Delta] \in H_{n-1}(F_v)$. Let $r : (F_v, \phi) \hookrightarrow$

$(F_v, F_v \setminus \mathring{B})$ be the inclusion. The relative extension $\tau_{\omega_v} : H_{n-1}(F_v, F_v \setminus \mathring{B}) \to H_n(T_v, F_v)$ is also defined since $F_v \setminus \mathring{B}$ is strong deformation retract of $T_i \setminus \mathring{B}$ by the proof of 3. The naturality of the extension 1.3 gives the following commutative diagram:

$$\begin{tikzcd}
H_{n-1}(F_v) \arrow[r, "r_*"] \arrow[dr, "\tau_{\omega_v}"] & H_{n-1}(F_v, F_v \setminus \mathring{B}) \arrow[r, "exc_*", "\simeq"'] \arrow[d, "\tau_{\omega_v}"] & H_{n-1}(F, F') \arrow[d, "\tau_{\omega_v}"] \\
 & H_n(T_v, F_v) \arrow[r, "\simeq"] & H_n(T, F)
\end{tikzcd}$$

Now $x \in H_{n-1}(F_v)$ is transformed into $\langle x, s \rangle \cdot c \in H_{n-1}(F, F')$. The result now follows by using the previous lemma. $\square$

Theorem 5.0.4 (The PICARD-LEFSCHETZ FORMULA)**.** *If $q \neq n-1$ then $\pi_1(G^*, b)$ acts trivially on $H_q(Y_b)$. For $q = n-1$ the elementary path w_i, acts by*

$$(w_i)_*(x) = x + (-1)^{n(n+1)/2} \langle x, \delta_i \rangle \delta_i, \quad x \in H_{n-1}(Y_b).$$

Proof. Let $f : Y_+ \to D_+$ with $D_+^* = D_+ \setminus \{t_1, \ldots t_r\}$. We prove that the extension along the path w_i is

$$\tau_{w_i} : H_{n-1}(Y_b) \to H_n(Y_+, Y_b), \tau_{w_i}(x) = -(-1)^{n(n-1)/2} \langle x, \delta_i \rangle \cdot \Delta_i. \tag{5.0.3}$$

$$\begin{tikzcd}
H_{n-1}(Y_b) \arrow[rr, "(l_i)_*"] \arrow[d, "\tau_{\omega_i}"] \arrow[dr, "\tau_{\omega_i}"] & & H_{n-1}(F_i) \arrow[dl, "\tau_{\omega_i}"'] \arrow[d, "\tau_{\omega_i}"] \\
H_n(Y_+, Y_b) \arrow[r, "\simeq"] & H_n(Y_+, L) & H_n(T_i, F_i) \arrow[l]
\end{tikzcd}$$

The lowers triangles commute because the extension is natural by 1.3. $\tau_{w_i} : H_{n-1}(Y_b) \to H_n(Y_+, L)$ is $T_{w_i} = T_{l_i^{-1}\omega_i l_i} = T_{l_i^{-1}} \circ \omega_{i*} \circ l_{i*} + T_{l_i}$. Since $l_i \subset L$ the first and the third summands are 0. Therefore $\tau_{w_i} = \tau_{\omega_i} \circ l_i$ and hence the upper triangle also commutes. Using 1.3 we get the Picard-Lefschetz formula. $\square$

Chapter 6

The Monodromy and Hard Lefschetz Theorem.

6.1 Monodromy Theorem.

We now establish the connection between Hard Lefschetz theorem and monodromy. Let $f : Y \to G$ be as in 2.1.2, where Y is the modification of the projective manifold X along the axis of the pencil of hyperplanes $\{H_t\}_{t\in G}$, and $f(y)$ is the hyperplane H_t through y. Let $I \subset H_{n-1}(Y_b)$ be the module of invariant cycles as defined in 3.3.8, the following result gives the reason why this submodule is called invariant submodule.

Theorem 6.1.1. *The module $I \subset H_{n-1}(Y_b)$ consists of the cycles which are invariant under the action of $\pi_1(G^*, b)$.*

Proof. Since $\pi_1(G^*, b) = \langle [w_1], \dots, [w_r] \mid [w_1] \cdot [w_2] \cdots [w_r] = 1 \rangle$ where w_i are elementary paths. Now $y \in H_{n-1}(Y_b)$ is invariant under the action of pi_1 iff

$$y = (w_i)_*(y) = y \pm \langle y, \delta_i \rangle \delta_i, \quad \langle y, \delta_i \rangle \quad for\ i = 1, \dots, r.$$

We also have by 3.3.11, $I = \{y \in H_{n-1}(X_b) \mid \langle y, x \rangle = 0\ for\ every\ x \in V\}$. Since $\delta_1, \dots \delta_r$ generate V we have $I = \{y \in H_{n-1}(X_b) \mid \langle y, \delta_i \rangle = 0\ for\ i = 1, \dots, r\}$.

