Cohomology of Representations and Langlands Functoriality

A thesis

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by

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Dedicated to

My Parents and My Sister

Abstract

Let G be a real semi-simple Lie group. Let Γ be an arithmetic subgroup of the group G. Suppose that F is a finite-dimensional representation of G. One of the objects of interest is the cohomology group $H^*(\Gamma, F)$. In particular, determining when these groups are non-zero and computing cohomology classes of these groups. It is well known that these groups have interpretations using relative Lie algebra cohomology of the group G with respect to a compact subgroup K. This interpretation gives us a relation between the cohomology groups $H^*(\Gamma, F)$ and a finite subset of the set of representations of G. Here we obtain some non-vanishing results for the cohomology classes for the group $\mathrm{GL}(N)$. We use the principle of Langlands functoriality to compute these classes. We start with π, a *'nice'* representation of a classical group G, and use Local Langlands correspondence to transfer π to a representation, $\iota(\pi)$, of an appropriate $\mathrm{GL}(N)$ and ask whether $\iota(\pi)$ contribute to the cohomology groups $H^*(\Gamma, F)$. We characterize when a tempered representation of a classical group G transfers to a cohomological representation of $\mathrm{GL}(n)$. This is summarized in Theorem 4.2.5. We also start with a cohomological representation of $\mathrm{Sp}(4, \mathbb{R})$ and ask when the transferred representation of $\mathrm{GL}(5, \mathbb{R})$ is cohomological. We obtain a complete result in the case of representations with trivial coefficients. This is summarized in Theorem 5.5.2.

Certificate

Certified that the work incorporated in the thesis entitled "*Cohomology of Representations and Langlands Functoriality*", submitted by *Makarand Sarnobat* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Date: February 27, 2019

Prof. A. Raghuram

Thesis Supervisor

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

Date: February 27, 2019

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Contents

Chapter 1

Introduction

Let G be a connected semi-simple real Lie group. Let K be a maximal compact subgroup of G and Γ a torsion free arithmetic subgroup of G. Suppose F is a finite-dimensional complex representation of G. Let $S_K^G = G/K$ be the associated symmetric space. We are interested in the Eilenberg-Maclane cohomology groups $H^*(\Gamma, F)$. Let $\Omega_{S_K^G}(F)$ be the space of smooth F-valued differential forms on S_K^G. Let $\Omega_{S_K^G}(F)^\Gamma$ be the subspace of Γ-invariant differential forms in $\Omega_{S_K^G}(F)$, where the action of Γ is obtained in the obvious way. Then, we have a canonical isomorphism between $H^*(\Gamma, F)$ and the cohomology groups obtained from the sequence of $\Omega^*_{S_K^G}(F)^\Gamma$.(See [23]).

Furthermore, using the projection map from $\Gamma\backslash G(\mathbb{R})$ to $\Gamma\backslash S_G^K$, we can identify the complex $\Omega_{S_G^K}(F)^\Gamma$ with the co-chain complex of the relative Lie algebra cohomology $C^*(\mathfrak{g}, \mathfrak{k}; C^\infty(\Gamma\backslash G(\mathbb{R})) \otimes F)$, where $C^\infty(\Gamma\backslash G(\mathbb{R}))$ is the space of all smooth complex valued functions on $\Gamma\backslash G(\mathbb{R})$, $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K, respectively. This identification leads to a canonical isomorphism [23]

$$H^*(\mathfrak{g}, \mathfrak{k}; C^\infty(\Gamma\backslash G(\mathbb{R})) \otimes F) \longrightarrow H^*(\Gamma, F). \tag{1.1}$$

Let $L^2(\Gamma\backslash G(\mathbb{R}))$ be the space of smooth square integrable functions on

$\Gamma\backslash G(\mathbb{R})$ and $L_0^2(\Gamma\backslash G(\mathbb{R}))$ be the space of cusp forms in $L^2(\Gamma\backslash G(\mathbb{R}))$, i.e., all functions f such that Xf is square integrable for all $X \in \mathfrak{U}(\mathfrak{g})$ and

$$\int_{(U(\mathbb{R})\cap\Gamma)\backslash U(\mathbb{R})} f(ug)du = 0 \ ; \ \text{for all } g \in G(\mathbb{R}),$$

where U is the unipotent radical of a proper parabolic subgroup P in G. The inclusion $L_0^2(\Gamma\backslash G(\mathbb{R})) \longrightarrow L^2(\Gamma\backslash G(\mathbb{R}))$ induces a homomorphism in the cohomology groups

$$H^*(\mathfrak{g}, \mathfrak{k}; L_0^2(\Gamma\backslash G(\mathbb{R})) \otimes F) \longrightarrow H^*(\mathfrak{g}, \mathfrak{k}; L^2(\Gamma\backslash G(\mathbb{R})) \otimes F).$$

This map is injective and the image can be thought of as a subgroup of $H^*(\Gamma, F)$. The image of $H^*(\mathfrak{g}, \mathfrak{k}; L_0^2(\Gamma\backslash G(\mathbb{R}))\otimes F)$ is denoted by $H^*_{cusp}(\Gamma, F)$. For $H^*_{cusp}(\Gamma, F)$, we have a Matsushima type decomposition

$$H^*_{cusp}(\Gamma, F) = \bigoplus_{H_\pi} m(H_\pi, \Gamma) H^*(\mathfrak{g}, \mathfrak{k}; H_\pi \otimes F), \tag{1.2}$$

where H_π are the representations of G which occur in the cuspidal part of the discrete spectrum with finite multiplicities. Note that the above decomposition is a finite direct sum. This isolates a finite set of representations of G depending on the finite-dimensional representation F (see [23]). Thus, the study of $(\mathfrak{g}, K)$-cohomology of representations of G is related to the study of cuspidal representations of G.

As an example, let $G = \mathrm{SL}(2, \mathbb{R})$ and $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. Let V_k be the finite-dimensional representation of G of dimension k. Then 1.2 reduces to the Eichler-Shimura isomorphism

$$H^1_{cusp}(\Gamma, V_k) = S^+_{k+1}(\Gamma) \oplus S^-_{k+1}(\Gamma),$$

where $S^\pm_{k+1}(\Gamma)$ denotes the space of holomorphic and anti-holomorphic cusp forms with respect to Γ; and the dimension of $S_{k+1}^{\pm}(\Gamma)$ denotes the multiplicities $m(D_k^{\pm}, \Gamma)$ of the discrete series representations of $G = \mathrm{SL}(2, \mathbb{R})$ occurring in $L_0^2(\Gamma \backslash G(\mathbb{R}))$. One sees that such an isomorphism lets us view functions (modular forms in this case) on $\mathrm{SL}(2, \mathbb{R}) \backslash \mathrm{SO}(2)$, which satisfy certain transformation properties with respect to Γ, as elements of the cohomology groups. This gives us a different viewpoint towards the same objects which is useful at times.

The aim of the thesis is to obtain non-vanishing results for the $(\mathfrak{g}, K)$-cohomology groups with respect to some representations of the group $G = \mathrm{GL}(n)$, over the real and the complex numbers. We now describe the ideas that we used to obtain these results. We know that Langlands functoriality allows us (under certain conditions), starting with a representation of a group G, to obtain a representation of a different group G'. To make things more precise, let us consider a particular example. Let $G = \mathrm{Sp}(2n, \mathbb{R})$. Then ${}^L G^{\circ} = \mathrm{SO}(2n+1, \mathbb{C})$ is the connected component of the Langlands dual group. Given any representation π of G, the Local Langlands correspondence attaches to this representation a *Langlands parameter* which is a homomorphism from the Weil group of $\mathbb{R}$ to the Langlands dual group. Using the fact that the inclusion $\iota : \mathrm{SO}(2n+1, \mathbb{C}) \hookrightarrow \mathrm{GL}(2n+1, \mathbb{C})$ is a L-map, we obtain a parameter for a representation of $\mathrm{GL}(2n+1, \mathbb{R})$. Given that we can obtain a representation of $\mathrm{GL}(2n+1, \mathbb{R})$ from a representation of $\mathrm{Sp}(2n, \mathbb{R})$, we may ask which of these representations of $\mathrm{GL}(2n+1)$ are cohomological? We consider the following 2 questions:

1. Consider the tempered representations of the group G and transfer them to obtain representations of $\mathrm{GL}(N)$ and check whether these representation are cohomological? This is the question which we will be taking up in Chapter 4. A complete answer for this question is noted in Theorem 4.2.5. We obtain a characterization for tempered represen-

tations of G which are transferred to a cohomological representation of $\mathrm{GL}(N)$.

2. David Vogan and Gregg Zuckerman in 1984 classified unitary cohomological representations of a real connected semi-simple Lie group (See [27]). Considering this, we start with a cohomological representation of G and transfer this representation to $\mathrm{GL}(n)$ and then ask whether the resulting representation is cohomological or not. That is we ask whether the property of being *'cohomological'* is preserved under the Langlands transfer. We take up this direction in the final chapter of the thesis. We will work out the case when $G = \mathrm{Sp}(4, \mathbb{R})$. We list down all the cohomological representations of G according to the Vogan-Zuckerman classification and compute their transfers to $\mathrm{GL}(5, \mathbb{R})$. In this case also we have a complete answer which is noted in 5.5.2.

The first instance of studying cohomological representations along with the functoriality was observed in a paper of Labesse and Schwermer in 1986 (see [17]). They considered the symmetric power transfer from $\mathrm{GL}(2)$ to $\mathrm{GL}(3)$ and proved that if we start with a cohomological representation of $\mathrm{GL}(2)$ the representation obtained by transferring to $\mathrm{GL}(3)$ is also cohomological. Further, if we know which finite-dimensional representation of $\mathrm{GL}(2)$ is relevant then we can also tell which finite-dimensional representation is relevant for the transferred representation. In loc.cit., non-trivial cuspidal cohomology classes for $\mathrm{SL}(3)$ over some number fields were also constructed.

This result was further generalized by Raghuram in his 2016 paper (see [20]), where a representation of $\mathrm{GL}(2)$ is transferred to a representation of $\mathrm{GL}(n)$ using the symmetric power transfer and a similar observation was made. We study how much this observation can be generalized.

A standard assumption which is made when one studies the special values of L-functions attached to a representation π is that π is cohomological.

Generally, one is looking at global representations, i.e., representations of $G(\mathbb{A})$, where $\mathbb{A}$ is the ring of adèles attached to a number field F. Cohomologicalness of a global representation just boils down to its cohomologicalness at the archimedean places. This thesis addresses questions which are local in nature and are related to the archimedean places. Results similar to the results obtained here have been used to study special values of L-functions. Some of the results in this direction can be found in [1], [20] and [21].

Chapter 2 introduces the groups that we would be dealing with, their corresponding Lie algebras and some theory of Lie algebras. We then list a class of finite-dimensional representations which will be relevant to the questions which we deal with here. Also, we make a list of the set of roots, positive roots and half sum of positive roots for further use. The main references for this chapter are [12], [14] and [15].

Chapter 3 introduces some basic representation theory, cohomological representations, Local Langlands correspondence and Langlands functoriality. All these will be done in the local setting where the base field is $\mathbb{R}$ or $\mathbb{C}$. We characterize discrete series representations for the classical Lie groups. We also make explicit the Local Langlands Correspondence for the case when $G = \mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$. For details in representation theory and cohomology of representations the reader can refer to [3], [14] and [15]. In case of Local Langlands correspondence and Langlands functoriality one can refer to [6], [7] and [18]. In the case of $\mathrm{GL}(n)$, this is given quite explicitly in [16].

Chapter 4 deals with transferring tempered representations from a classical group to an appropriate $\mathrm{GL}(N)$. The N depends on the group G and the list can be found in Table 4. This chapter characterizes tempered representations which are transferred to cohomological representations and the main result is stated in the form of Thm 4.2.5.

Chapter 5 deals with the special case $G = \mathrm{Sp}(4, \mathbb{R})$. This uses the Vogan-

Zuckerman classification of unitary irreducible cohomological representations of G (see [27]). We then transfer these representations to $\mathrm{GL}(5,\mathbb{R})$ and study the cohomological properties of the transferred representations. The main result of this chapter is noted in Thm 5.5.2. The theorem talks about representations which are cohomological with respect to the trivial coefficients. We also make a conjecture at the end of the chapter about representations which are cohomological with respect to non-trivial coefficients.

Chapter 2

Lie Groups and Lie Algebras

In this chapter we introduce the groups which are relevant to us. We will list down some of the relevant finite-dimensional representations of these Lie groups using their corresponding Lie Algebras.

2.1 Lie Theory

We start this chapter by defining some Lie groups and their corresponding Lie algebras. The aim of this chapter is to list some finite-dimensional representations of these groups using the Lie Algebras. Towards that, we will go through some theory of Lie algebras which will be relevant to us. As a reference, the reader may refer to [12], [14] and [15].

All the groups considered here are real Lie groups. Let's recall the definitions of the classical Lie groups that will be considered in this article. Let I_n be the $n \times n$ identity matrix. Define $J_n := \text{anti-diag}(1, \dots, 1)$, i.e., $J_n(i,j) = \delta_{i,n-j+1}$, and let $J'_{2n} := \text{anti-diag}(J_n, -J_n)$. Let $p + q = n$; we will often assume that $p \geq q \geq 1$. Let $I_{p,q} = \text{diag}(1, \dots, 1, -1, \dots, -1)$ be the $n \times n$ diagonal matrix with p many 1's and q many -1's. Put $J_{p,q} = \text{anti-diag}(J_q, I_{p-q}, J_q)$. Let ${}^tA$ and $A^* = \overline{{}^tA}$ denote the transpose and the conjugate-transpose of A, respectively; where, conjugation is either in $\mathbb{C}$ or in $\mathbb{H}$ as the case might be. Define:

$$\begin{aligned}
\mathrm{Sp}(2n,\mathbb{R}) &:= \{A \in \mathrm{GL}(2n,\mathbb{R}) \ : \ {}^tA\, J'_{2n}\, A \ = \ J'_{2n}\},\\
\mathrm{Sp}(p,q) &:= \{A \in \mathrm{GL}(n,\mathbb{H}) \ : \ A^*\, J_{p,q}\, A \ = \ J_{p,q}\},\\
\mathrm{SO}(p,q) &:= \{A \in \mathrm{SL}(n,\mathbb{R}) \ : \ {}^tA\, J_{p,q}\, A \ = \ J_{p,q}\},\\
\mathrm{U}(p,q) &:= \{A \in \mathrm{GL}(p+q,\mathbb{C}) \ : \ A^*\, J_{p,q}\, A \ = \ J_{p,q}\},\\
\mathrm{SO}^*(2n) &:= \{A \in \mathrm{SU}(n,n) \ : \ {}^tA \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} A = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}\},
\end{aligned}$$

where in the last definition, $\mathrm{SU}(n,n) = \mathrm{U}(n,n) \cap \mathrm{SL}(2n,\mathbb{C})$.

The finite-dimensional representations of these groups which are important to us are obtained by the restriction of irreducible finite-dimensional representations of their corresponding complexified Lie groups. These finite dimensional representations of the complexified Lie groups are in bijection with the irreducible finite-dimensional representations of the Lie algebras [12], [14] and [15]. Also, the *discrete series* representations of these Lie groups are parameterized by the set of highest weights modulo an equivalence relation (see [3]). We will deal with the discrete series representations in the next chapter.

The task at hand now, is to characterize the irreducible finite-dimensional representations of the Lie algebras. We will now set up the necessary notations. For details, the reader is referred to [3], [12], [14] and [15].

2.1.1 Root Space Decomposition

Root space decomposition, as the name suggests, decomposes the Lie algebra $\mathfrak{g}$ into to root spaces with respect to a Cartan subalgebra of $\mathfrak{g}$. This is similar to the familiar simultaneous eigenspace decomposition with respect to commuting matrices. This is useful in classification of Lie algebras, though we will not pursue this direction here. We would use this in parameterizing the irreducible finite-dimensional representations as well as the discrete series representations. In this section, we assume that the base field is $\mathbb{C}$.

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a maximal abelian subalgebra consisting of *ad*-semisimple elements. The linear transformation $ad\ X : \mathfrak{g} \to \mathfrak{g}$ is semi-simple (same as diagonalizable in this case) for all $X \in \mathfrak{h}$, and $\mathfrak{h}$ is abelian, hence, they can be simultaneously diagonalized allowing us to decompose $\mathfrak{g}$ into simultaneous root spaces, i.e, $\mathfrak{g}$ can be written as a direct sum of subspaces of the form,

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : ad\ H(X) = \alpha(H)X \text{ for all } H \in \mathfrak{h}\},$$

where α runs over all elements of $\mathfrak{h}^*$. Denote by $\Phi(\mathfrak{g}, \mathfrak{h})$ the set of non-zero elements $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq 0$. The elements of $\Phi(\mathfrak{g}, \mathfrak{h})$ are called roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The subspace $\mathfrak{g}_0$ is nothing but the centralizer of $\mathfrak{h}$. Therefore, we have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_\alpha.$$

This is known as the root space decomposition of $\mathfrak{g}$.

Example 2.1.1. *Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{h}$ be the set of all trace 0 diagonal matrices. That is,*

$$\mathfrak{h} = \{\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} : \sum a_i = 0\}$$

Then the set $\Phi(\mathfrak{g}, \mathfrak{h}) = \{e_i - e_j; 1 \leq i, j \leq n, i \neq j\}$ is a root system corresponding to $\mathfrak{g}$ with respect to $\mathfrak{h}$. This is called a root system of type A_{n-1}.

We have the following decomposition for $\mathfrak{sl}_n(\mathbb{C})$*:*

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{e_i - e_j \in \Phi(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_{e_i - e_j},$$

where $\mathfrak{g}_{e_i-e_j}$ *is spanned by the matrix* E_{ij} *which has* 1 *in the* $(ij)^{th}$ *place and* 0 *elsewhere.*

In particular, for $\mathfrak{sl}_2$*, we have the roots as* $\Phi(\mathfrak{g},\mathfrak{h}) = \{e_1 - e_2, e_2 - e_1\}$. *Here* $\mathfrak{h} = \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$. $\mathfrak{g}_{e_1-e_2} = \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rangle$ *and* $\mathfrak{g}_{e_2-e_1} = \langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rangle$

Example 2.1.2. *We have similar decompositions for Lie algebras of type* B, C *and* D*, corresponding to the odd orthogonal groups, the symplectic groups and the even orthogonal groups, respectively. For the explicit computations concerning root space decompositions and the corresponding root spaces, the reader is referred to [15].*

1. *For* $\mathfrak{so}_{2n+1}(\mathbb{C})$*, the set of roots are*

$$\Phi(\mathfrak{g},\mathfrak{h}) = \{\pm e_i \pm e_j; 1 \leq i \neq j \leq n\} \cup \{\pm e_k; 1 \leq k \leq n\}.$$

   This is a root system of type B_n.

2. *For* $\mathfrak{sp}_{2n}(\mathbb{C})$*, the set of roots are*

$$\Phi(\mathfrak{g},\mathfrak{h}) = \{\pm e_i \pm e_j; 1 \leq i \neq j \leq n\} \cup \{\pm 2e_k; 1 \leq k \leq n\}.$$

   This is a root system of type C_n.

3. *For* $\mathfrak{so}_{2n}(\mathbb{C})$*, the set of roots are*

$$\Phi(\mathfrak{g},\mathfrak{h}) = \{\pm e_i \pm e_j; 1 \leq i \neq j \leq n\}.$$

   This is a root system of type D_n.

The set of roots classify the set of simple Lie algebras. The roots also have an axiomatic definition. For a finite-dimensional real vector space V, a finite subset Φ of V is called a root system if the following conditions hold:

- Φ spans V.
- The transformation $s_\alpha(\beta) = \beta - \frac{2\langle\beta,\alpha\rangle}{|\alpha|^2}\alpha$ preserves the set Φ for all α, where $|\cdot|$ is the usual norm and $\langle\cdot,\cdot\rangle$ is the usual inner product on V.
- $\frac{2\langle\beta,\alpha\rangle}{|\alpha|^2} \in \mathbb{Z}$ whenever $\alpha, \beta \in \Phi$.

Let W be the subgroup of $\mathrm{GL}(V)$ generated by s_α's where α runs over Φ, i.e., $\mathrm{W} = \langle s_\alpha;\ \alpha \in \Phi\rangle$. This is called the **Weyl group** of Φ. Note that since s_α fixes the set Φ, it is a subset of the permutation group of Φ. Since Φ is finite, W is finite as well. We will call the dimension of V, the **rank** of Φ.

Definition 2.1.3. *A subset Δ of Φ is called a* ***base*** *of Φ if*

- *Δ is a basis of V,*
- *every root β can be written as $\beta = \sum\limits_{\alpha\in\Delta} k_\alpha\alpha$; such that the k_α are all either non-negative or non-positive integers.*

The elements of Δ are called **simple** roots. This allows us to define the set of positive and negative roots as follows: Let $\beta = \sum_{\alpha\in\Delta} k_\alpha\alpha$; if all the k_α are positive (negative) then we say that the root β is positive (negative), and write $\beta \succ 0$ ($\beta \prec 0$). We write Φ^+ for all the positive roots, Φ^- for all the negative roots; $\Phi = \Phi^+ \sqcup \Phi^-$.

Define a hyperplane $P_\alpha = \{\beta \in V : \langle\beta,\alpha\rangle = 0\}$ which is orthogonal to the root α. There are finitely many hyperplanes which divide V into finitely many regions which are the connected components of $V - \bigcup_{\alpha\in\Phi} P_\alpha$. These are called the **Weyl chambers**. An element β of V is called **regular** if $\beta \in V - \bigcup_{\alpha\in\Phi} P_\alpha$. We have the following theorem which describes the action of the Weyl group on V and the set of bases. Let Δ is a base of Φ. Fixing a base of Φ makes a particular Weyl chamber stand out. This chamber $\{\beta \in V : \langle \beta, \alpha \rangle > 0 \ \ \forall \ \ \alpha \in \Delta\}$ is called the **fundamental Weyl chamber** relative to Δ.

Theorem 2.1.4. *(see [12],* 10.3*) Let Δ be a base of Φ.*

1. *If $\gamma \in V$ is regular, then there exists a $\sigma \in \mathrm{W}$ such that $\langle \sigma(\gamma), \alpha \rangle > 0$ for all $\alpha \in \Delta$. In other words, given a regular γ, we can translate γ into the fundamental Weyl chamber using* W*.*

2. *If Δ' is another base of Φ, then there exists $\sigma \in \mathrm{W}$ such that $\sigma(\Delta') = \Delta$. So* W *also acts transitively on the set of bases.*

3. *If α is a root, then there exists a $\sigma \in \mathrm{W}$ such that $\sigma(\alpha) \in \Delta$.*

4. W *is generated by s_α for $\alpha \in \Delta$.*

5. *If $\sigma(\Delta) = \Delta, \sigma \in \mathrm{W}$, then $\sigma = 1$, i.e.,* W *acts simply transitively on the set of bases.*

We will have $V = \mathfrak{h}^*$, where $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra. In this case, given a Lie algebra $\mathfrak{g}$, we can do a root space decomposition of $\mathfrak{g}$ with respect to a $\mathfrak{h}$. Then the set $\Phi(\mathfrak{g}, \mathfrak{h}) = \{\alpha \in \mathfrak{h}^* : \mathfrak{g}_\alpha \neq 0\}$ satisfies all the axioms of a root system.

Remark 2.1. *Let $\mathfrak{g}$ be a semi-simple Lie algebra with $\mathfrak{h}$ a Cartan subalgebra. Let $\Phi(\mathfrak{g}, \mathfrak{h})$ be the set of roots and Δ be a base. For each $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$, let x_α be a non-zero element of $\mathfrak{g}_\alpha$. Then there exists a non-zero element $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that x_α, y_α and $[x_\alpha, y_\alpha] = h_\alpha$ generate a* 3 *dimensional Lie algebra which is isomorphic to $\mathfrak{sl}_2$. See [12] for details. This observation will transform the question of understanding finite-dimensional representations of $\mathfrak{g}$ to understanding finite-dimensional representations of $\mathfrak{sl}_2$.*

2.1.2 Universal Enveloping Algebras

Before defining weights and characterizing representations, we introduce the universal enveloping algebra, $\mathfrak{U}(\mathfrak{g})$, of a Lie algebra $\mathfrak{g}$. The universal enveloping algebra plays an important role in the theory of representations. An important fact is that the representations of $\mathfrak{g}$ are in one-one correspondence with the left modules over $\mathfrak{U}(\mathfrak{g})$.

