Coset CFT and Bosonic String Propagation in de Sitter Space-time

MS-thesis

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Certificate

This is to certify that this dissertation entitled “Coset CFT and Bosonic String Propagation in de Sitter space-time” towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Vivek Vishwakarma at International Centre for Theoretical Sciences, Bengaluru under the supervision of Prof. Spenta R Wadia, Professor Emeritus, Physics Department” during the academic year 2018-19.

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Declaration

I hereby declare that the matter embodied in the report entitled "Coset CFT and Bosonic String Propagation in de Sitter Spacetime" are the results of the work carried out by me at the Department of Physics, International Centre for Theoretical Physics, Bengaluru, under the supervision of Prof. Spenta R. Wadia and the same has not been submitted elsewhere for any other degree.

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Abstract

In this report, I will discuss the studies on various topics that has been made towards constructing boson string theory in de Sitter background. We study the basic notions of the de Sitter space-time and metric. We will also look at the Penrose diagrams and compare different coordinate systems. For constructing a string theory, one requires a conformal field theory because of the vanishing beta function. We will go over the basics of the conformal group and conformal field theory in d-dimensions. Similar analysis will be carried out for CFT in 2-dimensions and it leads to the Virasoro algebra. We will also look at the WZW model with Lie algebraic symmetry for which the affine Lie algebra arises as a spectrum generating algebra. We will also work out the Virasoro algebra for the model. Hilbert space of the theory contains infinitely many representation of the Virasoro algebra. But for the theory describing a physical system, energy eigenvalues has to be positive. Since the generators of the Virasoro algebra are energy eigenstates. So, the representation has to be unitary and conditions for unitary representation is reviewed here.

To progress further, we look at the SL(2,R) WZW model briefly. Since 3-dimensional de Sitter space-time is not isomorphic to any non-compact group manifold, machinery of the non-compact WZW model can be applied fully. However, dS_d is isomorphic to a coset and will be proved here. We will also look at the Virasoro algebra for the coset space model.

Introduction

Phenomena due to the effects of the gravity at the quantum level are very few. The path taken for searching such phenomena goes towards the high energy scales which include experiments done at LHC and also towards the cosmological scales which includes the cosmic microwave background (CMB) radiation and the black holes. Experiments at high energies so far has not provided any hints for any such phenomenon. However, observations at the cosmological scales looks promising especially CMB.

CMB is the heat map of the early universe which is the result of the quantum mechanical fluctuation at the birth of the universe. Assuming the big bang model for the evolution of the universe. Early universe is a good platform for the observation of phenomena involving quantum gravity. At the birth of the universe, it was the soup of matter and energy at very high temperature which cooled down as it expanded. From observations, it is evident that our universe has undergone two periods of exponential expansion with a period of reheating in between. The first period of expansion occur at the birth of the universe. At that time, universe has a large cosmological constant and a small radius of curvature. At the exponential expansion phase, the space-time metric is approximated by de Sitter space-time. de Sitter space-time is the maximally symmetric solution of the vacuum Einstein equations with positive cosmological constant. So, to get the theory of quantum gravity consistent with the observations, one needs to incorporate de Sitter background into the formulation.

Most promising formulation for the theory of quantum gravity is the string theory. String theory is well defined for the asymptotically flat[1,2] and anti de Sitter space-time[3,4] which have non-positive cosmological constant. String theory in curved background with positive cosmological constant suffers with some issues due to the structure of the space-time[5] and has not yet formulated consistently. In this report, we would like to summarize the studies made towards constructing the bosonic string theory in the de Sitter background.

The organization of the report is as follows. In the first chapter, we are going to review the basic notions of de Sitter space-time and its metric. We will define the de Sitter space-time in the embedding picture. We will look at the global coordinates, static coordinates and flat slicing coordinates and their corresponding metric and their conformal diagrams. We will try to deduce the isometry group from the embedding picture.

In chapter 2, we will review conformal field theory in d-dimensions. We will look at the conformal group in d-dimensions and derive the generators and their commutation relations. We will also derive the correlation function for the d-dimensional conformal field theory. We will also consider 2-dimensional free bosonic CFT and derive the operator product expansion of the fields and stress tensors and consequently the Virasoro algebra for the CFT.

In chapter 3, we will review the SO(3) Wess-Zumino-Witten model and the Sugawara construction for the current algebra. We will start with the SO(3) non-linear sigma model and will add the Wess-Zumino term to make the conformally invariant theory. We will derive the currents and the corresponding current algebra. We will also review the Sugawara construction and derive the Virasoro algebra for the model. We will also look at the constraints required for the unitary representations for the Virasoro and Kac-Moody algebra. We will also look at some aspects of non-compact WZW model.

In chapter 4, we will also look at some aspects of SL(2,R) WZW model. Since, de Sitter space-time is not isomorphic to any non-compact group for three dimensions. We will prove that de Sitter space-time is isomorphic to a coset of SO(d,1). We will also look at the Virasoro representation of the coset space model.

Chapter 1

de Sitter Space-time

In this chapter we will review the notions of d-dimensional de Sitter spacetime[6,7] and metric associated with it. We will look at metrics in different coordinate systems and its transformations laws. The main motivation for this is to introduce horizon in the metric. We will also compare different metrics using conformal diagrams and guess out the isometry group of de Sitter spacetime from the embedding picture. We will work with metric sign (-1, 1, …, 1)

1. de Sitter spacetime as embedding

Embedding picture of d-dimensional de Sitter spacetime is intuitively more clearer. It is defined as the d-dimensional timelike hyperboloid embedded in (d+1)-dimensional Minkowski spacetime, $R^{1,d}$. Let $(X_0, X_1, \dots, X_{d-1}, X_d)$ be the coordinates. Then,

$$-X_0^2 + X_1^2 + \cdots + X_{d-1}^2 + X_d^2 = l^2 \tag{1.1}$$

Where l is the radius. The metric corresponding to it is

$$ds^2 = -dX_0^2 + dX_1^2 + \cdots + dX_d^2 \tag{1.2}$$

2. Global coordinates

It is more convenient to express the hyperboloid in global coordinates. The transformation law is given by

$$X_0 = l sinh\left(\frac{t}{l}\right) \quad , \qquad X_i = l cosh\left(\frac{t}{l}\right) z_i \tag{2.1}$$

Where $i = 1, \dots, d$, z_i is constrained on the unit sphere, $z_i^2 = 1$, and $-\infty < t < \infty$. The metric can be obtained by substituting the variation of coordinates in above metric equation, giving

$$ds^2 = -dt^2 + l^2 cosh^2(t/l) d\Omega_{d-1}^2 \tag{2.2}$$

These coordinates covers the entire hyperboloid and hence called global coordinates. We can foliate the spatial (d-1)-sphere by (d-2)-sphere. The metric will be

$$ds^2 = -dt^2 + l^2 cosh^2(t/l)(d\theta^2 + sin^2\theta d\Omega_{d-2}^2) \tag{2.3}$$

Here $0 < \theta < \pi$.

Here the spacelike slice of the metric is a compact manifold. Hence, global coordinates are also called closed slicing of de Sitter. Global de Sitter can be viewed as infinitely large (d-1)-sphere contracting to a point and then expanding again to infinitely large (d-1)-sphere. $\theta = 0$ represent the north pole and $\theta = \pi$ the south pole which are just points. Other values of θ represent whole (d-2)-sphere. To draw the Penrose diagram (or the conformal diagram) we need the coordinate system in which

1) Massless particles travel at 45^0 in (t, θ)-plane (angle between time axis and null geodesic)
2) Covers the whole spacetime
3) Timelike slice is compact

Imposing the first condition,

$$-dt^2 + l^2 cosh^2\left(\frac{t}{l}\right) d\theta^2 = \Omega^2(\sigma)(-d\sigma^2 + d\theta^2)$$

Comparing the coefficients of infinitesimal displacements one gets $\Omega^2(\sigma) = l^2 cosh^2\left(\frac{t}{l}\right)$ and $dt = \Omega(\sigma) d\sigma$. So,

$$dt = \Omega(\sigma) d\sigma \qquad \Rightarrow \qquad dt = l \cosh\left(\frac{t}{l}\right) d\sigma$$

Integrating on both sides

$$\int \frac{dt}{l \cosh(t/l)} = \int d\sigma$$

$$\tan\left(\frac{\sigma}{2}\right) = \tanh\left(\frac{t}{2l}\right)$$

Where $-\pi/2 < \sigma < \pi/2$. Then the metric becomes

$$ds^2 = \frac{l^2}{cos^2\sigma}[-d\sigma^2 + d\theta^2 + sin^2\theta d\Omega_{d-2}^2] \qquad (2.4)$$

which satisfies all three conditions enlisted.

The Penrose diagram for the space-time is the square in the (σ, θ)-plane. Both coordinates have range π. $\theta \in (0, \pi)$ and $\sigma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Fixed σ spatial slices of the Penrose diagram are S^{d-1} on the constant X_0 slices of the embedding picture. I^+ is the future conformal boundary and I^- is the past conformal boundary. Any observer will approach them in the infinite future and infinite past respectively. It takes infinite affine time to reach from I^- to I^+. The left $(\theta = \pi)$ and right $(\theta = 0)$ edges of the Penrose diagram represent points as the radius of S^{d-2} vanishes. These are the poles of S^{d-1}. Null geodesics are the diagonals of the square.

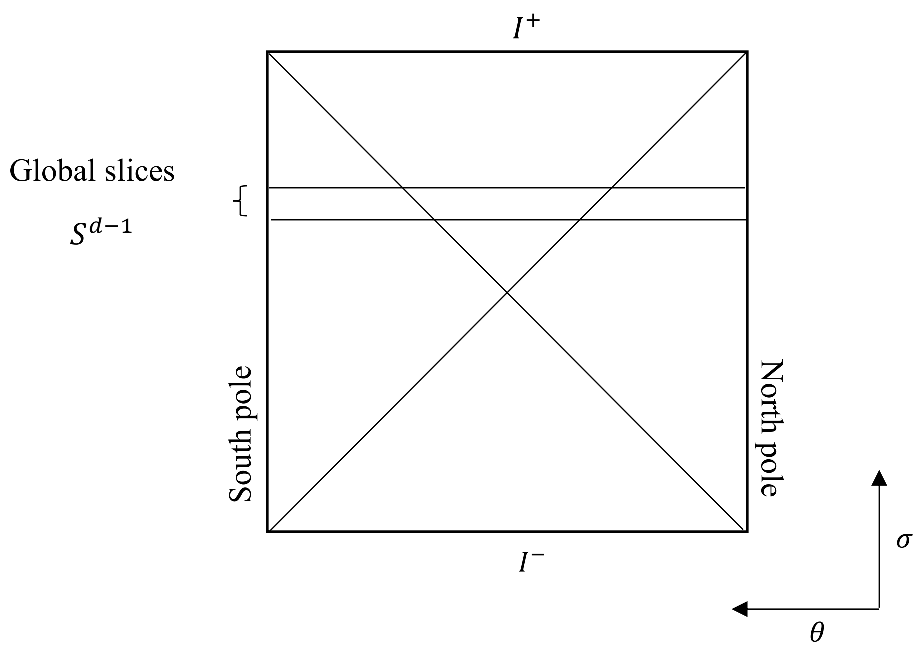


Fig.1. Penrose diagram in global coordinates

Space-time metric in the global coordinates at the far past ($t \to -\infty$) and far $t \to \infty$ future can be approximated as

$$ds^2 = -dt^2 + l^2 cosh^2(t/l) d\Omega_{d-1}^2$$

As $t \to -\infty$,

$$\cosh\left(\frac{t}{l}\right) = \left(e^{\frac{t}{l}} + e^{-\frac{t}{l}}\right)/2 \sim \frac{e^{-\frac{t}{l}}}{2}$$

Metric will be

$$ds^2 \sim -dt^2 + l^2 \frac{e^{-\frac{2t}{l}}}{4} d\Omega_{d-1}^2 \tag{2.5}$$

Similarly, as $t \to \infty$, metric will be

$$ds^2 \sim -dt^2 + l^2 \frac{e^{\frac{2t}{l}}}{4} d\Omega_{d-1}^2 \tag{2.6}$$

de Sitter metric has a compact spatial slice and non-compact time coordinate. de Sitter space do not have any spatial asymptotes. It only has temporal asymptotes at infinite past I^- and at infinite future I^+.[5]

Next two coordinate systems will introduce the notion of the horizons.

