Developing better Signal-Noise Discriminators for Gravitational Wave Signals from Compact Binary Coalescences in a Network of LIGO - like Detectors

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Certificate

This is to certify that this dissertation entitled **Developing better Signal-Noise Discriminators for Gravitational Wave Signals from Compact Binary Coalescences in a Network of LIGO - like Detectors** towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by **Sourath Tarun Ghosh** at Inter University Centre for Astronomy and Astrophysics (IUCAA), Pune, under the supervision of **Dr.Sukanta Bose**, Professor at Inter University Centre for Astronomy and Astrophysics (IUCAA), Pune, during the academic year 2018-2019.

15.3.19

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This thesis is dedicated to my dear parents

Declaration

I hereby declare that the matter embodied in the report entitled **Developing better Signal-Noise Discriminators for Gravitational Wave Signals from Compact Binary Coalescences in a Network of LIGO - like Detectors** are the results of the work carried out by me at the Department of Physics, Inter University Centre for Astronomy and Astrophysics, Pune, under the supervision of **Dr.Sukanta Bose** and the same has not been submitted elsewhere for any other degree.

Ghosh 20.3.2019

Sourath Tarun Ghosh

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Abstract

Even with the best noise canceling techniques, the data in the ground-based gravitational wave detectors are often plagued with non-stationary noise transients (glitches). These noise transients severely affect the precision of the matched filtering technique used to detect CBC signals [8],[6]. Hence, additional statistical tests (discriminators) are required. In 2017, a unified χ^2 formalism was proposed in Ref. [1], which is a mathematical framework for all single detector χ^2 distributed tests constructed in the context of CBC searches. This formalism, gave a procedure to construct a plethora of χ^2 discriminator tests and also gave a way to quantify the efficiency of a test at discriminating a certain glitch type. Consequently, it showed that previously known tests like the Allen's χ^2 tests [2] are special cases of the formalism. While, the authors of Ref. [1] also hint at a way to extend the formalism to the coherent multidetector case, they leave this case open for research.

In this thesis we explore the case of a coherent network of multiple detectors. We interpret the previously known Null SNR test (Ref. [5]) as a part of the extended unified χ^2 formalism. Consequently, we also construct other Null SNR-like tests which are constructed using subsets of the Null Space, while the Null SNR is constructed using the whole Null Space. In addition to these tests we propose a class of Network chi-squared tests, $\chi^2_{general}$ (statistically independent from Null SNR-like tests), that are derived from the basis vectors used in the single detector χ^2 tests in all the detectors.

The Null SNR like tests can sometimes be weak at discriminating double (multiple) coincident glitches, while $\chi^2_{general}$ does not face this issue. Also, unlike Null SNR-like tests, $\chi^2_{general}$ tests do not exploit the information contained in the detector beam pattern functions. Hence a network χ^2 which is an addition of a null SNR like test and a $\chi^2_{general}$ test should address both the weaknesses.

In addition to the theory, we have also numerically tested some of these discriminators. One such illustration is presented in this thesis

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Chapter 1

Introduction

1.1 A brief introduction to Gravitational Waves

In the year 1915, Einstein proposed a theory of gravity known as General Relativity. Using the assumptions of the equivalence principle and the assumption that speed of light is constant in all frames of references, this theory proposes a set of equations (analogous to Newtons inverse square law for gravitation) known as Einstein's Field equations. General Relativity teaches us to view gravitation due to a mass as the curvature in the space-time metric caused due to that mass rather than a force field. The Einstein's equations relate the object's mass (through the stress energy tensor) and the curvature created by that mass (through the Einstein tensor). **Gravitational waves are traveling wave solutions to the Einstein's equations**.

The Einstein's field equations are given by:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}R = 8\pi T_{\alpha\beta} \tag{1.1}$$

where, $G_{\alpha\beta}$ is the Einstein Tensor, $R_{\alpha\beta}$ is the Ricci Tensor, $R = R^{\alpha\beta}R_{\alpha\beta}$ is the Ricci scalar and $T_{\alpha\beta}$ is the Stress Energy Tensor.

The Ricci Tensor is related to the Riemann curvature tensor by,

$$R_{\alpha\beta} = R^{\gamma}_{\alpha\gamma\beta} \tag{1.2}$$

and the Riemann curvature tensor is related to the metric tensor $g_{\alpha\beta}$ through Christoffel connections.

For nearly flat metrics we can approximate them by a perturbation over the flat metric $\eta_{\alpha\beta}$. That is,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{1.3}$$

where

$$|h_{\alpha\beta}| \ll 1\,. \tag{1.4}$$

Note that, we choose,

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,. \tag{1.5}$$

To a good approximation, the Gravitational waves detected on Earth are traveling wave vaccuum solutions of the Einstein's equations in the weak field limit. This is because the source is far away, $T_{\alpha\beta} \approx 0$ at earth, and the deviations from the flat metric are very small.

We define three quantities,

$$h = h^{\alpha}_{\alpha} \tag{1.6}$$

$$\bar{h}^{\alpha\beta} = h^{\alpha}_{\beta} - \frac{\eta^{\alpha\beta}}{2} h \tag{1.7}$$

$$\therefore \bar{h} = -\, h\,. \tag{1.8}$$

The Riemann and Einstein tensors in weak field limit looks like

$$R_{abcd} = \frac{1}{2}(h_{ad,bc} + h_{bc,ad} - h_{ac,bd} - h_{bd,ac}) \tag{1.9}$$

$$G_{ab} = -\frac{1}{2}\left(\bar{h}^{,c}_{ab,c} + \eta_{ab}\bar{h}^{,cd}_{cd} - \bar{h}^{,c}_{bc,a} - \bar{h}^{,c}_{ac,b}\right) . \tag{1.10}$$

We now utilize gauge freedom to impose

$$\bar{h}^{ab}_{,b} = 0 . \tag{1.11}$$

In this gauge,

$$G^{ab} = -\frac{1}{2}\Box\bar{h}^{ab} . \tag{1.12}$$

Therefore the weak field Einstein equations become

$$\Box\bar{h}^{ab} = -16\pi T^{ab} . \tag{1.13}$$

We can now find the weak field metric by using the fact that newtons law holds in the limiting case $|\phi|, v << 1$. Note: $\partial_j = v_j\partial_0 \implies \Box = \nabla + O(v^2\nabla^2)$.
Hence, to lowest order we have

$$\nabla^2\bar{h}^{00} = -16\pi\rho . \tag{1.14}$$

Comparing with Newton's equation $\nabla^2\phi = 4\pi\rho$, we get

$$\bar{h}^{00} = -4\phi \tag{1.15}$$

Other components $\bar{h}^{ab}$ are negligible, hence

$$h = \bar{h}^{00} = -4\phi \tag{1.16}$$

$$\therefore h^{00} = -2\phi \tag{1.17}$$

$$h^{ii} = -2\phi . \tag{1.18}$$

Gravitational Waves that we detect are the traveling wave vacuum solutions of equation 1.13 . In other words they are solutions to

$$\Box\bar{h}^{ab} = 0 . \tag{1.19}$$

These solutions are plane waves of the form

$$\bar{h^{ab}} = A^{ab} \exp(ik_b x^b) \tag{1.20}$$

subject to $k^2 = 0$ This means that k is a null vector and the wave travels at the speed of light. This also implies here is no dispersion as phase velocity=group velocity =1, because

$$\omega^2 = |\vec{k}|^2 \,. \tag{1.21}$$

Now from the gauge condition (equation 1.11) we get

$$A^{\alpha\beta} k_\beta = 0 \,. \tag{1.22}$$

We can use the gauge freedom to further restrict the amplitude further. It is shown in [7] that we can impose

$$A^\alpha_\alpha = 0 \tag{1.23}$$

$$A_{\alpha\beta} U^\beta = 0 \,. \tag{1.24}$$

where U is the 4-velocity vector. Equations 1.22, 1.23 and 1.24 are called Transverse Traceless (TT) gauge. This is analogous to the residual gauge symmetry in Electromagnetism. For a photon in Coulomb gauge we can additionally set $A_0 = 0$ from 1.23 and 1.8 we get (as $h = 0$)

$$\bar{h}^{TT}_{ab} = h^{TT}_{ab} \,. \tag{1.25}$$

Now if the axes are oriented such that the wave travels along the +z axis then$A_{xx} = -A_{yy}$ and $A_{xy} = A_{yx}$.
Hence,

$$h_{xx}(t) = -\,h_{yy}(t) = h_+(t) \tag{1.26}$$

$$h_{xy}(t) = -\,h_{yx}(t) = h_\times(t) \,. \tag{1.27}$$

Hence the metric perturbation in the TT gauge for a gravitational wave travelling in the +z direction is,

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(t) & h_\times(t) & 0 \\ 0 & h_\times(t) & -h_+(t) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{1.28}$$

We can find the change in proper distance between two masses at $(0,0,0)$ and $(\epsilon,0,0)$ caused by an incomimg gravitational wave in +z direction,

$$\Delta l = \int |ds^2|^{1/2} \tag{1.29}$$

$$= \int_0^\epsilon |g_{xx}|^{1/2} \tag{1.30}$$

$$\approx |g_{xx}(x=0)|\epsilon \tag{1.31}$$

$$\approx \left(1 + \frac{1}{2} h_{xx}^{TT}\right) \epsilon \, . \tag{1.32}$$

We also see that on applying geodesic equation, and taking the only relevant connections Γ^α_{00} that

$$\frac{dU^\alpha}{d\tau} = 0 \, . \tag{1.33}$$

This is because in TT gauge $\Gamma^\alpha_{00} = 0$. So initially if a particle is at rest, it will remain at rest at the same coordinate in TT guage during the impact of the wave.

We can use the geodesic deviation equation to estimate the deviation in the world line of two particles . Now considering the geodesic deviation equation in a locally inertial frame, the equation reduces to

$$\frac{d^2}{d\tau^2} \xi^a = R^a_{bcd} U^b U^c \xi d = \epsilon R^a_{00x} = -\epsilon R^a_{0x0} \tag{1.34}$$

$$R^x_{0x0} = R_{x0x0} = -\frac{1}{2} h^{TT}_{xx,00} \tag{1.35}$$

$$R^y_{0y0} = R_{y0y0} = -\frac{1}{2} h^{TT}_{yy,00} = -R^x_{0x0} \tag{1.36}$$

$$R^y_{0x0} = R_{y0x0} = -\frac{1}{2} h^{TT}_{xy,00} \tag{1.37}$$

$$\frac{\partial^2}{\partial t^2}\xi^y = \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h_{yy}^{TT} = -\frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h_{xx}^{TT} = \frac{\partial^2}{\partial t^2}\xi^x \tag{1.38}$$

$$\frac{\partial^2}{\partial t^2}\xi^y = \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h_{xy}^{TT} = \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h_{yx}^{TT} = \frac{\partial^2}{\partial t^2}\xi^x\,. \tag{1.39}$$

There are two direct ways of detecting gravitational waves that have been proposed till date. First one being the resonant bar detector and the second on being the laser interferometer method (which has been far more successful than the first method).

The resonant bar detector works on the principle that a continuous media can be treated as a spring mass system. Now since the external gravitational wave is a plane wave of frequency Ω, it acts as external periodic forcing. The amplitude of induced vibration can be measured. The problem with the bar detector is that it can only detect signals with frequancies which are very close to the resonant frequency. Also there is lot of external noise as compared to the laser interferometer which will be discussed in the next section.

1.2 An Interferometric Detector and its Beam Pattern Functions

The detection of gravitational waves using a Michelson Interferometer is by far the best detection scheme. In absence of a gravitational wave the length of each arm is equal.When an incoming gravitational wave hits the detector, it stretches one arm while compressing the other. The difference in the strain in each arm as a function of time is the data received at the detector. **A gravitational wave signal can be written concisely as,**

$$h(t) = F_+(\theta,\phi,\chi)h_+(t) + F_\times(\theta,\phi,\chi)h_\times(t) \tag{1.40}$$

where θ,ϕ,χ are the polar, azimuthal and polarization angles of the incoming gravitational wave and $F_+(\theta,\phi,\chi)$, $F_\times(\theta,\phi,\chi)$ are the **plus and cross beam Pattern functions** respectively.

There are few coordinate systems/frame of references to keep in mind while estimating the beam pattern functions:

1. **The Source coordinate system/Source frame of reference for Compact Binary Coalescences** (X_s, Y_s, Z_s)**:** The Z_s axis is along the total angular momentum of the binary. The X_s axis is along the separation vector of the binary masses at the coalescence time t_c and the Y_s axis is along the direction which creates a right handed coordinate system.

2. **The wave coordinate system/Radiation frame of reference** (X_w, Y_w, Z_w)**:** This coordinate system is attached to the incoming wave. The Z_w axis is along the direction of propagation of the wave. The $(X_w), (Y_w)$ axes are chosen such that the coordinate system is right handed.

3. **The Detector coordinate system/frame of reference** (X_d, Y_d, Z_d)**:** The X_d axis is defined such that the arms of the detector make an angle of 45 degrees and 135 degrees in the counterclockwise orientation. The Y_d axis is defined such that the arms of the detector make an angle of -45 degrees and 45 degrees in the counterclockwise orientation and the Z_d axis is defined such that the system of coordinates is right handed.

Thus, the spatial perturbation matrix ($h_{\mu\nu}$ with the row and column corresponding to the time coordinate removed) in the wave frame takes the form

$$H_w(t) = \begin{pmatrix} h_+(t) & h_\times(t) & 0 \\ h_\times(t) & -h_+(t) & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{1.41}$$

The corresponding perturbation matrix in the detector frame H can be obtained by performing a coordinate transformation using an appropriate Euler rotation $O(\theta, \phi, \chi)$

$$H(t; \theta, \phi, \chi) = O^\dagger(\theta, \phi, \chi) H_w(t) O(\theta, \phi, \chi) . \tag{1.42}$$

Also, in detector frame of reference, unit vectors along the arms are:

$$\hat{l}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{1.43}$$

$$\hat{l}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} . \tag{1.44}$$

Let $\overrightarrow{l}_1$ and $\overrightarrow{l}_2$ represent the arms of the detector.

From the section 1.1, the change in arm lengths is given by

$$\overrightarrow{\delta l_1(t)} = \frac{1}{2} H(t;\theta,\phi,\chi)\, \overrightarrow{l_1} \tag{1.45}$$

$$\overrightarrow{\delta l_2(t)} = \frac{1}{2} H(t;\theta,\phi,\chi)\, \overrightarrow{l_2} . \tag{1.46}$$

Hence the strains in both the arms are given by

$$\frac{||\overrightarrow{\delta l_1}(t)||}{||\overrightarrow{l_1}||} = \frac{1}{2} \frac{\hat{l}_1^\dagger H(t;\theta,\phi,\chi)\, \overrightarrow{l_1}}{||\overrightarrow{l_1}||} = \frac{1}{2} \hat{l}_1^\dagger H(t;\theta,\phi,\chi) \hat{l}_1 \tag{1.47}$$

$$\frac{||\overrightarrow{\delta l_2}(t)||}{||\overrightarrow{l_2}||} = \frac{1}{2} \frac{\hat{l}_2^\dagger H(t;\theta,\phi,\chi)\, \overrightarrow{l_1}}{||\overrightarrow{l_2}||} = \frac{1}{2} \hat{l}_2^\dagger H(t;\theta,\phi,\chi) \hat{l}_2 . \tag{1.48}$$

The gravitational wave signal received in the detector is the difference in strain values of the detector.

$$h(t) = \frac{||\overrightarrow{\delta l_1}(t)||}{||\overrightarrow{l_1}||} - \frac{||\overrightarrow{\delta l_2}(t)||}{||\overrightarrow{l_2}||} . \tag{1.49}$$

The signal can be rewritten as,

$$h(t) = \frac{1}{2} \left(\hat{l}_1^\dagger O^\dagger(\theta,\phi,\chi) H_w(t) O(\theta,\phi,\chi) \hat{l}_1 - \hat{l}_2^\dagger O^\dagger(\theta,\phi,\chi) H_w(t) O(\theta,\phi,\chi) \hat{l}_2 \right) \tag{1.50}$$

$$= (O_{11}O_{21} - O_{12}O_{22}) h_+(t) + (O_{11}O_{22} + O_{12}O_{21}) h_\times . \tag{1.51}$$

Hence, from equation 1.40 we get

$$F_+(\theta,\phi,\chi) = (O_{11}(\theta,\phi,\chi)O_{21}(\theta,\phi,\chi) - O_{12}(\theta,\phi,\chi)O_{22}(\theta,\phi,\chi)) \tag{1.52}$$

$$= -\frac{1}{2}\left(1+\cos^2\theta\right)\cos 2\phi\cos 2\chi - \cos\theta\sin 2\phi\sin 2\chi \tag{1.53}$$

$$F_\times(\theta,\phi,\chi) = (O_{11}(\theta,\phi,\chi)O_{22}(\theta,\phi,\chi) + O_{12}(\theta,\phi,\chi)O_{21}(\theta,\phi,\chi)) \tag{1.54}$$

$$= +\frac{1}{2}\left(1+\cos^2\theta\right)\cos 2\phi\sin 2\chi - \cos\theta\sin 2\phi\cos 2\chi\,. \tag{1.55}$$

1.3 Compact Binary Coalescences (CBCs)

Compact Binary Coalescences are binary systems of either two Black Holes, two Neutron stars or one Black hole and one Neutron star, that coalesce to form one object. This object can either be a Black hole or a Neutron star.

