Unraveling the Physics of Quantum Hall Edge States

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by

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Certificate

This is to certify that this dissertation entitled **Unraveling the Physics of Quantum Hall Edge States** towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Sandeep Joy at Indian Institute of Science Education and Research under the supervision of Sreejith G J, Assistant Professor, Department of Physics, during the academic year 2018-2019.

Sreejith G J

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Sreejith G J
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I dedicate my thesis to Indira and Joy

Declaration

I hereby declare that the matter embodied in the report entitled **Unraveling the Physics of Quantum Hall Edge States** are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Sreejith G J and the same has not been submitted elsewhere for any other degree.

Sandeep Joy

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Abstract

The primary goal of this thesis is to understand the edge states of quantum Hall systems. In the initial part of the thesis, we explore the integer quantum Hall edge states and bosonize them. The fact that IQHE is a non-interacting phenomenon makes it easier to connect the microscopic picture with the effective field theory. After that, we looked at the edge states of fractional quantum Hall system from the framework of Chern-Simons theory. There is also a microscopic understanding of edge excitation in FQHE. Connecting these two descriptions of the edge is challenging. We gained some insight in this direction by studying both pictures. In the final section, we studied the robustness of these edge states using the null vector criterion and an equivalent of the same. Several example systems were explored in order to come up with simple models of quantum Hall edge interfaces that are numerically tractable.

Contents

Chapter 1

Introduction

The discovery of the fractional quantum Hall effect changed the then existing perspective on phases of matter. Strongly interacting two-dimensional electron liquid in FQHE cannot be characterized by a local order parameter. Landau's idea of order parameter to describe phases of the matter completely breaks down in this scenario. They possess certain kind of global ordering characteristics known as topological order. There are many more theoretical models where topological order is predicted to exist, such as quantum spin liquids. However, the fact that FQHE is so robust that it can be manipulated by the experimentalist stirred a lot of interest and progress in the field.

FQHE have a finite energy gap in the bulk and gapless excitations along the boundary. These gapless excitations are bosonic modes which can be modeled as Luttinger liquids. The Lagrangian for the edge can be understood as arising from gauge invariance of the Chern Simons Lagrangian which describes the bulk of the quantum Hall system. These bosonic states can be related to the fermionic field through bosonization relations. The resulting fermionic correlation functions lead to non-trivial quantization of IV characteristics in tunneling experiments.

There have been creative proposals for topological quantum computing based on the quantum Hall edge-superconductor interfaces. The current enthusiasm is driven by the actual realization of such interfaces in graphene FQH experiments. Coupled edge states are immune to the localization and gap opening as long as there is no coupling or tunneling between states with opposite chirality. But it is essential to find suitable candidates which are stable even in the presence of backscattering as it is closer to real systems. This motivates us to pursue a thorough investigation of coupled quantum Hall edges states.

We begin this thesis by reviewing the main ideas of the constructive bosonization approach. We next move to look at the IQH edge states. We bosonized the same following different approximation schemes. Bosonizing and understanding the coupled IQH edge states give us an idea on how to proceed with the FQH edge states. Unfortunately, in FQHE there is no single particle picture; hence motivating the constructive bosonization is exigent. Instead, we begin by reviewing the Chern-Simons picture of bulk phase and derive the boundary modes. The vertex operator defined using these bosonic modes can give rise to fermionic degrees of freedom. Analysis of the stability of coupled edge states using fermionic picture is presented. The algebraic properties of K matrix describing the bulk phase carries information regarding the possibility of gapping out the spectrum of edge states.

Chapter 2

Bosonization

Paradigmatic idea by Landau, the so-called Landau-Fermi liquid theory fails to capture the physics of a one-dimensional interacting fermionic system. The constrained spatial dimension makes interacting one-dimensional systems highly correlated, compared to its counterparts in higher dimensions [10]. The perturbation theory is incapable of elucidating the physics there. However, we can solve this interacting problem with the help of bosonization technique. The interacting fermion problem which is quartic in terms of fermions is converted to a quadratic problem in the language of bosons under this mapping, which will enable us to exactly calculate the spectrum. The resulting theory is known as Tomonaga-Luttinger Liquid theory [11].

The technique of bosonization involves the identification of the Hilbert space of fermions with bosons. This will help us establish an operator level identity between them. This tool will help us along the way as The edge modes of two-dimensional quantum Hall systems is one-dimensional. Hence we make an extensive review of the subject here.

2.1 Fermion Field

We assume that our system can be described in terms of a set of fermionic annihilation and creation operators which satisfy the given canonical anti-commutation rules.

$$\left\{c_k^\dagger, c_{k'}^\dagger\right\} = \{c_k, c_{k'}\} = 0 \quad \& \quad \left\{c_k, c_{k'}^\dagger\right\} = \delta_{k.k'} \tag{2.1}$$

The allowed momenta are quantized and take the following values due to the periodic boundary condition of the fermion fields.

$$k = \frac{2\pi}{L} \times n_k, \qquad n_k \in \mathbb{Z} \tag{2.2}$$

The allowed momenta are not bounded from below. The unbounded nature of momenta will be important when we define the bosonic operators in a later section. Having already defined the fermion operators in momentum basis, we can now define the fermion fields in the position basis as follows:

$$\psi(x) = \sqrt{\frac{2\pi}{L}} \sum_{k=-\infty}^{\infty} e^{-ikx} c_k \tag{2.3}$$

$$\psi^\dagger(x) = \sqrt{\frac{2\pi}{L}} \sum_{k=-\infty}^{\infty} e^{ikx} c_k^\dagger \tag{2.4}$$

The canonical anti-commutation of the above field operator can be shown as;

$$\{\psi(x), \psi(x')\} = \left\{\psi^\dagger(x), \psi^\dagger(x')\right\} = 0 \tag{2.5}$$

$$\left\{\psi(x), \psi^\dagger(x')\right\} = 2\pi \sum_{n\in\mathcal{Z}} \delta\left(x - x' - nL\right) \tag{2.6}$$

2.2 Vacuum State, Normal Ordering and Number Operator

The vacuum state of the above fermion has the following properties,

$$\begin{aligned} c_k|0\rangle_0 &= 0 \quad \text{for} \quad k > 0 \text{ (i.e. } n_k > 0) \\ c_k^\dagger|0\rangle_0 &= 0 \quad \text{for} \quad k \leq 0 \text{ (i.e. } n_k \leq 0) \end{aligned} \tag{2.7}$$

Now we define a formal and very useful procedure in field theory, the normal ordering. Given a product of c_k and $c_k^\dagger$,the fermion normal ordering is defined as, all c_k with ≥ 0 and all $c_k^\dagger$ with $k < 0$ is moved to the right of all operators so that;

$$: ABC... := ABC... - {}_0\langle 0|ABC...|0\rangle_0 \quad \forall\, A, B, C, ... \in \{c_k; c_k^\dagger\} \tag{2.8}$$

Let $\hat{N}$ be the operator that counts the number of occupied states with respect to $|0\rangle_0$;

$$\hat{N} = \sum_{k=-\infty}^{\infty} : c_k^\dagger c_k := \sum_{k=-\infty}^{\infty} \left[c_k^\dagger c_k - {}_0\langle 0| c_k^\dagger c_k |0\rangle_0 \right] \tag{2.9}$$

N particle Hilbert space will be the set of states with the same $\hat{N}$ eigenvalues N. Different states in this set is marked by different configurations of particle-hole excitations, any of which will be represented by $|N\rangle$. Moreover, for given N, let $|N\rangle_0$ be the state with no particle-hole excitations.

2.3 Density Operator

Let us calculate the electron density $: \psi^\dagger(x)\psi(x) :$,

$$\begin{aligned} : \psi^\dagger(x)\psi(x) : &= \frac{2\pi}{L} \sum_{k,k'=-\infty}^{\infty} e^{i(k-k')x} : c_k^\dagger c_{k'} : \\ &= \frac{2\pi}{L} \sum_{q,k=-\infty}^{\infty} e^{iqx} : c_{k+q}^\dagger c_k : \\ &= \frac{2\pi}{L} \sum_{q\neq 0} e^{iqx} \sum_k c_{k+q}^\dagger c_k + \frac{2\pi}{L} \sum_k : c_k^\dagger c_k : \\ &= \frac{2\pi}{L} \sum_{q\neq 0} e^{iqx} \rho(q) + \frac{2\pi}{L} \hat{N} \end{aligned} \tag{2.10}$$

In the penultimate step, we removed the normal-ordering of the first term, because it is redundant when two fermion operators have different k indices. We have defined an important linear combination of operators that creates particle-hole like excitations in the last step,

$$\rho(q) \equiv \sum_k c_{k+q}^\dagger c_k \tag{2.11}$$

Lets calculate its commutators

$$\begin{aligned} [\rho(p), \rho(q)] &= \sum_k \left[c_{k+p}^\dagger c_k, c_{k'+q}^\dagger c_{k'} \right] \\ &= \sum_{k'} \left(c_{k'+p+q}^\dagger c_{k'} - c_{k'+q}^\dagger c_{k'-p} \right) \end{aligned} \tag{2.12}$$

If $p \neq -q$,

$$[\rho(p), \rho(q)] = \rho(p+q) - \rho(p+q) = 0 \tag{2.13}$$

However if $p = -q$,

$$[\rho(p), \rho(q)] = \sum_{k'} \left(: c^\dagger_{k'} c_{k'} : + \left\langle c^\dagger_{k'} c_{k'} \right\rangle_0 - : c^\dagger_{k'+q} c_{k'+q} : - \left\langle c^\dagger_{k'-p} c_{k'-p} \right\rangle_0 \right) \tag{2.14}$$

We can now make the shift $k' - p \to k$ within the normal ordering sign because it will not introduce infinite quantities. Once the two terms cancel, we are left with

$$[\rho(p), \rho(-p)] = \sum_{k'} \left(\left\langle c^\dagger_{k'} c_{k'} \right\rangle_0 - \left\langle c^\dagger_{k'-p} c_{k'-p} \right\rangle_0 \right) = -n_p \tag{2.15}$$

Thus,

$$[\rho(p), \rho(q)] = -n_p \delta_{p,-q} \tag{2.16}$$

The set of k being unbounded from below was a sufficient criterion for the above derivation. Now let us define some operators so that this looks exactly like a bosonic commutation relation.

$$\begin{aligned} b_p &\equiv \frac{-i}{\sqrt{n_p}} \rho(-p) = \frac{-i}{\sqrt{n_p}} \sum_k c^\dagger_{k-p} c_k \qquad (p > 0) \\ b^\dagger_p &\equiv \frac{i}{\sqrt{n_p}} \rho(p) \quad = \frac{i}{\sqrt{n_p}} \sum_k c^\dagger_{k+p} c_k \qquad (p > 0) \end{aligned} \tag{2.17}$$

such that,

$$[b_p, b_q] = 0 = \left[b^\dagger_p, b^\dagger_q\right] \quad \& \quad \left[b_p, b^\dagger_q\right] = \delta_{p,q} \tag{2.18}$$

then

$$\begin{aligned} : \psi^\dagger(x) \psi(x) : &= \frac{2\pi}{L} \sum_{q>0} \left(e^{iqx} \rho(q) + e^{-iqx} \rho(-q) \right) + \frac{2\pi}{L} \hat{N} \\ &= \frac{2\pi}{L} \sum_{p>0} i\sqrt{n_p} \left[e^{-ipx} b_p - e^{ipx} b^\dagger_p \right] \end{aligned} \tag{2.19}$$

We can also show the following,

$$\left[b_p, \hat{N}\right] = 0 \quad \& \quad \left[b^\dagger_p, \hat{N}\right] = 0 \quad \forall\, p \tag{2.20}$$

which implies that $b_p|N\rangle_0 = 0$. This means that, there are no particle-hole like excitations in the N particle ground state. Any particle-hole excited state within an N particle Hilbert space can be constructed by acting an arbitrary linear combination of the $b_p^\dagger$ represented by $f\left[\left\{b_k^\dagger\right\}\right]$ on $|N\rangle_0$ [12].

2.4 Klein Factor

Klein factor, $\hat{F}$, is the ladder operator which connects different fermion particle number states. This is an important ingredient in bosonization, since no combination of the bosonic operators can perform this task. Klein factors also make sure that different fermionic species will anticommute if there are multiple species of them. It can be defined using its action on an arbitrary state and its commutators.

$$\begin{aligned} \hat{F}^\dagger|N\rangle \equiv \hat{F}^\dagger f\left[\left\{b_k^\dagger\right\}\right]|N\rangle_0 &\equiv f\left[\left\{b_k^\dagger\right\}\right]c_{N+1}^\dagger|N\rangle_0 = f\left[\left\{b_k^\dagger\right\}\right]|N+1\rangle_0 \\ \hat{F}|N\rangle \equiv \hat{F} f\left[\left\{b_k^\dagger\right\}\right]|N\rangle_0 &\equiv f\left[\left\{b_k^\dagger\right\}\right]c_{N}|N\rangle_0 \quad = f\left[\left\{b_k^\dagger\right\}\right]|N-1\rangle_0 \end{aligned} \tag{2.21}$$

$$\left[b_q, F^\dagger\right] = \left[b_q^\dagger, F^\dagger\right] = \left[b_q, F\right] = \left[b_q^\dagger, F^\dagger\right] = 0 \quad \forall \quad q \tag{2.22}$$

$\hat{F}^\dagger$ commute through $f\left[\left\{b_k^\dagger\right\}\right]$, and adds an electron to the lowest unoccupied level. This results in a new ground state, to be precise, $c_{N+1}^\dagger|N\rangle_0$. The particle-hole excitation acts on this state. Thus $\hat{F}^\dagger|N\rangle$ has same set of particle hole excitations with a different ground state having an extra electron. As a consequence of the definition of the Klein factor, we can derive the following commutation relations,

$$\begin{aligned} \left\{F, F^\dagger\right\} &= 2\mathbb{I} \\ \left[F, \hat{\mathbb{N}}\right] &= F \\ \left[F^\dagger, \hat{\mathbb{N}}\right] &= -F^\dagger \end{aligned} \tag{2.23}$$

2.5 Fermionic Operator in terms of Bosonic Operator

Let us calculate the commutator between ψ and the bosonic operator.

$$\begin{aligned} [b_p, \psi(x)] &= \frac{i}{\sqrt{n_p}} e^{ipx} \psi(x) \\ \left[b_p^\dagger, \psi(x)\right] &= \frac{-i}{\sqrt{n_p}} e^{-ipx} \psi(x) \end{aligned} \tag{2.24}$$

Define $\alpha_q(x) \equiv \frac{i}{\sqrt{n_p}} e^{ipx}$. Now let us look at the action of $[b_p, \psi(x)]$ on N particle ground state $|N\rangle_0$.

$$[b_p, \psi(x)] \, |N\rangle_0 = b_p \psi(x) |N\rangle_0 = \alpha_p(x) \psi |N\rangle_0 \tag{2.25}$$

That is, $\psi|N\rangle_0$ is an eigenstate of b_p with eigenvalue $\alpha_p(x) \; \forall \; p > 0$, hence it has a coherent-state representation. Since $\psi(x)|N\rangle_0 \propto |N-1\rangle_0$, it must be that

$$\psi(x)|N\rangle_0 = e^{\left[\sum_p \alpha_p(x) b_k^\dagger\right]} \hat{F} \hat{\Lambda}(x) |N\rangle_0 \tag{2.26}$$

where $\hat{F}$ is the Klein factor used to increase or decrease the fermion number as bosonic operator can't do this. $\hat{\Lambda}$ is a phase counting operator which can be calculated to be $\sqrt{\frac{2\pi}{L} e^{-i\frac{2\pi N x}{L}}}$. If we want to find action of ψ on a general state in terms of the bosonic operators we have to invoke some results which we will prove in the appendix. We know that any of the possible particle-hole excitation can be obtained by acting $f\left[\left\{b_k^\dagger\right\}\right]$ on $|N\rangle_0$.

$$|N\rangle = f\left[\left\{b_k^\dagger\right\}\right] |N\rangle_0 \tag{2.27}$$

So,

$$\psi_0(x)|N\rangle = \psi_0(x) f\left[\left\{b_k^\dagger\right\}\right] |N\rangle_0 \tag{2.28}$$

Using 2.25 we can write

$$\begin{aligned} \psi(x) b_k^\dagger &= \left[b_k^\dagger - \alpha_k^*(x)\right] \psi(x) \\ \Rightarrow \psi(x) \left[b_k^\dagger\right]^n &= \left[b_k^\dagger - \alpha_k^*(x)\right]^n \psi(x) \\ \Rightarrow \psi(x) f\left[\left\{b_k^\dagger\right\}\right] &= f\left[\left\{b_k^\dagger - \alpha_k^*(x)\right\}\right] \psi(x) \end{aligned} \tag{2.29}$$

So,

$$\begin{aligned}
\psi(x)|N\rangle &= f\left\{\left[b_k^\dagger - \alpha_k^*(x)\right]\right\}\psi(x)|N\rangle_0 \\
&= f\left\{\left[b_k^\dagger - \alpha_k^*(x)\right]\right\}e^{\left[\sum_p \alpha_p(x) b_k^\dagger\right]}|N\rangle_0 \\
&= \hat{F}\hat{\Lambda}(x)e^{\left[\sum_k \alpha_k(x) b_k^\dagger\right]} f\left\{\left[b_k^\dagger - \alpha_k^*(x)\right]\right\}|N\rangle_0
\end{aligned} \tag{2.30}$$

Making use of the following identity

$$e^{\left[-\sum_k \alpha_k^*(x) b_k^\dagger\right]} b_k^\dagger e^{\left[\sum_k \alpha_k^*(x) b_k^\dagger\right]} = b_k^\dagger - \alpha_k^*(x) \tag{2.31}$$

We get

$$\Rightarrow e^{\left[-\sum_k \alpha_k^*(x) b_k\right]} f\left[\left\{b_k^\dagger\right\}\right] e^{\left[\sum_k \alpha_k^*(x) b_k\right]} = f\left[\left\{b_k^\dagger - \alpha_k^*(x)\right\}\right] \tag{2.32}$$

Finally,

$$\begin{aligned}
\psi(x)|N\rangle &= \hat{F}\hat{\Lambda}(x)e^{\left[\sum_k \alpha_k(x) b_k^\dagger\right]} f\left\{\left[b_k^\dagger - \alpha_k^*(x)\right]\right\}|N\rangle_0 \\
&= \hat{F}\hat{\Lambda}(x)e^{\left[\sum_k \alpha_k(x) b_k^\dagger\right]} e^{\left[-\sum_k \alpha_k^*(x) b_k\right]} f\left[\left\{b_k^\dagger\right\}\right] e^{\left[\sum_k \alpha_k^*(x) b_k\right]}|N\rangle_0 \\
&= \hat{F}\hat{\Lambda}(x)e^{\left[\sum_k \alpha_k(x) b_k^\dagger\right]} e^{\left[-\sum_k \alpha_k^*(x) b_k\right]}|N\rangle
\end{aligned} \tag{2.33}$$

This is the bosonization formula, where fermionic annihilation operator is written in terms of the bosons, $\hat{F}$ and $\hat{N}$.

