p-adic L-functions for Modular Forms

A thesis submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

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April, 2014

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This is to certify that this thesis entitled "p-adic L-functions for Modular Forms" submitted towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune, represents work carried out by Punya Plaban Satpathy under the supervision of Prof. A. Raghuram.

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Dedicated to 09'

Acknowledgments

I would like to thank my supervisor, Prof. A. Raghuram, for the patient guidance, encouragement and advice he has provided throughout my time as his student. I have been extremely lucky to have a supervisor who cared so much about my work, and who responded to my questions and queries so promptly. I am also grateful to all faculty members of the Department of Mathematics at IISER Pune who have helped me directly or indirectly. At last, I would like to thank Jeeten Patel for helping me with latex without which writing of this thesis would not have been possible.

Abstract

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In this thesis we give an exposition of various known techniques of constructing p-adic L-functions in different cases. The key idea behind this is constructing a non-archimedean measure and then performing a p-adic Mellin transform of a suitable p-adic character. There is also the alternative approach introduced by Iwasawa of finding a p-adic holomorphic power series interpolating the special values of a Dirichlet L-function. We explain the prerequisites and expand upon above ideas to give a comprehensive view.

Contents

Chapter 1

Introduction

The theory of p-adic L-functions may be said to have begun when Euler discovered that the Riemann zeta function $\zeta(s)$ assumes rational values (essentially Bernoulli numbers) at negative integers. Some two centuries later, Kummer was led to look at Bernoulli numbers when he discovered that the question of whether a given prime p was "regular" depended upon the p-adic behaviour of certain Bernoulli numbers. He discovered the "Kummer congruences" between Bernoulli numbers, which could be naively described in terms of zeta values as follows;

"If $k \geq 1$ is an integer not divisible by $p-1$, then $\zeta(1-k)$ is p-integral, and the value of $\zeta(1-k)$ mod p depends only on the value of k mod $p-1$."

In the late 1950's Kubota and Leopoldt constructed a p-adic analogue of the Riemann zeta-function using the technique of p-adic Mellin transform, which p-adically interpolated the values of the Riemann zeta-function at the negative odd integers. This also lead to an alternate explanation of the congruence satisfied by Bernoulli numbers, discovered by Kummer.

Later Manin in [Ma2] used Mazur's p-adic integral to construct p-adic analogues of the Mellin transform for modular forms of arbitrary weight (for the full modular group) and showed that the values of these functions at integer points of the critical strip coincide with the classical values (upto elementary factors), so he was able to construct p-adic analogues of L-functions attached to modular forms.

In the subsequent chapters we give an overview of these various approaches towards constructing p-adic L-functions.

In Chapter 2, starting with basic p-adic analysis, non-archimedean distributions are constructed out of Bernoulli numbers. They are regularized to get an appropriate measure, which is then used to obtain p-adic analogue for the Riemann zeta function. Later there is a discussion on p-adic (holomorphic) functions which are defined by convergent power series and which take pre-assigned values at $s = 0, 1, 2, \ldots$. These results then lead to a p-adic extension for the Dirichlet L-functions $L(s, \chi)$, where χ is a Dirichlet chracter. In the end there is a brief discussion on associating power series to non-archimedean measures on $\mathbb{Z}_p$ based on [Ka].

In Chapter 3, Jacquet-Langlands theory is discussed in some detail; results from represenation theory of p-adic groups are given, specifically results about spherical representations, such as relating the satake parameters to the Hecke eigenvalues for spherical principal series, deriving the formula for spherical Whittaker function are discussed as well as the notion of a local L-function together with its functional equation. Then these local results are used to give global results of Jacquet-Langlands theory such as proving the functional equation for the global L-function of a cuspidal automorphic representation. At the end of the chapter, a comparision is given between the Jacquet-Langlands L-function and the classical L-function attached holomorphic cusp forms.

In the last chapter, following Manin we first describe the construction of non-archimedean measures associated with a holomorphic cusp form on $SL(2, \mathbb{Z})$, which is also assumed to be a simultaneous eigenfunction of all the Hecke operators. Then we use this measure to p-adically interpolate the critical values of the classical L-function attached to the cusp form and we explicate this construction in the case of the Ramanujan Δ-function.

Chapter 2

p-adic extension of $\zeta(s)$ and $L(s,\chi)$

2.1 p-adic analysis and p-adic zeta function

In the late 18th century Kummer discovered some interesting congruence relations satisfied by Bernoulli numbers. Later Kubota and Leopoldt re-interpreted those formulae using the so called p-adic zeta function, describing which is the aim of this section.

2.1.1 Open sets in $\mathbb{Q}_p$ and p-adic Distributions

The aim of this section is to introduce the basic objects needed for the study of p-adic zeta functions, the non-archimedian measures associated to Bernoulli polynomials.

Sets of type $a + p^N\mathbb{Z}_p = \{x \in \mathbb{Q}_p; |x-a|_p \leq p^{-N}\}$, denoted as $a + (p^N)$ where $a \in \mathbb{Q}_p$ and $N \in \mathbb{Z}$, form a basis for the open sets in $\mathbb{Q}_p$. So any open set in $\mathbb{Q}_p$ is a union of these kind of open sets, from now on open set of the type $a + p^N\mathbb{Z}_p$ will be called an interval. These intervals are also closed because of the p-adic topology.

An open set in $\mathbb{Q}_p$ is compact if it can be written as finite union of intervals (the intervals can be made disjoint). The proof is easy and follows from the fact that for a compact set any given open cover has a finite subcover.

A locally constant function plays the same role in p-adic analysis as the role played by step functions in real analysis while defining the concept of a Riemann integral.

Let X and Y be two topological spaces, $f : X \to Y$ is called locally constant if for any $x \in X$, $\exists$ a neighbourhood U of x in X such that f is constant on U.

For example the function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ given by $f(x)$ = first digit in the p-adic expansion of x, then f is clearly locally constant because f has constant value on the neighbourhood U of x, given by $U = x + p\mathbb{Z}_p$

Clearly a locally constant function is continuous.

p-adic distributions

In this section X will always denote a *compact-open* subset of $\mathbb{Q}_p$ such as $\mathbb{Z}_p$ or $\mathbb{Z}_p^\times$.

A p-adic distribution μ on X is a $\mathbb{Q}_p$ linear map from the $\mathbb{Q}_p$-vector space of locally constant functions on X to $\mathbb{Q}_p$, the value of μ at a locally constant function f is denoted as $\int f\mu$.

Equivalently one can define a p-adic distribution μ on X as an additive map from set of *compact-open* subsets of X to $\mathbb{Q}_p$.

Proposition 2.1.1. *Every map μ from the set of intervals contained in X to $\mathbb{Q}_p$ which satisfies*

$$\mu(a + (p^N)) = \sum_{b=0}^{p-1} \mu(a + bp^N + (p^{N+1})),$$

whenever $a + (p^N) \in X$ extends uniquely to a p-adic distribution on X.

Proof. Refer to the Proposition in §2.3, [Ko]. □

A p-adic distribution on X is called a measure if its values on *compact-open* subsets of X are bounded by some real constant, i.e., $|\mu(U)|_p \leq B$ for all *compact-open* $U \subset X$ ($B \in \mathbb{R}$).

Some examples of p-adic distributions are $Haar$ distribution and $Mazur$ distribution given by $\mu_{Haar}(a + (p^N) = \frac{1}{p^N}$ and $\mu_{Mazur}(a + (p^N) = \frac{a}{p^N} - \frac{1}{2}$, but one should notice that the values of these two distributions grow p-adically to ∞ as $N \to \infty$, hence they are not qualified as measures, this can also be realised from the following example.