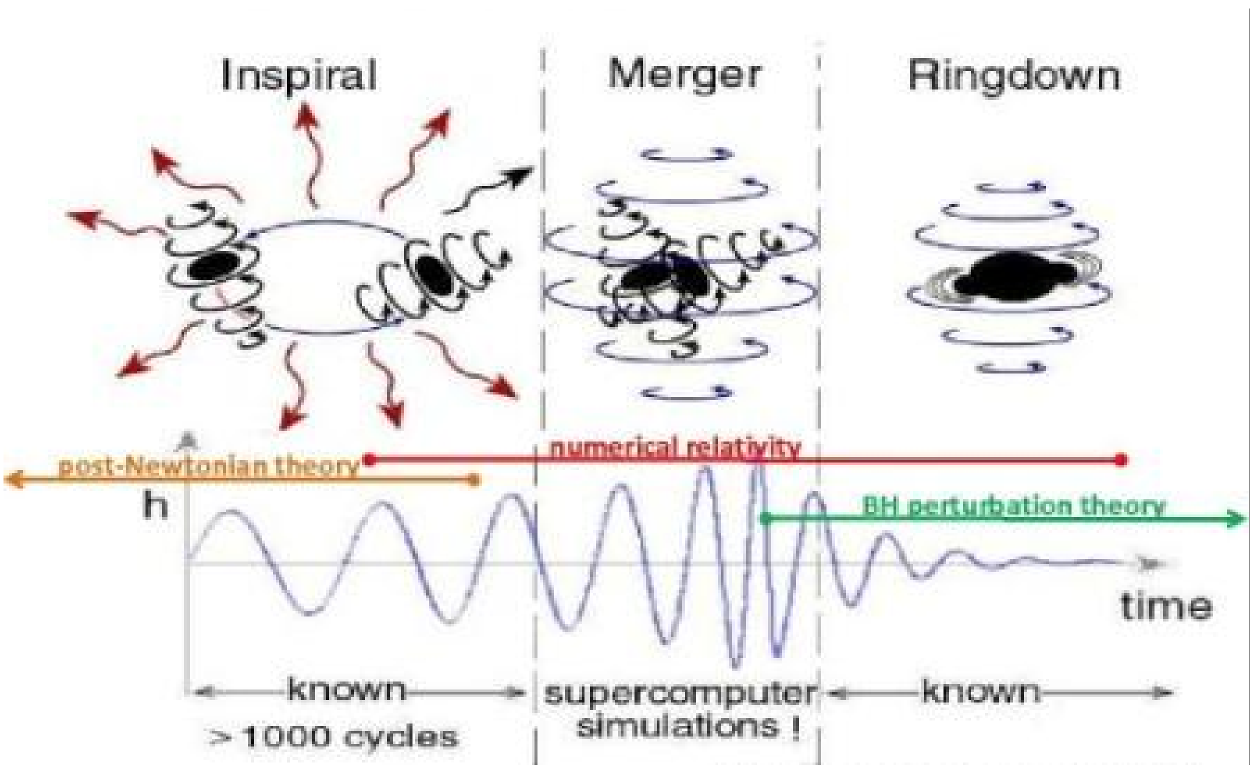


Figure 1.1: A sample Black hole Binary waveform. Picture taken from [9] with credits to Kip Thorne

One can estimate the waveform of a CBC by finding the (approximate) solution to Einstein's equation. The waveform, as shown in Fig. 1.1, can be split into three regions:

- Inspiral waveform: This waveform corresponds to the time span from about 1000 cycles before merger to about 10 cycles before merger. In this time period the space-time fabric is not heavily perturbed, hence this waveform is estimated by analytically approximating Einstein's equations in the weak field limit.

- Merger waveform: This waveform corresponds to the time span from about 10 cycles before merger to the time of merger. In this time period the space-time is highly perturbed and only numerical methods can be used to solve the Einstein's equations.

- Ringdown waveform: This is the part of the waveform corresponding to the time after the merger. The merger results in a single perturbed black hole or a single perturbed neutron star. Perturbation theory is used to estimate this part of the waveform.

The most general CBC waveform depends on 17 parameters but in this thesis we assume that the orbits are circular the masses of the binary are not spinning. This reduces the number of signal parameters to 9 (listed in Table. 1.1).

Parameters	Symbols
Binary Component Masses	M_1 and M_2
Coalescence Time	t_c
Coalescence Phase	ϕ_c
Distance to Source	D
Inclination Angle: The angle between the source angular momentum vector and the direction of incoming wave	ι
Polarization Angle (in essence only one parameter)	• ψ: From source frame to radiation frame • χ: From radiation frame to detector frame
Sky Directions: (Polar angle, azimuthal angle)	$(\theta, .\phi)$

Table 1.1: List of important Compact Binary Coalescence Parameters

The signal from a CBC can be written in the general form,

$$\tilde{h}(f) = Af^{-\frac{7}{6}} \exp\left(-i\Psi(f)\right) \tag{1.56}$$

where $\Psi(f)$ is the phase that researchers try to approximate. A depends on $M_1, M_2, \theta, \phi, \psi, \iota$ and D and the phase $\Psi(f)$ depends on M_1, M_2, t_c and ϕ_c. In simplest approximation, known as the Newtonian waveform,

$$\Psi_N(f; t_c, \tau_0, \phi_c) = 2\pi f t_c + \frac{6\pi f_s \tau_0}{5} \left(\frac{f}{f_s}\right)^{-5/3} - \phi_c - \frac{\pi}{4}\,. \tag{1.57}$$

Here, f_s is the seismic cutoff frequency ($f_s = 10$Hz in advanced LIGO detectors while $f_s = 40$Hz for initial LIGO detectors and the simulations in the thesis), τ_0 is the chirp time defined by,

$$\tau_0 = \frac{5}{256\pi f_s} \left(\pi \mathcal{M} f_s\right)^{-5/3} \text{seconds} \tag{1.58}$$

$$\approx 1393 \left(\frac{f_s}{10Hz}\right)^{-8/3} \left(\frac{\mathcal{M}}{M_\odot}\right)^{-5/3} \text{seconds}, \tag{1.59}$$

where $\mathcal{M} = \mu^{2/5} M_{tot}^{3/5}$ is the chirp mass, μ is the reduced mass and M_{tot} is the total mass of the binary.

1.4 Data Analysis Conventions and Matched Filtering

The conventions for Fourier transforms and inverse Fourier transforms are

$$\tilde{V}(f) = \int_{-\infty}^{\infty} V(t) \exp\left(-2\pi i f t\right) dt \text{ and} \tag{1.60}$$

$$V(t) = \int_{-\infty}^{\infty} \tilde{V}(f) \exp\left(2\pi i f t\right) df \,. \tag{1.61}$$

Definition 1.4.1. *The inner product of two time series $a(t)$ and $b(t)$ is defined in terms of their Fourier transforms and the detector power spectral density as*

$$(a(t), b(t)) = \int_{-\infty}^{\infty} \frac{\tilde{a}(f)\tilde{b}^*(f)}{S(f)} df \,. \tag{1.62}$$

The data time series in the detector is real. Hence, both $a(t)$ and $b(t)$are real.Therefore, $\tilde{a}(-f) = a^*(f)$, $\tilde{b}(-f) = b^*(f)$ and $S(-f) = S(f)$. So, the inner product can be re-written as

$$(a(t), b(t)) = \int_0^{\infty} \frac{\tilde{a}(f)\tilde{b}^*(f) + \tilde{a}^*(f)\tilde{b}(f)}{S(f)} df = 2\Re\left(\int_0^{\infty} \frac{\tilde{a}(f)\tilde{b}^*(f)}{S(f)} df\right) \,. \tag{1.63}$$

In a Network of M detectors, the network data vector is an $M \times 1$ vector, with the i^{th} entry being a time series corresponding to the i^{th} detector.

1.5 Background Noise Characteristics

The noise background of the detectors $(n(t))$ is assumed to be a second order stationary, zero-mean Gaussian random variable. Hence,

$$\begin{aligned} \langle n(t) \rangle =& 0 & (1.64) \\ \langle n(t) n(t') \rangle =& C(t - t') \,. & (1.65) \end{aligned}$$

Taking Fourier transforms of the above equations we get,

$$\begin{aligned} \langle \tilde{n}(f) \rangle =& 0 & (1.66) \\ \langle \tilde{n}(f) \tilde{n}^*(f') \rangle =& S(f) \delta(f - f') \,, & (1.67) \end{aligned}$$

where $S(f)$ is the Fourier transform of $C(t)$.
In data analysis, $C(t)$ is known as the two-point correlation function and $S(f)$ is known as the **two-sided power spectral density**. Note the some authors write the inner product in terms of the one sided power spectral density, which is half of the two-sided power spectral density.

The probability density function of a Gaussian random process $n(t)$ is given by,

$$p_n(n(t)) = A \exp\left(-(n(t), n(t))/2\right) \,, \tag{1.68}$$

with A being the normalization constant.

1.6 Matched Filtering

With the inner product defined in section 1.4, we can define a CBC template waveforms normalized to σ as

$$\tilde{\mathbf{h}}(f) = \sigma \frac{f^{-7/6} \exp\left(-i\psi(f)\right)}{\sqrt{2 \int_0^\infty f^{-7/3} df}} \,. \tag{1.69}$$

In other words,

$$(\mathbf{h}(t), \mathbf{h}(t)) = \sigma^2 \,. \tag{1.70}$$

In case we have only one detector, we can choose $\sigma = 1$.
Hence,

$$(\mathbf{h}(t), \mathbf{h}(t)) = 1\,. \tag{1.71}$$

Matched Filtering is a data analysis technique used to determine whether there is a signal of known waveform $h(t) = A\mathbf{h}(t)$ (in our case a CBC) present in the data $s(t)$ or not. The problem can be formulated as a hypothesis testing problem in statistics.

Null Hypothesis $\mathcal{H}_0$: Signal is not present :$s(t) = n(t)$.
Alternate Hypothesis $\mathcal{H}_1$: Signal is present: $s(t) = n(t) + h(t) \implies n(t) = s(t) - A\mathbf{h}(t)$.

The likelihood ratio that $\mathcal{H}_1$ is true given that the data is $s(t)$ is defined as

$$\lambda(\mathcal{H}_1|s) = \frac{p\left(s|\mathcal{H}_1\right)}{p\left(s|\mathcal{H}_0\right)} \tag{1.72}$$

$$= \frac{p\left(n(t) = s(t) - A\mathbf{h}(t)\right)}{p\left(n(t) = s(t)\right)} \tag{1.73}$$

$$= \exp(s(t), \mathbf{h}(t)) \exp\left(-(\mathbf{h}(t), \mathbf{h}(t)/2\right) \tag{1.74}$$

where the last step is obtained using eq.1.68. A certain threshold on likelihood ratio is agreed upon by researchers above which they consider the data as possible candidate for a CBC signal.

One can see that the likelihood ratio increases monotonically with the inner product $(s(t), \mathbf{h}(t))$. Hence any threshold on the likelihood function corresponds to a threshold on this inner product.The inner product

$$(s(t), \mathbf{h}(t)) = 2\Re \int_0^\infty \frac{\tilde{s}^*(f)\mathbf{h}(f)}{S(f)} df \tag{1.75}$$

is known as the match between the data $s(t)$ and the signal template $\mathbf{h}(t)$. The ratio $\frac{\mathbf{h}(f)}{S(f)}$ is known as the match filter.

In the matched filtering procedure, the data is time-slided with a bank of match filters (each filter corresponding to a different combination of CBC template parameters). The time at which the inner product is maximum is taken to be the time of coalescence. Also, to speed up computation, instead of spanning all possible coalescence phases, the template bank consists of templates with only $\phi_c = 0$, known as h_0 templates and $\phi_c = \pi/2$ known as $h_{\pi/2}$ templates. This is because a template with a general coalescence phase,

$$h_{\phi_c} = \cos\phi_c h_0 + \sin\phi_c h_{\pi/2}\,. \tag{1.76}$$

Hence, the optimal match filter output, known as Signal to Noise Ratio (SNR) for h_{ϕ_c} becomes,

$$\rho = \sqrt{(s(t), h_0(t))^2 + (s(t), h_{\pi/2}(t))^2}\,. \tag{1.77}$$

1.7 CBC Waveform and Templates

The general CBC signal can be rewritten in terms of $h_0(t)$ and $h_{\pi/2}(t)$ as

$$h_+(t) = \mathcal{A}^1 h_0(t) + \mathcal{A}^3 h_{\pi/2}(t) \tag{1.78}$$

$$h_\times(t) = \mathcal{A}^2 h_0(t) + \mathcal{A}^4 h_{\pi/2}(t) \tag{1.79}$$

where,

$$\mathcal{A}^1 = A_+ \cos 2\phi_c \cos 2\psi - A_\times \sin 2\phi_c \sin 2\psi \tag{1.80}$$

$$\mathcal{A}^2 = A_+ \cos 2\phi_c \sin 2\psi + A_\times \sin 2\phi_c \cos 2\psi \tag{1.81}$$

$$\mathcal{A}^3 = -A_+ \sin 2\phi_c \cos 2\psi - A_\times \cos 2\phi_c \sin 2\psi \tag{1.82}$$

$$\mathcal{A}^4 = -A_+ \sin 2\phi_c \sin 2\psi + A_\times \cos 2\phi_c \cos 2\psi \tag{1.83}$$

and,

$$A_+ = \frac{D_0}{D}\left(\frac{1+\cos^2\iota}{2}\right) \tag{1.84}$$

$$A_\times = \frac{D_0}{D}\cos\iota\,. \tag{1.85}$$

Therefore, the CBC waveform can be rewritten as,

$$h(t) = \mathcal{A}^\mu h_\mu(t) \tag{1.86}$$

where,

$$h_1(t) = F_+ h_0(t) \quad (1.87)$$

$$h_2(t) = F_\times h_0(t) \quad (1.88)$$

$$h_3(t) = F_+ h_{\pi/2}(t) \quad (1.89)$$

$$h_4(t) = F_\times h_{\pi/2}(t)\,. \quad (1.90)$$

1.8 Noise Transients (Glitches)

The term ‘glitch’ refers to the **transient non-stationary component** of the noise. The above mentioned matched filtering technique used to extract signal from noise, is optimal under the assumption that the background noise is stationary (equation 1.65). Hence, these glitches can fool the procedure even when few cycles of the glitch match with the cycles of the trigger template, causing the SNR to be high.

Many glitches can be modelled as Sine-Gaussian (SG) functions. Hence, for the thesis I have chosen this to be the model for the glitch.

Definition 1.8.1. *The SG glitch is defined as*

$$g(t) = A\sin(2\pi f_0 t)\exp\left(\frac{2\pi f_0 t}{Q}\right)^2 \quad (1.91)$$

where f_0 is the central frequency, Q is the quality factor and A is the amplitude of the glitch.

1.9 Some Useful Theorems in Statistics

The following are some useful theorems of Statistics used in this thesis.[10]

Theorem 1.9.1. *If X is a $N(\mu, 1)$ random variable, then $Y = X^2$ is a non-central χ^2 random variable with one degree of freedom and μ^2 as the non-central parameter.*

Corollary 1.9.2. *If X is a $N(0, 1)$ random variable, then $Y = X^2$ is a χ^2 random variable with one degree of freedom.*

Theorem 1.9.3. *If* X_i, $i = 1$ *to* N, *are independent non-central* χ^2 *with* p_i *degrees of freedom and non-centrality parameters* λ_i *respectively, then* $X = \sum_{i=1}^{N} X_i$ *is a non-central* χ^2 *random variable with* $p = \sum_{i=1}^{N} p_i$ *degrees of freedom and non-central parameter* $\lambda = \sum_{i=1}^{N} \lambda_i$.

Corollary 1.9.4. *If* X_i $i = 1$ *to* N *are independent* χ^2 *with* p_i *degrees of freedom then* $X = \sum_{i=1}^{N} X_i$ *is* χ^2 *random variable with* $p = \sum_{i=1}^{N} p_i$ *degrees of freedom.*

Chapter 2

Unified χ^2 formalism

As mentioned in the previous sections, non-stationary noise transients (glitches) can give a high SNR value and hence fool the matched filtering test. These transients occur very frequently in the data streams. Hence, to discriminate against them a class of tests known as χ^2 tests are used. The statistic computed by this test follows a χ^2 distribution when the data consists of pure stationary Gaussian noise or Gravitation wave signal and stationary gaussian noise. It acquires a non-central parameter in presence of a glitch.

In ref.[1] the authors provide a formalism which describes a general way to generate and quantify the effectiveness of all χ^2 tests.

2.1 The Formalism

The setting of this formalism is in an (ideally infinite dimensional) Hilbert space, $D = L_2([0,T],\mu)$, whose members are all possible data vectors (time series) of length T (the observation time).