2.6 Bosonic Field Operators

Let us define some useful bosonic fields,

$$\varphi(x) \equiv -i\sum_{p>0} \alpha_p^*(x) \times e^{-\frac{ap}{2}} b_p = -\sum_{p>0} \frac{e^{-ipx}}{\sqrt{n_p}} \times e^{-\frac{ap}{2}} \times b_p \tag{2.34}$$

$$\varphi^\dagger(x) \equiv i\sum_{p>0} \alpha_p(x) \times e^{-\frac{ap}{2}} b_p^\dagger = -\sum_{p>0} \frac{e^{ipx}}{\sqrt{n_p}} \times e^{-\frac{ap}{2}} \times b_p^\dagger \tag{2.35}$$

and their hermitian combination,

$$\Phi \equiv \varphi(x) + \varphi^\dagger(x) = -\sum_{p>0} \frac{e^{-\frac{ap}{2}}}{\sqrt{n_p}}\left[e^{-ipx} \times b_p + e^{ipx} \times b_p^\dagger\right] \tag{2.36}$$

The unnormal ordered product that might be appearing in the theory is regularised by introducing the infinitesimal parameter $a > 0$. We can think of $\frac{1}{a}$ as a cut-off for momenta differences that occurs in Φ. We can observe that bosonic fields are periodic in x. Let us calculate the derivative of Φ,

$$\begin{aligned}\partial_x \Phi &= -\sum_{p>0} \frac{e^{-\frac{ap}{2}}}{\sqrt{n_p}} \left[-ipe^{-ipx} \times b_p + ipe^{ipx} \times b_p^\dagger\right] \\ &= \frac{2\pi}{L} \sum_{p>0} e^{-\frac{ap}{2}} i\sqrt{n_p} \left[e^{-ipx} \times b_p - e^{ipx} \times b_p^\dagger\right] \\ &= \rho(x) - \frac{2\pi}{L}\hat{N} \quad (\text{for} \quad a \to 0)\end{aligned} \tag{2.37}$$

Frequently the following commutation relations will be required. I will be just stating them here as it can be derived easily.

$$[\varphi(x), \varphi(x')] = \left[\varphi^\dagger(x), \varphi^\dagger(x')\right] = 0 \tag{2.38}$$

$$\begin{aligned}\left[\varphi(x), \varphi^\dagger(x')\right] &= -ln\left(1 - e^{-i\frac{2\pi}{L}((x-x')-ia)}\right) \\ &\xrightarrow{L\to\infty} -ln\left(i\frac{2\pi}{L_x}((x-x') - ia\right)\end{aligned} \tag{2.39}$$

$$\begin{aligned}[\Phi(x), \Phi(x')] &\xrightarrow{L\to\infty} -2i\left[\arctan\left(\frac{x-x'}{a}\right) - \frac{\pi}{L}(x-x')\right] \\ &\xrightarrow{L=\infty, a\to 0} -i\pi\epsilon(x-x')\end{aligned} \tag{2.40}$$

$$\begin{aligned}[\Phi(x), \partial_{x'}\Phi(x')] &\xrightarrow{L\to\infty} 2\pi i\left[\frac{\frac{a}{\pi}}{(x-x')^2 + a^2} - \frac{1}{L}\right] \\ &\xrightarrow{L=\infty, a\to 0} 2\pi i\left[\delta(x-x') - \frac{1}{L}\right]\end{aligned} \tag{2.41}$$

Using the above field operators we can rewrite the bosonization identity

$$\psi = \hat{F}\sqrt{\frac{2\pi}{L}} e^{-i\frac{2\pi N x}{L}} e^{-i\varphi^\dagger(x)} e^{-i\varphi(x)} \tag{2.42}$$

With the help of Baker-Hausdorff formula we can rewrite the above expression as follows

$$\psi(x) = \frac{\hat{F}}{a} e^{-i\frac{2\pi\hat{N}x}{L_x}} \times e^{-i\Phi(x)} \tag{2.43}$$

The diverging factor makes it obvious that the above expression is not normal ordered.

2.7 Multi-species Klein Factors

So far we have discussed the bosonization of single species fermion. In order to extend this idea to many species we have to introduce the Klein factors for each of them. Let us consider the case with two kinds of fermions. To maintain the fermionic characteristic of the fermion fields, the action of Klein factors on a state has to defined carefully. If there are two species, the ground state $|N_1, N_2\rangle_0$, is a tensor product of $|N_1\rangle_0$ and $|N_2\rangle_0$ ground states.

$$\begin{aligned} |N_1, N_2\rangle_0 &\equiv |N_1\rangle_0 \otimes |N_2\rangle_0 \\ &= c^{1\dagger}_{N_1} c^{1\dagger}_{N_1-1} ... c^{1\dagger}_1 |0\rangle_0 \otimes c^{2\dagger}_{N_2} c^{2\dagger}_{N_2-1} ... c^{2\dagger}_1 |0\rangle_0 \end{aligned} \tag{2.44}$$

We have to define F_2 such that it can pick up the sign resulting from anti-commuting it past the first species of fermionic operators.

$$\begin{aligned} F_1 |N_1, N_2\rangle_0 &\equiv |N_1 - 1, N_2\rangle_0 \\ F_1^\dagger |N_1, N_2\rangle_0 &\equiv |N_1 + 1, N_2\rangle_0 \\ F_2 |N_1, N_2\rangle_0 &\equiv (-1)^{N_1} |N_1, N_2 - 1\rangle_0 \\ F_1^\dagger |N_1, N_2\rangle_0 &\equiv (-1)^{N_1} |N_1, N_2 + 1\rangle_0 \end{aligned} \tag{2.45}$$

If there are more than two species, say M, then we should define a particular ordering for the species

$$|N_1, N_2, ..., N_M\rangle_0 \equiv |N_1\rangle_0 \otimes |N_2\rangle_0 \otimes ... \otimes |N_M\rangle_0 \tag{2.46}$$

So that

$$\begin{aligned} F_\nu |N_1, N_2, ..., N_\nu, ..., N_M\rangle_0 &\equiv (-1)^{\sum_{i<\nu} \hat{\mathbb{N}}} |N_1, N_2, ..., N_\nu - 1, ..., N_M\rangle_0 \\ F_\nu^\dagger |N_1, N_2, ..., N_\nu, ..., N_M\rangle_0 &\equiv (-1)^{\sum_{i<\nu} \hat{\mathbb{N}}} |N_1, N_2, ..., N_\nu + 1, ..., N_M\rangle_0 \end{aligned} \tag{2.47}$$

So on an arbitrary state, action of Klein factor will be

$$\begin{aligned} F_\nu |\vec{N}\rangle &= F_\nu f\left[\left\{b_k^\dagger\right\}\right] |N_1, .., N_\nu, .., N_M\rangle_0 = f\left[\left\{b_k^\dagger\right\}\right] (-1)^{\sum_{i<\nu} \hat{\mathbb{N}}} |N_1, .., N_\nu - 1, .., N_M\rangle_0 \\ F_\nu^\dagger |\vec{N}\rangle &= F_\nu^\dagger f\left[\left\{b_k^\dagger\right\}\right] |N_1, .., N_\nu, .., N_M\rangle_0 = f\left[\left\{b_k^\dagger\right\}\right] (-1)^{\sum_{i<\nu} \hat{\mathbb{N}}} |N_1, .., N_\nu + 1, .., N_M\rangle_0 \end{aligned} \tag{2.48}$$

The phase factor keeps track of the number of signs picked up while acting with a fermion operator c_k^ν on $|\vec{N}\rangle_0$ to obtain a different $|\vec{N}'\rangle_0$. The Klein factors still commute with different b_q^ν's.

$$\left[b_q^\nu, F_{\nu'}^\dagger\right] = \left[b_q^{\nu+}, F_{\nu'}^\dagger\right] = \left[b_q^\nu, F_{\nu'}\right] = \left[b_q^{\nu+}, F_{\nu'}^\dagger\right] = 0 \qquad \forall \qquad \nu, \nu' \& q \tag{2.49}$$

ν is the species index. Different commutation relations between Klein factors and number operators are,

$$\begin{aligned}
&\left\{F_\nu, F_{\nu'}^\dagger\right\} = 2\delta_{\nu,\nu'} \\
&\{F_\nu, F_{\nu'}\} = \left\{F_\nu^\dagger, F_{\nu'}^\dagger\right\} = 0, \quad if \quad \nu \neq \nu' \\
&\left[F_\nu, \hat{N}_{\nu'}\right] = \delta_{\nu,\nu'} F_\nu \\
&\left[F_\nu^\dagger, \hat{N}_{\nu'}\right] = -\delta_{\nu,\nu'} F_\nu^\dagger
\end{aligned} \tag{2.50}$$

2.8 Notion of $\hat{F} = e^{-i\hat{\theta}}$

Since the spectrum of $\hat{N}$ is unbounded from above or below, $\hat{F}$ is unitary:$\hat{F}^{-1} = \hat{F}^\dagger$, hence we can write,

$$\hat{F} = e^{-i\hat{\theta}} \tag{2.51}$$

Since $\left[F, \hat{\mathbb{N}}\right] = F$,

$$\left[e^{-i\hat{\theta}}, \hat{\mathbb{N}}\right] \equiv e^{-i\hat{\theta}} \tag{2.52}$$

Using the identity; if $C \equiv [A, B]$, and $[A, C] = [B, C] = 0$,

$$\left[A, e^B\right] = Ce^B \tag{2.53}$$

Identifying $A = \hat{N}$, $B = -i\hat{\theta}$, $C = -1$, we get

$$\left[\hat{\mathbb{N}}, -i\hat{\theta}\right] = -1 \qquad \text{or} \qquad \left[\hat{\mathbb{N}}, i\hat{\theta}\right] = 1 \tag{2.54}$$

Thus the phase operator $\hat{\theta}$ is conjugate to $\hat{\mathbb{N}}$. We should remember that $\hat{\theta}_\nu$ is only defined up to modulo 2π. The only meaningful operator is $e^{i\hat{\theta}_\nu}$. Now we want to know how two $\hat{\theta}_\nu$

will commute.

$$
\begin{aligned}
\{F_\nu, F_{\nu'}\} &= \left\{e^{-i\hat{\theta}_\nu}, e^{-i\hat{\theta}_{\nu'}}\right\} \\
&= e^{-i\hat{\theta}_\nu} e^{-i\hat{\theta}_{\nu'}} + e^{-i\hat{\theta}_{\nu'}} e^{-i\hat{\theta}_\nu} \\
&= e^{-i\left(\hat{\theta}_\nu + \hat{\theta}_{\nu'}\right)} \left(e^{-\frac{1}{2}\left[\hat{\theta}_\nu, \hat{\theta}_{\nu'}\right]} + e^{-\frac{1}{2}\left[\hat{\theta}_{\nu'}, \hat{\theta}_\nu\right]}\right) \\
&= e^{-i\left(\hat{\theta}_\nu + \hat{\theta}_{\nu'}\right)} \left(e^{-\frac{1}{2}\left[\hat{\theta}_\nu, \hat{\theta}_{\nu'}\right]} + e^{\frac{1}{2}\left[\hat{\theta}_\nu, \hat{\theta}_{\nu'}\right]}\right)
\end{aligned}
\tag{2.55}
$$

If the above commutator has to be zero, the natural choices are

$$
\left[\hat{\theta}_\nu, \hat{\theta}_{\nu'}\right] = i\pi \tag{2.56}
$$

to make it consistent we will choose the following

$$
\left[\hat{\theta}_\nu, \hat{\theta}_{\nu'}\right] = i\pi \mathrm{sign}(\nu - \nu') \tag{2.57}
$$

We see that the ordering of the $|\vec{N}\rangle$ is ingrained into the commutation of $\hat{\theta}$ operators.

2.9 Klein Factor as Zero Mode

Let us include the Klein factor into the bosonic field,

$$
\begin{aligned}
\psi(x) &= \frac{\hat{F}}{a} e^{-i\frac{2\pi\hat{N}x}{L_x}} \times e^{-i\Phi(x)} \\
&= \frac{e^{-i\hat{\theta}}}{a} e^{-i\frac{2\pi\hat{N}x}{L_x}} \times e^{-i\Phi(x)} \\
&= \frac{1}{a} e^{-i\frac{2\pi x}{L}} e^{-i\left[\hat{\theta} + \frac{2\pi\hat{N}x}{L} + \Phi(x)\right]}
\end{aligned}
\tag{2.58}
$$

Let us define a new bosonic field as,

$$
\tilde{\Phi}(x) \equiv \hat{\theta} + \frac{2\pi\hat{N}x}{L} + \Phi(x) \tag{2.59}
$$

Now let us calculate its commutator

$$\begin{aligned}\left[\tilde{\Phi}_\mu(x), \tilde{\Phi}_\lambda(x')\right] &= \left[\hat{\theta}_\mu + \frac{2\pi\hat{N}_\mu x}{L} + \Phi_\mu(x), \hat{\theta}_\lambda + \frac{2\pi\hat{N}_\lambda x'}{L} + \Phi_\lambda(x')\right] \\ &= \left[\hat{\theta}_\mu, \hat{\theta}_\lambda\right] + \left[\Phi_\mu(x), \Phi_\lambda(x')\right] \\ &= i\pi\text{sign}(\nu - \lambda) - \delta_{\mu\lambda} i\pi\epsilon(x - x')\end{aligned} \tag{2.60}$$

2.10 Linear Combination of Bosonic Fields

In the future when we diagonalize a bosonic Hamiltonian we will have to create new linear combinations of them. We want to identify the conditions on the linear transformation such that vertex operators constructed out of them are fermions. Let us define,

$$\chi_i = \sum_j A_{ij} \tilde{\Phi}_j \tag{2.61}$$

To answer this let us calculate the commutators of the χ's.

$$\begin{aligned}\left[\chi_\mu(x), \chi_\nu(y)\right] &= \left[\sum_i A_{\mu i}\tilde{\Phi}_i, \sum_j A_{\nu j}\tilde{\Phi}_j\right] \\ &= \sum_{i,j} A_{\mu i} A_{\nu j} \left[\tilde{\Phi}_i, \tilde{\Phi}_j\right] \\ &= \sum_{i,j} A_{\mu i} A_{\nu j} \left[i\pi\text{sign}(\nu - \lambda) - \delta_{\mu\lambda} i\pi\epsilon(x - x')\right] \\ &= i\pi \left[\text{sign}(\nu - \lambda) \sum_{i \neq j} A_{\mu i} A_{\nu j} - \delta_{\mu\lambda} i\pi\epsilon(x - x') \sum_i A_{\mu i} A_{\nu i}\right]\end{aligned} \tag{2.62}$$

The criterion for χs to be valid bosonic field are

$$\begin{aligned}\sum_i A_{\mu i} A_{\nu i} &= \delta_{\mu,\lambda} \\ \sum_{i \neq j} A_{\mu i} A_{\nu j} sgn(i - j) &= sgn(\mu - \nu)\end{aligned} \tag{2.63}$$

2.11 Open Problems

The treatment of Klein factor as zero mode has to be done carefully. Multiplying Θ with arbitrary number will makes the Klein factor ill defined. Another point to remember is that if we want to refermionize a bosonic system we have introduce the Klein factor there diligently. These kind of details are problem specific . For example, if we have only one species of fermion and a particle number conserving Hamiltonian we can get rid of Klein factor easily without affecting the physics.

Chapter 3

Understanding the Edge States of IQHE

3.1 Landau Levels

3.1.1 Basics of the Landau Gauge

Hamiltonian of an electron moving in $x - y$ plane in the presence an external magnetic field $\vec{B} = B_0\hat{z}$ is given by

$$H_0 = \frac{1}{2m}\left(\vec{P} + e\vec{A}\right)^2 \tag{3.1}$$

We will work with the Landau gauge which is suitable for rectangular geometry,

$$\vec{A} = \begin{bmatrix} -B_0 y \\ 0 \end{bmatrix} \tag{3.2}$$

Substituting $\vec{A}$ in the equation 3.1 we get,

$$H_0 = \frac{1}{2m}\left(p_x - eB_0 y\right)^2 + \frac{1}{2m}p_y^2 \tag{3.3}$$

Since Hamiltonian is cyclic in x coordinate, we can seek for eigenfunctions which are also eigenstates of $\hat{p_x}$. These are nothing but plane waves in x direction. This inspire us to consider the following ansatz as eigenfunction using separation of variables,

$$\Psi_k(x, y) = e^{ikx}\psi_k(y) \tag{3.4}$$

When this Hamiltonian acts on the before mentioned wave function, we see that the operator p_x just get replaced by its eigenvalue $\hbar k$. The Hamiltonian thus becomes,

$$\begin{aligned} H_0\Psi_k(x,y) &= \left\{\frac{1}{2m}p_y^2 + \frac{1}{2m}\left(-i\hbar\partial_x - eB_0 y\right)^2\right\}\Psi_k(x,y) \\ &= \left\{\frac{1}{2m}p_y^2 + \frac{1}{2m}\left(\hbar k - eB_0 y\right)^2\right\}\Psi_k(x,y) \\ &= \left\{\frac{1}{2m}p_y^2 + \frac{1}{2}m\omega_c^2\left(y - y_0\right)^2\right\}\Psi_k(x,y) \\ &\equiv H_k\Psi_k(x,y) \end{aligned} \tag{3.5}$$

where $\omega_c \equiv \frac{eB_0}{m}$ is the cyclotron frequency, $l_B \equiv \sqrt{\frac{\hbar}{eB_0}}$ is the magnetic length and $y_0 \equiv l_B^2 k_x$. This Hamiltonian is just the harmonic oscillator with eigenstates,

$$\Psi_{n,k}(x,y) = \frac{1}{\sqrt{L_x}} \times e^{ikx} \times \frac{i^n}{\sqrt{2^n n!\sqrt{\pi}l_B}} e^{-\frac{(y-y_0)^2}{2l_B^2}} \times H_n\left(\frac{y-y_0}{l_B}\right) \tag{3.6}$$

with H_n being the Hermite polynomial. They depend on two quantum numbers $n \in \mathbb{N}$ and $k \in \mathbb{R}$. Note that the momentum in x direction, $\hbar k$, has turned into the position of the harmonic oscillator in y direction, which is now centered at $y_0 = kl_B^2$.