3. Flat Slicing

The transformation law is given by

$$X_0 = lsinh\left(\frac{t}{l}\right) + \frac{r^2}{2l}e^{t/l}, \quad X_1 = lcosh\left(\frac{t}{l}\right) - \frac{r^2}{2l}e^{\frac{t}{l}}, \quad X_i = e^{t/l}z_i \tag{3.1}$$

Where $i = 2, \ldots, d, -\infty < t < \infty$ and $r^2 = \sum_i z_i^2$. The corresponding metric is given by,

$$ds^2 = -dt^2 + e^{2t/l}d\vec{z}^2 \tag{3.2}$$

Here $d\vec{z}^2$ is the flat metric on R^{d-1}. Here

$$X_0 + X_1 > 0$$

This coordinate system only covers half of the spacetime. Here t is not the global time.

Comparing the global and flat slicing coordinates-

Writing the spatial flat slice of the metric in the spherical coordinates

$$ds^2 = -dt^2 + e^{\frac{2t}{l}}(dr^2 + r^2 d\Omega_{d-2}^2)$$

And the metric in the global coordinates,

$$ds^2 = -dt_g{}^2 + l^2 cosh^2(t_g/l)(d\theta^2 + sin^2\theta d\Omega_{d-2}^2)$$

t_g is the global time coordinate. This gives

$$re^{\frac{t}{l}} = lcosh\left(\frac{t_g}{l}\right)sin\theta \qquad e^{\frac{t}{l}} = \cos\theta\cosh\left(\frac{t_g}{l}\right) + \sinh\left(\frac{t_g}{l}\right) \tag{3.3}$$

The second expression is obtained by equating the sum $X_0 + X_1$ obtained by the transformation laws in both coordinates.

At late times, the time coordinate in both the coordinate system becomes same, $t_g \sim t$. The Penrose diagram for the coordinate system is draw using above relations between two coordinates system. Penrose diagram has the same labelling as one above.

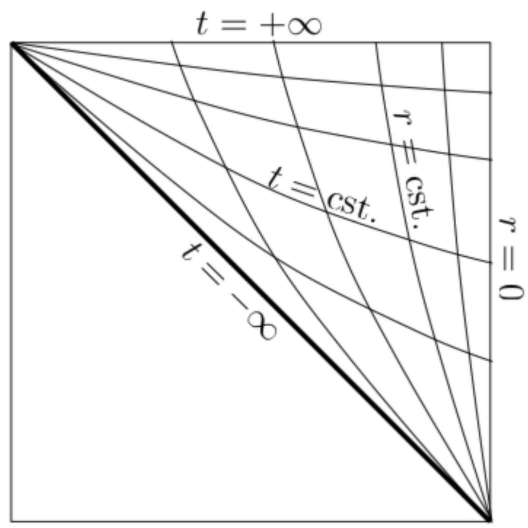


Fig. 2. Penrose diagram in Flat Slicing Coordinates. Figure taken from ref.[19]

Flat slicing covers only half of the Penrose diagram (upper triangular part). Timelike worldline will exit the flat slicing in the past in finite affine time unless they are sitting at the north pole.

4. Static Patch

In above two coordinate system, the metric is time dependent. There is a time-like killing vector fields for de Sitter spacetime. So, there can be a coordinate system in which metric has no explicit time-dependence. The transformation law is given by

$$X_0 = \sqrt{l^2 - r^2}\sinh\left(\frac{t}{l}\right), \qquad X_1 = \sqrt{l^2 - r^2}\cosh\left(\frac{t}{l}\right), \qquad X_i = rz_i \tag{4.1}$$

Where $0 < r < l\,,\ i = 2, \ldots, d$ and $\vec{z}^2 = 1$. The corresponding metric is

$$ds^2 = -\left(1 - \frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{1 - \frac{r^2}{l^2}} + r^2 d\Omega_{d-2}^2 \tag{4.2}$$

The horizon is at $r = l$. The Penrose diagram for static coordinates is

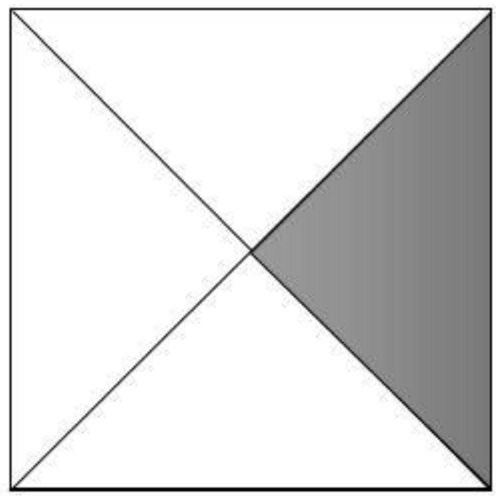


Fig. 3. Penrose diagram for the static coordinates. The shaded region is the causal patch or static patch. It is the area accessible to a single observer and it is the region of space-time covered by static coordinates

This is the causal patch of an observer sitting at the north pole $r = 0$ or $\theta = 0$ in the global coordinates. This is the region of de Sitter space-time accessible to a single observer. Bifurcate Killing horizon is $r_s = l$. Other three patches can also be covered by independent static coordinate system.

Comparing static coordinates and flat slicing coordinates-

Let (t_s, r_s) be the static coordinates and (t_f, r_f) be the flat slicing coordinates. Metric for the static patch is

$$ds^2 = -\left(1 - \frac{{r_s}^2}{l^2}\right) d{t_s}^2 + \frac{d{r_s}^2}{1 - \frac{{r_s}^2}{l^2}} + {r_s}^2 d\Omega_{d-2}^2$$

And the flat slicing metric is

$$ds^2 = -d{t_f}^2 + e^{\frac{2t_f}{l}} (d{r_f}^2 + {r_f}^2 d\Omega_{d-2}^2)$$

Comparing the coefficients of then S^{d-2} metric in both cases gives

$$r_s = r_f e^{\frac{t_f}{l}} \tag{4.3}$$

Comparing the embedding definitions for both coordinates system

(4.4)

$$e^{-\frac{2t_s}{l}} = e^{-\frac{2t_f}{l}} - \frac{{r_f}^2}{l^2}$$

Eq. (4.3) can be used to find the cosmological horizon of an observer at the north pole in the flat slicing coordinates. It is at $r_s = l$ and is equal to

$$r_f = le^{-\frac{t_f}{l}}$$

5. Analytic Continuation

Consider the embedding picture of de Sitter spacetime in coordinates $(X_0, X_1, \dots, X_{d-1}, X_d)$. It is a timelike hyperbola with radius l. Analytic continuation along the time direction

$$X_0 \to iX_{d+1}$$

Gives a d-sphere, S^d, with radius l embedded in $R^{d+!}$

$$X_1^2 + X_2^2 + \cdots + X_d^2 + X_{d+1}^2 = l^2 \tag{5.1}$$

In global coordinates, this transformation gives the metric for d-sphere

$$ds^2 = l^2(d\Theta^2 + sin^2\Theta \mathrm{d}\Omega_{d-1}^2) \tag{5.2}$$

Where $\Theta = \frac{t}{l} + \frac{\pi}{2}$. There is a shift by $\frac{\pi}{2}$ which changes cosine to sine. The time like slice at $t = 0$ is the equator of S^d.

6. Isometries of de Sitter

Consider the embedding picture of the de Sitter spacetime. The isometries of de Sitter will be those that preserve hyperboloid in $R^{d,1}$ space. These isometries are the rotations

$$M_{ij} = X_i\partial_j - X_j\partial_i \tag{6.1}$$

And boosts

$$K_i = X_0\partial_i + X_i\partial_0 \tag{6.2}$$

Together, these generate the isometry group SO(d,1). No. of generator for the group is

$\frac{d(d-1)}{2}$ rotations $+d$ boost $= \frac{d(d+1)}{2}$ generators.

Chapter 2

Conformal Field theory

In this chapter, we define notions of local conformal transformation in d dimensional space-time[8]. We will consider infinitesimal conformal transformations and derive the coordinate transformation laws and generators of d-dimensional conformal group. We will also derive functional form for correlation functions for a d-dimensional conformal field theory. Then we work with conformal field theory in two dimensions and derive the Virasoro algebra.