Consider the function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ given by $f(x) = x$, let's try to integrate it w.r.t the $Haar$ distribution, we will use the fact that for any N,

$$\mathbb{Z}_p = \bigcup_{a=0}^{p^N-1} (a + (p^N))$$

and let's find out the Riemann sum

$$S_{N,x_{a,N}} = \sum_{a=0}^{p^N-1} f(x_{a,N})\mu(a+(p^N))$$

if we choose $x_{a,N} = a$ for each N then we get $S_{N,x_{a,N}} = \frac{p^N-1}{2}$ which goes to $\frac{-1}{2}$ p-adically as $N \to \infty$. On the other hand if we choose $x_{a,N} = a + p^N b$, where b is a fixed p-adic integer, then we will get $S_{N,x_{a,N}} = \frac{p^N-1}{2} + b$ which goes to $b - \frac{1}{2}$, p-adically as $N \to \infty$. Since the Riemann sum is not unique, hence f isn't integrable.

Now we will give an example of a p-adic distribution which can be regularized to make it bounded and hence will give us a p-adic measure on $\mathbb{Z}_p$. This distribution is the so called *Bernoulli* distribution. To define this p-adic distribution, we will first define the Bernoulli polynomials. The k-th Bernoulli polynomial is defined as

$$F(x,t) = F(t)e^{xt} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!},$$

where $F(t) = t/(e^t - 1)$, The first few Bernoulli polynomials are given as:

$$B_0(x) = 1,\, B_1(x) = x - \frac{1}{2},\, B_2(x) = x^2 - x + \frac{1}{6}.$$

Using the k-th Bernoulli polynomial defined above, let us define a map μ_k to be

$$\mu_k(a+(p^N)) = p^{N(k-1)}B_k\left(\frac{a}{p^N}\right) \; for\ any\ a \in \{0,1,2,\ldots,p^N-1\}.$$

Note that $\mu_0 = \mu_{Haar}$ and $\mu_1 = \mu_{Mazur}$.

Proposition 2.1.2. *μ_k extends to a distribution on $\mathbb{Z}_p$.*

Proof. See proposition, §2.4, [Ko]. □

This distribution μ_k is called the k-th Bernoulli distribution. Note that μ_k is not a measure for any nonnegative integer k. However, there is a method to regularize the Bernoulli distribution to obtain a measure.

Definition 2.1.3. *Let α be any rational integer, not equal to one and that p does not divide α. We define the k-th regularized Bernoulli distribution on $\mathbb{Z}_p$ as*

$$\mu_{k,\alpha}(U) = \mu_k(U) - \alpha^{-k}\mu_k(\alpha U).$$

We must show that $\mu_{k,\alpha}$ is well-defined. It is clear that the sum of two distributions is a distribution and that for any $\alpha \in \mathbb{Z}_p$ and a distribution μ on $\mathbb{Z}_p$, $\alpha\mu$ is a distribution as well. Thererfore, we only need to show that $\mu(\alpha U)$ is a distribution. It can be shown as follows: Write U as a finite disjoint union of intervals, say $\{U_i\}_{i=1}^n$. Then $x \in \alpha U \iff x/\alpha \in U \iff x/\alpha \in U_i$ for a unique $i \iff x \in \alpha U_i$ for a unique i. This proves that αU is a disjoint union of $\{\alpha U_i\}_{i=1}^n$, and that $\mu(\alpha U) = \sum \mu(\alpha U_i)$.

For $\alpha \in \mathbb{Z}_p$, let $\{\alpha\}_N$ be the unique rational integer such that $0 \le \{\alpha\}_N \le p^N - 1$ and $\{\alpha\}_N \equiv \alpha \pmod{p^N}$. If $U = a + (p^N)$ for some $a \in \{0, 1, 2, \ldots, p^N - 1\}$, then

$$\begin{aligned}
\alpha U &= \{x \in \mathbb{Z}_p : x/\alpha \in U\} = \{x \in \mathbb{Z}_p : |x/\alpha - a|_p \le p^{-N}\} \\
&= \{x \in \mathbb{Z}_p : |1/\alpha|_p |x - \alpha a|_p \le p^{-N}\} \\
&= \{\alpha a\}_N + (p^N)
\end{aligned}$$

Now it will be easier for us to find $\mu_{k,\alpha}$ for each k. First let $k = 0$, then $\mu_k = \mu_{Haar}$, so $\mu_{0,\alpha}(a+(p^N)) = \mu_{Haar}(a+(p^N)) - \alpha^{-0}\mu_{Haar}(\{\alpha a\}_N + (p^N)) = \frac{1}{p^N} - \frac{1}{p^N} = 0$. Hence $\mu_{0,\alpha}(U) = 0$ for any open-compact subset U of $\mathbb{Z}_p$.

Proposition 2.1.4. *$\mu_{k,\alpha}$ is a measure for any rational integer α, not divisible by p and for all $k \ge 1$.*

Proof. We will be proving this only for $k = 1$. For more details see Theorem 5, §2.5, [Ko]. Note that:

$$\begin{aligned}
\mu_{1,\alpha}(a + (p^N)) &= \mu_1(a + (p^N)) - \alpha^{-1}\mu_1(\{\alpha a\}_N + (p^N)) \\
&= B_1\Big(\frac{a}{p^N}\Big) - \alpha^{-1} B_1\Big(\frac{\{\alpha a\}_N}{p^N}\Big) \\
&= \frac{a}{p^N} - \frac{1}{2} - \frac{1}{\alpha}\Big(\frac{\{\alpha a\}_N}{p^N} - \frac{1}{2}\Big) \\
&= \frac{1}{\alpha}\Big[\frac{\alpha a}{p^N}\Big] + \frac{1}{2}\Big(\frac{1}{\alpha} - 1\Big).
\end{aligned}$$

We claim that this is a measure. It is enough to show that $|\mu_{1,\alpha}(a + (p^N))|_p$ is bounded for any $a \in \{0, 1, \ldots, p^N - 1\}$, because any compact-open subset of $\mathbb{Z}_p$ is a finite disjoint union of intervals. Note that α is not divisible by p, i.e., α is in $\mathbb{Z}_p^\times$, therefore $1/\alpha \in \mathbb{Z}_p^\times$ as well. So if $p \ne 2$ then $1/2(1/\alpha - 1) \in \mathbb{Z}_p$. If $p = 2$, we may write $1/\alpha = 1 + \sum_{i=1}^\infty a_i 2^i$ with $a_i \in \{0, 1\}$ since $1/\alpha \in \mathbb{Z}_p^\times$, hence $1/2(1/\alpha - 1)$ is again in $\mathbb{Z}_p$.

Since $\frac{1}{\alpha}[\frac{\alpha a}{p^N}]$ is also in $\mathbb{Z}_p$, it follows that $\mu_{1,\alpha}(a+(p^N)) \in \mathbb{Z}_p$ for any interval $a+(p^N)$. It follows that $\mu_{1,\alpha}$ is a measure. More precisely, $|\mu_{1,\alpha}(U)|_p \leq 1$. □

Definition 2.1.5. *Let μ be a p-adic measure on a compact-open set X in $\mathbb{Q}_p$, and $f : X \to \mathbb{Q}_p$ a continuous function. Then the N-th Riemann sum is defined as*

$$S_{N,x_{a_i,N}}(f) = \sum_{i=1}^{m} f(x_{a_i,N})\mu(a_i + (p^N)),$$

where X is expressed as a disjoint union of $\{a_i + (p^N)\}_{i=1}^m$ and $x_{a_i,N}$ are arbitrary points in $a_i + (p^N)$ for each i.

Theorem 2.1.6. *$lim_{N\to\infty} S_{N,x_{a_i,N}}$ in $\mathbb{Q}_p$, and it is independent of the choice of $\{x_{a_i,N}\}$.*

We define $\int f\mu$ to be this limit in the above theorem. Note that it is well-defined by the theorem.

