Spectral Graph Theory

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled Spectral Graph Theory towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Amol Sahebrao Hinge at Indian Institute of Science Education and Research under the supervision of Dr. Chandrasheel Bhagwat, Assistant Professor, Department of Mathematics, during the academic year 2018-2019.

Dr. Chandrasheel Bhagwat

Committee:

Dr. Chandrasheel Bhagwat

Dr. Kaneenika Sinha

This thesis is dedicated to my grandparents

Declaration

I hereby declare that the matter embodied in the report entitled Spectral Graph Theory are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Chandrasheel Bhagwat and the same has not been submitted elsewhere for any other degree.

Amol Sahebrao Hinge

Acknowledgments

Firstly, I would like to express my sincere gratitude to my supervisor Prof. Chandrasheel Bhagwat for continuously guiding and encouraging me throughout my thesis. I gained greatly from his insightful understanding of the subject through the numerous discussions we have had. I am grateful to my TAC member Prof. Kaneenika Sinha for supporting me throughout the thesis. I am thankful to my family for encouraging me throughout my life. Finally, I thank all my friends for making my life at IISER delightful.

Abstract

In this thesis, we will state and prove the relationship between distribution of primes and Laplacian spectrum of a natural number network. We also look at Laplacian spectrum of an arithmetic network and observed some interesting pattern for k-th highest eigenvalue of a Laplacian matrix of an arithmetic network. We showed that the degree distribution of natural number network and arithmetic network follows power law which means both networks are scale free networks.

Contents

Introduction

Spectral graph theory plays a significant role in a variety of areas such as number theory and discrete mathematics. Spectral graph theory is the study of the properties of graph using eigenvalues and eigenvectors of a matrix associated with a graph. Generally this matrix is adjacency matrix or Laplacian matrix. In this thesis we mainly work with Laplacian matrix. The graphs which we have studied are natural number network and arithmetic network. A natural number network is a graph with vertices labelled as $1, 2, \cdots, n$ and the adjacency is defined by divisibility relation. An arithmetic network is a graph with vertices labelled according to arithmetic progression and the adjacency is defined by divisibility relation.

In Chapter 1, we have stated the definition of Laplacian matrix, spectrum of a graph and star shape graph which will be used in further chapters. In Chapter 2, we have established the relationship between Laplacian spectrum of a natural number network and number of prime numbers between $n/2$ to n. As the highest eigenvalue of a Laplacian matrix of a natural number network with n vertices is n, therefore we have calculated the highest eigenvalues of Laplacian matrix of an arithmetic networks and we have observed a pattern between the highest eigenvalue and number of vertices. We have also calculated k-th highest eigenvalue and observed some pattern between the k-th highest eigenvalue and the number of vertices which is discussed in section 3.1. We have not yet proved this result theoretically but we have done several numerical experiment and all these experiments support the result which is stated in Section 3.1. In [1], W. N. Anderson and T. D. Morley gave a basic upper bound for eigenvalues of the Laplacian matrix for a general graph. In [2], Li Jiong-Sheng and Zhang Xiao-Dong improved the basic upper bound given by W.N.Anderson and J.D.Morley which we have discussed in section 3.2.2. The improvised upper bound given by Li Jiong-Sheng and Zhang Xiao-Dong is in terms of three highest degrees of a graph. In section 3.3, we have given expression for three highest degrees of an arithmetic network to estimate the upper bound on the highest eigenvalue of a Laplacian matrix of an arithmetic network.

In Chapter 4, we have analysed the degree distribution of a network. In [3], S. M. Shekatkar, C. Bhagwat and G. Ambika have analysed the degree distribution of a natural number network. As this degree distribution follows power law, this network is called a scale free network. In section 4.2, we have done the same analysis for different arithmetic networks and we have observed that the degree distribution of arithmetic networks follows power law and hence these networks are also scale free networks.

Chapter 1

Preliminaries

1.1 Laplacian matrix

Given a graph G with n vertices, its Laplacian matrix L_G is defined as :

$$L_G = D_G - A_G$$

where D_G and A_G are degree diagonal matrix and adjacency matrix of the graph G respectively.

$$D_G(i,j) = \begin{cases} d_i & : \text{if i = j} \\ 0 & : \text{otherwise} \end{cases}$$

$$A_G(i,j) = \begin{cases} 1 & : \text{if i and j are adjacent} \\ 0 & : \text{otherwise} \end{cases}$$

1.2 Spectrum of a graph

The spectrum of a graph is defined as the multiset of eigenvalues of the Laplacian matrix or adjacency matrix corresponding to a graph. In this thesis, we would be working with Laplacian matrix of a graph. Laplacian spectrum of a graph is the multiset of eigenvalues of Laplacian matrix of a graph.

1.3 Star shape graph

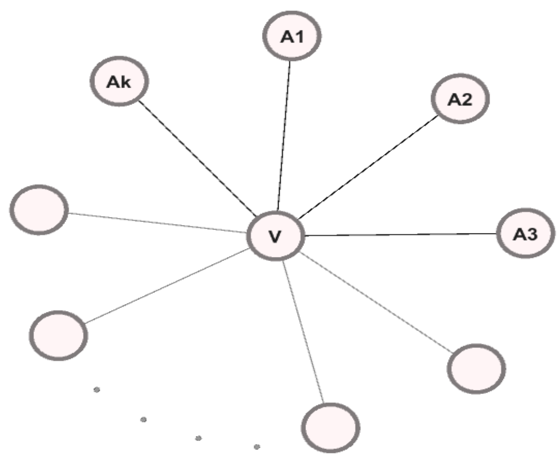

Figure 1.1: G_S

The star shape graph G_S consists of $(n+1)$ vertices, k components namely A_1,A_2,......,A_k where each A_i is finite and connected for all $1 \leq i \leq k$, each A_i has n_i vertices and the centre vertex V is adjacent to all other n vertices in G_S and A_1,A_2,......,A_k are disconnected components in $G_S \setminus \{V\}$.

1.4 Connectedness of a graph

A graph is said to be connected if there is a path between every pair of vertex. From every vertex to any other vertex, there should be some path to traverse.

1.5 Diameter of a graph

Diameter of a graph is the greatest distance between any pair of vertices. To find the diameter of a graph, first find the shortest path between each pair of vertices. The greatest length of any of these paths is the diameter of the graph.

Chapter 2

Laplacian spectrum of a network

2.1 Natural number network

Definition 2.1.1. *A natural number network G_n is a simple undirected graph on n vertices defined as follows:*
Let vertex set and edge set of G_n be V_{G_n} and E_{G_n} respectively.

- $V_{G_n} = \{1, 2, 3, \ldots, n\}$
- $E_{G_n} = \{\{j, k\}$ *is an edge if* $j < k$ *and* $j|k$ *where* $j, k \in V_{G_n}\}$

Example 1. *Natural number network with 10 vertices.*

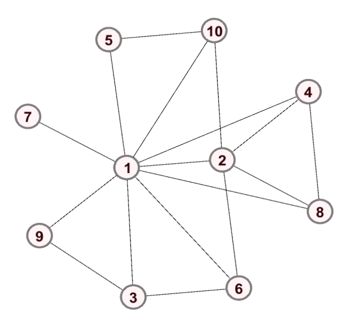

Figure 2.1: G_{10}

Remarks 2.1.1.

1. *1 is adjacent to* $x \; \forall \; x \in V_{G_n}$ *and* $x \neq 1$.

2. G_n *is connected.*

3. *Diam*$(G_n) = 2 \;\; \forall \;\; n \geq 3$.

4. *Consider the graph* G_n. *The vertex* k *is adjacent only to the vertex* $1. \iff k$ *is a prime and* $n/2 < k \leq n$.

Let $S = \{p : 1 \leq p \leq \text{n and p is prime}\}$. Using prime counting function $\pi(x)$ which counts the number of prime numbers less than or equal to x (asymptotically), we can deduce that the number of primes p such that $n/2 < p \leq n$ is equal to $\pi(n) - \pi(n/2)$ as $n \to \infty$, where $\pi(n) = \#\{p : 1 \leq p \leq \text{n and p is prime}\}$ as $n \to \infty$. The multiplicity of an eigenvalue 1 in the Laplacian spectrum of G_n increases as we increase the size of the natural number network.