3.1.2 Lowest Landau Level (LLL) Approximation

Let us begin by defining the Fermionic field operator in the lowest Landau level,

$$\psi^{LLL}(x,y) = \sum_{k=-\infty}^{\infty} \Psi_{0,k}(x,y) c_k \tag{3.7}$$

The Fermionic operator $c_k^\dagger$ creates an electron in the lowest Landau level , localized around $y_0 = kl_B^2$ in the y direction . These operators obey the canonical anti-commutation relation,

$$\left\{c_k^\dagger, c_{,k'}^\dagger\right\} = \{c_k, c_{k'}\} = 0 \qquad \& \qquad \left\{c_k, c_{k'}^\dagger\right\} = \delta_{k.k'} \tag{3.8}$$

Now we pick a rectangle of dimension $L_x \times L_y$. Having a finite length in x direction is equivalent to the having periodic boundary condition. So $k$we are quantized as follows,

$$k = \frac{2\pi}{L_x} \times n_k, \qquad n_k \in \mathbb{Z} \tag{3.9}$$

For a finite sample restricted to $0 \leq y \leq L_y$ we would expect the allowed k values to range between $0 \leq k \leq \frac{L_y}{l_B^2}$. That is,

$$
\begin{aligned}
0 \leq \frac{2\pi}{L_x} \times n_k &\leq \frac{L_y}{l_B^2} \\
0 \leq n_k &\leq \frac{L_x \times L_y}{2\pi l_B^2} \equiv N_\phi
\end{aligned} \tag{3.10}
$$

In the fully filled Landau level configuration, all momentum eigenstates are occupied between $k = 0$ to $k = \frac{2\pi N_\phi}{L_x} = \frac{L_y}{l_B^2}$.

3.2 Fermi Liquid Theory of IQHE Edge States

In the presence of smooth confining potential, energy of the lowest Landau level momentum eigenstates are given by

$$\epsilon(k) = \frac{1}{2}\hbar\omega_c + V(x_k) \tag{3.11}$$

The bulk excitations have an energy cost of $\hbar\omega_c$. Edge excitation can be created by promoting an electron from lower to higher momenta state. Since $V(k + \Delta K) - V(k) \longrightarrow 0$ in the thermodynamic limit, edge excitations are gapless. In terms of $\epsilon(k)$ we can write the non-interacting Hamiltonian in second quantized picture as;

$$H = \sum_k \epsilon\,(k)\, c_k^\dagger c_k \tag{3.12}$$

This is the Hamiltonian of chiral Fermi liquid because all the low energy excitations propagates in the same direction which is decided by the slope of the confining potential.

$$v_F = \frac{\partial \epsilon(k)}{\partial k} \propto \nabla V \tag{3.13}$$

This agrees with classical notion of skipping orbitals.

3.3 Bosonization of Edge states in the LLL

3.3.1 Fermionic Field in the Thermodynamic Limit

In the thermodynamics limit $L_y \to \infty$, up to the leading non-trivial order, filled Landau levels can be approximated by a infinite Dirac sea as shown below. We will choose the edge at $y = L_y$.

$$\psi^{LLL}(x,y) = \sum_{k=-\infty}^{\infty} \frac{e^{i\frac{2\pi n_k}{L_x}x}}{\sqrt{L_x\sqrt{\pi}l_B}} \times e^{-\frac{\left(L_y - \frac{2\pi n_k}{L_x}\times l_B^2\right)^2}{2l_B^2}} c_k \tag{3.14}$$

Now we will change the summation index as $n_k = N_\phi + n_{k'}$. So $\psi^{LLL}(x,y)$ becomes

$$\begin{aligned}
\psi^{LLL}(x,y) &= \sum_{n_{k'}=-\infty}^{\infty} \frac{e^{i\frac{2\pi\times(N_\phi+n_{k'})}{L_x}x}}{\sqrt{L_x\sqrt{\pi}l_B}} e^{-\frac{\left(L_y - \frac{2\pi\times(N_\phi+n_{k'})}{L_x}\times l_B^2\right)^2}{2l_B^2}} c_{k'+k_{N_\phi}} \\
&= \sum_{n_{k'}=-\infty}^{\infty} \frac{e^{i\frac{2\pi\times(N_\phi+n_{k'})}{L_x}x}}{\sqrt{L_x\sqrt{\pi}l_B}} e^{-\frac{\left(\frac{2\pi n_{k'}}{L_x}\times l_B^2\right)^2}{2l_B^2}} c_{k'+k_{N_\phi}} \\
&= \frac{e^{i\frac{2\pi N_\phi}{L_x}x}}{\sqrt{L_x\sqrt{\pi}l_B}} \sum_{n_{k'}=-\infty}^{\infty} e^{i\frac{2\pi n_{k'}}{L_x}x} \times e^{-\frac{\left(\frac{2\pi n_{k'}}{L_x}\right)^2\times l_B^2}{2}} c_{k'+k_{N_\phi}}
\end{aligned} \tag{3.15}$$

Now we take the limit $L_y \to \infty$, i.e. $N_\phi \to \infty$. In this limit the Gaussian term will provide a smooth ultraviolet cutoff $|k| < \frac{1}{l_B}$. To leading order in l_B^2, we can remove this cutoff, thereby obtaining the effective chiral fermion field.

$$\psi_0(x) \equiv lim_{L_y\to\infty}\sqrt{2\pi\sqrt{\pi}l_B} \times \psi^{LLL}(x,y) = e^{ik_{N_\phi}x}\sqrt{\frac{2\pi}{L_x}}\sum_{k=-\infty}^{\infty} e^{ikx}\tilde{c}_k \tag{3.16}$$

and we will introduce the shifted operators $\tilde{c}_{k'} \equiv c_{k'+k_{N_\phi}}$. The operator $\tilde{c}_k$ annihilates a particle of momentum k from a state above the fully filled Landau Level. After taking both limit and shift we have got a Fermi sea, where momentum eigenstates are filled up to $k = 0$. There is no y in the argument of ψ_0 , since it is understood that we are interested in the edge. Also we add the normalization factor of $\frac{1}{\sqrt{2\pi\sqrt{\pi}l_B}}$ to the definition of the Hilbert space for ease of working. This whole approximation is valid until and unless we are interested in the states near the edge. We can completely forget what was there in the bulk and happily think about the edge with an optimistic eye.

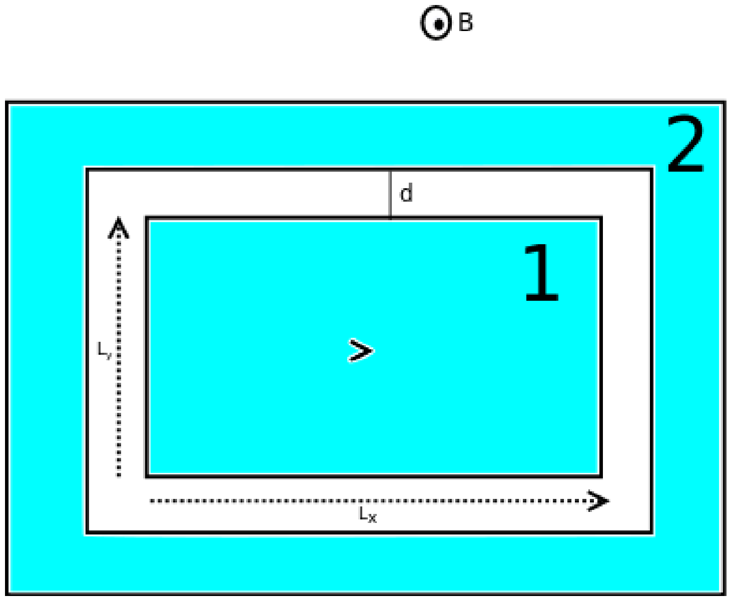

Figure 3.1: Geometry of coupled QHE system

3.4 Coupled Edge States

Now let us look at coupled integer quantum hall edge states. Here we will consider two interacting edges in the Corbino geometry as shown in the figure 3.1. The edge state of the coupled IQHE system are $y = L_{1y}$(denoted as ψ_1) and at $y = L_{1y} + d$ (denoted as ψ_2) respectively.

$$\psi_1(x) = \frac{e^{ik_{N_\phi}x}}{\sqrt{L_{1x}}} \sum_{k=-\infty}^{\infty} e^{ikx}\tilde{c}_k \tag{3.17}$$

where

$$k = \frac{2\pi}{L_{1x}} n_k \quad , \quad N_\phi \equiv \frac{L_{1x} \times L_{1y}}{2\pi l_B^2} \quad \& \quad k_{N_\phi} = \frac{2\pi}{L_{1x}} \times N_\phi = \frac{L_{1y}}{l_B^2} \tag{3.18}$$

and

$$\tilde{c}_k \equiv c_{k+k_{N_\phi}} \tag{3.19}$$

If we are interested in edge states at $y = L_{1y} + d$, then calculation for the second edge will be modified. Since

$$L_{1y} + d \leq k l_B^2 \leq L_{2y} + L_{1y} + d \quad \& \quad k = \frac{2\pi n_k}{L_{2x}} \tag{3.20}$$

$$\Rightarrow \frac{L_{2x}(L_{1y} + d)}{2\pi l_B^2} \leq n_k \leq \frac{L_{2x}(L_{2y} + L_{1y} + d)}{2\pi l_B^2} \tag{3.21}$$

with the definition $k_\Delta \equiv \frac{d}{l_B^2}$, we get

$$\widetilde{\psi}_2(x) = \frac{e^{i(k_{N_\phi}+k_\Delta)\times x}}{\sqrt{L_{2x}}} \sum_{k=-\infty}^{\infty} e^{i\frac{2\pi n_k}{L_{2x}}x} \tilde{c}_{k+k_\Delta} \tag{3.22}$$

In the second edge, momentum eigenstates for $k + k_\Delta \geq 0$ are filled, whereas for $k + k_\Delta < 0$ are empty in the ground state. This is reversed in the case of the first edge. So we define instead $k' \equiv -(k + k_\Delta)$. Then

$$\widetilde{\psi}_2(x) = \frac{e^{i(k_{N_\phi}+2k_\Delta)\times x}}{\sqrt{L_{2x}}} \sum_{k'=-\infty}^{\infty} e^{-ik'x} \tilde{c}_{-k'} \tag{3.23}$$

We define a second species of fermions such that

$$\check{c}_k \equiv \tilde{c}_{-k} \tag{3.24}$$

$$\widetilde{\psi}_2(x) = \frac{e^{i(k_{N_\phi}+2k_\Delta)\times x}}{\sqrt{L_{2x}}} \sum_{k=-\infty}^{\infty} e^{-ikx} \check{c}_k \tag{3.25}$$

For the convenience let us define the following,

$$\widetilde{\psi}_2(x) = e^{i2k_\Delta x} \times \frac{e^{i(k_{N_\phi}+2k_\Delta)\times x}}{\sqrt{L_{2x}}} \sum_{k=-\infty}^{\infty} e^{-ikx} \check{c}_k \equiv e^{i2k_\Delta x} \times \psi_2(x) \tag{3.26}$$

The relative phase differences becomes important when we discuss the interactions, hence I will not discuss the implication of this term here. The dispersion relation of these two edge modes are given by

$$\epsilon_1(k) = v_1 k \qquad \& \qquad \epsilon_2(k) = v_2 k \tag{3.27}$$

where v_1 and v_2 are the slope of the confining potential around at the Fermi level. We can have different slope for the two edge since the confining potential can have any shape. The first edge is right moving (slope of the potential is positive) whereas the second edge is left moving (slope of the potential is positive). Effectively now we have two interacting chiral edge states.

3.5 Bosonization Dictionary

Given below are the important formulas from bosonization dictionary for the two modes

$$
\begin{aligned}
\psi_{1,2}(x) &= e^{ik_{N_\phi}x}\sqrt{\frac{2\pi}{L_x}}\sum_{k=-\infty}^{\infty} e^{\pm ikx}c_k^{1,2} \\
\{\psi_{1,2}(x),\psi_{1,2}(x')\} &= \left\{\psi_{1,2}^\dagger(x),\psi_{1,2}^\dagger(x')\right\} = 0 \\
\left\{\psi_{1,2}(x),\psi_{1,2}^\dagger(x')\right\} &= \sum_{n\in\mathcal{Z}}\delta\left(x-x'-nL_x\right) \\
\rho_{1,2}(q) &= \sum_k c_{k+q}^{1,2\dagger}c_k^{1,2} \\
\rho_{1,2}(p) &= \begin{cases} -i\sqrt{n_p}b_p^{1,2+} & p>0 \\ i\sqrt{n_p}b_{-p}^{1,2} & p<0 \end{cases} \\
:\psi_{1,2}^\dagger(x)\psi_{1,2}(x): &= \frac{2\pi}{L_x}\sum_{q\neq 0}e^{\mp iqx}\rho(q) + \frac{2\pi}{L_x}\hat{N} \\
\varphi_{1,2}(x) &\equiv -i\sum_{p>0}\alpha_p^*(\mp x)\times e^{-\frac{ap}{2}}b_p^{1,2} = -\sum_{p>0}\frac{e^{\pm ipx}}{\sqrt{n_p}}\times e^{-\frac{ap}{2}}\times b_p^{1,2} \\
\varphi_1^\dagger(x) &\equiv i\sum_{p>0}\alpha_p(\mp x)\times e^{-\frac{ap}{2}}b_p^{1,2\dagger} = -\sum_{p>0}\frac{e^{\mp ipx}}{\sqrt{n_p}}\times e^{-\frac{ap}{2}}\times b_p^{1,2\dagger} \\
\Phi_{1,2} &\equiv \varphi_{1,2}(x)+\varphi_{1,2}^\dagger(x) = -\sum_{p>0}\frac{e^{-\frac{ap}{2}}}{\sqrt{n_p}}\left[e^{\pm ipx}\times b_p^{1,2}+e^{\mp ipx}\times b_p^{1,2\dagger}\right] \\
\psi_{1,2}(x) &= \hat{F}_{1,2}\sqrt{\frac{2\pi}{L}}e^{\pm i\frac{2\pi Nx}{L}}e^{\left[\sum_k\alpha_k(\mp x)b_k^\dagger\right]}e^{\left[-\sum_k\alpha_k^*(\mp x)b_k\right]} \\
\psi_{1,2}(x) &= \hat{F}_{1,2}\sqrt{\frac{2\pi}{L}}e^{\pm i\frac{2\pi Nx}{L}}e^{-i\varphi_{1,2}^\dagger(x)}e^{-i\varphi_{1,2}(x)}
\end{aligned}
$$

$$:\psi_{1,2}^\dagger(x)\psi_{1,2}(x): = \mp\partial_x\Phi_{1,2} + \frac{2\pi}{L}\hat{N}_{1,2}$$

The commutation relations of the bosonic modes are given by

$$
\begin{aligned}
\left[\varphi_{1,2}(x),\varphi_{1,2}(x')\right] &= \left[\varphi_{1,2}^\dagger(x),\varphi_{1,2}^\dagger(x')\right] = 0 \\
\left[\varphi_{1,2}(x),\varphi_{1,2}^\dagger(x')\right] &= -ln\left(1-e^{-i\frac{2\pi}{L}(\mp(x-x')-ia)}\right)
\end{aligned}
$$

$$[\Phi_{1,2}(x), \Phi_{1,2}(x')] \approx -2\dot{i}\left[\mp\text{arctan}\left(\frac{x-x'}{a}\right) \pm \frac{\pi}{L}(x-x')\right]$$
$$[\Phi_1(x), \partial_{x'}\Phi_1(x')] \approx 2\pi\dot{i}\left[\mp\delta(x-x') \pm \frac{1}{L}\right]$$

3.6 Bosonized Hamiltonain

Now let us look at the bosonized version of the above linear dispersion Hamiltonian.

$$H_{KE} = \sum_\nu \sum_{k=-\infty}^{\infty} v_\nu k : c_k^{\nu+} c_k^\nu := \sum_\nu v_\nu \int_0^{L_x} \frac{dx}{2\pi} : \psi_\nu^\dagger(x)\left(-i\partial_x\right)\psi_\nu(x) : \tag{3.28}$$

To write the bosonized version of the Hamiltonian density we make a diligent use of the bosonization formula and point splitting.

$$H_{KE} = \sum_\nu \left[\frac{v_\nu}{2}\int_0^{L_x} dx : (\partial_x\Phi_\nu)^2 : + \frac{\pi}{L_x} v_\nu \times \hat{N}_\nu(\hat{N}_\nu + 1)\right] \tag{3.29}$$