The universal enveloping algebra of a Lie algebra $\mathfrak{g}$, is a pair $(\mathfrak{U}(\mathfrak{g}), i)$ where $\mathfrak{U}(\mathfrak{g})$ is an associative algebra with 1 together with a map i which is an inclusion of $\mathfrak{g}$ into $\mathfrak{U}(\mathfrak{g})$ which satisfies:

$$i([x,y]) = i(x)i(y) - i(y)i(x)$$

and given any algebra homomorphism $\phi : \mathfrak{g} \to A$ into an associative algebra A, ϕ factors through $\mathcal{U}(\mathfrak{g})$, i.e., the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "i"] \arrow[dr, "\phi"'] & \mathfrak{U}(\mathfrak{g}) \arrow[d, dashed, "\tilde{\phi}"] \\ & A \end{tikzcd}$$

This is the universal property of $\mathfrak{U}(\mathfrak{g})$. This is a categorical way to define the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ and hence $\mathfrak{U}(\mathfrak{g})$ is unique once we prove its existence. The universal enveloping algebra can be realized as a quotient of the tensor algebra.

Let $\mathrm{T}(\mathfrak{g}) = \bigoplus\limits_{n=0}^{\infty} \mathfrak{g}^{\otimes n}$ denote the tensor algebra of $\mathfrak{g}$. Denote by I the ideal of $\mathrm{T}(\mathfrak{g})$ generated by elements of the form

$$\langle x \otimes y - y \otimes x - [x,y] : x, y \in \mathfrak{g} \rangle.$$

Then $\mathfrak{U}(\mathfrak{g})$ is isomorphic to the quotient $\mathrm{T}(\mathfrak{g})/\mathrm{I}$. The following result will shed some light on the structure of $\mathfrak{U}(\mathfrak{g})$.

Theorem 2.1.5 (Poincaré-Birkhoff-Witt Theorem). *(see [12]) Let us denote by* π *the quotient map* $\mathrm{T}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$*. Suppose* $(x_1, x_2, \dots)$ *is an ordered basis of* $\mathfrak{g}$*. Then the elements of the form* $x_{i_1} x_{i_2} \dots x_{i_m} = \pi(x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_m})$*, where* $m \in \mathbb{Z}$ *and* $i_1 \leq i_2 \leq \cdots \leq i_m$*; along with* 1*, give a basis for* $\mathfrak{U}(\mathfrak{g})$*.*

2.1.3 The theory of weights

Weights, defined using the root system of $\mathfrak{g}$, play an important role in the characterization of the finite-dimensional representations of $\mathfrak{g}$.

We will continue working with an abstract vector space V, even though we will eventually specialize to the case when $V = \mathfrak{g}$.

Let Φ be a root system of V with Weyl group W. Let Λ be the set of all elements λ of V such that $2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$, for all $\alpha \in \Phi$. Observe that the set Λ is actually a subgroup of V containing Φ. Moreover, $\lambda \in \Lambda$ if and only if $2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$, for all $\alpha \in \Delta$ for a base Δ of Φ. We call a weight λ ***dominant*** if all the integers $2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ are non-negative and ***strongly dominant*** if all the integers $2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ are positive. The set of all dominant weights lies in the closure of the fundamental Weyl chamber relative to Δ, while strongly dominant weights are contained in the interior of the fundamental Weyl chamber.

Suppose G is a semi-simple classical group with the complexified Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is semisimple. A dominant integral weight for the Lie group G is defined as a dominant integral weight for the corresponding Lie algebra. For the types B_n, C_n and D_n, we shall list all the dominant integral weights. We also give a list for the split unitary groups. In each of these cases, we describe a root system corresponding to the chosen Cartan subalgebra, along with a base and set of positive roots, and compute the half sum of positive roots ρ.

1. For type B_n: The split group in this case is $\mathrm{SO}(n+1, n)$. The diagonal

torus T consists of all matrices of the form

$$\mathrm{diag}(t_1,\ldots,t_n,1,t_n^{-1},\ldots,t_1^{-1}) = \begin{pmatrix} t_1 & & & & & & \\ & \ddots & & & & & \\ & & t_n & & & & \\ & & & 1 & & & \\ & & & & t_n^{-1} & & \\ & & & & & \ddots & \\ & & & & & & t_1^{-1} \end{pmatrix},$$

with $t_j \in \mathbb{R}^\times$. The complexified Lie algebra of $\mathrm{SO}(n+1,n)$ is $\mathfrak{so}_{2n+1}$ and the Lie algebra of T denoted $\mathfrak{t}_\mathbb{C}$ has matrices of the form $\mathrm{diag}(a_1,\ldots,a_n,0,-a_n,\ldots,-a_1)$, where $a_j \in \mathbb{C}$. Denote by e_i, the character on the torus which picks out a_i. Then we fix the following data:

- $\Phi(\mathfrak{g},\mathfrak{t}) = \{\pm e_i \pm e_j : i < j\} \cup \{\pm e_i\}$.
- The base $\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n\}$.
- The set of positive roots $\Phi^+ = \{e_i \pm e_j\} \cup \{e_i\}$.
- The half sum of positive roots $\rho = (n-\frac{1}{2})e_1 + (n-\frac{3}{2})e_2 + \cdots + \frac{1}{2}e_n$. We will also write ρ as the n-tuple $(n-\frac{1}{2}, n-\frac{3}{2}, \ldots, \frac{1}{2})$.

2. For type C_n: The split group in this case is $\mathrm{Sp}(2n,\mathbb{R})$. The diagonal torus T consists of all matrices of the form:

$$\mathrm{diag}(t_1,\ldots,t_n,t_n^{-1},\ldots,t_1^{-1}),$$

with $t_j \in \mathbb{R}^\times$. The complexified Lie algebra of $\mathrm{Sp}(2n,\mathbb{R})$ is denoted $\mathfrak{sp}_{2n}$ and the Lie algebra $\mathfrak{t}_\mathbb{C} = \{\mathrm{diag}(a_1,\ldots,a_n,-a_n,\ldots,-a_1) : \ a_j \in \mathbb{C}\}$. Denote by e_i, the character on the torus which picks out a_i. Then we fix the following data:

- $\Phi(\mathfrak{g}, \mathfrak{t}) = \{\pm e_i \pm e_j : i < j\} \cup \{\pm 2e_i\}$.
- The base $\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, 2e_n\}$.
- The set of positive roots $\Phi^+ = \{e_i \pm e_j\} \cup \{2e_i\}$.
- The half sum of positive roots $\rho = ne_1 + (n-1)e_2 + \cdots + 1e_n$. We will also write ρ as the n-tuple $(n, n-1, \ldots, 1)$.

3. For type D_n: The split group in this case is $\mathrm{SO}(n, n)$. The diagonal torus T consists of all matrices of the form:

$$\mathrm{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}),$$

with $t_j \in \mathbb{R}^\times$. The complexified Lie algebra of $\mathrm{SO}(2n, \mathbb{R})$ is denoted $\mathfrak{so}_{2n}$ and the Lie algebra $\mathfrak{t}_\mathbb{C}$ of T has matrices of the form $\mathrm{diag}(a_1, \ldots, a_n, -a_n, \ldots, -a_1)$ with $a_j \in \mathbb{C}$. Denote by e_i, the character on the torus which picks out a_i. Then we fix the following data:

- $\Phi(\mathfrak{g}, \mathfrak{t}) = \{\pm e_i \pm e_j : i < j\}$.
- The base $\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_{n-1} + e_n\}$.
- The set of positive roots $\Phi^+ = \{e_i \pm e_j\}$.
- The half sum of positive roots $\rho = (n-1)e_1 + (n-2)e_2 + \cdots + 1e_{n-1}$. We will also write ρ as the n-tuple $(n-1, n-2, \ldots, 1, 0)$.

4. The unitary groups: When n is even take the unitary group as $\mathrm{U}(\frac{n}{2}, \frac{n}{2})$, and let $p = n/2$. For a maximal torus T of $\mathrm{U}(\frac{n}{2}, \frac{n}{2})$ take all matrices of the form

$$\{\mathrm{diag}(t_1, \ldots, t_p, \bar{t}_p^{-1}, \ldots, \bar{t}_1^{-1}) : t_i \in \mathbb{C}^\times\}.$$

The complexified Lie algebra of the unitary groups is denoted by $\mathfrak{u}_n$ and the Lie algebra $\mathfrak{t}_\mathbb{C} = \mathrm{diag}(a_1, \ldots, a_p, -\bar{a}_p, \ldots, -\bar{a}_1) :\ a_j \in \mathbb{C}$. When n is odd we consider $\mathrm{U}(\frac{n+1}{2}, \frac{n-1}{2})$, and put $p = (n+1)/2$ and $q = (n-1)/2$. For $T \in \mathrm{U}(\frac{n+1}{2}, \frac{n-1}{2})$ choose all matrices of the form

$$\{\mathrm{diag}(t_1, \ldots, t_{p-1}, t_p, \bar{t}_{p-1}^{-1}, \ldots, \bar{t}_1^{-1}) : \ t_i \in \mathbb{C}^{\times}, \ 1 \leq i \leq p-1, \ t_p \in S^1\},$$

hence $\mathfrak{t}_{\mathbb{C}} = \mathrm{diag}(a_1, \ldots, a_{p-1}, a_p, -\bar{a}_{p-1}, \ldots, -\bar{a}_1) : \ a_j \in \mathbb{C}$. In either case, denote by e_i, the character on the torus which picks out the i^{th} entry of the diagonal matrix. We have:

- $\Phi(\mathfrak{g}, \mathfrak{t}) = \{e_i - e_j : i \neq j, \ 1 \leq i, j \leq n\}$.
- The base $\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n\}$.
- The set of positive roots $\Phi^+ = \{e_i - e_j : i < j\}$.
- The half sum of positive roots $\rho = (\frac{n-1}{2})e_1 + (\frac{n-3}{2})e_2 + \cdots + (\frac{1-n}{2})e_n$. We will also write ρ as the n-tuple $(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2})$.

In all the above cases (1) – (4), a weight $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$, will be written as an n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$ and this will stand for $\lambda_1 e_1 + \cdots + \lambda_n e_n$. We now describe the dominant integral weights.

Example 2.1.6. 1. *For type A_n: A dominant integral weight λ is given by a string of $n-1$ integers, say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$, such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1}$.*

2. *For type B_n: A dominant integral weight λ is given by a string of n integers, say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, such that $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \geq 0$.*

3. *For type C_n: A dominant integral weight λ is given by a string of n integers, say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, such that $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \geq 0$.*

4. *For type D_n: A dominant integral weight λ is given by a string of n integers, say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, such that $\lambda_1 \geq \lambda_2 \cdots \geq |\lambda_n|$.*

We will now classify the finite-dimensional representation of the above mentioned Lie algebras.

2.1.4 Finite-dimensional representations

Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra of a semi-simple Lie algebra $\mathfrak{g}$. We start by classifying the finite-dimensional representations of $\mathfrak{sl}_2$. The representations of $\mathfrak{sl}_2$ are important as stated in Remark 2.1.

Representations of $\mathfrak{sl}_2$

Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have the following relations:

$$[h, x] = 2x; \quad [h, y] = -2y; \quad [x, y] = h.$$

Suppose V is a finite dimensional $\mathfrak{g}$-module. Note that $\mathfrak{h} = \langle h \rangle$ is a Cartan subalgebra of $\mathfrak{g}$. Therefore we have a decomposition of V into eigenspaces of h of the form $V_\lambda = \{v \in V : h \cdot v = \lambda v\}$ for $\lambda \in \mathbb{C}$. Whenever V_λ is nonzero, we call it the weight space of V and λ the weight. We have the following lemma.

Lemma 2.1.7. *[12],* 7.1 *Let $v \in V_\lambda$. Then $x \cdot v \in V_{\lambda+2}$ and $y \cdot v \in V_{\lambda-2}$.*

Proof. $h \cdot (x \cdot v) = (hx) \cdot v = [h, x]v + (xh)v = (2 + \lambda)v$. This proves that $x \cdot v \in V_{\lambda+2}$.
Similarly, $h \cdot (y \cdot v) = (hy) \cdot v = [h, y]v + (yh)v = (-2 + \lambda)v$. This proves that $y \cdot v \in V_{\lambda-2}$. □

Observe that since V is finite-dimensional and $V = \bigoplus_{\lambda\in\mathbb{C}} V_\lambda$, there exists a λ such that $V_\lambda \neq 0$ but $V_{\lambda+2} = 0$. Thus for any vector $v \in V_\lambda$, $x \cdot v = 0$. Such a v is called a maximal vector.

Lemma 2.1.8. *[12], 7.2 Let V be an irreducible $\mathfrak{g}$-module and let v_0 be a maximal vector Put $v_{-1} = 0$ and $v_i = \frac{1}{i!}y^i \cdot v_0$; $i \geq 0$. Then we have,*

1. $h \cdot v_i = (\lambda - 2i)v_i$,
2. $y \cdot v_i = (i + 1)v_{i+1}$,
3. $x \cdot v_i = (\lambda - i + 1)v_{i-1}; i \geq 0$.

We have the following classification theorem for representations of $\mathfrak{sl}_2(\mathbb{C})$.

Theorem 2.1.9. *[12], 7.2 Let V be an irreducible module for $\mathfrak{g}$.*

1. *Relative to h, V is a direct sum of weight spaces V_μ, $\mu = m, m-2, \ldots, -(m-2), -m$, where $dimV = m + 1$ and $dimV_\mu = 1$ for all μ.*
2. *Up to scaling, V has a unique maximal vector with weight m.*
3. *There is a unique irreducible module of $\mathfrak{g}$ of dimension $m + 1$.*

Proof. For a proof, see [12], section 7.2. □

Thus, we observe that the finite-dimensional representations of $\mathfrak{sl}_2$ are characterized by non-negative integers.

We now classify the finite-dimensional representations of a semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}$ be such a Lie algebra. Let V be a finite-dimensional $\mathfrak{g}$-module. Then a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ acts diagonally on V. So we can write $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda$, where

$$V_\lambda = \{v \in V : h \cdot v = \lambda(h)v;\ \forall h \in \mathfrak{h}\}.$$

The spaces V_λ are called **weight spaces** of V with respect to $\mathfrak{h}$ and λ is the corresponding weight.

Example 2.1.10. *If $V = \mathfrak{g}$ is viewed as a $\mathfrak{g}$-module via the adjoint action, then the weight space decomposition coincides with the root space decomposition.*

Note that such a decomposition of a $\mathfrak{g}$-module may not be possible when the space V is infinite-dimensional. Nevertheless, we can always form a subspace $V' = \sum\limits_{\lambda \in \mathfrak{h}^*} V_\lambda$. We then have the following result.

Theorem 2.1.11. *[12],* 20.1 *Let V be an arbitrary $\mathfrak{g}$-module. Then,*

1. *$\mathfrak{g}_\alpha$ maps V_λ to $V_{\lambda+\alpha}$.*
2. *The sum $V' = \sum\limits_{\lambda \in \mathfrak{h}^*} V_\lambda$ is direct and V' is a $\mathfrak{g}$-submodule.*
3. *If $dim V < \infty$, then $V = V'$.*

Let $\mathfrak{g}, \mathfrak{h}$ be as above. Let $\Phi(\mathfrak{g}, \mathfrak{h})$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and Δ be a base of $\Phi(\mathfrak{g}, \mathfrak{h})$. Let V be a $\mathfrak{g}$-module. Call a non-zero vector $v^+ \in V_\lambda$ a **maximal vector** if v^+ is killed by $\mathfrak{g}_\alpha$ for all $\alpha \in \Phi^+$ i.e. $\mathfrak{g}_\alpha \cdot v^+ = 0$ for all $\alpha \in \Phi^+$. The existence of a maximal vector is ambiguous and it may not even exist for an infinite-dimensional $\mathfrak{g}$-module V. But if V is finite-dimensional then the existence of a maximal vector in V is clear and further the line through v^+ is stabilized by the Borel subalgebra $B(\Delta) = \mathfrak{h} \oplus \bigoplus\limits_{\alpha \succ 0} \mathfrak{g}_\alpha$.

We will study a class of $\mathfrak{g}$-modules which will encompass the finite-dimensional representations of $\mathfrak{g}$. Let v^+ be a maximal vector of V with weight λ. Let $V = \mathfrak{U}(\mathfrak{g}) \cdot v^+$. This is called the ***standard cyclic module*** of weight λ. We call λ the **highest weight** of V. The structure of such a module is well known and is summarized in the theorem below.

Recall from Remark 2.1 that, given $x_\alpha \in \mathfrak{g}_\alpha$, there exist $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that x_α, y_α and $[x_\alpha, y_\alpha] = h_\alpha$ generate a 3 dimensional subalgebra isomorphic to $\mathfrak{sl}_2$.

Theorem 2.1.12. *[12],* 20.2 *Let V be a standard cyclic $\mathfrak{g}$-module, with maximal vector $v^+ \in V_\lambda$. Let $\Phi^+ = \{\beta_1, \beta_2, \ldots, \beta_m\}$ be the set of positive roots. Then,*

1. *V is spanned by the vectors $y_{\beta_1}^{i_1} y_{\beta_2}^{i_2} \ldots y_{\beta_m}^{i_m} v^+$ with $i_j \in \mathbb{Z}$; in particular, V is a direct sum of its weight spaces. (Note the similarity of this basis with the basis we had for representations of $\mathfrak{sl}_2$.)*

2. *The weights occurring in V are of the form $\mu = \lambda - \sum_{i=1}^{l} k_i \alpha_i$ for $k_i \in \mathbb{Z}^+$, $\alpha_i \in \Delta$, that is all the weights satisfy $\mu \prec \lambda$. Note that this justifies the terminology* highest weight.

3. *For each $\mu \in \mathfrak{h}^*$, $dim V_\mu < \infty$. Also, $dim V_\lambda = 1$.*

4. *Each $\mathfrak{g}$-submodule of V is a direct sum of weight spaces.*

5. *V is an indecomposable $\mathfrak{g}$-module, with a unique maximal submodule and a corresponding irreducible quotient.*

6. *Every nonzero homomorphic image of V is also a standard cyclic $\mathfrak{g}$-module of weight λ.*

Corollary 2.1.13. *[12],* 20.2 *Let V be as above. If V is irreducible, then there is a unique maximal vector up to scalar multiplication.*

Let us now construct a standard cyclic $\mathfrak{g}$-module with highest weight λ. We observe that a standard cyclic module of weight λ, has a $B(\Delta)$ submodule which is 1-dimensional and spanned by a maximal vector. Let D_λ be a 1-dimensional vector space spanned by v^+. Define the $B = B(\Delta)$ action on D_λ as

$$(h + \sum_{\alpha \succ 0} x_\alpha) \cdot v^+ = \lambda(h) v^+.$$

Note that we have defined the action of $\mathfrak{h}$ and have extended it to B trivially. This makes D_λ into a B module as well as a $\mathfrak{U}(B)$-module. Now define

$$Z(\lambda) = \mathfrak{U}(\mathfrak{g}) \bigotimes_{\mathfrak{U}(B)} D_\lambda.$$

$Z(\lambda)$ is a left $\mathfrak{U}(\mathfrak{g})$-module via the standard left action of $\mathfrak{U}(\mathfrak{g})$. It is not difficult to see that $Z(\lambda)$ is a standard cyclic module of highest weight λ, with the maximal vector $1 \otimes v^+$. Let $Y(\lambda)$ be a maximal submodule of $Z(\lambda)$. Then $V(\lambda) = Z(\lambda)/Y(\lambda)$ is an irreducible quotient. Thus we have constructed an irreducible standard cyclic module of weight λ. The next result shows that such a standard cyclic module is unique.

Proposition 2.1.14. *[12],* 20.3 *Let V, W be irreducible standard cyclic $\mathfrak{g}$-modules of weight λ. Then V is isomorphic to W.*

Now we will address the question of whether such a cyclic module is finite-dimensional or not. If yes, when? Let $\mathfrak{g}, \mathfrak{h}, \Phi, \Delta$ be as before. Let $h_{\alpha_i} = h_i$ be a basis of $\mathfrak{h}$ such that h_i's are part of copies of $\mathfrak{sl}_2$ for each root $\alpha_i \in \Delta$.

Theorem 2.1.15. *[12],* 21.1 *Let V be a finite-dimensional module of weight λ. Then $\lambda(h_i)$ is a non-negative integer for all i.*

Proof. The proof immediately follows from the fact that, for any representation of $\mathfrak{sl}_2$, the highest weight is a non-negative integer. $\square$

This implies that the weight λ of a standard cyclic module has to be a highest weight of $\mathfrak{g}$ with respect to Δ. The next result says that this is actually a necessary and sufficient condition.

Theorem 2.1.16. *[12],* 21.2 *The map $\lambda \to V(\lambda)$ induces a one to one correspondence between the set of dominant integral weights and isomorphism classes of finite dimensional $\mathfrak{g}$-modules.*

This gives us a way to parameterize the finite-dimensional representations of $\mathfrak{g}$ using the highest weights. So for a Lie algebra of type A_n, B_n, C_n, D_n, the set of finite-dimensional representations are in bijection with the corresponding string of integers as mentioned in Example 2.1.6.

This will take care of the semi-simple Lie algebras and groups which are considered here. Apart from these we need the finite-dimensional representations of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ as real groups. The finite-dimensional representations of $\mathrm{GL}(n, \mathbb{R})$ are parameterized by a string of n integers $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_i \geq \lambda_{i+1}$. And finally, for the real reductive group $\mathrm{GL}(n, \mathbb{C})$, the finite-dimensional representations are parameterized by $(\lambda, \lambda^*) = ((\lambda_1, \ldots, \lambda_n), (\lambda_1^*, \ldots, \lambda_n^*))$ with $\lambda_i \geq \lambda_{i+1}$ and $\lambda_i^* \geq \lambda_{i+1}^*$. The representation of $\mathrm{GL}(n, \mathbb{C})$ having highest weight (λ, λ^*) is defined as follows: For $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n$, suppose M_λ denotes the irreducible representation of the complex (algebraic) group $\mathrm{GL}(n, \mathbb{C})$ then we have

$$M_{(\lambda,\lambda^*)} \ = \ M_\lambda \otimes \overline{M_{\lambda^*}} \ = \ M_\lambda \otimes \overline{M_\lambda^\vee},$$

which is an algebraic representation of the real group $\mathrm{GL}(n, \mathbb{C})$.

Chapter 3

Representation Theory and Langlands Functoriality

This chapter is broadly divided into two parts. We will introduce what are called *'cohomological representations'* and give some examples of cohomological representations for the classical groups, $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$. In the second part, we introduce Langlands functoriality which gives us an algorithm to *'transfer'* representations of a group (an L-packet to be precise) to some other group. The second part will make these concepts precise.

All the representations considered here are complex representations. The groups will be real groups unless mentioned specifically. The main reference for the first part of the chapter will be [3]. The second part can be found in [18]. For a general exposition see [4], [6], [7] and [8].

3.1 Representation Theory

We will now introduce some infinite-dimensional representations of Lie groups. In particular, we will be interested in the tempered representations and the discrete series representations of G.

3.1.1 General definitions

Let G be a real semi-simple Lie group. Let K be a maximal compact subgroup of G. A representation of G is a homomorphism from G to $\mathrm{GL}(V)$ which is continuous in the sense that the corresponding map $G \times V \to V$ is continuous. Usually, V is a Hilbert space. We will also assume that these representations are unitary.

Definition 3.1.1. *A representation π of G is called admissible if*

$$\pi|_K = \bigoplus_{\sigma \in \hat{K}} m_\sigma \sigma,$$

and all the m_σ's are finite.

Let (π, V) be a representation of G. For $v \in V$, we let $c_v : G \to V$ be given by $c_v(g) = \pi(g) \cdot v$. We say that a vector $v \in V$ is smooth or C^∞ if the map c_v is C^∞. We denote the set of all smooth vectors of V by V^∞. Note that V^∞ is stable under the action of G. A vector $v \in V$ is called K-finite if v is contained in a finite-dimensional K-invariant subspace of V. Let the space of K-finite vectors of V be denoted by V_K. If V_K^∞ denotes the intersection of V_K and V^∞, then V is admissible if $V_K^\infty = V$. When V is an Hilbert space define the function: $c_{v,w} : G \to \mathbb{C}$ by $c_{v,w}(g) = \langle \pi(g) \cdot v, w \rangle$, for $v, w \in V$. These are called the *coefficients* of the representation.

We will now introduce the notion of a $(\mathfrak{g}_0, K)$-module. Let G, K be as above. The Lie algebras of G and K will be denoted by $\mathfrak{g}_0$ and $\mathfrak{k}_0$ and their complexifications by $\mathfrak{g}$ and $\mathfrak{k}$.