2.1.1 Laplacian spectrum of a natural number network

In G_n, let d_i be the degree of i^{th} vertex which is also the degree of vertex i in G_n. Note that G_n is a simple undirected graph, so there are no multiple edges or self loops in the graph. Let us define a matrix L_{G_n} for G_n as follows :

$$L_{G_n}(i,j) = \begin{cases} d_i & : \text{if i = j} \\ -1 & : \text{if i and j are adjacent} \\ 0 & : \text{otherwise} \end{cases}$$

Theorem 2.1.1. *The number of prime numbers* p *such that* $n/2 < p \leq n. =$ *Multiplicity of 1 in the Laplacian spectrum of* G_n.
This theorem is proved using the following lemmas:

Lemma 2.1.2. *:* $U_n = \{$*All composite numbers less than or equal to* n *and all prime numbers less than or equal to* $n/2\}$ *forms a connected component.*

Proof *: We will use induction to prove this lemma,*

Base case , n=4

Clearly, the component containing vertex 2 and vertex 4 forms a connected component. That implies U_4 is connected.

Assume that the lemma is true for n.

Then U_n forms a connected component in G_n. Note that the number of vertices in G_n is n. Add vertex $(n+1)$ to G_n and add corresponding edges as well. Now we will show that U_{n+1} is connected.

Case 1 *: $(n+1)$ is a prime number.*

Clearly $(n+1)$ is greater than $n/2$ and it is a prime number, so the vertex $(n+1)$ will just get connected vertex 1 in the graph G_{n+1}, which implies that vertex $(n+1)$ will form a component with only one vertex and the component U_{n+1} will remain same as component U_n which we know that is connected. Hence the component U_{n+1} form a connected component.

Case 2 *: $(n+1)$ is a composite number.*

As $(n+1)$ is a composite number $\exists$ atleast one $x \in G_{n+1}$ such that x is adjacent to $(n+1)$. Now if we show that atleast one factor of $(n+1)$ is less than or equal to $n/2$ then Case 2 is done.

Claim *: At least 1 factor of $(n+1)$ is less than or equal to $n/2$.*

Proof by contradiction, suppose that both the factors of $(n+1)$ are greater than $n/2$. Let us say, $(n+1) = ab$, where $a > n/2$ and $b > n/2$. Therefore $(n+1) > n^2/4$, which is false for every $n \geq 5$. Hence the vertex $(n+1)$ is connected to one of the vertices in U_n and we know that U_n is already connected which makes U_{n+1} also a connected component. Therefore component U_{n+1} is a connected component in G_{n+1}.

$\square$

Lemma 2.1.3. *There are $(l+1)$ components in G_n namely $P_1, P_2, ..., P_l$ and U_n, where l is the number of prime numbers p such that $n/2 < p \leq n$.*

Proof *: We know that the number of prime numbers p such that $n/2 < p \leq n$ is equal to l, which implies there will be l degree 1 vertices in G_n and each vertex corresponds to one component as it is only connected to vertex 1 and not to any other vertex in G_n. It implies that there are l single vertex components in G_n and in Lemma 2.1.2, we have already shown that all the remaining vertices except vertex 1 forms a single connected component which we have represented as U_n therefore there are $(l+1)$ connected components in G_n and we call each singleton component as $P_1, P_2, ..., P_l$.*

$\square$

Lemma 2.1.4. *The graph G_n is an example of the graph G_S.*

Proof *: The vertex 1 in G_n is analogous to vertex V in G_S as 1 is adjacent to $x \, \forall \, x \in V_G$ and $x \neq 1$. From Lemma 2.1.3 components $P_1, P_2, ..., P_l$ and U_n are $(l+1)$ finite and connected components of G_n and $P_1, P_2, ..., P_l$ and U_n are disconnected components in $G_n \setminus \{1\}$. Therefore the graph G_n is an example of the graph G_S.*

□

Lemma 2.1.5. *Let G be a simple connected graph with n vertices, G^c be its compliment and K be the complete graph on these n vertices. Eigenvalues of L_G are $\lambda_1 = 0 \leq \lambda_2 \leq \cdots \leq \lambda_n$, eigenvalues of L_{G^c} are $\mu_1 = 0 \leq \mu_2 \leq \cdots \leq \mu_n$ and L_K is the Laplacian matrix of K. Then,*

$$\lambda_i + \mu_{n+2-i} = n \; ; \; \forall \, i = 2, 3, \cdots, n.$$

Proof *: We can observe that $L_G + L_{G^c} = L_K$ and with the help of basic linear algebra we can also see that L_K and L_G commute. Now we will show that L_G and L_{G^c} commute.*

$$L_{G^c} = L_K - L_G$$

$$L_G L_{G^c} = L_G L_K - L_G L_G \tag{2.1}$$

$$L_{G^c} L_G = L_K L_G - L_G L_G \tag{2.2}$$

From eq. (2.1) and eq. (2.2) we have shown that L_G and L_{G^c} commute. We also know that if two matrices commute then they have same eigenvector. Here, L_K and L_G commute and L_G and L_{G^c} also commute. Hence, L_G, L_{G^c}, L_K have same eigenvector(Eigenvalues corresponding to this same eigenvector may be different). We can observe that,

$$L_K L_G = L_G L_K = nL_G$$

Let $\bar{v}$ be the eigenvalue of L_G corresponding to non-zero eigenvalue λ.

$$L_G \bar{v} = \lambda \bar{v}$$

$$L_K L_G \bar{v} = \lambda L_K \bar{v}$$

$$nL_G \bar{v} = \lambda L_K \bar{v}$$

$$n\lambda \bar{v} = \lambda L_K \bar{v}$$

$$L_K \bar{v} = n\bar{v}$$

For any eigenvector $\bar{v}$ of L_K, the eigenvalue corresponding to $\bar{v}$ is n. Therefore,

$$L_k \text{ have eigenvalues } \{0, \underbrace{n, n, \cdots, n}_{(n-1)times}\}$$

$$\begin{aligned} L_{G^c} + L_G &= L_K \\ L_{G^c}\bar{v} + L_G\bar{v} &= L_K\bar{v} \\ \lambda_i\bar{v} + \mu_j\bar{v} &= n\bar{v} \end{aligned}$$

For each eigenvector $\bar{v}$ corresponding to eigenvalue n we will get $\lambda_i + \mu_j = n$. The multiplicity of eigenvalue n in the Laplacian spectrum of K is $(n-1)$, hence there will be $(n-1)$ pairs of $\lambda_i + \mu_j = n$. Let us arrange the values of λ_i's in ascending order.

$$\begin{aligned} \lambda_2 + \mu_j &= n \\ \lambda_3 + \mu_j &= n \\ \vdots \quad \vdots \quad &\vdots \\ \lambda_n + \mu_j &= n \end{aligned}$$

This will arrange μ_j's in descending order.

$$\begin{aligned} \lambda_2 + \mu_n &= n \\ \lambda_3 + \mu_{n-1} &= n \\ \vdots \quad \vdots \quad &\vdots \\ \lambda_n + \mu_2 &= n \end{aligned}$$

Therefore,

$$\lambda_i + \mu_{n+2-i} = n \ ; \ \forall\, i = 2, 3, \cdots, n.$$

$\square$

Lemma 2.1.6. *Let G_S be the star shape graph and A_i's are the components of G_S as defined in section 2.1 $\forall\ 1 \leq i \leq k$ and the eigenvalues of L_{A_i} be 0,$\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{i(n_i-1)}$ $\forall\ 1 \leq i \leq k$.*

Then the eigenvalues of the L_{G_S} are given by :

$$\{0, \underbrace{1, 1, 1, \cdots, 1}_{(k-1)times}, (n+1)\} \amalg \{(\lambda_{ij}+1) : 1 \leq i \leq k \ , \ 1 \leq j \leq (n_i - 1) \ and \ \lambda_{ij} \neq 0\}$$

$$L_{G_S} = \begin{bmatrix} n & -1 & \cdots & \cdots & \cdots & -1 \\ -1 & L_{A_1} + \mathbb{I} & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & L_{A_i} + \mathbb{I} & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ -1 & 0 & 0 & 0 & 0 & L_{A_k} + \mathbb{I} \end{bmatrix}$$