The inner product in this space is identical to the one mentioned in section 1.4

Let's suppose $\mathbf{h}$ is the maximum SNR template which was triggered by the matched filtering procedure (henceforth, known as the triggered template). We consider the space ($\mathcal{S}$) which is a p dimensional subspace of the space ($\mathcal{N}_h$) orthogonal to the vector h. Mathematically,

$$\mathcal{N}_{\mathbf{h}} = \mathcal{D} - \{\mathbf{h}\} = \mathcal{D} \cap \{\mathbf{h}\}^c \text{ and} \tag{2.1}$$

$$\mathcal{S} \subset \mathcal{N}_{\mathbf{h}} \,. \tag{2.2}$$

The χ^2 test corresponding to the template $\mathbf{h}$ and the subspace $\mathcal{S}$ is just the $\mathcal{L}_2$ norm of the projection of the data vector on the subspace. In other words, if $\{u_1(t), u_2(t), \ldots, u_p(t)\}$ are a set of ortho-normal basis vectors of $\mathcal{S}$,

$$\chi^2_{\mathcal{S}}(s(t)) = ||s(t)_{\mathcal{S}}||^2 = \sum_{i=1}^{p} (s(t), u_i(t))^2 \,. \tag{2.3}$$

Now, for any non-stochastic vector $\lambda(t)$

- $(n(t) + \lambda(t), u_i(t))$ is a $\mathcal{N}((\lambda, u_i(t)), 1)$ random variable.

- Hence $(n(t) + \lambda(t), u_i)^2$ is a non-central χ^2 with one degree of freedom and non-central parameter $(\lambda, u_i)^2$.

- Therefore $\chi^2_{\mathcal{S}}(s(t) = n(t) + \lambda(t)) = \sum_{i=1}^{p}(n(t) + \lambda(t), u_i(t))^2$ is a non-central χ^2 with p degrees of freedom non-central parameter being $\sum_{i=1}^{p}(\lambda(t), u_i)^2 = ||\lambda(t)_{\mathcal{S}}||^2$.

Therefore, non-central parameter vanishes in presence of signal, i.e., $\lambda(t) = A\mathbf{h}(t)$. For a particular realization of a glitch, i.e., $\lambda(t) = g(t)$ the non-central parameter is the square of the projection of the glitch onto the subspace $\mathcal{S}$.
$\mathcal{N}$ is an infinite dimensional space orthogonal to $\mathbf{h}$, so we have infinitely many choices of $\mathcal{S}$. Each of those choices produces a unique χ^2 discriminator for the trigger template h.

In summary, to every template in the template bank, one can associate an orthogonal subspace (out of many choices) $\mathcal{S}$ to create the discriminator. Curiously, this matches exactly with the **Fibre-Bundle structure** in differential geometry as illustrate in Fig. 2.1.

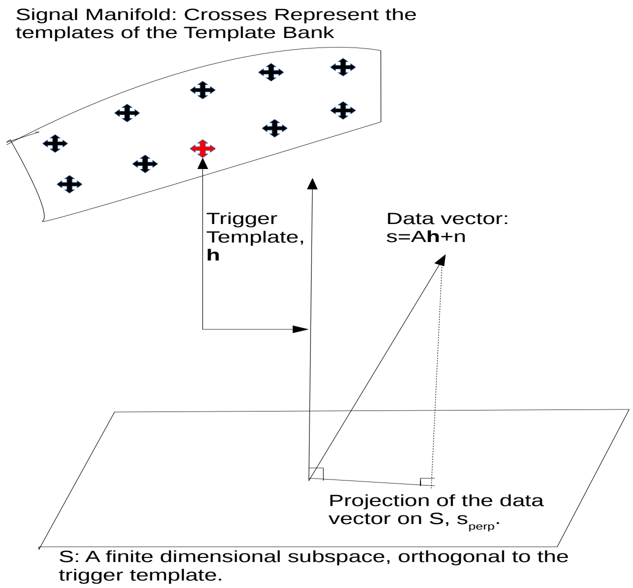


Figure 2.1: Geometrical Picture of the Unified χ^2 Formalism

A threshold of how many standard deviations (of the χ^2 distribution with p degrees of freedom) should the non-central parameter be in order to classify the data as a glitch is agreed upon by researchers. Here we choose the threshold to be 3 standard deviations.
The mean and standard deviation of a χ^2 random variable with p degrees of freedom are

$$\mu_{\chi^2_p} = p \tag{2.4}$$
$$\sigma_{\chi^2_p} = \sqrt{2p}\,. \tag{2.5}$$

Hence, if

$$\chi^2(s(t)) > p + 3\sqrt{2p} : \; s(t) \text{ is classified as a glitch,} \tag{2.6}$$
$$\chi^2(s(t)) \leq p + 3\sqrt{2p} : \; s(t) \text{ is not classified as a glitch.} \tag{2.7}$$

For discriminating a particular glitch from the template **h** a good choice of an orthogonal subspace $\mathcal{S}$ would be **a low dimensional subspace (for lower standard deviation of the expected/reference χ^2) with a high value of projection of the glitch on it.**

To check the efficiency of discriminator test, a plot of χ^2 per degree of freedom versus SNR is generated for injected glitches, injected signals, and pure Gaussian noise. As an example, a plot for the Ambiguity χ^2 test, taken from Ref [1], is shown in fig. 2.1. One can see that there is a clear separation between the χ^2 values corresponding to glitches and the χ^2 values corresponding to stationary Gaussian noise or signal plus stationary Gaussian noise.

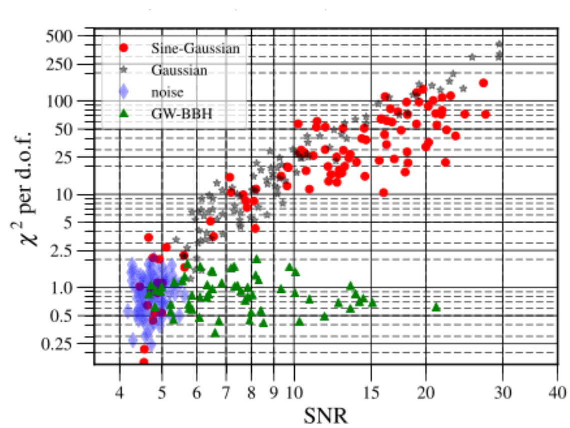


Figure 2.2: An example, taken from Ref [1] of testing the Ambiguity χ^2 Discriminator for triggers of binary black hole injections (Green), non-stationary Gaussian glitches, Sine Gaussian glitches, Stationary Gaussian noise (Blue).

2.2 Construction of a General χ^2

The Unified χ^2 formalism gives a procedure to construct a χ^2 discriminator for the trigger template h, from any arbitrary set of vectors $\{v_\alpha | \alpha = 1, 2, ..., n\}$:

I. First remove components parallel to h from all the vectors (v_α) to form new set of vectors $\{\Delta v_\alpha | \alpha = 1, 2, ..., n\}$,

$$\Delta v_\alpha = v_\alpha - (v_\alpha, \mathbf{h})\mathbf{h} \;\; \alpha = 1 \text{ to } n. \tag{2.8}$$

The span of this new set of vectors creates a **unique** p dimensional space $\mathcal{P}$ orthogonal to the template **h**, where $p \leq n$.

II. Ortho-normalize the set $\{\Delta v_\alpha | \alpha = 1, 2, ..., n\}$ to get a set of **basis vectors** $\{u_\alpha | \alpha = 1, 2, .., p\}$ of $\mathcal{P}$.

III. The χ^2 discriminator, which is the L_2 norm of the projection of the data vector onto $\mathcal{P}$ is given by

$$\chi^2_{\mathcal{P}}(s(t)) = ||s(t)_{\mathcal{P}}||^2 = \sum_{\alpha=1}^{p} (s(t), u_\alpha(t))^2 \,. \tag{2.9}$$

One of the many ways to perform this orthonormalization and obtain the χ^2 is the covariance matrix method.

(i) Let,

$$\Delta c_\alpha(s) = (s(t), \Delta v_\alpha(t)) \,. \tag{2.10}$$

(ii) The covariance matrix is defined as,

$$C_{\alpha\beta} = \langle \Delta c_\alpha(n) \Delta c_\beta(n) \rangle \,. \tag{2.11}$$

We can prove that,

$$C_{\alpha\beta} = \Delta v_\alpha(n) \Delta v_\beta(n) \,. \tag{2.12}$$

(iii) The χ^2 discriminator can be alternatively written as,

$$\chi^2_{\mathcal{P}}(s(t)) = \Delta c_\alpha(s) \left[C^{-1}\right]^{\alpha\beta} \Delta c_\beta(s) \,. \tag{2.13}$$

Note that if the set $\{v_\alpha\}$ is not linearly independent, i.e., $p < n$, C will not be invertible. C will have $n - p$ orthonormal eigenvectors having zero eigenvalue. In such a scenario, we need to remove these eigenvectors and reconstruct a new covariance matrix which has only non-zero eigenvalues.

2.3 Allen's χ_t^2 (The traditional χ^2)

This discriminator, given in Ref. [2], is one of the most commonly used discriminator in the data analysis pipeline of the gravitational wave detector. It uses the fact that even though the glitch has significant match with the template and resembles the match of true signal with template, when the frequency spectrum is divided into several bands, the match of the glitch with the template in each band would be different from the match of the true signal and template in that band. In other words, while the total power of the glitch from time 0 to T resembles the power of the signal from 0 to T, the power of the glitch in smaller time intervals would in general be different from the power of the signal.
The procedure to construct the Allen's χ^2 is as follows:

- Let **h** to be a normalized template waveform which got triggered in the matched filtering procedure.Split the frequency bands into p parts$\{\Delta f_\alpha|\alpha = 1 \text{ to } p\}$ $||\mathbf{h}|| = 1$
- Define the p waveform vectors:

$$\begin{aligned} \tilde{v}_\alpha(f) &= \tilde{h}(f) \quad f \in \Delta f_\alpha, && (2.14)\\ &= 0 \qquad \text{otherwise.} && (2.15)\\ & && (2.16) \end{aligned}$$

  Where,

$$|v_\alpha|^2 = q_\alpha \tag{2.17}$$

$$\sum_{\alpha=1}^{p} q_\alpha = 1\,. \tag{2.18}$$

- Define:

$$\Delta v_\alpha = v_\alpha - q_\alpha h\,. \tag{2.19}$$

- The Allen's χ^2 discriminator is then defined by

$$\chi_t^2(s(t)) = \sum_{\alpha=1}^{p} \frac{(s(t), \Delta v_\alpha)^2}{q_\alpha}\,. \tag{2.20}$$

- This χ^2 has $p-1$ degrees of freedom, due to the constraint,

$$\sum_{\alpha=1}^{p} \Delta v_\alpha = 0\,. \tag{2.21}$$

One can notice that $(\Delta v_\alpha, h) = 0$, i.e., Δv_α are orthogonal to h and thus span a subspace orthogonal to h.

In this thesis, after orthonormalizing $\{\Delta v_\alpha\}$, one set of basis we obtained are,

$$u_1 = \left(\frac{v_1}{q_1} - \frac{v_2}{q_2}\right)\left(\sqrt{\frac{1}{q_1} + \frac{1}{q_2}}\right)^{-1} \tag{2.22}$$

$$u_2 = \left(\frac{v_1 + v_2}{q_1 + q_2} - \frac{v_3}{q_3}\right)\left(\sqrt{\frac{1}{q_1 + q_2} + \frac{1}{q_3}}\right)^{-1} \tag{2.23}$$

$$\vdots \tag{2.24}$$

$$u_{p-1} = \left(\frac{v_1 + v_2 + \ldots + v_{p-1}}{q_1 + q_2 + \ldots + q_{p-1}} - \frac{v_p}{q_p}\right)\left(\sqrt{\frac{1}{q_1 + q_2 + \ldots + q_{p-1}} + \frac{1}{q_p}}\right)^{-1}\,. \tag{2.25}$$

We verify that the χ^2 discriminator generated by these basis vectors is indeed the Allen's χ^2 discriminator. That is,

$$\chi^2(s(t)) = \sum_{\alpha=1}^{p-1} (s(t), u_\alpha(t))^2 = \sum_{\alpha=1}^{p} \frac{(s(t), \Delta v_\alpha)^2}{q_\alpha} = \chi_t^2(s(t))\,. \tag{2.26}$$

2.4 Ambiguity χ^2

The ambiguity function takes two templates as inputs and gives the inner product as the output, i.e.,

$$\mathcal{H}_{\alpha\beta} = (h_\alpha, h_\beta)\,. \tag{2.27}$$

The Ambiguity χ^2 exploits the fact that the projection of the glitch on various templates of the template bank would be different when compared to the projection of the triggered template on those templates. The Ambiguity χ^2 discriminator is also designed with the intent to reduce the computational cost of the χ^2 test. In this test for a given trigger template h, p other templates h_1, h_2,...,h_p are used to construct the χ^2. The reduction in computational cost comes from the fact that the inner product between two templates, also known as ambiguity functions are already pre-computed before any data analysis is done. Also, the inner product between data and all templates are already computed as a part of the matched filtering test.

The procedure to construct the ambiguity χ^2 follows directly from the general formalism mentioned in section 2.2:

(i) Define ambiguity functions, $\mathcal{H}_{0\alpha} = (h_0(0), h_\alpha)$ and $\mathcal{H}_{\pi/2\alpha} = (h_{\pi/2}(0), h_\alpha)$.

(ii) $\Delta h_\alpha = h_\alpha - h_0(0)\mathcal{H}_{0\alpha} - h_{\pi/2}(0)\mathcal{H}_{\pi/2\alpha}$ (removing components parallel to h_0 and $h_{\pi/2}$).

(iii) $\Delta c_\alpha(x) = (x, \Delta h_\alpha)$.

(iv) $C_{\alpha\beta} = (\Delta h_\alpha, \Delta h_\beta) = (h_\alpha, h_\beta) - \mathcal{H}_{0\alpha}\mathcal{H}_{0\beta} - \mathcal{H}_{\pi/2\alpha}\mathcal{H}_{\pi/2\beta}$.

(v) $\chi^2 = \Delta c_\alpha [C^{-1}]^{\alpha\beta} \Delta c_\alpha$.

(vi) This procedure has a computationally cost of $O(p^2)$, since the only additional cost is to invert the covariance matrix.

2.5 Effect of Mismatch between Signal and Template Parameters

An inevitable error that creeps into the matched filtering, and consequently the χ^2 discriminators is the slight difference (mismatch) of the incoming signal and the template that gets selected by the matched filtering procedure.
There are two causes of mismatch:

1. Practically, the template bank is a discrete (not continuous) set of templates and the parameters vary in small discrete steps. Hence, **the template that perfectly matches the incoming gravitational wave (known as mismatch template) might be in between neighboring templates of the template bank.**

2. Templates are approximate solutions to the Einstein's Equation, while the incoming Gravitational wave signal is an exact solution. Improving these approximations is an active area of research in the field of Gravitational wave Source modelling. For simplicity, in this thesis, we **assume that templates are exact solutions to Einstein's Equation.**

Let θ denote all physical parameters of the template. In a template bank, the mismatch parameter ϵ is defined as the maximum possible deviation from unity the inner product between the trigger template $h(t;\theta)$ and the template correponding to incoming gravitational wave $h(t;\theta+\Delta\theta)$ can have. In other words,

$$1-\epsilon = \min\left[(h(t;\theta), h(t;\theta+\Delta\theta))\right] = \min\left[\mathcal{H}(\theta,\Delta\theta)\right] . \tag{2.28}$$

Here $\mathcal{H}(\theta,\Delta\theta)$ denotes the ambiguity function. The minimum is taken over all possible trigger templates and the respective deviations the GW signal template can have from these trigger templates. From section 2.1 we know that the component of the incoming signal template orthogonal to the trigger template will induce a non-central parameter in the χ^2.