Definition 3.1.2. *A $(\mathfrak{g}_0, K)$-module is a real or complex vector space which is a $\mathfrak{g}_0$-module, which is a locally K-finite and semi-simple K-module such that the operations of $\mathfrak{g}_0$ and K are compatible in the following sense:*

- $\pi(k) \cdot (\pi(X) \cdot v) = \pi(Ad\ k(X)) \cdot \pi(k) \cdot v \quad (k \in K;\ X \in \mathfrak{U}(\mathfrak{g}_0);\ v \in V),$

- *if F is a K-stable finite-dimensional subspace of V, then the representation of K on F is differentiable and has $\pi|_{\mathfrak{k}}$ as its differential.*

A $(\mathfrak{g}_0, K)$-module is *admissible* if it is admissible as a K-module. Let V be a $(\mathfrak{g}_0, K)$-module in which every K-stable finite-dimensional subspace is semi-simple with respect to K. Then, the subspace V_K of K-finite vectors in V is semi-simple as a K-module and is stable under $\mathfrak{g}_0$. Thus V_K is a $(\mathfrak{g}_0, K)$-module.

A $(\mathfrak{g}_0, K)$-module (π, V) is called *unitary* if V has a positive non-degenerate scalar product $(\cdot, \cdot)$ which is invariant under K and infinitesimally invariant under $\mathfrak{g}_0$, i.e, we have

- $(\pi(k) \cdot v, \pi(k) \cdot w) = (v, w)$, for $v, w \in V$ and $k \in K$.
- $(\pi(x) \cdot v, w) + (v, \pi(x) \cdot w) = 0$ for $x \in \mathfrak{g}_0$, and $v, w \in V$.

We denote the category of $(\mathfrak{g}_0, K)$-modules by $\mathfrak{C}_{\mathfrak{g}_0, K}$. We say that a $(\mathfrak{g}_0, K)$-module has infinitesimal character χ if there exists an algebra homomorphism $\chi : \mathrm{Z}(\mathfrak{g}_0) \to \mathbb{C}$ such that $z \cdot v = \chi(z) \cdot v$ for all $z \in \mathrm{Z}(\mathfrak{g}_0)$ and $v \in V$, where $\mathrm{Z}(\mathfrak{g}_0)$ is the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_0)$. Note that if V is a G-representation then the subspace V_K^∞ of K-finite smooth vectors in V is a $(\mathfrak{g}_0, K)$-module. We note that given a representation of the Lie group G, we can construct the $(\mathfrak{g}_0, K)$-module V_K^∞ consisting of K-finite smooth vectors in V. This subspace is dense in V and classifies unitary representations (see [4]). We will now introduce a more restricted category of $(\mathfrak{g}_0, \mathfrak{k}_0)$-modules. A $\mathfrak{g}_0$-module V, is called a $(\mathfrak{g}_0, \mathfrak{k}_0)$*-module*, if V is locally $\mathfrak{k}_0$-finite (i.e., every vector $v \in V$ is $\mathfrak{k}_0$-finite) and is semi-simple as a $\mathfrak{k}_0$-module. As before, we will call a $(\mathfrak{g}_0, \mathfrak{k}_0)$-module V *admissible* if V is semi-simple as a $\mathfrak{k}_0$-module.

Example 3.1.3. *Let G be a connected Lie group, $\mathfrak{g}_0$ be its Lie algebra and $\mathfrak{k}_0$ be the Lie algebra of a compact subgroup of G. Then every $(\mathfrak{g}_0, K)$-module* *is a* $(\mathfrak{g}_0, \mathfrak{k}_0)$*-module.*

We will be working in the category $\mathfrak{C}_{\mathfrak{g}_0,\mathfrak{k}_0}$. We will now introduce cohomological representations in this category.

3.1.2 Cohomology of Representations

Let V be a $\mathfrak{g}_0$-module. Let $\mathfrak{k}_0$ be a subalgebra of $\mathfrak{g}_0$. Define:

$$C^q(\mathfrak{g}_0, \mathfrak{k}_0; V) = Hom_{\mathfrak{k}_0}(\wedge^q(\mathfrak{g}_0 \backslash \mathfrak{k}_0), V),$$

where the action of $\mathfrak{k}_0$ on $\wedge^q(\mathfrak{g}_0 \backslash \mathfrak{k}_0)$ is induced by the adjoint representation. The map $d : C^q \to C^{q+1}$ is defined by

$$df(x_0, \ldots, x_q) = \sum_i (-1)^i x_i \cdot f(x_0, \ldots, \hat{x_i}, \ldots, x_q) + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_0, \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots, x_q).$$

Then C^q along with the map $d : C^q \to C^{q+1}$ defines a chain complex. The cohomology groups of this chain complex are denoted by $H^i(\mathfrak{g}_0, \mathfrak{k}_0; V)$. The cohomology groups H^i may also be viewed as Ext functors, and are called the relative Lie algebra cohomology of $\mathfrak{g}_0$ with respect to the subalgebra $\mathfrak{k}_0$.

Definition 3.1.4. *Let* (π, V) *be a representation of a real reductive Lie group* G*. By a slight abuse of notation, we also denote the corresponding* $(\mathfrak{g}_0, \mathfrak{k}_0)$*-module by* V*. We say that* V *is* ***cohomological*** *if there exists a finite-dimensional representation* F *such that at least one of the cohomology groups* $H^i(\mathfrak{g}_0, \mathfrak{k}_0; V \otimes F^*)$ *is non-zero, where* F^* *denotes the dual of* F*.*

Cohomological representations play an important part of the theory of automorphic forms. Some of the applications were mentioned in the introduction. 'Cohomologicalness' may be thought of as an appropriate generalization of 'holomorphic' in the context of modular forms. We will now introduce an important class of cohomological representations. Let G be a semisimple Lie group. Let K be a maximal compact subgroup of G. Further assume that rank $G =$ rank K, equivalently a maximal torus of K is also a maximal torus of G. Let T be such a maximal torus.

Let $\Phi(\mathfrak{g}, \mathfrak{t})$ (resp. $\Phi_K(\mathfrak{k}, \mathfrak{t})$) be the set of roots of $\mathfrak{g}$ (resp. $\mathfrak{k}$) with respect to $\mathfrak{t}$. Let W and W_K be the Weyl groups of G and K respectively. Denote by $P(\Phi)$ the set of weights of Φ. A weight Λ of Φ is called ***regular*** if for all $\alpha \in \Delta$ we have $\langle \Lambda, \alpha \rangle \ > 0$. Note that this is in line with the definition given in Chapter 2. A representation (π, V) of G is called a discrete series representation if π is unitary and the coefficients $c_{v,w}$ are square integrable, i.e., $c_{v,w}$ are in $L^2(G)$.

Theorem 3.1.5 ([15] Thm 9.6)**.** *For an irreducible unitary representation (π, V) of G, the following statements are equivalent:*

1. *All coefficients are in $L^2(G)$.*
2. *Some nonzero K-finite coefficient is in $L^2(G)$.*
3. *π is equivalent to a direct summand of the right regular representation of G on $L^2(G)$.*

Observe that the Weyl group W_K acts on the set of regular weights of G. The discrete series representations of G correspond bijectively to the orbits of W_K in the set of regular elements in $P(\Phi)$. Furthermore, the bijection is canonical. Let ω_Λ be the class of representations associated to a regular element $\Lambda \in P(\Phi)$. The elements of ω_Λ have infinitesimal character χ_Λ (see [3] II. 5.1).

We note that $\Lambda = \lambda + \rho$ for some $\lambda \in X^*(T) = \mathrm{Hom}(T, \mathbb{C}^\times)$, and where ρ is the half sum of positive roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Suppose that F is a finite-dimensional representation of G with highest weight λ, then the infinitesimal character of F is χ_Λ. The following is a well known fact:

Theorem 3.1.6 (II.5.3 [3]). *Let* $(\pi, V) \in \omega_\Lambda$. *Let* H *be the underlying* $(\mathfrak{g}_0, K)-$*module of* $K-$*finite vectors in* V. *Let* (σ, F) *be a finite-dimensional irreducible representation of* G.
Then,

1. *If the highest weight of* (σ, F) *is not* $\Lambda - \rho$, *then* $dim H^i(\mathfrak{g}_0, \mathfrak{k}_0; H \otimes F^*) = 0$, *for all* i.

2. *If the highest weight of* (σ, F) *is* $\Lambda - \rho$, *then* $dim H^i(\mathfrak{g}_0, \mathfrak{k}_0; H \otimes F^*) = \delta_{i,q}$ *where* $q = (dim G/K)/2$.

Thus ω_Λ is the set of discrete series representations of G having the same infinitesimal character. The theorem above states that the discrete series representations in ω_Λ are cohomological with respect to the finite-dimensional representation with highest weight λ, where $\Lambda = \lambda + \rho$.

Another important class of representations of a real reductive group is a slightly larger class which is the class of tempered representations. Tempered representations are almost discrete series in the following sense:

Definition 3.1.7. *A representation* (π, V) *of* G *is called* ***'tempered'*** *if the* K*-finite matrix coefficients of the representation* π *are in* $L^{2+\epsilon}$ *for all* $\epsilon > 0$.

Tempered representations play an important role in the Langlands classification of irreducible admissible representations of a group G. The tempered representations are the building blocks for representations of G. To elaborate, every irreducible admissible representation G can be obtained via parabolic induction. Further, the representation which we use on the Levi subgroup of the parabolic is tempered. So in some sense, if we know all the tempered representations, we know all the representations of the group G.

3.2 Langlands Dual and Functoriality

In this section we give a brief introduction to the theory put forth by Langlands and what we mean by functoriality. The main references for this section are [2], [6], [7], [18].

Langlands theory is like a bridge connecting Galois representations and representations of algebraic groups over the adeles. The connecting objects being the L-functions, which are analytic objects. There are two parts to the program. One is the global theory and the other is the local theory. Though these are mostly conjectures in the global setting some of the local conjectures have been proved. Langlands himself proved the local conjectures when the field is real or complex numbers. To state the conjectures precisely we need to introduce some jargon. We concern ourselves with the local case with the base field being real or complex numbers. For details see [2], [6], [7], [8] and [18].

3.2.1 Weil Group

The first ingredient is the so called *Weil group*, W_F, which depends on the base field. We assume $F = \mathbb{R}$ or $\mathbb{C}$. The Weil group of $\mathbb{R}$ is defined as the non-split extension of $\mathbb{C}^\times$ by $\mathbb{Z}/2\mathbb{Z} = Gal(\mathbb{C}/\mathbb{R})$, i.e., we have a short exact sequence

$$0 \longrightarrow \mathbb{C}^\times \longrightarrow W_\mathbb{R} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

which does not split. The Weil group can also be described as follows. $W_\mathbb{R} = \mathbb{C} \sqcup j\mathbb{C}$ where j acts on $\mathbb{C}$ as $jzj^{-1} = \bar{z}$ and $j^2 = -1$. This group will play an important role in the Langlands correspondence, on the Galois side of the correspondence. The Weil group of $\mathbb{C}$, $W_\mathbb{C}$, is just $\mathbb{C}^\times$.

The Weil group can be defined for other local fields as well. The Weil-Deligne group may also be considered at times [2] and [18].

We will be interested in finite-dimensional representations of the Weil group, that are also “admissible homomorphisms". But before defining admissible homomorphisms. We first define the L-group or the Langlands dual group of a group G.

3.2.2 Langlands dual group

Let G be a connected reductive real group. Let T be a maximal torus of G. Let $X^*(T)$ be the set of characters of T, i.e., $X^*(T) = \mathrm{Hom}(T, \mathbb{C}^\times)$ and $X_*(T)$ be the set of co-characters of T, i.e., $X_*(T) = \mathrm{Hom}(\mathbb{C}^\times, T)$. Let Φ be a set of roots and $\Phi^\wedge$ be the set of co-roots. Given a group G and a torus T, we associate the quadruple $\Psi(G,T) = (X^*(T), \Phi, X_*(T), \Phi^\wedge)$ to the pair G, T. This association gives us a bijection between the isomorphism classes of connected reductive groups over $\mathbb{C}$ and the set of quadruples satisfying the axioms of the root datum modulo the choice of the torus. Further, this bijection is canonical. On the set of root datum there is an involution given by

$$\Psi(G,T) = (X^*(T), \Phi, X_*(T), \Phi^\wedge) \mapsto (X_*(T), \Phi^\wedge, X^*(T), \Phi).$$

Given G and T as above associate the root datum, $\Psi(G,T)$, as above. Applying the involution to $\psi(G,T)$ obtain $\Psi' = (X_*(T), \Phi^\wedge, X^*(T), \Phi)$. Such a Ψ' will be the root datum of a connected complex reductive group. This group will be denoted by ${}^LG^\circ$ (see [2]). We give some examples in the table below.

The Langlands dual group is defined as the semi-direct product of ${}^LG^\circ$ with $\mathbb{Z}/2\mathbb{Z} = Gal(\mathbb{C}/\mathbb{R})$. The cases that are considered here will almost always have the trivial action of the Galois group. The only case where we have a non-trivial action is when $G = \mathrm{U}(p,q)$. We will elaborate the semi-direct product for $\mathrm{U}(p,q)$ while we do some explicit computations in the next chapter. For a general definition, the reader is referred to [18].

G	${}^L G^\circ$
$\mathrm{GL}(n,\mathbb{R})$	$\mathrm{GL}(n,\mathbb{C})$
$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{SO}(2n+1,\mathbb{C})$
$\mathrm{Sp}(p,q),\ p+q=n$	$\mathrm{SO}(2n+1,\mathbb{C})$
$\mathrm{GSp}(2n,\mathbb{R})$	$\mathrm{GSp}(2n,\mathbb{C})$
$\mathrm{SO}(p,q),\ p+q=2n+1$	$\mathrm{Sp}(2n,\mathbb{C})$
$\mathrm{SO}(p,q),\ p+q=2n$	$\mathrm{SO}(2n,\mathbb{C})$
$\mathrm{SO}^*(2n)$	$\mathrm{SO}(2n,\mathbb{C})$
$\mathrm{U}(p,q),\ p+q=n$	$\mathrm{GL}(n,\mathbb{C})$

Along with the above data for a group G, we also fix a base Δ for the root system of G with respect to T. Fixing a base is the same as fixing a Borel subgroup of G. Once we have done this, we call the parabolic subgroups containing this Borel subgroup as relevant.

3.2.3 Local Langlands Correspondence

We can now define the set of admissible homomorphisms. An admissible homomorphism α, is a homomorphism from W_F to the Langlands dual group ${}^L G$ of G. Let Γ_F denote the Galois group of $\bar{F}$ over F (recall that our F is $\mathbb{R}$ or $\mathbb{C}$).

Definition 3.2.1. *Let $\alpha : W_F \to {}^L G$ be a homomorphism with the following properties:*

1. α is a homomorphism over Γ_F, i.e.,

$$\begin{tikzcd} W_F \arrow[rr, "\alpha"] \arrow[dr] & & {}^L G \arrow[dl] \\ & \Gamma_F & \end{tikzcd}$$

commutes.

2. *α is continuous and maps semi-simple elements to semi-simple elements.*

3. *The image of α is contained in a relevant parabolic subgroup of ${}^LG$.*

An α which satisfies these conditions is called an admissible homomorphism.

Call α_1 and α_2 equivalent if they differ by an inner automorphism by an element of ${}^LG^\circ$. We will denote the equivalence classes of admissible homomorphisms associated to a group G by $\Phi(G)$.

The other side of the Langlands correspondence is the set of equivalence classes of irreducible admissible representations of the group G. This will be denoted by $\Pi(G)$. Langlands conjectured and proved that there exists a finite fibered surjective map from $\Pi(G)$ to $\Phi(G)$ such that the following properties are satisfied. Denote by Π_α the finite non-empty subset of $\Pi(G)$ associated to α. These finite subsets of $\Pi(G)$ will be called L-packets (see [2] and [18]).

1. Let α and β be two admissible homomorphisms in $\Phi(G)$. Then Π_α and Π_β are disjoint subsets of $\Pi(G)$.

2. Representations in Π_α share the same central character.

3. If α and β differ by an element of $H^1(W_F, Z^\wedge)$, i.e., $\alpha = \phi\beta$ for some $\phi \in H^1(W_F, Z^\wedge)$, then $\Pi_\alpha = \{\pi_\phi \otimes \pi : \pi \in \Pi_\beta\}$.

4. Suppose $\eta : H \to G$ has abelian kernel and co-kernel, $\alpha \in \Phi(G)$ and $\beta = \eta\alpha$. Then the pull back of any $\pi \in \Pi_\alpha$ to H is a direct sum of finitely many irreducible elements of Π_β.

5. Let $\alpha \in \Pi(G)$. If one element $\pi \in \Pi_\alpha$ is square-integrable modulo the center of G then all the elements of Π_α are. Furthermore, this happens only if the image of W_F under α in ${}^LG^\circ$ is not contained in any proper parabolic subgroup of ${}^LG^\circ$.

6. If one element of Π_α is tempered then all the elements are. This happens if and only if the image of W_F in ${}^LG^\circ$ is relatively compact in ${}^LG^\circ$.

The $\alpha \in \Phi(G)$ corresponding to a representation (or an L-packet) π (or Π_α) is called the *Langlands parameter* of π.

Suppose that G is a real (or complex) group such that G has discrete series representations then the above properties imply that all the discrete series representations having the same infinitesimal character lie in the same L-packet [18]. The above correspondence is more commonly known as the *'Local Langlands Correspondence'*.

Suppose that there is a L-map (L-homomorphism) $\eta : {}^LG \to {}^LH$, Langlands conjectured that given a representation, π, of G one should be able to lift π to a representation of H. Langlands conjectured this in the setting of an adèlic group but proved the local result for the base fields $\mathbb{R}$ and $\mathbb{C}$. This method of lifting representations given a map between the corresponding L-groups is known as *Langlands Functoriality.* We will be considering a special case when the L-map is an inclusion. For example, suppose $G = \mathrm{Sp}(2n, \mathbb{R})$. From Table **??**, we know that the connected component of the Langlands dual group for G is ${}^LG^\circ = \mathrm{SO}(2n+1, \mathbb{C})$. We can then consider the inclusion map $\mathrm{SO}(2n+1) \hookrightarrow \mathrm{GL}(2n+1, \mathbb{C}) = {}^L\mathrm{GL}(2n+1, \mathbb{R})^\circ$. Thus, composing a Langlands parameter with this inclusion we obtain an element of $\Phi(\mathrm{GL}(2n+1, \mathbb{R}))$. This will correspond to an L-packet for the group $\mathrm{GL}(2n+1, \mathbb{R})$. The following diagram should help in visualizing this phenomenon:

$$\begin{tikzcd} \Pi(\mathrm{Sp}(2n,\mathbb{R})) \arrow[r, dashed, "\iota"] \arrow[d] & \Pi(\mathrm{GL}(2n+1,\mathbb{R})) \arrow[d] \\ \Phi(\mathrm{Sp}(2n,\mathbb{R})) \arrow[r, "i"] & \Phi(\mathrm{GL}(2n+1,\mathbb{R})) \end{tikzcd}$$

where the vertical arrows are surjective and are given by the Local Langlands

Correspondence, the bottom right arrow is induced from the inclusion $i : \mathrm{SO}(2n+1, \mathbb{C}) \to \mathrm{GL}(2n+1, \mathbb{C})$ and the top arrow is such that the diagram commutes. Note that the top arrow is not exactly a map or a function. This just gives us a association between L-packets of the two groups in consideration. We will call the L-packet obtained in this way a transfer of the representation which we started with. Note that the L-packets for $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$ are singletons.

Since we plan to transfer representations of classical groups to $\mathrm{GL}(n)$, it will be fruitful to discuss the case of $G = \mathrm{GL}(n, \mathbb{R})$ and $G = \mathrm{GL}(n, \mathbb{C})$ in some detail.

3.2.4 Local Langlands correspondence for $\mathrm{GL}(n, \mathbb{R})$

In this section, we list the irreducible admissible representations of $\mathrm{GL}(n, \mathbb{R})$. Further, the representations of $W_{\mathbb{R}}$ will be constructed and a bijection between the admissible homomorphisms of $W_{\mathbb{R}}$ and the irreducible admissible representations of $\mathrm{GL}(n, \mathbb{R})$ will be made explicit (see [16]).

We know that ${}^L G^\circ$ for $G = \mathrm{GL}(n, \mathbb{R})$ is $\mathrm{GL}(n, \mathbb{C})$. Thus we want admissible homomorphisms from $W_{\mathbb{R}}$ to $\mathrm{GL}(n, \mathbb{C})$.

To begin with, let us compute all the one dimensional representations of $W_{\mathbb{R}}$. Recall that $W_{\mathbb{R}} = \mathbb{C} \bigsqcup j\mathbb{C}$, the action of j on $\mathbb{C}$ is given by complex-conjugation and $j^2 = -1$. Now, any homomorphism ϕ from $\mathbb{C}^\times$ to itself has the following form:

$$\phi : z \mapsto z^\mu \bar{z}^\nu,$$

where the difference $\mu - \nu$ should be an integer. This is forced because on the circle of unit one in $\mathbb{C}^\times$ the continuous homomorphisms are given by integers. Suppose that the image of j in $\mathbb{C}^\times$ is w. We have

$$\phi(z) = \phi(j\bar{z}j^{-1}) = w\phi(\bar{z})w^{-1} = \phi(\bar{z}).$$

This forces $\mu = \nu$. We also have

$$w^2 = \phi(j^2) = \phi(-1) = 1.$$

Thus, $w = \pm 1$. Hence, any one-dimensional representation of $W_{\mathbb{R}}$ is parameterized by a complex number and a sign. Thus we have:

$$\begin{aligned}
(\mu, +): \quad \phi(z) = |z|^{\mu} \quad \phi(j) = 1; \\
(\mu, -): \quad \phi(z) = |z|^{\mu} \quad \phi(j) = -1.
\end{aligned}$$

Now, we classify the irreducible two-dimensional representations of $W_{\mathbb{R}}$. Let (ϕ, V) be a two-dimensional irreducible representation of $W_{\mathbb{R}}$. Choose a basis $v, u \in V$ such that the image of $\mathbb{C}^{\times}$ contains 2×2 matrices which are diagonal. Suppose that

$$\phi(z) = \begin{pmatrix} z^{\mu}\bar{z}^{\nu} & 0 \\ 0 & z^{\alpha}\bar{z}^{\beta} \end{pmatrix}.$$

Note that, since we started with an irreducible representation of $W_{\mathbb{R}}$ either $\mu \neq \alpha$ or $\nu \neq \beta$. This is forced: Else any invariant one-dimensional subspace of V which is invariant under $\phi(j)$ will be a sub-representation of $W_{\mathbb{R}}$. The image of j is not diagonal and put $\phi(j) \cdot v = u$. Hence v and u are linearly independent. Further,

$$\phi(z) \cdot u = \phi(j\bar{z}j^{-1}) \cdot u = \phi(j)\phi(\bar{z}) \cdot v = z^{\mu}\bar{z}^{\nu} \cdot u.$$

A straightforward computation shows that $\mu = \beta$ and $\nu = \alpha$. Since we have $\phi(j)^{-1} = (-1)^{\mu-\nu}\phi(j)$ on the span of u and v, can write the representation

ϕ as,

$$\phi(z) \cdot v = z^\mu \bar{z}^\nu \cdot v, \quad \phi(j) \cdot v = u$$
$$\phi(z) \cdot u = z^\nu \bar{z}^\mu \cdot v, \quad \phi(j) \cdot u = (-1)^{\mu-\nu} v.$$

Thus, any two-dimensional irreducible representation of $W_\mathbb{R}$ is given by 2 complex numbers such that the difference is a positive integer, or equivalently a pair (l, t) such that $l \in \mathbb{Z}_+$ and t is a complex number. The dictionary to go from one parameterization to another is $l = \mu - \nu$ and $2t = \mu + \nu$. In terms of the latter parameterization, the two-dimensional representation looks like:

$$\phi(re^{i\theta}) \cdot v = r^{2t} e^{il\theta} \cdot v, \quad \phi(j) \cdot v = u$$
$$\phi(re^{i\theta}) \cdot u = r^{2t} e^{-il\theta} \cdot v, \quad \phi(j) \cdot u = (-1)^l v.$$

Lemma 3.2.2. *(see [16]) Every finite-dimensional semi-simple representation of $W_\mathbb{R}$ can be written as a sum of one and two-dimensional representations of $W_\mathbb{R}$.*

Proof. For details, see [16]. □

Since ${}^L\mathrm{GL}(n, \mathbb{R})^\circ = \mathrm{GL}(n, \mathbb{C})$, the complex n-dimensional representations of $W_\mathbb{R}$ is the same as the set $\Phi(\mathrm{GL}(n, \mathbb{R}))$. Langlands also gave a characterization of the set of all the irreducible admissible representations $\Pi(\mathrm{GL}(n, \mathbb{R}))$. We will now describe this set and eventually give a bijection between $\Phi(G)$ and $\Pi(G)$ for $G = \mathrm{GL}(n, \mathbb{R})$.