Assume that two edges have velocities that are equal in magnitude, $v_1 = -v_2 = v$ [14, 15, 16]. Now consider the case where the two edges start interacting with each other. As a first step let us identify the interaction terms that are possible. General local interaction are of the form $\psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x)$. Let us see the possible terms that arise;

$$\begin{aligned}
V_1 &= \sum_\nu \psi_\nu^\dagger(x)\psi_\nu^\dagger(x)\psi_\nu(x)\psi_\nu(x) + h.c. \\
V_2 &= \psi_2^\dagger(x)\psi_1^\dagger(x)\psi_2(x)\psi_1(x) + h.c. \\
V_3 &= e^{-i2k_\Delta x} \times \psi_2^\dagger(x)\psi_1^\dagger(x)\psi_1(x)\psi_1(x) + h.c. \\
V_4 &= e^{-i4k_\Delta x} \times \psi_2^\dagger(x)\psi_2^\dagger(x)\psi_1(x)\psi_1(x) + h.c.
\end{aligned} \tag{3.30}$$

The term V_1 corresponds to the forward scattering of the fermions and V_2 represent the backward scattering. Let us first consider only V_1 and V_2 type interaction

$$H_{int} = \int_0^{L_x} \frac{dx}{2\pi}\left[\sum_\nu \frac{g_4^\nu}{2} : \left(: \psi_\nu^\dagger(x)\psi_\nu(x) :\right)^2 : + g_2 : \Big(: \psi_2^\dagger(x)\psi_2(x) :: \psi_1^\dagger(x)\psi_1(x) : \Big) :\right] \tag{3.31}$$

Let us assume that $g_4^1 = g_4^2 = g_4$. In the bosonic picture this Hamiltonian becomes

$$H_{int} = \frac{2\pi}{L_x}\left[\frac{g_4}{2} \times \left(\hat{N}_1^{\ 2} + \hat{N}_2^{\ 2}\right) + g_2 \hat{N}_1 \hat{N}_2\right] + \int_0^{L_x} \frac{dx}{2\pi}\left[\frac{g_4}{2}\sum_\nu : (\partial_x \phi_\nu)^2 : - g_2 : (\partial_x \phi_1)(\partial_x \phi_1) :\right] \tag{3.32}$$

We can immediately see that in bosonic language, g_4 just renormalizes the fermi velocity. The total hamiltonian now becomes

$$\begin{aligned} H &= H_0 + H_{int} \\ &= \frac{v}{2}(1+\bar{g}_4)\int_0^{L_x} dx \left[\sum_\nu : (\partial_x \phi)^2 : -2\frac{\bar{g}_2}{1+\bar{g}_4} : (\partial_x \phi_2)(\partial_x \phi_1) :\right] \\ &\quad + \frac{2\pi v}{L_x}(1+\bar{g}_4)\left[\sum_\nu \hat{N}_\nu^{\ 2} + 2\frac{\bar{g}_2}{1+\bar{g}_4}\hat{N}_1 \hat{N}_2\right] \end{aligned} \tag{3.33}$$

with

$$\begin{aligned} \bar{g}_2 &= \frac{g_2}{2\pi v} \\ \bar{g}_4 &= \frac{g_4}{2\pi v} \end{aligned} \tag{3.34}$$

In the bosonic picture, interacting edge states corresponds to a quadratic Hamiltonian which can be readily diagonalized with the help of Bogoliubov transformation. Defining K to be

$$K = \sqrt{\frac{1+\bar{g}_4-\bar{g}_2}{1+\bar{g}_4+\bar{g}_2}} \tag{3.35}$$

we can write the new bosonic fields as

$$\begin{aligned} \chi_1(x) &= \frac{1}{2}\left[\sqrt{K} + \frac{1}{\sqrt{K}}\right]\phi_1 - \frac{1}{2}\left[\sqrt{K} - \frac{1}{\sqrt{K}}\right]\phi_2 \\ \chi_2(x) &= \frac{1}{2}\left[\sqrt{K} + \frac{1}{\sqrt{K}}\right]\phi_2 - \frac{1}{2}\left[\sqrt{K} - \frac{1}{\sqrt{K}}\right]\phi_1 \end{aligned} \tag{3.36}$$

In terms of these new fields, the diagonalized Hamiltonian is

$$H = \frac{u}{2}\int_0^{L_x} \frac{dx}{2\pi}\sum_\nu : (\partial_x \chi_\nu)^2 : \tag{3.37}$$

where

$$u = v_F \sqrt{\left(1+\bar{g}_4\right)^2 - \bar{g}_2^2} \tag{3.38}$$

Now I have got two uncoupled bosnons. I can refermionize these and get new fermions.These are not the bare fermions that we started with, rather a linear combinations of them.

3.7 Adding a Tunneling Term

Another physical process that can happen at the interface of the edge would be tunneling between the two modes. Hamiltonian density corresponding to this will be

$$H_{tun} = \int dx \int dx' \left(\psi_1^\dagger \left(x'\right) V \left(x'-x\right) \psi_2 \left(x\right) + h.c.\right) \tag{3.39}$$

Let us consider $V\left(x\right) = \Delta\delta\left(x-x'\right)$, which corresponds to a uniformly distributed scatterer.

$$H_{tun} = \Delta \int dx \left(\psi_1^\dagger \left(x\right) \psi_2 \left(x\right) + hc\right) \tag{3.40}$$

The bosonized version of this Hamiltonian is

$$H_{tun} = 2\Delta \int dx \cos\left(\left[\phi_2 - \phi_1\right]\right) \tag{3.41}$$

After the addition of this term the Hamiltonian corresponds to the Sine-Gordon Hamiltonian. The tunneling terms breaks the chirality of the new bosons. This term is equivalent to the mass term in the fermionic picture which leads to the gap opening.

3.8 Bosonization of IQHE Edge States : Other Approaches

Let us again consider a two dimensional free electron gas under uniform magnetic field, $\vec{B} = B_0\hat{z}$. In this section we will work with the symmetric gauge choice which is suited for circular geometry. In the symmetric gauge vector potential can be written as,

$$\vec{A} = -\frac{1}{2}(\hat{r}\times\vec{B}) = \frac{B_0}{2}\begin{bmatrix} -y \\ x \end{bmatrix} \tag{3.42}$$

The eigenfunction in this gauge choice is given by

$$\eta(z,\bar{z})_{n,m} = \frac{(-1)^n}{l_B\sqrt{2\pi}}\sqrt{\frac{n!}{2^m(m+n)!}}e^{\left(-\frac{|z|^2}{4l_B^2}\right)}\left(\frac{z}{l_B}\right)^m L_n^m\left(\frac{|z|^2}{2}\right) \tag{3.43}$$

With $n = 0,1,2,3,4...\nu$ and $m = 0,1,2,3,4...N_\phi - 1$. Here $N_\phi = An_B$ is the degeneracy of the each Landau level, with $n_B = \frac{1}{2\pi l_B^2}$ and A is the area of the system. The Fermionic field operator can be written in the Landau level basis as

$$\psi^\dagger(z) = \sum_{n,m}\eta^*_{n,m}(z)\,c^\dagger_{n,m} \tag{3.44}$$

The Fermionic operator $c^\dagger_{n,m}$ creates an electron in the Landau level n, with guiding center m. They obey the anti-commutation relation,

$$\left\{c^\dagger_{n,m}, c^\dagger_{n',m'}\right\} = \{c_{n,m}, c_{n',m'}\} = \left\{c_{n,m}, c^\dagger_{n',m'}\right\} = \delta_{n,n'}\delta_{m.m'} \tag{3.45}$$

If we restrict the Hilbert space to the lowest Landau level n=0 only Fermionic field operators become,

$$\psi^\dagger(z) = \sum_m \eta^*_m(z)\,c^\dagger_m \tag{3.46}$$

The lowest Landau level wave function are holomorphic function of z (we will stop using the Landau level index n). They are given by,

$$\eta_m = \frac{1}{l_B\sqrt{2\pi m! 2^m}}\left(\frac{z}{l_B}\right)^m e^{\left(-\frac{|z|^2}{4l_B^2}\right)} \tag{3.47}$$

With the modified definition of the field operators we can calculate the anti-commutation relation between the fields.

$$\left\{\psi_0(z), \psi_0^\dagger(z)\right\} = \frac{1}{2\pi}\times e^{\left(-\frac{|z-z'|^2}{4l_B^2}\right)}\times e^{\left(-\frac{\left[z\bar{z'}-z'\bar{z}\right]}{4l_B^2}\right)} \tag{3.48}$$

3.8.1 Density Operator - Apporach 1

A linear combination of the operators that creates particle-hole like excitations is defined as follows

$$\rho(n) = \int dz\,d\bar{z}\frac{1}{l_B\sqrt{2\pi n!\sigma^n}}\times\left(\frac{z}{l_B}\right)^n\times e^{\left(-\frac{|z|^2}{\sigma^2 l_B^2}\right)}\times\rho(z) \tag{3.49}$$

They are called density operators because, $\rho(z)$ which is defined below is the real space density of the electron at the position z.

$$\begin{aligned}\rho(z) &= \psi_0^\dagger(z)\psi_0(z) \\ &= \sum_{m,m'} \eta_m^*(z)\eta_{m'}(z)c_m^\dagger c_{m'}\end{aligned} \tag{3.50}$$

Substituting the above expression into the $\rho(n)$ and performing the integral we get the following expression

$$\begin{aligned}\rho(n) &= \frac{1}{l_B^3\sqrt{2\pi}}\sum_m \frac{\frac{1}{2}\times\left(\frac{2\sigma^2 l_B^2}{\sigma^2+2}\right)^{m+n}\times(m+n)!}{\sqrt{2^m 2^{m+n}\sigma^n n! m!(m+n)!}\times l_B^{2(m+n)}} c_m^\dagger c_{n+m} \\ &= \frac{1}{2\times l_B^3\times\sqrt{2\pi}}\times\left(\frac{2}{\sigma}\right)^{\frac{n}{2}}\times\left(\frac{\sigma^2}{\sigma^2+2}\right)^n\times\frac{1}{\sqrt{n!}}\sum_m\left(\frac{\sigma^2}{\sigma^2+2}\right)^m\times\sqrt{\frac{(m+n)!}{m!}}c_m^\dagger c_{n+m}\end{aligned} \tag{3.51}$$

For the time being let us take the factor σ to be $\sqrt{2}$. So the above equation reduces to the following.

$$\rho(n) = \frac{1}{2\times l_B^3\times\sqrt{2\pi}}\times\frac{1}{2^{\frac{3n}{4}}}\times\frac{1}{\sqrt{n!}}\sum_m\frac{1}{2^m}\times\sqrt{\frac{(m+n)!}{m!}}c_m^\dagger c_{n+m} \tag{3.52}$$

This construction is not suitable because action of density operator on any states in the Hilbert space will not create a uniform superposition of the states. One would naively expect the other way. It is the factor 2^m that create this issue. No value of σ can remove the (spurious) factor of 2^m from the denominator. For the time being we stop here and attribute this factor to the curvature. If σ was tending to infinity we could have avoided this extra unexpected factor. But this will make the inverse transformation complicated. Hence we will take an alternative approach.

3.8.2 Density Operator - A different approach

We now define a linear combination of the operators that create particle-hole like excitations in a different way. These are also called density operators.

$$\rho(n) = 2^{\frac{n}{2}}\times\int dz\,d\bar{z}\left(\frac{z}{l_B}\right)^{-n}\times\rho(z) \tag{3.53}$$

These are the representation of the lowest Landau level projected density on the spherical harmonics subset $((\bar{z})^{-n})$. Substituting the $\rho(z)$ expression into the definition of $\rho(n)$ we get the following.

$$\rho(n) = \left(\frac{1}{l_B\sqrt{2\pi}}\right)^2 \times 2^{\frac{n}{2}} \sum_{m,m'} \int dz\, d\bar{z}\, \frac{1}{\sqrt{2^m 2^{m'} m! m'!}} \frac{\bar{z}^{-n}\bar{z}^m z^{m'}}{l_B^{m+m'-n}} e^{\left(-\frac{|z|^2}{2l_B^2}\right)} c_m^\dagger c_{m'} \tag{3.54}$$

The integral involved is a standard Gamma function integral which can be easily calculated with the help of Mathematica.

$$\begin{aligned}\rho(n) &= \frac{1}{l_B^2} \times 2^{\frac{n}{2}} \sum_{m'} \frac{\frac{1}{2} \times \left(\frac{1}{2l_B^2}\right)^{-(1+m')} \times (m')!}{\sqrt{2^{m'+n}2^{m'}(m'+n)!m'!} \times l_B^{2m'}} c_{m'+n}^\dagger c_{m'} \\ &= \sum_m \sqrt{\frac{m!}{(m+n)!}} c_{m+n}^\dagger c_m\end{aligned} \tag{3.55}$$

This operator has so many interesting properties.

1. The action of this operator $\rho(n)$ on LLL wave function, will raise the angular momentum by n, creating the excitation at the edge of the system when $n \ll N_\phi$.

2. Notice that $\rho(-n) \equiv \rho^\dagger(n)$

Let us now calculate some commutators between these density operators.

$$\begin{aligned}\left[\rho\left(n\right), \rho\left(n'\right)\right] &= \sum_{m,m'} \sqrt{\frac{m!}{(m+n)!}} \sqrt{\frac{m'!}{(m'+n')!}} \left[c_{m+n}^\dagger c_m, c_{m'+n'}^\dagger c_{m'}\right] \\ &= -\sum_{j=-n}^{-1} \sqrt{\frac{j!}{(j+n+n')!}} c_{j+n'+n}^\dagger c_j\end{aligned} \tag{3.56}$$

When $n = -n'$ the ground state expectation value is $-n$. If $n \neq -n'$ then their ground state expectation value vanishes.

$$\langle[\rho(n), \rho(n')]\rangle_0 = -n\delta_{n.n'} \tag{3.57}$$

This expression will enable us to define the bosonic operators in terms of the fermionic operators. If we define b_n and $b_n^\dagger$ by

$$
\begin{aligned}
b_n &\equiv \frac{1}{\sqrt{n}}\rho(-n) = \frac{1}{\sqrt{n}}\sum_m \sqrt{\frac{m!}{(m+n)!}} c_m^\dagger c_{m+n} \quad (n>0) \\
b_n^\dagger &\equiv \frac{1}{\sqrt{n}}\rho(n) = \frac{1}{\sqrt{n}}\sum_m \sqrt{\frac{m!}{(m+n)!}} c_{m+n}^\dagger c_m \quad (n>0)
\end{aligned}
\tag{3.58}
$$

If we approximate their commutation relation with the ground state expectation value

$$
\begin{aligned}
&\left[b_n, b_{n'}^\dagger\right] \backsimeq \left\langle \left[b_n, b_{n'}^\dagger\right] \right\rangle = \delta_{n.n'} \\
&[b_n, b_{n'}] \backsimeq \langle [b_n, b_{n'}] \rangle = 0 = \left[b_n^\dagger, b_n^\dagger\right] \backsimeq \left\langle \left[b_n^\dagger, b_n^\dagger\right] \right\rangle
\end{aligned}
\tag{3.59}
$$

We can say that the b_n and $b_n^\dagger$ are approximate bosonic operators. As with cases of other examples of bosonization, they don't satisfy the commutation relation strictly; nevertheless their ground state expectation values do.

Chapter 4

FQH Edge States

In the last chapter, we looked at the IQH edge states and bosonized them. The bosonized operators were used to map the coupled edge system to the standard Tomonaga-Luttinger liquid. The present chapter aims to do the same for FQH states. Unfortunately, we can't follow the same steps that we did for the IQHE case here. Instead, we will start from Chern Simons theory and derive the boundary bosonic theory. We will begin by recapitulating the ideas of Chern-Simons theory. Then we will proceed to the hierarchical construction and finally go to the boundary modes.

4.1 Chern-Simons Theory

The Chern-Simons theory in $2+1$ dimensional Minkowski spacetime is described by the action

$$S_{CS}\left[A\right]=\frac{k}{4\pi}\int d^3x\ \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \tag{4.1}$$

where A_μ is the Chern-Simons gauge field. Under the gauge transformation, $A_\mu \longrightarrow A_\mu+\partial_\mu\omega$, we have

$$S_{CS}\left[A\right]\longrightarrow S_{CS}\left[A\right]+\frac{k}{4\pi}\int d^3x\ \epsilon^{\mu\nu\rho}\left[A_\mu\partial_\nu\partial_\rho\omega+\partial_\mu\omega\partial_\nu A_\rho+\partial_\mu\omega\partial_\nu\partial_\rho\omega\right] \tag{4.2}$$

The first and third term in the additional part is identically zero due to the anti symmetry property of Levi-Civita symbol. The second term can be written as a total derivative

$$\epsilon^{\mu\nu\rho}\partial_\mu\left(\omega\partial_\nu A_\rho\right)=\epsilon^{\mu\nu\rho}\partial_\mu\omega\partial_\nu A_\rho+\epsilon^{\mu\nu\rho}\omega\partial_\mu\partial_\nu A_\rho=\epsilon^{\mu\nu\rho}\partial_\mu\omega\partial_\nu A_\rho \tag{4.3}$$

That is, change in the action under a gauge transformation is a total derivative given by

$$\delta S_{SC}[A] = \frac{k}{4\pi}\int d^3x\, \partial_\mu \left(\epsilon^{\mu\nu\rho}\omega\partial_\nu A_\rho\right) \tag{4.4}$$

We can discard this term in the absence of a boundary. Let us calculate the current arises from this Lagrangian.

$$\begin{aligned} J_i &= \frac{\delta S_{CS}[A]}{\delta A_i} \\ &= \frac{k}{2\pi}\epsilon^{i\nu\rho}\partial_\nu A_\rho = -\frac{k}{2\pi}\epsilon^{ij}E_j \end{aligned} \tag{4.5}$$

From the quantization of Chern-Simons theory we know that k is $\frac{\nu e^2}{\hbar}$ for some integer ν[13]. This corresponds to the hall conductivity of ν filled Landau levels.

$$\sigma_{xy} = \frac{k}{2\pi} = \frac{\nu e^2}{2\pi\hbar} \tag{4.6}$$

It means that Chern-Simons theory can capture the basic physics of the integer quantum Hall effect.