$\square$

Theorem 6.1.2 (MONODROMY THEOREM). *Let $\pi = \pi_1(G^*, b)$. For the homology with coefficients from a field the following results are equivalent :*

1. *The Hard Lefschetz Theorem*
2. *V is a non-trivial simple π-module or $V = 0$*
3. *$H_{n-1}(Y_b)$ is a semi-simple π- module.*

Proof.

$$2 \Rightarrow 3$$

$I \cap V \subset V$ is a π-invariant submodule of V. $I \cap V = 0$ or $I \cap V = V$ since V is simple. But since I acts non-trivially on V and trivially on $I \cap V$ we have $I \cap V = 0$ and hence by 3.3.1 we have $H_{n-1}(Y_b) = V \oplus I$ is a direct sum of simple and semi-simple π-module, and hence is a semi-simple π-module.

$$3 \Rightarrow 1$$

We show $3 \Rightarrow 6$. It suffices to show that the map $I \to I^\vee$ given by $z \to \langle z, - \rangle$, is epimorphic where $I^\vee$ is the dual of the module I : Let $\varphi \in I^\vee$, since $H_{n-1}(Y_b)$ is semi-simple we have $I \oplus M = H_{n-1}(Y_b)$ fro some π-invariant submodule $M \subset H_{n-1}(Y_b)$, so φ can be extended to a linear form ψ on $H_{n-1}(Y_b)$ as follows:

$$\psi(x + y) = \varphi(x), x \in I,\ y \in M.$$

Since $\langle -, - \rangle$ is non-degenerate on $H_{n-1}(Y_b)$ there is unique $z \in H_{n-1}(Y_b)$ with $\langle z, x + y \rangle = \varphi(x)$. Let $\alpha \in \pi$, then $\langle \alpha z, x + y \rangle = \langle z, \alpha^{-1}(x + y) \rangle = \varphi(x)$, and hence $z = \alpha z$ for every $\alpha \in \pi$, i.e $z \in I$ and $\varphi(x) = \langle z, x \rangle$ for every $x \in I$.

$$1 \Rightarrow 2$$

We show $5 \Rightarrow 2$. By 5 V is generated by $\delta_1, \ldots \delta_r$. Let $F \neq 0$ be a π-invariant submodule of V, and let $x \in F$ be a non-zero element. There is δ_μ with $\langle x, \delta_\mu \rangle \neq 0$. By Picard-Lefschetz-formula we have $(w_\mu)_*(x) = x \pm \langle x, \delta_\mu \rangle \delta_\mu$. Thus π acts non trivially on F and $\delta_\mu \in F$, but then F, contains all of V by the following result. $\square$

Lemma 6.1.3. *For any two vanishing cycles δ_ν, δ_μ there is an $\alpha \in \pi$ such that $\alpha \cdot \delta_\mu = \pm\delta_\nu$.*

Proof. The proof of the lemma is deferred to next section. $\square$

6.2 Zariski's Theorem

Let $G \subset \mathbb{P}_N$ be a projective line in general position with respect to a hypersurface $X \subset \mathbb{P}_N$. That is G avoids the singularities of X and intersects X transversally then $G \cap X = \{t_1, \dots, t_r\}$ is finite and r =degree of X.

Theorem 6.2.1. *The embedding $G \backslash X \hookrightarrow \mathbb{P}_N \backslash X$ induces an epimorphism of the fundamental groups.*