Now,

$$h(t;\theta+\Delta\theta) = \mathcal{H}(\theta,\Delta\theta)h(t;\theta) + h(t;\theta+\Delta\theta)_{\perp} \qquad (2.29)$$

$$\therefore h(t;\theta+\Delta\theta)_{\perp} = h(t;\theta+\Delta\theta) - \mathcal{H}(\theta,\Delta\theta)h(t;\theta) \qquad (2.30)$$

$$\therefore ||h(t;\theta+\Delta\theta)_{\perp}||^2 = 1 - (\mathcal{H}(\theta,\Delta\theta))^2 = (1+\mathcal{H}(\theta,\Delta\theta))\,(1-\mathcal{H}(\theta,\Delta\theta)) < 2\epsilon\,. \qquad (2.31)$$

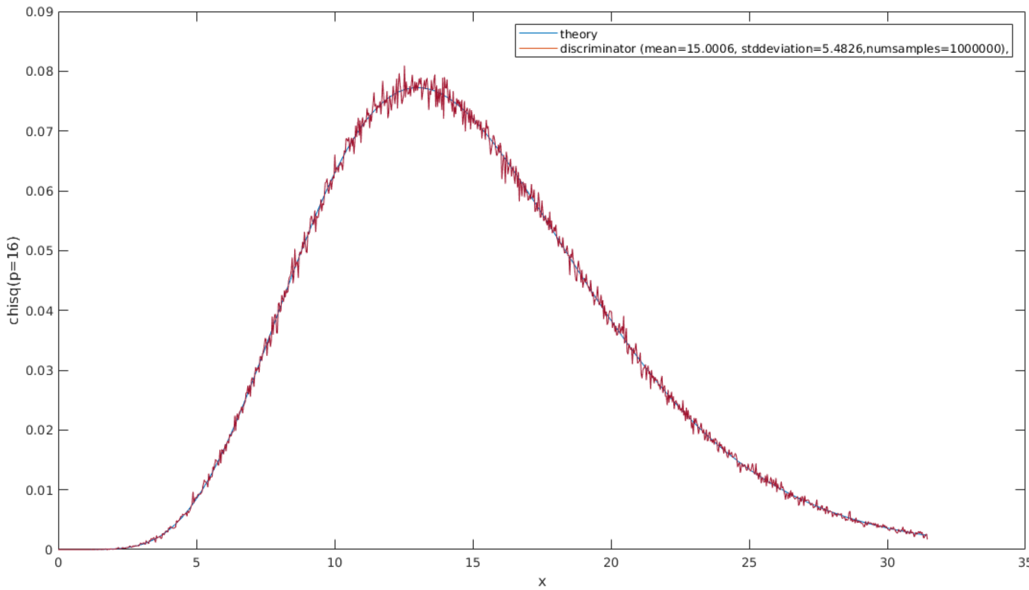
Therefore ,

$$||h(t;\theta+\Delta\theta)_{\mathcal{S}}||^2 < ||h(t;\theta+\Delta\theta)_{\perp}||^2 < 2\epsilon\,. \qquad (2.32)$$

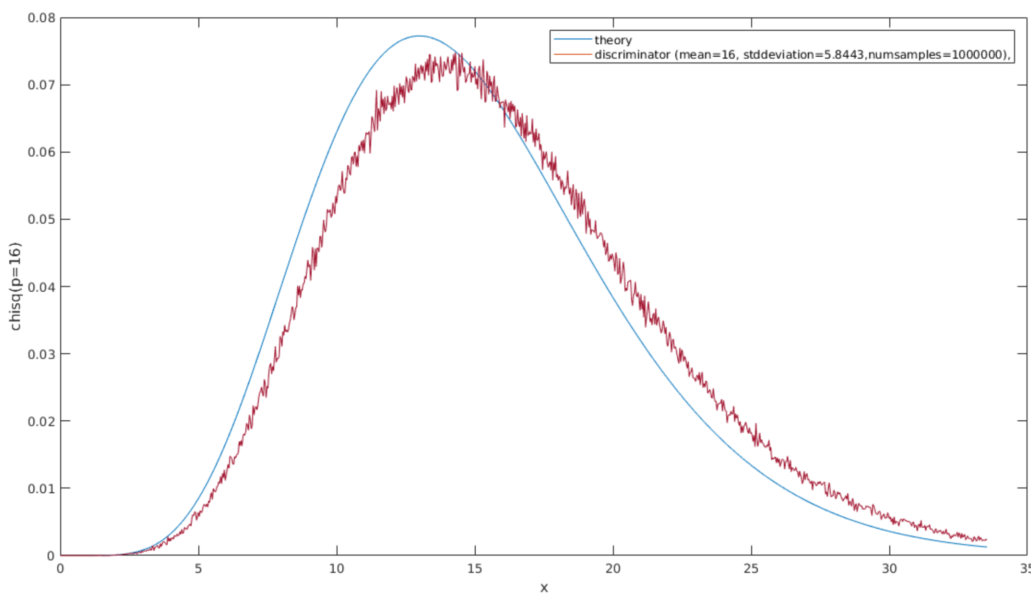
Let the amplitude of the incoming GW signal be A,

$$||Ah(t;\theta+\Delta\theta)_{\mathcal{S}}||^2 < 2A^2\epsilon\,. \qquad (2.33)$$

$\chi^2_{\mathcal{S}}(s(t) = Ah(t;\theta+\Delta\theta)+n)$ will be a non-central χ^2 with the non-central character less than $2A^2\epsilon$. Fig.2.5 illustrates the effect of signal-template parameter mismatch. We see in fig. 2.5.[b] the non-central character induced due to the mismatch.



[a]



[b]

Figure 2.3: [a] Perfect match between Template and Signal in Allen's χ^2 test with 16 bands.[b] Effect of small Mismatch between Template and Signal in Allen's χ^2 test with 16 bands.

Chapter 3

Coherent Network of M detectors

As of today, there are three operational kilometer-scale gravitational wave detectors namely LIGO Hanford, LIGO Livingston and VIRGO. In near future there are several other detectors coming up like Japan's KAGRA detector (in 2020), LIGO India (around 2024). Such a big network of detectors allows for what is known as a coherent analysis. This analysis aims to exploit the fact that, since the detectors are fixed on Earth, an incoming Gravitational Wave from a given direction fixes the beam pattern functions, and time delays in the arrival time of the Gravitational waves in all detectors. Therefore, by estimating the beam pattern functions and time delays, the source direction can be estimated. These features also open doors to new consistency tests which can be used to discriminate signals from noise.

3.1 Network Coordinate System

The coordinate system used for a network of detectors in this thesis is the centre of earth coordinate system (x, y, z). The origin is at the centre of the earth, the z axis points to the North Pole, the x-axis connects the origin to the point of intersection of the longitude passing through Greenwich and the equatorial plane. The y axis is chosen to create a right handed coordinate system.

In a system of multiple detectors, the gravitational wave does not reach all the detectors at the same time. The delay in the coalescence time measured at these detectors needs to be considered. We choose the reference time as the time measured in the Network frame of reference. In this frame, if R is the radius of the Earth, $\hat{r}^x$ is the unit vector along the position vector of the x$^{\text{th}}$ detector, and $\hat{r}$ is the unit vector along direction from which the gravitational wave is incoming, the difference in the coalescence time measured in the network frame and the detector frame is given by:

$$t - t^x = \delta t^x(\hat{r}^x, \theta, \phi, \chi) = \frac{R(\hat{r^x} \cdot \hat{r})}{c}\,. \tag{3.1}$$

Also, given the sky-direction and polarization angles of an incoming signal in the network frame (θ, ϕ, χ) the corresponding angles in the the detector frame $(\theta^x, \phi^x, \chi^x)$ are uniquely determined. Hence, if $\overrightarrow{\alpha^x}$ represents the position and orientation of the x$^{\text{th}}$ detector, the x$^{\text{th}}$ detector beam pattern functions can be recast in terms of (θ, ϕ, χ) and $\overrightarrow{\alpha^x}$. In other words,

$$F^x_{+,\times}(\theta^x, \phi^x, \chi^x) = F^x_{+,\times}\left(\overrightarrow{\alpha^x}, \theta, \phi, \chi\right)\,. \tag{3.2}$$

Hence, the signal in the x$^{\text{th}}$ detector is given by

$$h^x(t) = \mathcal{A}^\mu(D, \psi, \phi_c, \iota) h^x_\mu(t) \tag{3.3}$$

where,

$$h^x_1(t) = F^x_+ h_0(t^x) \tag{3.4}$$
$$h^x_2(t) = F^x_\times h_0(t^x) \tag{3.5}$$
$$h^x_3(t) = F^x_+ h_{\pi/2}(t^x) \tag{3.6}$$
$$h^x_4(t) = F^x_\times h_{\pi/2}(t^x)\,. \tag{3.7}$$

3.2 Data Analysis Conventions for a Network of Detectors

For a system of M detectors, the detector data vectors is represented by a $M\times$ one column vector with the x$^{\text{th}}$ component being the data in the x$^{\text{th}}$ detector.

$$\overrightarrow{s(t)} = \begin{bmatrix} s^1(t) \\ s^2(t) \\ \vdots \\ s^M(t) \end{bmatrix}_{M\times 1} . \tag{3.8}$$

Definition 3.2.1 (Network Inner Product)**.** *The inner product of two network (M detectors) time series is the sum of the inner products of the individual detector data vectors.That is,*

$$\left(\overrightarrow{a(t)}, \overrightarrow{b(t)}\right) = \sum_{x=1}^{M} \int_{-\infty}^{\infty} \frac{\tilde{a^x}(f)\tilde{b^x}^*(f)}{S_x(f)} df = \sum_{x=1}^{M} \left(a^x(t), b^x(t)\right)_x . \tag{3.9}$$

The noises in each detectors are assumed to be independent second order stationary and zero mean Gaussian. Hence,

$$\langle \tilde{n}^x(f) \rangle = 0 \tag{3.10}$$

$$\langle \tilde{n}^x(f)(\tilde{n}^y(f'))^* \rangle = \delta^{xy} S^x(f) \delta(f - f') . \tag{3.11}$$

Similarly the network beam pattern functions $\{+, \times\}$ are defined as,

$$\overrightarrow{F_{+,\times}} = \begin{bmatrix} F^1_{+,\times} \\ F^2_{+,\times} \\ \vdots \\ F^M_{+,\times} \end{bmatrix}_{M\times 1} . \tag{3.12}$$

The network signal vector can be written as,

$$\overrightarrow{h(t)} = \begin{bmatrix} h^1(t) \\ h^2(t) \\ \vdots \\ h^M(t) \end{bmatrix} = \mathcal{A}^\mu(D,\psi,\phi_c,\iota) \begin{bmatrix} h^1_\mu(t) \\ h^2_\mu(t) \\ \vdots \\ h^M_\mu(t) \end{bmatrix} = \mathcal{A}^\mu(D,\psi,\phi_c,\iota)\overrightarrow{h_\mu(t)} \tag{3.13}$$

or alternatively,

$$\overrightarrow{h(t)} = \overrightarrow{F_+} h_+(t) + \overrightarrow{F_\times} h_\times(t)\,. \tag{3.14}$$

Note on template normalization in each detector: In a system of multiple detectors, the power spectral densities in each detector will in general be different. Hence we cannot normalize h_0 and $h_{\pi/2}$ to 1 in every detector. So we let the be normalized to σ^x in the xth detector. That is,

$$(h_0(t), h_0(t))_x = (h_{\pi/2}(t), h_{\pi/2}(t))_x = (\sigma^x)^2 \tag{3.15}$$

$$(h_0(t), h_{\pi/2}(t))_x = 0 \tag{3.16}$$

This can be written more concisely in the form, for $i, j \in \{0, \pi/2\}$ and for all detectors ($\forall x = 1, 2, \ldots, M$),

$$\frac{(h_i(t), h_j(t))_x}{(\sigma^x)^2} = \delta_{ij}\,. \tag{3.17}$$

3.3 Generalization of the Unified χ^2 Formalism to the Coherent Multidetector Case

Having defined the conventions for a network of detectors one can easily generalize the unified χ^2 formalism to the case of multiple detectors, as follows [1]:

- The Hilbert space of data vectors is

$$\mathcal{D}_{network} = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \ldots \oplus \mathcal{D}_M\,, \tag{3.18}$$

  where $\mathcal{D}_x$ is the Hilbert space corresponding to the x$^{\text{th}}$ detector.

- With the network inner product defined earlier one needs to find a finite dimensional subspace $\mathcal{S}_{network} \in \mathcal{D}_{network}$, which is orthogonal to $\overrightarrow{h_\mu(t)}$, $\mu = 1, 2, 3, 4$.

- The χ^2 (like in the single detector) is just the L_2 norm of the network data vector projected onto $\mathcal{S}_{network}$.

3.3.1 Example: Sum of individual detector χ^2 (χ^2_{add})

Lets consider that each detector performs a single detector χ^2 with p degrees of freedom.Let the basis of the orthogonal subspace,corresponding to the maximum SNR template of the x$^{\text{th}}$ detector be denoted by,

$$u_{xk}(t) \text{ where } k = 1 \text{ to } p\,.$$

Hence,

$$\frac{(u_{xk}(t), u_{xl}(t))_x}{(\sigma^x)^2} = \delta_{kl}\,. \tag{3.19}$$

We can create a network χ^2 discriminator by simply adding the χ^2 test at each detector. That is, using equation 3.19, the network χ^2 discriminator is

$$\chi^2_{add} = \sum_{x=1}^{M} \chi^2_x \tag{3.20}$$

$$= \sum_{x=1}^{M} \sum_{k=1}^{p} \left(s^x(t), \frac{u_{xk}(t)}{\sigma^x} \right)^2_x . \tag{3.21}$$

We know that χ^2_{add} has to be a χ^2 with pM degrees of freedom since the noises in each detector are independent. As a check, we can obtain a basis set of $p \times M$ vectors corresponding to this χ^2 (proof in next section):

$$\overrightarrow{v_{xk}(t)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \frac{u_{xk}(t)}{\sigma^x} \\ \vdots \end{bmatrix}_{M\times 1} \quad \text{(Non-zero entry only in the x}^{\text{th}}\text{ position)} \tag{3.22}$$

That is,

$$\chi^2_{add} = \sum_{x=1}^{M} \sum_{k=1}^{p} \left(\overrightarrow{s(t)}, \overrightarrow{v_{xk}(t)} \right)^2 . \tag{3.23}$$

The orthogonal subspace $\mathcal{S}_{Network}$ is the span of the vectors $\overrightarrow{w_{xk}(t)}$.

3.4 Coherent SNR: The Appropriate Generalization of Single detector SNR

The network data vector is given by,

$$\overrightarrow{s(t)} = \overrightarrow{n(t)} + \overrightarrow{h(t)} . \tag{3.24}$$

The log-likelihood ratio for the x$^{\text{th}}$ detector is given by

$$\log \Lambda^x(h) = (s^x, h^x) - (h, h)_x/2 . \tag{3.25}$$

The noises in each detector are assumed to be independent. Hence, the conditional probabilities multiply and the network likelihood ratio is given by

$$\Lambda_{Network} = \prod_{x=1}^{M} \frac{P(s^x|h^x(t))}{P(s^x|h^x(t)=0)} = \prod_{x=1}^{M} \Lambda^x(h) . \tag{3.26}$$

Therefore,

$$\log \Lambda_{Network} = \sum_{x=1}^{M} \log \Lambda^x = \sum_{x=1}^{M} (s^x, h^x) - (h, h)_x/2 \tag{3.27}$$

$$= (\overrightarrow{s}, \overrightarrow{h}) - (\overrightarrow{h}, \overrightarrow{h})/2 . \tag{3.28}$$

Let C be a symmetric 4×4 matrix defined by,

$$C_{\mu\nu} = (\overrightarrow{h_\mu}, \overrightarrow{h_\nu}) . \tag{3.29}$$

Then,

$$\ln \Lambda_{Net} =(\overrightarrow{s}, A^{\mu}\overrightarrow{h_{\mu}}) - \frac{1}{2}(A^{\mu}\overrightarrow{h_{\mu}}, A^{\nu}\overrightarrow{h_{\nu}}) \tag{3.30}$$

$$=A^{\mu}(\overrightarrow{s}, \overrightarrow{h_{\mu}}) - \frac{1}{2}A^{\mu}C_{\mu\nu}A^{\nu}\,. \tag{3.31}$$

We need to maximize the network log-likelihood, as follows:

$$\frac{\partial}{\partial A^{\alpha}} \ln \Lambda_{Net} =0 \tag{3.32}$$

$$\implies \frac{\partial}{\partial A^{\alpha}} \left(A^{\mu}(\overrightarrow{s}, \overrightarrow{h_{\mu}}) - \frac{1}{2}A^{\mu}C_{\mu\nu}A^{\nu} \right) =0\,. \tag{3.33}$$

Invoking the fact that C is symmetric, we get

$$(\overrightarrow{s}, \overrightarrow{h_{\alpha}}) - \frac{1}{2}\left(\delta^{\mu}_{\alpha}C_{\mu\nu}A^{\nu} + A^{\mu}C_{\mu\nu}\delta^{\nu}_{\alpha}\right) = 0 \implies (\overrightarrow{s}, \overrightarrow{h_{\alpha}}) = C_{\alpha\nu}A^{\nu}\,. \tag{3.34}$$

Inverting, by multiplying the equation with $C^{\mu\alpha}$ we get

$$C^{\mu\alpha}(\overrightarrow{s}, \overrightarrow{h_{\alpha}}) =C^{\mu\alpha}C_{\alpha\nu}A^{\nu} \tag{3.35}$$

$$\implies C^{\mu\alpha}(\overrightarrow{s}, \overrightarrow{h_{\alpha}}) =\delta^{\mu}_{\nu}A^{\nu} \tag{3.36}$$

$$\implies C^{\mu\alpha}(\overrightarrow{s}, \overrightarrow{h_{\alpha}}) =A^{\mu}\,. \tag{3.37}$$

The **Coherent SNR** for the network is defined as,

$$\rho^2_{coh} =2\ln \Lambda_{Net,Max} \tag{3.38}$$

$$=2M^{\mu\nu}(\overrightarrow{s}, \overrightarrow{h_{\nu}})(\overrightarrow{s}, \overrightarrow{h_{\mu}}) \tag{3.39}$$

$$-(\overrightarrow{s}, \overrightarrow{h_{\alpha}})C^{\mu\alpha}M_{\mu\nu}(\overrightarrow{s}, \overrightarrow{h_{\beta}})C^{\nu\beta} \tag{3.40}$$

$$=2(\overrightarrow{s}, \overrightarrow{h_{\nu}})C^{\mu\nu}(\overrightarrow{s}, \overrightarrow{h_{\mu}})(\overrightarrow{s}, \overrightarrow{h_{\nu}})C^{\mu\nu}(\overrightarrow{s}, \overrightarrow{h_{\mu}}) \tag{3.41}$$

$$=(\overrightarrow{s}, \overrightarrow{h_{\nu}})C^{\mu\nu}(\overrightarrow{s}, \overrightarrow{h_{\mu}})\,. \tag{3.42}$$