4.2 Effective Theory of Laughlin States

Extending the argument from previous section we can propose the following Lagrangian, which will explain the Laughlin $\frac{1}{m}$ Hall states.

$$S_{eff}[a, A] = \frac{e^2}{\hbar}\int d^3x \left\{\frac{1}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho - \frac{m}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho\right\} \tag{4.7}$$

where a_μ is the $U(1)$ gauge field and A_μ is the external field. The first term is a mixed Chern-Simon term which comes from the coupling of A_μ and J_μ. Let us calculate the Hall conductivity from the above Lagrangian. Let us first integrate out the a_μ. The equation of motion (EOM) of a_μ is given by

$$\frac{F_{\alpha\mu}}{m} = f_{\alpha\mu} \tag{4.8}$$

Up to some gauge transformation, solution to the above equation is $a_\mu = \frac{A_\mu}{m}$. Substituting this back to the action we get,

$$S_{eff}[A] = \frac{e^2}{\hbar} \int d^3x \left\{ \frac{1}{4\pi m} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right\} \tag{4.9}$$

From this action we can read off the Hall conductivity as $\sigma_{xy} = \frac{e^2}{2\pi\hbar}\frac{1}{m}$, which is required for the Laughlin state. To find the charge of the quasi-particles, let's add a term to Lagrangian where emergent gauge field a_μ couples to its own current.

$$\Delta S = \int d^3x \, a_\mu j^\mu \tag{4.10}$$

To ensure gauge invariance, j^μ must be conserved, i.e. $\partial_\mu j^\mu = 0$. Let's keep the background field A_μ to be zero. The EOM of a_μ is

$$\frac{me^2}{2\pi\hbar} f_{\sigma\lambda} = \epsilon_{\sigma\lambda\beta} j^\beta \tag{4.11}$$

The most simple possible current would be an electron sitting at the origin with charge e

$$j^\mu = \big(e\delta^2(x), 0, 0\big) \tag{4.12}$$

the EOM will read as

$$\frac{me^2}{2\pi\hbar} f_{12} = e\delta^2(x) \tag{4.13}$$

The Chern-Simons term attaches a flux of $\frac{\hbar}{em}$ to the particle. The electron current J^0 in this background is

$$\begin{aligned} J^0 &= \frac{\delta S_{CS}[A]}{\delta A_0} \\ &= \frac{e^2}{2\pi\hbar} f_{12} = \frac{e}{m}\delta^2(x) \end{aligned} \tag{4.14}$$

4.3 Chern-Simons Theory with a Boundary

Let us consider a Hall state with a filling fraction $\frac{1}{m}$ which has a boundary at $y = 0$, i.e. $y > 0$ is vacuum and $y < 0$ is the Hall state. When we showed that Chern-Simons theory is gauge invariant we dropped the total derivative term saying that it vanishes at the boundary. Also when we derived the EOM we discarded the boundary term. We can't be doing that anymore. Let us re-consider the gauge transformation, $a_\mu \longrightarrow a_\mu + \partial_\mu \omega$, we have

$$
\begin{aligned}
S_{CS}[a] &\longrightarrow S_{CS}[a] - \frac{m}{4\pi}\int d^3x\, \epsilon^{\mu\nu\rho}\left[a_\mu \partial_\nu \partial_\rho \omega + \partial_\nu \omega \partial_\nu a_\mu + \partial_\mu \omega \partial_\nu \partial_\rho \omega\right] \\
&\longrightarrow S_{CS}[a] - \frac{m}{4\pi}\int d^3x\, \partial_\mu \left(\epsilon^{\mu\nu\rho} \omega \partial_\nu a_\rho\right)
\end{aligned}
\tag{4.15}
$$

In particular along the boundary

$$
\begin{aligned}
\delta S_{SC}[a]|_{y=0} &= -\frac{m}{4\pi}\int_{y=0} d^3x\, \partial_\mu \left(\epsilon^{\mu\nu\rho} \omega \partial_\nu a_\rho\right) \\
&= -\frac{m}{4\pi}\int_{y=0} dxdt\, \omega \left(\partial_0 a_1 - \partial_1 a_0\right)
\end{aligned}
\tag{4.16}
$$

We need additional degrees of freedom on the edge to cancel this surface term, and obtain a properly gauge invariant theory. Let us look at the variation of the action to find out what is the extra degree of freedom/term that will do the necessary.

$$
\begin{aligned}
\delta S_{CS}[a] &= -\frac{m}{4\pi}\int d^3x\, \left[\epsilon^{\mu\nu\rho} \delta a_\mu \partial_\nu a_\rho + \epsilon^{\mu\nu\rho} a_\mu \left(\partial_\nu \delta a_\rho\right)\right] \\
&= -\frac{m}{4\pi}\int d^3x\, \left[\epsilon^{\mu\nu\rho} \delta a_\mu f_{\nu\rho} + \epsilon^{\mu\nu\rho} \partial_\nu \left(a_\mu \delta a_\rho\right)\right]
\end{aligned}
\tag{4.17}
$$

The EOM would be $f_{\nu\rho} = 0$ if the boundary term vanished. i.e.

$$
\begin{aligned}
\delta S_{CS}[a]\,|_{y=0} &= 0 \\
\Longrightarrow\; -\frac{m}{4\pi}\int\limits_{y=0} d^3x\, \left[\epsilon^{\mu\nu\rho} \partial_\nu \left(a_\mu \delta a_\rho\right)\right] &= 0 \\
\int\limits_{y=0} dtdx\, \left[a_0 \delta a_1 - a_1 \delta a_0\right] &= 0 \\
\Longrightarrow\; a_0 \delta a_1 - a_1 \delta a_0 &= 0 \\
\Longrightarrow\; \left(a_0 - c a_1\right)|_{y=0} &= 0
\end{aligned}
\tag{4.18}
$$

Let us choose the gauge condition $a_0 = 0$ to find out the dynamics of these extra degrees of freedom. The EOM of the original Lagrangian with respect to a_0 now gives a constraint

$$
\epsilon_{ij} \partial_i a_j = 0 \tag{4.19}
$$

This can be solved by substituting $a_i = \partial_i \phi$. Substituting this back to the original action we get

$$\begin{aligned} S_{CS}[a] &= -\frac{m}{4\pi}\int d^3x\ \epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho \\ &= -\frac{m}{4\pi}\int d^3x\ \left(a_2\partial_0 a_1 - a_1\partial_0 a_2\right) = -\frac{m}{4\pi}\int d^3x\ \left(\partial_2\phi\partial_0\partial_1\phi - \partial_1\phi\partial_0\partial_2\phi\right) \\ &= -\frac{m}{4\pi}\int d^3x\ \left(\partial_2\left(\phi\partial_0\partial_1\phi\right) - \partial_1\left(\phi\partial_0\partial_2\phi\right)\right) \\ &= \frac{m}{4\pi}\int d^3x\ \left(\nabla\times\vec{V}\right)_z = \frac{m}{4\pi}\int\left(\vec{V}.d\vec{l}\right)dt \end{aligned} \tag{4.20}$$

where $\vec{V} = (\phi\partial_0\partial_1\phi, \phi\partial_0\partial_2\phi)$ and we used the Stokes's theorem in the last step. Resulting action at the boundary is,

$$\begin{aligned} S_{edge}[\phi] &= \frac{m}{4\pi}\int \phi\partial_t\partial_x\phi\, dxdt \\ &= -\frac{m}{4\pi}\int \partial_x\phi\partial_t\phi\, dxdt \end{aligned} \tag{4.21}$$

This gauge cannot be used to describe the physical edge excitations, since the Hamiltonian is zero. The edge excitations of the FQH has finite non-universal velocities. We use a different gauge which is

$$a_t - ca_x = 0 \tag{4.22}$$

It is convenient to choose a new coordinate system that satisfy

$$\tilde{x} = x + ct, \qquad \tilde{t} = t, \qquad \tilde{y} = y \tag{4.23}$$

The gauge potential under this coordinate transformation changes to

$$\implies \tilde{a}_{\tilde{t}} = a_t - ca_x, \qquad \tilde{a}_{\tilde{x}} = a_x, \qquad \tilde{a}_{\tilde{y}} = a_y \tag{4.24}$$

Let us see what happens to our original action by repeating the previous discussion

$$\begin{aligned} S &= -\frac{m}{4\pi}\int d^3x'\ \epsilon^{\mu\nu\rho}a'_{\mu}\partial_{\nu'}a'_{\rho} \\ &= -\frac{m}{4\pi}\int d^3\tilde{x}\ \left(\tilde{a}_{\tilde{y}}\partial_{\tilde{t}}\tilde{a}_{\tilde{x}} - \tilde{a}_{\tilde{x}}\partial_0\tilde{a}_{\tilde{y}}\right) \\ &= -\frac{m}{4\pi}\int \partial_{\tilde{x}}\phi\partial_{\tilde{t}}\phi\, d\tilde{x}d\tilde{t} \end{aligned} \tag{4.25}$$

In terms of original physical coordinates action has the form

$$S = -\frac{m}{4\pi}\int \partial_x\phi\left(\partial_t\phi - c\partial_x\phi\right)\, dxdt \tag{4.26}$$

The EOM of this boundary Lagrangian is given by

$$\partial_t\partial_x\phi - c\partial_x^2\phi = 0 \tag{4.27}$$

If we define a new field $\rho = \frac{\partial_x\phi}{2\pi}$ the EOM becomes

$$\partial_t\rho - c\partial_x\rho = 0 \tag{4.28}$$

The solution to the above equation is of the form $\rho(x-ct)$ which corresponds to a right moving wave. In short, Chern-Simons theory has a chiral scalar field at the boundary. Now let us quantize this theory.

$$\begin{aligned} \phi(x,t) &= \frac{1}{\sqrt{L}}\sum_{k=-\infty}^{\infty}\phi_k(t)e^{ikx} \\ \rho(x,t) &= \frac{1}{\sqrt{L}}\sum_{k=-\infty}^{\infty}\rho_k(t)e^{ikx} \end{aligned} \tag{4.29}$$

where $k = \frac{2\pi n_k}{L} \quad n_k \in \mathbb{Z}$. Since $\rho = \frac{\partial_x\phi}{2\pi}$ we get

$$\rho_k(t) = \frac{ik}{2\pi}\phi_k(t) \tag{4.30}$$

and

$$
\begin{aligned}
H &= \int dx\, \mathcal{H} \\
&= \frac{mc}{4\pi} \int dx\ \left(\partial_x \phi\right)^2 \\
&= 2\pi mc \sum_{k>0}^{\infty} \rho_k \rho_{-k}
\end{aligned}
\tag{4.31}
$$

If we define ρ_k to be the canonical coordinate, the canonical conjugate π_k satisfies the Hamilton's equation

$$
\begin{aligned}
\dot{\pi}_q &= -\frac{\partial \mathcal{H}}{\partial \rho_q} \\
&= -\frac{2\pi m}{iq} \dot{\rho}_{-q}
\end{aligned}
\tag{4.32}
$$

If we impose the canonical quantization we get

$$
\begin{aligned}
&[\rho_k, \pi_{k'}] = i\delta_{k,k'} \\
\Longrightarrow &[\rho_k, \rho_{k'}] = \frac{k}{2\pi m} \delta_{k+k'}
\end{aligned}
\tag{4.33}
$$

Using this we can find out the other commutation relations

$$
[\rho_k, \phi_{k'}] = \frac{i}{m} \delta_{k+k'} \tag{4.34}
$$

$$
[\phi_k, \phi_{k'}] = \frac{2\pi}{m} \frac{1}{k} \delta_{k+k'}, k \neq 0 \tag{4.35}
$$

These commutation relations can be written in the position basis as follows

$$
\begin{aligned}
[\phi(x), \phi(x')] &= \frac{\pi i}{m} \mathrm{sgn}(x - x') \\
[\rho\left(x\right), \phi\left(x'\right)] &= \frac{i}{m} \delta\left(x - x'\right) \\
[\rho\left(x\right), \phi\left(x'\right)] &= -\frac{i}{2\pi m} \partial_x \delta\left(x - x'\right)
\end{aligned}
\tag{4.36}
$$

4.4 Charged Excitations as Vertex Operators

The following commutation relation from previous section tells us that

$$
\begin{aligned}
\left[\rho\left(x\right),\phi\left(x'\right)\right] &= \frac{i}{m}\delta\left(x-x'\right) \\
\Longrightarrow \left[\rho\left(x\right),e^{im\phi\left(x'\right)}\right] &= -e^{im\phi\left(x'\right)}\delta\left(x-x'\right)
\end{aligned}
\tag{4.37}
$$

which implies $e^{im\phi(x')}$ removes a unit charge and $e^{-im\phi(x')}$ adds one. To see the mutual statistics of this operator, let us calculate its canonical commutator

$$
\begin{aligned}
\left\{e^{im\phi(x)},e^{im\phi(x')}\right\} &= e^{im(\phi(x)+\phi(x'))}\left\{1+e^{-m^2[\phi(x),\phi(x')]}\right\} \\
&= e^{im(\phi(x)+\phi(x'))}\left\{1+e^{-m^2\left[\frac{\pi i}{m}\mathrm{sgn}(x-x')\right]}\right\} \\
&= e^{im(\phi(x)+\phi(x'))}\left\{1+e^{-i\pi m\mathrm{sgn}(x-x')}\right\} \\
&= 0 \qquad \text{since} \quad m\equiv 2\mathbb{Z}+1
\end{aligned}
\tag{4.38}
$$

Expoentail of a field operator is called vertex operator in conformal field theory. To make the definition precise we will define the fermion operator to be the following

$$
\begin{aligned}
\psi\left(x\right) &\equiv \quad \eta e^{im\phi(x)} \\
\psi^{\dagger}\left(x\right) &\equiv \quad \eta e^{-im\phi(x)}
\end{aligned}
\tag{4.39}
$$

where η is a constant which depends on the ultraviolet cut-off of the theory.

4.5 Hierarchy Construction and K-Matrices

In the previous sections we looked at the Chern-Simons theory and understood how it explains the fractional quantum Hall effect for the filling fration $\frac{1}{m}$, where $m \in 2\mathbb{Z}+1$. We then derived the bosons living at the edge of the Hall state. Now we are going to repeat the same exercise for other filling fractions too. We introduce a new gauge field $\tilde{a}_\mu$which represent the quantum Hall state formed by the quasi-particles. The quasi-particle current is given by

$$
j_\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu\tilde{a}_\rho
\tag{4.40}
$$

The new Chern-Simons action which includes the dynamics of $\tilde{a}_\mu$ is given by

$$S_{eff}\left[a,\tilde{a},A\right]=\frac{e^2}{\hbar}\int d^3x\ \left\{\frac{1}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho-\frac{m}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho+\frac{1}{2\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu\tilde{a}_\rho-\frac{\tilde{m}}{4\pi}\epsilon^{\mu\nu\rho}\tilde{a}_\mu\partial_\nu\tilde{a}_\rho\right\} \tag{4.41}$$

To find out the new Hall conductivity we will integrate out $\tilde{a}_\mu$ first then a_μ. EOM of $\tilde{a}_\mu$ is

$$\begin{aligned}\tilde{f}_{\mu\nu}&=\frac{1}{\tilde{m}}f_{\mu\nu}\\ \implies\ \tilde{a}_\mu&=\frac{1}{\tilde{m}}a_\mu\end{aligned} \tag{4.42}$$

Substituting $\tilde{a_\mu}$ we get

$$S_{eff}\left[a,A\right]=\frac{e^2}{\hbar}\int d^3x\ \left\{\frac{1}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho-\frac{1}{4\pi}\left(m-\frac{1}{\tilde{m}}\right)\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho\right\} \tag{4.43}$$

i.e. the Hall conductivity is

$$\sigma_{xy}=\frac{e^2}{2\pi\hbar}\frac{1}{m-\frac{1}{\tilde{m}}} \tag{4.44}$$

Similar to the last section, we can find the charge of the quasi particles by coupling them to a_μ or $\tilde{a}_\mu$. For a static hole coupled to a_μ the EOM (keeping A_μ zero) is

$$\epsilon^{\beta\nu\rho}\left(\tilde{m}\tilde{f}_{\mu\alpha}-f_{\mu\alpha}\right)=0 \tag{4.45}$$

Again considering $j^\beta=(e\delta^2(x),0,0)$,we get

$$\begin{aligned}\frac{e^2}{2\pi\hbar}\left(mf_{12}-\tilde{f}_{12}\right)&=e\delta^2(x)\\ \tilde{m}\tilde{f}_{12}-f_{12}&=0\end{aligned} \tag{4.46}$$

Solving the above two equations we get

$$f_{12}=\frac{2\pi\hbar}{e}\frac{1}{m-\frac{1}{\tilde{m}}}\delta^2(x) \tag{4.47}$$