Proof. Choose a point b in $G \setminus X$. All the lines through b form a subspace $\check{\mathbb{P}}_{N-1}$ of $\check{\mathbb{P}}_N$. Let $a \in \check{\mathbb{P}}_{N-1}$ so that $G_a = G$. The point b in $\mathbb{P}_N$ is blown up:

$$Q = \{(x, z) \in \mathbb{P}_N \times \check{\mathbb{P}}_{N-1} | x \in G_z\}.$$

We have two projections

$$\mathbb{P}_N \xleftarrow{p} Q \xrightarrow{f} \check{\mathbb{P}}_{N-1}.$$

Now,

$$p^{-1}(b) = \{b\} \times \check{\mathbb{P}}_{N-1}$$

and

$$p : Q \setminus p^{-1}(b) \simeq \mathbb{P}_N \setminus \{b\}.$$

Let $Y = p^{-1}(X)$, $p^{-1}(b) \cap Y = \emptyset$ since $b \notin X$. By 1.3.7 the second projection $f : Q \to \check{\mathbb{P}}_{N-1}$ fibres Q locally trivially with typical fibre G. Let $C \subset \check{\mathbb{P}}_{N-1}$ be the algebraic subset of set of lines through b which are not in general position with X. The pair $Q^* = Q \setminus f^{-1}(C)$, $Y^* = Y \setminus f^{-1}(C)$ is locally trivially fibred by f over $\check{\mathbb{P}}_{N-1} \setminus C$. Y^* is smooth and $f|_{Y^*}$ has a maximal rank everywhere, and hence $Q^* \setminus Y^*$ is fibred locally trivially over $\check{\mathbb{P}}_{N-1} \setminus C$ with typical fibre $G \setminus X$. Consider the following commutative diagram, the upper line is the exact homotopy sequence of this fibration, we show i_* is epimorphic by showing that there exists a $\beta \in \pi_1(Q^* \setminus Y^*, (b,a))$ with $f_*(\beta) = 1$ for every $\alpha \in \pi_1(\mathbb{P}_N \setminus X)$

$$\begin{tikzcd}
\pi_1(G \setminus X, b) \arrow[r] \arrow[ddr, "i_*"'] & \pi_1(Q^* \setminus Y^*, (b,a)) \arrow[r, "f_*"] \arrow[d, "j_*"] & \pi_1(\check{\mathbb{P}}_{N-1} \setminus C, a) \\
 & \pi_1(Q \setminus Y, (b,a)) \arrow[d, "p_*"] & \\
 & \pi_1(\mathbb{P}_N \setminus X, b). &
\end{tikzcd}$$

We now prove that p_* is epimorphic. Let $b' \neq b$, every element in $\pi_1(\mathbb{P}_N \setminus X, b')$ is given by a path which avoids b and such a path is uniquely lifted to $Q \setminus Y$ because $p : Q \setminus (Y \cup p^{-1}(b)) \simeq \mathbb{P}_N \setminus (X \cup \{b\})$. Similarly j_* is epimorphic: since $f^{-1}(C) \cap (Q \setminus Y)$ is of real codimension 2every path in $Q \setminus Y$ can homotopically be deformed avoiding $f^{-1}(C)$ and is contained in $Q^* \setminus Y^*$. Consider an arbitrary counterimage of α, $\beta' \in \pi_1(Q^* \setminus Y^*)$ but $f(\beta') \neq 1$. There is a path u in $\{b\} \times (\check{\mathbb{P}}_{n-1} \setminus C) \subset Q^* \setminus Y^*$ with $f_*(\beta') = [f \circ u]$. Then $\beta = \beta'[u]^{-1}$ is a counterimage of α with $f_*(\beta) = 1$ since $p \circ j \circ u$ is constant.

□

Let G_0 and G_1 be two lines in general position with respect to the hypersurface X, and $b \in G_0 \cap G_1$, $b \notin X$. Let v_0 and v_1 be elementary paths through b in $G_0 \setminus X$ and $G_1 \setminus X$.

Theorem 6.2.2. *If X is irreducible the homotopy classes of the elementary paths v_0 and v_1 are conjugates in $\pi_1(\mathbb{P}_N \setminus X, b)$.*

Proof. Let $Z \subset X$ be the proper algebraic set containing of all points x such that the line through x and b is not in general position. The points $c_0 \in G_0 \cap X$ and $c_1 \in G_1 \cap X$ be such that v_0 encircles c_0 and v_1 encircles c_1. Let w be a path in $X \setminus Z$ from c_0 to c_1, such a path exists since X is irreducible. The line through b and $w(t)$, $0 \leq t \leq 1$ be denoted by G_t, and $\Phi_t : \mathbb{C} \simeq G_t \setminus \{b\}$ be isomorphisms so that $\mathbb{C} \times [0,1] \to \mathbb{P}_N$, $(x,t) \to \Phi_t(z)$, is continuous. Let $\Phi_t^{-1}(w(t)) = w^*(t)$ Choose ρ small enough so that the disk G_t with centre $w(t)$ and radius ρ intersects X only in $w(t)$. The homotopy H between the paths $\omega_0(s) = w^*(0) + \rho \cdot e^{2\pi i s}$ and $\omega_1(s) = w^*(1) + \rho \cdot e^{2\pi i s}$ which encircle c_0 and c_1 respectively once is $H(t,s) = \Phi_t(w^*(t) + \rho \cdot e^{2\pi i s})$. And hence v_0 and v_1 are conjugates in $\pi_1(\mathbb{P}_N \setminus X, b)$. □

We now give the proof of 6.1.3.