Consider the following representations of $\mathrm{GL}(1, \mathbb{R})$ and $\mathrm{GL}(2, \mathbb{R})$. For $\mathrm{GL}(1, \mathbb{R})$; $t \in \mathbb{C}$:

$$(t, +) : \ 1 \otimes |\cdot|^t$$
$$(t, -) : \ sgn \otimes |\cdot|^t$$

For $\mathrm{GL}(2,\mathbb{R})$: Denote by D_l, the discrete series representation of $\mathrm{GL}(2,\mathbb{R})$ with lowest non-negative K type being the character $r_\theta \mapsto e^{-i(l+1)\theta}$, where $r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and central character $a \mapsto \mathrm{sgn(a)^{l+1}}$. Then for $\mathrm{GL}(2,\mathbb{R})$ the representations which are relevant are parameterized by a positive integer l and a complex number t:

$$(l,t) : D_l \otimes |det(\cdot)|^t.$$

In either of the above cases, the $|\cdot|$ is the usual modulus on $\mathbb{R}$.

Theorem 3.2.3. *[16] Let $G = \mathrm{GL}(n,\mathbb{R})$. Let $n = \sum n_i$ be a partition of n such that each n_i is either 1 or 2. Let $D = \Pi_{i=1}^r \mathrm{GL}(n_i)$ be the block diagonal subgroup of G. Let σ_i be a representation of $\mathrm{GL}(n_i)$ which is one of the above types. Extend this representation of D to the block upper triangular matrices B. Then, define $I(\sigma_1,\ldots,\sigma_r) = Ind_B^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$, where B is the block upper triangular subgroup of G. Then,*

1. *If $n_1^{-1}Re(t_1) \geq n_2^{-1}Re(t_2) \geq \cdots \geq n_r^{-1}Re(t_r)$, then $I(\sigma_1,\ldots,\sigma_r)$ has a unique irreducible quotient and is denoted by $J(\sigma_1,\ldots,\sigma_r)$.*

2. *The representations $J(\sigma_1,\ldots,\sigma_r)$ exhaust irreducible admissible representations up to infinitesimal equivalence.*

3. *Two such representations are equivalent if and only if they correspond to the same partition and the representations σ_i are permuted.*

Langlands essentially proved a much general result which says that such a statement is true for any real reductive group with B's and σ_i's replaced appropriately. See [28] for details.

Now, having listed all the admissible homomorphism for $\mathrm{GL}(n,\mathbb{R})$ and all the irreducible admissible representations we will now give a bijection between the two sets. Note that any n-dimensional representation of $W_{\mathbb{R}}$ breaks up as a direct sum of one and two-dimensional representations. For representations of $\mathrm{GL}(n,\mathbb{R})$, we induce from representations of $\mathrm{GL}(1,\mathbb{R})$ and $\mathrm{GL}(2,\mathbb{R})$. The way we have listed the representations in both the cases make the bijection obvious. The bijection is given by sending a representation of $\mathrm{GL}(1,\mathbb{R})$ to one dimensional representations of $W_{\mathbb{R}}$, i.e. $(t,\pm)\mapsto(\mu,\pm)$, and as for the representation of $\mathrm{GL}(2,\mathbb{R})$, the same parameters used for classifying these parameterize the two-dimensional representations of $W_{\mathbb{R}}$. Thus, we have:

Theorem 3.2.4. *[16] For $G=\mathrm{GL}(n,\mathbb{R})$, the association above is a well defined bijection between the set of all equivalence classes of n-dimensional semisimple complex representations of $W_{\mathbb{R}}$ and the set of all equivalence classes of irreducible admissible representations of $\mathrm{GL}(n,\mathbb{R})$.*

We write down explicitly the parameters for the discrete series representations of $\mathrm{GL}(2,\mathbb{R})$ for further reference.

Parameters of discrete series for $\mathrm{GL}(2,\mathbb{R})$

A discrete series representation of $\mathrm{GL}(2,\mathbb{R})$ is parameterized by a positive integer l. We denote the corresponding representation by D_l. Consider the representation $D_{l,t}:=D_l\otimes|\cdot|^t$ of $\mathrm{GL}(2,\mathbb{R})$. The Langlands parameter of $D_{l,t}$ is given by a homomorphism $\phi(D_{l,t}):W_{\mathbb{R}}\to\mathrm{GL}(2,\mathbb{C})$, which is

$$\phi(D_{l,t})(z)=\begin{pmatrix}(z\bar{z})^t(\frac{z}{\bar{z}})^{\frac{l}{2}} & 0\\ 0 & (z\bar{z})^t(\frac{z}{\bar{z}})^{-\frac{l}{2}}\end{pmatrix},\quad \phi(D_{l,t})(j)=\begin{pmatrix}0 & (-1)^l\\ 1 & 0\end{pmatrix}.$$

3.2.5 Local Langlands correspondence for $\mathrm{GL}(n, \mathbb{C})$

We now give a similar analysis for $\mathrm{GL}(n, \mathbb{C})$. The story here is less complicated. For $\mathrm{GL}(n, \mathbb{C})$ we always induce from the upper triangular matrices and the building blocks are representations of $\mathrm{GL}(1, \mathbb{C})$. These representations are of the form

$$z \mapsto [z]^l |z|_{\mathbb{C}}^t, \qquad \text{with } l \in \mathbb{Z};\ t \in \mathbb{C},$$

where $[z] = \frac{z}{|z|}$ and $|z|_{\mathbb{C}} = |z|^2$. The Weil group in this case is $W_{\mathbb{C}} = \mathbb{C}^\times$. Analogous to Theorem 3.2.3, we have the following result for $G = \mathrm{GL}(n, \mathbb{C})$:

Theorem 3.2.5. *[16] Let $G = \mathrm{GL}(n, \mathbb{C})$. Let σ_i's be representations of $\mathrm{GL}(1, \mathbb{C})$ of the above type. As before, define $I(\sigma_1, \ldots, \sigma_n) = Ind_B^G(\sigma_1, \ldots, \sigma_n)$.*

1. *Suppose that the parameters t_i of σ_i are such that $Re(t_1) \geq Re(t_2) \geq \cdots \geq Re(t_n)$, then $I(\sigma_1, \ldots, \sigma_n)$ has a unique irreducible quotient and is denoted by $J(\sigma_1, \ldots, \sigma_n)$.*

2. *The representations $J(\sigma_1, \ldots, \sigma_n)$ exhaust all the irreducible admissible representations of G, up to infinitesimal equivalence.*

3. *Two such representations are isomorphic if and only if the σ_i's are permuted.*

This gives all the irreducible admissible representations of $\mathrm{GL}(n, \mathbb{C})$. The Weil group of $\mathbb{C}$ is $\mathbb{C}^\times$ and any semisimple representation of $W_{\mathbb{C}}$ is a sum of one-dimensional representations. The one-dimensional representations of $W_{\mathbb{C}}$ are given by

$$z \mapsto z^\mu \bar{z}^\nu$$

such that $\mu, \nu \in \mathbb{C}$ and the difference $\mu - \nu \in \mathbb{Z}$. For the purpose here, it will be more convenient to write these representations as

$$z \mapsto [z]^l |z|_{\mathbb{C}}^t = z^{t+\frac{l}{2}} \bar{z}^{t-\frac{l}{2}}.$$

Writing the representations in this form makes the bijection clear and we have:

Theorem 3.2.6. *[16] For $G = \mathrm{GL}(n,\mathbb{C})$, the association above is a well defined bijection between the set of all equivalence classes of n-dimensional semisimple complex representations of $W_{\mathbb{C}}$ and the set of all equivalence classes of irreducible admissible representations of* $\mathrm{GL}(n,\mathbb{C})$.

Apart from this, it will also be helpful to classify which of the above representations are tempered and cohomological for both $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$. Recall that a Langlands parameter of a tempered representation has the property that the image of the Weil group ($W_{\mathbb{R}}$ or $W_{\mathbb{C}}$) in ${}^LG^{\circ}$ is bounded. See [18].

3.2.6 Tempered cohomological representations of $\mathrm{GL}(n,\mathbb{R})$

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a dominant integral weight for $\mathrm{GL}(n,\mathbb{R})$; then $\lambda_j \in \mathbb{Z}$; $\lambda_1 \geq \cdots \geq \lambda_n$. Assume that λ is a *pure* weight, i.e., there exists an integer w, called the purity weight of λ such that $\lambda_i + \lambda_{n-i+1} = \mathsf{w}$. Define an n-tuple of integers $\ell = \ell(\lambda) = (\ell_1, \ldots, \ell_n)$ by $\ell = 2\lambda + 2\rho_n - \mathsf{w}$, where ρ_n is half the sum of positive roots of $\mathrm{GL}(n,\mathbb{R})$. Then $\ell_i = 2\lambda_i + n - 2i + 1 - \mathsf{w}$ for $1 \leq i \leq n$. Let's note the parity condition:

$$\ell_i \;\equiv\; n + 1 - \mathsf{w} \pmod 2. \tag{3.1}$$

When n is even, let P be the standard parabolic subgroup of type $(2, 2, \ldots, 2)$. Then define

$$J_\lambda \;=\; \mathrm{Ind}_{P(\mathbb{R})}^{\mathrm{GL}(n,\mathbb{R})}(D(\ell_1)|det|^{\frac{\mathsf{w}}{2}} \otimes \cdots \otimes D(\ell_{n/2})|det|^{\frac{\mathsf{w}}{2}}).$$

We know that J_λ is irreducible, essentially tempered and cohomological with respect to the finite-dimensional representation $M_\lambda^\vee$ [20]. When n is odd, we let P be the standard parabolic subgroup of type $(2, 2, \ldots, 2, 1)$, and for any sign character $\epsilon : \mathbb{R}^\times \to \{\pm 1\}$, define

$$J_\lambda^\epsilon \;=\; \mathrm{Ind}_{P(\mathbb{R})}^{\mathrm{GL}(n,\mathbb{R})}(D(\ell_1)|det|^{\frac{\mathsf{w}}{2}} \otimes \cdots \otimes D(\ell_{(n-1)/2})|det|^{\frac{\mathsf{w}}{2}} \otimes \epsilon|det|^{\frac{\mathsf{w}}{2}}).$$

We know that J_λ^ϵ is irreducible, essentially tempered and cohomological with respect to the finite-dimensional representation $M_\lambda^\vee$. For a pure weight λ, if $\pi \in \mathrm{Coh}(\mathrm{GL}(n, \mathbb{R}), \lambda^\vee)$ is essentially tempered, then $\pi = J_\lambda$ if n is even, and $\pi = J_\lambda^\epsilon$ for some ϵ if n is odd.

Since we are only concerned with the tempered representations here we make the following remark.

Remark 3.1. *It's easy to see that the representation J_λ (or J_λ^ϵ) is tempered if and only if the purity weight $\mathsf{w} = 0$. When n is odd we do not have any conditions on ϵ.*

3.2.7 Tempered cohomological representations of $\mathrm{GL}(n, \mathbb{C})$

A dominant-integral weight for $\mathrm{GL}(n, \mathbb{C})$, as a real Lie group, is of the form $\lambda = ((\lambda_1, \ldots, \lambda_n), (\lambda_1^*, \ldots, \lambda_n^*))$ with $\lambda_j, \lambda_j^* \in \mathbb{Z}$, $\lambda_1 \geq \cdots \geq \lambda_n$, $\lambda_1^* \geq \cdots \geq \lambda_n^*$. We say λ is pure if there exists an integer w such that $\lambda_i + \lambda_{n-i+1}^* = \mathsf{w}$. For such a pure weight, define two strings of half-integers $a = (a_1, \ldots, a_n) = \lambda + \rho_n$ and $b = (b_1, \ldots, b_n) = \mathsf{w} - \lambda - \rho_n$. Now, define the representation J_λ induced from the Borel subgroup $B(\mathbb{C})$ of $\mathrm{GL}(n, \mathbb{C})$ as

$$J_\lambda \;=\; \mathrm{Ind}_{B(\mathbb{C})}^{\mathrm{GL}(n,\mathbb{C})}(z^{a_1}\bar{z}^{b_1} \otimes \cdots \otimes z^{a_n}\bar{z}^{b_n}).$$

We know that J_λ is irreducible, essentially tempered and cohomological with respect to the finite-dimensional representation $M_\lambda^\vee$ [20]. For a pure weight λ, if $\pi \in \mathrm{Coh}(\mathrm{GL}(n, \mathbb{C}), \lambda^\vee)$ is essentially tempered, then $\pi = J_\lambda$.

Remark 3.2. *J_λ is tempered if and only if the purity weight* $\mathsf{w} = 0$.

Chapter 4

Transfer of tempered representations

This chapter proves a result about the cohomological properties of representations of $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$ which are obtained by transferring tempered representations of classical groups. We will be considering the following transfers:

G	${}^L G^\circ$	Transferred to
$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{SO}(2n+1)$	$\mathrm{GL}(2n+1,\mathbb{R})$
$\mathrm{Sp}(p,q); p+q=n$	$\mathrm{SO}(2n+1)$	$\mathrm{GL}(2n+1,\mathbb{R})$
$\mathrm{SO}(p,q) : p+q=2n+1$	$\mathrm{Sp}(2n)$	$\mathrm{GL}(2n,\mathbb{R})$
$\mathrm{SO}(p,q); p+q=2n; p,q$ even	$\mathrm{SO}(2n)$	$\mathrm{GL}(2n,\mathbb{R})$
$\mathrm{SO}^*(2n)$	$\mathrm{SO}(2n)$	$\mathrm{GL}(2n,\mathbb{R})$
$\mathrm{U}(p,q); p+q=n$	$\mathrm{GL}(n,\mathbb{C})$	$\mathrm{GL}(n,\mathbb{C})$

The question that we want to address is the following:

Let π be a tempered representation of one of the groups G in the above table. Langlands functoriality gives us a representation $\iota(\pi)$ of an appropriate $\mathrm{GL}(N)$. Is $\iota(\pi)$ cohomological? If yes, with respect to which finite-dimensional representation? We answer this question completely in this chapter.

4.1 Transfer of finite-dimensional representations

We will compute the transfer of finite-dimensional representations of a split classical group G to the appropriate $\mathrm{GL}(N, \mathbb{R})$ or $\mathrm{GL}(N, \mathbb{C})$. We need some preliminaries.

Let G be a connected reductive algebraic group over $\mathbb{R}$ with a split torus, say T. Let $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ and $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ be the group of characters and co-characters, respectively. The root datum of G is given by $(X^*(T), \Phi, X_*(T), \Phi^\wedge)$, where Φ is the set of roots of G with respect to T, and $\Phi^\wedge$ the set of co-roots. The connected component of the Langlands dual of G is the connected complex reductive group ${}^L G^\circ$ whose root datum is $(X_*(T), \Phi^\wedge, X^*(T), \Phi)$.

There is a natural non-degenerate bilinear pairing between $X^*(T)$ and $X_*(T)$ given by $\langle\ ,\ \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$ such that

$$\phi \circ \psi(z) \;=\; z^{\langle \phi, \psi \rangle}, \quad \phi \in X^*(T), \quad \psi \in X_*(T).$$

This gives an isomorphism $X^*(T) \cong \mathrm{Hom}(X_*(T), \mathbb{Z})$. Tensoring by $\mathbb{C}^\times$ we get

$$X^*(T) \otimes \mathbb{C}^\times \;\cong\; \mathrm{Hom}(X_*(T), \mathbb{Z}) \otimes \mathbb{C}^\times \;\cong\; \mathrm{Hom}(X_*(T), \mathbb{C}^\times),$$

where the last isomorphism is given by $\xi \otimes a \mapsto a^{\xi()}$, where $a^{\xi()}(\psi) = a^{\xi(\psi)}$. Let ${}^L T^\circ$ be a maximal torus of ${}^L G^\circ$. The natural map ${}^L T^\circ \to \mathrm{Hom}(X^*({}^L T^\circ), \mathbb{C}^\times)$ is an isomorphism. Thus we have

$${}^L T^\circ \;\cong\; \mathrm{Hom}(X^*({}^L T^\circ), \mathbb{C}^\times) \;=\; \mathrm{Hom}(X_*(T), \mathbb{C}^\times) \;=\; X^*(T) \otimes \mathbb{C}^\times. \quad (4.1)$$

Therefore, if $\chi \in X^*(T)$ then $\chi \otimes 1 \in {}^LT^\circ$, giving us a way to identify the weights of T with elements of the dual torus.

Given the natural inclusion $\iota : {}^LG^\circ \to \mathrm{GL}_N(\mathbb{C})$, we may take the dual torus ${}^LT^\circ$ to sit inside ${}^LD^\circ$ the diagonal matrices in $\mathrm{GL}_N(\mathbb{C})$, where D is the diagonal torus in $\mathrm{GL}_N(\mathbb{R})$ or $\mathrm{GL}_N(\mathbb{C})$. We define a "transfer of weights" (by a slight abuse of terminology), denoted as $\lambda \mapsto \iota(\lambda)$, from $X^*(T) \to X^*(D)$ such that the diagram

$$\begin{tikzcd} {}^LT^\circ \arrow[r, "\iota"] & {}^LD^\circ \\ X^*(T) \arrow[u] \arrow[r, "\iota"] & X^*(D) \arrow[u] \end{tikzcd}$$

commutes; the vertical arrows come from (4.1).

We will now compute the transfers of finite-dimensional representations of G (from Table 4), to an appropriate $\mathrm{GL}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$. We know that for a semi-simple group, the finite-dimensional representations of G are classified by the highest weights of the corresponding Lie algebra $\mathfrak{g}$, which are listed in Chapter 2. We recall them here.

1. For type B_n: A dominant integral weight is given by a string of n integers, say $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, such that $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \geq 0$.

2. For type C_n: A dominant integral weight is given by a string of n integers, say $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, such that $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \geq 0$.

3. For type D_n: A dominant integral weight is given by a string of n integers, say $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, such that $\lambda_1 \geq \lambda_2 \cdots \geq |\lambda_n|$.

4. And lastly, for the groups $\mathrm{U}(p,q)$; $p+q=n$: A highest weight is given by a string of n integers $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

We denote by M_λ the finite-dimensional representation of G with highest weight λ and $\phi(M_\lambda) : W_\mathbb{R} \to {}^LG$ the corresponding Langlands parameter. Note that such an M_λ is obtained as the Langlands quotient of the induced representation $Ind_B^G(\otimes_{i=1}^n \chi_{\lambda_i} |\cdot|^{\rho_i})$, where ρ_i is the i^{th} coefficient of ρ which is the half sum of positive roots corresponding to the Borel B and $\chi_{\lambda_i}(x) = x^{\lambda_i}$.

4.1.1 $\mathrm{Sp}(2n, \mathbb{R})$ to $\mathrm{GL}(2n+1, \mathbb{R})$:

The parameter $\phi(M_\lambda)$ is given by: $\phi(M_\lambda)(z) = (z\bar{z})^{\lambda+\rho}$, i.e.,

$$\phi(M_\lambda)(z) = \begin{pmatrix} (z\bar{z})^{\lambda_1+n} & & & & & & \\ & \ddots & & & & & \\ & & (z\bar{z})^{\lambda_n+1} & & & & \\ & & & 1 & & & \\ & & & & (z\bar{z})^{-\lambda_n-1} & & \\ & & & & & \ddots & \\ & & & & & & (z\bar{z})^{-\lambda_1-n} \end{pmatrix},$$

$$\phi(M_\lambda)(j) = \mathrm{Diag}((-1)^{\lambda_1}, \dots, (-1)^{\lambda_n}, 1, (-1)^{-\lambda_n}, \dots, (-1)^{-\lambda_1}).$$

Thus the transfer of the finite-dimensional representation M_λ to $\mathrm{GL}(2n+1, \mathbb{R})$, is the unique irreducible quotient of

$$Ind_{B(\mathbb{R})}^{\mathrm{GL}(2n+1,\mathbb{R})}(\otimes_{i=1}^n \chi_{\lambda_i} |\cdot|^{\rho_i} \otimes 1 \otimes_{i=n+1}^{2n+1} \chi_{-\lambda_{2n+1-i}} |\cdot|^{\rho_i}).$$

This representation is the finite-dimensional representation $\iota(M_\lambda) = M_{\lambda'}$, of $\mathrm{GL}(2n+1, \mathbb{R})$ with highest weight, $\lambda' = (\lambda_1, \dots, \lambda_n, 0, -\lambda_n, \dots, -\lambda_1)$.

Hence we have the following proposition:

Proposition 4.1.1. *For $G = \mathrm{Sp}(2n, \mathbb{R})$, the transfer of a finite dimensional representation, M_λ, of G to $\mathrm{GL}(2n+1, \mathbb{R})$ is the finite dimensional representation $M_{\lambda'}$ where $\lambda' = \iota(\lambda) = (\lambda_1, \dots, \lambda_n, 0, -\lambda_n, \dots, -\lambda_1)$..*

4.1.2 From $\mathrm{SO}(n+1,n)$ to $\mathrm{GL}(2n,\mathbb{R})$:

We have ${}^LG^\circ = \mathrm{Sp}(2n,\mathbb{C})$, for $G = \mathrm{SO}(n+1,n)$. The Langlands parameter $\phi(M_\lambda)$ is given by:

$$\begin{aligned}\phi(M_\lambda)(z) &= \mathrm{diag}(a_1,\ldots,a_n,a_n^{-1},\ldots,a_1^{-1}), \text{ where } a_i = (z\bar{z})^{\lambda_i+n-\frac{2i-1}{2}}, \text{ and}\\ \phi(M_\lambda)(j) &= \mathrm{diag}((-1)^{\lambda_1},\ldots,(-1)^{\lambda_n},(-1)^{\lambda_n},\ldots,(-1)^{\lambda_1}).\end{aligned}$$

The resulting representation of $\mathrm{GL}(2n,\mathbb{R})$ is the unique irreducible quotient of

$$\mathrm{Ind}_{B_{2n}(\mathbb{R})}^{\mathrm{GL}(2n,\mathbb{R})}(\chi_{\lambda_1}|\cdot|^{\frac{2n-1}{2}}\otimes\cdots\otimes\chi_{\lambda_1}|\cdot|^{\frac{1}{2}}\otimes\chi_{-\lambda_n}|\cdot|^{-\frac{1}{2}}\otimes\cdots\otimes\chi_{-\lambda_1}|\cdot|^{-\frac{2n-1}{2}}),$$

which one knows to be the finite-dimensional representation of $\mathrm{GL}(2n,\mathbb{R})$ with highest weight $\iota(\lambda) = (\lambda_1,\ldots,\lambda_n,-\lambda_n,\ldots,-\lambda_1)$. We have:

Proposition 4.1.2. *For $G = \mathrm{SO}(n+1,n)$ the transfer of M_λ to $\mathrm{GL}(2n,\mathbb{R})$ is the finite-dimensional representation $M_{\iota(\lambda)}$ with highest weight $\iota(\lambda) = (\lambda_1,\ldots,\lambda_n,-\lambda_n,\ldots,-\lambda_1)$.*

4.1.3 $\mathrm{SO}(n,n)$ to $\mathrm{GL}(2n,\mathbb{R})$:

We have ${}^LG^\circ = \mathrm{SO}(2n,\mathbb{C})$. See [9]. The parameter for M_λ is:

$$\begin{aligned}\phi(M_\lambda)(z) &= \mathrm{diag}(a_1,\ldots,a_n,a_n^{-1},\ldots,a_1^{-1}), \quad a_i = (z\bar{z})^{\lambda_i+n-i}, \ 1\le i\le n,\\ \phi(M_\lambda)(j) &= \mathrm{diag}((-1)^{\lambda_1},\ldots,(-1)^{\lambda_n},(-1)^{\lambda_n},\ldots,(-1)^{\lambda_1}).\end{aligned}$$

The transfer of M_λ is the unique irreducible quotient of

$$\mathrm{Ind}_{B_{2n}}^{\mathrm{GL}(2n,\mathbb{R})}(\chi_{\lambda_1}|\cdot|^{n-1}\otimes\chi_{\lambda_2}|\cdot|^{n-2}\otimes\cdots\otimes\chi_{\lambda_n}\otimes\chi_{-\lambda_n}\otimes\cdots\otimes\chi_{-\lambda_1}|\cdot|^{1-n}).$$

This representation is the same as

$$\iota(M_\lambda) = M_{\iota(\lambda)} \otimes |\cdot|^{-\frac{1}{2}},$$

where the $M_{\iota(\lambda)}$ is the finite-dimensional representation of $\mathrm{GL}(2n, \mathbb{R})$ with highest weight $\iota(\lambda) = (\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1)$.