1. Conformal Group in d-dimensions

Let $g_{\mu\nu}$ be the metric in d-dimensional spacetime. Conformal transformation of the coordinates is an invertible mapping $x \to x'$ which leaves metric invariant upto the scale

$$g_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) \tag{1.1}$$

Consider the infinitesimal transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$. Then the metric transforms as

$$g_{\mu\nu} \to g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \tag{1.2}$$

Since the transformation is a conformal transformation, this implies

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu} \tag{1.3}$$

Taking the trace on both sides gives

$$f(x) = \frac{2}{d} \partial_\rho \epsilon^\rho \tag{1.4}$$

For simplicity, consider $g_{\mu\nu} = \eta_{\mu\nu}$ where $\eta_{\mu\nu} = diag(1,1, \dots ,1)$. Taking extra derivative on both sides of equation (1.3) and taking linear combinations of equations obtained by permutation of the indices and contracting the indices on the derivatives gives

$$(d-1)\partial^2 f(x) = 0$$

For $d = 1$ above equation imposes no constraint on $f(x)$. $d = 2$ case will be considered later. For $d \geq 3$, we get $\partial_\mu \partial_\nu f(x) = 0$ and

$$f(x) = A + B_\mu x^\mu \tag{1.5}$$

Where A and B_μ are constant. This implies

(1.6)

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \qquad\qquad c_{\mu\nu\rho} = c_{\mu\rho\nu}$$

First term in the R.H.S of the above equation represent infinitesimal translation. Substituting linear term in the R.H.S. in equation (1.3) gives

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b_\rho^\rho \eta_{\mu\nu}$$

This implies

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu} \qquad\qquad m_{\mu\nu} = -m_{\nu\mu}$$

$b_{\mu\nu}$ is the sum of the anti-symmetric part (when indices are different then R.H.S vanishes) and a pure trace part. Anti-symmetric part represent infinitesimal rigid rotations and the pure trace part represent infinitesimal scale transformation. Now substituting the quadratic part in eq. (1.3) one gets

$$c_{\mu\nu\rho} = \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho + \eta_{\nu\rho}b_\mu$$

Where $b_\mu \equiv \frac{1}{d} c^\sigma_{\sigma\mu}$. The corresponding infinitesimal transformation

$$x'^\mu = x^\mu + 2(x.b)x^\mu - b^\mu x^2$$

So, the infinitesimal transformations are

$$\begin{aligned} x'^\mu &= x^\mu + a^\mu & \qquad x'^\mu &= M^\mu_\nu x^\nu \\ x'^\mu &= \alpha x^\mu & \qquad x'^\mu &= \frac{x^\mu - b^\mu x^2}{1 - 2b.x + b^2x^2} \end{aligned} \tag{1.7}$$

The last equation corresponds to the special conformal transformation. The generators of the conformal transformation are

$$\begin{gathered} P_\mu = -i\partial_\mu \\ D = -ix^\mu\partial_\mu \\ L_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \\ K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu) \end{gathered} \tag{1.8}$$

The above generators satisfy the following commutation relations,

$$\begin{aligned} \left[D, P_\mu\right] = DP_\mu - P_\mu D &= -\left(x^\mu\partial_\mu\partial_\mu - \partial_\mu\left(x^\mu\partial_\mu\right)\right) \\ &= \partial_\mu = iP_\mu \end{aligned} \tag{1.9}$$

Similarly,

$$\left[D, K_\mu\right] = -iK_\mu \qquad\qquad \left[K_\mu, P_\mu\right] = 2i\left(\eta_{\mu\nu}D - L_{\mu\nu}\right) \tag{1.10}$$

$$[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \qquad [P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu)$$

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} - \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\mu\rho}L_{\nu\sigma} + \eta_{\mu\sigma}L_{\nu\rho})$$

Using the suitable linear combinations of the generators one can rewrite commutation relations as

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{bc}J_{ad}) \tag{1.11}$$

Where $a, b, c, d = 0,1, \dots, d, d+1$ and $\eta_{ab} = diag(1,1, \dots,1, -1)$ if spacetime is Euclidean.

And

$$J_{\mu\nu} = L_{\mu\nu} \qquad J_{d+!,\mu} = 1/2(P_\mu - K_\mu)$$
$$J_{d+!,0} = D \qquad J_{0,\mu} = 1/2(P_\mu + K_\mu) \tag{1.12}$$

The commutation relation here is similar to the commutation relation for SO(d+1,1) group. Also the number of generators for the conformal group is

$$1 + d + \frac{d(d-1)}{2} + d = \frac{(d+1)(d+2)}{2}$$

Which is same as no of generators for SO(d+1,1). So, conformal group is isomorphic to SO(d+1,1).

2. Energy Momentum Tensor

Consider the action

$$S = \int d^d x \, \mathcal{L}(\phi, \dot{\phi})$$

Under arbitrary infinitesimal transformation $x^\mu \to x^\mu + \epsilon^\mu$, action varies as

$$\delta S = \int d^d x \, T^{\mu\nu} \partial_\mu \epsilon_\nu \tag{2.1}$$

Where $T^{\mu\nu}$ is the energy momentum tensor assumed to be symmetric.

3. Correlation Functions

Consider a conformal field theory with action $S[\Phi]$ where Φ is the set of all functionally independent fields of the theory. The two point function of the fields ϕ_1, ϕ_2 is

$$< \phi_1(x_1)\phi_2(x_2) > = \frac{1}{Z}\int [d\Phi] \, \phi_1(x_1)\phi_2(x_2)\exp(-S[\Phi]) \tag{3.1}$$

Where Z is the partition function of the CFT.

Action is invariant under the conformal transformations. So, doing the coordinate transformation transforms two point function as

$$< \phi_1(x_1)\phi_2(x_2) > = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\Delta_2/d} < \phi_1(x'_1)\phi_2(x'_2) > \tag{3.2}$$

Scale transformations gives

$$< \phi_1(x_1)\phi_2(x_2) > = \lambda^{\Delta_1}\lambda^{\Delta_2} < \phi_1(\lambda x_1)\phi_2(\lambda x_2) > \tag{3.3}$$

Rotation and translation invariance gives

$$< \phi_1(x_1)\phi_2(x_2) > = f(|x_1 - x_2|) \tag{3.4}$$

Using above equations gives

$$< \phi_1(x_1)\phi_2(x_2) > = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$$

2-point function is also invariant under special conformal transformations. So,

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1 - 2bx + x^2b^2)^d}$$

Then the transformation law for the 2-point function gives

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{C_{12}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \frac{(\gamma_1\gamma_2)^{(\Delta_1+\Delta_2)/2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{3.5}$$

Where $\gamma_i = 1 - 2bx_i + {x_i}^2b^2$; $i = 1,2$. This gives $\Delta_1 = \Delta_2$ and the two point function will be

$$< \phi_1(x_1)\phi_2(x_2) > = \frac{C_{12}}{|x_1 - x_2|^{2\Delta_1}} \tag{3.6}$$

Similarly, the three point function is

$$< \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) > = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_3+\Delta_2-\Delta_1} x_{31}^{\Delta_1+\Delta_3-\Delta_2}} \tag{3.7}$$

Where $x_{ij} = |x_i - x_j|$

Four point function or higher are functions of conformal invariant cross-ratios (e.g. $\frac{x_{12}x_{34}}{x_{13}x_{24}}$).

4. Conformal Group in 2-dimensions

Consider the conformal field theory for free boson in 2-dimensional Minkowski space-time

(4.1)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi = \frac{1}{2}\partial_0\phi\partial_0\phi - \frac{1}{2}\partial_1\phi\partial_1\phi$$

Using the light cone coordinates

$$z = \frac{x^0 + ix^1}{\sqrt{2}} \qquad \bar{z} = \frac{x^0 - ix^1}{\sqrt{2}}$$

$$dz = \frac{\partial z}{\partial x^0}dx^0 + \frac{\partial \bar{z}}{\partial x^1}dx^1 = \frac{1}{\sqrt{2}}dx^0 + \frac{i}{\sqrt{2}}dx^1 \tag{4.2}$$

$$d\bar{z} = \frac{\partial z}{\partial x^0}dx^0 - \frac{\partial \bar{z}}{\partial x^1}dx^1 = \frac{1}{\sqrt{2}}dx^0 - \frac{i}{\sqrt{2}}dx^1$$

The action is

$$S = \int d^2x\frac{1}{2}\partial_\mu\phi\partial_\mu\phi = \int dzd\bar{z}\,\partial_z\phi\partial_{\bar{z}}\phi \tag{4.3}$$

Under the conformal transformation $z \to w(z)$ and $\bar{z} \to \overline{w}(\bar{z})$, we get

$$dw = \left(\frac{dw}{dz}\right)dz \qquad d\overline{w} = \left(\frac{d\overline{w}}{d\bar{z}}\right)d\bar{z}$$

$$\frac{d}{dw} = \left(\frac{dz}{dw}\right)\frac{d}{dz} \qquad \frac{d}{d\overline{w}} = \left(\frac{d\bar{z}}{d\overline{w}}\right)\frac{d}{d\bar{z}}$$

Then, the action is

$$S = \int dzd\bar{z}\,\partial_z\phi\partial_z\phi = \int dwd\overline{w}\left(\frac{dz}{dw}\right)\left(\frac{d\bar{z}}{d\overline{w}}\right)\left(\frac{dw}{dz}\right)\left(\frac{d\overline{w}}{d\bar{z}}\right)\partial_w\phi\partial_{\overline{w}}\phi$$

$$= \int dwd\overline{w}\,\partial_w\phi\partial_w\phi$$

And hence invariant under conformal transformation.

To get the generators of the conformal transformations, consider infinitesimal conformal transformations

$$z \to z' = z + \epsilon(z) \qquad \bar{z} \to \bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}) \tag{4.4}$$

Where $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ can be expressed as Laurent series in the respective variables. Consider the transformation of the field $\phi(z,\bar{z})$,

$$\phi'(z',\bar{z}') = \phi(z,\bar{z})$$

$$= \phi(z',\bar{z}') - \epsilon(z)\partial\phi - \bar{\epsilon}(\bar{z})\bar{\partial}\phi$$

$$\delta\phi = -\epsilon(z)\partial\phi - \bar{\epsilon}(\bar{z})\bar{\partial}\phi$$

$$= \sum_n c_n l_n \phi(z,\bar{z}) + \overline{c_n}\, \overline{l_n} \phi(z,\bar{z}) \tag{4.5}$$

Where $l_n = -z^{n+1}\partial_z$ and $\overline{l_n} = -\bar{z}^{n+1}\bar{\partial}_{\bar{z}}$ are the generators satisfying the commutation relations

$$[l_n, l_m] = (n-m) l_{n+m} \qquad [\overline{l_n}, \overline{l_m}] = (n-m)\bar{l}_{n+m}$$
$$[l_n, \bar{l}_m] = 0 \tag{4.6}$$

So, the conformal algebra is the direct sum of two isomorphic algebra.

Each of these two infinite dimensional algebra contains a finite dimensional subalgebra generated by l_{-1}, l_0 and l_1 (similar generators for other algebra). This sub-algebra is associated with the conformal group. Comparing with the definitions of the generators, $l_{-1} = -\partial_z$ generates translations in the complex plane, $l_0 = -z\partial_z$ generates rotations and scale transformations and $l_1 = -z^2\partial_z$ generates special conformal transformations.

5. Mode Expansion of Field

Consider the above free boson theory with cylindrical boundary conditions such that $-\infty < x^0 < \infty$ and $x^1 \sim x^1 + 2\pi$. Then

$$\mathcal{L} = \partial_\mu\phi\partial_\mu\phi \qquad \text{with} \quad \phi(x^0,0) = \phi(x^0,2\pi) \tag{5.1}$$

Field satisfying the above boundary condition can be Fourier expanded as

$$\phi(x^0,x^1) = \sum_{n\epsilon\mathbb{Z}} e^{inx^1} f_n(x^0) \tag{5.2}$$

The equation of motion can be obtained from the Lagrangian is

$$\partial_\mu\partial_\mu\phi = \partial_0\partial_0\phi - \partial_1\partial_1\phi = 0 \tag{5.3}$$

Substituting the Fourier expansion of ϕ in above equation, we get

$$\partial_0^2\phi = -n^2\phi$$

Or

$$\partial_0^2 f_n = -n^2 f_n$$

This is the equation of motion for simple harmonic oscillator and its solutions are well-known.