As vertex V in G_S is connected to every other vertex in G_S, we can say that the graph G_S is connected. Therefore the smallest eigenvalue of L_{G_S} is 0. (Proof of this is given in [4])
Let $\bar{v}_i$ represents the eigenvector of L_{A_i} corresponding to the eigenvalue λ_{ij} $\forall$ $1 \leq i \leq k$ and $1 \leq j \leq (n_i - 1)$.

$$\begin{aligned} L_{A_i}\bar{v} &= \lambda_{ij}\bar{v} \\ (L_{A_i} + \mathbb{I})\bar{v} &= L_{A_i} \ \bar{v} + \mathbb{I}\bar{v} \\ (L_{A_i} + \mathbb{I})\bar{v} &= (\lambda_{ij} + 1)\bar{v} \end{aligned}$$

We have showed that if L_{A_i} have eigenvalue λ_{ij} and eigenvector $\bar{v}_i$, then $(L_{A_i} + \mathbb{I})$ will have eigenvalue $(\lambda_{ij} + 1)$ and eigenvector $\bar{v}_i$ and G_S will also have eigenvalue $(\lambda_{ij} + 1)$ with eigenvector $\bar{V}$ which will look like following,
Let,

$$v_i = \begin{bmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{in_i} \end{bmatrix}_{n_i \times 1}$$

Then,

$$\bar{V} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{i1} \\ x_{i2} \\ \vdots \\ x_{in_i} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(n+1)\times 1}$$

Therefore G_S *will have eigenvalue* $(\lambda_{ij}+1)$ *with eigenvector* $\bar{V}$ *and which is true* $\forall\ 1 \leq i \leq k\ ,\ 1 \leq j \leq (n_i{-}1)$. *Therefore we have showed that the set*

$$\{(\lambda_{ij}+1) : 1 \leq i \leq k\ ,\ 1 \leq j \leq (n_i{-}1) \text{ and } \lambda_{ij} \neq 0\}$$

is a set of eigenvalues of L_{G_S}.
G_S *is simple connected graph with (n+1) vertices and we can easily see from Figure 1.1 that* ${G_S}^c$ *will have 2 connected components. Therefore the eigenvalues of* $L_{{G_S}^c}$ *are* $\mu_1 = 0 \leq \mu_2 = 0 \leq \mu_3 \leq \cdots \leq \mu_{n+1}$ *and eigenvalues of* L_{G_S} *are* $\lambda_1 = 0 \leq \lambda_2 \leq \cdots \leq \lambda_{n+1}$. *Using Lemma* (2.1.5),

$$\begin{aligned} \mu_2 + \lambda_{n+1} &= n+1 \\ \lambda_{n+1} &= n+1 \end{aligned}$$

Therefore the highest eigenvalue of L_{G_S} *is* $(n+1)$.
Let $\bar{u_i}$ *be the eigenvector of* L_{A_i} *corresponding to eigenvalue 0. Therefore,*

$$\bar{u_i} = \begin{bmatrix} x_i \\ x_i \\ \vdots \\ x_i \end{bmatrix}_{n_i \times 1}$$

$$L_{A_i}\bar{u_i} = 0\bar{u_i}$$
$$(L_{A_i} + \mathbb{I})\bar{u_i} = 1\bar{u_i}$$

We have showed that if L_{A_i} *have eigenvalue* 0 *and eigenvector* $\bar{u_i}$*, then* $(L_{A_i} + \mathbb{I})$ *will have eigenvalue* 1 *and eigenvector* $\bar{u_i}$ *and* G_S *will also have eigenvalue* 1 *with eigenvector* $\bar{U}$ *which will look like following,*

$$\bar{U} = \begin{bmatrix} 0 \\ u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix}_{(n+1)\times 1}$$

We also know that eigenvector of L_{G_S} *corresponding eigenvalue 0 is a constant vector and it is orthogonal to every other non-zero eigenvector of* L_{G_S}*.*

$$\bar{U} \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 0$$

$$\sum_{i=1}^{k} n_i x_i = 0 \tag{2.3}$$

Equation (2.3) *implies that if we fix any* $(k-1)$ x_i*'s then the remaining one* x_i *is fixed, therefore degree of freedom for this equation is* $(n-1)$ *and hence the dimension of eigenspace related to eigenvalue 1 for matrix* L_{G_S} *is* $(n-1)$*. As the dimension of eigenspace is number of linearly independent eigenvectors corresponding to the eigenvalue, therefore the eigenvector* $\bar{U}$ *have* $(k-1)$ *chooses which are linearly independent which implies the multiplicity of eigenvalue 1 for matrix* L_{G_S} *is* $(k-1)$*. Hence the eigenvalues of the* L_{G_S} *are given by :*

$$\{0, \underbrace{1, 1, 1, \cdots, 1}_{(k-1)times}, (n+1)\} \ \amalg \ \{(\lambda_{ij}+1) : 1 \leq i \leq k \ , \ 1 \leq j \leq (n_i - 1) \ and \ \lambda_{ij} \neq 0\}$$

$\square$

We hereby provide the proof of Theorem 2.1.1 *using the lemmas stated above.*

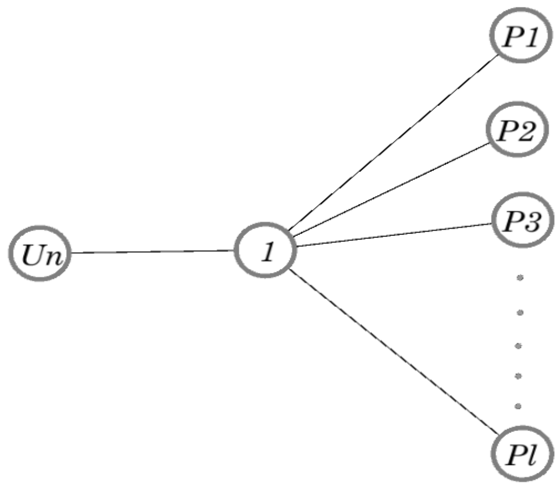

Figure 2.2: G_n

Proof of Theorem 2.1.1*:*

In Lemma 2.1.4*, we have proved that the graph* G_n *is an example of the graph* G_S*. From Figure 2.2* $P_1, P_2, ..., P_l$ *and* U_n *are* $(l + 1)$ *connected components of* G_n*. From Lemma* 2.1.6*, the multiplicity of eigenvalue 1 in the Laplacian spectrum of* G_S *is equal to* $(k - 1)$*, which indeed is equal to number of connected components of* G_S *minus 1. In natural number network* G_n *there are* $(l+1)$ *connected components, hence the multiplicity of eigenvalue 1 in the Laplacian spectrum of* G_n *is* l*, which indeed is equal to number of prime numbers* p *such that* $n/2 < p \leq n$*.*

$\therefore$ *Number of prime numbers* p *such that* $n/2 < p \leq n$*.* $=$ *Multiplicity of 1 in the Laplacian spectrum of* G_n*.* $\square$

Remark 2.1.2. *According to Bertrand's postulate* $\exists$ *atleast one prime number between* $\frac{n}{2}$ *and* n*. From Theorem* 2.1.1 *and Bertrand's postulate we can observe that the second lowest eigenvalue of* L_{G_n} *is 1 as there is atleast one prime number between* $\frac{n}{2}$ *and* n*.*

2.2 Arithmetic Network

Definition 2.2.1. *Arithmetic network $G_{a,d,n}$ is a simple undirected graph on n vertices defined as follows : Let vertex set and edge set of $G_{a,d,n}$ are $V_{G_{a,d,n}}$ and $E_{G_{a,d,n}}$ respectively.*

- $V_{G_{a,d,n}} = \{dk + a : (a, d) = 1 \text{ and } k = 1, 2, 3, \ldots, n-1\}$
- $E_{G_{a,d,n}} = \{\{dk_1 + a, dk_2 + a\} \text{ is an edge if } k_1 < k_2 \text{ and } (dk_1 + a) \mid (dk_2 + a)\}$

Let us start with simple example of the graph that is $G_{1,4,n}$.