Proposition 2.1.7. *Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be the function $f(x) = x^{k-1}$ (k is a fixed positive integer). Let X be a compact-open subset of $\mathbb{Z}_p$. Then*

$$\int\limits_X 1\mu_{k,\alpha} = k \int\limits_X f\mu_{1,\alpha}$$

Proof. See Proposition, §2.6, [Ko]. □

Thus, we claim that the expression $\int\limits_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\alpha}$ can be interpolated p-adically. We know that if $|f(x) - x^{k-1}|_p \leq \varepsilon$ for $x \in \mathbb{Z}_p^\times$, then

$$\Big| \int\limits_{\mathbb{Z}_p^\times} f\mu_{1,\alpha} - \int\limits_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\alpha} \Big|_p \leq \varepsilon$$

(recall that $|\mu_{1,\alpha}(U)|_p \leq 1$ for all compact-open U.) Choose for f the function $x^{k'-1}$ where $k' \equiv k \pmod{p-1}$ and $k' \equiv k \pmod{p^N}$(writing this as one congruence $k' \equiv k$ $(\bmod\ (p-1)p^N)$. Then we have

$$|x^{k'-1} - x^{k-1}|_p \leq \frac{1}{p^{N+1}}$$

for $x \in \mathbb{Z}_p^\times$. Thus,

$$\Big| \int_{\mathbb{Z}_p^\times} f\mu_{1,\alpha} - \int_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\alpha}\Big|_p \leq \frac{1}{p^{N+1}}.$$

We conclude that, for any fixed $s_0 \in \{0,1,2,\ldots,p-2\}$, by letting k run through $S_{s_0} := \{s \in \mathbb{Z}^+ | s \equiv s_0 \, mod\,(p-1)\}$, we can extend the function of k given by $\int_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\alpha}$ to a continuous function of p-adic integer s given by :

$$\int_{\mathbb{Z}_p^\times} x^{s_0+s(p-1)-1}\mu_{1,\alpha}$$

But we are originally trying to p-adically interpolate the numbers $-\frac{1}{k}\int 1\mu_{B,k}$ which are the values of $\zeta(1-k)$ (for $k>0$). We just saw that we can interpolate,

$$\int_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\alpha} = \frac{1}{k}\int_{\mathbb{Z}_p^\times} 1\mu_{k,\alpha}.$$

Let's relate the two numbers:

$$\begin{aligned}
\frac{1}{k}\int_{\mathbb{Z}_p^\times} 1\mu_{k,\alpha} &= \frac{1}{k}\mu_{k,\alpha}(\mathbb{Z}_p^\times) \\
&= \frac{1}{k}(\alpha^{-k}-1)(1-p^{k-1})B_k \\
&= (\alpha^{-k}-1)(1-p^{k-1})(-\frac{1}{k}\int_{\mathbb{Z}_p} 1\mu_{B,k})
\end{aligned}$$

So, we will interpolate the numbers $(1-p^{k-1})(-\frac{B_k}{k})$:

$$(1-p^{k-1})(-\frac{B_k}{k}) = \frac{1}{\alpha^{-k}-1}\int_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\alpha}. \tag{2.1.8}$$

Definition 2.1.9. *Let α be a rational integer that is not equal to one and not divisible by p. For any positive integer k, we define*

$$\zeta_p(1-k) := (1-p^{k-1})(-\frac{B_k}{k})$$

so that, by the preceding paragraph

$$\zeta_p(1-k) = \frac{1}{\alpha^{-k}-1} \int_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\alpha}.$$

Note that this expression on the right does not depend on α, i.e, if $\beta \in \mathbb{Z}$, $p \nmid \beta$, $\beta \neq 1$, then $(\beta^{-k}-1)^{-1} \int_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\beta} = (\alpha^{-k}-1)^{-1} \int_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\alpha}$, since both equal to $(1-p^{k-1})(-B_k/k)$.

Theorem 2.1.10 (Kummer)**.** *The following holds*

1. *If $(p-1) \nmid k$ then $\frac{B_k}{k}$ is a p-adic integer.*
2. *If $(p-1) \nmid k$ and $k \equiv k'$ (mod $(p-1)p^N$) then*

$$(1-p^{k-1})\frac{B_k}{k} \equiv (1-p^{k'-1})\frac{B_{k'}}{k'} \; mod \; p^{N+1}$$

Proof. If $k=1$, then $|B_k/k|_p = 1$ for any $p>2$. For the case $k>1$, choose α such that $2 \leq \alpha \leq p-1$ and $p-1$ is the smallest positive integer satisfying $\alpha^{p-1}-1 \equiv 0$ (mod p). Note that such α exists since α is a p-adic unit and it can be identified with a $(p-1)$th root of unity. By the choice of α and the hypothesis, $\alpha^{-k}-1 \not\equiv 0$ (mod p), and so $\alpha^{-k}-1$ is a p-adic unit. Therefore

$$\left|\frac{B_k}{k}\right|_p = |1-p^{k-1}|_p^{-1}|\alpha^{-k}-1|_p^{-1}\left|\int_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\alpha}\right|_p \leq |\mu_{1,\alpha}(\mathbb{Z}_p^\times)| \leq 1.$$

To prove the second statement, let α be as chosen earlier. Since k is congruent to k' mod $(p-1)p^N$, we have $\alpha^k \equiv \alpha^{k'}$ (mod p^{N+1}). So we have $\int x^{k-1}\mu_{1,\alpha} \equiv \int x^{k'-1}\mu_{1,\alpha}$ (mod p^{N+1}). Hence the proof is completed by eq. 2.1.8. □

Definition 2.1.11. *For any $\alpha \in \mathbb{Z}$ with $\alpha \neq 1$ and $p \nmid \alpha$ and for a fixed integer $s_0 \in \{0,1,2,\ldots,p-2\}$, $\zeta_{p,s_0}(s)$ is defined as*

$$\zeta_{p,s_0}(s) = \frac{1}{\alpha^{-(s_0+(p-1)s)}-1} \int_{\mathbb{Z}_p^\times} x^{s_0+(p-1)s-1}\mu_{1,\alpha}$$

for any p-adic integer s, except at $s=0$ in case of $s_0=0$.

Note that $\zeta_{p,s_0}(s)$ is continuous except where s_0 and s are both zero. This can be shown in the same manner as the continuity of $\zeta_p(1-k)$. One can also show that $\zeta_{p,s_0}(k)$ does not depend on the choice of α for any k in A_{s_0}. This is because $\zeta_p(1-k) = \zeta_{p,s_0}(k_0)$ where $k = s_0 + (p-1)k_0$ for some $s_0 \in \{0, 1, 2, \dots, p-2\}$ and $k_0 \in \mathbb{Z}$ with $k_0 > 0$. It follows from the continuity of $\zeta_{p,s_0}(s)$ and the density of A_{s_0} that $\zeta_{p,s_0}(s)$ is independent of the choice of α.

We also note that $\zeta_p(t)$ has a pole at $t = 1$, by taking $k = 0$ (and so $s_0 = k_0 = 0$ as well) in $\zeta_p(1-k) = \zeta_{p,s_0}(k_0)$.

2.2 p-adic extension for Dirichlet L-function

2.2.1 Dirichlet Characters and Generalized Bernoulli Numbers

We have to first define the notion of a Dirichlet character. Let n be a positive integer, then a map $\chi : \mathbb{Z} \to \mathbb{C}$ is called a Dirichlet character of modulus n if

1. $a \equiv a' (mod n) \Rightarrow \chi(a) = \chi(a')$,
2. $\chi(ab) = \chi(a)\chi(b)$ for $\forall a, b \in \mathbb{Z}$,
3. $\chi(a) \neq 0 \Longleftrightarrow (a, n) = 1$.

Let χ' be another Dirichlet character of modulus m and let n be a multiple of m. Now define $\chi : \mathbb{Z} \to \mathbb{C}$ by $\chi(a) = \chi'(a)$ if $(a, n) = 1$ and $\chi(a) = 0$ if $(a, n) > 1$. Now χ is easily seen to be a Dirichlet character modulus n, which is said to be induced from χ'. A Dirichlet character χ modulus n is said to be primitive, if it is not induced from any characher χ' modulus m with $m < n$. In this case n is called the conductor of χ and is denoted by f_χ.