This gives us charge of the quasiparticle. Using the above ideas we can write a general theory for a hierarchical fractional quantum Hall state. It is given by the action

$$S_K[a^i,A]=\frac{e^2}{\hbar}\int d^3x\ \left[-\frac{1}{4\pi}\epsilon^{\mu\nu\rho}a^i_\mu K_{ij}\partial_\nu a^j_\rho+\frac{1}{2\pi}\epsilon^{\mu\nu\rho}A_\mu q_i\partial_\nu a^j_\rho\right] \tag{4.48}$$

where i runs from 1 to N. The Lagrangian depends on K matrix which specifies the Chern-Simons coupling. It is a symmetric integer valued matrix with K_{11} being odd and K_{II} being even for $I \neq 1$. The vector $\vec{q}$ specifies the charge of the gauge field which was $(1, 0)$ in the previous case. This vector species the linear combination of current that is viewed as the electron current. From the K matrix and $\vec{q}$ we can extract out the physical information such as conductivity and charge of the quasi-particles. The hall conductivity will be given by

$$\nu = \vec{q}^T K^{-1} \vec{q} \tag{4.49}$$

To find out the charge of the quasi particle excitations we will add a term to the Lagrangian which couples the emergent gauge field with its own current

$$\mathcal{L}_{new} = l_i a_\mu^i j^\mu \tag{4.50}$$

Charge of the corresponding quasiparticle is given by

$$Q_{\vec{l}} = \left(q^T K^{-1} \vec{l} \right) \tag{4.51}$$

4.6 Equivalent K Matrices

We have seen that K matrix and charge vector $\vec{q}$ defines a hierarchical state. Now we pose the question whether a new Lagranian created by the linear combination of gauage field corresponds to same FQH system? To make sure that the charge quantization remains valid we need the $SL(N, \mathbb{Z})$ transformation. Accordingly,

$$\tilde{K} = S^T K S \qquad \tilde{\vec{q}} = S^T \vec{q} \tag{4.52}$$

4.7 Edge Modes of Hierarchial State

Let us now calculate the edge modes of a hierarchial quantum Hall state. The first step would be to diagonalize the K-matrix so that the derivation for a single component Chern-Simons edge mode will follow. Let us note that K is orthogonally diagonalizable since it is symmetric. So,

$$\begin{aligned} & D = S^T K S \qquad \text{where} \quad D_{IJ} = \delta_{IJ}\lambda_J \quad \text{and} \quad SS^T = \mathbb{I} \\ \implies & K = SDS^T \end{aligned} \tag{4.53}$$

defining $S^T_{MJ} a^J_\rho = b^M_\rho$ and substituting the above expression for K in the action we get

$$\begin{aligned} S_K &= -\frac{1}{4\pi}\int d^3x \ \left[\epsilon^{\mu\nu\rho} a^I_\mu K_{IJ} \partial_\nu a^J_\rho\right] \\ &= -\frac{1}{4\pi}\int d^3x \ \left[\epsilon^{\mu\nu\rho} a^I_\mu S_{IL} D_{LM} S^T_{MJ} \partial_\nu a^J_\rho\right] \\ &= -\sum_L \frac{\lambda_L}{4\pi}\int d^3x \ \left[\epsilon^{\mu\nu\rho} b^L_\mu \partial_\nu b^L_\rho\right] \end{aligned} \tag{4.54}$$

For each b^L_μ we can proceed with previous edge theory calculation and obtain the scalar field

$$\begin{aligned} b^L_0 &= 0 \\ \epsilon_{ij}\partial_i a_j &= 0 \\ \implies b^L_i &= \partial_i \varphi^L \end{aligned} \tag{4.55}$$

So at the boundary resulting action is

$$S_{edge}\left[\phi\right] = -\sum_L \frac{\lambda_L}{4\pi}\int \partial_t \varphi^L \partial_x \varphi^L dxdt \tag{4.56}$$

This can be converted to the following action by a change of coordinates

$$S = -\sum_L \frac{\lambda_L}{4\pi}\int \partial_x \varphi^L \left(\partial_t \varphi^L + c^L \partial_x \varphi^L\right) \, dxdt \tag{4.57}$$

Define $\phi^L = S_{LM}\varphi^M$, edge action in terms of new fields is

$$\begin{aligned} S &= -\sum_L \frac{\lambda_L}{4\pi}\int \partial_x \varphi^L \left[\partial_t \varphi^L + c^L \partial_x \varphi^L\right] \, dxdt \\ &= -\frac{1}{4\pi}\int \left[K_{PN}\partial_x \phi^P \partial_t \phi^N + V_{PN}\partial_x \phi^M \partial_x \phi^N\right] \, dxdt \end{aligned} \tag{4.58}$$

where

$$V_{PN} = S_{PL}\widetilde{D}_{LM} S^T_{MN}, \qquad \widetilde{D}_{LM} \equiv c^M D_{LM} \tag{4.59}$$

the matrix V is symmetric. It must be positive definite so that the Hamiltonian will be bounded from below. This tells me that positive eigenvalue of K corresponds to left moving boson and negative eigenvalue of K corresponds to right moving boson. The commutator of the bosonic fields can be found as follows

$$\begin{aligned} &[\varphi^M(x), \partial_{x'}\varphi^N(x')] = 2\pi i D^{-1}_{MN}\delta(x-x') \\ \implies &[\phi^\alpha(x), \partial_{x'}\phi^\beta(x')] = i2\pi \left(SDS^T\right)^{-1}_{\alpha\beta}\delta(x-x') = i2\pi K^{-1}_{\alpha\beta}\delta(x-x') \end{aligned} \tag{4.60}$$

Defining $\rho_\alpha = \frac{\partial_x \phi}{2\pi}$ to be the charge density of the α^{th} condensate. We get the following commutator

$$[\rho_\alpha(k), \rho_\beta(k')] = \left(K^{-1}\right)_{\alpha\beta}\frac{1}{2\pi}k\delta_{k+k'} \tag{4.61}$$

The electron density at the edge can be shown to be

$$\rho_e = q_\alpha \rho_\alpha \tag{4.62}$$

Let us look at a general vertex operator $e^{il_i\phi_i}$. It has the following commutation relation with the charge density.

$$\left[\rho_\alpha(x), e^{il_\beta\phi_\beta}\right] = l_\beta\left(K^{-1}\right)_{\beta\alpha}\delta(x-x') \tag{4.63}$$

This implies that total charge of the quasiparticle is given by

$$Q_l = q^T K^{-1} l \tag{4.64}$$

For this operator to corresponds to an electron we need the following condition.

$$l_i = K_{ij}L_j \quad q_i l_i = 1 \tag{4.65}$$

Integrating commutator in equation 4.60 we can find out the commutation relation between the fields. The integration constant can be arbitrary. We have to choose one smartly so that the vertex operators will satisfy the anti-commutation rule, which is given by

$$[\phi_i(x), \phi_j(x')] = i\pi\left(K^{-1}_{ij}\text{sgn}(x-x') + \Theta_{ij}\right) \tag{4.66}$$

where,

$$\begin{aligned} \Theta_{ij} &\equiv K^{-1}_{ik} L_{kl} K^{-1}_{lj} \\ L_{ij} &\equiv \operatorname{sgn}(i-j)\left(K_{ij} + q_i q_j\right) \end{aligned} \tag{4.67}$$

The extra term Θ_{ij} can be thought of as Klein factor which will make sure that fermionic operators will anticommute. Let us see why this is the case. Consider two fermion field operators given by

$$\psi_\alpha(x) = e^{iK_{\alpha\mu}\phi_\mu} \qquad \psi_\beta(x) = e^{iK_{\beta\nu}\phi_\nu} \tag{4.68}$$

Their anticommutatition can be calculated to be

$$\begin{aligned} \psi_\alpha(x)\,\psi_\beta(x') &= e^{iK_{\alpha\mu}\phi_\mu} e^{iK_{\beta\nu}\phi_\nu} \\ &= e^{iK_{\beta\nu}\phi_\nu} e^{iK_{\alpha\mu}\phi_\nu} e^{-K_{\alpha\mu}K_{\beta\nu}[\phi_\mu,\phi_\nu]} \\ &= \psi_\beta(x')\,\psi_\alpha(x)\, e^{-i\left[K_{\alpha\beta}\operatorname{sgn}(x-x') + L_{\alpha\beta}\right]} \\ &= (-1)^{q_\alpha q_\beta}\,\psi_\beta(x')\,\psi_\alpha(x) \end{aligned} \tag{4.69}$$

4.8 K-Matrices for Multi-Layer FQHE

So far we have discussed about single-layer FQH system. We can use the the K-matrix formalism to represent the multilayer quantum hall systems too. We will be doing this for the Halperin's generalization of the Laughlin wave function. I will elucidate this for a bilayer, extension to many layers will be straight forward. Consider the following bilayer quantum Hall wave function

$$\prod_{i<j}\left(z_{1i} - z_{1j}\right)^l \prod_{i<j}\left(z_{2i} - z_{2j}\right)^n \prod_{i,j}\left(z_{1i} - z_{2j}\right)^m e^{-\frac{1}{4l_B^2}\left(\sum_i |z_{1i}|^2 + \sum_j |z_{2j}|^2\right)} \tag{4.70}$$

The second layer electron seems to be bound to the first layer quasihole excitations. Repeating the similar arguments as that of hierarchical state we can propose the effective field theory with the following K matrix and integer charge vector $\vec{q}$.

$$K = \begin{bmatrix} l & n \\ n & m \end{bmatrix} \qquad \vec{q} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.71}$$

Chapter 5

Stability of Quantum Hall Edge States

So far in this text, we have seen that quantum Hall systems can host edge modes which can be chiral or non-chiral, depending on the structure of bulk. In this chapter we will explore the conditions under which the edge states of the quantum Hall states can be gapped. From the ideas of chiral Luttinger liquid, we know that no back-scattering term can gap out two 1D fermion channel of same chirality. We would expect the same to hold for the edge excitations of a FQH system. But what if they have zero net chirality? Is it always possible to gap out such a system? The primary motivation behind this chapter is the paper Protected Edge States Without Symmetry by Michael Levin [17]. We will explore the so-called Null Vector Criterion in great details and try to elucidate the ideas with few examples.

In the last chapter we have seen that edge theory of a fractional quantum Hall state can be described by the following Lagrangain

$$\mathcal{L} = \frac{1}{4\pi}\left(\partial_t \Phi^T K \partial_x \Phi - \partial_x \Phi^T V \partial_x \Phi\right) \tag{5.1}$$

where Φ is $2N$ component chiral bosonic field, K is the $2N \times 2N$ non-singular coupling symmetric integer matrix and V is the $2N \times 2N$ non-universal velocity matrix which is also symmetric.

Let us assume that we have n_L left and n_R right movers respectively in the edge, i.e. $2N = n_L + n_R$. If we have unequal number of left and right movers it is not possible to gap out the systems simultaneously. The backscattering between different modes will always gap out in equal number. This claim can be augmented using the following argument. The thermal Hall conductance of a set of chiral luttinger liquid is given by

$$K_H = (n_L - n_R)\,\frac{\pi^2 k_B^2}{3\hbar}T \tag{5.2}$$

Since $n_L - n_R$ is given by the signature of K matrix, it's an invariant quantity. If we have unequal number of opposite chiral modes we will always have a non-zero thermal Hall conductance. This implies we always have gapless modes in the system [18]. But what happens if we have an equal number of them? Haldane Null vector criterion gives a necessary condition so that system will gap out[19].

5.1 Null Vector Criterion

Assume that we have N right movers and N left movers. The null vector criterion states that if there exist N integer vectors Λ_i such that, ${\Lambda_i}^T K \Lambda_j = 0 \quad \forall \quad \Lambda_i, \Lambda_j$, the system can be fully gapped out. As a first step towards the proof I will define the following matrix

$$M \equiv \begin{bmatrix} \Lambda_1 & \Lambda_2 & . & . & . & \Lambda_N \end{bmatrix}_{2N\times N} \tag{5.3}$$

According to Smith Normal Form [20] the above matrix can be cast into the following form

$$M = SDR \tag{5.4}$$

where S a is $2N \times 2N$ uni-modular integer matrix, R is an $N \times N$ uni-modular integer matrix and D is a $2N\times$ N integer matrix of the following form

$$D = \begin{pmatrix} d_1 & 0 & ... & 0 \\ 0 & d_2 & ... & 0 \\ & & & \\ 0 & 0 & ... & d_N \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & ... & 0 \end{pmatrix} \equiv \begin{pmatrix} \tilde{D}_{N\times N} \\ 0_{N\times N} \end{pmatrix} \tag{5.5}$$

Since Λ_i's are independent vectors, the rank of D must be N, so that none of the $d_i = 0$. Now let us do the following change of variable

$$\Phi \to S^{-1}\Phi \tag{5.6}$$

Under this change of variable, the K matrix and the Λ will transform in the following way

$$\begin{aligned} K' &= S^T K S \\ \Lambda' &= S^{-1}\Lambda \end{aligned} \tag{5.7}$$

This will make the K and M have the following form

$$\begin{aligned} M' &= S^{-1}M = S^{-1}SDR \\ &= \begin{pmatrix} \tilde{D} \\ 0 \end{pmatrix} R = \begin{pmatrix} \tilde{D}R \\ 0 \end{pmatrix} \end{aligned} \tag{5.8}$$

This implies that last N components of Λ_i are zero in this basis. Since $M^T K M = 0$

$$\begin{aligned} \implies M^T K M &= M'^T S^T \left(\left(S^T\right)^{-1} K' S^{-1} \right) S M' \\ &= M'^T K' M' = \begin{pmatrix} R^T \tilde{D}^T & 0 \end{pmatrix} K' \begin{pmatrix} \tilde{D}R \\ 0 \end{pmatrix} \end{aligned} \tag{5.9}$$

Since K is symmetric matrix K' will also symmetric. The new K matrix can be written in the form of block partitioned matrix given below.

$$\begin{aligned} K' &= \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \quad \text{with} \quad A^T = A \quad \& \quad C^T = C \\ \implies M^T K M &= \begin{pmatrix} R^T \tilde{D}^T & 0 \end{pmatrix} \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} \tilde{D}R \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} R^T \tilde{D}^T & 0 \end{pmatrix} \begin{pmatrix} A\tilde{D}R \\ B^T \tilde{D}R \end{pmatrix} \\ &= R^T \tilde{D}^T A \tilde{D} R \\ \implies A &= 0 \end{aligned} \tag{5.10}$$

i.e. K' will be of the form

$$K' = \begin{pmatrix} 0 & A \\ A^T & B \end{pmatrix} \qquad \text{with} \quad B^T = B \tag{5.11}$$

Then

$$\left(K'\right)^{-1} = \begin{bmatrix} -\left(A^T\right)^{-1} BA^{-1} & \left(A^T\right)^{-1} \\ A^{-1} & 0 \end{bmatrix} \tag{5.12}$$

Let us see what are the commutation relation between the new bosonic fields

$$\begin{aligned} \left[\Phi_I'(x), \partial_y \Phi_J'(y)\right] &= \left[S_{I\alpha}^{-1}\Phi_\alpha(x), S_{J\beta}^{-1}\partial_y\Phi_\beta(y)\right] \\ &= S_{I\alpha}^{-1}S_{J\beta}^{-1}\left[\Phi_\alpha(x), \partial_y\Phi_\beta(y)\right] \\ &= S_{I\alpha}^{-1}S_{J\beta}^{-1}\left(i2\pi K_{\alpha\beta}^{-1}\delta(x-y)\right) \\ &= i2\pi\left(S^T K S\right)_{IJ}^{-1}\delta(x-y) \\ &= i2\pi\left(K'\right)_{IJ}^{-1}\delta(x-y) \end{aligned} \tag{5.13}$$

In the next step we will do another change of variable so that we will get N non-chiral Luttinger liquids.

$$\widetilde{\Phi} \longrightarrow T\Phi' \tag{5.14}$$

where

$$T = \begin{bmatrix} 0 & A \\ 1 & \frac{1}{2}\left(A^{-1}\right)^T B \end{bmatrix} \tag{5.15}$$

if we write

$$\Phi' = \begin{bmatrix} \Phi_1' \\ \Phi_2' \end{bmatrix} \tag{5.16}$$

then

$$\widetilde{\Phi} = \begin{bmatrix} A\Phi_2' \\ \Phi_1' + \frac{1}{2}\left(A^{-1}\right)^T B\Phi_2' \end{bmatrix} \equiv \begin{bmatrix} \theta_I \\ . \\ \theta_N \\ \varphi_1 \\ . \\ \varphi_N \end{bmatrix} \equiv \begin{bmatrix} \vec{\theta} \\ \vec{\varphi} \end{bmatrix} \tag{5.17}$$

Let us calculate the commutation relations between θ and φ.

$$\begin{aligned}
\left[\theta_I(x), \partial_y \theta_J(y)\right] &= A_{I\alpha} A_{J\beta} \left[\Phi'_{2\alpha}(x), \partial_y \Phi'_{2\beta}(y)\right] \\
&= A_{I\alpha} A_{J\beta} \left[\Phi'_{\alpha+N}(x), \partial_y \Phi'_{\beta+N}(y)\right] \\
&= A_{I\alpha} A_{J\beta} \left[i2\pi \left(K'\right)^{-1}_{N+\alpha, N+\beta} \delta(x-y)\right] \\
&= 0
\end{aligned} \tag{5.18}$$

$$\begin{aligned}
\left[\varphi_I(x), \partial_y \varphi_J(y)\right] = &\left[\Phi'_{1I}(x) + \frac{1}{2}\left(\left(A^{-1}\right)^T B \Phi'_2\right)_I(x), \partial_y \Phi'_{1J}(y) + \frac{1}{2}\partial_y \left(\left(A^{-1}\right)^T B \Phi'_2\right)_J(y)\right] \\
&= \left[\Phi'_{1I}(x), \partial_y \Phi'_{1J}(y)\right] + \left[\Phi'_{1I}(x), \frac{1}{2}\partial_y \left(\left(A^{-1}\right)^T B \Phi'_2\right)_J(y)\right] \\
&\quad + \left[\frac{1}{2}\left(\left(A^{-1}\right)^T B \Phi'_2\right)_I(x), \partial_y \Phi'_{1J}(y)\right] \\
&\quad + \left[\frac{1}{2}\left(\left(A^{-1}\right)^T B \Phi'_2\right)_I(x), \frac{1}{2}\partial_y \left(\left(A^{-1}\right)^T B \Phi'_2\right)_J(y)\right] \\
&= \left[\Phi'_I(x), \partial_y \Phi'_J(y)\right] + \frac{1}{2}\left(\left(A^{-1}\right)^T B\right)_{J\beta} \left[\Phi'_I(x), \partial_y \Phi'_{N+\beta}(y)\right] \\
&\quad + \frac{1}{2}\left(\left(A^{-1}\right)^T B\right)_{I\alpha} \left[\Phi'_{N+\alpha}(x), \partial_y \Phi'_J(y)\right] \\
&\quad + \frac{1}{4}\left(\left(A^{-1}\right)^T B\right)_{I\alpha} \left(\left(A^{-1}\right)^T B\right)_{J\beta} \left[\Phi'_{N+\alpha}(x), \partial_y \Phi'_{N+\beta}(y)\right]
\end{aligned} \tag{5.19}$$