- $V_{G_{1,4,n}} = \{1, 5, 9, \cdots, 4n-3\}$
- $E_{G_{a,d,n}} = \{\{4k_1 + 1, 4k_2 + 1\}$ is an edge if $k_1 < k_2$ and $(4k_1 + 1) \mid (4k_2 + 1)\}$

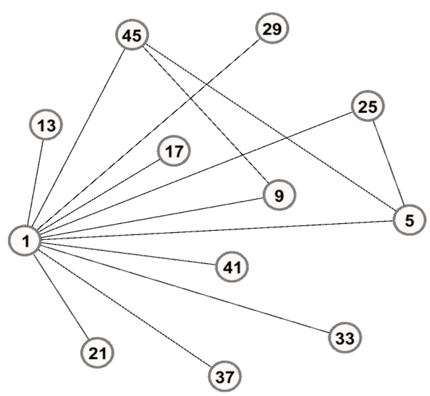

Figure 2.3: $G_{1,4,12}$

Notice that $G_{1,1,n} = G_n$. The natural number network is an example of arithmetic network.

Chapter 3

Laplacian spectrum of an arithmetic network

3.1 k-th highest eigenvalue of $L_{G_{a,d,n}}$

Laplacian spectrum of an arithmetic network is defined as the multiset of eigenvalues of $L_{G_{a,d,n}}$. We have calculated the eigenvalues of $L_{G_{a,d,n}}$ for different values of n for fixed values of a, d. We also vary values of a, d to observe how highest eigenvalues changes according to the values of a, d. We have plotted the graph of highest eigenvalue of $L_{G_{a,d,n}}$ verses the number of vertices and following are our observations.

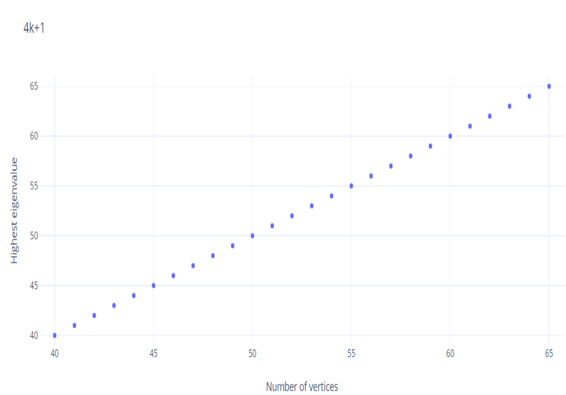

Figure 3.1: Zoomed version for 4k+1

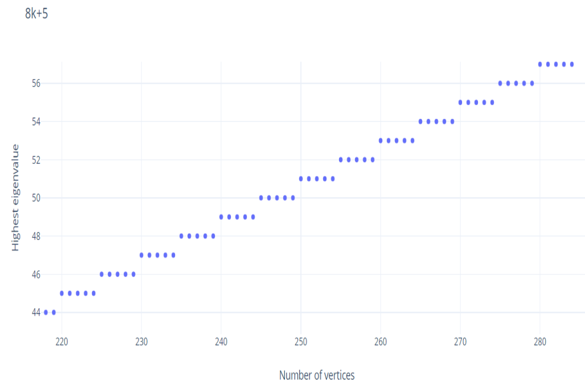

Figure 3.2: Zoomed version for 8k+5

Let the highest eigenvalue of a $L_{G_{a,d,n}}$ with n vertices be $H_1(G_{a,d,n})$. Observations for highest eigenvalue of $L_{G_{a,d,n}}$,

1. $H_1(G_{a,d,al+1}) \sim H_1(G_{a,d,al+2}) \sim \cdots \sim H_1(G_{a,d,al+a})$
2. $H_1(G_{a,d,a+n}) - H_1(G_{a,d,n}) \sim 1$.

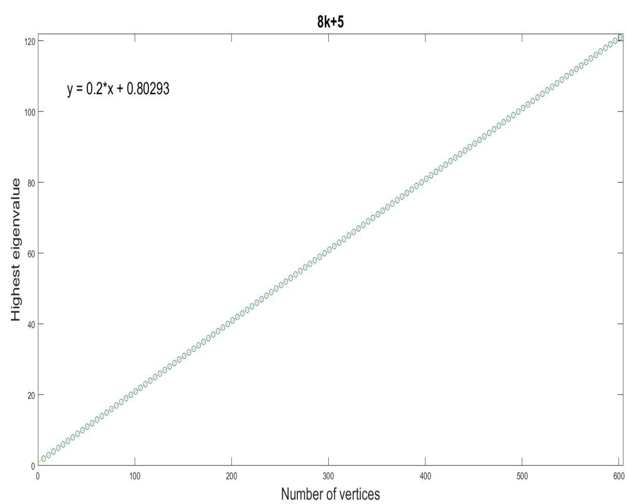

Figure 3.3: **First value plot**. From the group of every five points in Figure 3.2, we have plotted first point from every group against the corresponding value of number of vertices. The equation of the graph is given by $y = \frac{1}{5}x + \frac{4}{5} + 0.00293$. The equation of first value plot for general arithmetic network $(G_{a,d,n})$ is given by $y = \frac{1}{a}x + \frac{a-1}{a} + c$.

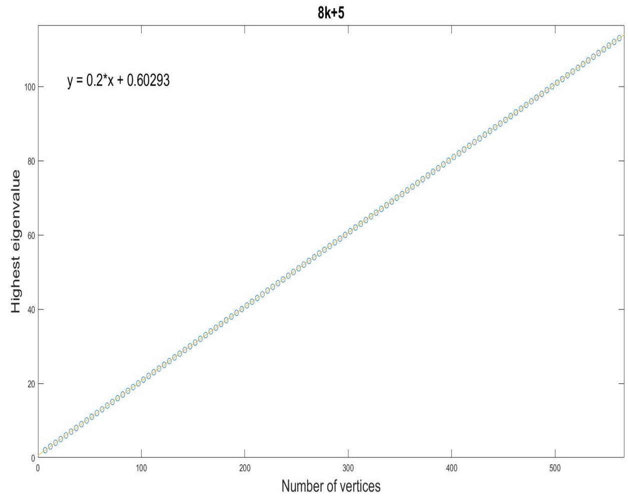

Figure 3.4: **Second value plot**. From the group of every five points in Figure 3.2, we have plotted second point from every group against the corresponding value of number of vertices. The equation of the graph is given by $y = \frac{1}{5}x + \frac{3}{5} + 0.00293$. The equation of second value plot for general arithmetic network $(G_{a,d,n})$ is given by $y = \frac{1}{a}x + \frac{a-2}{a} + c$.

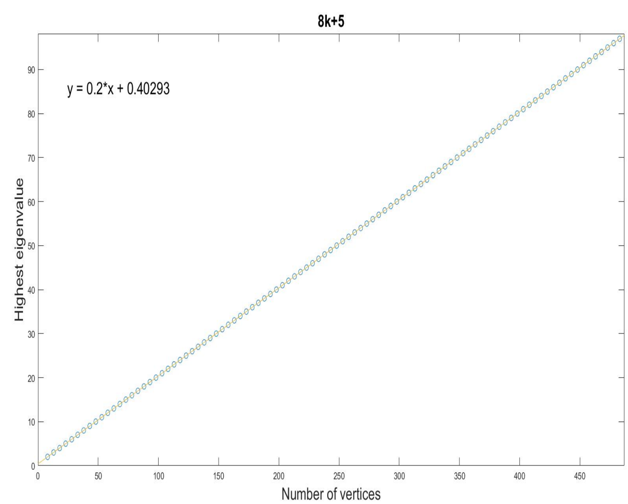

Figure 3.5: **Third value plot**. From the group of every five points in Figure 3.2, we have plotted third point from every group against the corresponding value of number of vertices. The equation of the graph is given by $y = \frac{1}{5}x + \frac{2}{5} + 0.00293$. The equation of third value plot for general arithmetic network $(G_{a,d,n})$ is given by $y = \frac{1}{a}x + \frac{a-3}{a} + c$.

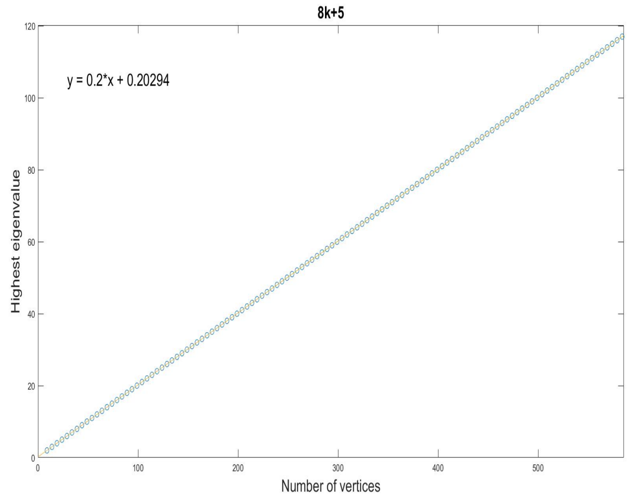

Figure 3.6: **Fourth value plot**. From the group of every five points in Figure 3.2, we have plotted fourth point from every group against the corresponding value of number of vertices. The equation of the graph is given by $y = \frac{1}{5}x + \frac{1}{5} + 0.00293$. The equation of fourth value plot for general arithmetic network $(G_{a,d,n})$ is given by $y = \frac{1}{a}x + \frac{a-4}{a} + c$.