For $\lambda = (\lambda_1, \ldots, \lambda_{n-1}, \lambda_n)$, define $\lambda^\# = (\lambda_1, \ldots, \lambda_{n-1}, -\lambda_n)$. The Langlands parameters for M_λ and $M_{\lambda^\#}$ are not conjugates by an inner automorphism of ${}^L G^\circ$ but they are conjugate by an element of $\mathrm{GL}(2n, \mathbb{C})$. Thus M_λ and $M_{\lambda^\#}$ are transferred to the same representation of $\mathrm{GL}(2n, \mathbb{R})$. Thus we have:

Proposition 4.1.3. *For $G = \mathrm{SO}(n, n)$ the transfer of M_λ is $M_{\iota(\lambda)} \otimes |\cdot|^{-\frac{1}{2}}$, where $M_{\iota(\lambda)}$ is the finite-dimensional representation of $\mathrm{GL}(2n, \mathbb{R})$ with highest weight $\iota(\lambda) = (\lambda_1, \ldots, \lambda_{n-1}, |\lambda_n|, -|\lambda_n|, -\lambda_{n-1}, \ldots, -\lambda_1)$.*

4.1.4 Unitary groups to $\mathrm{GL}(n, \mathbb{C})$:

Similar to the case of even orthogonal groups, we need to subdivide into two cases depending on the parity of n. Recall that a dominant integral weight of $\mathrm{GL}(n, \mathbb{C})$ is given by $\lambda = ((\lambda_1, \ldots, \lambda_n), (\lambda_1^*, \ldots, \lambda_n^*))$ with $\lambda_i, \lambda_i^* \in \mathbb{Z}$ and $\lambda_i \geq \lambda_{i+1}, \lambda_i^* \geq \lambda_{i+1}^*$.

n even.

In this case we transfer from the unitary group $\mathrm{U}(\frac{n}{2}, \frac{n}{2})$. The Langlands dual group is $\mathrm{GL}(n, \mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}$ and ${}^L G^\circ = \mathrm{GL}(n, \mathbb{C})$. The absolute rank of $U(\frac{n}{2}, \frac{n}{2})$ is n. Let M_λ be the finite-dimensional representation of $\mathrm{U}(\frac{n}{2}, \frac{n}{2})$ with highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$. Let $p = \frac{n}{2}$. The restriction of the Langlands parameter $\phi(M_\lambda)$ to $\mathbb{C}^\times$ is given by

$$\phi(M_\lambda)(z) = \mathrm{diag}(a_1, \ldots, a_p, \bar{a}_p^{-1}, \ldots, \bar{a}_1^{-1}),$$

where, for $1 \leq i \leq p$, we have

$$a_i \;=\; z^{\lambda_i+\rho_i}\bar{z}^{-\lambda_{n-i+1}-\rho_{n-i+1}}.$$

We will transfer representations of $\mathrm{U}(\frac{n}{2}, \frac{n}{2})$ to representations of $\mathrm{GL}(n, \mathbb{C})$ using stable base change (see [13]). The dual for $\mathrm{U}(\frac{n}{2}, \frac{n}{2})$ as noted above is $\mathrm{GL}(n, \mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}$, where the action of j, the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$, is given by $g \mapsto \Phi_n{}^t g^{-1}\Phi_n^{-1}$ where $(\Phi_n)_{ij} = (-1)^{i-1}\delta_{i,n-j+1}$. Now, ${}^L R_{\mathbb{C}\backslash\mathbb{R}} GL_n = \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}$, where the semidirect product acts by interchanging the two copies of $\mathrm{GL}(n, \mathbb{C})$. Define the map BC as:

$$BC : {}^L\mathrm{U}(\frac{n}{2}, \frac{n}{2}) \hookrightarrow {}^L R_{\mathbb{C}\backslash\mathbb{R}} GL_n(\mathbb{C}), \quad (g, 1) \mapsto (g, \theta(g), 1), \quad (g, \theta) \mapsto (g, \theta(g), \theta).$$

Note that $R_{\mathbb{C}\backslash\mathbb{R}} GL_n(\mathbb{C}) = \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$. Hence the parameter obtained by stable base change gives us a representation of $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$. Such a transferred representation of $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$ determines a representation of $\mathrm{GL}(n, \mathbb{C})$ with the Langlands parameter obtained by projecting on the first component. Similarly, a representation π of $\mathrm{GL}(n, \mathbb{C})$, with Langlands parameter $\Phi(\pi)$, determines a representation of $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$ with parameter $(\Phi(\pi), \theta(\Phi(\pi)))$. This gives us a way to go back and forth between representations of $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$, hence transferring a representation of $\mathrm{U}(\frac{n}{2}, \frac{n}{2})$ to $\mathrm{GL}(n, \mathbb{C})$.

Thus the representation of $\mathrm{GL}(n, \mathbb{C})$ obtained by stable base-change of the finite-dimensional representation M_λ has inducing data

$$z^{\lambda_1+\frac{n-1}{2}}\bar{z}^{-\lambda_n+\frac{n-1}{2}} \otimes \cdots \otimes z^{\lambda_p+\frac{1}{2}}\bar{z}^{-\lambda_{p+1}+\frac{1}{2}} \otimes z^{\lambda_{p+1}-\frac{1}{2}}\bar{z}^{-\lambda_p-\frac{1}{2}} \otimes \cdots \otimes z^{\lambda_n-\frac{n-1}{2}}\bar{z}^{-\lambda_1-\frac{n-1}{2}}.$$

The transferred representation of $\mathrm{GL}(n,\mathbb{C})$ is the finite-dimensional representation with highest weight $\iota(\lambda) = ((\lambda_1,\ldots,\lambda_n),(-\lambda_n,\ldots,-\lambda_1))$ which has purity weight 0. A low-dimensional example might help the reader to see the finer details:

Example 4.1.4. Consider $G = \mathrm{U}(2,2)$. Take a dominant integral weight $\lambda = (\lambda_1,\ldots,\lambda_4)$ and recall that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$. The torus of $\mathrm{U}(2,2)$ has the form $T = \{\mathrm{diag}(a_1, a_2, \bar{a}_2^{-1}, \bar{a}_1^{-1}) : a_i \in \mathbb{C}^\times\}$. The Langlands parameter for M_λ on $\mathbb{C}^\times$ is

$$\begin{pmatrix} z^{\lambda_1+\frac{3}{2}}\bar{z}^{-\lambda_4+\frac{3}{2}} & & & \\ & z^{\lambda_2+\frac{1}{2}}\bar{z}^{-\lambda_3+\frac{1}{2}} & & \\ & & z^{\lambda_3-\frac{1}{2}}\bar{z}^{-(\lambda_2+\frac{1}{2})} & \\ & & & z^{\lambda_4-\frac{3}{2}}\bar{z}^{-(\lambda_1+\frac{3}{2})} \end{pmatrix}.$$

This is the parameter for the finite-dimensional representation of $\mathrm{GL}(4,\mathbb{C})$ with highest weight $\iota(\lambda) = ((\lambda_1,\ldots,\lambda_4),(-\lambda_4,\ldots,-\lambda_1))$.

n odd.

In this case we transfer from the unitary group $\mathrm{U}(\frac{n+1}{2},\frac{n-1}{2})$. The Langlands dual group is $\mathrm{GL}(n,\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}$ and ${}^LG^\circ = \mathrm{GL}(n,\mathbb{C})$. Let M_λ be the finite-dimensional representation of $\mathrm{U}(\frac{n+1}{2},\frac{n-1}{2})$ with highest weight $\lambda = (\lambda_1,\ldots,\lambda_n)$. We will use stable base change as in the previous case to obtain a representation of $\mathrm{GL}(n,\mathbb{C})$. For convenience let $p = \frac{n+1}{2}$ and $q = \frac{n-1}{2}$. Observe that $p = q+1$. Then the Langlands parameter $\phi(M_\lambda)$ is given by:

$$\phi(M_\lambda)(z) = \mathrm{diag}(a_1,\ldots,a_q,a_p,\bar{a}_q^{-1},\ldots,\bar{a}_1^{-1}),$$

where

$$a_i \;=\; z^{\lambda_i+\rho_i}\bar{z}^{-\lambda_{n-i+1}-\rho_{n-i+1}}.$$

Thus we observe that the transferred representation has inducing data:

$$z^{\lambda_1+\frac{n-1}{2}}\bar{z}^{-\lambda_n+\frac{n-1}{2}} \otimes \cdots \otimes z^{\lambda_q+1}\bar{z}^{-\lambda_{n-q+1}+1} \otimes z^{\lambda_p}\bar{z}^{-\lambda_p} \otimes z^{\lambda_{n-q+1}-1}\bar{z}^{-\lambda_q-1}$$
$$\otimes \cdots \otimes z^{\lambda_n-\frac{n-1}{2}}\bar{z}^{-\lambda_1-\frac{n-1}{2}},$$

and, as we had seen this earlier, the transferred representation is finite-dimensional with highest weight $\iota(\lambda) = ((\lambda_1, \ldots, \lambda_n), (\lambda_1^*, \ldots, \lambda_n^*))$ which has purity 0.

Example 4.1.5. *Consider $G = U(3,2)$. M_λ be the finite dimensional representation of G with highest weight $\lambda = \lambda_1 \geq \cdots \geq \lambda_5$.*

The Langlands parameter of M_λ on $\mathbb{C}^\times$ is given by:

$$z \mapsto \mathrm{diag}(\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \mathrm{a}_4, \mathrm{a}_5)$$

where

$$\begin{aligned}
a_1 &= |z|^{\lambda_1+\lambda_5}(\tfrac{z}{|z|})^{\lambda_1-\lambda_5+4} \\
a_2 &= |z|^{\lambda_2+\lambda_4}(\tfrac{z}{|z|})^{\lambda_2-\lambda_4+2} \\
a_3 &= (\tfrac{z}{\bar{z}})^{4\lambda_3} \\
a_4 &= |z|^{\lambda_2+\lambda_4}(\tfrac{z}{|z|})^{-(\lambda_2-\lambda_4+2)} \\
a_5 &= |z|^{\lambda_1+\lambda_5}(\tfrac{z}{|z|})^{-(\lambda_1-\lambda_5+4)}
\end{aligned}$$

4.2 Transfer of Discrete Series representations

The discrete series representations with infinitesimal character χ_Λ have non-zero $(\mathfrak{g}, \mathfrak{k})$-cohomology with respect to the finite dimensional representation with highest weight $\Lambda - \rho$. This section is devoted to calculating the transfers of the discrete series representations to appropriate $\mathrm{GL}(N)$ (as in Table 4) and checking whether the transferred representations are cohomological or not, and if they are cohomological then with respect to which finite-dimensional representation.

4.2.1 $\mathrm{Sp}(p, q)$ to $\mathrm{GL}(2n+1, \mathbb{R})$:

Let $G = \mathrm{Sp}(p, q); p + q = n; p \geq q$. The split rank of $\mathrm{Sp}(p, q)$ is q.

$$\rho = ne_1 + (n-1)e_2 + \cdots + e_n.$$

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in X^*(T)$. Let the discrete series representation of $\mathrm{Sp}(p, q)$ corresponding to $\lambda + \rho$ be denoted by π_λ.

We observe that $\mathrm{Sp}(p, q)$ are all in the same inner class along with $\mathrm{Sp}(2n, \mathbb{R})$. Hence they have the same Langlands dual group. Recall that ${}^L G^\circ$ is $\mathrm{SO}(2n+1)$. The Langlands parameter $\phi(\pi_\lambda)$ is given by:

$$\phi(\pi_\lambda)(z) = \mathrm{Diag}(a_1, \ldots, a_n, 1, a_n^{-1}, \ldots, a_1^{-1}),$$

where,

$$a_i = (\frac{z}{\bar{z}})^{\lambda_i + n - i + 1}.$$

Also,

$$\phi(\pi_\lambda)(j) = \mathrm{AntiDiag}((-1)^{2(n+\lambda_1)}, \ldots, (-1)^{2(1+\lambda_n)}, 1, \ldots, 1) = \mathrm{Antidiag}(1).$$

Similar to the calculations in section 4, this gives us a representation of

$\mathrm{GL}(2n+1,\mathbb{R})$. The inducing data for this representation is

$$D_{2(n+\lambda_1)} \otimes D_{2((n-1)+\lambda_2)} \otimes \cdots \otimes D_{2(1+\lambda_n)} \otimes 1,$$

which can be read off from the parameter itself.

This representation of $\mathrm{GL}(2n+1,\mathbb{R})$ is tempered as well as cohomological. See Section 3.2.6). Furthermore, the representation is cohomological with respect to

$$\begin{aligned} \mu &= \frac{(2(\lambda_1+n),\ldots,2(\lambda_n+1),0,-2(\lambda_n+1),\ldots,-2(\lambda_1+n))}{2} - \rho_{2n+1} \\ &= (\lambda_1,\ldots,\lambda_n,0,-\lambda_n,\ldots,\lambda_1) \\ &= \lambda'. \end{aligned}$$

Hence, we have the following proposition:

Proposition 4.2.1. *Suppose π_λ is a discrete series representation of* $\mathrm{Sp}(p,q)$. *Then*

$$\pi_\lambda \in \mathrm{Coh}(\mathrm{Sp}(p,q),\lambda^\vee) \quad \textit{and} \quad \iota(\pi_\lambda) \in \mathrm{Coh}(\mathrm{GL}(2n+1,\mathbb{R}),\iota(\lambda)^\vee).$$

4.2.2 Odd $\mathrm{SO}(p,q)$; $p+q=2n+1$ to $\mathrm{GL}(2n,\mathbb{R})$:

Since $p+q$ is odd, all the groups $\mathrm{SO}(p,q)$ are inner forms of each other. Hence they have the same Langlands dual group and ${}^LG^\circ = \mathrm{Sp}(2n,\mathbb{C})$.

$$\rho = (n-\frac{1}{2})e_1 + (n-\frac{3}{2})e_2 + \cdots + \frac{1}{2}e_n.$$

Let π_λ be a discrete series representation of G. Then, the Langlands parameter $\phi(\pi_\lambda)$ of this representation is given by:

$$\begin{aligned}\phi(\pi_\lambda)(z) &= \mathrm{Diag}(a_1, \ldots, a_n, a_n^{-1}, \ldots, a_1^{-1}), \text{ where } a_i = (\frac{z}{\bar{z}})^{\lambda_i + \frac{2n+1-2i}{2}} \text{ and} \\ \phi(\pi_\lambda)(j) &= \mathrm{AntiDiag}((-1)^{2(\lambda_1+n-\frac{1}{2})}, \ldots, (-1)^{2(\lambda_n+\frac{1}{2})}, 1, \ldots, 1).\end{aligned}$$

Thus the transferred representation of $\mathrm{GL}(2n, \mathbb{R})$, of a discrete series representation has inducing data,

$$D_{2(\lambda_1+(n-\frac{1}{2}))} \otimes D_{2(\lambda_2+(n-\frac{3}{2}))} \otimes \cdots \otimes D_{2(\lambda_n+\frac{1}{2})}.$$

This representation of $\mathrm{GL}(2n, \mathbb{R})$ is tempered and cohomological. It is a straightforward calculation as in the previous case to compute the finite-dimensional representation with respect to which $\iota(\pi_\lambda)$ is cohomological. The finite-dimensional representation is of highest weight $\lambda' = (\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1)$.

Thus we have the following proposition:

Proposition 4.2.2. *Suppose π_λ is a discrete series representation of $\mathrm{SO}(p, q)$. Then*

$$\pi_\lambda \in \mathrm{Coh}(\mathrm{SO}(p, q), \lambda^\vee) \quad \textit{and} \quad \iota(\pi_\lambda) \in \mathrm{Coh}(\mathrm{GL}(2n, \mathbb{R}), \iota(\lambda)^\vee).$$

4.2.3 Even $\mathrm{SO}(p, q)$; $p + q = 2n$ to $\mathrm{GL}(2n, \mathbb{R})$:

There are two inner classes for the even orthogonal groups, which are as follows:

- $\mathrm{SO}(p, q)$ such that p and q are both even, and $\mathrm{SO}^*(2n)$.
- $\mathrm{SO}(p, q)$ such that p and q are both odd.

We observe that $\mathrm{SO}(p, q)$ has a discrete series if and only if the group is in the first inner class. So henceforth we assume that p and q are even or

$G = \mathrm{SO}^*(2n)$.

$$\rho = (n-1)e_1 + (n-2)e_2 + \cdots + 1e_{n-1}.$$

Now there are two cases to be considered here:

- n is even.
- n is odd.

When n is even the inner class having discrete series representation contains $\mathrm{SO}(n,n)$, where, as when n is odd, the class contains $\mathrm{SO}(n+1,n-1)$. The Langlands dual groups are $\mathrm{SO}(2n,\mathbb{C})$ and $O(2n,\mathbb{C})$ respectively. For simplicity, we will denote any group from the inner class of $\mathrm{SO}(n,n)$ as SO_{2n} and any group from the inner class of $\mathrm{SO}(n+1,n-1)$ as SO'_{2n}.

Let π_λ be a discrete series representation of SO_{2n} or SO'_{2n}. Then, the Langlands parameter $\phi(\pi_\lambda)$ of this representation is given by:

$$\begin{aligned} \phi(\pi_\lambda)(z) &= \mathrm{Diag}(a_1,\ldots,a_n,a_n^{-1},\ldots,a_1^{-1}) \text{ where } a_i = (\frac{z}{\bar{z}})^{\lambda_i+n-i} \text{ and} \\ \phi(\pi_\lambda)(j) &= \mathrm{AntiDiag}((-1)^{2(\lambda_1+n-1)},\ldots,(-1)^{2\lambda_n},1,\ldots,1). \end{aligned}$$

The transferred representation, $\iota(\pi_\lambda)$, of $\mathrm{GL}(2n,\mathbb{R})$, of the discrete series representation has inducing data,

$$D_{2(\lambda_1+n-1)} \otimes \cdots \otimes D_{2\lambda_n}.$$

We note that $\iota(\pi_\lambda) \otimes |\cdot|^{\frac{1}{2}}$ is cohomological with respect to the finite dimensional representation $M_{\lambda'}$, where $\lambda' = (\lambda_1,\ldots,\lambda_n,-\lambda_n,\ldots,-\lambda_1)$.

Thus we have the following proposition:

Proposition 4.2.3. *Suppose π_λ is a discrete series representation of*

$G = \mathrm{SO}_{2n}$ *or* $G = \mathrm{SO}'_{2n}$. *Then*

$$\pi_\lambda \in \mathrm{Coh}(G, \lambda^\vee) \quad \textit{and} \quad \iota(\pi_\lambda \otimes |\cdot|^{\frac{1}{2}}) \in \mathrm{Coh}(\mathrm{GL}(2n, \mathbb{R}), \iota(\lambda)^\vee).$$

4.2.4 $\mathrm{U}(p, q)$; $p + q = n$ to $\mathrm{GL}(n, \mathbb{C})$:

Let $p \geq q$. All the groups $\mathrm{U}(p, q)$ are inner forms of each other. Thus they have the same Langlands dual group which is $\mathrm{GL}(n, \mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}$, where the action of j, the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$, is given by $g \mapsto \Phi_n \, {}^t g^{-1} \Phi_n^{-1}$. Here, $\rho = (\frac{n-1}{2})e_1 + (\frac{n-3}{2})e_2 + \cdots + (-\frac{n-1}{2})e_n$. Let π_λ be a discrete series representation of $\mathrm{U}(p, q)$. Then, the Langlands parameter $\phi(\pi_\lambda)$ is given by:

$$\phi(\pi_\lambda)(z) = \mathrm{diag}(a_1, \ldots, a_n), \quad a_i = (z/\bar{z})^{\lambda_i + \frac{n-2i+1}{2}}.$$

The representation of $\mathrm{GL}(n, \mathbb{C})$ obtained by transferring π_λ via stable base change is:

$$\mathrm{Ind}_{B_n(\mathbb{C})}^{\mathrm{GL}(n,\mathbb{C})} \left((\tfrac{z}{\bar{z}})^{a_1} \otimes \cdots \otimes (\tfrac{z}{\bar{z}})^{a_n} \right),$$

where $a_i = \lambda_i + (n - 2i + 1)/2$. From Sect. 3.2.7, this representation of $\mathrm{GL}(n, \mathbb{C})$ is tempered, and cohomological with respect to the dual of the finite-dimensional representation with highest weight given by $(\lambda, \lambda^\vee) := ((\lambda_1, \ldots, \lambda_n), (-\lambda_n, \ldots, -\lambda_1))$ which is exactly $\iota(\lambda)$, the weight of the finite-dimensional representation transferred from $\mathrm{U}(\frac{n}{2}, \frac{n}{2})$ or $\mathrm{U}(\frac{n+1}{2}, \frac{n-1}{2})$, as might be the case.

Let's note that the weight $\iota(\lambda)$ does not give a self-dual coefficient system as in the cases of transfers from the other classical groups we have dealt with, however, as one might expect for stable base change from unitary groups, it gives a conjugate-self-dual representation. For $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n$, suppose M_λ denotes the irreducible representation of the complex (algebraic) group $\mathrm{GL}(n, \mathbb{C})$ then we have

$$M_{\iota(\lambda)} \;=\; M_\lambda \otimes \overline{M_{\lambda^\vee}} \;=\; M_\lambda \otimes \overline{M_\lambda^\vee},$$

which is now an algebraic representation of the real group $\mathrm{GL}(n, \mathbb{C})$. Clearly, $M_{\iota(\lambda)}^\vee = M_\lambda^\vee \otimes \overline{M_\lambda} = M_{\lambda^\vee} \otimes \overline{M_\lambda}$. Hence, $M_{\iota(\lambda)}^\vee$ is the representation corresponding to the weight $(\lambda^\vee, \lambda)$. Therefore, for the identity $M_{\iota(\lambda)}^\vee = M_{\iota(\lambda)^\vee}$ to hold, if we define:

$$\iota(\lambda)^\vee = (\lambda, \lambda^\vee)^\vee := (\lambda^\vee, \lambda).$$

then we have:

Proposition 4.2.4. *Suppose π_λ is a discrete series representation of $\mathrm{U}(p, q)$. Then*

$$\pi_\lambda \in \mathrm{Coh}(\mathrm{U}(p,q), \lambda^\vee) \quad \textit{and} \quad \iota(\pi_\lambda) \in \mathrm{Coh}(\mathrm{GL}(n, \mathbb{C}), \iota(\lambda)^\vee),$$

where, $\iota(\lambda) = ((\lambda_1, \ldots, \lambda_n), (-\lambda_n, \ldots, -\lambda_1)) = (\lambda, \lambda^\vee)$.

4.2.5 The main results:

We summarize the results of all the above into the following theorem:

Theorem 4.2.5. *1. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a dominant integral weight for $\mathrm{Sp}(2n, \mathbb{R})$. Then, as representations of $\mathrm{GL}(2n+1, \mathbb{R})$, we have:*

$$\iota(M_\lambda) = M_{\iota(\lambda)}, \quad \iota(\lambda) = (\lambda_1, \ldots, \lambda_n, 0, -\lambda_n, \ldots, -\lambda_1) = \iota(\lambda)^\vee.$$

Let G be in the inner class of $\mathrm{Sp}(2n, \mathbb{R})$, and if π_λ is a discrete series representation of G in $\mathrm{Coh}(G, \lambda^\vee)$, then

$$\iota(\pi_\lambda) \in \mathrm{Coh}(\mathrm{GL}(2n+1, \mathbb{R}), \iota(\lambda)^\vee).$$

2. *Let* $\lambda = (\lambda_1, \dots, \lambda_n)$ *be a dominant integral weight of* $\mathrm{SO}(n+1, n)$. *Then, as representations of* $\mathrm{GL}(2n, \mathbb{R})$, *we have:*

$$\iota(M_\lambda) = M_{\iota(\lambda)}, \quad \iota(\lambda) = (\lambda_1, \dots, \lambda_n, -\lambda_n, \dots, -\lambda_1) = \iota(\lambda)^\vee.$$

Let G *be in the inner class of* $\mathrm{SO}(n+1, n)$, *and if* π_λ *is a discrete series representation of* G *in* $\mathrm{Coh}(G, \lambda^\vee)$, *then*

$$\iota(\pi_\lambda) \in \mathrm{Coh}(\mathrm{GL}(2n, \mathbb{R}), \iota(\lambda)^\vee).$$

3. *Let* $\lambda = (\lambda_1, \dots, \lambda_n)$ *be a dominant integral weight of* $G = \mathrm{SO}(n, n)$. *Then, as representations of* $\mathrm{GL}(2n, \mathbb{R})$ *we have:*

$$\iota(M_\lambda) = M_{\iota(\lambda)} \otimes |\cdot|^{-\frac{1}{2}}, \quad \iota(\lambda) = (\lambda_1, \dots, \lambda_{n-1}, |\lambda_n|, -|\lambda_n|, -\lambda_{n-1}, \dots, -\lambda_1).$$

If $G = \mathrm{SO}_{2n}$ *or* SO'_{2n}, *and* π_λ *is a discrete series representation of* G *in* $\mathrm{Coh}(G, \lambda)$, *then*

$$\iota(\pi_\lambda) \otimes |\cdot|^{\frac{1}{2}} \in \mathrm{Coh}(\mathrm{GL}(2n, \mathbb{R}), \iota(\lambda)^\vee),$$

which may also be written as

$$\iota(\pi_\lambda) \in \mathrm{Coh}(\mathrm{GL}(2n, \mathbb{R}), \iota(M_\lambda)^\vee).$$

4. *Let* $\lambda = (\lambda_1, \dots, \lambda_n)$ *be a dominant integral weight for* $\mathrm{U}(\frac{n}{2}, \frac{n}{2})$ *or* $\mathrm{U}(\frac{n+1}{2}, \frac{n-1}{2})$. *Then, as representations of* $\mathrm{GL}(n, \mathbb{C})$, *we have:*

$$\iota(M_\lambda) = M_{\iota(\lambda)}, \quad \iota(\lambda) = ((\lambda_1, \dots, \lambda_n), (-\lambda_n, \dots, -\lambda_1)).$$

Let G be in the inner class of $\mathrm{U}(\frac{n}{2}, \frac{n}{2})$ *or* $\mathrm{U}(\frac{n+1}{2}, \frac{n-1}{2})$ *depending on the parity of n, and if π_λ is a discrete series representation of G in* $\mathrm{Coh}(G, \lambda^\vee)$*, then*

$$\iota(\pi_\lambda) \in \mathrm{Coh}(\mathrm{GL}(n, \mathbb{C}), \iota(\lambda)^\vee).$$

5. *For any of the above groups in* $(1) - (4)$*, the transfer of a tempered representation π is cohomological if and only if π is a discrete series representation of G.*

We have answered one of the two questions that were posed in the introduction. In the next chapter, we will address the second question that we had posed namely:

If we start with a cohomological representation of G, then does it transfer to a cohomological representation of $\mathrm{GL}(n)$*?*

We will assume in the next chapter that $G = \mathrm{Sp}(4, \mathbb{R})$.