Proof. w_μ and w_ν be elementary paths belonging to δ_μ and δ_ν respectively. By 6.2.2, $[w_\mu]$ and $[w_\nu]$ are conjugates in $\check{\mathbb{P}}_N \setminus \check{X}$, and since $\pi_1(G^*) \to \check{\mathbb{P}}_N \setminus \check{X}$ is surjective there is a path u in G^* such that

$$[u] \cdot [w_\mu][u]^{-1} = [w_\nu].$$

Let $p_2 : W \setminus p_2^{-1}(\check{X}) \to \check{\mathbb{P}}_N \setminus \check{X}$ be locally trivially fibre bundle as in 2.1. $f^* : Y^* \to G^*$ is fibre bundle and hence the action of $\pi_1(G^*)$ on $H_{n-1}(Y_b)$ factors through $\pi_1(\check{\mathbb{P}}_N \setminus \check{X})$ and thus

$$u_* \circ w_{\mu_*} = w_{\nu_*} \circ u_*.$$

Let $x \in H_{n-1}(Y_b)$ be arbitrary element, then by Picard-Lefschetz-formula,

$$\langle x, \delta_\mu \rangle u_*(\delta_\mu) = \langle u_*(x), \delta_v \rangle \delta_v.$$

Since the intersection form is non degenerate by Poincaré duality, either $\delta_\mu = 0$ and hence $\delta_\nu = 0$ or there is x such that $\langle x, \delta_\mu \rangle \neq 0$ which is $u_*(\delta_\mu) = c \cdot \delta_v$. Now $\langle u_*(x), \delta_v \rangle \delta_v = \langle u_*(x), u_*(\delta_\mu) \rangle u_*(\delta_\mu) = c^2 \langle u_*(x), \delta_v \rangle \delta_v$ which implies $c = \pm 1$.

□

Chapter 7

Singular points of Complex hypersurfaces

In this chapter we give a brief study of singular points of complex hypersurfaces. We Let $f \in \mathbb{C}[z_1, \ldots z_n]$ be a non-constant polynomial, and let $V \subset \mathbb{C}^{n+1}$ denote the zero set of f. The aim of this chapter is to study the topology of V in neighborhood of some point z_0.

7.1 Brauner's Construction

Let V be hypersurface as above. Let $K = V \cap S_\epsilon$ where ϵ is small enough. Then the topology of V with the disk bounded by S_ϵ is closely related to the topology of K in the sense that if $D_\epsilon = \{z \mid ||z - z_0|| \leq \epsilon\}$ and if z_0 is a non-singular or an isolated singular point of V then for small ϵ $D_\epsilon \cap V$ is homeomorphic to cone over K. By cone over K we mean, $Cone(K) = \{tk + (1-t)z_0 \mid k \in K 0 \leq t \leq 1\}$.

Definition 7.1.1. Given a manifold M and a sub-manifold N we say that N can be knotted in M if there exists an embedding of N in M which is not isotopic to N.

Proposition 7.1. Let $z_0 \in V$ be regular point of f then $K = V \cap S_\epsilon$ is an unknotted sphere in S_ϵ, for small enough ϵ.

Proof. The smooth function $r(z) = ||z - z_0||^2$ restricted to non-singular points of V has non-degenerate critical point at z_0 and so $r(z) = u_1^2 + \dots u_k^2$ in local coordinates $u_1, \dots u_k$. Hence K is diffeomorphic to to sphere $\{(u_1, \dots, u_k) | u_1^2 + \dots u_k^2 = \epsilon^2\}$ □

On the other hand if z_0 is not a regular point of then the embedding can be knotted as illustrated in the following example.

Proposition 7.2. Let $f(z_1, z_2) = z_1^p + z_2^q$ be a polynomial with $p, q \geq 2$ and co-prime, here origin is the critical point of f. Then the intersection of $V = f^{-1}(0)$ with a sphere S_ϵ centered at the origin is a "torus knot" of the type (p, q) in the 3-sphere.