So,

$$f_n(x^0) = a_n e^{inx^0} + b_n e^{-inx^0} \qquad n \neq 0$$

And

$$f_0(x^0) = px^0 + q \tag{5.4}$$

So, the Fourier expansion of ϕ is

$$\phi(x^0,x^1) = q + px^0 + \sum_{n\in\mathbb{Z}, n\neq 0} \left(a_n e^{inx^0} + b_n e^{-inx^0}\right) e^{inx^1}$$

For quantizing the theory, promote field ϕ to the operator,

$$\phi(x^0, x^1) = q + 2px^0 + i \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{n}\left(a_n e^{in(x^0+x^1)+} - \tilde{a}_n e^{-in(x^0-x^1)}\right) \qquad (5.5)$$

Additional numerical coefficients are added to write the resultant commutation relations of creation and annihilation operators in more suitable forms.

The conjugate momentum of the theory is

$$\pi(x^0, x^1) = \partial_0 \phi \qquad (5.6)$$

And imposing the equal time commutation relations

$$[\phi(x^0, x^1), \pi(x^0, y^1)] = i\delta(x^1 - y^1)$$
$$[\phi(x^0, x^1), \phi(x^0, y^1)] = 0 \qquad (5.7)$$
$$[\pi(x^0, x^1), \pi(x^0, y^1)] = 0$$

Now, using the mode expansion of field ϕ and conjugate momentum π and taking the Fourier moments of above commutation relations, one get

$$[a_n, a_m] = [\tilde{a}_n, \tilde{a}_m] = n\delta_{n+m,0}$$
$$[a_n, \tilde{a}_m] = 0 \qquad (5.8)$$
$$[q, p] = i$$

Going over to Euclidean space-time (replace $x^0 \to -i\tau$) and use the conformal coordinates get

$$z = e^{i(x^1 - i\tau)} \qquad \bar{z} = e^{-i(i\tau + x^1)} \qquad (5.9)$$

Then

$$\phi(z, \bar{z}) = q - iplog(z\bar{z}) + i \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{n}(a_n z^{-n} + \tilde{a}_n \, \bar{z}^{-n}) \qquad (5.10)$$

$\phi(z, \bar{z})$ is not a holomorphic (or anti-holomorphic) function because $logz$ (or $log\bar{z}$) is not the holomorphic (or anti-holomorphic) function.

6. Vacuum Expectation Value of Fields

Defining the vacuum state as

$$a_k|0> = 0 \qquad \text{for } k > 0 \qquad (6.1)$$

and also satisfying

$$p|0> = 0 \qquad (6.2)$$

To evaluate the vacuum expectation value we require normal ordering of operators. Normal ordering orders creation operators to the left and annihilation operators to the right. Vacuum expectation value of the normal ordered operator vanishes. Also, the normal ordering of the zero mode operators is

$$:pq: = qp$$

While evaluating the vacuum expectation values one only need to take into account contributions from commutation relations.

The mode expansion of the fields are expressed in terms of the conformal coordinates in which time part denotes the radial direction and space part represent the angular direction in the complex plane. Vacuum state is defined at the origin. So, the time ordering of operators in Minkowski coordinates becomes radial ordering in the conformal coordinates. See in the figure.

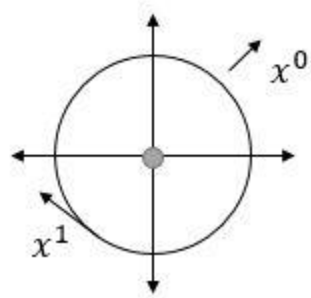


Fig. 4. Space-time coordinates in the complex plane after the change of variables

Assuming $\phi(z,\bar{z})$ and $\phi(w,\bar{w})$ are scalar fields such that $|z| > |w|$. Then, using the Wicks theorem for two fields one gets

$$R\big(\phi(z,\bar{z})\phi(w,\bar{w})\big) =: \phi(z,\bar{z})\phi(z,\bar{z}): + < 0\big|R\big(\phi(z,\bar{z})\phi(w,\bar{w})\big)\big|0 >$$

$$= :\phi(z,\bar{z})\phi(w,\bar{w}): - i[p,q]logz\bar{z} +$$

$$\{\Big[i\textstyle\sum_{n>0}\frac{1}{n}a_n z^{-n}, i\sum_{m<0}\frac{1}{m}a_m w^{-m}\Big] + \text{anti-holomorphic terms}\}$$

Using the commutation relations of the creation and annihilation operators, get

$$\left[i\sum_{n>0}\frac{1}{n}a_n z^{-n}, i\sum_{m<0}\frac{1}{m}a_m w^{-m}\right] = -\sum_{n>0,m<0}\frac{1}{nm}n\delta_{n+m,0}z^{-n}w^{-m}$$

Similarly for the commutation relations with anti-holomorphic terms. Finally, we get

$$R\big(\phi(z,\bar{z})\phi(w,\bar{w})\big) = :\phi(z,\bar{z})\phi(w,\bar{w}): - \log z\bar{z}$$

$$+\left\{\sum_{n>0}\frac{1}{n}\Big(\frac{w}{z}\Big)^n + \sum_{n>0}\frac{1}{n}\Big(\frac{\bar{z}}{\bar{w}}\Big)^n\right\} \tag{6.3}$$

This sum converges for $|z| > |w|$ and it is the logarithmic series. So,

$$= :\phi(z,\bar{z})\phi(w,\bar{w}): - \log z\bar{z} - \log\left(1 - \frac{w}{z}\right) - \log\left(1 - \frac{\bar{z}}{\bar{w}}\right)$$

$$= :\phi(z,\bar{z})\phi(w,\bar{w}): - \log(z-w) - \log(\bar{z}-\bar{w}) \tag{6.4}$$

Since the vacuum expectation values contains logarithmic singularities. A more standard result will be the operator product of $\partial\phi(z,\bar{z})\partial\phi(w,\bar{w})$. So,

$$\partial\phi(z,\bar{z})\partial\phi(w,\bar{w}) = -\frac{1}{(z-w)^2} + :\partial\phi(z,\bar{z})\partial\phi(w,\bar{w}): \tag{6.5}$$

Now as $z \to w$, normal ordered part remain finite. So, the main contribution comes from singular part and is

$$\partial\phi(z)\partial\phi(w) \sim -\frac{1}{(z-w)^2} \tag{6.6}$$

7. Operator Product Expansions

Stress tensor of a theory is an important quantity. For a classical field theory, stress tensor is given by $T_{zz}(z) = -\frac{1}{2}\partial\phi(z)\partial\phi(z)$. If we quantize the field, the product of two operators become singular and we require operator ordering to define the quantum stress tensor

$$T_{zz}(z) = -\frac{1}{2}:\partial\phi(z)\partial\phi(z): \tag{7.1}$$

: (…) : is the normal ordering.

Now consider the operator product of $T_{zz}(z)\partial\phi(w)$. To compute this, we will use the Wick's theorem[8,12],

$$\begin{aligned} R\big(\partial\phi(z_1)\ldots\partial\phi(z_n)\big) &= \ :\partial\phi(z_1)\ldots\partial\phi(z_n): + \\ &\sum_{per} <0|R\big(\partial\phi(z_1)\partial\phi(z_2)\big)|0>:\partial\phi(z_3)\ldots\partial\phi(z_n): + \cdots \\ &+\sum_{per} <0|R\big(\partial\phi(z_1)\partial\phi(z_2)\big)|0> \ \ldots <0|R\big(\partial\phi(z_{n-1})\partial\phi(z_n)\big)|0> \end{aligned} \tag{7.2}$$

Above expression is valid for even number of fields. For odd number of fields, one field is left unpaired in the last summation of above expression.

Now using above expression for three fields get

$$:\partial\phi(z)\partial\phi(z):\partial\phi(w) = :\partial\phi(z)\partial\phi(z)\partial\phi(w): + 2\partial\phi(z)\left(-\frac{1}{(z-w)^2}\right)$$

Now as $z \to w$, we can do a Taylor of the field $\partial\phi(z)$ around w. So, above operator product will become

$$T_{zz}(z)\partial\phi(w) = \frac{1}{(z-w)^2}\partial\phi(w) + \frac{1}{z-w}\partial^2\phi(w) + \text{regular terms} \tag{7.3}$$

Similarly, for operator product of $T_{zz}(z)T_{ww}(w)$ one gets

$$T_{zz}(z)T_{ww}(w) = \frac{1}{4} : \partial\phi(z)\partial\phi(z) :: \partial\phi(w)\partial\phi(w):$$

$$= \frac{1}{2(z-w)^4} + \frac{2}{(z-w)^2}T_{ww}(w) + \frac{1}{z-w}\partial_w T_{ww}(w) \tag{7.4}$$

For any general theory, this operator product will be

$$= \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2}T_{ww}(w) + \frac{1}{z-w}\partial_w T_{ww}(w) \tag{7.5}$$

Here c is the central charge. It also represent number of fields for boson CFT. The central charge c is related to the energy scale and hence term with the central charge produces conformal anomaly.

8. Virasoro Algebra

Consider the holomorphic stress tensor component, $T_{zz}(z) = T(z)$. We can do a mode expansion of the stress tensor as

$$T(z) = \sum_n z^{-n-2} L_n \qquad \Rightarrow \qquad L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1} T(z) \tag{8.1}$$

Where L_n are the mode operators of stress tensor. Similar expression is for anti-holomorphic component of the stress tensor.

Now, the generator of the conformal transformations i.e., conformal charge is defined as

$$Q_\epsilon = \frac{1}{2\pi i}\oint dz\, \epsilon(z) T(z) \tag{8.2}$$

Also note that the current for the theory is $J(z) = \epsilon(z)T(z)$. Now $\epsilon(z)$ is the holomorphic function and hence can be expressed as a Laurent series. So, using the mode expansion for stress tensor and $\epsilon(z)$ one gets

$$Q_\epsilon = \sum_n \epsilon_n L_n \tag{8.3}$$

Also,

$$\delta\phi(z) = -[Q_\epsilon, \phi(z)] \tag{8.4}$$

The modes of the stress tensor are the generators of local conformal transformations on the Hilbert space. So evaluating the commutation relations of the modes gives the algebra,

$$[L_n, L_m] = \oint \left[\frac{dz}{2\pi i}\frac{dw}{2\pi i} - \frac{dw}{2\pi i}\frac{dz}{2\pi i}\right] z^{n+1}T(z)w^{m+1}T(w)$$

Deforming the contour such that it goes around singularity $z = w$. Using the Cauchy integral formula and general expression for the operator product of the stress tensors we get,

$$[L_n, L_m] = \frac{c}{12}(n+1)n(n-1)\delta_{n+m,0} + (n-m)L_{n+m} \tag{8.5}$$

Same calculation can be performed taking anti-holomorphic part and we get

$$[\bar{L}_n, \bar{L}_m] = \frac{c}{12}(n+1)n(n-1)\delta_{n+m,0} + (n-m)\bar{L}_{n+m} \tag{8.6}$$

And

$$[L_n, \bar{L}_m] = 0 \tag{8.7}$$

Again, the algebra of the local conformal transformation is the direct sum of two Virasoro algebra.