Chapter 5

Transfer from $\mathrm{Sp}(4,\mathbb{R})$ to $\mathrm{GL}(5,\mathbb{R})$

In the previous chapter, we used Langlands functoriality to transfer a tempered representation of a classical group G to an appropriate $\mathrm{GL}(N)$ and studied the cohomological properties of these representations. Vogan-Zuckerman in 1984 classified the unitary cohomological dual of a semi-simple group G (see [27]). In this chapter, we ask the following question: *"Let π be an irreducible unitary cohomological representation of $G = \mathrm{Sp}(4,\mathbb{R})$ which is cohomological with respect to constant coefficients; denote by $\iota(\pi)$ the representation of $\mathrm{GL}(5,\mathbb{R})$ obtained by transferring π; is $\iota(\pi)$ cohomological?"* If so, with respect to which finite-dimensional representation of $\mathrm{GL}(5,\mathbb{R})$?

5.1 Vogan-Zuckerman Classification

5.1.1 Parameters and the Classification

We will briefly recall the Vogan-Zuckerman classification for unitary cohomological representations. The parameterization for the irreducible unitary cohomological representations is not the same one as used by Langlands but we do have a way to obtain the corresponding data in terms of Langlands classification. Let G be a connected real semi-simple Lie group with finite center. Let $\mathfrak{g}_0$ be the Lie algebra of G and $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$. Let $K \subseteq G$ be a maximal compact subgroup of G and θ the corresponding Cartan involution of G. Then, we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p},$$

where $\mathfrak{k}$ is the $+1$ eigenspace of θ and $\mathfrak{p}$ the -1 eigenspace.

For any irreducible unitary representation (π, V), the Harish-Chandra module, V_K^∞, associated with it is the space of all smooth K-finite vectors in V. Harish-Chandra in 1953 proved the following result:

Theorem 5.1.1. *(Harish-Chandra, [11]) V_K^∞ is irreducible as a $\mathfrak{g}$ module and determines π up to unitary equivalence.*

This reduces our study of irreducible unitary representations of G to studying irreducible $(\mathfrak{g}, K)$-modules. Vogan-Zuckerman described precisely these $(\mathfrak{g}, K)$-modules. We need two parameters: a θ- stable parabolic subalgebra $\mathfrak{q}$ and an admissible homomorphism λ on the Levi of $\mathfrak{q}$.

We construct the θ-stable parabolic subalgebra as follows: Let $x \in i\mathfrak{k}_0$. Since K is compact, $ad(x) : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable with real eigenvalues. Then define,

$$\begin{aligned}
\mathfrak{q} &= \text{sum of non-negative eigen-spaces of } ad(x), \\
\mathfrak{l} &= \text{the zero eigen-space of } ad(x) = \text{centralizer of } x.\ , \\
\mathfrak{u} &= \text{sum of positive eigen-spaces.}
\end{aligned}$$

Then $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is the Levi decomposition of $\mathfrak{q}$. We also have $\mathfrak{l}_0 = \mathfrak{g}_0 \cap \mathfrak{l}$. Since $\theta(x) = x$, the subalgebras $\mathfrak{q}, \mathfrak{l}, \mathfrak{u}$

are all invariant under the Cartan involution θ. The parabolic subalgebras obtained in this way are called *θ-stable parabolic subalgebras* of $\mathfrak{g}$. This gives us the first parameter of the two.

Let $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ be a Cartan subalgebra containing ix. Then $\mathfrak{t} \subseteq \mathfrak{l}$. For any subspace $\mathfrak{f}$, which is stable under $ad(\mathfrak{t})$, let $\Delta(\mathfrak{f}, \mathfrak{t}) = \{\alpha_1, \dots, \alpha_r\}$ be the set of roots of $\mathfrak{t}$ occurring in $\mathfrak{f}$. Note that the set $\Delta(\mathfrak{f}, \mathfrak{t})$ may have multiplicities. Define

$$\rho(\mathfrak{f}) = \frac{1}{2} \sum_{\alpha_i \in \Delta(\mathfrak{f})} \alpha_i.$$

Let $L \subseteq G$ be the connected subgroup of G with Lie algebra $\mathfrak{l}_0$. A representation $\lambda : \mathfrak{l} \to \mathbb{C}$ is called admissible if

- λ is a differential of a unitary character (also denoted by λ) of L.
- If $\alpha \in \Delta(\mathfrak{u})$, then $\langle \alpha, \lambda|_{\mathfrak{t}} \rangle \geq 0$.

A θ-stable parabolic subalgebra $\mathfrak{q}$ along with an admissible character of $\mathfrak{l}$ gives us an irreducible unitary cohomological representation of G. Given $\mathfrak{q}$ and an admissible λ, define

$$\mu(\mathfrak{q}, \lambda) = \text{Representation of } K \text{ of highest weight } \lambda|_{\mathfrak{t}} + 2\rho(\mathfrak{u} \cap \mathfrak{p}).$$

We are now in a position to state the classification result.

Theorem 5.1.2 ([27] Theorem 5.3)**.** *Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra and let $\lambda : \mathfrak{l} \to \mathbb{C}$ be an admissible character. Then there is a unique irreducible $\mathfrak{g}$-module $A_{\mathfrak{q}}(\lambda)$ such that:*

1. *The restriction of $A_{\mathfrak{q}}(\lambda)$ to $\mathfrak{k}$ contains $\mu(\mathfrak{q}, \lambda)$.*
2. *The center $Z(\mathfrak{g})$ of the universal enveloping algebra acts by $\chi_{\lambda+\rho}$ on $A_{\mathfrak{q}}(\lambda)$.*

3. *If a representation of highest weight δ of $\mathfrak{k}$ appears in the restriction of $A_{\mathfrak{q}}(\lambda)$, then*

$$\delta = \lambda|_{\mathfrak{t}} + 2\rho(\mathfrak{u}\cap\mathfrak{p}) + \sum_{\alpha\in\Delta(\mathfrak{u}\cap\mathfrak{p})} n_\alpha \alpha,$$

with n_α's non-negative integers.

This classifies all the irreducible unitary cohomological representations of the Lie group G. The representation $A_{\mathfrak{q}}(\lambda)$ has non-trivial cohomology with respect to the finite-dimensional representation of G with highest weight λ.

Remark 5.1. *$A_q(\lambda)$ is the discrete series representation if and only if $\mathfrak{l} \subseteq \mathfrak{k}$. Further, $A_{\mathfrak{q}}$ is a tempered representation if and only if $[\mathfrak{l},\mathfrak{l}] \subseteq \mathfrak{k}$.*

5.1.2 θ-stable subalgebras for $\mathrm{Sp}(2n,\mathbb{R})$

We will now parameterize the θ-stable parabolic subalgebras of $\mathrm{Sp}(2n,\mathbb{R})$. Let $\mathrm{Sp}(2n,\mathbb{R}) = \{A \in \mathrm{GL}(2n,\mathbb{R}) : {}^tAJA = J\}$, where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Let $\mathfrak{g}_0$ be the corresponding Lie algebra. Let

$$\mathfrak{h}_0 = \left\{ \begin{pmatrix} & & & x_1 & & \\ & 0 & & & \ddots & \\ & & & & & x_n \\ -x_1 & & & & & \\ & \ddots & & & 0 & \\ & & -x_n & & & \end{pmatrix} \middle| x_i \in \mathbb{R} \right\}.$$

Let $K = \{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A, B \in \mathrm{GL}(n,\mathbb{R}), A^tB = {}^tBA, A^tA + B^tB = I_n\}$ be a maximal compact subgroup of $\mathrm{Sp}(2n,\mathbb{R})$ and W_K be the Weyl group of K. Any element of W_K acts on an element of $i\mathfrak{h}_0$ by permuting the entries x_i.

We have the following result to aid us in listing all the θ-stable subalgebras of $\mathrm{Sp}(4,\mathbb{R})$.

Lemma 5.1.3. *The following sets are in* $1-1$ *correspondence:*

1. *{open, polyhedral root cones in* $i\mathfrak{h}/W_K$ *}*

2. *{ordered partitions of n:* $n = \sum_{j=1}^{s}(n_j+m_j)+m$ *with* $n_j, m_j, m, s \geq 0, n_j+m_j > 0$*}*

Proof. Let $x = (x_1,\ldots,x_n) \in i\mathfrak{h}/W_K$. Since W_K acts by permuting the coordinates of x, we can assume that $x_1 \geq x_2 \geq \cdots \geq x_r > 0 > x_{r+1} \geq x_{r+2} \geq \cdots \geq x_n$. This can also be expressed as follows:

$$x = (s,\ldots,s,s{-}1,\ldots,s{-}1,\ldots,1,\ldots,1,0,\ldots,0,-1,\ldots,-1,\ldots,-s,\ldots,-s),$$

where the number of $j > 0$ occurring in the above representation is n_j, the number of $j < 0$ occurring in the above representation is m_j, and the number of zeros is m, with $n = \sum_{j=1}^{s}(n_j+m_j)+m$ with $n_j, m_j, m, s \geq 0, n_j+m_j > 0$. This gives us a bijection between the two sets above. □

Let $\mathfrak{Q}$ be the set of all θ-stable parabolic subalgebras of $\mathfrak{g}$. The group K acts on the set $\mathfrak{Q}$ via the adjoint action due to which we get a finite set of θ-stable parabolic subalgebras $\mathfrak{Q}/K$. The following lemma gives us a bijection between $\mathfrak{Q}/K$ and open polyhedral root cones in $i\mathfrak{h}/W_K$.

Lemma 5.1.4. *Every* $x \in i\mathfrak{h}/W_K$ *defines a* θ*-stable parabolic subalgebra* $\mathfrak{q}_x$ *by setting* $\mathfrak{q}_x = \mathfrak{l}_x + \mathfrak{u}_x$*, where*

$$\mathfrak{l}_x = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta(\mathfrak{g},\mathfrak{h}),\alpha(x)=0} \mathfrak{g}_\alpha; \quad \mathfrak{u}_x = \bigoplus_{\alpha\in\Delta(\mathfrak{g},\mathfrak{h}),\alpha(x)>0} \mathfrak{g}_\alpha.$$

Two θ*-stable parabolic subalgebras* $\mathfrak{q}_x, \mathfrak{q}_y$ *are equal if and only if* x *and* y *are in the same open polyhedral root cone.*

Conversely, up to conjugacy be K, any θ-stable parabolic subalgebra $\mathfrak{q}$ is $\mathfrak{q} = \mathfrak{q}_x$ for some $x \in i\mathfrak{h}/W_K$.

Two θ-stable parabolic subalgebras, $\mathfrak{q}_1, \mathfrak{q}_2$ are said to be equivalent if $\Delta(\mathfrak{u}_1 \cap \mathfrak{p}) = \Delta(\mathfrak{u}_2 \cap \mathfrak{p})$, i.e. if the non-compact parts in the unipotent radical of the parabolic subalgebras are equal.

5.1.3 Langlands data for $A_q(\lambda)$

Let us see how to obtain the inducing data of $A_\mathfrak{q}(\lambda)$, such that the representation $A_\mathfrak{q}(\lambda)$ is a Langlands quotient of the corresponding induced representation; here $A_\mathfrak{q}(\lambda)$ be a representation of G which is irreducible unitary and cohomological which is obtained as in [27]. Fix a maximally split θ-stable Cartan subgroup $H = TA$ of L (corresponding to the Levi part $\mathfrak{l}$ of the θ-stable parabolic subalgebra $\mathfrak{q}$) and an Iwasawa decomposition $L = (L \cap K)AN^L$. Put

$$\begin{aligned} MA &= \text{Langlands decomposition of centralizer of } A \text{ in } G, \\ \nu &= (\frac{1}{2} \text{ sum of roots of } \mathfrak{a} \text{ in } \mathfrak{n}^L) + \lambda|_\mathfrak{a} \quad \in \mathfrak{a}^*. \end{aligned}$$

Now, let P be any parabolic subgroup of G such that P has Levi factor MA satisfying $\langle Re(\nu), \alpha\rangle \geq 0$ for all roots α of $\mathfrak{a}$ in $\mathfrak{n}^L$. It remains to describe the discrete series representation σ of M. The Harish-Chandra parameter of σ is $\rho^+ + \lambda|_\mathfrak{t} + \rho(\mathfrak{u})$; where ρ^+ is half sum of positive roots of $\mathfrak{t}$ in $\mathfrak{m} \cap \mathfrak{l}$ and $\rho(\mathfrak{u})$ is half sum of roots of $\mathfrak{t}$ in $\mathfrak{u}$. The only difficulty here is that if M is not connected then the Harish-Chandra parameter does not completely define the discrete series representation of M. We fix this as follows:

Let

$$\begin{aligned} \mu^M(\mathfrak{q}, \lambda) &= \text{Representation of } M \cap K \text{ of extremal weight} \\ &\quad \lambda|_{\mathfrak{t}} + 2\rho(\wedge^{dim(\mathfrak{u}\cap\mathfrak{p})}(\mathfrak{u} \cap \mathfrak{p}))|_{\mathfrak{t}}. \end{aligned}$$

Let σ be the discrete series representation with lowest $M \cap K$ type $\mu^M(\mathfrak{q}, \lambda)$. This completely determines the discrete series representation of M. And thus we have constructed the Langlands inducing data for the representation $A_{\mathfrak{q}}(\lambda)$.

We now address the case of $\mathrm{Sp}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$.

5.2 SL(2, ℝ) to GL(3, ℝ)

In this section, as a warm-up example we will study the cohomological properties of representations of $\mathrm{GL}(3, \mathbb{R})$ which are obtained by transferring unitary cohomological representations of $\mathrm{SL}(2, \mathbb{R})$. We will compute the unitary cohomological representations of $\mathrm{SL}(2, \mathbb{R})$ using the Vogan-Zuckerman classification. We denote by $\mathfrak{sl}(2, \mathbb{C})$ the complexified Lie algebra of $\mathrm{SL}(2, \mathbb{R})$. Let $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ be the basis of $\mathfrak{sl}(2, \mathbb{R})$.

Let $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\phi(A) = -{}^tA$ and $\theta = int(w) \circ \phi$. Then θ is the Cartan involution on $\mathfrak{sl}(2, \mathbb{C})$. Note that:

$$\mathfrak{sl}(2, \mathbb{C}) = \langle H \rangle \oplus \langle X, Y \rangle,$$

where $\mathfrak{k} = \langle H \rangle$ is the $+1$ eigen space of θ and $\mathfrak{p} = \langle X, Y \rangle$ is the -1 eigen space of θ.

There are 3 $\theta-$stable parabolic subalgebras of $\mathfrak{sl}(2, \mathbb{C})$ corresponding to

0, H and $-H$.

1. Corresponding to 0: This gives the full algebra of $q_0 = \mathfrak{sl}(2,\mathbb{C})$. Also, $\mathfrak{l} = \mathfrak{sl}(2,\mathbb{C})$.

2. Corresponding to H: The parabolic subalgebra is

$$\mathfrak{q}_1 = \langle H \rangle \oplus \langle X \rangle,$$

   where $\langle H \rangle$ is $\mathfrak{l}$ and $\mathfrak{u} = \langle X \rangle$.

3. Corresponding to $-H$: The parabolic subalgebra is

$$\mathfrak{q}_2 = \langle H \rangle \oplus \langle Y \rangle,$$

   where $\langle H \rangle$ is $\mathfrak{l}$ and $\mathfrak{u} = \langle Y \rangle$.

Note that the only possible admissible character λ for $\mathfrak{q}_0$ is $\lambda = 0$. This gives rise to the trivial representation of $\mathrm{SL}(2,\mathbb{R})$. This representation is transferred to the trivial representation of $\mathrm{GL}(3,\mathbb{R})$ which is cohomological with respect to the trivial coefficients. Observe that the Levi parts of both $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are contained in $\mathfrak{k}$. Thus the cohomological representations $A_{\mathfrak{q}_1}(\lambda)$ and $A_{\mathfrak{q}_2}(\lambda)$ are discrete series representations with highest weight λ. The Langlands parameter for a representation of $\mathrm{SL}(2,\mathbb{R})$ is a homomorphism from the Weil group of $\mathbb{R}$ to $\mathrm{PGL}(2,\mathbb{C})$. The parameter $\phi(D_n)$ for the discrete series representation of $\mathrm{SL}(2,\mathbb{R})$ is given by

$$z \mapsto \begin{pmatrix} (\frac{z}{\bar{z}})^{\frac{n}{2}} & 0 \\ 0 & 1 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

To compute the transfer of the discrete series representations of $\mathrm{SL}(2,\mathbb{R})$ to $\mathrm{GL}(3,\mathbb{R})$, we embed $\mathrm{PGL}(2,\mathbb{C})$ into $\mathrm{GL}(3,\mathbb{R})$ via the 3- dimensional representation induced by GL$(2,\mathbb{C})$ taking

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \begin{pmatrix} \frac{a}{b} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{b}{a} \end{pmatrix}.$$

The image of PGL$(2,\mathbb{C})$ can be identified with SO$(3) \subset$ GL$(3,\mathbb{C})$ which preserves the quadratic form $\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Thus one observes that the transfer of a discrete series representation of SL$(2,\mathbb{R})$, with highest weight n, to GL$(3,\mathbb{R})$ has Langlands parameter

$$z \mapsto \begin{pmatrix} (\frac{z}{\bar{z}})^{\frac{n}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (\frac{z}{\bar{z}})^{\frac{-n}{2}} \end{pmatrix}; \qquad j \mapsto \begin{pmatrix} 0 & 0 & (-1)^n \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We know that this corresponds to a cohomological representation of GL$(3,\mathbb{R})$ which is cohomological with respect to the finite dimensional representation with highest weight $(n, 0, -n)$ 4.2.5. We have already seen that the transfer of M_n, the finite dimensional representation of SL$(2,\mathbb{R})$ with highest weight n, transfers to the finite dimensional representation of GL$(3,\mathbb{R})$ with highest weight $(n, 0, -n)$ 4.2.5. Thus we have the following result:

Proposition 5.2.1. *Let π be an irreducible unitary cohomological representation of* SL$(2,\mathbb{R})$ *with respect to the finite dimensional representation M. Then the representation of* GL$(3,\mathbb{R})$*, $\iota(\pi)$, obtained by the Langlands transfer is cohomological with respect to $\iota(M)$.*

Remark 5.2. *Note that the only irreducible unitary cohomological representations of* SL$(2,\mathbb{R})$ *are the discrete series representations and the trivial representation.*

5.3 $G = \mathrm{Sp}(4,\mathbb{R})$

5.3.1 Basis for the Lie algebra

We will change notations for the group $G = \mathrm{Sp}(4,\mathbb{R})$ as follows. This will aid us in doing explicit computations. Let $G = \mathrm{Sp}(4,\mathbb{R})$, $\mathfrak{g}$ be the corresponding complexified Lie algebra.

Let $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. Then $\mathrm{Sp}(4,\mathbb{R}) = \{A \in \mathrm{GL}(4,\mathbb{R}) : {}^tAJA = J\}$ and $\mathfrak{g} = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M(4,\mathbb{C}) : A = -{}^tD, B = {}^tB, C = {}^tC\}$. We have the following multiplication table for the Lie algebra of $\mathfrak{g}$ (see [19]):

	Z	Z'	N_+	N_-	X_+	X_-	P_{1+}	P_{1-}	P_{0+}	P_{0-}
Z	0	0	N_+	$-N_-$	$2X_+$	$-2X_-$	P_{1+}	$-P_{1-}$	0	0
Z'	0	0	$-N_+$	N_-	0	0	P_{1+}	$-P_{1-}$	$2P_{0+}$	$-2P_{0-}$
N_+	$-N_+$	N_-	0	$Z'-Z$	0	$-P_{1-}$	$2X_+$	$-2P_{0-}$	P_{1+}	0
N_-	N_-	$-N_-$	$Z-Z'$	0	$-P_{1+}$	0	$-2P_{0+}$	$2X_-$	0	P_{1-}
X_+	$-2X_+$	0	0	P_{1+}	0	Z	0	N_+	0	0
X_-	$2X_-$	0	P_{1-}	0	$-Z$	0	N_-	0	0	0
P_{1+}	$-P_{1+}$	$-P_{1+}$	$-2X_+$	$2P_{0+}$	0	$-N_-$	0	$Z+Z'$	0	N_+
P_{1-}	P_{1-}	P_{1-}	$2P_{0-}$	$-2X_-$	$-N_+$	0	$-Z-Z'$	0	N_-	0
P_{0+}	0	$-2P_{0+}$	$-P_{1+}$	0	0	0	0	$-N_-$	0	Z'
P_{0-}	0	$2P_{0-}$	0	$-P_{1-}$	0	0	$-N_+$	0	$-Z'$	0

where the basis elements are:

$$Z = -i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Z' = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$N_+ = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & -i \\ -1 & 0 & -i & 0 \\ 0 & i & 0 & 1 \\ i & 0 & -1 & 0 \end{pmatrix}, \quad N_- = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & i \\ -1 & 0 & i & 0 \\ 0 & -i & 0 & 1 \\ -i & 0 & -1 & 0 \end{pmatrix},$$

$$X_+ = \frac{1}{2}\begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_- = \frac{1}{2}\begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_{1+} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & i \\ 1 & 0 & i & 0 \\ 0 & i & 0 & -1 \\ i & 0 & -1 & 0 \end{pmatrix}, \quad P_{1-} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & -i \\ 1 & 0 & -i & 0 \\ 0 & -i & 0 & -1 \\ -i & 0 & -1 & 0 \end{pmatrix},$$

$$P_{0+} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & -1 \end{pmatrix}, \quad P_{0-} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & -1 \end{pmatrix}.$$

Note that, we have the Cartan decomposition for $\mathfrak{g} = \mathfrak{sp}(4) = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k} = \langle Z, Z', N_+, N_- \rangle$ and $\mathfrak{p} = \langle X_+, X_-, P_{1+}, P_{1-}, P_{0+}, P_{0-} \rangle$.

5.3.2 Parabolic subgroups of $G = \mathrm{Sp}(4, \mathbb{R})$

The Langlands parameter for a representation can be read off from the inducing data of the representation. The inducing data includes a parabolic subgroup of G, a representation of the Levi part of the parabolic and a character on the Lie algebra of a split torus inside the parabolic. Since, we want to transfer the representations of $G = \mathrm{Sp}(4, \mathbb{R})$ to $\mathrm{GL}(5, \mathbb{R})$, we would like to know the Langlands parameters for the representations $A_{\mathfrak{q}}(\lambda)$'s. It is well known that the parabolic subgroups containing a Borel subgroup are in bijection with the subsets of the base corresponding to the Borel (see [26]). It will be convenient if we list down the parabolic subgroups of $\mathrm{Sp}(4,\mathbb{R})$ beforehand.