For a primitive Dirichlet character χ mod f, define the kth-generalised Bernoulli number (denoted by $B_{k,\chi}$) via the following function

$$F_\chi(t) = \sum_{a=1}^{f} \chi(a) \frac{te^{at}}{e^{ft} - 1} = \sum_{k=0}^{\infty} B_{k,\chi} \frac{t^k}{k!} .$$

Observe that

$$\begin{aligned}\sum_{k=0}^{\infty} B_{k,\chi}\frac{t^k}{k!} &= \sum_{a=1}^{f}\chi(a)\frac{te^{at}}{e^{ft}-1}\\ &= \sum_{a=1}^{f}\chi(f-a)\frac{te^{(f-a)t}}{e^{ft}-1}\\ &= \chi(-1)\sum_{a=1}^{f}\chi(a)\frac{-te^{-at}}{e^{-ft}-1}\\ &= \chi(-1)\sum_{k=0}^{\infty}(-1)^k B_{k,\chi}\frac{t^k}{k!},\end{aligned}$$

where the second last step is obtained by using the fact that χ is periodic mod f and $\chi(-a) = \chi(-1)\chi(a)$. Hence $B_{k,\chi} = 0$ for $k \not\equiv \delta \pmod 2$ if $\chi \neq 1$ (where $\delta = 0$ if $\chi(-1) = 1$ and $\delta = 1$ if $\chi(-1) = -1$).

Let χ be a Dirichlet character, then the *Dirichlet L-function* associated to χ is defined by the formula

$$L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}.$$

If we take $\chi = 1$, then $L(s,1) = \zeta(s)$. Since $|\chi(n)| \leq 1$ for all $n \in \mathbb{Z}$, by comparing with the zeta function we obtain the following result

Proposition 2.2.1. *$L(s,\chi)$ converges absolutely for $Re(s) > 1$ and has the following Euler product expansion*

$$L(s,\chi) = \prod_p \frac{1}{1-\chi(p)p^{-s}} \ .$$

Theorem 2.2.2. *For a Dirichlet character χ and any positive integer k,*

$$L(1-k,\chi) = -\frac{B_{k,\chi}}{k} \ . \tag{2.2.3}$$

Proof. Refer to Theorem 1, §2, [Iw]. □

2.2.2 Interpolating $L(1-k,\chi)$ via p-adic power series

Let p be a prime number and let Ω_p be an algebraic closure of $\mathbb{Q}_p$. In the following section, we shall fix both p and Ω_p and consider p-adic functions which are defined on sufficiently large domains in Ω_p and take values in the same field Ω_p. Let $L(s;\chi)$ be the classical L-function of Dirichlet. The main problem of this section is to find a suitable p-adic function which may be regarded as a p-adic analogue of the classical function $L(s;\chi)$. To solve this problem, Kubota-Leopoldt looked for a p-adic meromorphic function which takes the same values as $L(s;\chi)$ at $s=0,-1,-2,\ldots$ observing that by Theorem 2.2.2, these values of $L(s;\chi)$ are algebraic numbers and hence, may be considered as elements of the algebraically closed field Ω_p.

In [KL], Kubota-Leopoldt obtained such a function $f(s)$, although the condition $f(n)=L(n,\chi)$ for $n=0,-1,-2,\ldots,$ had to be modified slightly, and they called it the p-adic L-function for the Dirichlet character χ. In this chapter following Iwasawa, we shall first study p-adic (holomorphic) functions which are defined by convergent power series and which take pre-assigned values at $s=0,1,2,\ldots$ Using the results thus obtained, we shall then discuss the existence and the uniqueness of the function $f(s)$ as mentioned above.

Throughout this section, let K be a finite extension of $\mathbb{Q}_p$, and let

$$K[[x]] := \{A(x)=\sum_{i=0}^{\infty} a_i x^i : a_i \in K\}$$

the set of all power series. We say that $A(x)$ converges at s to mean that $|a_i s^i|_p \to 0$ as $i\to\infty$. Now let us define

$$\parallel A \parallel = \parallel \sum_{i=0}^{\infty} a_i x^i \parallel = \ sup_i |a_i|_p$$

and $P_K=\{A(x)\in K[[x]] : \parallel A \parallel < \infty\}$. Then $\parallel . \parallel$ is a norm on P_K and one can show that P_K is complete with respect to this norm.

Define $\binom{x}{n}$, for any non-negative integer n, to be the polynomial of degree n given by,

$$\binom{x}{n} = \frac{x(x-1)(x-2)\ldots(x-n+1)}{n!}$$

This is clearly continous on $\mathbb{Z}_p$.

Proposition 2.2.4. *Let n be a non-negative integer, and write it to the base p, i.e., $n = \alpha_0 + \alpha_1 p + \cdots + \alpha_t p^t$, with $0 \le \alpha_i \le p-1$ for all i, and let $S_n := \sum_{i=0}^{t} \alpha_i$. Then,*

$$\left\| \binom{x}{n} \right\| \le p^{(n-S_n)/(p-1)} .$$

Proof. Refer to Lemma 3, §3, [Iw]. □

Theorem 2.2.5. *Let $0 < r < p^{-1/p-1}$ and $A(x) = \sum\limits_{i=0}^{\infty} a_i \binom{x}{i}$ with $a_i \in K$ and $|a_i| \le Mr^i$, for some M and for all i. Then $A(x) \in P_K$ and the radius of convergence of $A(x)$ is at least $(rp^{-1/p-1})^{-1}$.*

Proof. Let

$$A_k(x) = \sum_{n=0}^{k} a_n \binom{x}{n}.$$

Then $A_k(x)$ is a polynomial of degree k, so it can be also be written as $A_k(x) = \sum_{n=0}^{\infty} a_{k,n} x^n$, with $a_{k,n} = 0$ for all $n > k$. By proposition 2.2.4 we have

$$\left\| a_n \binom{x}{n} \right\| < |a_n|_p p^{(n-S_n)/(p-1)} < Mr^n p^{(n-S_n)/(p-1)} < M(rp^{1/p-1})^n$$

and so $A_k(x)$ is in P_K .

Also, $\{A_k(x)\}_{k=1}^{\infty}$ is Cauchy because

$$\| A_l - A_k \| \le \ max_{k<n\le l} \left(\left\| a_n \binom{x}{n} \right\| \right) \ < \ M(rp^{1/p-1})^{k+1},$$

and this converges to 0 as $k, l \to \infty$. By the choice of $A_k(x)$, it is clear that $\{A_k(x)\}$ converges to $A(x)$, and the limit is in P_K by the completeness of P_K .

Now, we wish to prove the last assertion. Write $A(x)$ as $A(x) := \sum_{n=0}^{\infty} a_{0,n} x^n$. Then, $\because \{A_k(x)\} = \sum_{n=0}^{k} a_n \binom{x}{n} \to \sum_{n=0}^{\infty} a_{0,n} x^n = A(x)$, $a_{k,n} \to a_{0,n}$ as $k \to \infty$ for each n. Let $n < k$, then

$$|a_{k,n}|_p \ = \ |a_{k,n} - a_{n-1,n}|_p \ \le \| A_k - A_{n-1} \| < \ M(rp^{1/p-1})^n$$

and by $k \to \infty$ we obtain $|a_{0,n}|_p < M(rp^{1/p-1})^n$, hence if $|x|_p < (rp^{1/p-1})^{-1}$ then $|a_{0,n} x^n|_p \to 0$ as $n \to \infty$.

□

Theorem 2.2.6. *Let $\{b_i\}_{i=1}^{\infty} \subset K$ and define $c_n := \sum_{i=0}^{n} \binom{n}{i}(-1)^{n-i} b_i$. If there exists $M > 0$ such that $|c_n|_p < Mr^n$ for some r with $0 < r < p^{-1/p-1}$, then $A(x) := \sum_{n=0}^{\infty} c_n \binom{x}{n}$ is in P_K and $A(k) = b_k$ for all $k = 1, 2, 3, \ldots$.*

Let q be an integer such that:

$$q := \begin{cases} p & \text{if } p \neq 2 \\ 4 & \text{if } p = 2. \end{cases}$$

We will fix this notation for the rest of this section. Now, let a be in $\mathbb{Z}_p^{\times}$ and write $a = \sum_{i=0}^{\infty} \alpha_i p^i$ with $0 \leq \alpha_i \leq p-1$ and $\alpha_0 \neq 0$. First, suppose $p \neq 2$. Since α_0 is not divisible by p, we have $\alpha_0^{p-1} \equiv 1 \pmod{p}$. Therefore we can identify α_0 with a primitive $(p-1)$ -th root of unity. Let $\omega(a)$ be such a $(p-1)$-th root of unity. (This is the *Teichmüller* representative of α_0.) If $p = 2$, take an element such that $a = 1 + \alpha_1 2 + \sum_{i=2}^{\infty} \alpha_i 2^i$ with $\alpha_i \in \{0, 1\}$, and similarly identify $1 + \alpha_1 2$ with $\{\pm 1\}$. Also, let $\langle a \rangle := a/\omega(a)$. In this way, any p-adic unit a can be written as $\omega(a)\langle a \rangle$. We also note that $\langle a \rangle$ is an element in $1 + q\mathbb{Z}_p$. Let us extend ω to $\mathbb{Z}_p$, by setting $\omega(a) = 0$ for all $a \notin \mathbb{Z}_p^{\times}$. Then, clearly this is a Dirichlet character of conductor q.