Now we define $C_+, C_\times, C_{+\times}$ as

$$C_+ = \sum_{x=1}^{M} (\sigma^x F_+^x)^2 = \left(\overrightarrow{F_+ h_0}, \overrightarrow{F_+ h_0}\right) \tag{3.43}$$

$$C_\times = \sum_{x=1}^{M} (\sigma^x F_\times^x)^2 = \left(\overrightarrow{F_\times h_0}, \overrightarrow{F_\times h_0}\right) \tag{3.44}$$

$$C_{+\times} = \sum_{x=1}^{M} (\sigma^x F_+^x)(\sigma^x F_\times^x) = \left(\overrightarrow{F_+ h_0}, \overrightarrow{F_\times h_0}\right) \;. \tag{3.45}$$

Therefore, from the definition of $C_{\mu\nu}$,

$$C_{\mu\nu} = \sum_{x=1}^{M} \left(h_\mu^x, h_\nu^x\right)_x \tag{3.46}$$

$$\therefore C_{\mu\nu} = \begin{pmatrix} C_+ & C_{+\times} & 0 & 0 \\ C_{+\times} & C_\times & 0 & 0 \\ 0 & 0 & C_+ & C_{+\times} \\ 0 & 0 & C_{+\times} & C_\times \end{pmatrix} . \tag{3.47}$$

The polarization angle comes into picture while relating the source coordinate system to the wave coordinate system (ψ) and also while relating the wave coordinate system to the detector coordinate system (χ). Hence, we can choose (χ) such that the matrix C is diagonal. The ψ changes according to the wave coordinate system generated (the dominant polarization frame) by this choice of (χ). In the dominant polarization frame,

$$C_{+\times} = 0\,. \tag{3.48}$$

Hence, in the dominant polarization frame, $C_{\mu\nu}$ is **diagonal**. Therefore,

$$C_{\mu\nu} = \begin{pmatrix} C_+ & & & \\ & C_\times & & \\ & & C_+ & \\ & & & C_\times \end{pmatrix} \tag{3.49}$$

$$C^{\mu\nu} = \begin{pmatrix} 1/C_+ & & & \\ & 1/C_\times & & \\ & & 1/C_+ & \\ & & & 1/C_\times \end{pmatrix} . \tag{3.50}$$

Consequently,

$$\rho_{coh}^2 =(\overrightarrow{s}, \overrightarrow{h_{mu}})C^{\mu\nu}(\overrightarrow{s}, \overrightarrow{h_{nu}}) \tag{3.51}$$

$$=\frac{1}{C_+}\left[(\overrightarrow{s}, \overrightarrow{F_+ h_0})^2 + (\overrightarrow{s}, \overrightarrow{F_+ h_{\pi/2}})^2\right]$$
$$+\frac{1}{C_\times}\left[(\overrightarrow{s}, \overrightarrow{F_\times h_0})^2 + (\overrightarrow{s}, \overrightarrow{F_\times h_{\pi/2}})^2\right] . \tag{3.52}$$

Now we define f_+^x and $f_\times^x$ as

$$f_{+,\times}^x = \frac{\sigma^x F_{+,\times}^x}{\sqrt{C_{+,\times}}} . \tag{3.53}$$

Therefore,

$$\overrightarrow{f_{+,\times}} \cdot \overrightarrow{f_{+,\times}} =\frac{\sum_{x=1}^M (\sigma^x F_{+,\times}^x)^2}{C_{+,\times}} \tag{3.54}$$

$$=\frac{C_{+,\times}}{C_{+,\times}} = 1 , \tag{3.55}$$

and

$$\overrightarrow{f_+} \cdot \overrightarrow{f_\times} =\frac{\sum_{x=1}^M \sigma^x F_x \sigma^x F_\times}{\sqrt{C_+ C_\times}} \tag{3.56}$$

$$=\frac{C_{+\times}}{\sqrt{C_+ C_\times}} \tag{3.57}$$

$$= 0 . \tag{3.58}$$

Therefore $\overrightarrow{f_+}$ and $\overrightarrow{f_\times}$ are $M \times 1$ orthonormal vectors in the dominant polarization frame. Now,

$$\frac{(\overrightarrow{s}, \overrightarrow{F_{+,\times} h_{0,\pi/2}})^2}{C_{+,\times}} = \left(\sum_{x=1}^{M} \left(s^x, h_{0,\pi/2} \frac{F^x_{+,\times}}{\sqrt{C_{+,\times}}}\right)\right)^2 \tag{3.59}$$

$$= \left(\sum_{x=1}^{M} \left(s^x, h_{0,\pi/2} \frac{f^x_{+,\times}}{\sigma^x}\right)\right)^2 \tag{3.60}$$

$$= \left(\sum_{x=1}^{M} \left(s^x, h_{0,\pi/2} \frac{f^x_{+,\times}}{\sigma^x}\right)\right) \left(\sum_{y=1}^{M} \left(s^y, h_{0,\pi/2} \frac{f^y_{+,\times}}{\sigma^y}\right)\right) \tag{3.61}$$

$$= \sum_{x,y=1}^{M} \left(s^x, \frac{h_{0,\pi/2}}{\sigma^x}\right) f^x_{+,\times} f^y_{+,\times} \left(s^y, \frac{h_{0,\pi/2}}{\sigma^y}\right) . \tag{3.62}$$

Now, we define two $M \times 1$ vectors $\overrightarrow{c_0}$, $\overrightarrow{c_{\pi/2}}$ as

$$\overrightarrow{c_{0,\pi/2}} = \begin{bmatrix} \left(s^1, \frac{h_{0,\pi/2}}{\sigma^1}\right) \\ \left(s^2, \frac{h_{0,\pi/2}}{\sigma^2}\right) \\ \vdots \\ \left(s^M, \frac{h_{0,\pi/2}}{\sigma^M}\right) \end{bmatrix} . \tag{3.63}$$

Therefore,

$$\frac{(\overrightarrow{s}, \overrightarrow{F_{+,\times} h_{0,\pi/2}})^2}{C_{+,\times}} = \left(\overrightarrow{c_{0,\pi/2}} \cdot \overrightarrow{f_{+,\times}}\right)^2 \tag{3.64}$$

and ρ^2_{coh} can be rewritten as

$$\rho^2_{coh} = \sum_{i=0,\pi/2} \left(\overrightarrow{c_i} \cdot \overrightarrow{f_+}\right)^2 + \left(\overrightarrow{c_i} \cdot \overrightarrow{f_\times}\right)^2 . \tag{3.65}$$

The coincident square-SNR ρ^2_{coinc} is just the sum of square-SNR in each dectector,that is

$$\rho^2_{coinc} = \sum_{x=1}^{M} \rho^2_x \tag{3.66}$$

$$= \sum_{x=1}^{M} \frac{(s^x, h_0)^2 + (s^x, h_{\pi/2})^2}{(\sigma^x)^2} \tag{3.67}$$

$$= \sum_{i=0,\pi/2} ||\overrightarrow{c_i}||^2 \,. \tag{3.68}$$

Chapter 4

Null SNR and other Network χ^2 Discriminators

4.1 The Concept of Null Streams

We see from equation 3.14 that the signal in every detector is the linear combination of the plus and cross polarizations of the Gravitational wave. Hence, in the ideal case where there in no noise, for a system of $M > 2$ detectors, we can use two equations to solve for $h_+(t)$ and $h_\times(t)$ and these values of $h_+(t)$ and $h_\times(t)$ should satisfy the remaining $M-2$ equations [3].

In presence of noise, this statement translates to stating that we can construct $M-2$ different linear combinations of the signals in each detector, such that, the signal part in each linear combination is zero.
Mathematically, this is done in the following way.

1) Using any ortho-normalization procedure one can find a set of $M-2$ vectors $\{\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_{M-2}}\}$ orthonormal to both $\overrightarrow{F_+}$ and $\overrightarrow{F_\times}$.

2) Using these vectors one can define a matrix,

$$B = \begin{bmatrix} \overrightarrow{e_1}^\dagger \\ \overrightarrow{e_2}^\dagger \\ \vdots \\ \overrightarrow{e_{M-2}}^\dagger \end{bmatrix}_{M-2} . \tag{4.1}$$

3) Operating B on the network data vector we get,

$$\overrightarrow{z(t)}_{M-2\times 1} = B\overrightarrow{s(t)} = B\left(\overrightarrow{F_+}h_+(t) + \overrightarrow{F_\times}h_\times(t) + \overrightarrow{n(t)}\right) = B\overrightarrow{n(t)} . \tag{4.2}$$

The $M-2$ entries, $\{\overrightarrow{z_\alpha} = \overrightarrow{e_\alpha}^\dagger\overrightarrow{s(t)}|\alpha = 1,2,\ldots,M-2\}$ in $\overrightarrow{z(t)}$ are linear combinations of individual detector data without any signal component in them and are hence called **Null Streams** and the set $\{\overrightarrow{e_\alpha}|\alpha = 1,2,\ldots,M-2\}$ are independent **Null Directions**.
In presence of glitches, we expect some residue other than Gaussian noise in the null stream. As signals get cancelled out in the null stream, this excess power in the null stream due to glitches, can be used to discriminate signals from glitches.
In order to generate a statistic which exploits this fact, the authors of [4] first whiten the data and construct whitened null streams, $\overrightarrow{z_w(t)}$.
Consequently squaring each data point of all the null streams and adding , they create a statistic,

$$E_{null} = \sum_{\alpha=1}^{M-2}\sum_{f=0}^{N-1}|\tilde{z}_\alpha(f)|^2 \tag{4.3}$$

which is χ^2 distributed with $2N(M-2)$ degrees of freedom. where N is the number of data points in the time series.

We see that this formalism can be used for even unmodelled sources as nowhere is the template waveform function ever used.

Also, for the case of CBC signals, one can use the information of the template waveform to add more discriminatory power to the null stream formalism. This is illustrated in the following sections.

4.2 Null SNR and its Representation in the Unified χ^2 Formalism

The **Null SNR** is defined as,

$$\rho_N^2 = \rho_{coinc}^2 - \rho_{coh}^2 = \sum_{i=0,\pi/2} ||\overrightarrow{c_i}||^2 - \left(\overrightarrow{c_i} \cdot \overrightarrow{f_+}\right)^2 - \left(\overrightarrow{c_i} \cdot \overrightarrow{f_\times}\right)^2 \tag{4.4}$$

where $\overrightarrow{f_+}$ and $\overrightarrow{f_\times}$ are $M \times 1$ orthonormal vectors. Hence, we can find a set of $M-2$ orthonormal vectors (denoted by $\overrightarrow{e_\alpha}, \alpha = 1, M-2$) that along with $\overrightarrow{f_+}$ and $\overrightarrow{f_\times}$ form a orthonormal basis set of $\mathbb{R}^M$.

The set $\{\overrightarrow{e_\alpha} | \alpha = 1, 2, \ldots, M-2\}$ is **equivalent** the null directions defined in the previous section under the condition that the power spectral densities of the detectors are same, i.e., in the case we can choose $\sigma^x = 1$ for all detectors.

Therefore,

$$||\overrightarrow{c_i}||^2 = \left(\overrightarrow{c_i} \cdot \overrightarrow{f_+}\right)^2 + \left(\overrightarrow{c_i} \cdot \overrightarrow{f_\times}\right)^2 + \sum_{\alpha=1}^{M-2} \left(\overrightarrow{c_i} \cdot \overrightarrow{e_\alpha}\right)^2 . \tag{4.5}$$

Therefore the Null SNR can be rewritten as,

$$\rho_N^2 = \sum_{i=0,\pi/2} \sum_{\alpha=1}^{M-2} \left(\overrightarrow{c_i} \cdot \overrightarrow{e_\alpha}\right)^2 \tag{4.6}$$

$$\therefore \rho_N^2 = \sum_{i=0,\pi/2} \sum_{\alpha=1}^{M-2} \left(\sum_{x=1}^{M} \left(s^x, \frac{e_\alpha^x h_i}{\sigma^x} \right) \right)^2 . \tag{4.7}$$

Next we define $2(M-2)$ vectors $\overrightarrow{b_{\alpha i}}$ (of dimension $M \times 1$) where $\alpha = 1$ to $(M-2)$ and $i = 0, \pi/2$ by

$$\overrightarrow{b_{\alpha i}} = h_i \begin{bmatrix} e_\alpha^1/\sigma^1 \\ e_\alpha^2/\sigma^2 \\ \vdots \\ e_\alpha^M/\sigma^M \end{bmatrix}_{M \times 1} . \tag{4.8}$$

Therefore,

$$\rho_N^2 = \sum_{i=1,\pi/2} \sum_{\alpha=1}^{M-2} \left(\overrightarrow{s}, \overrightarrow{b_{\alpha i}}\right)^2 . \tag{4.9}$$

Note that,

$$(\overrightarrow{b_{\alpha i}}, \overrightarrow{b_{\beta j}}) = \sum_{x=1}^{M} \frac{e_\alpha^x e_\beta^x (h_i, h_j)_x}{(\sigma^x)^2} \tag{4.10}$$

$$= \sum_{x=1}^{M} e_\alpha^x e_\beta^x \delta_{ij} \tag{4.11}$$

$$= \delta_{\alpha\beta}\delta_{ij} . \tag{4.12}$$

Hence, $b_{\alpha i}$ form a set of $2(M-2)$ orthonormal vectors. Now, we prove $\left(\overrightarrow{s}, \overrightarrow{b_{\alpha i}}\right)$ are mutually independent unit zero-mean Gaussians in presence of pure Gaussian noise.
Proof:

Linear combination of independent Gaussian random variables is a Gaussian random variable. Hence $\left(\overrightarrow{s}, \overrightarrow{b_{\alpha i}}\right)$ are Gaussian random variables. Now,

$$\overrightarrow{s(t)} = \overrightarrow{F_+ h_+(t)} + \overrightarrow{F_\times h_\times(t)} + \overrightarrow{n(t)} \tag{4.13}$$

$$\therefore \left\langle \left(\overrightarrow{s}, \overrightarrow{b_{\alpha i}}\right) \right\rangle = \left\langle \left(\overrightarrow{F_+ h_+} + \overrightarrow{F_\times h_\times} + \overrightarrow{n}, \overrightarrow{b_{\alpha i}}\right) \right\rangle . \tag{4.14}$$

Now,

$$\left\langle \left(\overrightarrow{n}, \overrightarrow{b_{\alpha i}}\right) \right\rangle = 0 \tag{4.15}$$

and

$$\left(\overrightarrow{F_{+,\times}h_j}, \overrightarrow{b_{\alpha i}}\right) = \sum_{x=1}^{M} \left(F^x_{+,\times}h_j, b^x_{\alpha i}\right) \tag{4.16}$$

$$= \sqrt{C_{+,\times}} \sum_{x=1}^{M} \left(\frac{f^x_{+,\times}h_j}{\sigma^x}, \frac{h_i e^x_\alpha}{\sigma^x}\right) \tag{4.17}$$

$$= \sqrt{C_{+,\times}} \sum_{x=1}^{M} f^x_{+,\times} e^x_\alpha \left(\frac{(h_j, h_i)_x}{(\sigma^x)^2}\right) \tag{4.18}$$

$$= \sqrt{C_{+,\times}} \left(\sum_{x=1}^{M} f^x_{+,\times} e^x_\alpha\right) \delta_{ij} \tag{4.19}$$

$$= \sqrt{C_{+,\times}} \delta_{ij} \left(\overrightarrow{f_{+,\times}} \cdot \overrightarrow{e_\alpha}\right) \tag{4.20}$$

$$= 0\,. \tag{4.21}$$

Hence,

$$\left(\overrightarrow{F_+h_+} + \overrightarrow{F_\times h_\times}, \overrightarrow{b_{\alpha i}}\right) = 0 \tag{4.22}$$

$$\therefore \left\langle \left(\overrightarrow{s}, \overrightarrow{b_{\alpha i}}\right) \right\rangle = 0\,. \tag{4.23}$$

This proves that mean of $\left(\overrightarrow{s}, \overrightarrow{b_{\alpha i}}\right)$ is zero. From eq 4.21,

$$\left\langle \left(\overrightarrow{s}, \overrightarrow{b_{\alpha i}}\right) \left(\overrightarrow{s}, \overrightarrow{b_{\beta j}}\right) \right\rangle \tag{4.24}$$

$$= \left\langle \left(\overrightarrow{F_+h_+} + \overrightarrow{F_\times h_\times} + \overrightarrow{n}, \overrightarrow{b_{\alpha i}}\right) \left(\overrightarrow{F_+h_+} + \overrightarrow{F_\times h_\times} + \overrightarrow{n}, \overrightarrow{b_{\beta j}}\right) \right\rangle \tag{4.25}$$

$$= \left\langle \left(\overrightarrow{n}, \overrightarrow{b_{(\alpha,i)}}\right) \left(\overrightarrow{n}, \overrightarrow{b_{(\beta,j)}}\right) \right\rangle\,. \tag{4.26}$$