For $\mathrm{Sp}(4,\mathbb{R})$, there are 4 subsets of the base for the root system which is $\{e_1 - e_2, 2e_2\}$. Thus there are 4 parabolic subgroups of $\mathrm{Sp}(4,\mathbb{R})$. One of them being the group itself which corresponds to the full base. This leaves 3 proper parabolic subgroups of $\mathrm{Sp}(4,\mathbb{R})$ which are:

1. **Minimal parabolic:** $B = M_0A_0N_0$, which corresponds to the empty subset of the base, where
$M_0 = \{\begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 0 \\ 0 & 0 & 0 & \epsilon_2 \end{pmatrix} : \epsilon_i \in \{\pm 1\}\}$,
$A_0 = \{\mathrm{diag}(a, b, a^{-1}, b^{-1}) : a, b \in \mathbb{R}^{\times}_{>0}\}$, and
$N_0 = \{n(x_0, x_1, x_2, x_3) = \begin{pmatrix} 1 & 0 & x_1 & x_2 \\ 0 & 1 & x_2 & x_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x_0 & 1 \end{pmatrix}\} \subset \mathrm{Sp}(4)$.

2. **Siegel parabolic:** $P_S = M_SA_SN_S$, which corresponds to the subset $\Sigma = \{e_1 - e_2\}$ of the base, where
$M_S = \{\begin{pmatrix} m & 0 \\ 0 & {}^tm^{-1} \end{pmatrix} : m \in SL^{\pm}(2,\mathbb{R})\}$,
$A_S = \{\mathrm{diag}(a, a, a^{-1}, a^{-1}) : a > 0\}$, and
$N_S = \{\begin{pmatrix} 1_2 & x \\ 0 & 1_2 \end{pmatrix} : x = {}^tx \in M_2(\mathbb{R})\}$.

3. **Jacobi Parabolic:** $P_J = M_JA_JN_J$, which corresponds to the subset $\Sigma = \{2e_2\}$ of the base, where

$$M_J = \{ \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & \epsilon & 0 \\ 0 & c & 0 & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \epsilon = \pm 1 \}$$

$A_J = \{\mathrm{diag}(a, 1, a^{-1}, 1) : a \in \mathbb{R}^{\times}_{>0}\}$, and

$N_J = \{n(x_0, x_1, x_2, 0) : x_i \in \mathbb{R}\}$.

5.3.3 θ-stable parabolic subalgebras of $\mathfrak{sp}(4)$

We list all the θ-stable parabolic subalgebras $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{sp}(4)$ along with the possible admissible $\lambda : \mathfrak{l} \to \mathbb{C}$ which can be obtained from a highest weight of $\mathfrak{h}$. We note that a highest weight of $\mathfrak{h}$ can be extended to an admissible character of $\mathfrak{l}$ if and only if $\lambda|_{\mathfrak{h} \cap [\mathfrak{l}_0, \mathfrak{l}_0]}$ and $\lambda|_{\mathfrak{a}} = 0$ where the subalgebra $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{sp}(4, \mathbb{R})$ (see [10]). Along with the θ-stable parabolic subalgebras and their corresponding admissible characters, we will also simultaneously list down some useful data for each θ-stable parabolic subalgebra, which will come in handy when we compute the Langlands parameters. To make the list we use Lemma 5.1.4 and Theorem 5.1.3.

1. $x = 0$ corresponding to the partition $2 = 2$.

   The θ-stable parabolic subalgebra corresponding to x is:

   $$\mathfrak{q}_1 = \mathfrak{sp}_4(\mathbb{C}) + 0$$

   The Levi part is: $\mathfrak{l} = \mathfrak{sp}_4(\mathbb{C})$.

   $\mathfrak{l}_o = \mathfrak{l} \cap \mathfrak{sp}_4(\mathbb{R}) = \mathfrak{sp}_4(\mathbb{R})$.

   $[\mathfrak{l}_o, \mathfrak{l}_o] = \mathfrak{l}_o$.

   So $\mathfrak{h} \cap [\mathfrak{l}_o, \mathfrak{l}_o] = \mathfrak{h}$.

   Therefore, λ of $\mathfrak{h}$ can be extended to get an admissible character of $\mathfrak{l}$ if and only if $\lambda = 0$. This θ- stable parabolic subalgebra corresponds to the parabolic subgroup G.

2. $x = -Z - 2Z'$ corresponding to the partition $2 = (0+1)+(0+1)+0$.
   The θ-stable parabolic subalgebra corresponding to x is:

$$\mathfrak{q}_2 = \langle Z, Z' \rangle + \langle N_+, X_-, P_{1-}, P_{0-} \rangle$$

   The Levi part is: $\mathfrak{l} = \langle Z, Z' \rangle$.
   $\mathfrak{l}_\circ = \mathfrak{l} \cap \mathfrak{sp}_4(\mathbb{R}) = \langle iZ, iZ' \rangle = \mathfrak{h}$.
   $[\mathfrak{l}_\circ, \mathfrak{l}_\circ] = 0$.
   So $\mathfrak{h} \cap [\mathfrak{l}_\circ, \mathfrak{l}_\circ] = 0$.
   Therefore, any highest weight λ of $\mathfrak{h}$ is an admissible character of $\mathfrak{l} = \mathfrak{h}$. This θ- stable parabolic subalgebra corresponds to the parabolic subgroup B.

3. $x = 2Z - Z'$ corresponding to the partition $2 = (1+0)+(0+1)+0$.
   The θ-stable parabolic subalgebra corresponding to x is:

$$\mathfrak{q}_3 = \langle Z, Z' \rangle + \langle N_+, X_+, P_{1+}, P_{0-} \rangle$$

   The Levi part is: $\mathfrak{l} = \langle Z, Z' \rangle$.
   $\mathfrak{l}_\circ = \mathfrak{l} \cap \mathfrak{sp}_4(\mathbb{R}) = \langle iZ, iZ' \rangle = \mathfrak{h}$.
   $[\mathfrak{l}_\circ, \mathfrak{l}_\circ] = 0$.
   So $\mathfrak{h} \cap [\mathfrak{l}_\circ, \mathfrak{l}_\circ] = 0$.
   Therefore, any highest weight λ of $\mathfrak{h}$ is an admissible character of $\mathfrak{l} = \mathfrak{h}$. This θ- stable parabolic subalgebra corresponds to the parabolic subgroup B.

4. $x = 2Z' - Z$ corresponding to the partition $2 = (0+1)+(1+0)+0$.

The θ-stable parabolic subalgebra corresponding to x is:

$$\mathfrak{q}_4 = \langle Z, Z' \rangle + \langle N_-, X_-, P_{1+}, P_{0+} \rangle$$

The Levi part is: $\mathfrak{l} = \langle Z, Z' \rangle$.
$\mathfrak{l}_\circ = \mathfrak{l} \cap \mathfrak{sp}_4(\mathbb{R}) = \langle iZ, iZ' \rangle = \mathfrak{h}$.
$[\mathfrak{l}_\circ, \mathfrak{l}_\circ] = 0$.
So $\mathfrak{h} \cap [\mathfrak{l}_\circ, \mathfrak{l}_\circ] = 0$.
Therefore, any highest weight λ of $\mathfrak{h}$ is an admissible character of $\mathfrak{l} = \mathfrak{h}$. This θ- stable parabolic subalgebra corresponds to the parabolic subgroup B.

5. $x = 2Z + Z'$ corresponding to the partition $2 = (1 + 0) + (1 + 0) + 0$.
The θ-stable parabolic subalgebra corresponding to x is:

$$\mathfrak{q}_5 = \langle Z, Z' \rangle + \langle N_+, X_+, P_{1+}, P_{0+} \rangle$$

The Levi part is: $\mathfrak{l} = \langle Z, Z' \rangle$.
$\mathfrak{l}_\circ = \mathfrak{l} \cap \mathfrak{sp}_4(\mathbb{R}) = \langle iZ, iZ' \rangle = \mathfrak{h}$.
$[\mathfrak{l}_\circ, \mathfrak{l}_\circ] = 0$.
So $\mathfrak{h} \cap [\mathfrak{l}_\circ, \mathfrak{l}_\circ] = 0$.
Therefore, any highest weight λ of $\mathfrak{h}$ is an admissible character of $\mathfrak{l} = \mathfrak{h}$. This θ- stable parabolic subalgebra corresponds to the parabolic subgroup B.

6. $x = Z + Z'$ corresponding to the partition $2 = (2 + 0) + 0$.
The θ-stable parabolic subalgebra corresponding to x is:

$$\mathfrak{q}_6 = \langle Z, Z', N_+, N_- \rangle + \langle X_+, P_{1+}, P_{0+} \rangle$$

Note that $\mathfrak{u} \cap \mathfrak{p} = \mathfrak{u}$ which is also equal to the intersection of the unipotent part of $\mathfrak{q}_5$ and $\mathfrak{p}$. Thus $\mathfrak{q}_6$ is equivalent to $\mathfrak{q}_5$, and the corresponding $A_{\mathfrak{q}}(\lambda)$'s are isomorphic.

7. $x = -(Z + Z')$ corresponding to the partition $2 = (0 + 2) + 0$.

   The θ-stable parabolic subalgebra corresponding to x is:

$$\mathfrak{q}_7 = \langle Z, Z', N_+, N_- \rangle + \langle X_-, P_{1-}, P_{0-} \rangle$$

   Note that $\mathfrak{u} \cap \mathfrak{p} = \mathfrak{u}$ which is also equal to the intersection of the unipotent part of $\mathfrak{q}_2$ and $\mathfrak{p}$. Thus $\mathfrak{q}_7$ is equivalent to $\mathfrak{q}_2$, and the corresponding $A_{\mathfrak{q}}(\lambda)$'s are isomorphic.

8. $x = Z$ corresponding to the partition $2 = (1 + 0) + 1$.

   The θ-stable parabolic subalgebra corresponding to x is:

$$\mathfrak{q}_8 = \langle Z, Z', P_{0+}, P_{0-} \rangle + \langle N_+, X_+, P_{1+} \rangle$$

   The Levi part is: $\mathfrak{l} = \langle Z, Z', P_{0+}, P_{0-} \rangle$.
   $\mathfrak{l}_\circ = \mathfrak{l} \cap \mathfrak{sp}_4(\mathbb{R}) = \langle iZ, iZ', P_{0+} + P_{0-}, i(P_{0+} - P_{0-}) \rangle$.
   $[\mathfrak{l}_\circ, \mathfrak{l}_\circ] = \langle 2i(P_{0+} - P_{0-}), 2(P_{0+} + P_{0-}), -2iZ' \rangle$.
   So $\mathfrak{h} \cap [\mathfrak{l}_\circ, \mathfrak{l}_\circ] = \langle \mathfrak{h}_2 = iZ' \rangle$.
   Therefore, a highest weight λ of $\mathfrak{h}$ can be extended to get an admissible character of $\mathfrak{l}$ if and only if $\lambda(\mathfrak{h}_2) = 0$. This θ- stable parabolic subalgebra corresponds to the parabolic subgroup P_J.

9. $x = -Z'$ corresponding to the partition $2 = (0 + 1) + 1$.

   The θ-stable parabolic subalgebra corresponding to x is:

$$\mathfrak{q}_9 = \langle Z, Z', X_+, X_- \rangle + \langle N_+, P_{1-}, P_{0-} \rangle$$

   The Levi part is: $\mathfrak{l} = \langle Z, Z', X_+, X_- \rangle$.

$\mathfrak{l}_\circ = \mathfrak{l} \cap \mathfrak{sp}_4(\mathbb{R}) = \langle iZ, iZ', X_+ + X_-, i(X_+ - X_-)\rangle$.
$[\mathfrak{l}_\circ, \mathfrak{l}_\circ] = \langle iZ, 2i(X_- - X_-), 2(X_+ - X_-)\rangle$.
So $\mathfrak{h} \cap [\mathfrak{l}_\circ, \mathfrak{l}_\circ] = \langle \mathfrak{h}_1 = iZ\rangle$.
Therefore, a highest weight λ of $\mathfrak{h}$ can be extended to get an admissible character of $\mathfrak{l}$ if and only if $\lambda = (0, \lambda)$. Note that this integral weight is conjugate under the Weyl group to an integral weight of the form $\lambda = (\lambda, 0)$. This θ- stable parabolic subalgebra corresponds to the parabolic subgroup P_J.

10. $x = Z - Z'$ corresponding to the partition $2 = (1+1) + 0$.
The θ-stable parabolic subalgebra corresponding to x is:

$$\mathfrak{q}_{10} = \langle Z, Z', P_{1+}, P_{1-}\rangle + \langle N_+, X_+, P_{0-}\rangle$$

The Levi part is: $\mathfrak{l} = \langle Z, Z', P_{1+}, P_{1-}\rangle$.
$\mathfrak{l}_\circ = \mathfrak{l} \cap \mathfrak{sp}_4(\mathbb{R}) = \langle iZ, iZ', P_{1+} + P_{1-}, i(P_{1+} - P_{1-})\rangle$.
$[\mathfrak{l}_\circ, \mathfrak{l}_\circ] = \langle i(P_{1+} - P_{1-}), -(P_{1+} + P_{1-}), -2i(Z + Z')\rangle$.
So $\mathfrak{h} \cap [\mathfrak{l}_\circ, \mathfrak{l}_\circ] = \langle \mathfrak{h}_1 + \mathfrak{h}_2\rangle$.
Therefore, a highest weight λ of $\mathfrak{h}$ can be extended to get an admissible character of $\mathfrak{l}$ if and only if $\lambda(\mathfrak{h}_1 + \mathfrak{h}_2) = 0$ i.e. $\lambda(\mathfrak{h}_1) = -\lambda(\mathfrak{h}_2)$. Note that such an integral weight is conjugate to an integral weight of the form (λ, λ). This θ- stable parabolic subalgebra corresponds to the parabolic subgroup P_S.

Thus this gives us a list of all the unitary, irreducible and cohomological representations of $G = \mathrm{Sp}(4,\mathbb{R})$, in terms of the θ-stable parabolic subalgebras and the possible admissible representations in each case.

We summarize the θ-stable parabolic subalgebras and the relevant data as below:

Parabolic subagebras	Corresponding Parabolic subgroups	Possible highest weight λ
$\mathfrak{q}_1$	G	$\lambda = 0$
$\mathfrak{q}_2 \sim \mathfrak{q}_7, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5 \sim \mathfrak{q}_6$	B	Any λ
$\mathfrak{q}_8$	P_J	$\lambda = (\lambda, 0)$
$\mathfrak{q}_9$	P_J	$\lambda = (\lambda, 0)$
$\mathfrak{q}_{10}$	P_S	$\lambda = (\lambda, \lambda)$

5.4 Parabolic subgroups of $\mathrm{SO}(5,\mathbb{C})$

For $G = \mathrm{Sp}(4,\mathbb{R})$, we know that ${}^LG^\circ = SO(5,\mathbb{C})$. Recall that, for a given representation π of G the Langlands parameter is a map $\phi(\pi) : W_\mathbb{R} \to {}^LG$ and the image of $W_\mathbb{R}$ under ϕ is contained in a parabolic subgroup of ${}^LG^\circ$. Hence, we write down explicitly the parabolic subgroups of $SO(5)$. For $\mathrm{SO}(5,\mathbb{C})$, the voice of the bilinear form is $J = \text{anti-diag}(1,-1,1,-1,1)$. Then the maximal torus for $\mathrm{SO}(5,\mathbb{C})$ contains elements of the form $\mathrm{diag}(a,b,1,b^{-1},a^{-1})$. For $\mathrm{SO}(5,\mathbb{C})$, we have 3 proper parabolics which are enumerated below:

1. **Minimal parabolic:** $B = M_0A_0N_0$, which corresponds to the empty subset of the base, where $M_0 = \{I_5\}$, A_0 is the subset of the diagonal matrices of the form $A_0 = \{\mathrm{diag}(a,b,1,b^{-1},a^{-1}) : a,b \in \mathbb{C}^\times\}$, and
$$N_B = \{\begin{pmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\} \subset SO(5).$$

2. **Siegel parabolic:** $P_S = M_S A_S N_S$ corresponding to the subset $\Sigma = \{e_1 - e_2\}$ of the base, where

$$M_S = \{\begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A \end{pmatrix} : A \in SL(2, \mathbb{C})\},$$

$A_S = \{\mathrm{diag}(a, a, 1, a^{-1}, a^{-1}) : a \in \mathbb{C}^\times\}$, and

$$N_S = \{\begin{pmatrix} 1 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\} \subset SO(5).$$

3. **Jacobi Parabolic:** $P_J = M_J A_J N_J$ corresponding to the subset $\Sigma = \{e_2\}$ of the base, where

$$M_J = \{\begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix} : A \in \mathrm{SO}(3)\} \in SO(5),$$

$A_J = \{\mathrm{diag}(a, 1, 1, 1, a^{-1}) : a \in \mathbb{C}^\times\}$, and

$$N_J = \{\begin{pmatrix} 1 & * & * & * & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\} \subset SO(5).$$

We can now compute the Langlands parameters for $A_{\mathfrak{q}}(\lambda)$'s and compute their transfers to representations of $\mathrm{GL}(5, \mathbb{R})$.

5.5 Cohomological representations with trivial coefficients

In this section, assume that $\lambda = 0$. So we will be looking at A_q, where $q = \mathfrak{l} + \mathfrak{u}$ is a θ-stable parabolic subalgebra, such that $\lambda = 0$ can be extended to an admissible character of $\mathfrak{l}$. Let $\mathfrak{Q}(\lambda)$ be the set of all non-equivalent $\mathfrak{q}$'s such that λ can be extended to an admissible character of $\mathfrak{q}$. When $\lambda = 0$, then $\mathfrak{Q}(\lambda)$ consists of all the 8 nonequivalent θ-stable parabolic subalgebras listed in 5.3.3. The main result of this chapter is Theorem 5.5.2.

5.5.1 Trivial and the Discrete Series representations

For $q_1 = \mathfrak{sp}_4(\mathbb{R})$, the representation A_q is the trivial representation of $\mathrm{Sp}(4,\mathbb{R})$ which is transferred to the trivial representation of $\mathrm{GL}(5,\mathbb{R})$.

From Remark 5.1, we note that A_q is the discrete series representations if q is one of the following:

- $q_2 = \langle Z, Z' \rangle \oplus \langle N_+, X_-, P_{1-}, P_{0-} \rangle$,
- $q_3 = \langle Z, Z' \rangle \oplus \langle N_+, X_+, P_{1+}, P_{0-} \rangle$,
- $q_4 = \langle Z, Z' \rangle \oplus \langle N_-, X_-, P_{1+}, P_{0+} \rangle$,
- $q_5 = \langle Z, Z' \rangle \oplus \langle N_+, X_+, P_{1+}, P_{0+} \rangle$.

The transfer of these representations has been dealt with in Theorem 4.2.5 and we know that the transfer of these representations is cohomological with respect to the trivial representation of $\mathrm{GL}(5,\mathbb{R})$. Note that, the θ stable parabolic subalgebras $\mathfrak{q}_6$ and $\mathfrak{q}_7$, give different realizations of $A_{\mathfrak{q}_5}$ and $A_{\mathfrak{q}_2}$ respectively since the corresponding θ stable parabolic subalgebras are equivalent.

This leaves us with 3 θ-stable parabolic subalgebras and their corresponding cohomological representations. The remaining parabolic subalgebras are:

- $q_8 = \langle Z, Z', P_{0+}, P_{0-}\rangle \oplus \langle N_+, X_+, P_{1+}\rangle$
- $q_9 = \langle Z, Z', X_+, X_-\rangle \oplus \langle N_+, P_{1-}, P_{0-}\rangle$
- $q_{10} = \langle Z, Z', P_{1+}, P_{1-}\rangle \oplus \langle N_+, X_+, P_{0-}\rangle$

Before we compute the transfers of the 3 remaining representations, we recall a result of Speh which classifies unitary irreducible cohomological representations of $\mathrm{GL}(n, \mathbb{R})$.

5.5.2 Speh's Classification

We describe the cohomological representations of $\mathrm{GL}(n, \mathbb{R})$ for any n as given by B. Speh [25]. We will need some special representations of $\mathrm{GL}(2n, \mathbb{R})$ which are not induced from any parabolic subgroup and which are cohomological.

Let $G = \mathrm{GL}(2n, \mathbb{R}), n > 1$. Let $C_n = TA$ be the Cartan subgroup containing matrices of the form:

$$\begin{pmatrix} cos\phi_1 & sin\phi_1 & & & \\ -sin\phi_1 & cos\phi_1 & & & \\ & & \ddots & & \\ & & & cos\phi_n & sin\phi_n \\ & & & -sin\phi_n & cos\phi_n \end{pmatrix} \begin{pmatrix} a_1 & & & & \\ & a_1 & & & \\ & & \ddots & & \\ & & & a_n & \\ & & & & a_n \end{pmatrix}.$$

Then the roots system $\Phi(\mathfrak{c}, \mathfrak{g})$ is of type A_{n-1} with each root occurring 4 times. Let Φ^+ be the set of positive roots and set $\rho_n = \sum\limits_{\alpha \in \Phi^+} 2\alpha$.
Let $P = M_n A_n N$ be the parabolic subgroup where N is determined by the set of positive roots Φ^+. Then the connected component M_n° of M_n is isomorphic to n copies of $SL(2, \mathbb{R})$ and T_n is isomorphic to a product of n copies of $O(2)$.

Let $\chi(k); k > 0$ be the quasi-character of C_n such that the restriction of χ, to each $SO(2)$ component, is $e^{2\pi ik}$ and the restriction to A_n is $exp(\frac{1}{2}\rho_n)$. Define

$$I(k) = J(\chi(k)),$$

where $J(\chi(k))$ is the Langlands quotient of the induced representation $\mathrm{Ind}_P^{\mathrm{GL}(2n)}(\pi(k) \otimes \chi(k))$ and $\pi(k) = D_k \otimes D_k \otimes \cdots \otimes D_k$ is a representation of $M = \mathrm{SL}_{\pm}(2, \mathbb{R})^n$. If $G = \mathrm{GL}(2, \mathbb{R})$, then put $I(k)$ to be the discrete series representation D_k of $\mathrm{GL}(2, \mathbb{R})$.

Having defined these representations, we now state the following result which will classify all the cohomological representations of $\mathrm{GL}(n, \mathbb{R})$.

Let $(n_0, n_1, \ldots, n_r)$ be a partition of n with $n_0 \geq 0$ and $n_i = 2m_i$ for all $1 \leq i \leq r$ and all the n_i are positive. Let $P = MAN$ be the parabolic corresponding to the partition $(n_0, \ldots, n_r)$. Then we know that

$$M = \prod_{i=0}^{r} \mathrm{SL}_{\pm}(n_i, \mathbb{R})$$

. Let $k_i = n - \sum\limits_{j=i+1}^{r} n_j - m_i$. Then, define $I(k_i)$ on $\mathrm{SL}_{\pm}(n_i, \mathbb{R})$. Define the induced representation

$$\mathrm{Ind}_P^G(\otimes_{i=1}^r I(k_i) \otimes \chi_{n_0} \otimes \chi_0),$$

where χ_{n_0} and χ_0 are trivial representations of $\mathrm{SL}_{\pm}(n_0, \mathbb{R})$ and AN respectively. Then we have,

Theorem 5.5.1. *(see [25]) The induced representation*

$$\mathrm{Ind}_P^G(\otimes_{i=1}^r I(k_i) \otimes \chi_{n_0} \otimes \chi_0)$$

is irreducible and classifies all the unitary, irreducible representations of $\mathrm{GL}(n, \mathbb{R})$ *which have cohomology with trivial coefficients.*

Even though this gives us a list of all the irreducible, unitary and cohomological representations of $\mathrm{GL}(n,\mathbb{R})$, we would like to state this result in a more usable way. We now will give the Langlands inducing data for these representations.

With notations as above, choose a Cartan subgroup $C_{(n-n_0)/2}$ in MA with the following properties:

- $C_{(n-n_0)/2} \cap \mathrm{SL}_{\pm}(n_l,\mathbb{R})$ is the fundamental Cartan subgroup of $\mathrm{SL}_{\pm}(n_l,\mathbb{R})$ for $l \geq 1$, and
- $C_{(n-n_0)/2} \cap \mathrm{SL}_{\pm}(n_0,\mathbb{R})$ is the split Cartan subgroup of $\mathrm{SL}_{\pm}(n_0,\mathbb{R})$.

Then we can decompose $C_{(n-n_0)/2}$ as $T_{(n-n_0)/2}A_{(n-n_0)/2}$ with the following properties:

- $T_{(n-n_0)/2} = \prod_{l=0}^{r} T_{n_l/2}$ with $T_{n_l/2} = T_{(n-n_0)/2} \cap \mathrm{SL}_{\pm}(n_l,\mathbb{R})$ for $l \geq 0$, and
- $A_{(n-n_0)/2} = A \prod_{l=0}^{r} A_{n_l}$ with $A_{n_l} = A_{(n-n_0)/2} \cap \mathrm{SL}_{\pm}(n_l,\mathbb{R})$.