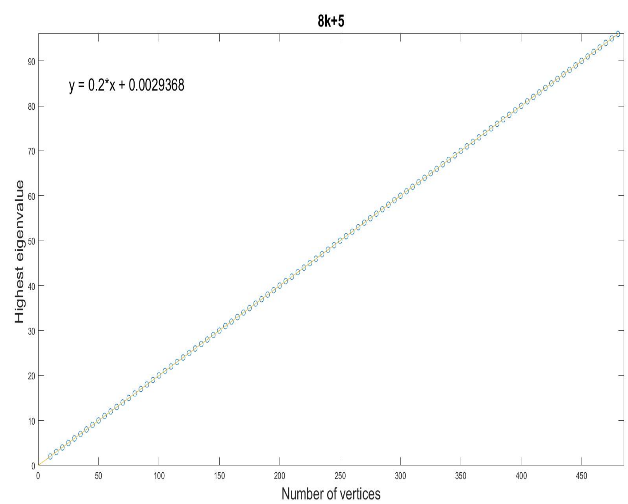

Figure 3.7: **Fifth value plot**. From the group of every five points in Figure 3.2, we have plotted fifth point from every group against the corresponding value of number of vertices. The equation of the graph is given by $y = \frac{1}{5}x + 0.00293$. The equation of fifth value plot for general arithmetic network $(G_{a,d,n})$ is given by $y = \frac{1}{a}x + \frac{a-5}{a} + c$.

There are five different graphs in the case of $8k+5$ as the value of a in $8k+5$ is 5. In general case $i.e.$ $dk+a$ we will have a different plots.

We have also plotted the graph between 2nd highest eigenvalue of $L_{G_{a,d,n}}$ and the number of vertices for different values of a, d and following are our observations:

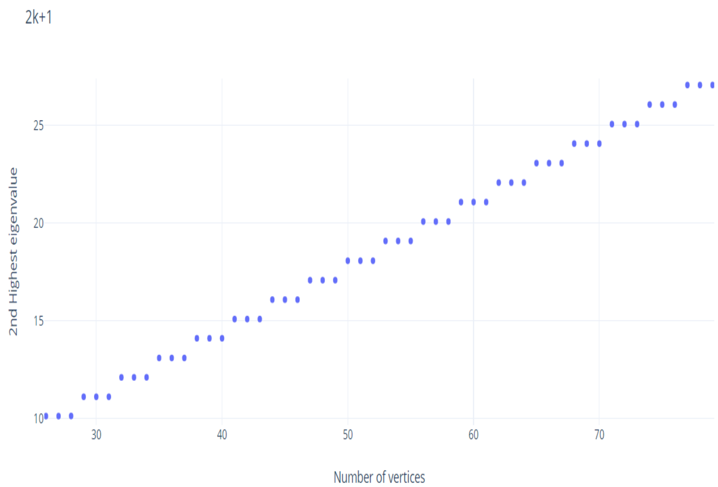

Figure 3.8: Zoomed version for 2k+1

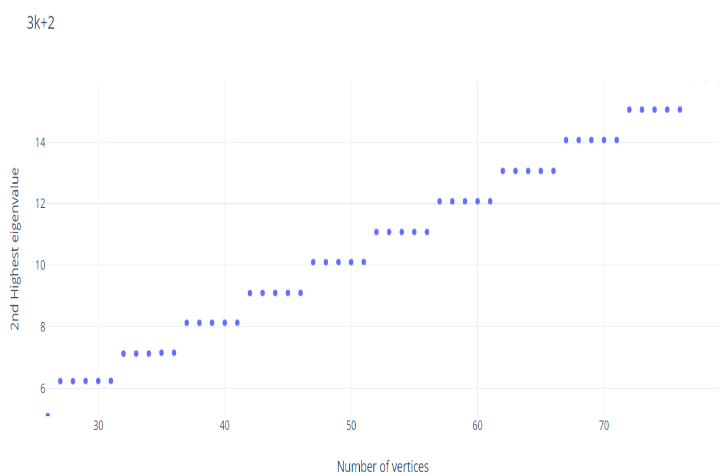

Figure 3.9: Zoomed version for 3k+2

Let 2nd highest eigenvalue of a $L_{G_{a,d,n}}$ be $H_2(G_{a,d,n})$. Observations for 2nd highest eigenvalue of $L_{G_{a,d,n}}$,

1. $H_2(G_{a,d,(a+d)l+2}) \sim H_2(G_{a,d,(a+d)l+3}) \sim \cdots \sim H_1(G_{a,d,(a+d)l+(a+d+1)})$.
2. $H_2(G_{a,d,(a+d+n)}) - H_2(G_{a,d,n}) \sim 1$.

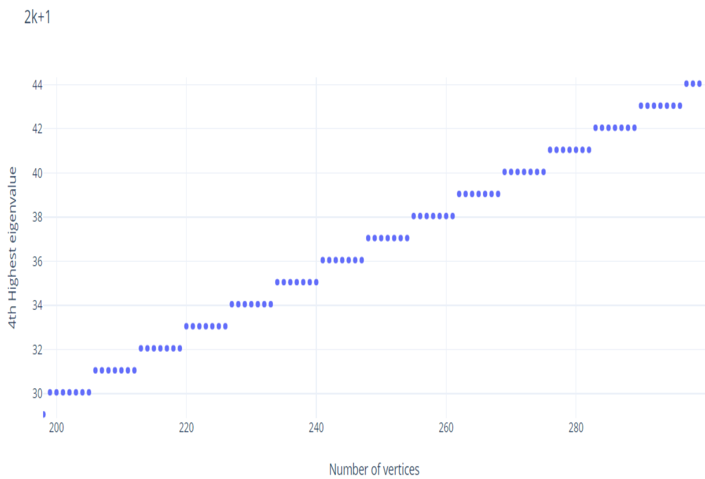

Figure 3.10: Zoomed version for 2k+1

Let k-th highest eigenvalue of a $L_{G_{a,d,n}}$ be $H_k(G_{a,d,n})$. Observations for k-th highest eigenvalue of $L_{G_{a,d,n}}$,

1. $H_k(G_{a,d,[a+(k-1)d]l+k}) \sim H_k(G_{a,d,[a+(k-1)d]l+(k+1)}) \sim \cdots \sim H_k(G_{a,d,[a+(k-1)d](l+1)+(k-1))})$.
2. $H_k(G_{a,d,(a+d(k-1)+n)}) - H_k(G_{a,d,n}) \sim 1$.

Note that $X \sim Y$ indicates $|X - Y| \leq 0.01$

3.2 An upper bound on eigenvalues of Laplacian matrix of a graph

3.2.1 Basic Upper Bound

In [1], W. N. Anderson and T. D. Morley gave a basic upper bound for eigenvalues of the Laplacian matrix of a graph G in terms of the degrees of the vertices of graph. Their result is the following:

Theorem 3.2.1. *: For a given graph G, let $d_1 \geq d_2 \geq \cdots \geq d_n$ be a degree sequence and λ is an eigenvalue of L(G), then*

$$\lambda \leq (d_1 + d_2) \tag{3.1}$$

3.2.2 Improved Upper Bound

In [2], Li Jiong-Sheng and Zhang Xiao-Dong improved the basic upper bound given by W.N.Anderson and J.D.Morley. Their result is the following:

Theorem 3.2.2. *For a given graph G, let $d_1 \geq d_2 \geq \cdots \geq d_n$ be a degree sequence and λ is an eigenvalue of L_G of G, then*

$$\lambda \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)}$$

To prove this we need following definitions and lemmas:

Definition 3.2.1. *Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set $E = \{e_1, e_2, \cdots, e_m\}$. For every edge $e_j = (v_x, v_y)$ assume one end of an edge to be positive end and other end to be negative end, then the oriented incidence matrix Q of a graph G is an $(n \times m)$ matrix defined as follows:*

$$Q(i, j) = \begin{cases} 1 & : \text{ if } v_i \text{ is the positive end of } e_j \\ -1 & : \text{ if } v_i \text{ is the negative end of } e_j \\ 0 & : \text{ otherwise} \end{cases}$$

We can notice that $QQ^T = L_G$.