For any Dirichlet character χ, of conductor f, we define $\chi_n := \chi \cdot \omega^{-n}$ where ω is a Dirichlet character defined as above and n is any positive integer. Let f_n be the conductor χ_n. Then, f_n must be a factor of fq. But f also must be a factor of $f_n q$ since $\chi = \chi_n \cdot \omega^n$, and so f_n differs from f by only a power of p. Therefore, if a is a rational integer such that $(a, p) = 1$ then $(a, f) = (a, f_n)$, and it follows that $\chi_n(a) = \chi(a)\omega(a)^{-n}$ for any such a.

Let $K = \mathbb{Q}_p(\chi)$ and define

$$b_k := (1 - \chi_k(p)p^{k-1})B_{k,\chi_k}, \ \ c_k := \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} b_i$$

Also let $A_\chi(x) := \sum_{n=0}^{\infty} c_n \binom{x}{n}$. We define *p-adic Dirichlet L-function* as

$$L_p(s, \chi) = \frac{1}{s-1} A_\chi(1-s).$$

We claim that this converges in $\{s \in \bar{\mathbb{Q}}_p : 0 < |s-1|_p < (p^{1/(p-1)})^{-1}|q|_p\}$. However, it requires a bit more work to prove that this function is even well-defined.

Proposition 2.2.7. *In $\mathbb{Q}_p(\chi)$, $B_{k,\chi} = \lim_{n\to\infty} \frac{1}{p^n f} S_{k,\chi}(p^n f)$, where $S_{k,\chi}(n) = \sum_{a=1}^{n} \chi(a) a^k$*

Proof. Refer to Lemma 1, §2, [Iw]. □

Proposition 2.2.8. *$c_k \equiv 0$ (mod $q^{k-2}f^{-1}$) for all $k = 1, 2, 3, \ldots$.*

Proof. Refer to Lemma 3, §4, [Iw]. □

This proposition say $|c_k|_p \leq |q^{-2}f^{-1}|_p|q|_p^k$. Thus by taking $r = |q|_p(< p^{-1/(p-1)})$ and $M = |q^{-2}f^{-1}|_p$ in Theorem 2.1, one can show that A_χ is well-defined in $P_{\mathbb{Q}_p(\chi)}$ and that it converges at s for $\{s \in \bar{\mathbb{Q}}_p : 0 < |s-1|_p < (p^{1/(p-1)}|q|_p)^{-1}\}$.

Proposition 2.2.9. *For a Dirichlet character χ and any positive integer k,*

$$\begin{aligned} L_p(1-k,\chi) &= (1-\chi_k(p)p^{k-1})\Big(-\frac{B_{k,\chi_k}}{k}\Big) \\ &= (1-\chi_k(p)p^{k-1})L(1-k,\chi_k) \end{aligned}$$

Proof. This follows directly from the definition and Theorem 2.2.2, because,

$$\begin{aligned} L_p(1-k,\chi) = -\frac{1}{k}A_\chi(k) &= -\frac{1}{k}(1-\chi_k(p)p^{k-1})B_{k,\chi_k} \\ &= (1-\chi_k(p)p^{k-1})L(1-k,\chi_k). \end{aligned}$$

□

2.3 Power series and p-adic measures

The following discussion is taken from Katz [Ka].

Theorem 2.3.1 (Mahler)**.** *Let R be p-adically complete and separated. Then any $f \in Contin(\mathbb{Z}_p, R)$ can be uniquely written*

$$f(x) = \sum_{n\geq 0} a_n \binom{x}{n}, \;\; a_n \in R, \;\; a_n \to 0. \tag{2.3.2}$$

The a_n may be recovered as the higher difference of f:

$$a_n = (\Delta^n f)(0) = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}f(i).$$

Conversely, any series

$$\sum_{n\geq 0} a_n \binom{x}{n}, \;\; a_n \in R, \;\; a_n \to 0$$

converges to an element of $Contin(\mathbb{Z}_p, R)$*.*

Proposition 2.3.3. *An* R*-valued measure* μ *on* $\mathbb{Z}_p$ *is uniquely determined by the sequence* $b_n(\mu) = \int_{\mathbb{Z}_p} \binom{x}{n} d\mu$ *of elements of* R*, and any sequence* $\{b_n\}$ *defines a* R*-valued measure* μ *by the formula*

$$\int_{\mathbb{Z}_p} f(x)d\mu = \sum_{n\geq 0} a_n b_n = \sum_{n\geq 0} b_n(\Delta^n f)(0).$$

Proposition 2.3.4. *Suppose that* p *is not a zero divisor in* R*. Then an* R*-valued measure* μ *on* $\mathbb{Z}_p$ *is uniquely determined by the sequence* $m_n(\mu) \in R$ *of its moments*

$$m_n(\mu) = \int_{\mathbb{Z}_p} x^n d\mu.$$

A sequence $\{m_n\}$ *of elements of* R *arises as the moments of an* R*-valued measure* μ *if and only if the quantities*

$$b_n := \sum_{i=0}^{n} c_{i,n} m_i,$$

(a priori in $R[1/p]$*), where* $c_{i,n}$ *is defined as*

$$\binom{x}{n} = \frac{x(x-1)\dots(x-n+!)}{n!} = \sum_{i=0}^{n} c_{i,n} x^i \tag{2.3.5}$$

all lie in R*, in which case we have*

$$\int_{\mathbb{Z}_p} \binom{x}{n} d\mu = b_n$$

Let us denote by A the subring of $\mathbb{Q}_p(T)$ consisting of all ratios $P(T)/Q(T)$ with $P, Q \in \mathbb{Z}_p[T]$ and $Q(1) \in \mathbb{Z}_p^\times$. One can show that A is the localization of the ring $\mathbb{Z}_p[T]$ at the maximal ideal $(p, T-1)$.

Theorem 2.3.6. *Given any element* $F(T) \in A$*, there is a* $\mathbb{Z}_p$*-valued measure* μ_F *on* $\mathbb{Z}_p$ *whose moments are given by the formula*

$$\int_{\mathbb{Z}_p} x^k d\mu_F = (T\frac{d}{dT})^k (F)|_{T=1} \tag{2.3.7}$$

Proof. Following Proposition 2.3.4, we must show that

$$\sum_{i=0}^{n} c_{i,n}(T\frac{d}{dT})^i(F)|_{T=1} \in \mathbb{Z}_p \; for \; n = 0, 1, 2, \ldots$$

or equivalently that (by using eq. 2.3.5)

$$\binom{T\frac{d}{dT}}{n}(F)|_{T=1} \in \mathbb{Z}_p \; for \; n = 0, 1, 2, \ldots$$

In fact we will prove a much stronger statement $\because$ any element of the ring A is of the form $P(T)/Q(T)$ with $P, Q \in \mathbb{Z}_p[T]$ and $Q(1) \in \mathbb{Z}_p^\times$, it is enough to prove that the operators $\binom{T\frac{d}{dT}}{n}$ act stably on the ring A, i.e.,

$$\binom{T\frac{d}{dT}}{n}(F) \in A$$

First Notice that,

$$\binom{T\frac{d}{dT}}{n}(T^a) = \sum_{i=0}^{n} c_{i,n}(T\frac{d}{dT})^i(T^a) = (\sum_{i=0}^{n} c_{i,n}a^i)T^a = \binom{a}{n}T^a$$

Hence the operators act stably on $\mathbb{Z}_p[T]$.