Representation theory for the Virasoro algebra will be discussed in the next chapter.

Chapter 3

Compact Wess-Zumino-Witten Model

In this chapter, we will start with the non-linear sigma model[8,9]. Then we modify the action by adding the Wess-Zumino term to get the conformal field theory with additional conserved currents generating affine Lie algebra. We will work out this Lie algebra for the theory. We will also work out the Virasoro algebra using the Sugawara construction for the theory. We will also look at the conditions for the highest weight unitary representation for both algebras.

1. Non-Linear Sigma Model

Let $g(x)$ matrix bosonic field living on the group manifold G. The action is

$$S_0 = \frac{1}{4a^2}\int d^2x\ Tr(\partial^\mu g^{-1}\partial_\mu g) \tag{1.1}$$

where a^2 is a positive dimensionless coupling constant. Let g be the Lie algebra associated with the group manifold. For the action to be real, $g(x)$ must be in the unitary representation and let t^a be the matrix representation of the Lie algebra generators such that

$$Tr(t^a t^b) = 2\delta_{a,b} \tag{1.2}$$

where $[t^a, t^b] = \sum_c i\, f_{abc} t^c$

This trace Tr is related to the usual trace by

$$Tr = \frac{1}{x_{rep}} Tr'$$

Here x_{rep} is the Dynkin index of the represention.

Looking at the positivity of the action. Since $g(x)$ is unitary, then

$$(g^{-1}\partial_\mu g)^\dagger = (\partial_\mu g^{-1})g = -g^{-1}(\partial_\mu g)$$

is anti-Hermitian. Also, $\partial_\mu g^{-1} = -g^{-1}(\partial_\mu g)g.$ So,

$$Tr\big(\partial^\mu g^{-1}\partial_\mu g\big) = Tr(-g^{-1}(\partial_\mu g)g(\partial_\mu g)) = Tr((g^{-1}\partial_\mu g)^\dagger g(\partial_\mu g)) \geq 0 \tag{1.3}$$

Lagrangian density is positive which ensures the positivity of the action.

The non-linear sigma model is conformal invariant classically. After quantization, the dimensionless coupling acquires a scale dependence. So, the beta function is non-zero and hence not conformal invariant.

This can also be seen classically, at the level of current. Conformal field theory has holomorphic factorization property for the field. So, the conserved current must simply factorize into holomorphic and anti-holomorphic part. However for the above theory, we don't get such current.[8]

2. Wess-Zumino Witten Model

For simplicity, consider the group manifold G to be compact. To enhance the symmetry of the action, we add a Wess-Zumino term to above action

$$\Gamma = -\frac{i}{24\pi}\int d^3y\,\epsilon^{\alpha\beta\gamma}Tr(\tilde{g}^{-1}\partial_\alpha\tilde{g}\tilde{g}^{-1}\partial_\beta\tilde{g}\tilde{g}^{-1}\partial_\gamma\tilde{g}) \tag{2.1}$$

Here, the integral is over a three dimensional manifold, whose boundary is the compactification of original 2-dimensional space. $\tilde{g}$ is the extension of the field g to the 3-dimensional manifold[9,10]. The extension of $\tilde{g}$ is not unique and there are two choices for the compact manifold as can be seen in the figure.

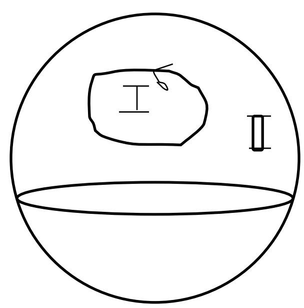


Fig. 5. Figure depicts the two choices of field extension for the theory on the sphere. Similar generalizations are carried forward to higher dimensions.

This gives the ambiguity in the definition of the Γ. The ambiguity is defined as the difference between the two choices.

$$\Delta\Gamma = -\frac{i}{24\pi}\left\{\int_{V_1} d^3y - \int_{V_2} d^3y\right\}\epsilon^{\alpha\beta\gamma}Tr(\tilde{g}^{-1}\partial_\alpha\tilde{g}\tilde{g}^{-1}\partial_\beta\tilde{g}\tilde{g}^{-1}\partial_\gamma\tilde{g})$$

Two choices have opposite orientations which changes the sign of one of the integral. Then, the range of integral extend over to whole 3-dimensional manifold. For the model at hand, this 3-dimensional manifold is topologically equivalent to the 3-sphere. So,

$$\Delta\Gamma = -\frac{i}{24\pi}\int_{S^3} d^3y\, \epsilon^{\alpha\beta\gamma} Tr(\tilde{g}^{-1}\partial_\alpha \tilde{g}\tilde{g}^{-1}\partial_\beta \tilde{g}\tilde{g}^{-1}\partial_\gamma \tilde{g}) \tag{2.2}$$

This ambiguity also requires that

$$e^{-\Gamma} = e^{-(-\Gamma')} \qquad \Rightarrow \qquad e^{\Gamma+\Gamma\prime} = 1$$

Negative sign is due to opposite orientation. So, the ambiguity $\Delta\Gamma$ is the integral multiple of $2\pi i$ and is the quantization condition for the model. Choice of the trace ensures that $\Delta\Gamma$ is defined modulo $2\pi i$. We also need the topological classification of mapping from 3-sphere into G as we have family of such mappings. So, normalizing $\Delta\Gamma$ such that any coupling constant multiplying Γ is quantized (integer).

Consider the action

$$S = S_0 + k\Gamma \tag{2.3}$$

here k is an integer. Taking the variation of action w.r.t. the field g such that $g \to g + \delta g$. Variation of the S_0 is

$$\delta S_0 = \frac{1}{2a^2}\int d^2x\, Tr(g^{-1}\delta g \partial^\mu(g^{-1}\partial_\mu g)) \tag{2.4}$$

Now the variation of the Wess-Zumino term can be written as the total derivative. So, applying the Gauss theorem, the 3-dimensional integral reduces to the 2-dimensional integral and is given be[11]

$$\delta\Gamma = \frac{i}{8\pi}\int d^2x\, \epsilon^{\mu\nu} Tr(g^{-1}\delta g \partial_\mu(g^{-1}\partial_\nu g)) \tag{2.5}$$

Then, the full equation of motion is given by

$$\partial^\mu(g^{-1}\partial_\mu g) + \frac{ika^2}{4\pi}\epsilon^{\mu\nu}\partial_\mu(g^{-1}\partial_\nu g) = 0 \tag{2.6}$$

Changing the variables to the complex coordinates $(z, \bar{z})$,

$$z = x^0 + ix^1 \qquad\qquad \bar{z} = x^0 - ix^1 \tag{2.7}$$

Also using $\epsilon_{z\bar{z}} = \frac{1}{2}$ and $\partial^z = 2\partial_{\bar{z}}$ one gets

$$\left(1 + \frac{ka^2}{4\pi}\right)\partial_z(g^{-1}\partial_{\bar{z}} g) + \left(1 - \frac{ka^2}{4\pi}\right)\partial_{\bar{z}}(g^{-1}\partial_z g) = 0 \tag{2.8}$$

This equation is also valid for non-compact WZW model in 2 dimensions except for minor changes in coefficient.

For $a^2 = \frac{4\pi}{k}$, we get the desired conservation law

$$\partial_{\bar{z}}(g^{-1}\partial_z g) = 0 \tag{2.9}$$

Since a^2 is positive here, k must be positive. We also get another solution for $a^2 = -\frac{4\pi}{k}$ in which k is negative and it corresponds to the conservation of the dual currents.

Solution to the above equation is of the form

$$g(z, \bar{z}) = f(z)h(\bar{z}) \tag{2.10}$$

Where $f(z)$ and $h(\bar{z})$ are arbitrary functions.

The separate conservation of the currents

$$\begin{aligned} \partial_{\bar{z}} J_z(z) = \partial_{\bar{z}}(\partial_z g g^{-1}) = 0 \\ \partial_z J_{\bar{z}}(\bar{z}) = \partial_z(g^{-1}\partial_{\bar{z}} g) = 0 \end{aligned} \tag{2.11}$$

implies the invariance under

$$g(z, \bar{z}) \to \Omega(z) g(z, \bar{z}) \bar{\Omega}^{-1}(\bar{z}) \tag{2.12}$$

here $\Omega(z)$ and $\Omega^{-1}(\bar{z})$ are arbitrary matrices in G. Under infinitesimal transformations

$$\Omega(z) = 1 + \omega(z) \qquad\qquad \bar{\Omega} = 1 + \bar{\omega}(\bar{z}) \tag{2.13}$$

$g(z, \bar{z})$ transforms as

$$\delta g = \omega g - g\bar{\omega} = \delta_\omega g + \delta_{\bar{\omega}} g \tag{2.14}$$

For $a^2 = \frac{4\pi}{k}$, the variation of the action under infinitesimal transformation is

$$\begin{aligned} \delta S = \frac{k}{2\pi} \int d^2x\, Tr\big(g^{-1}\delta g \partial_z (g^{-1}\partial_{\bar{z}} g)\big) \\ = \frac{k}{2\pi} \int d^2x\, Tr\big(\omega(z)\partial_{\bar{z}}(\partial_z g g^{-1}) - \bar{\omega}(\bar{z})\partial_z(g^{-1}\partial_{\bar{z}} g)\big) \end{aligned} \tag{2.15}$$

which clearly vanishes. So, the global G×G invariance of the sigma model is extended to the local G(z) ×G($\bar{z}$) invariance.