Choose a cuspidal parabolic subgroup Q containing $C_{(n-n_0)/2}$ and the upper triangular matrices and write, for $0 \leq l \leq r$, $2\rho_l$ for the sum of positive roots of $(\mathfrak{sl}(n_l,\mathbb{R}),\mathfrak{a}_l)$ for the sum of positive roots determined by Q. Let $\chi(n) \in \hat{C}_{(n-n_0)/2}$ be such that the following holds:

- $\chi(n)|_A = \chi_0$,
- $\chi(n)|_{A_0} = \rho_0$
- $\chi(n)|_{T_0}$ is trivial
- $\chi(n)|_{A_l} = \frac{1}{2}\rho_l$ for $l\rangle$,
- $\chi(n)|_{T_l}$ is a product of factors $exp((n - \sum_{i=l+1} n_i - m_l)2\pi i)$.

Then, $\mathrm{Ind}_P^G(\otimes_{i=1}^{r} I(k_i) \otimes \chi_{n_0} \otimes \chi_0) \cong J(\chi(n))$ (see [25]).

We will use this result when we check whether the transferred representations are cohomological or not.

5.5.3 Case of the Jacobi θ stable subalgebra

We now deal with the 2 representations of $\mathrm{Sp}(4,\mathbb{R})$ corresponding to the θ-stable subalgebras $\mathfrak{q}_8$ and $\mathfrak{q}_9$ listed in 5.5.1.

As noted earlier q_8 corresponds to the parabolic subgroup P_J. We will now compute the Langlands data for A_{q_8}.

$\mathfrak{q}_8 = \langle Z, Z', P_{0+}, P_{0-}\rangle \oplus \langle N_+, X_+, P_{1+}\rangle$, where $\langle Z, Z', P_{0+}, P_{0-}\rangle = \mathfrak{l}$ and $\langle N_+, X_+, P_{1+}\rangle = \mathfrak{u}$. Recall that $\lambda = 0$.

We choose a maximally split Cartan subgroup H inside L. The Levi L is isomorphic to $\mathrm{GL}(1,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. The Lie algebra corresponding to $H = TA$ is $\langle Z, P_{0+} + P_{0-}\rangle$. Note that the Lie algebra of T is generated by Z and for A is $P_{0+} + P_{0-} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

Now consider,

$$Cent_G(A) = MA.$$

Then M is isomorphic to $\mathrm{SL}(2,\mathbb{R}) \times \{\pm 1\}$. To compute the Langlands parameter for the representation $A_{\mathfrak{q}_8}$, we need a parabolic subgroup $P = MAN$ of $G = \mathrm{Sp}(4,\mathbb{R})$, a discrete series representation on M and a character ν of $\mathfrak{a}$. For the parabolic, choose any parabolic subgroup of G which has Levi factor MA. The Jacobi parabolic P_J is one such subgroup. This corresponds to the subset $\Sigma = \{2e_2\}$ of the base.

Thus, we know that the representation $A_{\mathfrak{q}_8}$ is obtained as the Langlands quotient of a representation which is induced from the Jacobi parabolic P_J. The character on $\mathfrak{a}$ is obtained by restricting ρ_L to $\mathfrak{a}$. Hence

$$\nu = \rho_L|_{\mathfrak{a}} = \frac{1}{2}(0,2) = (0,1).$$

Now for the discrete series representation of M: The Harish-Chandra parameter for the representation of $M^\circ = \mathrm{SL}(2, \mathbb{R})$, the connected component of M, is given by $\rho(u) + \rho(M \cap L)$ where ρ is computed with respect to $\mathfrak{t}$. Observe that, $M \cap L = \{\pm 1\}$ which implies that $\rho(m \cap L) = 0$. We have $\mathfrak{u} = \langle N_+, X_+, P_{1+}\rangle$. Thus

$$\rho(u) = \frac{1}{2}((1 + 2 + 1), 0) = (2, 0).$$

The only question remains is whether the representation on $\{\pm 1\} \subset M$ is the trivial one or the sign character. We compute this as follows:

The Lie algebra of M is $\mathfrak{m} = \langle Z, P_{0+} + P_{0-}, X_+, X_-\rangle$. Then $M \cap K = \{\pm 1\} \times \mathrm{SO}(2)$. The discrete series representation on $M \cap K$ is the representation with highest weight given by the formula
$2\rho \wedge^{\dim \mathfrak{u}\cap\mathfrak{p}} (\mathfrak{u} \cap \mathfrak{p})|_t$ which in this case is $(2 + 1) = 3$. Thus the character on $\{\pm 1\}$ is given by $sgn : -1 \mapsto -1$. Note that this computation gives us the discrete series representation on the $\{\pm 1\}$ as well as the $\mathrm{SL}(2, \mathbb{R})$ of the Levi part.

Now we compute the Langlands parameter for $A_{\mathfrak{q}_8}$.

Note that ${}^L\mathrm{Sp}(4, \mathbb{R})^\circ$ is $\mathrm{SO}(5, \mathbb{C})$. Thus a Langlands parameter is a map from $W_\mathbb{R}$, the Weil group of $\mathbb{R}$, to $\mathrm{SO}(5, \mathbb{C})$ which we will then compose with the inclusion into $\mathrm{GL}(5, \mathbb{C})$ to get a representation of $\mathrm{GL}(5, \mathbb{R})$. Since the representation $A_{\mathfrak{q}_8}$ is induced from the parabolic P_J, the image of $W_\mathbb{R}$ should lie inside the corresponding parabolic subgroup of $P_J \subseteq \mathrm{SO}(5, \mathbb{C})$.

The transfer of $A_{\mathfrak{q}_8}$ to $\mathrm{GL}(5, \mathbb{R})$ is the Langlands quotient of the following induced representation:

$$\mathrm{Ind}_P^G(D_4 \otimes \chi_1\epsilon \otimes \chi_{-1}\epsilon \otimes \epsilon)$$

where P is the $(2, 1, 1, 1)$ parabolic subgroup of $\mathrm{GL}(5, \mathbb{R})$ and $\chi_n(x) = x^n$, since the Langlands parameter for $A_{\mathfrak{q}_8}$ is given by

$$z \mapsto \begin{pmatrix} (z\bar{z}) & 0 & 0 & 0 & 0 \\ 0 & (\frac{z}{\bar{z}})^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & (z\bar{z})^{-1} & 0 \\ 0 & 0 & 0 & 0 & (\frac{z}{\bar{z}})^{-2} \end{pmatrix}; \quad j \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Observe that the Langlands quotient of $\mathrm{Ind}_P^G(D_4 \otimes \chi_1\epsilon \otimes \chi_{-1}\epsilon \otimes \epsilon)$ is isomorphic to $\mathrm{Ind}_P^G(D_4 \otimes \epsilon)$ where P is the $(2,3)$ parabolic of $\mathrm{GL}(5,\mathbb{R})$, and ϵ is the sign representation of $\mathrm{GL}(3,\mathbb{R})$. This follows from the fact that for the Borel B of $\mathrm{GL}(3,\mathbb{R})$, the Langlands quotient of $\mathrm{Ind}_B^{\mathrm{GL}(3,\mathbb{R})}(|\cdot|\otimes\epsilon\otimes|\cdot|^{-1})$, is ϵ.

Since the transferred representation is induced from the $(2,1,1,1)$ parabolic and we only have one factor of $\mathrm{GL}(2,\mathbb{R})$ in the inducing data, we consider the representation corresponding to the partition $5 = 3 + 2$ of $\mathrm{GL}(5,\mathbb{R})$ in terms of Speh's classification [25]. For the partition $n = 5 = 3+2$, we have $n_0 = 3, n_1 = 2, m_1 = 1$. The representation which is cohomological corresponding to this partition is obtained as a Langlands quotient of the $(2,1,1,1)$ parabolic. The discrete series representation on the $\mathrm{GL}(2,\mathbb{R})$ part of the Levi is given by $exp(n - \sum_{i=2} n_i - m_1)$, which is $n = 4$ since for $i > 1$, $n_i, m_i = 0$. Thus we observe that the representation which occurs in Speh's classification is $\mathrm{Ind}_P^G(D_4 \otimes 1)$. Thus, the transferred representation obtained from $A_{\mathfrak{q}_8}$ is an ϵ twist of$\mathrm{Ind}_P^G(D_4 \otimes 1)$. Hence, the transfer is unitary and we can appeal to Speh and observe that the transferred representation does not occur in the classification of Speh. Hence the transfer of $A_{\mathfrak{q}_8}$ is not a cohomological representation of $\mathrm{GL}(5,\mathbb{R})$.

A similar computation for the parabolic subalgebra $\mathfrak{q}_9$ shows that the representations $A_{\mathfrak{q}_8}$ and $A_{\mathfrak{q}_9}$ transfer to the same representation of $\mathrm{GL}(5,\mathbb{R})$.

Thus, the transfer of $A_{\mathfrak{q}_8}$ and $A_{\mathfrak{q}_9}$ to representations of $\mathrm{GL}(5,\mathbb{R})$ are not cohomological.

5.5.4 Case of Siegel θ-stable parabolic subalgebra

The last case left is the case when the θ-stable parabolic subalgebra is

$$\mathfrak{q}_{10} = \langle Z, Z', P_{1+}, P_{1-}\rangle \oplus \langle N_+, X_+, P_{0-}\rangle,$$

where $\langle Z, Z', P_{1+}, P_{1-}\rangle = \mathfrak{l}$ and $\langle N_+, X_+, P_{0-}\rangle = \mathfrak{u}$. Let $\lambda = 0$.

We choose a maximally split Cartan subgroup H inside L. The L is isomorphic to $\mathrm{GL}(1,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$. The Lie algebra corresponding to $H = TA$ is $\langle Z - Z', P_{1+} + P_{1-}\rangle$. Note that the Lie algebra of T is generated by $Z - Z'$ and for A is $P_{1+} + P_{1-} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$.

Now consider,

$$Cent_G(A) = MA.$$

Then M is isomorphic to $\mathrm{SL}(2,\mathbb{R})\times\{\pm 1\}$. To compute the Langlands parameter for the representation $A_{\mathfrak{q}_{10}}$, we need a parabolic subgroup $P = MAN$ of $G = \mathrm{Sp}(4,\mathbb{R})$, a discrete series representation on M and a character ν of $\mathfrak{a}$. For the parabolic, choose any parabolic subgroup of G which has Levi factor MA. Siegel parabolic is such a parabolic. This parabolic subgroup corresponds to the subset $\Sigma = \{e_1 - e_2\}$ of the base.

Thus, the representation $A_{\mathfrak{q}_{10}}$ is obtained as the Langlands quotient of a representation which is induced from the Siegel parabolic. Now we compute the other two parameters. The character on $\mathfrak{a}$ is obtained by restricting ρ_L

to $\mathfrak{a}$. Thus

$$\nu = \rho_L|_{\mathfrak{a}} = \frac{1}{2}(2,2) = (1,1).$$

Now for the discrete series representation of M: The Harish-Chandra parameter for the representation of $M^\circ = \mathrm{SL}(2,\mathbb{R})$, the connected component of M, is given by $\rho(u) + \rho(M \cap L)$ where ρ is computed with respect to $\mathfrak{t}$. Observe that, $M \cap L = \{\pm 1\}$ which implies that $\rho(m \cap L) = 0$. We have $\mathfrak{u} = \langle N_+, X_+, P_{0-}\rangle$. Thus

$$\rho(u) = \frac{1}{2}(2+2+2, 2+2+2) = (3,3).$$

The only question remains is whether the representation on $\{\pm 1\} \subset M$ is the trivial one or the sign character. We compute this as follows:

Note that $M \cap K = \{\pm 1\} \times \mathrm{SO}(2)$. The discrete series representation on $M \cap K$ is the representation with highest weight given by the formula $2\rho \wedge^{\dim \mathfrak{u}\cap\mathfrak{p}} (\mathfrak{u} \cap \mathfrak{p})|_t$ which in this case is $(2+2) = 4$. Thus the character on $\{\pm 1\}$ is the trivial character. Now we compute the Langlands parameter for $A_{\mathfrak{q}_{10}} \cong \mathrm{Ind}_{P_S}^G(D_3|det|^{\frac{1}{2}})$.

Since ${}^L\mathrm{Sp}(4,\mathbb{R})^\circ$ is $\mathrm{SO}(5,\mathbb{C})$, a Langlands parameter is a map from $W_{\mathbb{R}}$, the Weil group of $\mathbb{R}$, to $\mathrm{SO}(5,\mathbb{C})$ which we will then compose with the inclusion into $\mathrm{GL}(5,\mathbb{C})$ to get a representation of $\mathrm{GL}(5,\mathbb{R})$. Since the representation $A_{\mathfrak{q}_{10}}$ is induced from the parabolic P_S, the image of $W_{\mathbb{R}}$ should go inside P_S which is a parabolic subgroup of $\mathrm{SO}(5,\mathbb{C})$, corresponding to the subset $\Sigma = \{e_2\}$ of the base. The Langlands parameter for $A_{\mathfrak{q}_{10}}$ is given by

$$z \mapsto \mathrm{diag}(z^2\bar{z}^{-1}, z^{-1}\bar{z}^2, 1, z\bar{z}^{-2}, z^{-2}\bar{z});$$

and

$$j \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus the transfer of $A_{\mathfrak{q}_{10}}$ to $\mathrm{GL}(5,\mathbb{R})$ is the Langlands quotient of the following induced representation:

$$\mathrm{Ind}_P^G(D_3|det|^{\frac{1}{2}} \otimes 1 \otimes D_3|det|^{-\frac{1}{2}})$$

where P is the $(2,1,2)$ parabolic subgroup of $\mathrm{GL}(5,\mathbb{R})$. We need to analyze whether this representation occurs in the Speh's classification of unitary irreducible cohomological representations of $\mathrm{GL}(5,\mathbb{R})$.

We consider the partition $5 = 1 + 4$. Using notations from 5.5.2, we have $n_0 = 1, n_1 = 4$ and $m_1 = 2$. The representation occurring in Speh's classification corresponding to this partition is $\mathrm{Ind}_{(1,4)}^{\mathrm{GL}(5,\mathbb{R})}(1 \otimes I(3))$.

Now we must compute the corresponding Langlands data for this representation. Appealing to 5.5.2, we note that for the character $\chi(3)$ on T_2^0 given by $\chi(3)(e^{i\theta_1}, e^{i\theta_2}) = e^{3i(\theta_1+\theta_2)}$ and $\chi(3)|_{A^2} = exp(\frac{\rho_2}{2}) = \frac{a_1}{a_2}$,

$$I(3) = J(\chi(3)).$$

But as a representation of $\mathrm{GL}(4,\mathbb{R})$, $J(\chi(3)) = \mathrm{Ind}(D_3|det|^{\frac{1}{2}} \otimes D_3|det|^{-\frac{1}{2}})$. Thus we note that $\mathrm{Ind}_{(1,4)}^{\mathrm{GL}(5,\mathbb{R})}(1 \otimes I(3)) = \mathrm{Ind}_P^G(D_3|det|^{\frac{1}{2}} \otimes 1 \otimes D_3|det|^{-\frac{1}{2}})$. Hence, the transfer of $A_{\mathfrak{q}_{10}}$ occurs in the classification of Speh and is hence cohomological.

5.5.5 Summary

Thus, to summarize we have:

Theorem 5.5.2. *Let π be an irreducible unitary representation of* $\mathrm{Sp}(4,\mathbb{R})$ *such that π has non-vanishing cohomology with trivial coefficients. Let $\iota(\pi)$ denote the transferred representation of π to* $\mathrm{GL}(5,\mathbb{R})$*. Then $\iota(\pi)$ is cohomological with trivial coefficients if π is one of the following:*

1. *π is the trivial representation,*
2. *π is a discrete series representation of* $\mathrm{Sp}(4,\mathbb{R})$*,*
3. *π is induced from the Siegel parabolic.*

5.6 Cohomological representations with Non-Trivial coefficients

In this section, we will let $\lambda = (\lambda_1, \lambda_2),\ \lambda_1 \geq \lambda_2 \geq 0$ be a non-zero highest weight of $\mathrm{Sp}(4,\mathbb{R})$. We will consider the following sub cases:

- $\lambda_1 = \lambda_2 \neq 0$,
- $\lambda_1 > \lambda_2 \neq 0$,
- $\lambda_2 = 0$.

5.6.1 $\lambda = (\lambda, \lambda)$, with $\lambda \neq 0$

In this case, we note that the θ-stable parabolic subalgebras which are relevant are $\mathfrak{q}_2 \sim \mathfrak{q}_7, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5 \sim \mathfrak{q}_6$ and $\mathfrak{q}_{10}$. Note that $A_{\mathfrak{q}_i}$ for $2 \leq i \leq 7$ are discrete series representations. Thus, from 4.2.5 we know that these transfer to cohomological representations of $\mathrm{GL}(5,\mathbb{R})$.

The other representations left are the representations arriving from the parabolic $\mathfrak{q}_{10}$. As we have already seen before these representations are obtained as the Langlands quotient of a representation which is induced from the Siegel parabolic of $G = \mathrm{Sp}(4, \mathbb{R})$. We compute the Langlands parameters for the representations $A_{q_{10}}(\lambda)$, as before. We note that the discrete series representation on M_S is given by $2\lambda+3$. The character on $\mathfrak{a}$ does not change and is still given by

$$\nu = \rho_L|_{\mathfrak{a}} = \frac{1}{2}(2,2) = (1,1).$$

Thus, the representation $A_{\mathfrak{q}_{10}}(\lambda)$ is the irreducible Langlands quotient of the induced representation $\mathrm{Ind}_{P_S}^{G}(D_{2\lambda+3}|det|^{\frac{1}{2}})$. We note that the Langlands parameter of $A_{\mathfrak{q}_{10}(\lambda)}$ is given by

$$z \mapsto \mathrm{diag}(z^{\lambda+2}\bar{z}^{-\lambda-1}, z^{-\lambda-1}\bar{z}^{\lambda+2}, 1, z^{\lambda+1}\bar{z}^{-\lambda-2}, z^{-\lambda-2}\bar{z}^{\lambda+1})$$

and

$$j \mapsto \begin{pmatrix} 0 & (-1)^{2\lambda+3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & (-1)^{2\lambda+3} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus, the transfer of $A_{\mathfrak{q}_{10}(\lambda)}$ to $\mathrm{GL}(5, \mathbb{R})$ is obtained as a Langlands quotient of

$$\mathrm{Ind}_{(2,1,2)}^{\mathrm{GL}(5,\mathbb{R})}(D_{2\lambda+3}|det|^{\frac{1}{2}} \otimes 1 \otimes D_{2\lambda+3}|det|^{-\frac{1}{2}}).$$

The question whether this representation of $\mathrm{GL}(5, \mathbb{R})$ is cohomological or not does not seem to have an easy answer since the main ingredient, which is the Speh's classification for cohomological representations of $\mathrm{GL}(n, \mathbb{R})$ with non-trivial coefficients is not available. The expectation is that this representation is cohomological.

5.6.2 $\lambda = (\lambda_1, \lambda_2)$, with $\lambda_2 \neq 0$ and $\lambda_1 > \lambda_2$

In this case, we note that the θ-stable parabolic subalgebras which are relevant are $\mathfrak{q}_2 \sim \mathfrak{q}_7, \mathfrak{q}_3, \mathfrak{q}_4$ and $\mathfrak{q}_5 \sim \mathfrak{q}_6$. For these subalgebras the Levi parts, $\mathfrak{l}$, are contained in $\mathfrak{k}$ and hence the representations A_q are the discrete series representations. From 4.2.5, we know that the transfer of these representations are cohomological.

Thus we have:

Proposition 5.6.1. *Let $\lambda = (\lambda_1, \lambda_2)$, $\lambda_2 \neq 0$. Then the transfer of $A_{\mathfrak{q}}(\lambda)$ to* $\mathrm{GL}(5, \mathbb{R})$ *is cohomological.*

5.6.3 $\lambda = (\lambda, 0)$

The θ-stable parabolic subalgebras which are relevant are $\mathfrak{q}_2 \sim \mathfrak{q}_7, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5 \sim \mathfrak{q}_6, \mathfrak{q}_8$ and $\mathfrak{q}_9$. Out of these 6 θ-stable parabolic subalgebras, $\mathfrak{q}_2 \sim \mathfrak{q}_7, \mathfrak{q}_3, \mathfrak{q}_4$ and $\mathfrak{q}_5 \sim \mathfrak{q}_6$ correspond to the discrete series representations as before and we know that these transfer to cohomological representations of $\mathrm{GL}(5, \mathbb{R})$ from 4.2.5.

This leaves us with the representations $A_{\mathfrak{q}_8}(\lambda)$, $A_{\mathfrak{q}_9}(\lambda)$. Note that for the θ-stable parabolic subalgebra $\mathfrak{q}_8$, $\lambda|_{\mathfrak{t}} = \lambda$ and $\lambda|_{\mathfrak{a}} = 0$. These observations along with the calculations in section 5.5.3 imply that the Langlands parameter for the representation $A_{\mathfrak{q}_8}(\lambda)$ is given by:

$$z \mapsto \mathrm{diag}(z\bar{z}, (\frac{z}{\bar{z}})^{\frac{\lambda+2}{2}}, 1, (z\bar{z})^{-1}, (\frac{z}{\bar{z}})^{-\frac{\lambda+2}{2}})$$

$$j \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (-1)^{\lambda+2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, we observe that the transfer of $A_{\mathfrak{q}_8}(\lambda)$ is obtained by taking the Langlands quotient of:

$$\mathrm{Ind}_P^G(D_{\lambda+2} \otimes \chi_1 \epsilon \otimes \chi_{-1}\epsilon \otimes \epsilon),$$

where P is the $(2,1,1,1)$-parabolic subgroup of $\mathrm{GL}(5,\mathbb{R})$, $\chi_n(x) = x^n$ and ϵ is the sign character on $\mathbb{R}^{\times}$.

A similar calculation as above shows that the transfer of $A_{\mathfrak{q}_9}(\lambda)$ is also the Langlands quotient of

$$\mathrm{Ind}_P^G(D_{\lambda+2} \otimes \chi_1 \epsilon \otimes \chi_{-1}\epsilon \otimes \epsilon),$$

where P is as above. The question whether this representation of $\mathrm{GL}(5,\mathbb{R})$ is cohomological or not does not seem to have an easy answer since the main ingredient, which is the Speh's classification for cohomological representations of $\mathrm{GL}(n,\mathbb{R})$ with non-trivial coefficients is not available at the moment. The expectation is that this representation is not cohomological. We now summarize these observations.

5.6.4 Summary

Finally, to summarize the results we put everything in a tabular form. The table completely answers which unitary, irreducible cohomological representations of $G = \mathrm{Sp}(4,\mathbb{R})$ are transferred to cohomological representations of $\mathrm{GL}(5,\mathbb{R})$ in the $\lambda = 0$ case. In the non-trivial coefficients case, it seems like a difficult question at the moment since no analogous result of Speh's classification for cohomological representations with non-trivial coefficients seem to exist. One hopes to prove this and obtain a complete result for the case $G = \mathrm{Sp}(4,\mathbb{R})$.

Representation	Corresponding Parabolic	λ	Transfer cohomological or not
$A_{\mathfrak{q}_1}$	B	$\lambda = 0$	Cohomological
$A_{\mathfrak{q}_2} \cong A_{\mathfrak{q}_7}, A_{\mathfrak{q}_3}$ $A_{\mathfrak{q}_4}, A_{\mathfrak{q}_5} \cong A_{\mathfrak{q}_6}$	B	Any λ	Cohomological
$A_{\mathfrak{q}_8}$	P_J	$\lambda = 0$ $\lambda = (\lambda, 0)$	Not Cohomological Expected to be not cohomological
$A_{\mathfrak{q}_9}$	P_J	$\lambda = 0$ $\lambda = (\lambda, 0)$	Not Cohomological Expected to be not cohomological
$A_{\mathfrak{q}_{10}}$	P_S	$\lambda = 0$ $\lambda = (\lambda, \lambda)$	Cohomological Expected to be cohomological

Considering the observations made above, we make the following conjecture:

Conjecture 1. *Let π be an irreducible unitary representation of $\mathrm{Sp}(4,\mathbb{R})$ such that π has non-vanishing cohomology. Let $\iota(\pi)$ denote the transferred representation of π to $\mathrm{GL}(5,\mathbb{R})$. Then $\iota(\pi)$ is cohomological if π is one of the following:*

1. *π is the trivial representation,*
2. *π is a discrete series representation of $\mathrm{Sp}(4,\mathbb{R})$,*
3. *π is induced from the Siegel parabolic.*

Further, if π is cohomological with respect to the finite dimensional representation M_λ, then $\iota(\pi)$ is cohomological with respect to $\iota(M_\lambda)$.

It will be interesting to work out other examples, to try and find a general guiding principle for the transfer of cohomological representations from a classical group to an appropriate $\mathrm{GL}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$.

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