Let us calculate each commutator separately

$$\begin{aligned}
\left[\Phi'_{1I}(x), \partial_y \Phi'_{1J}(y)\right] &= i2\pi \left(K'\right)^{-1}_{IJ} \delta(x-y) \\
&= -i2\pi \left(\left(A^T\right)^{-1} B A^{-1}\right)_{IJ}
\end{aligned} \tag{5.20}$$

$$\begin{aligned}
\frac{1}{2}\left(\left(A^{-1}\right)^T B\right)_{J\beta} \left[\Phi'_{1I}, \partial_y \Phi'_{N+\beta}(y)\right] &= \frac{1}{2}\left(\left(A^{-1}\right)^T B\right)_{J\beta} \left[i2\pi \left(K'\right)^{-1}_{I,N+\beta} \delta(x-y)\right] \\
&= i2\pi \left[\frac{1}{2}\left(\left(A^{-1}\right)^T B\right)_{J\beta} \left(A^T\right)^{-1}_{I\beta}\right] \delta(x-y) \\
&= i2\pi \left[\frac{1}{2}\left(\left(A^{-1}\right)^T B A^{-1}\right)_{IJ}\right] \delta(x-y)
\end{aligned} \tag{5.21}$$

$$
\begin{aligned}
\frac{1}{2}\left(\left(A^{-1}\right)^{T}B\right)_{I\alpha}\left[\Phi'_{N+\alpha}\left(x\right),\partial_{y}\Phi'_{J}\left(y\right)\right] &= \frac{1}{2}\left(\left(A^{-1}\right)^{T}B\right)_{I\alpha}\left[i2\pi\left(K'\right)^{-1}_{N+\alpha,J}\delta\left(x-y\right)\right] \\
&= i2\pi\left[\frac{1}{2}\left(\left(A^{-1}\right)^{T}B\right)_{I\alpha}A^{-1}_{\alpha J}\right]\delta\left(x-y\right) \\
&= i2\pi\left[\frac{1}{2}\left(\left(A^{-1}\right)^{T}BA^{-1}\right)_{Ij}\right]\delta\left(x-y\right)
\end{aligned}
\tag{5.22}
$$

$$
\begin{aligned}
\left[\Phi'_{N+\alpha}\left(x\right),\partial_{y}\Phi'_{N+\beta}\left(y\right)\right] &= \left[i2\pi\left(K'\right)^{-1}_{N+\alpha,N+\beta}\delta\left(x-y\right)\right] \\
&= \frac{1}{4}\left(\left(A^{-1}\right)^{T}B\right)_{I\alpha}\left(\left(A^{-1}\right)^{T}B\right)_{J\beta}\times 0 \\
&= 0
\end{aligned}
\tag{5.23}
$$

i.e. $\left[\varphi_{I}\left(x\right),\partial_{y}\varphi_{J}\left(y\right)\right]=0$

$$
\begin{aligned}
\left[\theta_{I}\left(x\right),\partial_{y}\varphi_{J}\left(y\right)\right] &= \left[\left(A\Phi'_{2}\right)_{I}\left(x\right),\partial_{y}\left(\Phi'_{1}+\frac{1}{2}\left(A^{-1}\right)^{T}B\Phi'_{2}\right)_{J}\left(y\right)\right] \\
&= \left[A_{I\alpha}\Phi'_{2\alpha}\left(x\right),\partial_{y}\Phi'_{1J}\left(y\right)\right]+\left[A_{I\alpha}\Phi'_{2\alpha}\left(x\right),\frac{1}{2}\left(\left(A^{-1}\right)^{T}B\right)_{J\beta}\partial_{y}\Phi'_{2\beta}\left(y\right)\right] \\
&= A_{I\alpha}\left[\Phi'_{N+\alpha}\left(x\right),\partial_{y}\Phi'_{J}\left(y\right)\right]+A_{I\alpha}\left(\frac{1}{2}\left(\left(A^{-1}\right)^{T}B\right)_{J\beta}\right)\left[\Phi'_{N+\alpha}\left(x\right),\partial_{y}\Phi'_{N+\beta}\left(x\right)\right] \\
&= A_{I\alpha}\left[i2\pi\left(K'\right)^{-1}_{N+\alpha,J}\delta\left(x-y\right)\right]+A_{I\alpha}\left(\frac{1}{2}\left(\left(A^{-1}\right)^{T}B\right)_{J\beta}\right)\left[i2\pi\left(K'\right)^{-1}_{N+\alpha,N+\beta}\delta\left(x-y\right)\right] \\
&= A_{I\alpha}\left[i2\pi\left(A\right)^{-1}_{\alpha,J}\delta\left(x-y\right)\right] \\
&= i2\pi\delta_{IJ}\delta\left(x-y\right)
\end{aligned}
\tag{5.24}
$$

The above commutators hints that θ and φ are canonically conjugate variables. Let us see what form does the back-scattering term take in this new basis

$$
\begin{aligned}
\widetilde{\Lambda}_I^T \widetilde{K} \widetilde{\Phi} &= \Lambda_I' T^T \left(T^{-1}\right)^T K' \left(T^{-1}\right) \widetilde{\Phi} \\
&= \Lambda_I' K' T^{-1} \widetilde{\Phi} \\
&= \begin{bmatrix} R^T \tilde{D^T} & 0 \end{bmatrix} \begin{bmatrix} 0 & A \\ A^T & B \end{bmatrix} \begin{bmatrix} -\left(\frac{1}{2}\left(A^{-1}\right)^T B\right) A^{-1} & 1 \\ A^{-1} & 0 \end{bmatrix} \widetilde{\Phi} \\
&= \begin{bmatrix} R^T \tilde{D^T} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{2} B A^{-1} & A^T \end{bmatrix} \widetilde{\Phi} \\
&= \begin{bmatrix} R^T \tilde{D^T} & 0 \end{bmatrix} \widetilde{\Phi} \\
&= R^T \tilde{D^T} \vec{\theta}
\end{aligned}
\tag{5.25}
$$

So the back-scattering term becomes

$$
\begin{aligned}
\sum_{i=1}^{N} U_i \cos\left(\Lambda_i^T K \Phi\right) &= \sum_{i=1}^{N} U_i \cos\left(R^T \tilde{D^T} \vec{\theta}\right) \\
&= \sum_{i=1}^{N} U_i \cos\left(\sum_J R_{JI} d_J \theta_J\right)
\end{aligned}
\tag{5.26}
$$

We have got N non-chiral Luttinger liquids. The backscattering would have been same as those of standard Luttinger liquid if

$$
H_{backscattering} = \sum_{I=1}^{N} U_I \cos\left(d_I \theta_J\right) \tag{5.27}
$$

Let us make the following argument. In the low-energy limit this term in the Hamiltonian will lock the field to their local minima

$$
\sum_J R_{JI} d_J \theta_J = 2\pi s_i \tag{5.28}
$$

Since R is a uni-modular integer matrix, its inverse is also an integer matrix, i.e.

$$
d_J \theta_J = 2\pi \sum_J R_{JI}^{-1} s_i = 2\pi s_i' \tag{5.29}
$$

This tells us that standard non-chiral Luttinger liquid shares the same low energy constraint [21]. Note that we didn't worry about the Kinetic part of the Lagrangian, as we took the limit $U \to \infty$ at the end.

5.2 Microscopic Argument

In the previous section we have seen that null vector criterion is good way to decide a given quantum hall edge states can be gapped or not with a backscattering term. In this section we will discuss another equivalent criterion by Levin [17]. I will be providing an extensive proof of his theorem below.

Given a K matrix as described in the last section, all the modes can be gapped if and only if there exist a set of quasiparticles corresponding to integer vector set $\mathcal{M}$ such that the following two conditions hold.

1. Any two particle labelled by integer vectors m, m' in $\mathcal{M}$ should have a trivial mutual statistics.That is, $e^{i\theta_{mm'}} = 1$
2. If l is not in $\mathcal{M}$ then it will have non trivial statistics with at least one $m \in \mathcal{M}$, i.e., $e^{i\theta_{lm}} \neq 1$

We prove the above proposition by showing that the above conditions are equivalent to null vector criterion.

Theorem 5.2.1. *Let K be an integer $2N \times 2N$ symmetric matrix with vanishing signature, non-vanishing determinant and at least one odd element on the diagonal. There exists N linearly independent integer vectors $\{\Lambda_i; i = 1, 2...N\}$ satisfying $\Lambda_i^T K \Lambda_j = 0$ if and only if there is a set of integer vectors $\mathcal{M}$ satisfying the following two properties:*

1. *$m^T K^{-1} m'$ is an integer for any m , $m' \in \mathcal{M}$.*
2. *If l is not equivalent to any element of $\mathcal{M}$ then $m^T K^{-1} l$ is not an integer for some $m \in \mathcal{M}$.*

Proof. Before proceeding into the main proof let me point out some features of the set $\mathcal{M}$.

- The zero integer vector, $\vec{0}$ is always an element of $\mathcal{M}$.
- If $m \in \mathcal{M}$ then $m + Kx \in \mathcal{M}$. More precisely we divide the set $\mathcal{M}$ into equivalent classes s.t. two integer vectors $m, m' \in \mathcal{M}$ are equivalent if they satisfies $m' = m + Kx$ for some $x \in \mathbb{Z}^{2N}$

- If $l \notin \mathbb{Z}^{2N}$ then l is not equivalent to any element of $\mathcal{M}$, i.e. $m^T K^{-1} l$ is not an integer for some $m \in \mathcal{M}$. To see that consider $l \notin \mathbb{Z}^{2N}$. WLOG we can consider $l_i \notin \mathbb{Z}$ for some $i \in \{1, 2, ..., 2N\}$. If we choose $m = K\vec{e}_i \in \mathcal{M}$ then

$$\begin{aligned} m^T K^{-1} l &= \left(\vec{e}_i^T K\right) K^{-1} l \\ &= \vec{e}_i^T l \\ &= l_i \notin \mathbb{Z} \end{aligned} \tag{5.30}$$

We will first prove the claim in the forward direction. Assume that $\exists$ N linear independent integer vectors $\{\Lambda_i; i = 1, 2...N\}$ s.t. $\Lambda_i^T K \Lambda_j = 0$. We will explicitly construct the set of integer vectors, $\mathcal{M}$.

Forward Direction:

Let us perform a change of basis from standard basis to a new basis. We will again use the "Smith Canonical Form" idea to obtain new basis such that last N components of Λ_i are zero and K has the following form in that basis.

$$\tilde{K} = \begin{pmatrix} 0 & A \\ A^T & B \end{pmatrix} \quad \text{s.t} \quad B = B^T \tag{5.31}$$

where A and B are $N \times N$matrices. $\tilde{K}^{-1}$ can be calculated using the fact about Block matrix given in the appendix.

$$\tilde{K}^{-1} = \begin{bmatrix} -\left(A^T\right)^{-1} B A^{-1} & \left(A^T\right)^{-1} \\ (A)^{-1} & 0 \end{bmatrix} \tag{5.32}$$

I propose that all $\mathcal{M}$ are of the form

$$\begin{bmatrix} 0 \\ \vec{v} \end{bmatrix} \tag{5.33}$$

satisfies the above two criterion, where $\vec{v}$ is a N dimensional integer vector. Let us check whether $\mathcal{M}$ satisfies the first property. Consider $m, m' \in \mathcal{M}$.

$$\begin{aligned} m^T \tilde{K}^{-1} m' &= \begin{bmatrix} 0 & \vec{v}^T \end{bmatrix} \begin{bmatrix} -\left(A^T\right)^{-1} B A^{-1} & \left(A^T\right)^{-1} \\ (A)^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \vec{v'} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \vec{v}^T \end{bmatrix} \begin{bmatrix} \left(A^T\right)^{-1} \vec{v'} \\ 0 \end{bmatrix} = 0 \end{aligned} \tag{5.34}$$

In general up to modulo $\tilde{K}\mathbb{Z}^{2N}$ we get an integer.

$$m^T \tilde{K}^{-1} \left(m' + \tilde{K}\mathbb{Z}^{2N} \right) = m^T \tilde{K}^{-1} m' + m^T \mathbb{Z}^{2N} \tag{5.35}$$

which satisfies the first property. To prove the second property we will prove the contrapositive statement that if l be an integer vector such that $m^T K^{-1} l$ is an integer for all $m \in \mathcal{M}$ then l is equivalent to an element of $\mathcal{M}$. Consider

$$l = \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} \qquad \text{s.t.} \qquad m^T \tilde{K}^{-1} l \in \mathbb{Z} \qquad \forall m \in \mathcal{M} \tag{5.36}$$

$$\begin{aligned} m^T \tilde{K}^{-1} l &= \begin{bmatrix} 0 & \vec{v}^T \end{bmatrix} \begin{bmatrix} -\left(A^T\right)^{-1} B A^{-1} & \left(A^T\right)^{-1} \\ (A)^{-1} & 0 \end{bmatrix} \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} \\ &= \begin{bmatrix} \vec{v}^T (A)^{-1} & 0 \end{bmatrix} \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} \\ &= \vec{v}^T (A)^{-1} \vec{u}_1 \end{aligned} \tag{5.37}$$

$$\begin{aligned} &\implies \vec{v}^T (A)^{-1} \vec{u}_1 = p \in \mathbb{Z} \\ \implies &\sum_{i,j} v_i \left(A^{-1}\right)_{ij} \vec{u}_{1_j} = p \end{aligned} \tag{5.38}$$

In particular if $\vec{v} = \vec{e_k}$

$$\begin{aligned} \implies &\sum_j \left(A^{-1}\right)_{kj} \vec{u}_{1_j} = p_k \in \mathbb{Z} \\ &\implies \vec{u}_1 = A\vec{w} \quad \text{for some integer vector } \vec{w} \end{aligned} \tag{5.39}$$

This will enable us to write $\vec{l}$ as

$$\vec{l} = K. \begin{bmatrix} 0 \\ \vec{w} \end{bmatrix} + \begin{bmatrix} 0 \\ \vec{u}_2 - B\vec{w} \end{bmatrix} \tag{5.40}$$

This element evidently belongs to the class of $\mathcal{M}$. Hence we complete the proof in the forward direction.

Backward Direction:

Assume $\mathcal{M}$ to be a set of integer vector satisfying the above two properties. We want to construct $\{\Lambda_i; i = 1, 2...N\}$ satisfying the criterion that

$$\Lambda_i^T K \Lambda_j = 0 \quad \forall\, i, j \tag{5.41}$$

To that end let us consider the following set of integer vectors.

$$\mathcal{L} = \left\{m + Kx : m \in \mathcal{M}, x \in \mathbb{Z}^{2N}\right\} \tag{5.42}$$

It is clear that $\mathcal{L}$ is a discrete subgroup of $\mathbb{R}^{2N}$ under addition and indeed forms a lattice. This implies that $\mathcal{L}$ can be represented as follows

$$\mathcal{L} = U\mathbb{Z}^{2N} \tag{5.43}$$

where U is some $2N \times 2N$ integer matrix. Now, let us consider the matrix $P = U^T K^{-1} U$. We claim that it has the following properties.

1. It is symmetric. This follows from the fact that K is a symmetric matrix.

2. It has a vanishing signature. This follows from the fact that P and K^{-1} are congruent to each other. So both of them will have same signature, similarly for K and K^{-1}. We should then use the fact that K had a vanishing signature.

3. P has at least one odd entry in the diagonal.

4. P is an integer matrix. Consider the following bilinear form $y_1^T P y_2$ for some $y_1, y_2 \in \mathbb{Z}^{2N}$.

$$\begin{aligned} y_1^T P y_2 &= y_1^T U^T K^{-1} U y_2 \\ &= (m + Kx_1)^T K^{-1} (m' + Kx_2) \,\text{since}\, \mathcal{L} = U\mathbb{Z}^{2N} \\ &= m^T K^{-1} m' + m^T x_2 + x_1^T m' + x_1^T K x_2 \in \mathbb{Z} \end{aligned} \tag{5.44}$$

   In particular if we consider the standard basis of $\mathbb{R}^{2N}$ we can infer that the entries of P is indeed integers.

5. P has a determinant ± 1. Consider $w \notin \mathbb{Z}^{2N}$, the claim is that $v^T P w \notin \mathbb{Z}$ for some $v \in \mathbb{Z}^{2N}$.

$$\begin{aligned} v^T P w &= v^T \left(U^T K^{-1} U\right) w \\ &= (Uv)^T K^{-1} (Uw) \end{aligned} \tag{5.45}$$

where $Uv \in \mathcal{L}$. Uw is either not an element of $\mathbb{Z}^{2N}$ or it belongs to set of integer vectors which are inequivalent to $\mathcal{M}$. This follows from the fact that since U generate the lattice $\mathcal{L}$, U^{-1} is a well defined object. So if $Uw \in \mathbb{Z}^{2N}$, then it won't belong to $\mathcal{M}$. The second property of $\mathcal{M}$ assure the existence of such a v. This means that if $w \notin \mathbb{Z}^{2N} \implies Pw \notin \mathbb{Z}^{2N}$. The contrapositive of this statement would be if $w \in \mathbb{R}^{2N}$s.t. $Pw \in \mathbb{Z}^{2N} \implies w \in \mathbb{Z}^{2N}$. This will be used to prove that P forms a bijection between $\mathbb{Z}^{2N} \to \mathbb{Z}^{2N}$. Since $Px = y$ where $\forall y \in \mathbb{Z}^{2N} \exists$ an integer vector solution $x \in \mathbb{Z}^{2N}$implies that P is an onto function. To prove that P is into we use contradiction.

$$\begin{aligned}
&\exists\, x_1 \neq x_2 \in \mathbb{Z}^{2N} \text{ s.t. } Px_1 = Px_2 \\
&\quad \implies P(x_1 - x_2) = 0 && \text{,call } x_1 - x_2 = x \\
&\qquad\qquad\qquad\quad Px = 0 && \text{WLOG let assume } x_1 \text{ is the largest entry of } x \\
&\quad \implies P\left(\frac{1}{x_1}.x\right) = 0 && \text{call } \frac{1}{x_1}.x = \tilde{x}
\end{aligned}$$

$\tilde{x}$ is not an integer vector by construction, but $P\tilde{x} \in \mathbb{Z}^{2N}$. This lead to a contradiction which suggests that P is indeed into function. Now we can asserts that P^{-1} exists and it has to have integer entries since P forms a bijection between $\mathbb{Z}^{2N} \to \mathbb{Z}^{2N}$. Since P^{-1} is an integer matrix P has a determinant ± 1.