So, the action with $a^2 = \frac{4\pi}{k}$ is written as

$$S = \frac{k}{16\pi} \int d^2x\, Tr\big(\partial^\mu g^{-1} \partial_\mu g\big) + k\Gamma \tag{2.16}$$

3. Current Algebra

Let the currents be

$$J(z) = -kJ_z(z) = -k\partial_z g g^{-1}$$
$$\bar{J}(\bar{z}) = kJ_{\bar{z}}(\bar{z}) = kg^{-1}\partial_{\bar{z}} g \tag{3.1}$$

Then, the variation of action in complex coordinates ($d^2x = -\frac{i}{2}dzd\bar{z}$) is

$$\delta S = \frac{i}{4\pi}\int dz\, d\bar{z}\{\partial_z Tr[\omega(z)J(z)] + \partial_{\bar{z}} Tr[\bar{\omega}(\bar{z})\bar{J}(\bar{z})]\} \tag{3.2}$$

Integral involves total derivatives, applying Stokes theorem, get

$$\delta S = \frac{i}{4\pi}\oint dz\, Tr[\omega(z)J(z)] - \frac{i}{4\pi}\oint d\bar{z}\ Tr[\bar{\omega}(\bar{z})\bar{J}(\bar{z})] \tag{3.3}$$

Here we are taking the holomorphic contour to be counterclockwise and anti-holomorphic contour to be clockwise. With

$$J(z) = \textstyle\sum_a J^a(z)t^a \qquad \text{and} \qquad \omega(z) = \sum_a \omega^a(z)t^a$$
$$\bar{J}(\bar{z}) = \textstyle\sum_a \bar{J}^a(\bar{z})t^a \qquad \text{and} \qquad \bar{\omega}(\bar{z}) = \sum_a \bar{\omega}^a(\bar{z})t^a \tag{3.4}$$

here t^a are the generators in any representation and they are trace normalized as

$$Tr(t^a t^b) = 2\delta_{ab}$$

This gives

$$\delta S = \frac{i}{4\pi}\oint dz \sum_a \omega^a J^a - \frac{i}{4\pi}\oint d\bar{z} \sum_a \bar{\omega}^a \bar{J}^a$$

Let X be the correlation function of the fields or currents or stress tensors. Then the variation of X is given by the Ward's identity

$$\delta < X > = -\frac{1}{2\pi i}\oint dz \sum_a \omega^a < J^a X > + \frac{1}{2\pi i}\oint d\bar{z} \sum_a \bar{\omega}^a < \bar{J}^a X > \tag{3.5}$$

variation of the holomorphic current follows from eq.(3.1) and eq.(2.14) as

$$\begin{aligned}\delta_\omega J &= -k(\partial_z(\delta_\omega g)g^{-1} + \partial_z g\delta_\omega g^{-1})\\ &= -k(\partial_z(\delta_\omega g)g^{-1} - \partial_z g g^{-1}(\delta_\omega g)g^{-1})\\ &= -k(\partial_z\omega) - k(\omega\partial_z g g^{-1}) + k\partial_z g g^{-1}\omega\\ &= [\omega, J] - k(\partial_z\omega)\end{aligned} \tag{3.6}$$

Now using equation eq.(3.4) and the commutation relations of the generators t^a, get

(3.7)

$$\delta_\omega J^a = \sum_{b,c} i f_{abc} \omega^b J^c - k(\partial_z \omega^a)$$

Substituting in eq. (3.5) get

$$\delta_\omega J^b = -\frac{1}{2\pi i}\oint dz \sum_a \omega^a < J^a J^b > = \sum_{b,c} i f_{abc} \omega^b J^c - k(\partial_z \omega^a) \tag{3.8}$$

Now, comparing the contour integral with R.H.S and Cauchy integral theorem one gets

$$J^a(z)J^b(w) \sim \frac{k\delta_{ab}}{(z-w)^2} + \frac{\sum_c i f_{abc} J^c(w)}{z-w} \tag{3.9}$$

Similarly, the operator product expansion of $\partial J^a(z)J^b(w)$ is

$$\partial J^a(z)J^b(w) \sim -\frac{2k\delta_{ab}}{(z-w)^3} - \frac{\sum_c i f_{abc} J^c(w)}{(z-w)^2} \tag{3.10}$$

obtained by taking the derivative on both sides by z.

Expressing the current $J^a(z)$ as Laurent series

$$J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} J_n^a \qquad \Rightarrow \quad J_n^a = \frac{1}{2\pi i}\oint dz\, z^n J^a(z) \tag{3.11}$$

Proceeding in the similar fashion as with the case of Virasoro algebra, get

$$[J_n^a, J_m^b] = \sum_c i f_{abc} J_{n+m}^c + kn\delta_{ab}\delta_{n+m,0} \tag{3.12}$$

Similar calculations can be carried out for anti-holomorphic current, one gets

$$\delta_{\bar{\omega}} \bar{J}^b = [\bar{\omega}, \bar{J}] - k(\partial_{\bar{z}} \bar{\omega}) \tag{3.13}$$

And the commutation relation is

$$[\bar{J}_n^a, \bar{J}_m^b] = \sum_c i f_{abc} \bar{J}_{n+m}^c + kn\delta_{ab}\delta_{n+m,0} \tag{3.14}$$

Also, $\bar{\omega}(\bar{z})$ is independent z so,

$$\delta_{\bar{\omega}} J^a = 0 \tag{3.15}$$

And the commutation relation will be

$$[J_n^a, \bar{J}_m^b] = 0 \tag{3.16}$$

The two algebra are thus independent.

4. Sugawara Construction and Virasoro algebra

Classically, the stress tensor is given by $T(z) = \frac{1}{2k}\sum_a J^a(z)J^a(z)$. Upon quantization, the product of the operators at the same point becomes singular and requires certain prescription for ordering of the operators. So, the quantum stress tensor is defined as

$$T(z) = \gamma \colon \sum_a J^a(z)J^a(z) \colon \tag{4.1}$$

The constant γ cannot be fixed from the classical field theory as it is renormalized by quatum effects. However it can be fixed by requiring that the current J^a to be a primary field of dimension 1. To show this, we need to evaluate the operator product $T(z)J^b(w)$. So, consider the contraction

$$J^a(z)\colon J^b(w)J^b(w)\colon = \frac{1}{2\pi i}\oint_w dx \frac{[J^a(z)J^b(x)J^b(w) + J^b(x)J^a(z)J^b(w)]}{x-w}$$

$$= \frac{1}{2\pi i}\oint dx \left\{\left[\frac{k\delta_{ab}}{(z-x)^2} + \sum_c if_{abc}\frac{J^c(x)}{z-x}\right]\frac{J^b(w)}{x-w} + \frac{J^b(x)}{x-w}\left[\frac{k\delta_{ab}}{(z-w)^2} + \sum_c if_{abc}\frac{J^c(w)}{z-w}\right]\right\}$$

Now the operator product $J^b(x)J^c(w)$ only give terms of the form $(x-w)$ in the denominator. In the first part in the R.H.S. we need to use partial fractions to separate terms with denominators $(x-w), (z-x))$ or their powers. While in the second part of the R.H.S. we only have x dependent terms with denominator $(x-w)$ and its higher powers. This integral can be simply evaluated by the application of the Cauchy Integral formula. So, after using the operator product expansion of $J^b(x)J^c(w)$, get

$$= \frac{1}{2\pi i}\oint dx \{\frac{k\delta_{ab}}{(z-x)^2}J^b(w) + \frac{J^b(x)k\delta_{ab}}{(z-w)^2}$$
$$+ \sum_c if_{abc}\left[if_{cbd}\frac{J^d(w)}{x-w} + \frac{k\delta_{cd}}{(x-w)^2} + \colon J^c(w)J^b(w)\colon\right]\left(\frac{1}{z-x}\right)$$
$$+ \sum_c if_{abc}\frac{\colon J^b(w)J^c(w)\colon}{z-w}\} \left(\frac{1}{x-w}\right)$$

Since f_{cbd} is anti-symmetric, so, the product $f_{cbd}\delta_{cd}$ vanishes.Similarly,

$$\sum_c if_{abc}(\colon J^c(w)J^b(w)\colon + \colon J^b(w)J^c(w)\colon) = 0$$

Also, we have

$$-\sum_{b,c} f_{abc}f_{cbd} = \sum_{b,c} f_{abc}f_{dbc} = 2p\delta_{ad} \tag{4.2}$$

p is the dual Coxeter number which is half of the value of the Casimir in the adjoint representation. So, the end result is

$$J^a(z) : \sum_b J^b(w) J^b(w) := 2(k+p) \frac{J^a(w)}{(z-w)^2}$$

Now by inverting the order of the contracting fields and then multiplying by γ, one get

$$T(z)J^a(w) = 2\gamma(k+p) \frac{J^a(z)}{(z-w)^2}$$

$$= 2\gamma(k+p) \left[\frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} \right]$$

For $T(z)$ to be genuine energy momentum tensor, the coefficient of the second term in the R.H.S has to be 1. So,

$$\gamma = \frac{1}{2(k+p)} \tag{4.3}$$

And

$$T(z) = \frac{1}{2(k+p)} : \sum_a J^a(z) J^a(z) :$$

This also imply that the conformal dimension of the current J^a is 1. Now, comparing with the classical result we see that the quantum stress tensor is normal ordered and the factor γ is renormalized to $\frac{1}{2(k+p)}$.

Now evaluating the operator product expansion for the stress tensors in terms of the currents.

$$T(z)T(w) = \frac{1}{2(k+p)} \frac{1}{2\pi i} \oint dx \sum_a \{T(z)J^a(x)J^a(w) + J^a(x)T(z)J^a(w)\} \frac{1}{x-w}$$

Using the operator product expansion of $T(z)J^a(w)$ and $\partial J^a(z)J^b(w)$ and Cauchy integral formula, we get

$$T(z)T(w) = \frac{k\delta_{aa}}{2(k+p)} \frac{1}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial T(w) \tag{4.4}$$

here $\delta_{aa} = \dim(g)$ and the central charge in terms of k is defined as

$$c \equiv \frac{k\delta_{aa}}{k+p} \tag{4.5}$$

Sugawara stress tensor can be expressed in terms of the modes as

(4.6)

$$L_n = \frac{1}{2\left(k+\frac{c_2}{2}\right)} \sum_a \left\{ \sum_{m \le -1} J_m^a J_{n-m}^a + \sum_{m \ge 0} J_{n-m}^a J_m^a \right\}$$

Above expression is valid for $n \neq 0$. Since J_m^a and J_{n-m}^a commute, there is no need for operator ordering. However for $n = 0$ case, J_m^a and J_{-m}^a do not commute and hence require normal ordering (or any other ordering). So, it can also be written as

$$L_n = \frac{1}{2(k+p)} \sum_{a,m} :J_m^a J_{n-m}^a: \tag{4.7}$$

So, the complete Kac-Moody and Virasoro algebra for the model is

$$\begin{gathered} [L_n, L_m] = \frac{c}{12}(n+1)n(n-1)\delta_{n+m,0} + (n-m)L_{n+m} \\ [J_n^a, J_m^b] = \sum_c i f_{abc} J_{n+m}^c + kn\delta_{ab}\delta_{n+m,0} \\ [L_n, J_m^a] = -m J_{n+m}^a \end{gathered} \tag{4.8}$$

Full affine Lie algebra or Kac-Moody algebra is not a symmetry algebra since its generators does not commute with L_0. It is the spectrum generating algebra of the theory.

Note- This analysis also hold true for non-compact WZW model. The allowed representations of the algebras depends on the symmetry group of the model.

5. Unitary Representations of the Virasoro algebra

In the conformal field theory, the energy eigenstates are eigenstates of L_0 and $\bar{L}_0$[8]. For a theory describing a physical system, the energy eigenvalues has to be non-negative. To construct the representation of the Virasoro algebra we will follow the same approach as used to construct the representation of su(2) in the theory of angular momentum.