Proof. By solving the equations $z_1^p + z_2^q = 0$ and $|z_1|^2 + |z_2|^2 = 1$, we observe that $K = \{(\zeta e^{qi\theta}, \eta e^{(pi\theta + \pi i)/q}) |\ \theta \in [0, 2\pi]\} \subset T^2 = \{(z_1, z_2) \mid |z_1| = \zeta, |z_2| = \eta\}$. Hence K sweeps around the torus p times in one coordinate and q times in other. □

Torus knot of the type (2,3) is illustrated in 7.1

7.2 Fibration Theorem

We now state fibration theorem without proof (see [7] chapter 4 for the proof) which is useful in describing the topology of K.

Theorem 7.2.1. *If z_0 is any point of the complex hypersurface $V = f^{-1}(0)$ and if S_ϵ is sufficiently small sphere centered at z_0 then the mapping $\Phi(z) = f(z)/|f(z)|$ from $S_\epsilon \setminus K$ to the unit circle is the projection map of a smooth locally trivially fibre bundle. Each fibre $F_\theta = \Phi^{-1}(e^{i\theta}) \subset S_\epsilon \backslash K$.*

Now by using Morse theory one can prove that each fibre F_θ is parallelizable and is homotopic to a finite CW-complex of dimension n and K is $n-2$ connected. For the proof of these statements refer [7]

If we make the additional hypothesis that z_0 is an isolated critical point of f we can give a better description of of each fibre F_θ.

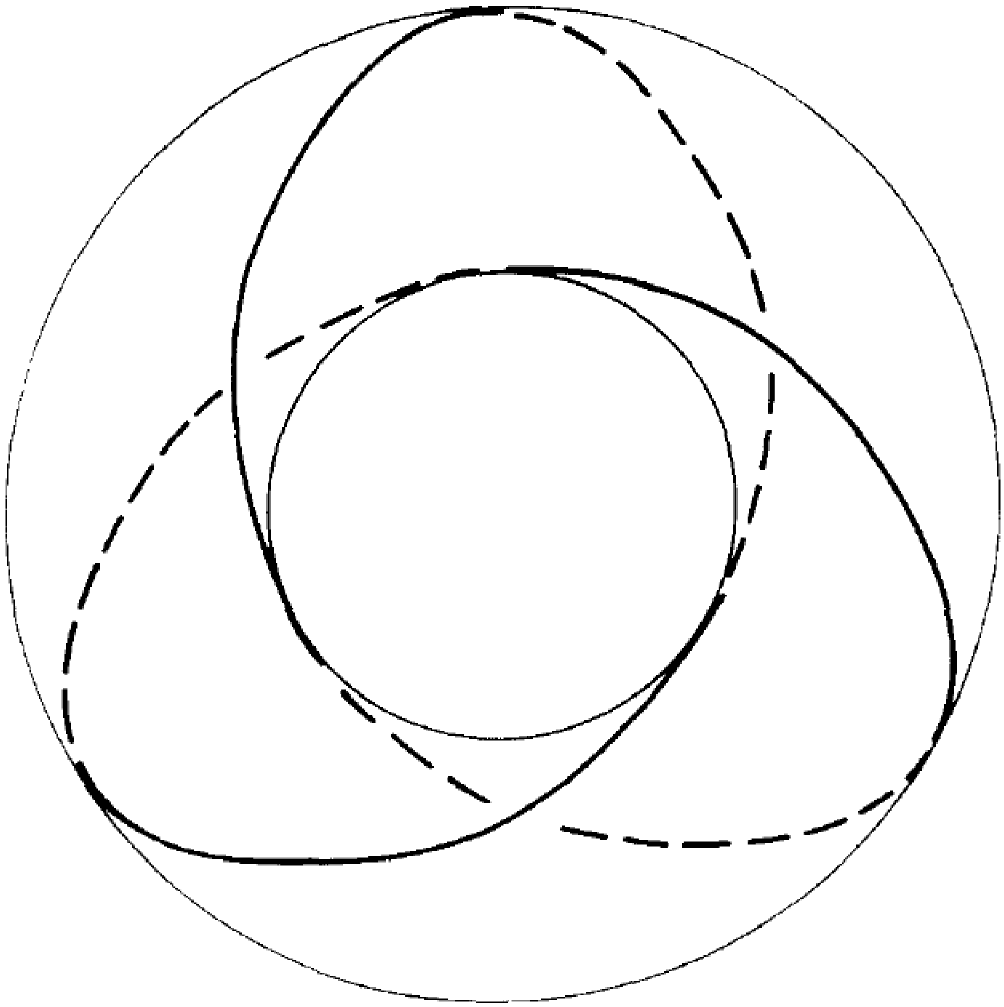

Figure 7.1: Torus knot of type (2,3)