Definition 3.2.2. *Given a graph* G*, its line graph* K *is a graph defined as follows: Each vertex of* K *represents an edge of a graph* G *and two vertices of* K *are adjacent if and only if their corresponding edges share a common endpoint in* G.

Example 2. *An example of a line graph* K *of a graph* G.

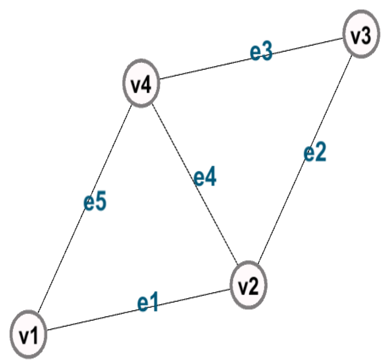

Figure 3.11: G

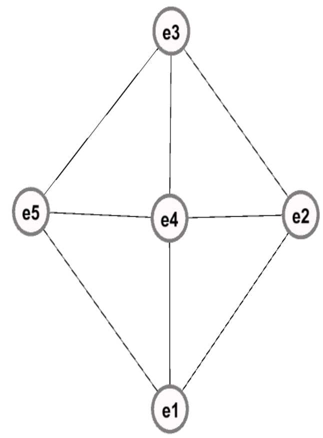

Figure 3.12: K

Notice that, if $e_j = (v_x, v_y)$ *then the degree of* e_j *in line graph* K *is given by,*

$$d(e_j) = d(v_x) + d(v_y) - 2$$

Lemma 3.2.3. *For a given graph* G*, let* $d_1 \geq d_2 \geq \cdots \geq d_n$ *be a degree sequence and let* B *be the adjacency matrix of the graph* K *of* G*. If* μ *is the highest eigenvalue of* B*, then*

$$\mu \leq \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)}$$

Proof of Lemma 3.2.3 *is given in [2]. We will assume that Lemma* 3.2.4 *is true and use it in proving the improved upper bound.*

Lemma 3.2.4. *For a given graph G, let Q be its oriented incidence matrix. Let B be the adjacency matrix of the line graph K of G. Then $|Q^TQ| = 2\mathbb{I}_m + B$, where $|A|$ stands for the matrix whose entries are absolute values of the entries of A.*

Proof: *Every main-diagonal entry of B is 0 as their there are no self edges in K. For (i, j) and $i \neq j$, the (i, j) entry of B will be 1 if the edges e_i and e_j and adjacent in graph G, and 0 otherwise. Whereas each main-diagonal entry of $|Q^TQ|$ is 2 and all other entries of $|Q^TQ|$ are same as entries of B. Therefore, $|Q^TQ| = 2\mathbb{I}_m + B$.* $\square$

We hereby provide the proof of Theorem 3.3.2,

Proof of Theorem 3.3.2: *Let the highest eigenvalue of a matrix A be $\mathcal{H}(A)$. From Lemma 3.2.3 and Lemma 3.2.4,*

$$\mathcal{H}(|Q^TQ|) \leq 2 + \mathcal{H}(B) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)}$$

We know that $\mathcal{H}(Q^TQ) \leq \mathcal{H}(|Q^TQ|)$ and (Q^TQ) and (QQ^T) have same non-zero eigenvalues.

$$\therefore \mathcal{H}(QQ^T) = \mathcal{H}(Q^TQ) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)}$$

$$\therefore \mathcal{H}(QQ^T) = \lambda \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)}$$

$$\therefore \lambda \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)} \tag{3.2}$$

$\square$

It is clear that (3.2) is a better upper bound than (3.1).

Corollary 3.2.5. *For a given graph G, let $d_1 \geq d_2 \geq \cdots \geq d_n$ be a degree sequence and λ is an eigenvalue of L(G) of G. If the two vertices of the largest degree are not adjacent to each other then,*

$$\lambda \leq d_1 + d_3$$

$\square$

Remarks 3.2.1. *In [1], W. N. Anderson and T. D. Morley proved the following statement, which is stronger than Theorem 3.2.1.*

Theorem 3.2.6. *For a given graph G, if λ is an eigenvalue of L_G, then*

$$\lambda \leq \max_{(u,v)\in E_G} \{d(u) + d(v)\}$$

In a similar way, we may prove the following result,

Theorem 3.2.7. *For a given graph G, if λ is an eigenvalue of L_G. Let*

$$a = \max_{(u,v)\in E_G} \{d(u) + d(v)\}$$

and let us assume that $(x, y) \in E_G$ is a edge such that,

$$a = \{d(x) + d(y)\}$$

Now lets denote,

$$b = \max_{(u,v)\in E_G-(x,y)} \{d(u) + d(v)\}$$

Then,

$$\lambda \leq 2 + \sqrt{(a-2)(b-2)}$$

3.3 Degree estimation for arithmetic network

Let $G_{a,d,n}$ be an arithmetic network as defined in definition (3.0.1).

- $V_{G_{a,d,n}} = \{dk + a : (a, d) = 1 \text{ and } k = 1, 2, 3, \ldots, n-1\}$

3.3.1 Expression for degree of $(dk + a)$

Let $D(dk + a)$ denotes the number of proper divisors of $(dk + a) \in V_{G_{a,d,n}}$ and $M(x)$ denotes the number of proper multiples of $(dk + a) \in V_{G_{a,d,n}}$.

$$\deg(dk + a) = D(dk + a) + M(dk + a)$$

Let us calculate the degree of seed vertex which is $\deg(a)$,

$$\deg(a) = D(a) + M(a)$$

Clearly, $D(a) = 0$. Now let us calculate the value of $M(a)$,

$$\begin{aligned}
M(a) &= \#\{k_1 : a|(dk_1 + a),\ 1 \leq k_1 \leq (n-1)\} \\
M(a) &= \#\{k_1 : a|dk_1,\ 1 \leq k_1 \leq (n-1)\} \\
M(a) &= \#\{k_1 : a|k_1,\ 1 \leq k_1 \leq (n-1)\} \qquad \cdots\cdots \{gcd(a,d) = 1\} \\
M(a) &= \#\{k_1 : k_1 = ac,\ 1 \leq k_1 \leq (n-1), c \in \mathbb{Z}^+\} \\
M(a) &= \#\{c :\ 1/a \leq c \leq (n-1)/a\} \\
\therefore M(a) &= \left\lfloor \frac{n-1}{a} \right\rfloor
\end{aligned}$$

$$\therefore \deg(a) = \left\lfloor \frac{n-1}{a} \right\rfloor$$

In a similar way, we can calculate the value of $M(dk+a)$ for general $0 \leq k \leq (n-1)$.

$$\begin{aligned}
M(dk+a) &= \#\{k_1 : (dk+a)|(dk_1 + a),\ (k+1) \leq k_1 \leq (n-1)\} \\
M(dk+a) &= \#\{k_1 : (dk+a)|d(k_1 - k),\ (k+1) \leq k_1 \leq (n-1)\} \\
M(dk+a) &= \#\{k_1 : (dk+a)|(k_1 - k),\ (k+1) \leq k_1 \leq (n-1)\} \qquad \cdots\cdots \{gcd(dk+a,d) = 1\} \\
M(dk+a) &= \#\{k_1 : (k_1 - k) = (dk+a)c,\ (k+1) \leq k_1 \leq (n-1), c \in \mathbb{Z}^+\} \\
M(dk+a) &= \#\{c :\ 1/a \leq c \leq (n-k-1)/(dk+a)\} \\
\therefore M(dk+a) &= \left\lfloor \frac{n-k-1}{dk+a} \right\rfloor
\end{aligned}$$

It is slightly difficult to calculate the value of $D(dk+a)$ for a general value of k. So we can find the value of $D(dk+a)$ for some of the cases.

$$D(d+a) = \begin{cases} 1 & : \text{ if a} = 1 \\ 0 & : \text{ otherwise} \end{cases}$$

$$D(2d+a) = \begin{cases} 1 & : \text{ if a} = 1,\, 2 \\ 0 & : \text{ otherwise} \end{cases}$$

$$\therefore \deg(d+a) = \begin{cases} \lfloor \frac{n-2}{d+a} \rfloor + 1 & : \text{ if a} = 1 \\ \lfloor \frac{n-2}{d+a} \rfloor & : \text{ otherwise} \end{cases}$$

$$\therefore \deg(2d+a) = \begin{cases} \lfloor \frac{n-3}{2d+a} \rfloor + 1 & : \text{ if a} = 1,\ 2 \\ \lfloor \frac{n-3}{2d+a} \rfloor & : \text{ otherwise} \end{cases}$$

Conjecture 1. *In an arithmetic network,* $\deg(a) \geq \deg(a + k_1 d)\ \forall\ k_1 = 1, 2, \cdots, (n-1)$.