Let $|0>$ be the vacuum state of the theory such that it is invariant under global conformal transformations. Then, it is annihilated by L_{-1}, L_0 and L_1 and their anti-holomorphic counterparts. This implies

$$\begin{aligned} L_n|0> &= 0 \qquad (n \ge -1) \\ \bar{L}_n|0> &= 0 \end{aligned} \tag{5.1}$$

Let $\phi(z,\bar{z})$ be the primary field. The asymptotic state is defined as

$$|h,\bar{h}> = \phi(0,0)|0> \tag{5.2}$$

Such that it satisfies

$$L_0|h,\bar{h}> = h|h,\bar{h}> \qquad \bar{L}_0|h,\bar{h}> = \bar{h}|h,\bar{h}> \tag{5.3}$$

And

$$L_n|h,\bar{h}>=0 \qquad\qquad \bar{L}_n|h,\bar{h}>=0 \tag{5.4}$$

for $n > 0$. All the states spanning the representation are obtained by repeatedly applying L_{-n} and $\bar{L}_{-n}$ to the asymptotic state $|h,\bar{h}>$. Representation is spanned by the states of the form.

$$L_{-n_1} \dots L_{-n_k}|h> \qquad\qquad 0 < n_1 < \cdots < n_k \tag{5.5}$$

This representation is called the highest weight representation with weight h. For simplicity, we will work with the generators from the holomorphic part only as the two Virasoro algebra are independent. The full representation will be the tensor product of the representations of both algebras.

Since, the theory describe a physical system. The inner product of the states has to be non-negative. This implies, that for any $n > 0$

$$< h|L_n L_{-n}|h> =< h|[L_n, L_{-n}] + L_{-n}L_n|h> \geq 0$$

L_n annihilates $|h>$ and using the commutation relation for the Virasoro algebra, get

$$< h|L_n L_{-n}|h> = < h|2nh + \frac{c}{12}(n-1)n(n+1)|h>$$

Assuming $< h|h> = 1$,

$$2nh + \frac{c}{12}(n-1)n(n+1) \geq 0$$

Case 1 - If $n = 1$, then $h \geq 0$

Case 2 – If n is very large, then

$$2nh + \frac{c}{12}(n-1)n(n+1) \sim \frac{c}{12}n^3 \geq 0$$

So, $c \geq 0$.

So, the necessary condition for the unitary representation is $c \geq 0$ and $h \geq 0$ [8,13,14].

Unitarity also gives imposes constraint

$$L_n^\dagger = L_{-n} \tag{5.6}$$

Let $M_N = M_N(c,h)$ be the matrix of the inner product of all the basis states. It is called Gram matrix. Then, the necessary and sufficient condition for the unitary representation is

$$\det\big(M_N(c,h)\big) \geq 0 \tag{5.7}$$

n is the level of the state. Level of a state $|i>$,

$$|i> = L_{-n_1} \dots L_{-n_r}|h>$$

is defined as an integer N such that $N = n_1 + \cdots + n_r$.

We can check the constraints imposed by the unitary conditions on states at lower levels i.e., $N = 0,1,2$.

Case 1 - $N = 0$

$$\det\big(M_0(c,h)\big) = < h|h > = 1 \geq 0$$

Case 2 - $N = 1$

$$\det\big(M_1(c,h)\big) = < h|L_1L_{-1}|h > = 2h \geq 0$$

This implies $h \geq 0$.

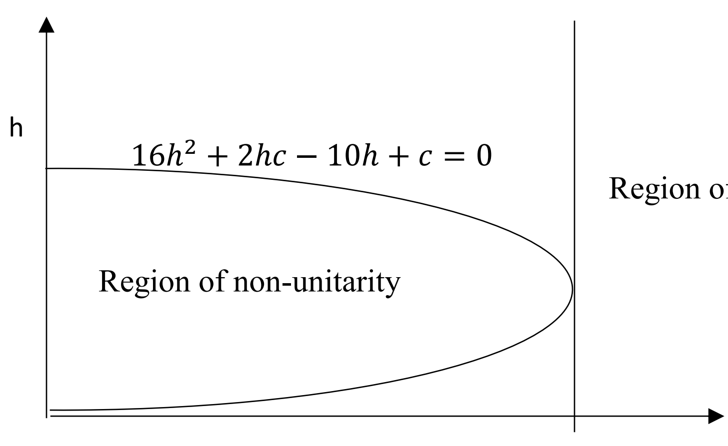
Case 3 - $N = 3$

$$\det\big(M_2(c,h)\big) = det\begin{bmatrix} < h|L_2L_{-2}|h > & < h|L_2L_{-1}^2|h > \\ < h|L_1^2L_{-2}|h > & < h|L_1^2L_{-1}^2|h > \end{bmatrix}$$

$$= \det\begin{bmatrix} 4h + \frac{c}{2} & 6h \\ 6h & 8h^2 + 4h \end{bmatrix}$$

$$= 2h(16h^2 + 2hc - 10h + c) \geq 0$$

From case 2, $h \geq 0$. So, $(16h^2 + 2hc - 10h + c) \geq 0$. The plot between c and h is



To progress further one need the formula for the determinant of the Gram matrix for any $N \in \mathbb{N}$. This is a well known result and is called the Kac determinant formula

$$\det\big(M_N(c,h)\big) = \prod_{k=1}^{N}\big(\psi_k(c,h)\big)^{P(n-k)} \tag{5.8}$$

Where

(5.9)

$$\psi_k(c,h) = \prod_{pq=k} \left(h - h_{p,q}(c)\right)$$

p , q range over positive integers and $P(n-k)$ is the integer counting the power and

$$h_{p,q}(c) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}$$

And m is given by

$$c = 1 - \frac{6}{m(m+1)} \tag{5.11}$$

$m = 2,3, \dots;\ p = 1,2, \dots, m-1\ ; q = 1,2, \dots, p.$

An irreducible highest weight representation is specified by the pair of numbers (c,h)[14]. The representation is unitary if $c \geq 1$ and $h \geq 0$ or

$$c = 1 - \frac{6}{m(m+1)} \qquad\qquad m = 2,3, \dots$$

and

$$h(c) = h_{p,q}(c) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}$$

$p = 1,2, \dots, m-1\ ; q = 1,2, \dots p.$

The values for c forms a discrete series when $c < 1.$ The representation for $m = 2,3$ and 4 are related to the Ising, tri-critical Ising and three states Potts model respectively. The values of h for corresponding m matches with the critical exponents of these models.

Similarly, the conditions for the unitary representaions of affine Lie algebra is given by-

An irreducible representation of affine Lie algebra $\hat{\mathrm{g}}$ is characterized by vacuum representation of g or its highest weight μ and the value of the central charge k. The necessary and sufficient conditions for the highest weight unitary representation of $\hat{\mathrm{g}}$ is

$$\frac{2k}{\psi^2} \in \mathbb{Z}$$

and

$$k > \psi\mu > 0$$

where $\frac{2k}{\psi^2}$ is the level of the representation of $\hat{\mathrm{g}}$ and ψ is the highest root of g.

The proof of this result can be found in [13].

Chapter 4

General WZW and Coset Models

The motivation for studying WZW model on non-compact spaces is that these models are conformal and can give the detail about the spectrum on the curved backgrounds[3,4]. We will found out that de Sitter space-time is isomorphic to a coset and we will prove that isomorphism. We will also look at the Virasoro algebra for the coset space model.

1. SL(2,R) WZW model

Anti-de Sitter spacetime is the maximal symmetric solution of the vacuum Einstein equations with negative cosmological constant. In the embedding picture in Lorentzian signature, (d+1)-dimensional anti-de Sitter spacetime is defined as the hyperboloid in $R^{d,2}$. Let $(X_0, X_1, \dots, X_d, X_{d+1})$ be the embedding coordinates.

$$-X_0^2 + X_1^2 + \cdots + X_d^2 - X_{d+1}^2 = l^2 \tag{1.1}$$

It is a non-compact manifold.

The metric is

$$ds^2 = -dX_0^2 + dX_1^2 + \cdots + dX_d^2 - dX_{d+1}^2 \tag{1.2}$$

In global coordinates[15],

$$\begin{aligned} X_0 &= R\cosh\rho\cos\tau \qquad & X_{d+1} &= R\cosh\rho\sin\tau \\ X_i &= R\sinh\rho\,\Omega_i \qquad & &\Big(i = 1,2,\dots,d\,;\ \sum_i \Omega_i^2 = 1\Big) \end{aligned} \tag{1.3}$$

here $\rho \in [0,\infty)$ and $\tau \in [0,2\pi)$. Then, the metric will be

$$ds^2 = R^2(-cosh^2\rho\, d\tau^2 + d\rho^2 + sinh^2\rho\, d\Omega^2) \tag{1.4}$$

SL(2,R)[16] is the multiplicative group of 2×2 real matrices with determinant 1 i.e.,

$$M = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, ad - bc = 1\,; a,b,c,d \in R \right\} \tag{1.5}$$

It is a simple non-compact Lie group. Let g be the group element of SL(2,R). Then it is of the form

$$g = e^{iu\sigma_2} e^{\rho\sigma_3} e^{iv\sigma_3} \tag{1.6}$$

where σ^i $(i = 1,2,3)$ are Pauli matrices and

$$u = \frac{1}{2}(t + \phi) \qquad v = \frac{1}{2}(t - \phi)$$

here t is the time and ϕ is the space coordinate.

There is another useful parametrization of the group element g.

$$g = \begin{bmatrix} X_3 + X_1 & X_0 - X_2 \\ -X_0 - X_2 & X_3 - X_1 \end{bmatrix} \tag{1.7}$$

Since $\det(g) = 1$,

$$X_3^2 + X_0^2 - X_1^2 - X_2^2 = 1 \tag{1.8}$$

which is a 3-dimensional hyperboloid and it is the group manifold of SL(2,R) and it is also the 3-dimensional AdS space-time in Lorentzian signature in the embedding picture. We can also look this from the coset of the group. Anti-de Sitter

So, SL(2,R) WZW model can be used to describe a string theory on the 3-dim AdS space-time. Field extension from a 2-dimensional surface to a 3-dimensional manifold for the 3-dim hyperboloid is unique[9,10]. This gives no quantization conditions for the coupling of the Wess-Zumino term. Then, the action for SL(2,R) WZW model is given by

$$S = \frac{k}{8\pi\alpha}\int d^2\sigma\ Tr(g^{-1}\partial g g^{-1}\partial g) + \frac{k}{24\pi}\int d^3y\, \epsilon^{\alpha\beta\gamma} Tr(g^{-1}\partial_\alpha g g^{-1}\partial_\beta g g^{-1}\partial_\gamma g)$$

here g is the element of SL(2,R) and k is not quantized.

De Sitter space-time is also non-compact and is isomorphic to $\frac{SO(d,1)}{SO(d-1,1)}$ which cannot be reduced to any group for $d = 3$ case. Also, consistent string propagation requires the total central charge of all the conformal field theories to be zero - $c_{CFT} + (-26) = 0.$ Hence we need to look for the coset construction. Just sigma model will not do.