Now one should notice that $\binom{T\frac{d}{dT}}{n} = T^n\frac{(\frac{d}{dT})^n}{n!}$ (one could easily prove it using induction, by observing that for $n = 1$ the statement holds trivially and if we assume it to be true for $n-1$, we could prove it for n by using the identity $\binom{a}{n+1} = \binom{a}{n}\frac{a-n}{n+1}$)

Then we see that Leibniz' formula is satisfied.

$$\begin{aligned}
\binom{T\frac{d}{dT}}{n}(F \cdot G) &= \frac{T^n}{n!}(\frac{d}{dT})^n(F \cdot G) \\
&= \frac{T^n}{n!}\sum_{i=0}^{n} F^{(i)}G^{(n-i)} = T^n \sum_{i+j=n} \frac{(\frac{d}{dT})^i)(F)}{i!} \cdot \frac{(\frac{d}{dT})^j)(F)}{j!} \\
&= \sum_{i+j=n} \binom{T\frac{d}{dT}}{i}(F) \cdot \binom{T\frac{d}{dT}}{j}(G).
\end{aligned}$$

Let $Q \in \mathbb{Z}_p[T]$ such that $Q(1) \in \mathbb{Z}_p^\times$. Applying Leibniz's formula to the product $Q \cdot \frac{1}{Q} = 1$, we find inductively that $\binom{T\frac{d}{dT}}{n}(\frac{1}{Q}) \in A$. Applying the same formula to the product $P \cdot \frac{1}{Q}$ then shows that A is stable by the action of $\binom{T\frac{d}{dT}}{n}$. $\square$

2.3.1 Examples of μ_F and relations to Iwasawa's approach

Now we will use Theorem 2.3.6 to find the measure μ_F for some specific functions.

If $i > n$, then we our convention is $\binom{n}{i} = 0$

1. Let $F(T) = T^n$. In that case

$$b_i(\mu) = \sum_{k=0}^{i} c_{i,k} m_k = \sum_{k=0}^{i} c_{i,k} (T\frac{d}{dT})^k (F)|_{T=1} = \binom{T\frac{d}{dT}}{i} T^n|_{T=1} = \binom{n}{i} T^n|_{T=1} = \binom{n}{i}$$

Then by Mahler's Theorem,

$$\int f d\mu_F = \sum_{i\geq 0} a_i b_i = \sum_{i\geq 0} a_i \binom{n}{i} = f(n)$$

2. Let $F(T) = (T-1)^n$. In that case

$$b_k(\mu) = \binom{T\frac{d}{dT}}{k} (T-1)^n|_{T=1} = \binom{T\frac{d}{dT}}{k} (\sum_{i=0}^{n} \binom{n}{i} T^i (-1)^{n-i})|_{T=1} = \sum_{i=0}^{n} \binom{n}{i} \binom{i}{k} (-1)^{n-i}$$

Then by Mahler's Theorem,

$$\begin{aligned} \int f d\mu_F &= \sum_{k\geq 0} a_k b_k \\ &= \sum_{k\geq 0} a_k \sum_{i=0}^{n} \binom{n}{i} \binom{i}{k} (-1)^{n-i} \\ &= \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} (\sum_{k\geq 0} a_k \binom{i}{k}) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} f(i) = (\Delta^n f)(0) = a_n. \end{aligned}$$

So we obtain the following result,

Proposition 2.3.8. *One can identify $\mathbb{Z}_p$ valued measures on $\mathbb{Z}_p$ with the elements of $\mathbb{Z}_p[[T-1]]$, the measure μ corresponding to the series $\sum b_n (T-1)^n$ is given by,*

$$\int_{\mathbb{Z}_p} f d\mu_F = \sum_{n\geq 0} a_n b_n \qquad (2.3.9)$$

where $f(x) = \sum_{n\geq 0} a_n \binom{x}{n}$.

The measures μ_F we considered above correspond exactly to the rational functions in $\mathbb{Z}_p[[T-1]]$. The multiplication of power series corresponds to convolution of measures on tha additive group $\mathbb{Z}_p$.

$$\int f(x)d(\mu * \nu) := \int \int f(x+t)d\mu(x)d\nu(t).$$

Chapter 3

L-functions on GL(2)

3.1 Main results of Hecke theory of Modular forms

Let $\Gamma = SL(2, \mathbb{Z})$ be the full modular group, defined by

$$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\, ad - bc = 1 \right\}.$$

One can show that the modular group is generated by the following two matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathit{and}\ S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The upper half plane is

$$H = \{z \in \mathbb{C} : \Im(z) > 0\},$$

where $\Im(z)$ is the imaginary part of z.

Then there is an action of Γ on H via,

$$\Gamma \times H \to H$$

$$(\gamma, z) \mapsto \gamma(z) = \frac{az + b}{cz + d},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Also define $j(\gamma, z) = cz + d$ in this case.

Definition 3.1.1. *For any integer k, let*

$$(f|_k\gamma)(z) = (det\gamma)^{\frac{k}{2}} j(\gamma, z)^{-k} f(\gamma(z)),$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Definition 3.1.2. *Let $k \in \mathbb{Z}$. A function $f : H \to \mathbb{C}$ is called modular form of weight k if*

1. *f is holomorphic on H;*

2. *$f|_k\gamma = f \quad \forall \gamma \in \Gamma$;*

3. *f is holomorphic at ∞.*

We will quickly explain the third condition about holomorphy of f at ∞. Since Γ contains the matrix $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ that acts on H via the mapping $z \mapsto z + 1$, any function f which has property (2) must satisfy the equation $f(z+1) = f(z)$ for any $z \in H$. Now set $D = \{q \in \mathbb{C} : |q| < 1\}$ and $D' = D\backslash\{0\}$. Then we have a holomorphic map

$$H \to D'$$

$$z \mapsto e^{2\pi i z} = q$$

and the map

$$g : D' \to \mathbb{C}$$

$$q \mapsto f\Big(\frac{log\, q}{2\pi i}\Big)$$

is well defined and $f(z) = g(e^{2\pi i z})$. Now if f is holomorphic on H, then g is holomorphic on D' and so g has a Laurent expansion $g(q) = \sum_{n\in\mathbb{Z}} a_n q^n$ for $q \in D'$.

The relation $|q| = e^{-2\pi\Im(z)}$ tells us that $q \to 0$ as $\Im(z) \to \infty$. We say that f is holomorphic at ∞ if g extends holomorphically to $q = 0$, i.e., if its Laurent series sums over $n \in \mathbb{N}$. This means that f has a *Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n q^n$$

with $a_n \in \mathbb{C}$ for all n.

The set of all modular forms of weight k is denoted with $M_k(\Gamma)$. If $f \in M_k(\Gamma)$ for some k, we set $f(\infty) = a_0$ if $\sum_{n=0}^{\infty} a_n q^n$ is the Fourier expansion of f . We say that f is a cusp form if $a_0 = 0$. The set of cusp forms of weight k will be denoted with $S_k(\Gamma)$.

Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \; mod \; N \right\}.$$

Let χ be a Dirichlet character mod N. We define a character χ of $\Gamma_0(N)$ by

$$\chi(\gamma) = \chi(d), \;\; \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N). \tag{3.1.3}$$

A modular form of type (k, χ) with respect to $\Gamma_0(N)$ is a function $f : H \to \mathbb{C}$ such that

1. f is holomorphic on H;
2. $f|_k\beta = \chi(\beta)f \quad \forall \beta \in \Gamma_0(N)$;
3. $f|_k\gamma$ is holomorphic at ∞ for all $\gamma \in \Gamma$.

If $a_0 = 0$ in the Fourier expansion of $f|_k\gamma$ for all $\gamma \in \Gamma$, we say that f is a cusp form of type (k, χ) with respect to $\Gamma_0(N)$. The set of modular forms and cusp forms of type (k, χ) with respect to $\Gamma_0(N)$ will be denoted by $M_k(N, \chi)$ and $S_k(N, \chi)$, respectively.