It is easy to prove above conjecture for $a = 1$.

Proof: $\deg(1) = (n-1)$ and $\deg(1 + k_1 d) \leq (n-1)\ \forall\ k_1 = 1, 2, \cdots, (n-1)$.

$$\therefore \deg(1) \geq \deg(1 + k_1 d)\ \forall\ k_1 = 1, 2, \cdots, (n-1)$$

$\square$

We tried to prove this conjecture for general a. However, we could not complete it.

Chapter 4

Degree distribution of a network

Definition 4.0.1. *A scale-free network is a network whose degree distribution follows a power law, at least asymptotically. The fraction $P(k)$ is defined as the number of nodes in the network having k connections to other nodes divided by the total number of nodes in the network. We say the network is scale free, if for large values of k*

$$P(k) \sim k^{-\gamma}$$

where γ is a parameter whose value typically lies in the range $2 \leq \gamma \leq 3$, however it may lie outside this range.

4.1 Degree distribution of a natural number network

We have already defined natural number network in chapter 1. The nodes in natural number network are $1, 2, \cdots, n$ and there is a edge between two nodes if either of the two divides the other. Recall that we represent natural number network with n vertices as G_n. As the sequence of natural numbers have natural ordering it is better to view natural natural network as growing network with the addition of a new node at time t defined as follows:

1. At time $t = 1$ we start with the node $n = 1$ and after every unit time interval we add the next node from G_n.

2. This added node connects to all existing nodes whose numbers divide it.

Now we will construct two natural number network with different sizes $n = 8$ and $n = 16$.

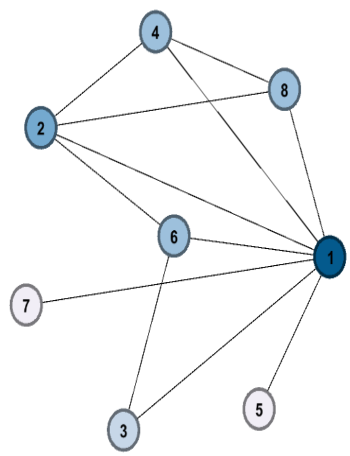

Figure 4.1: G_8

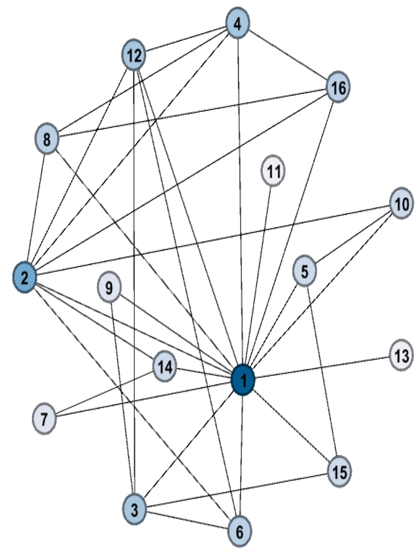

Figure 4.2: G_{16}

In Figure 4.1 and Figure 4.2 colour of a node represents degree of a node, darker the colour higher the degree.

In [3], S. M. Shekatkar, C. Bhagwat and G. Ambika have analysed the degree distribution of a natural number network. The natural number network is grown till the size of a network reaches $n = 2^{25} = 3, 35, 54, 432$. The resulting distribution of degrees follows a power law $(P(k) \sim k^{-\gamma})$ asymptotically. Using the method of maximum likelihood they have calculated the value of scaling index $\gamma \sim 2$. As the degree distribution of natural number network follows power law, we can say that natural number network is a scale free network.
As per our computing power we have grown the size of a natural number network till $n = 2^{12} = 4096$.

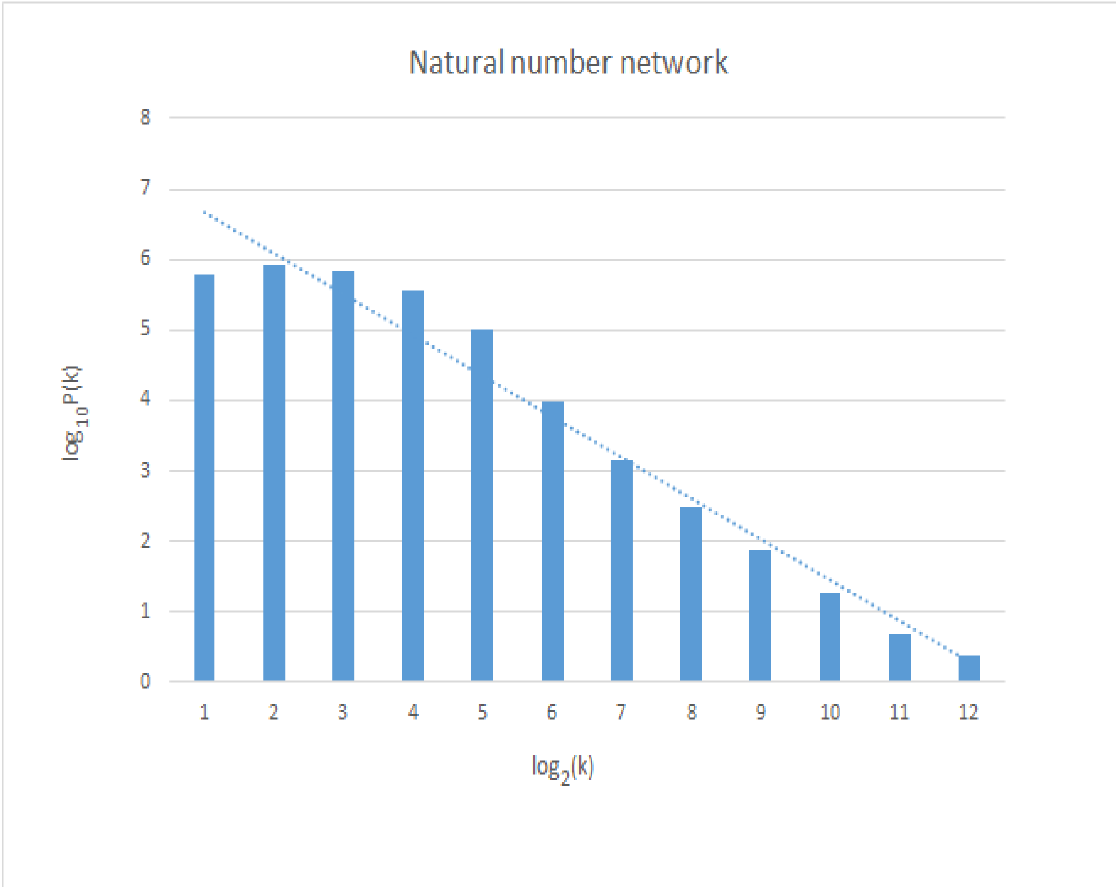

Figure 4.3: **Degree distribution of natural number network with logarithmic binning**. Sizes of successive bins are equal to successive positive powers of 2 and count in each bin is normalized by dividing by a bin width. The dotted line in the graph has slope $\gamma \sim 1.92$ and it is calculated using least square method.

4.2 Degree distribution of an arithmetic network

In this section, we will study the degree distribution of different arithmetic networks. The construction of a network and plotting of a graph is similar to what we have done in section 4.1. We will grow the size of different arithmetic network till $n = 2^{12} = 4096$ and calculate the value of scaling index γ.