Now

$$\left\langle \left(\overrightarrow{n}, \overrightarrow{b_{\alpha i}}\right) \left(\overrightarrow{n}, \overrightarrow{b_{\beta j}}\right) \right\rangle \tag{4.27}$$

$$= \left\langle \left(\sum_{x=1}^{M} \left(n^x, b^x_{(\alpha,i)}\right) \right) \left(\sum_{y=1}^{M} \left(n^y, b^y_{(\beta,j)}\right) \right) \right\rangle \tag{4.28}$$

$$= \left\langle \sum_{x,y=1}^{M} \left(n^x, b^x_{\alpha i}\right) \left(n^y, b^y_{\beta j}\right) \right\rangle \tag{4.29}$$

$$= \left\langle \sum_{x,y=1}^{M} \left(\int_{-\infty}^{\infty} \frac{n^x(f) h_i^*(f) e^x_\alpha}{\sigma^x S^x(f)} df \right) \left(\int_{-\infty}^{\infty} \frac{n^{y*}(f) h_j(f') e^y_\beta}{\sigma^y S^y(f')} df' \right) \right\rangle \tag{4.30}$$

$$= \sum_{x,y=1}^{M} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\langle n^x(f) n^{y*}(f') \rangle \, h_i^*(f) h_j(f') e^x_\alpha e^y_\beta}{\sigma^x \sigma^y S^x(f) S^y(f')} df df' \tag{4.31}$$

$$= \sum_{x,y=1}^{M} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\delta^{xy} \delta(f-f') h_i^*(f) h_j(f') e^x_\alpha e^y_\beta}{\sigma^x \sigma^y S^y(f')} df' df \tag{4.32}$$

$$= \sum_{x=1}^{M} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\delta(f-f') h_i^*(f) h_j(f') e^x_\alpha e^x_\beta}{(\sigma^x)^2 S^(f')} df' df \tag{4.33}$$

$$= \sum_{x=1}^{M} e^x_\alpha e^x_\beta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{h_i^*(f) h_j(f)}{(\sigma^x)^2 S^(f)} df \tag{4.34}$$

$$= \sum_{x=1}^{M} e^x_\alpha e^x_\beta (\delta_{ij}) \tag{4.35}$$

$$= (\overrightarrow{e_\alpha} \cdot \overrightarrow{e_\beta})(\delta_{ij}) \tag{4.36}$$

$$= \delta_{\alpha\beta} \delta_{ij} \, . \tag{4.37}$$

Hence,

$$\left\langle \left(\overrightarrow{s}, \overrightarrow{b_{\alpha i}}\right) \left(\overrightarrow{s}, \overrightarrow{b_{\beta j}}\right) \right\rangle = \left\langle \left(\overrightarrow{n}, \overrightarrow{b_{\alpha i}}\right) \left(\overrightarrow{n}, \overrightarrow{b_{\beta j}}\right) \right\rangle = \delta_{\alpha\beta} \delta_{ij} \, . \tag{4.38}$$

This proves that $\left(\overrightarrow{s}, \overrightarrow{b_{\alpha i}}\right)$ are independent and have unit variance. Hence from above we conclude that $\left(\overrightarrow{s}, \overrightarrow{b_{\alpha i}}\right)$ are mutually independent unit zero-mean Gaussians in presence of pure Gaussian noise.This implies that $\left(\overrightarrow{s}, \overrightarrow{b_{\alpha i}}\right)^2$ are independent χ^2 random variables with one degree of freedom and ρ^2_{Null} is a χ^2 random variable with $2(M-2)$ degrees of freedom.

In terms of the unified χ^2 formalism, If D^x are the individual detector Hilbert spaces,

$$D_{Network} = D_1 \oplus D_2 \oplus \ldots \oplus D_M \,. \tag{4.39}$$

With the above mentioned network inner product, the $2(M-2)$ dimensional space spanned by $\overrightarrow{b_{\alpha i}}$ is orthogonal to the following four vectors: $\overrightarrow{F_{+,\times} h_{0,\pi/2}}$. Hence, the space is also orthogonal to the signal. We call the $M-2$ dimensional space spanned by $\{\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_{M-2}}\}$ the Null space $(\mathcal{N})$, $\mathcal{N} \subset \mathbb{R}^M$.

$$\rho^2_{\mathcal{N}} = \sum_{i=0,\pi/2} \sum_{\alpha=1}^{M-2} (\overrightarrow{c_i} \cdot \overrightarrow{e_\alpha})^2 = \sum_{i=0,\pi/2} ||\overrightarrow{c_i}||^2_{\mathcal{N}} \,. \tag{4.40}$$

We can also create a statistic using only a l dimensional subset of the null space $\mathcal{N}_{sub}$. Let,

$$\mathcal{N}_{sub} = \mathrm{Span}\{\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_l}\} \subseteq \mathcal{N} \,. \tag{4.41}$$

$$\rho^2_{\mathcal{N}_{sub}} = \sum_{i=0,\pi/2} \sum_{\alpha=1}^{l} \left(\overrightarrow{s}, \overrightarrow{b_{\alpha i}}\right)^2 = \sum_{i=0,\pi/2} \sum_{\alpha=1}^{l} (\overrightarrow{c_i} \cdot \overrightarrow{e_\alpha})^2 = \sum_{i=0,\pi/2} ||\overrightarrow{c_i}||^2_{\mathcal{N}_{sub}} \,. \tag{4.42}$$

Hence, $\rho^2_{\mathcal{N}_{sub}}$ in presence of pure Gaussian noise or signal plus Gaussian noise is χ^2 distributed with $2l$ degrees of freedom.

Also note that: $\rho^2_{\mathcal{N}_{sub}}$ **can also be interpreted as sum of squares of the Euclidean norms of the projection of vectors** $\overrightarrow{c_0}$ **and** $\overrightarrow{c_{\pi/2}}$ **on** $\mathcal{N}_{sub}$**.**

4.3 $\chi^2_{general}$: A General way to Combine Individual Detector χ^2 Basis

Let $\overrightarrow{w_\alpha}$, $\alpha = 1$ to q be a set of q ortho-normal $M \times 1$ column vectors. Lets consider that each detector performs a single detector χ^2 with p degrees of freedom. Let the basis of the orthogonal subspace, corresponding to the maximum SNR template of the x$^{\text{th}}$ detector be denoted by,

$$u_{xk} \text{ where } k = 1 \text{ to } p\,.$$

Hence,

$$\frac{(u_{xk}, u_{xl})_x}{(\sigma^x)^2} = \delta_{kl}\,. \tag{4.43}$$

We can define $q \times p$ vectors

$$\overrightarrow{v_{\alpha k}} = \begin{bmatrix} \frac{w_\alpha^1 u_{1k}}{\sigma^1} \\ \frac{w_\alpha^2 u_{2k}}{\sigma^2} \\ \vdots \\ \frac{w_\alpha^M u_{Mk}}{\sigma^M} \end{bmatrix}\,. \tag{4.44}$$

Note that,

$$\begin{aligned}
(\overrightarrow{v_{\alpha k}}, \overrightarrow{v_{\beta l}}) =& \sum_{x=1}^{M} \frac{w_\alpha^x w_\beta^x (u_{xk}, u_{xl})_x}{(\sigma^x)^2} && (4.45)\\
=& \sum_{x=1}^{M} w_\alpha^x w_\beta^x \delta_{kl} && (4.46)\\
=& \delta_{\alpha\beta}\delta_{kl}\,. && (4.47)
\end{aligned}$$

Hence, $\overrightarrow{v_{\alpha k}}$ form a set of $p \times q$ orthonormal vectors. Now we define,

$$\chi^2_{general} = \sum_{k=1}^{p} \sum_{\alpha=1}^{q} (\overrightarrow{s}, \overrightarrow{v_{\alpha k}})^2 \tag{4.48}$$

and prove that $(\overrightarrow{s}, \overrightarrow{v_{\alpha k}})$ are mutually independent unit zero-mean Gaussian random variables in presence of pure Gaussian noise.
Proof:

Linear combination of independent Gaussian random variables is a Gaussian random variable. Hence $(\overrightarrow{s}, \overrightarrow{v_{\alpha k}})$ are Gaussians.

Therefore,

$$\begin{aligned}
\overrightarrow{s(t)} =& \overrightarrow{F_+ h_+(t)} + \overrightarrow{F_\times h_\times(t)} + \overrightarrow{n(t)} && (4.49)\\
\langle(\overrightarrow{s}, \overrightarrow{v_{\alpha k}})\rangle =& \left\langle \left(\overrightarrow{F_+ h_+} + \overrightarrow{F_\times h_\times} + \overrightarrow{n}, \overrightarrow{v_{\alpha k}} \right) \right\rangle\,. && (4.50)
\end{aligned}$$

Now,

$$\langle(\overrightarrow{n}, \overrightarrow{v_{\alpha k}})\rangle = 0 \tag{4.51}$$

and

$$\Big\langle \Big(\overrightarrow{F_{+,\times} h_j}, \overrightarrow{v_{\alpha k}}\Big) \Big\rangle = \sum_{x=1}^{M} \big\langle \big(F_{+,\times}^{x} h_j, v_{\alpha k}^{x}\big) \big\rangle \tag{4.52}$$

$$= \sqrt{C_{+,\times}} \sum_{x=1}^{M} \left(\frac{f_{+,\times}^{x} h_j}{\sigma^x}, \frac{u_{xi} w_{\alpha}^{x}}{\sigma^x} \right) \tag{4.53}$$

$$= \sqrt{C_{+,\times}} \sum_{x=1}^{M} f_{+,\times}^{x} w_{\alpha}^{x} \left(\frac{(h_j, u_{xi})_x}{(\sigma^x)^2} \right) \tag{4.54}$$

$$= \sqrt{C_{+,\times}} \left(\sum_{x=1}^{M} f_{+,\times}^{x} w_{\alpha}^{x} \right) (0) \tag{4.55}$$

$$= 0\,. \tag{4.56}$$

Hence,

$$\left(\overrightarrow{F_+ h_+} + \overrightarrow{F_\times h_\times}, \overrightarrow{v_{\alpha k}}\right) = 0 \tag{4.57}$$

$$\therefore \langle(\overrightarrow{s}, \overrightarrow{v_{\alpha k}})\rangle = 0\,. \tag{4.58}$$

This proves that mean of $(\overrightarrow{s}, \overrightarrow{v_{\alpha k}})$ is zero. From eq 4.56,

$$\langle(\overrightarrow{s}, \overrightarrow{v_{\alpha k}})\,(\overrightarrow{s}, \overrightarrow{v_{\beta l}})\rangle \tag{4.59}$$

$$= \Big\langle \left(\overrightarrow{F_+ h_+} + \overrightarrow{F_\times h_\times} + \overrightarrow{n}, \overrightarrow{v_{\alpha k}}\right) \left(\overrightarrow{F_+ h_+} + \overrightarrow{F_\times h_\times} + \overrightarrow{n}, \overrightarrow{v_{\beta l}}\right) \Big\rangle \tag{4.60}$$

$$= \langle(\overrightarrow{n}, \overrightarrow{v_{\alpha k}})\,(\overrightarrow{n}, \overrightarrow{v_{\beta l}})\rangle\ . \tag{4.61}$$

Now,

$$\langle(\overrightarrow{n}, \overrightarrow{v_{\alpha k}})\,(\overrightarrow{n}, \overrightarrow{v_{\beta l}})\rangle \tag{4.62}$$

$$= \left\langle \left(\sum_{x=1}^{M} (n^x, v^x_{\alpha k})\right) \left(\sum_{y=1}^{M} (n^y, v^y_{\beta l})\right) \right\rangle \tag{4.63}$$

$$= \left\langle \sum_{x,y=1}^{M} (n^x, v^x_{\alpha k})\,(n^y, v^y_{\beta l}) \right\rangle \tag{4.64}$$

$$= \left\langle \sum_{x,y=1}^{M} \left(\int_{-\infty}^{\infty} \frac{n^x(f) u^*_{xk}(f) w^x_\alpha}{\sigma^x S^x(f)} df\right) \left(\int_{-\infty}^{\infty} \frac{n^{y*}(f) u_{yl}(f') w^y_\beta}{\sigma^y S^y(f')} df'\right) \right\rangle \tag{4.65}$$

$$= \sum_{x,y=1}^{M} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\langle n^x(f) n^{y*}(f')\rangle\, u^*_{xk}(f) u_{yl}(f') w^x_\alpha w^y_\beta}{\sigma^x \sigma^y S^x(f) S^y(f')} df\, df' \tag{4.66}$$

$$= \sum_{x,y=1}^{M} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\delta^{xy}\delta(f-f') u^*_{xk}(f) u_{yl}(f') w^x_\alpha w^y_\beta}{\sigma^x \sigma^y S^y(f')} df'\, df \tag{4.67}$$

$$= \sum_{x=1}^{M} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\delta(f-f') u^*_{xk}(f) u_{xl}(f') w^x_\alpha w^x_\beta}{(\sigma^x)^2 S^(f')} df'\, df \tag{4.68}$$

$$= \sum_{x=1}^{M} w^x_\alpha w^x_\beta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{u^*_{xk}(f) u_{xl}(f)}{(\sigma^x)^2 S^(f)} df \tag{4.69}$$

$$= \sum_{x=1}^{M} w^x_\alpha w^x_\beta (\delta_{kl}) \tag{4.70}$$

$$= (\overrightarrow{w_\alpha} \cdot \overrightarrow{w_\beta})(\delta_{kl}) \tag{4.71}$$

$$= \delta_{\alpha\beta}\delta_{kl}\,. \tag{4.72}$$

Hence,

$$\langle(\overrightarrow{s}, \overrightarrow{v_{\alpha k}})\,(\overrightarrow{s}, \overrightarrow{v_{\beta l}})\rangle = \langle(\overrightarrow{n}, \overrightarrow{v_{\alpha k}})\,(\overrightarrow{n}, \overrightarrow{v_{\beta l}})\rangle = \delta_{\alpha\beta}\delta_{kl}\,. \tag{4.73}$$

This proves that $(\overrightarrow{s}, \overrightarrow{v_{\alpha k}})$ are independent and have unit variance. Hence from above we conclude that $(\overrightarrow{s}, \overrightarrow{v_{\alpha k}})$ are mutually independent unit zero-mean Gaussians in presence of pure Gaussian noise.This implies that $(\overrightarrow{s}, \overrightarrow{v_{\alpha k}})^2$ are independent χ^2 random variables with one degree of freedom and $\chi^2_{general}$ is a χ^2 random variable with pq degrees of freedom.

In terms of the unified χ^2 formalism, If D^x are the individual detector Hilbert spaces,

$$D_{Network} = D_1 \oplus D_2 \oplus \ldots \oplus D_M\,. \tag{4.74}$$

With the above mentioned network inner product, we can conclude from equation 4.56 that the pq dimensional space spanned by $\overrightarrow{v_{\alpha k}}$ is orthogonal to the following four vectors: $\overrightarrow{F_{+,\times}h_{0,\pi/2}}$. Hence, the space is also orthogonal to the signal, that is, $\overrightarrow{F_+h_+} + \overrightarrow{F_\times h_\times}$.

Now we prove that $\rho^2_{\mathcal{N}_{sub}}$ and $\chi^2_{general}$ are independent random variables,

$$\left\langle (\vec{s}, \overrightarrow{v_{\alpha k}}) \left((\vec{s}, \overrightarrow{b_{\beta i}} \right) \right\rangle = \left\langle (\vec{n}, \overrightarrow{v_{\alpha,k}}) \left((\vec{n}, \overrightarrow{b_{\beta i}} \right) \right\rangle \tag{4.75}$$

$$= \left\langle \left(\sum_{x=1}^{M} (n^x, v^x_{\alpha k}) \right) \left(\sum_{y=1}^{M} \left(n^y, b^y_{\beta i}\right) \right) \right\rangle \tag{4.76}$$

$$= \left\langle \sum_{x,y=1}^{M} (n^x, v^x_{\alpha k}) \left(n^y, b^y_{\beta i}\right) \right\rangle \tag{4.77}$$

$$= \left\langle \sum_{x,y=1}^{M} \left(\int_{-\infty}^{\infty} \frac{n^x(f) u^*_{xk}(f) w^x_\alpha}{\sigma^x S^x(f)} df \right) \left(\int_{-\infty}^{\infty} \frac{n^{y*}(f) h_i(f') e^y_\beta}{\sigma^y S^y(f')} df' \right) \right\rangle \tag{4.78}$$

$$= \sum_{x,y=1}^{M} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\langle n^x(f) n^{y*}(f') \rangle \, u^*_{xk}(f) h_i(f') w^x_\alpha e^y_\beta}{\sigma^x \sigma^y S^x(f) S^y(f')} df df' \tag{4.79}$$

$$= \sum_{x,y=1}^{M} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\delta^{xy} \delta(f-f') u^*_{xk}(f) h_i(f') w^x_\alpha e^y_\beta}{\sigma^x \sigma^y S^y(f')} df' df \tag{4.80}$$

$$= \sum_{x=1}^{M} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\delta(f-f') u^*_{xk}(f) h_i(f') w^x_\alpha e^x_\beta}{(\sigma^x)^2 S^(f')} df' df \tag{4.81}$$

$$= \sum_{x=1}^{M} w^x_\alpha e^x_\beta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{u^*_{xk}(f) h_i(f)}{(\sigma^x)^2 S^(f)} df \tag{4.82}$$

$$= 0\,. \tag{4.83}$$

Hence, $(\vec{s}, \overrightarrow{v_{\alpha,k}}) \left(\vec{s}, \overrightarrow{b_{\alpha i}} \right)$ are independent random variables. From this we can conclude that ,$\rho^2_{\mathcal{N}_{sub}}$ and $\chi^2_{general}$ are independent χ^2 random variables. Hence, their sum

$$\chi^2_{Network} = \rho^2_{\mathcal{N}_{sub}} + \chi^2_{general} \tag{4.84}$$

is χ^2 distributed with $2l + pq$ degrees of freedom.Moreover the subspace orthogonal to the signal is the union of the subspaces corresponding to $\rho^2_{\mathcal{N}_{sub}}$ and $\chi^2_{general}$.