Now suppose $f \in S_k(N, \chi)$ has the following fourier expansion at ∞

$$f(z) \;=\; \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

Then we put

$$Ł(s; f) \;=\; \sum_{n=1}^{\infty} a_n n^{-s}.$$

Since $a_n = O(n^{\frac{k}{2}})$ (see Corollary 2.1.6, §2.1, [Mi]), $L(s; f)$ converges absolutely and uniformly on any compact subset of $Re(s) > 1 + \frac{k}{2}$, so that it is holomorphic on $Re(s) > 1 + \frac{k}{2}$. We call $L(s; f)$ the Dirichlet series associated with f. For $N > 0$, we put

$$\Lambda_N(s; f) = (2\pi/\sqrt{N})^{-s}\varGamma(s)L(s; f), \quad \omega_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$$

Proposition 3.1.4. *For any element $f(z)$ of $S_k(N,\chi)$, $\Lambda_N(s;f)$ is holomorphic on the whole s-plane and satisfies the functional equation*

$$\Lambda_N(s;f) = i^k \Lambda_N(k-s; f|_k\omega_N).$$

Proof. See Corollary 4.3.7, §4.3, [Mi]. □

For $f \in S_k(N,\chi)$ as above and a Dirichlet character ψ, we put

$$f_\psi(z) = \sum_{n=0}^{\infty} \psi(n) a_n e^{2\pi i n z},$$

and

$$L(s;f,\psi) = \sum_{n=1}^{\infty} \psi(n) a_n n^{-s}.$$

Let $m = m_\psi$ be the conductor of ψ, and put

$$\Lambda_N(s;f,\psi) = (2\pi/m\sqrt{N})^{-s}\Gamma(s)L(s;f,\psi)$$

By definition,

$$L(s;f_\psi) = L(s;f,\psi)$$

$$\Lambda_{Nm^2}(s;f_\psi) = \Lambda_N(s;f,\psi)$$

Theorem 3.1.5. *Let $f(z)$ be an element of $S_k(N,\chi)$, and ψ a primitive Dirichlet character of conductor m. If (m, N) = 1, then $\Lambda_N(s;f,\psi)$ can be holomorphically continued to the whole s-plane, is bounded on any vertical strip, and satisfies the functional equation:*

$$\Lambda_N(s;f,\psi) = i^k C_\psi \Lambda_N(k-s; f|_k\omega_N, \overline{\psi})$$

for a constant C_ψ.

Proof. See Theorem 4.3.12, §4.3, [Mi]. □

Let us recall some facts about Hecke operators. For each prime p we consider the double coset

$$\Gamma_0(N)\begin{pmatrix} 1 & \\ & p \end{pmatrix}\Gamma_0(N) = \bigcup_j \Gamma_0(N)\gamma_j = \bigcup_{\substack{(a,N)=1\\ ad=p, a>0}} \bigcup_{b=0}^{d-1} \Gamma_0(N)\sigma_a \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

where $\sigma_a = \begin{pmatrix} a' & b' \\ & d' \end{pmatrix}$ in $SL(2, \mathbb{Z})$ is chosen congruent to $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ modulo N. Then for $f \in S_k(N, \chi)$, we define

$$\begin{aligned} T(p)f &= p^{k/2-1} \sum_j f|_k \gamma_j \\ &= p^{k-1} \sum_{\substack{a>0 \\ ad=p}} \sum_{b=0}^{d-1} \chi(a) f(\frac{az+b}{d}) d^{-k}. \end{aligned}$$

Suppose $f(z) \in S_k(N, \chi)$ and its fourier expansion (at ∞) is $f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$, and if $g(z) = (T(p)f)(z) = \sum_{n=1}^{\infty} a'(n) e^{2\pi i n z}$, then we have

$$a'(n) = \sum_{d|(n,p)} \chi(d) d^{k-1} a\Big(\frac{np}{d^2}\Big) = a(np) + \chi(p) p^{k-1} a\Big(\frac{n}{p}\Big). \tag{3.1.6}$$

This means that if $f(z)$ is a simultaneous eigenfunction for all $T(p)$, $(p, N) = 1$ i.e., there is a λ_p such that

$$T(p)f = \lambda_p f, \quad \forall (p, N) = 1.$$

Then,

$$a(np) + \chi(p) p^{k-1} a\Big(\frac{n}{p}\Big) = \lambda_p a(n) \quad \forall (p, N) = 1$$

In particular if $a(1) = 1$, then $a(p) = \lambda_p$ and we have $a(pq) = a(p)a(q)$ for all primes p and q.

Theorem 3.1.7 (Euler Product). *Under the hypothesis of Theorem 3.1.5, $L(s; f, \psi)$ has an Euler product expansion if and only if f is an eigenfunction of T(p) for all prime p; more precisely $T(p)(f) = c_p f$ for all p if and only if*

$$L(s; f, \psi) \;=\; \sum_{n=1}^{\infty} \psi(n) a_n n^{-s} \;=\; \prod_p (1 - c_p \psi(p) p^{-s} + \psi(p)^2 \chi(p) p^{k-1-2s}).$$

Proof. See Theorem 1.9, §1, [Ge]. □

3.2 Automorphic forms as functions on SL(2,$\mathbb{R}$)

Let $G = SL(2,\mathbb{R})$. In G consider the following subgroups.

$$\Big\{A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0\Big\}, \Big\{N = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\Big\}$$

and

$$\Big\{K = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} = r(\theta), 0 < \theta < 2\pi\Big\}$$

The group $B = NA$ acts transitively on H as $\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} i = x + iy$. Thus the upper half plane is identified with G/K, the stability subgroup of G at i being K.

Since $G = BK = NAK$, each $g \in G$ may be expressed as
$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} . \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$.

3.2.1 Maps from $S_k(\Gamma)$ *to* G

Assume k is even. Let $f \in S_k(\Gamma)$. Define $\phi_f : G \to \mathbb{C}$ as,
$\phi_f(g) = f(g(i))j(g,i)^{-k}$.

Now we know that for any $z = x + iy \in H, \exists g \in G$ such that $g(i) = z$ and $j(g,i) \neq 0$. Hence $\phi_f \equiv 0 \iff f \equiv 0$.

Then ϕ_f has the following properties.

1. $\phi_f(\gamma g) = \phi(g)\ \forall \gamma \in \Gamma$.

2. $\phi_f(gr(\theta)) = e^{-ik\theta}\phi(g)\ \forall r(\theta) \in K$

3. $\phi_f(g)$ is bounded in particular $\int\limits_{\Gamma\backslash G} |\phi(g)|^2 dg < \infty$, hence $\phi \in L^2(\Gamma\backslash G)$.

4. $\phi_f(g)$ is cuspidal i.e. for any $g \in G$ and $\sigma \in SL(2,\mathbb{Z})$

   $\int\limits_0^1 \phi\Big(\sigma \begin{pmatrix} 1 & xh \\ 0 & 1 \end{pmatrix} g\Big) dx = 0.$

Proof.

$$\begin{aligned}\phi_f(\gamma g) &= f(\gamma(g(i)))j(\gamma g, i)^{-k}\\ &= f(\gamma(g(i)))j(\gamma, g(i))^{-k}j(g,i)^{-k}\\ &= f(g(i)).j(g,i)^{-k}(\because\ f\ is\ modular\ w.r.t\ \Gamma)\\ &= \phi_f(g).\end{aligned}$$

So, ϕ_f can be regarded as a continuous function on $\Gamma\backslash G$.

$$\begin{aligned}\phi_f(gr(\theta)) &= f((gr(\theta))i).j(gr(\theta), i)^{-k}\\ &= f(g(i))j(g,i)^{-k}j(r(\theta),i)^{-k}\\ &= \phi_f(g).e^{-ik\theta} = e^{-ik\theta}\phi_f(g).\end{aligned}$$

Since $\phi_f(\gamma g) = \phi_f(g)$ and that f is a cusp form, it follows that $(Imz)^{k/2}|f(z)|$ is bounded.