2. de Sitter Spacetime and Coset

In the embedding picture, d-dimensional de Sitter space-time is a d-dimensional timelike hyperboloid. The isometry group of de Sitter is SO(d,1).

$$-X_0^2 + X_1^2 + \cdots + X_{d-1}^2 + X_d^2 = l^2$$

On analytic continuation, we obtain a d-sphere with isometry group SO(d+1). This isometry group is the group of rotation. From differential geometry, d-dimensional sphere is a homogeneous space and[18]

$$\frac{SO(d+1)}{SO(d)} \cong S^d \tag{2.1}$$

Proof- Start with any point $p \in S^d$, then we can reach to the north pole n of this sphere with successive rotations along different axis depending on the starting point. Now fix the north pole and rotate the sphere around frozen axis through centre and north pole then the isotropy group is $SO(d)$. Isotropy group is defined as

$$H := \{s \in G: sn = n\}$$

Then, by applying the characterization theorem -

Let G be a Lie group, M be a homogeneous G-space and let p be any point on M. The isotropy group H is the closed subgroup of G, and the map

$$F: \mathrm{G/H} \to \mathrm{M}$$

$$g\mathrm{H} \to g.p$$

is a diffeomorphism.

Proof of the above theorem can be found in the references[18].

One gets,

$$F: \frac{SO(d+1)}{SO(d)} \to S^d$$

is a diffeomorphism. Hence, proved.

In the Lorentzian signature, the de Sitter space-time time has isometry group $SO(d,1)$ which is the group of rotation and boost which leaves the time-like hyperbola invariant. de Sitter space-time is also isomorphic to the coset

$$\frac{SO(d,1)}{SO(d-1,1)} \cong dS_d \tag{2.2}$$

Proof- Let's start with any point p on dS_d. We want to reach at a point $n = (0, l, 0, \ldots, 0)$ on dS_d. $SO(d,1)$ is the group of rotations and boosts on d-dimensional time-like hyperbola. So, successive boost and rotations by the application of group elements can reach to point n. Note that the flow of time is not fixed, it can go forward as well as backward. Now, fixing the point n or freezing the axis passing through the center of the hyperboloid and point n. Then, the number of boost now possible is reduced by 1 and number of rotations is reduced by d-1. So, total number of boosts and rotations possible is

$$(d-1) \text{ boosts} + \left(\frac{d(d-1)}{2} - (d-1)\right) \text{ rotations} = \frac{d(d-1)}{2} \text{ generators}$$

It is the same number of boosts and rotations generators as in $SO(d-1,1)$. Hence, the isotropy group is $SO(d-1,1)$. Now applying the characterization theorem for the homogeneous spaces one get,

$$F : \frac{SO(d,1)}{SO(d-1,1)} \to dS_d$$

is a diffeomorphism and hence proved.

For $d = 3$, we get

$$\frac{SO(3,1)}{SO(2,1)} \cong dS_3$$

3. Virasoro Algebra of the Coset Space Model

Let G be the compact Lie group and g be the corresponding Lie algebra. The affine Kac-Moody algebra associated with g has generators J_n^a $(1 \leq a \leq \dim \mathrm{G},\ n \in \mathbb{Z})$ and the commutation relation[8]

$$[J_n^a, J_m^b] = i f^{abc} J_{n+m}^c + kn\delta^{ab}\delta_{n+m,0} \tag{3.1}$$

where the central element k is real and integral multiple of $\frac{1}{2}\psi^2$ in the highest weight representation and ψ is the highest root of g.

The representation of the Virasoro algebra can be obtained from Sugawara construction. Define the operator

$$\mathcal{L}_n^g = \frac{1}{2\left(k + \frac{c_\psi^g}{2}\right)} \sum_{i,m} :J_{n-m}^a J_m^a: \tag{3.2}$$

here c_ψ^g is the adjoint representation Casimir operator and :(…): is the normal ordering

$$:J_n^a J_{-n}^a: = J_{-n}^a J_n^a$$

Then, the commutation relation

$$\left[\mathcal{L}_n^g, J_m^a\right] = -mJ_{n+m}^a \tag{3.3}$$

c_ψ^g is related to structure constants as

$$\sum_{c,b=1}^{\dim g} f^{abc} f^{dbc} = c_\psi^g \delta_{ad} \tag{3.4}$$

The commutation relation for the Virasoro algebra is[8,17]

$$\left[\mathcal{L}_n^g, \mathcal{L}_m^g\right] = (n-m)\mathcal{L}_{n+m}^g + \frac{c}{12}(n-1)n(n+1)\delta_{n+m,0} \tag{3.5}$$

Central charge c of the Virasoro algebra is related to the central charge k of the Kac-Moody algebra as

$$c = \frac{2k(\dim \mathrm{g})}{c_\psi^g + 2k} \tag{3.6}$$

Now let H be the subgroup of G with algebra h. A left coset G/H is defined to be the set of elements of G such that

$$\frac{\mathrm{G}}{\mathrm{H}} = \{r \in \mathrm{H} | sr \in \mathrm{H}, \forall s \in \mathrm{G}\} \tag{3.7}$$

Choose a basis of the generators of g such that the first dim h constitute the basis for the generators of h. Then, defining

$$\mathcal{L}_n^h = \frac{1}{2(c_\psi^h/2 + k)} \sum_m \sum_{a=1}^{\dim h} :J_{n-m}^a J_m^a: \tag{3.8}$$

c_ψ^h is the adjoint representation Casimir operator of h and it is related to structure constants of h as

$$\sum_{c,b=1}^{\dim h} f^{abc} f^{dbc} = c_\psi^h \delta_{ad} \tag{3.9}$$

Expression similar to eq. (2.6) are obtained by restricting a, b, c and d between 1 and dim h.

$$[\mathcal{L}_n^h, J_m^a] = -m J_{n+m}^a \qquad (1 < a < \dim h) \tag{3.10}$$

Commutation relations for the coset G/H.

$$[\mathcal{L}_n^g - \mathcal{L}_n^h, J_m^a] = 0 \qquad (1 < a < \dim h) \tag{3.11}$$

obtained by using eq.(2.10). This means

$$\begin{aligned} &[\mathcal{L}_n^g - \mathcal{L}_n^h, \mathcal{L}_m^h] = 0 \\ &[\mathcal{L}_n^g - \mathcal{L}_n^h, \mathcal{L}_m^g - \mathcal{L}_m^h] = [\mathcal{L}_n^g, \mathcal{L}_m^g] - [\mathcal{L}_n^h, \mathcal{L}_m^h] \\ &= (n-m)(\mathcal{L}_{n+m}^g - \mathcal{L}_{n+m}^h) + \frac{c_g - c_h}{12}(n-1)n(n+1)\delta_{n+m,0} \end{aligned} \tag{3.12}$$

$\mathcal{L}_n^g, \mathcal{L}_n^h$ satisfies the Virasoro algebra, so does their difference. Let

$$K_n = \mathcal{L}_n^g - \mathcal{L}_n^h \tag{3.13}$$

These are the Virasoro generators for the coset G/H. The central charge will be

$$c = \frac{2k(\dim \mathrm{g})}{c_\psi^g + 2k} - \frac{2k(\dim \mathrm{h})}{c_\psi^h + 2k} \tag{3.14}$$

c must be non-negative. If h is not simple but $\mathrm{h} = \mathrm{h}_1 \oplus \mathrm{h}_2$ then the central charge will be

$$c = \frac{2k(\dim \mathrm{g})}{c_\psi^g + 2k} - \frac{2k(\dim \mathrm{h}_1)}{c_\psi^{h_1} + 2k} - \frac{2k(\dim \mathrm{h}_2)}{c_\psi^{h_2} + 2k} \tag{3.15}$$

Unitarity condition imposes the constraint

$$J_{-m}^a = J_m^{a\,\dagger} \tag{3.16}$$

Above analysis is carried out assuming compact Lie group. For any non-compact Lie group quotient with any closed subgroup, it may be possible that the spectrum of the coset CFT may not be unitary. It is also possible that the subgroup may introduce anomalies in the resultant theories. For some of the theories, it is possible to have unitary representaions (e.g. SL(2,C)/SU(2) mod R coset model)[20].

More work needs to be done towards showing that the resultant coset CFT for the de Sitter space-time have unitary representations of the algebras.

Conclusion

This bring to the end of the report. Cosmological observations seems more promising towards the detection of phenomena which is solely due to the quantum gravity. On the other hand, we have a string theory which is a theory of quantum gravity but with no experiment to verify. De Sitter space-time acts as a bridge with closes the gap a little bit. So, studying de Sitter space-time is the good candidate for the pedagogical study.

In the first chapter, we learn to visualize the de Sitter space-time as an embedding. We looked at global coordinates which cover whole space-time. We also learnt that the spatial part of the de Sitter metric is compact. So, there are no spatial infinities. However there are asymptotes along time direction at infinite past and infinite future. So, the standard gauge/gravity correspondence cannot simply be taken to de Sitter space-time. To introduce the horizon in the space-time, we looked at flat slicing and static coordinates. Flat slicing cover only half of the space-time. By comparing the global coordinate system with flat slicing coordinates, we obtained the trajectories of constant space and time coordinate of the flat slicing coordinate system. Static patch metric turns out be the metric with no explicit time dependence. It covers only one fourth of the Penrose diagram. It is the spacetime that is accessible to a single observer sitting at the north pole. Horizon of the space-time is found out to be at r=l. We also compared static and flat slicing coordinates and found out horizon with flat slicing coordinates. We also looked at and the isometry group and the Euclidean de Sitter spacetime. This introduced us to the basic notions of de Sitter space-time.

To understand the formalism of the string theory, we looked at the conformal group and conformal field theory in d dimensions. We started with the infinitesimal transformations for CFT with dimensions greater than 2 and found out finite number of generators for the conformal group and its algebra. However, when we worked with infinitesimal transformations for 2-dimensional CFTs, we obtained infinitely many generators. Along the way of deriving the Virasoro algebra for the theory, we also acquired mathematical tools like operator product expansion, mode expansion and radial quantization.

We also looked at the WZW model with Lie algebraic symmetry for which the affine Lie algebra is the spectrum generating algebra. We learnt that for compact WZW model, there is no unique field extensions and this caused the ambiguity in the Wess Zumino term. This further led to the quantization of the coupling. However, later we learnt that for non-compact WZW model, Wess-Zumino term do not acquire any ambiguity and coupling is not quantized. We also derived the affine Lie algebra for the model.

Since the generators of the Virasoro algebra turned out to be energy eigenstates, the representation for the Virasoro algebra has to be unitary for a theory describing the physical system. To get the foundation for constructing the boson string theory in de Sitter space-time, we explored SL(2,R) WZW model only to get sense of how anti-de Sitter spacetime is isomorphic to the group manifold of SL(2,R). However, it turned out that de Sitter space is isomorphic to the coset which do not reduce to any smaller subgroup for three dimensions. Lastly, we looked at the Virasoro algebra for the coset space model for the compact Lie Group G. More work needs to be done in showing that the coset CFT for de Sitter space-time has unitary representation for the algebras.

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