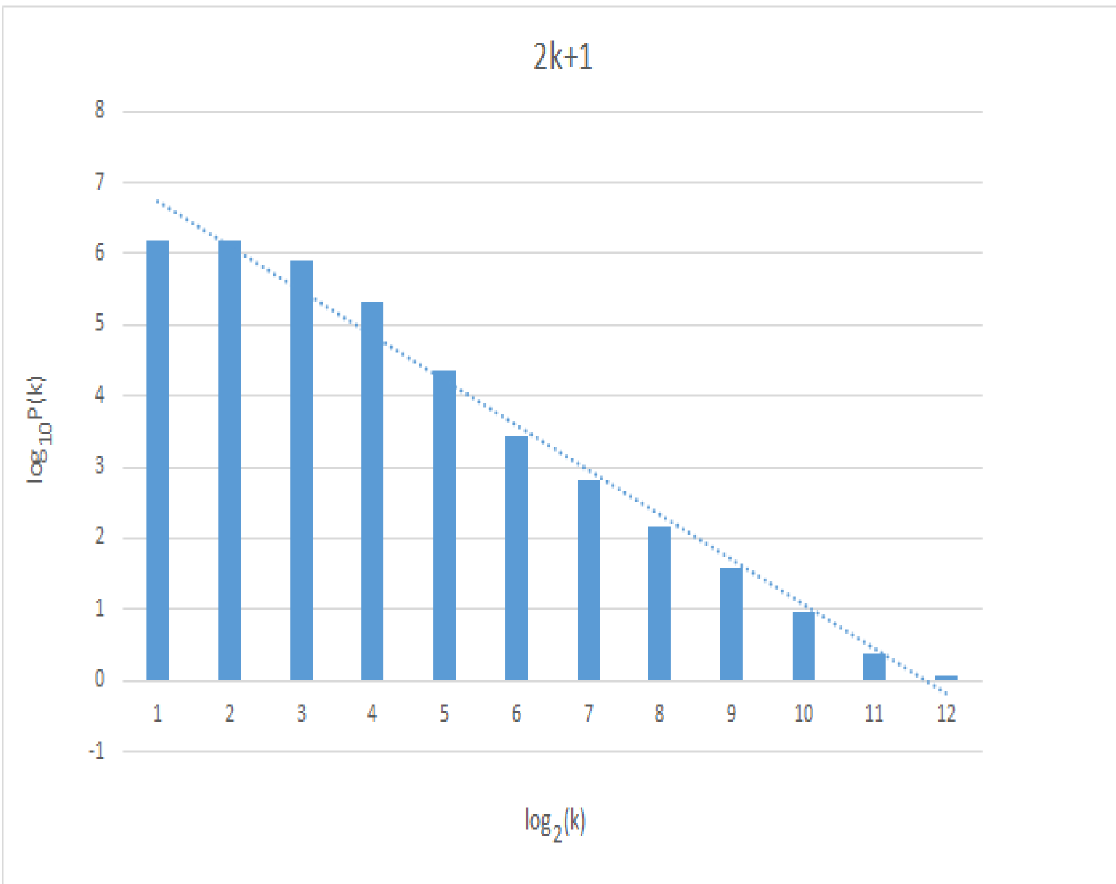

Figure 4.4: **Degree distribution of $G_{1,2,4096}$ network with logarithmic binning**. Sizes of successive bins are equal to successive positive powers of 2 and count in each bin is normalized by dividing by a bin width. The dotted line in the graph has slope $\gamma \sim 2.088$ and it is calculated using least square method.

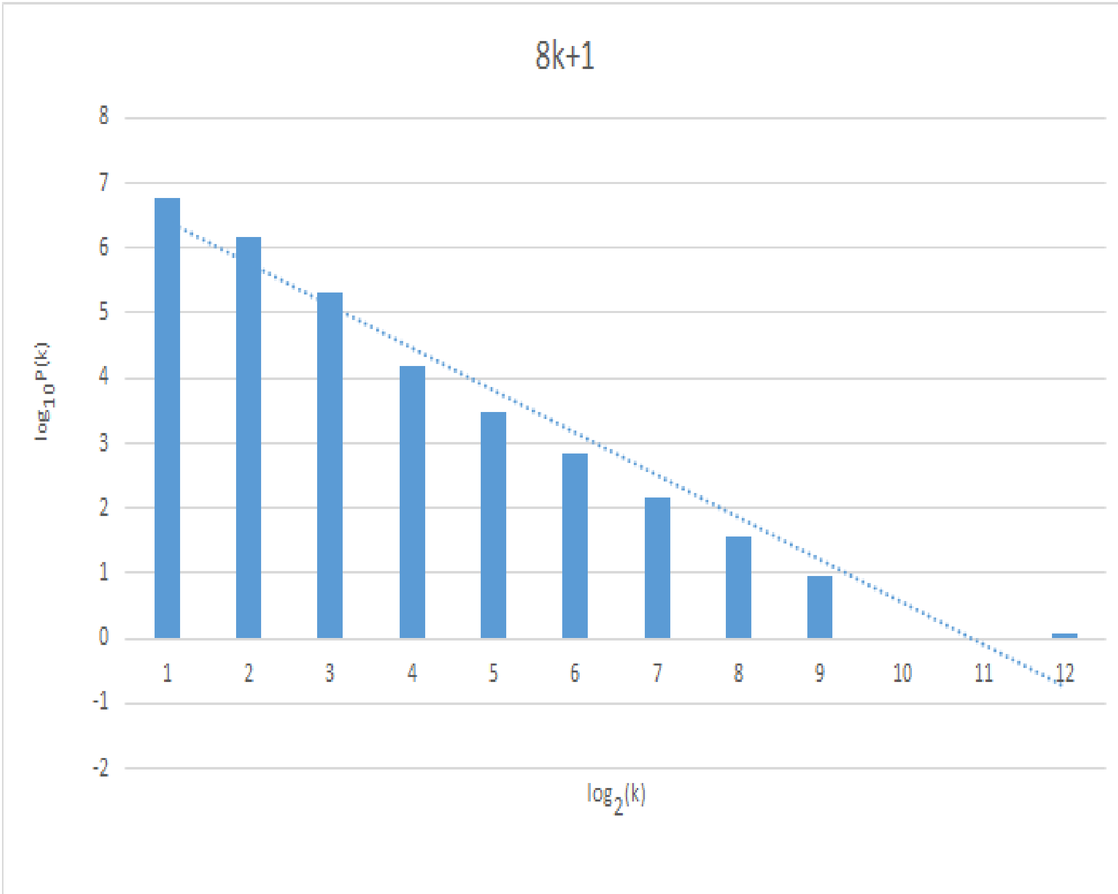

Figure 4.5: **Degree distribution of $G_{1,8,4096}$ network with logarithmic binning**. Sizes of successive bins are equal to successive positive powers of 2 and count in each bin is normalized by dividing by a bin width. The dotted line in the graph has slope $\gamma \sim 2.15$ and it is calculated using least square method.

Chapter 5

Conclusion

1. We proved that the number of prime numbers p such that $n/2 < p \leq n$ is equal to the multiplicity of 1 in the Laplacian spectrum of a natural number network.

2. We calculated the k-th highest eigenvalue of a Laplacian matrix for arithmetic networks using python programme and we observed the following:
   Let k-th highest eigenvalue of $L_{G_{a,d,n}}$ be $H_k(G_{a,d,n})$.

   (a) $H_k(G_{a,d,[a+(k-1)d]l+k}) \sim H_k(G_{a,d,[a+(k-1)d]l+(k+1)}) \sim \cdots \sim H_k(G_{a,d,[a+(k-1)d](l+1)+(k-1))})$.

   (b) $H_k(G_{a,d,(a+d(k-1)+n)}) - H_k(G_{a,d,n}) \sim 1$.

   Note that $X \sim Y$ indicates $|X - Y| \leq 0.01$.
   It will be interesting to explore the possibility of analytic proofs for these observations.

3. In an arithmetic network $G_{a,d,n}$, let a be a seed vertex then $\deg(a) \geq \deg(a+k_1 d)$ $\forall$ $k_1 = 1, 2, \cdots, (n-1)$. We proved above relation for $a = 1$ and attempted proving for general a, but we could not complete it. The expression for degree of vertex a is as follows:

$$\deg(a) = \left\lfloor \frac{n-1}{a} \right\rfloor$$

4. We showed that the degree distribution of a natural number network and an arithmetic network follows power law which means both networks are scale free networks.

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