The next step is to use the following theorem [22]

Theorem 5.2.2. *Suppose A,B are two symmetric, indefinite, integer matrices with determinant ± 1. Suppose in addition, they have same dimension and same signature and are either both even or both odd where an even matrix has only even elements on the diagonal, and an odd matrix has at least one odd element on the diagonal. Then there must exists an integer matrix W with a unit determinant such that $W^T AW = B$.*

Applying this theorem to the matrix P which is an odd matrix with vanishing signature we can deduct that we can always block diagonalize P into

$$W^T P W = \begin{bmatrix} \boldsymbol{I}_{N\times N} & 0 \\ 0 & -\boldsymbol{I}_{N\times N} \end{bmatrix} \tag{5.46}$$

where W is an integer matrix with $\det(W) = \pm 1$. Now we can construct the Λ_is using W.

Define $v_i = w_i + w_{i+N}$ for $i \in \{1, 2, ..., N\}$, where w_i is the i'th column of W. If we define

$$\Lambda_i = \det(K).K^{-1}Uv_i \tag{5.47}$$

We can see that this set contains N linearly independent integer vectors,

$$\begin{aligned}
\text{then } \Lambda_i^T K \Lambda_j &= \left(\det(K).v_i^T U^T K^{-1}\right) K \left(\det(K).K^{-1}Uv_j\right) \\
&= (\det(K))^2 \, v_i^T \left(U^T K^{-1} U\right) v_j \\
&= (\det(K))^2 \, v_i^T P v_j \\
&= (\det(K))^2 \left(w_i^T + w_{i+N}^T\right) P \left(w_j + w_{j+N}\right) \\
&= (\det(K))^2 \left(w_i^T P w_j + w_i^T P w_{j+N} + w_{i+N}^T P w_j + w_{i+N}^T P w_{j+N}\right) \\
&= (\det(K))^2 \left(\delta_{ij} + 0 + 0 - \delta_{ij}\right) \\
&= 0
\end{aligned} \tag{5.48}$$

Hence we proved the existence of Λ_i's. □

5.3 Examples

Let us make our understanding strong by providing multiple examples.

Example 1

$$K = \begin{bmatrix} 1 & 0 \\ 0 & -p \end{bmatrix} \quad \text{where } p \text{ is an positive odd integer and} \quad \vec{q} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{5.49}$$

$$\text{let} \quad \Lambda = \begin{bmatrix} a \\ b \end{bmatrix} \tag{5.50}$$

$$\Lambda^T K \Lambda = 0 \implies a^2 - pb^2 = 0 \tag{5.51}$$

1. If p is a perfect square then there exist a null vector for the above K matrix given by

$$\Lambda = \begin{bmatrix} \pm\sqrt{p}b \\ b \end{bmatrix} \quad b \in \mathbb{Z} \tag{5.52}$$

An example for this case would be given by p=9. This can be thought of as a $\frac{8}{9}$ FQH system or an interface between filling fraction 1 and $\frac{1}{9}$ state with opposite magnetic field. One of the perturbation that can gap out this system is given by choosing

$\Lambda^T = \begin{bmatrix} 3 & -1 \end{bmatrix}$. This process corresponds to annihilating one electron at one edge and creating 3 electrons at another edge. Even though this process does not conserve particle number this can be achieved by placing the interface close to superconductor. Let us see how things works out in Levin's picture. We have to find a set of integer vector such that it satisfies the two conditions. The following set of vector forms the inequivalent class of integer vectors with required properties.

$$\mathcal{M} = \{\begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 9 \end{bmatrix}\} \tag{5.53}$$

2. There are no integer solution to above equation whenever p is not a perfect square. No perturbation $U(\Lambda)$ can open up a gap in that case. An experimentally realizable case would $p = 3$, which corresponds to either $\frac{2}{3}$ hierarchical edge state or an interface between 1 and $\frac{1}{3}$ state with opposite magnetic field.

FQH system with filling fraction $\frac{2}{3}$ and $\frac{8}{9}$ provides one of the easiest examples where we can perform numerical simulations. Especially in $\nu = \frac{2}{3}$, it wold be interesting to numerically verify whether it is robust against the above mentioned perturbations.

Example 2

$$K = \begin{bmatrix} p & 0 \\ 0 & -p \end{bmatrix} \qquad \text{where } p \text{ is an positive odd integer and} \quad \vec{q} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{5.54}$$

It is easy to see that above K matrix has a null vector $\Lambda^T = \begin{bmatrix} 1 & -1 \end{bmatrix}$. This process corresponds to annihilating an electron in one edge and creating one in another edge. Since it conserves charge this can be achieved with out the help of a superconducting interface. The special case of $p = 1$ corresponds to edge states IQH system that has been studied using the constructive bosonization approach in the chapter 2.

Example 3

$$K = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \qquad \vec{q} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{5.55}$$

This corresponds to a $\nu = \frac{2}{5}$ quantum Hall states. This has both edge mode with same chirality. Hence it isn't possible to gap out the system simultaneously. Instead let us construct the following 4×4 matrix out of this

$$\tilde{K} = \begin{bmatrix} K & 0 \\ 0 & -K \end{bmatrix} \tag{5.56}$$

This has two right and two left moving modes.

$$\text{let} \qquad \Lambda = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \tag{5.57}$$

$$\Lambda^T K \Lambda = 0 \implies 3a^2 + 4ab + 3b^2 - 3c^2 - 4cd - 3d^2 = 0$$

This can be achieved by choosing $a = c$ and $b = d$. One of the choices for null vector will be

$$\Lambda = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{5.58}$$

The above analysis can be extended into a more general class of K matrices given by

$$\tilde{K} = \begin{bmatrix} K_{N\times N} & 0_{N\times N} \\ 0_{N\times N} & -K_{N\times N} \end{bmatrix} \tag{5.59}$$

This matrix has null vectors given by

$$\Lambda^i_\alpha = \delta_{\alpha,i} + \delta_{\alpha,i+N} \quad \alpha \in \{1, 2..., N\} \qquad j \text{ represnts the coordinates} \tag{5.60}$$

This implies two coupled FQH with magnetic field applied in opposite direction can always gap out the system simultaneously.

Chapter 6

Conclusion and Outlook

This project was aimed at understanding the relationship between the microscopic picture and the effective field theory of the quantum Hall edge states. We have gathered a thorough conceptual understanding of integer quantum Hall edge states and the field theoretical picture of the edge. We have constructively bosonized the edge fermions and related these to standard results of Tomonaga-Luttinger Hamiltonian. The second part of the thesis mainly dealt with understanding FQH edge states starting from the Chern-Simons theory formulation. We observed that it is possible to construct fermion operators out of the bosonic edge modes which live at the boundary. Appropriate perturbations which gap out the edge modes correspond to the backscattering between these fermions. This idea was utilized to check the stability of the edge states in the final chapter. This project can be taken forward in the following ways.

- We saw $\frac{2}{3}$ FQH state is very robust against backscattering. This can be modeled either as a coupled/bilayer quantum Hall state with filling fraction 1 and $-\frac{1}{3}$ or a hierarchical quantum Hall state effective filling fraction $\frac{2}{3}$. A numerical analysis of the $\frac{2}{3}$ quantum Hall edge states in both settings can be done to gain further insight into its robustness.

- The effect of disorder was never taken into account in the null vector criterion. Extending the argument in the presence of disorder would be a good way to proceed.

- In the null vector criterion, we assumed that we have a uniformly distributed scatterer. But we can experimentally construct systems where we have a single scatterer such as a coupled quantum dot or a periodic/aperiodic array of scatterers. Looking at such theoretical models can tell about stability beyond the null vector criterion.

- Levin's criterion can be connected to the braiding statistics of quasiparticles. Whether edge can be gapped or not depends on the presence of a set of quasiparticles that can be annihilated at the edge. These form the Lagrangian subgroup of possible quasi-particle excitations present in the bulk. Understanding this concept will give more insight into the physical mechanism which is responsible for the stability.

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Appendix A

Useful Identities

A.1 Poisson Summation

Consider the following sawtooth function,

$$f(x) = n - \frac{1}{2} - x \quad \text{if} \quad x \in (n-1, n) \quad \text{where} \quad n \in \mathbb{Z} \tag{A.1}$$

Our fourier series convention is given below,

$$f(x) = \sum_{k \in \mathbb{Z}} e^{i2\pi kx} f(k) \quad \text{and} \quad f(k) = \int_0^1 e^{i2\pi kx} f(x)\, dx \tag{A.2}$$

Fourier components of the sawtooth functions can be calculated as,

$$f(k) = \int_0^1 e^{i2\pi kx} \left(\frac{1}{2} - x\right) dx = \begin{cases} 0 & if\ k = 0 \\ \frac{1}{i2\pi k} & if\ k \neq 0 \end{cases} \tag{A.3}$$

Let us write the derivative of $f(x)$ using both its definition and fourier series representation. Using fourier series we get,

$$\begin{aligned} f(x) &= \sum_{k \in \mathbb{Z}, k \neq 0} \frac{e^{i2\pi kx}}{i2\pi k} \\ \implies \frac{df(x)}{dx} &= \sum_{k \in \mathbb{Z}} e^{i2\pi kx} - 1 \end{aligned} \tag{A.4}$$

Using the definition, the derivative will be,

$$\frac{df(x)}{dx} = -1 + \sum_{k\in\mathbb{Z}} \delta(x-k) \tag{A.5}$$

Comparing these results we get,

$$\sum_{k\in\mathbb{Z}} e^{i2\pi kx} = \sum_{n\in\mathbb{Z}} \delta(x-n) \tag{A.6}$$

A.2 Baker-Hausdorff Identity and It's Corollaries

A and B are operators; Define $[A,B]_{n+1} \equiv [[A,B]_n, B]$, and $[A,B]_0 \equiv A$. Now consider the following operator-valued function $f(y) \equiv e^{-yB}Ae^{yB}, y \in \mathbb{R}$.

$$\begin{aligned} \frac{df(y)}{dy} &= -e^{-yB}BAe^{yB} + e^{-yB}ABe^{yB} \\ &= e^{-yB}[A,B]e^{yB} \end{aligned} \tag{A.7}$$

By induction we can prove the following,

$$\frac{d^n f(y)}{dy^n} = e^{-yB}[A,B]_n e^{yB} \tag{A.8}$$

Now if we Taylor expand the function around the point $y = 0$ we get,

$$\begin{aligned} f(y) &= \sum_{n=0}^{\infty} \frac{y^n}{n!} \left(\frac{d^n F(y)}{dy^n} \right)_{y=0} \\ &= \sum_{n=0}^{\infty} \frac{y^n}{n!} [A,B]_n \end{aligned} \tag{A.9}$$

If $y = 1$ then

$$e^{-B}Ae^{B} = \sum_{n=0}^{\infty} \frac{[A,B]_n}{n!} \tag{A.10}$$

Case 1 : If $C \equiv [A,B]$, and $[A,C] = [B,C] = 0$ then

$$\begin{aligned} e^{-B}Ae^{B} &= A + C \\ \left[A, e^{B}\right] &= Ce^{B} \end{aligned} \tag{A.11}$$

By inducting on the equation we can show that

$$e^{-B}\,(A)^n\,e^{B} = (A+C)^n \tag{A.12}$$

Using the Taylor expansion of the function $F(A)$,

$$\begin{aligned}
F(A) &= \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(0) A^n \\
&= e^{B} \left(\sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(0)\,(A+C)^n \right) e^{-B} \\
&= e^{B} F(A+C) e^{-B} \\
\implies e^{-B} F(A) e^{B} &= F(A+C)
\end{aligned} \tag{A.13}$$

If $F(A) = e^{A}$ then,

$$\begin{aligned}
e^{-B} e^{A} e^{B} &= e^{A+C} \\
\implies \; e^{A} e^{B} &= e^{B} e^{A} e^{C}
\end{aligned} \tag{A.14}$$

Now, consider the real valued operator function $G(y) = e^{yA} e^{yB}, y \in \mathbb{R}$.

$$\begin{aligned}
\frac{dG(y)}{dy} &= e^{yA} A e^{yB} + e^{yA} B e^{yB} \\
&= e^{yA} e^{yB} A + e^{yA} e^{yB} yC + e^{yA} e^{yB} B \\
&= G(y)\,(A + yC + B)
\end{aligned} \tag{A.15}$$

The solution to above differential equation with boundary condition $G(0) = 1$ is,

$$\begin{aligned}
G(y) &= e^{y(A+B) + \frac{y^2}{2} C} \\
&= e^{y(A+B)} e^{\frac{y^2}{2} C}
\end{aligned} \tag{A.16}$$

Thus if we put $y = 1$ we get

$$e^{A} e^{B} = e^{A+B} e^{\frac{C}{2}} \tag{A.17}$$

Case 2 : If $[A, B] = DB$ and $[A, D] = [B, D] = 0$ then

$$\begin{aligned} AB &= B\,(A+D) \\ \Longrightarrow A^n B &= B\,(A+D)^n \\ F(A)B &= \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(0) A^n B \\ &= B \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(0)\,(A+D)^n \\ &= BF(A+D) \end{aligned} \tag{A.18}$$

If $F(A) = e^A$ then

$$e^A B = B e^{A+D} \tag{A.19}$$

Appendix B

Bogoliubov Transformation

Assume that we have a bosonic Hamiltonian of the following form

$$H = \epsilon_1 b_1^+ b_1 + \epsilon_2 b_2^+ b_2 + \lambda \left(b_1^+ b_2^+ + b_2 b_1 \right) \tag{B.1}$$

We can employ a simple trick to diagonalize this Hamiltonian. We define two new bosons

$$\begin{aligned} d_1^+ &= u b_1^+ + v b_2 \\ d_2^+ &= u b_2^+ + v b_1 \end{aligned} \tag{B.2}$$

For d_1 and d_2 to obey the canonical commutation relation we require $u^2 - v^2 = 1$. Thus above transformation can be parameterized by $u = \cosh\alpha$ and $v = \sinh\alpha$. From the above equations we can write the inverse transformation as,

$$\begin{aligned} b_1^+ &= u d_1^+ - v d_2 \\ b_2^+ &= u d_2^+ - v d_1 \end{aligned} \tag{B.3}$$

If we write the Hamiltonian in terms of new boson we get

$$\begin{aligned} H = {} & d_1^+ d_1 \left\{ \epsilon_1 u^2 + \epsilon_2 v^2 - 2uv\lambda \right\} + d_2^+ d_2 \left\{ \epsilon_1 v^2 + \epsilon_2 u^2 - 2uv\lambda \right\} + d_1^+ d_2^+ \left\{ -\epsilon_1 uv - \epsilon_2 uv + \lambda u^2 + \lambda v^2 \right\} \\ & + d_1 d_2 \left\{ -\epsilon_1 uv - \epsilon_2 uv + \lambda v^2 + \lambda u^2 \right\} + \epsilon_2 v^2 + \epsilon_1 v^2 - 2uv\lambda \end{aligned} \tag{B.4}$$

If we want to get this Hamiltonian in diagonal form, the coefficient of the cross terms must be zero. i.e.

$$-\epsilon_1 uv - \epsilon_2 uv + \lambda u^2 + \lambda v^2 = 0 \tag{B.5}$$

We know $2uv = \text{sinh}2\alpha$ and $u^2 + v^2 = \text{cosh}2\alpha$. i.e. $(\epsilon_1 + \epsilon_2)\, uv = \lambda\, (u^2 + v^2)$.

$$\begin{aligned} \Rightarrow \frac{(\epsilon_1 + \epsilon_2)}{2}\text{sinh}2\alpha &= \lambda\text{cosh}2\alpha \\ \text{tanh}2\alpha &= \frac{2\lambda}{\epsilon_1 + \epsilon_2} \equiv \widetilde{\lambda} \end{aligned} \tag{B.6}$$

We can write α in terms of $\widetilde{\lambda}$.

$$\alpha = \frac{1}{4} ln \left(\frac{1 + \widetilde{\lambda}}{1 - \widetilde{\lambda}} \right) \tag{B.7}$$

So,

$$\begin{aligned} \text{cosh}2\alpha &= \frac{1}{\sqrt{1 - \widetilde{\lambda}^2}} \\ \text{sinh}2\alpha &= \frac{\widetilde{\lambda}}{\sqrt{1 - \widetilde{\lambda}^2}} \end{aligned} \tag{B.8}$$

$$\begin{aligned} \Longrightarrow u = \text{cosh}\alpha &= \frac{1}{2} \left[\left(\frac{1 + \widetilde{\lambda}}{1 - \widetilde{\lambda}} \right)^{\frac{1}{4}} + \left(\frac{1 - \widetilde{\lambda}}{1 + \widetilde{\lambda}} \right)^{\frac{1}{4}} \right] \\ v = \text{sinh}\alpha &= \frac{1}{2} \left[\left(\frac{1 + \widetilde{\lambda}}{1 - \widetilde{\lambda}} \right)^{\frac{1}{4}} - \left(\frac{1 - \widetilde{\lambda}}{1 + \widetilde{\lambda}} \right)^{\frac{1}{4}} \right] \end{aligned} \tag{B.9}$$

By eliminating the crossed terms we get the following Hamiltonian upto a constant,

$$H = d_1^{+} d_1 \left\{ \epsilon_1 u^2 + \epsilon_2 v^2 - 2uv\lambda \right\} + d_2^{+} d_2 \left\{ \epsilon_1 v^2 + \epsilon_2 u^2 - 2uv\lambda \right\} \tag{B.10}$$