4.4 Compact Form of $\chi^2_{general}$

Similar to the case of ρ^2_N, we define p vectors, $\overrightarrow{c}_k \in \mathbb{R}^M$, $k = 1, 2, ..., p$ as

$$\overrightarrow{c}_k = \begin{bmatrix} \left(s^1(t), \frac{u_{1j}(t)}{\sigma^1}\right) \\ \left(s^2(t), \frac{u_{2j}(t)}{\sigma^2}\right) \\ \vdots \\ \left(s^M(t), \frac{u_{Mj}(t)}{\sigma^M}\right) \end{bmatrix}_{M\times 1} . \tag{4.85}$$

Also note again that $\overrightarrow{w_\alpha}$ are orthonormal basis vectors , $\alpha = 1, 2, \ldots, q$, span a q-dimensional space $\mathcal{Q}$, i.e.,

$$\mathcal{Q} = \text{Span}\{\overrightarrow{w_\alpha} | \alpha = 1 \text{ to } q\} \subseteq \mathbb{R}^M . \tag{4.86}$$

We see that

$$(\overrightarrow{c}_k \cdot \overrightarrow{w_\alpha})^2 = \left(\overrightarrow{s(t)}, \overrightarrow{v_{\alpha k}(t)}\right)^2 . \tag{4.87}$$

(and hence $(\overrightarrow{c}_k \cdot \overrightarrow{w_\alpha})^2$ is χ^2 distributed with one degree of freedom in presence of pure Gaussian noise or Signal plus pure Gaussian noise). Therefore,

$$\chi^2_{general}\left(\overrightarrow{s(t)}\right) = \sum_{k=1}^{p}\sum_{\alpha=1}^{q} (\overrightarrow{c}_k \cdot \overrightarrow{w_\alpha})^2 = \sum_{k=1}^{p} ||\overrightarrow{c}_k||^2_{\mathcal{Q}} . \tag{4.88}$$

Therefore $\chi^2_{general}$ can also be interpreted as sum of squares of the Euclidean norms of the projection of vectors $\overrightarrow{c_k}$ on $\mathcal{Q}$.

4.5 Special Cases

Some special cases of $\chi^2_{general}$ are as follows:

- Note that if $q = M$, for **any** choice of the set $\overrightarrow{w_\alpha}$, $\mathcal{Q} = \mathbb{R}^M$, and

$$\chi^2_{general}\left(\overrightarrow{s(t)}\right) = \sum_{k=1}^{p} ||\overrightarrow{c}_k||^2 = \sum_{x=1}^{M}\sum_{k=1}^{p}\left(s^x(t), \frac{u_{xk}(t)}{\sigma^x}\right)^2_x = \chi^2_{add} . \text{ (defined in 3.20)} \tag{4.89}$$

- The case where $\chi^2_{general}$ is the direct addition of the individual detector χ^2 of the first q detectors. In this case,

$$\overrightarrow{w_\alpha} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} ; \text{ (One in the } \alpha^{\text{th}} \text{ position, zero elsewhere, for } \alpha = 1 \text{ to } q) \tag{4.90}$$

Note that a individual detector χ^2 of the p$^{\text{th}}$ detector can be thought as a $\chi^2_{general}$ which uses only $\overrightarrow{w_p} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, with 1 only in the p$^{\text{th}}$ entry.

Henceforth, we represent the single detector χ^2 of the p$^{\text{th}}$ detector by χ^2_p.

- The case where the set $\{\overrightarrow{w_\alpha}\}$ is a subset of null and plus, cross directions, i.e., $\{\overrightarrow{f_+}, \overrightarrow{f_\times}, \overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_{M-2}}\}$. The χ^2 generated from set is called $\chi^2_{\mathcal{N}_{sub}}$. The χ^2 generated by the singleton subset:

  i) $\overrightarrow{f_+}$ is called χ^2_+.

  ii) $\overrightarrow{f_\times}$ is called $\chi^2_\times$.

  iii) $\overrightarrow{e_\alpha}$ is called $\chi^2_{\mathcal{N}_\alpha}$ $(\alpha = 1, 2, \ldots, M-2)$.

  Note again that the set $\{\overrightarrow{f_+}, \overrightarrow{f_\times}, \overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_{M-2}}\}$ forms an orthonormal basis of $\mathbb{R}^M$, hence, from 4.89,

$$\chi^2_+ + \chi^2_\times + \sum_{\alpha=1}^{M-2} \chi^2_{\mathcal{N}_\alpha} = \chi^2_{add}\,. \tag{4.91}$$

4.6 Synthetic Null and $+, \times$ detectors: An Alternate Interpretation of $\rho^2_{\mathcal{N}_{sub}}$ and $\chi^2_{\mathcal{N}_{sub}}$

Instead of viewing the system as a set of M physical detectors, one can rephrase the problem in terms of M synthetic (abstract) detectors, $\alpha = 1$ to $(M-2)$ represent synthetic null detectors, $\alpha = (M-1)$ and M represent the synthetic $+, \times$ detectors. The overwhitened data streams are given by,

$$o^x(f) = \frac{s^x(f)}{S^x(f)}\,. \tag{4.92}$$

Therefore ,

$$\text{Data in } \alpha^{th}\text{overwhitened synthetic detector} = \sum_{x=1}^{M} \frac{e^x_\alpha}{\sigma^x} o^x(f)\,. \tag{4.93}$$

$$\therefore \text{Noise in the overwhitened detector, } n_{\alpha,ow} = \sum_{x=1}^{M} \frac{e^x_\alpha}{\sigma^x S^x(f)} n^x(f)\,. \tag{4.94}$$

$$\therefore \langle n^*_{\alpha,ow}(f) n_{\beta,ow}(f')\rangle = \left\langle \left(\sum_{x=1}^{M} \frac{e^x_\alpha}{\sigma^x S^x(f)} n^{x*}(f)\right) \left(\sum_{y=1}^{M} \frac{e^y_\alpha}{\sigma^y S^y(f')} n^y(f')\right)\right\rangle \tag{4.95}$$

$$= \sum_{x,y=1}^{M} \frac{e^x_\alpha e^y_\beta \delta^{xy} \delta(f-f') S^x(f)}{\sigma^x \sigma^y S^x(f) S^y(f')} \tag{4.96}$$

$$= \left[\sum_{x=1}^{M} \frac{(e^x_\alpha)^2}{(\sigma^x)^2 S^x(f)}\right] \delta_{\alpha\beta} \delta(f-f')\,. \tag{4.97}$$

Therefore, the Power Spectral Density of the α^{th} detector is,

$$S_{N_\alpha}(f) = \left[\sum_{x=1}^{M} \frac{(e^x_\alpha)^2}{(\sigma^x)^2 S^x(f)}\right]^{-1} \tag{4.98}$$

and the synthetic detector stream is

$$s_{N_\alpha}(f) = \left(\sum_{x=1}^{M} \frac{e^x_\alpha s^x(f)}{\sigma^x S^x(f)}\right) S_{N_\alpha}(f)\,. \tag{4.99}$$

For a network of synthetic detectors,the network data vector is

$$\overrightarrow{s_N} = \begin{bmatrix} s_{N_1} \\ s_{N_2} \\ \vdots \\ s_{N_M} \end{bmatrix} . \tag{4.100}$$

We define $2(M-2)$ vectors, for $i = 0, \pi/2$ and $\alpha = 1$ to $M-2$

$$\overrightarrow{b_{N_{\alpha i}}} = h_i \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix}_{M\times 1} \quad \text{(Only } \alpha^{\text{th}} \text{ position is non-zero)} \tag{4.101}$$

In terms of synthetic detectors,

$$\left(\overrightarrow{s}, \overrightarrow{b}\right)^2 = \left(\sum_{x=1}^{M} \left(s^x, \frac{e^x_\alpha h_i}{\sigma^x}\right)_x\right)^2 \tag{4.102}$$

$$= \left(\sum_{x=1}^{M} \left(s^x \frac{e^x_\alpha}{\sigma^x}, h_i\right)_x\right)^2 \tag{4.103}$$

$$= \left(\sum_{x=1}^{M} \left(s^x \frac{e^x_\alpha}{\sigma^x}\right), h_i\right)^2_x \tag{4.104}$$

$$= (s_{N_\alpha}, h_i)^2_{N_\alpha} \tag{4.105}$$

$$= \left(\overrightarrow{s_N}, \overrightarrow{b_{N_{\alpha i}}}\right)^2_N . \tag{4.106}$$

Therefore $\left(\overrightarrow{s_N}, \overrightarrow{b_{N_{\alpha i}}}\right)^2_N$ is also χ^2 distributed with one degree of freedom, and

$$\rho^2_N = \sum_{i=0,\pi/2} \sum_{\alpha=1}^{M-2} \left(\overrightarrow{s}, \overrightarrow{b_{\alpha i}}\right)^2 = \sum_{i=0,\pi/2} \sum_{\alpha=1}^{M-2} \left(\overrightarrow{s_N}, \overrightarrow{b_{N_{\alpha i}}}\right)^2_N . \tag{4.107}$$

Therefore for a system of synthetic detectors,

$$D_{Network} = D_{N_1} \oplus D_{N_2} \oplus \ldots \oplus D_{N_M}\,. \tag{4.108}$$

and $b_{N(\alpha,i)}$ form a set of $2(M-2)$ orthonormal basis vectors for a space which is orthogonal to the signal vector in the synthetic detectors.

Similarly for χ^2_N, we can define a set of pM orthonormal vectors also orthonormal to the signal vector of the synthetic detectors by,

$$v_{N_{\alpha k}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ u_{\alpha k} \\ 0 \\ \vdots \end{bmatrix}_{M\times 1} \qquad (\text{ Only } \alpha^{\text{th}} \text{ position is non-zero}) \tag{4.109}$$

such that,

$$\chi^2_N = \sum_{k=1}^{p}\sum_{\alpha=1}^{M} (\overrightarrow{s}, \overrightarrow{v_{\alpha k}})^2 = \sum_{k=1}^{p}\sum_{\alpha=1}^{M} \left(\overrightarrow{s_{N_\alpha}}, \overrightarrow{v_{N_{\alpha k}}}\right)^2_N \tag{4.110}$$

4.7 Useful Properties of $\rho^2_{\mathcal{N}_{sub}}$ and $\chi^2_{general}$

From the expression of $\rho^2_{\mathcal{N}_{sub}}$, we can see that it is just composed of linear combinations of the square of projections of the data vector on the maximum SNR template, while $\chi^2_{general}$ is only composed of linear combinations of the projections of data vector on a subspace orthogonal to the maximum SNR template. Hence, for the two glitches g_1, g_2 in Fig. 4.7(1), $\rho^2_{\mathcal{N}_{sub}}$ will discriminate them with the same efficiency while $\chi^2_{general}$ will discriminate g_2 better than g_1.Similarly, for the two glitches g_1, g_2 in Fig. 4.7(2), χ^2_N will discriminate them with the same efficiency while $\rho^2_{\mathcal{N}_{sub}}$ will discriminate g_2 better than g_1

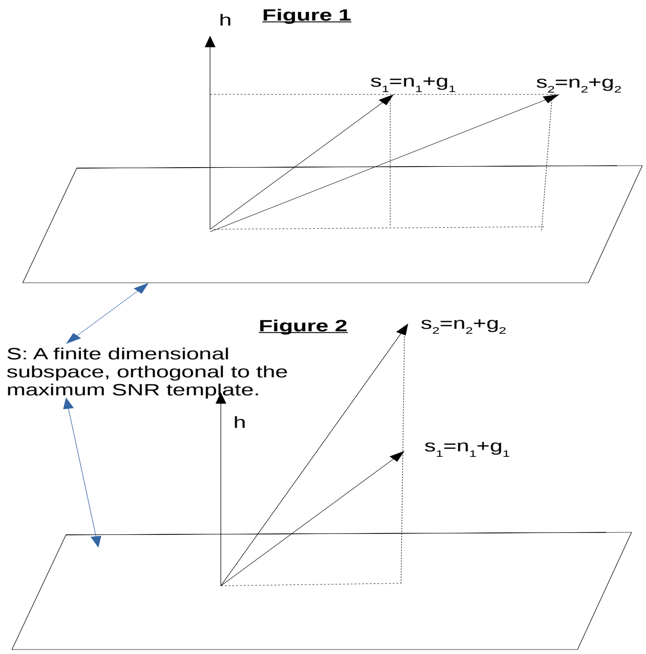


Figure 4.1: $\rho^2_{\mathcal{N}_{sub}}$ vs $\chi^2_{general}$(explanation provided in the accompanying text)

4.8 Double Coincident Glitches: Problem for Null stream Formalisms and $\rho^2_{\mathcal{N}_{sub}}$ resolved by $\chi^2_{general}$

Double coincident glitches are glitches that occur in multiple detectors at the same time. If they pass the matched filtering test, they can deceive the null stream formalism into believing that a signal has come from a certain direction. For illustrating this point, for simplicity, we consider a system of two identical detectors and one dominant polarization (say +). Then,

$$s^1 = F^1(\alpha^1, \theta, \psi, \chi)_+ h_+ + n^1 + g^1 \tag{4.111}$$

$$s^2 = F^2(\alpha^2, \theta, \psi, \chi)_+ h_+ + n^2 + g^2 \,. \tag{4.112}$$

Let,

$$g^1 = ||g^1_\parallel|| h_+ + g^1_\perp \tag{4.113}$$

$$g^2 = ||g^2_\parallel|| h_+ + g^2_\perp \,. \tag{4.114}$$

Therefore,

$$s^1 = \left(F^1(\alpha^1, \theta, \psi, \chi)_+ + ||g^1_\parallel||\right) h_+ + n^1 + g^1_\perp \tag{4.115}$$

$$s^2 = \left(F^2(\alpha^2, \theta, \psi, \chi)_+ + ||g^2_\parallel||\right) h_+ + n^2 + g^2_\perp \,. \tag{4.116}$$

Now we may be able to find beam pattern functions,

$$F'^1(\alpha^1, \theta', \psi', \chi') = \left(F^1(\alpha^1, \theta, \psi, \chi)_+ + ||g^1_\parallel||\right) \tag{4.117}$$

$$F'^2(\alpha^2, \theta', \psi', \chi') = \left(F^2(\alpha^2, \theta, \psi, \chi)_+ + ||g^2_\parallel||\right) \,. \tag{4.118}$$

Thereby leading us to believe that a source is there at network sky location parameters, θ', ψ', χ'. In such cases all null stream formalisms would fail, but the excess power of the null stream projected on the space orthogonal to the maximum SNR template, will be proportional to, $||F'^2 g^1_\perp - F'^1 g^2_\perp||^2$ which would in general be high and be brought out by $\chi^2_{general}$.

Note that if there is a glitch in only one detector, this would not be possible as the beam pattern function for the other detector would not change, that would inturn fix the beam pattern function of the first detector.

Chapter 5

An Illustration

In this chapter we have illustrated some (of the infinitely many possible) network χ^2 tests for the simple case of a known glitch waveform present in only one detector (The first detector) triggering a predetermined template corresponding to $30M_{\odot} - 30M_{\odot}$ black hole binary.

For the network χ^2 tests, we have chosen a 5 detector network located and oriented identical to way the LIGO-India, LIGO-Hanford, LIGO -Livingston, VIRGO, KAGRA network is on the Earth. For simplicity, we have assumed the detector power spectral densities to be constant, but different from one another.

For a system of 5 detectors, there are 3 null directions (3 Synthetic Null Detectors), one $\overrightarrow{f_+}$ (Synthetic plus detector) and one $\overrightarrow{f_\times}$ direction (Synthetic cross detector).

Out of the infinitely many network χ^2 tests, the tests illustrated in this section are as follows:

1. Null SNR tests in individual Synthetic null detectors ($\rho^2_{N_\alpha}|\alpha = 1, 2, 3$) and the Null SNR test for the whole Null Space (ρ^2_N).
2. The Allen's χ^2 test in each individual detector ($\chi^2_\alpha|\alpha = 1, 2, 3, 4, 5$) and their addition χ^2_{add}.
3. The Allen's χ^2 test in each individual synthetic detector ($\chi^2_{N_\alpha}|\alpha = 1, 2, 3, 4, 5$)and their sum χ^2_{Null}.

Further, in each network χ^2 test, there are three scenarios analyzed:

1. The system of detectors contains only pure gaussian noises in all five detectors. They are represented **blue circles** in the plots.