Theorem 7.2.2. *If z_0 is an isolated critical point of f, then each fibre F_θ is homotopic to $S^n \vee \ldots \vee S^n$ of n-spheres. Each fibre F_θ has the closure $\bar{F}_\theta = F_\theta \cup K$.*

Proof. We have $H_n(F_\theta)$ is a free abelian group since F_θ is homotopic to finite CW-complex of dimension n and torsion torsion elements would give rise to cohomology classes in dim $n+1$. Now by Hurewicz theorem we have finitely many maps $(S^n, basepoint) \to (F_\theta, basepoint)$ representing the basis and combining these yeild an isomorphism of homology groups of $S^n \vee \ldots \vee S^n \to F_\theta$ and using Whitehead's theorem it is a homotopy equivalence. □

Remark. Rank $H_n(F_\theta)$ is equal to middle Betti number of F_θ. ([7] see chapter 7.) Also each fibre F_θ has the closure $\bar{F}_\theta = F_\theta \cup K$.

Theorem 7.2.3. *The fibre F_θ has the homology of a point in dimension less than n, that is when $i < n$.*

Proof. By Alexander Duality $\tilde{H}^{2n-i}(\bar{F}_\theta) \simeq \tilde{H}_i(S_\epsilon \setminus \bar{F}_\theta)$ which is zero if $2n - i > n$. □

Theorem 7.2.4. *If the origin is an isolated critical point of f, then the fibres F_θ are not contractible, and the manifold $K = V \cap S_\epsilon$ is not an unknotted sphere in S_ϵ.*

Proof. Since F_θ is homotopic to $S^n \vee \ldots \vee S^n$ and the number of sphere is greater than 0 hence F_θ is not contractible. If K were topologically unknotted sphere in S_ϵ then $S_\epsilon \setminus K$ would be homotopic to circle. We give the explicit homotopy, Let $S_\epsilon = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + |z_2|^2 + \ldots + |z_n|^2 = 1\}$ let $K = \{(z_1, z_2, \ldots, z_n) \in S_\epsilon : |z_1|^2 + |z_2|^2 + \ldots + |z_{n-1}|^2 = 1\}$. We define the deformation retract of $S_\epsilon \setminus K$ to $\{(z_1, z_2, \ldots, z_n) \in S : |z_n|^2 = 1\} \subset S_\epsilon \setminus K$ by

$$f_t = \frac{(1-t)(z_1, z_2, \ldots, z_n) + t(0, 0, \ldots, 0, \frac{z_n}{|z_n|})}{|(1-t)(z_1, z_2, \ldots, z_n) + t(0, 0, \ldots, 0, \frac{z_n}{|z_n|})|}$$

Now by the exact homotopy sequence

$$\ldots \to \pi_{n+1}(S^1) \to \pi_n(F_0) \to \pi_n(S_n \setminus K) \to \ldots$$

would lead us to a contradiction. □

The next natural question to ask is if K is a topological sphere when the origin is an isolated critical point. The following theorem gives the criterion to say if K is a topological sphere.

Theorem 7.2.5. *If $n \neq 2$ then K is homeomorphic to S^{2n-1} if and only if K is a homology sphere.*

Proof. We give sketch of proof, now if $n \geq 3$, K is simple connected and the dimension of K is greater than or equal to 5, and hence we can apply generalized Poincaré hypothesis ([8]).

□

Remark. The statement is not true for $n = 2$. Consider the complex polynomial $f(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_3^5$. This 3-manifold is homology sphere, but $\pi_1(K)$ is isomorphic to $SL(2, \mathbb{Z}_5)$.

Conclusion

In this project we first looked at the existence of Lefschetz Pencil on a non singular complex projective variety. Along the way we encountered beautiful results like Lefschetz famous theorem on Homology of Hyperplane section, and Weak Lefschetz Theorem. We then see more intuitive and subtle Hard Lefschetz Theorem. We then viewed homotopy version of Lefschetz Theorem using Morse Theory, this is originally suggested by R.Thom and worked out by Andreotti-Frakel and Bott. We then saw Picard-Lefschetz Formula and Monodromy, it is a complex analog of Morse theory that studies the topology of a real manifold by looking at the critical points of a real function. At the end we saw the topology associated with singular points of Complex Hypersurfaces.

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