2. Along with pure Gaussian noise, there is an incoming signal from a certain randomly chosen direction ($\theta = 0.3142, \phi = 1.7593$) in the sky. Other parameters of the signal are; $m_1 = m_2 = 30M_\odot$, $\frac{D_0}{D} = 10$, $\phi_c = \pi/8, \psi = 0$, $\iota = 0$ and $t_c = 0$. They are represented **red circles** in the plots.

3. Along with pure Gaussian noise, there is a glitch in one detector whose amplitude is 20, central frequency is $f_0 = 80Hz$ and quality factor $Q = 28$. They are represented **black circles** in the plots.

In these plots the red horizontal line is the $3\sigma_{\chi^2}$ threshold line given by 2.6. 10 random realizations of Gaussian noise were used in each case.

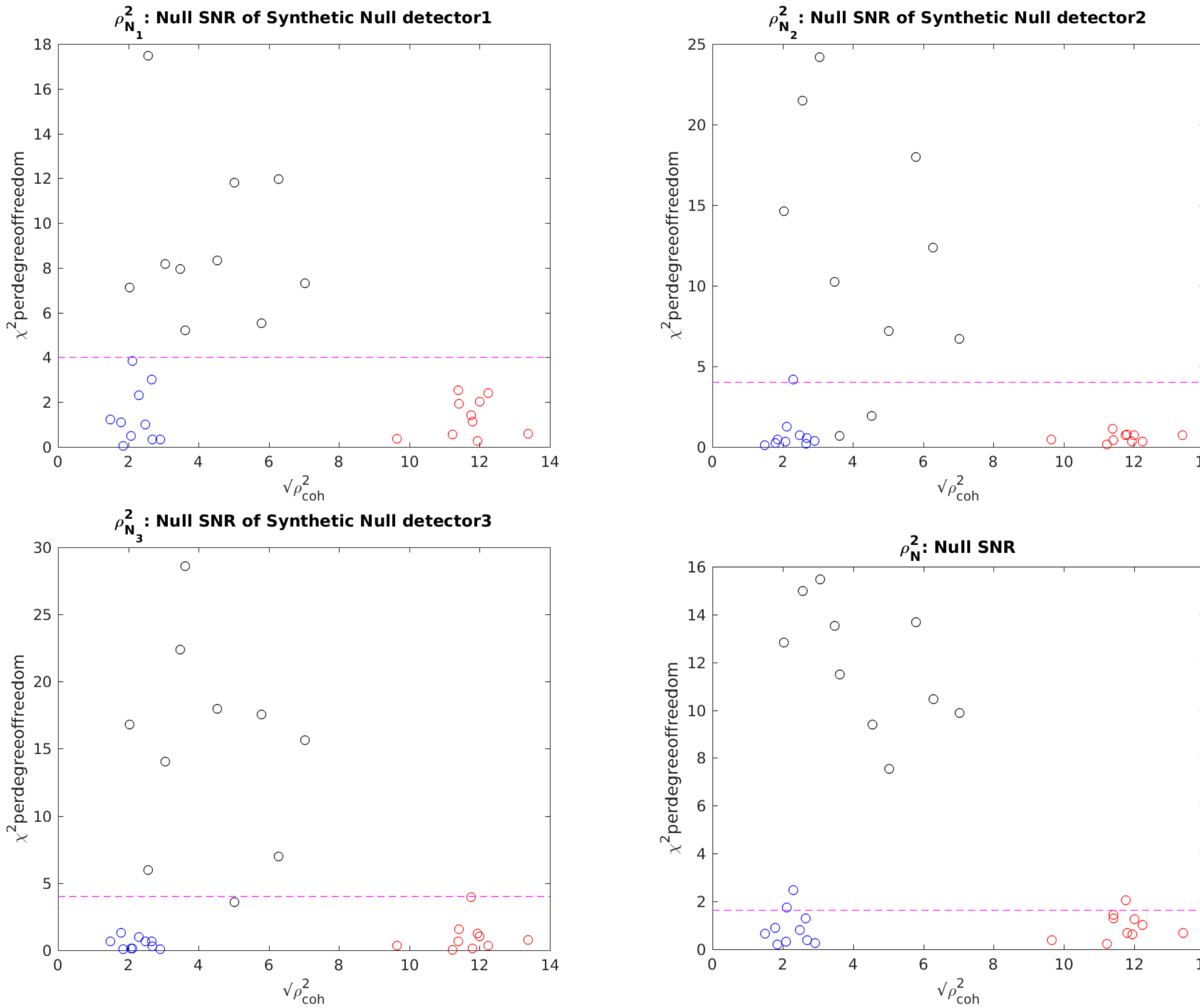


Figure 5.1: **The Null SNR Tests**: (i) Black circles: Glitch+Gaussian Noise (ii) Red circles: Signal+Gaussian Noise(iii) Blue circles: Pure Gaussian Noise (iv) Dashed Magenta line: 3 Sigma Threshold.

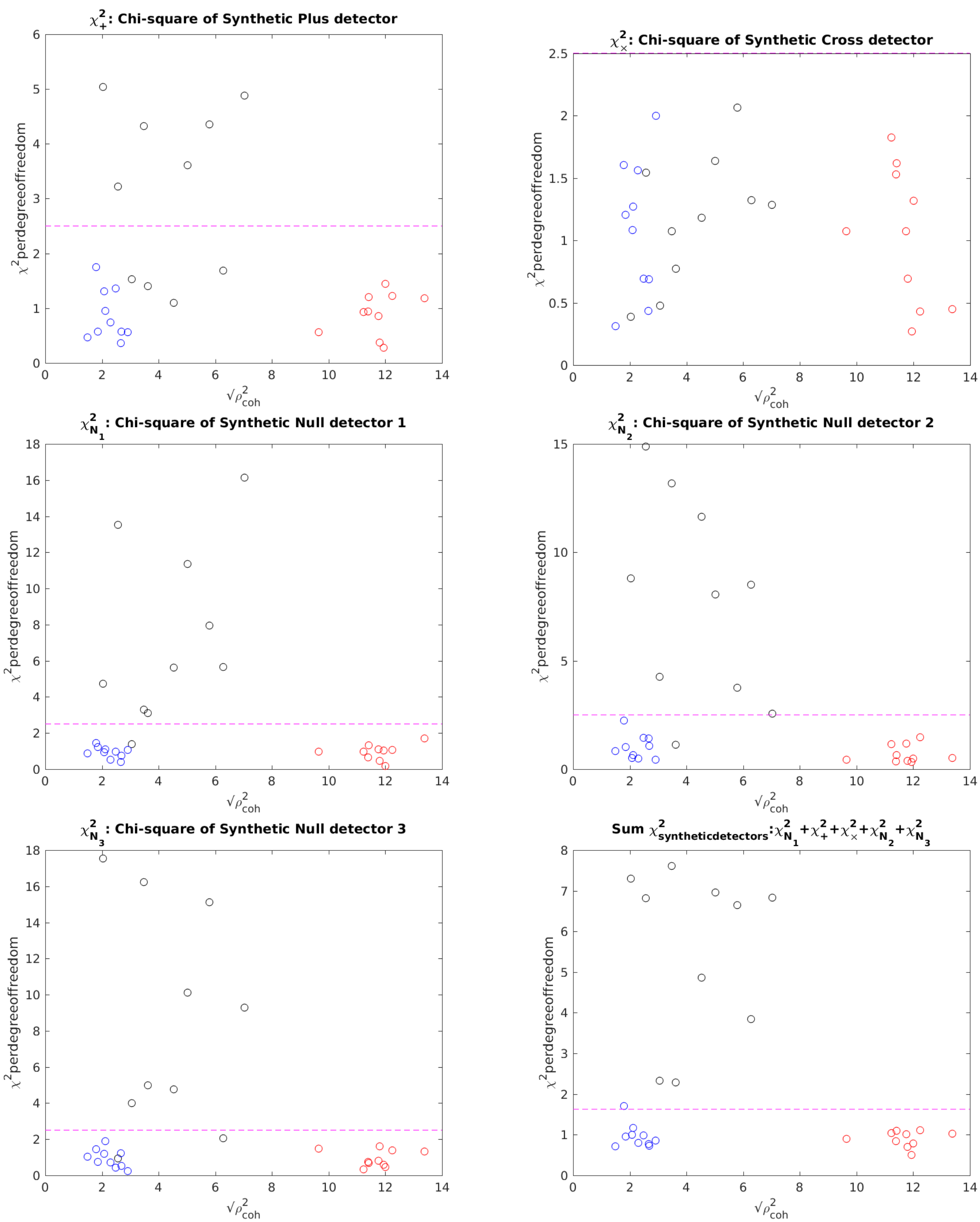


Figure 5.2: **Allen's χ^2 in individual Synthetic Detectors and their addition**: (i) Black circles: Glitch+Gaussian Noise (ii) Red circles: Signal+Gaussian Noise(iii) Blue circles: Pure Gaussian Noise (iv) Dashed Magenta line: 3 Sigma Threshold.

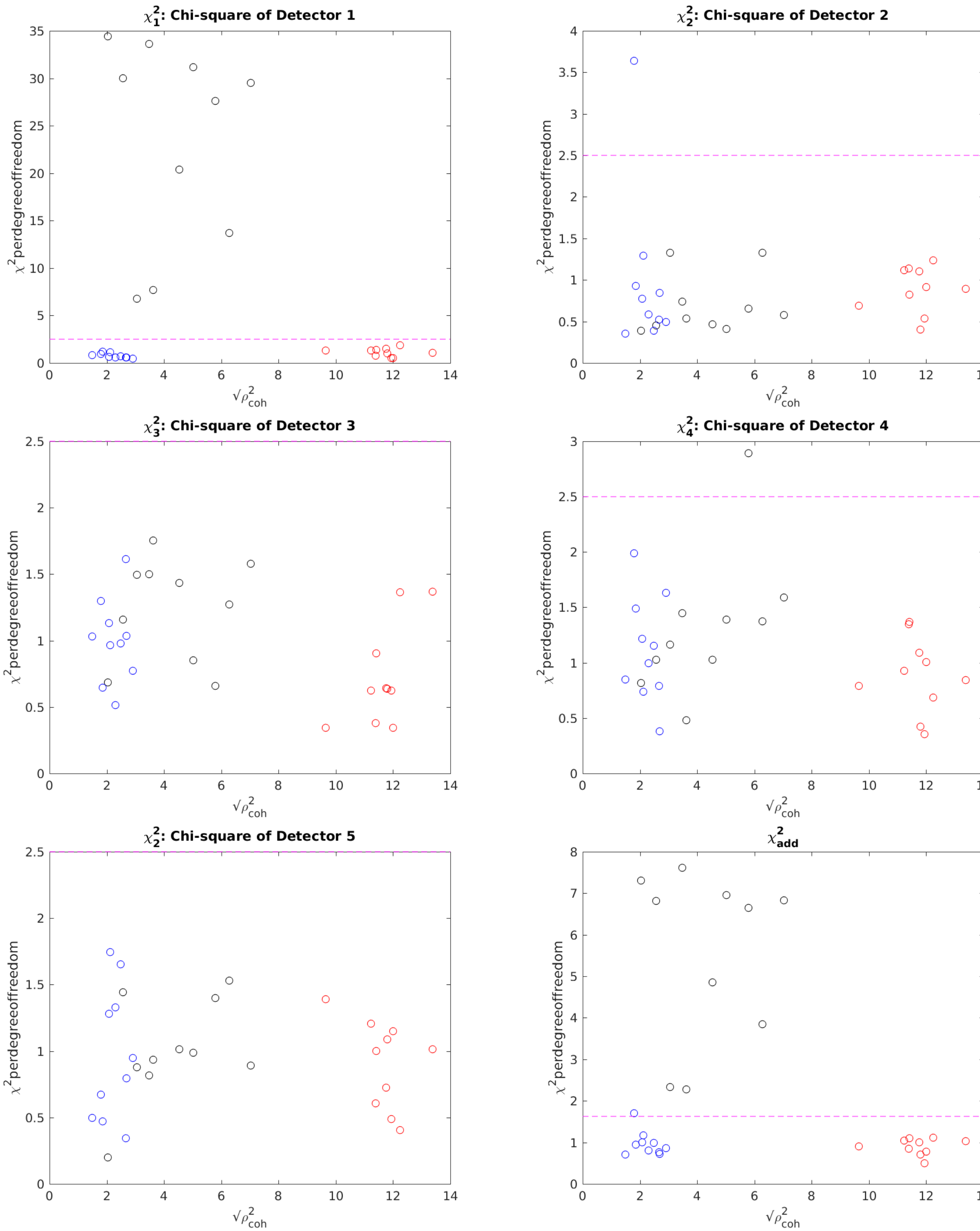


Figure 5.3: **Allen's χ^2 tests in Detectors and the addition of individual detector χ^2:** (i) Black circles: Glitch+Gaussian Noise (ii) Red circles: Signal+Gaussian Noise(iii) Blue circles: Pure Gaussian Noise (iv) Dashed Magenta line: 3 Sigma Threshold.

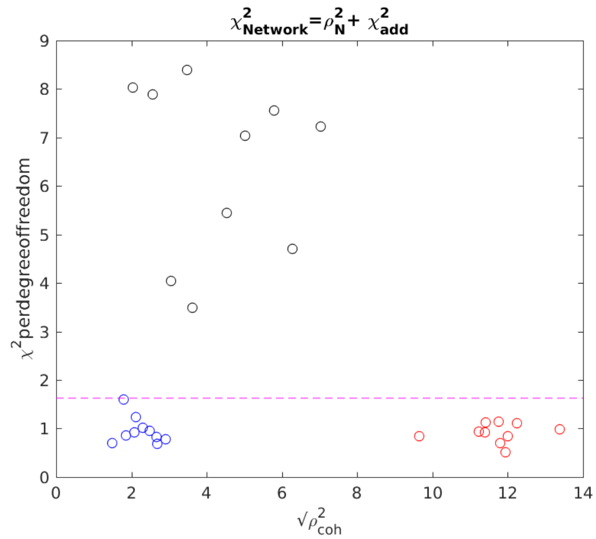


Figure 5.4: **A simple Network Discriminator:** $\chi^2_{Net} = \rho^2_N + \chi^2_{add}$: (i) Black circles: Glitch+Gaussian Noise (ii) Red circles: Signal+Gaussian Noise(iii) Blue circles: Pure Gaussian Noise (iv) Dashed Magenta line: 3 Sigma Threshold.

5.1 Observations and Remarks

There are several salient observations and conclusions that can be drawn from these plots:

- In all the plots, the cases of only pure noise and the cases of Signal plus pure Gaussian noise have mean value of χ^2 per degree of freedom as 1, which is expected because in both these two cases the discriminator follows a χ^2 distribution.
- Among the χ^2 tests (Fig. 5.3 and Fig. 5.2) we see that the best discrimination (χ^2 per degree of freedom for glitch) is given by the individual detector χ^2 for the first detector. This is expected that because the glitch was introduced only in the first detector. Furthermore, the individual χ^2 tests in the other detectors dont give any good results (see Fig. 5.3). Hence, in χ^2_{add} the discriminatory power is reduced when compared to the individual detector χ^2 for detector 1, χ^2_1 because the unnecessary degrees of freedom of other detectors are used.
- The Allen's χ^2 tests in synthetic detectors (Fig. 5.2) each have a performance which lies in between the performance of the individual Allen's χ^2 for the first detector and

the performances of the individual Allen's χ^2 of the other four detectors.

- We also verify equation 4.91, by checking that the plots produced by χ^2_{add} and by $\chi^2_+ + \chi^2_\times + \sum_{\alpha=1}^{M-2} \chi^2_{N_\alpha}$ are identical.
- For this example, the Null SNR tests (Fig. 5.1) and the χ^2_{Net} test (Fig. 5.4) also seem to do decently well at discriminating the glitch.

Chapter 6

Conclusion and Future Directions

I believe that the work carried out in this thesis can help fine tune the χ^2 discriminator tests for a system of detectors. For discriminating particular glitch, $\overrightarrow{g(t)}$ from the signal, $\overrightarrow{h(t)}$, a good χ^2 discriminator would correspond to taking the squared projection of the data vector on a low dimensional subspace orthogonal to the signal. To find this good subspace, there are few things one can tune:

1. The choice of individual detector basis vectors ($u_{\alpha k}$ defined in 3.19) used to construct $\chi^2_{general}$ (defined in equation 4.48).
2. The choice of the subspace of $\mathbb{R}^M$, $\mathcal{Q}$ (defined in 4.86), which is used to construct $\chi^2_{general}$ (equation 4.88).
3. The choice of the subspace of the null-space $\mathcal{N}$, $\mathcal{N}_{sub}$ (defined in 4.41), which is used to construct $\rho^2_{\mathcal{N}_{sub}}$ (defined in equation 4.42).

There are two questions, regarding matched filtering, that need to analyzed statistically in the context of a network of detectors.

- What templates does a given network glitch tend to trigger? and
- Which kind of network glitches tend to trigger a particular template?

The answers to these questions coupled with the above mentioned tuning should give a very powerful Network Signal-Glitch χ^2 discriminatory test which should be able to tackle the notorious classes of glitches.

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