$$\int\limits_G \phi_f(g) = \frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_0^{\infty}\int\limits_{-\infty}^{\infty}\phi(x,y,\theta)\frac{dxdy}{y^2}d\theta \tag{3.2.1}$$

Consequently,

$$\int\limits_{\Gamma\backslash G}|\phi_f(g)|^2dg = \iint_F |f(z)|^2y^k\frac{dxdy}{y^2}. \tag{3.2.2}$$

Where F is the fundamental domain for Γ. The right hand side of 3.2.2 is the norm of $f(z)$ w.r.t the Peterson inner product.

$$\begin{aligned}\int\limits_0^1 \phi_f\Big(\sigma\begin{pmatrix}1 & xh\\ 0 & 1\end{pmatrix}g\Big)dx &= \int\limits_0^1 f\Big(\sigma\begin{pmatrix}1 & xh\\ 0 & 1\end{pmatrix}g(i)\Big).j\Big(\sigma\begin{pmatrix}1 & xh\\ 0 & 1\end{pmatrix}g,i\Big)^{-k}dx\\ &= \int\limits_0^1 f(\sigma(z+hx))j(z+hx)^{-k}dx\\ &= \int\limits_0^1 (f|_k\sigma)(hx+z)dx\end{aligned}$$

Since this last expression is just the zeroth Fourier coefficient of f at the cusp

$s = \sigma(\infty)$, the result follows from the fact that f is a cusp form in the classical sense. $\square$

Let $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) - y\frac{\partial^2}{\partial x \partial \theta}$. Then, $\because \Delta \circ R(g) = R(g) \circ \Delta$, Δ acts via a scalar on a G-invariant irreducible subspace of $L^2(\Gamma\backslash G)$. Now let $A_k^2(\Gamma)$ denote the space of functions ϕ on G satisfying the following conditions

1. $\phi(\gamma g) = \phi(g)\ \forall \gamma \in \Gamma$,
2. $\phi(gr(\theta)) = e^{-ik\theta}\phi(g)$, $r(\theta) \in K$,
3. $\Delta\phi = -\frac{k}{2}(\frac{k}{2} - 1)\phi$,
4. ϕ is bounded and cuspidal.

Note that $A_k^2(\Gamma) \subset L^2(\Gamma\backslash G)$ by (4) ($\because \phi$ is bounded on G, by (1) ϕ is bounded on $\Gamma\backslash G$).

Theorem 3.2.3. *The formula*

$$\phi_f(g) = f(g(i))j(g,i)^{-k}$$

describes an isomorphism between $S_k(\Gamma)$ and $A_k^2(\Gamma)$.

Proof. See Proposition 2.1, §2, [Ge]. $\square$

3.3 Representations of GL(2) over a local field

Let F be a non-archimedean Local field, $\mathcal{O}$ its ring of integers, $\mathfrak{p}$ its unique maximal ideal and ϖ a generator of $\mathfrak{p}$. Let q be the cardinality of the residue field $\mathcal{O}/\mathfrak{p}$. We will denote $v : F \to \mathbb{Z} \cup \{\infty\}$ the valuation, defined by $v(0) = \infty$ and $v(\varpi^r u) = r$ if $r \in \mathcal{O}^\times$. We will denote by $\int\limits_F dx$ the additive Haar measure on F normalized so the volume of $\mathcal{O}$ is one, and $\int\limits_{F^\times} d^\times x$ the multiplicative Haar measure on $F^\times$ normalized so that the volume of $\mathcal{O}^\times$ is one. Thus $d^\times x = (1 - q^{-1})^{-1}|x|^{-1}dx$.

We will be interested in those representations of $GL(2,F)$ which are called smooth and admissible representations. A representation (π, V) is smooth if for any $v \in V$, the stabilizer of v in G is an open subgroup of G. A smooth representation is called admissible if for any open subgroup U of G, the space $V^U = \{v \in V | \pi(g)v = v\ \ \forall g \in U\}$ is finite-dimensional.

Theorem 3.3.1 (Local multiplicity one theorem). *Let (π, V) be an irreducible admissible representation of $GL(2, F)$, where F is a non-archimedean local field and let ψ be a non-trivial additive character of F. Then the space of linear functionals $\Lambda : V \to \mathbb{C}$ satisfying $\Lambda\Big(\pi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} v\Big) = \psi(x)\Lambda(v)$ (for all $v \in V$ and $x \in F$) is atmost one-dimensional.*

Now one can state this theorem in a slightly different manner using the notion of *Whittaker Model*.

Theorem 3.3.2 (Local multiplicity one,Equivalent form). *Let (π, V) be an irreducible admissible representation of $GL(2, F)$. Let ψ be a non-trivial additive character of F. Then there exists at most one space W of functions on $GL(2, F)$ such that if $w \in W$ then $w\Big(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\Big) = \psi(x)w(g)$ (for all $g \in GL(2, F)$ and $x \in F$) and such that W is closed under right translations by elements of $GL(2, F)$ and the resulting representation of $GL(2, F)$ is isomorphic to π.*

The space of functions W is called a Whittaker Model for the representation (π, V). One could use Frobenius reciprocity

$$Hom_N(\pi, \psi) = Hom_G(\pi, Ind_N^G(\psi)),$$

where $N = \Big\{\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mid x \in F\Big\}$ to conclude that specifying a Whittaker model is equivalent to specifying a Whittaker functional.

There is an important notion of a *Jacquet module* for an irreducible representation of $GL(2, F)$, which is of immense help when trying to analyse the irreducibility of these representations.

Let $B(F)$ denote the standard Borel subgroup of $GL(2, F)$, i.e., subgroup consisting of upper triangular matrices. Let (π, V) be a smooth representation of the Borel subgroup $B(F)$ of $GL(2, F)$. Let V_N be the vector subspace of V generated by elements of the form $\pi(u)v - v$ where $u \in N(F)$, $v \in V$. One can show that the quotient V/V_N (denoted by $J(V)$) is stable under the action of $T(F)$ (the subgroup of $GL(2, F)$ consisting of diagonal matrices), and consequently is a $T(F)$-module; it is called the *Jacquet Module* of the representation (π, V) and is denoted by $(\pi_N, J(V))$.

Let χ_1 and χ_2 be two quasicharacters of $F^\times$. Then we define a quasicharacter of $B(F)$ as follows

$$\chi\begin{pmatrix} a & b \\ & d \end{pmatrix} = \chi_1(a)\chi_2(d) \quad a, d \in F^\times. \tag{3.3.3}$$

Let $\mathcal{B}(\chi_1, \chi_2) = Ind_{B(F)}^{GL(2,F)}$ be the representation of $GL(2, F)$ obtained by inducing the one dimensional representation χ of $B(F)$. Then $\mathcal{B}(\chi_1, \chi_2)$ is an irreducible representation except in the following two cases.

1. If $\chi_1\chi_2^{-1}(y) = y^{-1}$ for all $y \in F^\times$, then $\mathcal{B}(\chi_1, \chi_2)$ has a one dimensional invariant subspace and the quotient representation is irreducible.

2. If $\chi_1\chi_2^{-1}(y) = y$ for all $y \in F^\times$, then $\mathcal{B}(\chi_1, \chi_2)$ has an irreducible subspace of codimension one.

When $\mathcal{B}(\chi_1, \chi_2)$ is irreducible it is called a Principal Series Representation. In the 2nd case the irreducible subspace of codimension one is called the *Steinberg* or *Special* representation.

Theorem 3.3.4. *Let (π, V) be an irreducible admissible representation of $GL(2, F)$. Then the dimension of the Jacquet module of V is atmost two dimensional. If it is non-zero, then π is isomorphic to a subrepresentation of $\mathcal{B}(\chi_1, \chi_2)$.*

Proof. See Proposition 4.7.1, §4.7, [Bu]. □

Now for the rest of the section we will be talking about spherical representation because they are some of the most important ingredients of an automorphic representation.

We choose a Haar measure on $GL(2, F)$ such that the measure of its maximal compact subgroup $K = GL(2, \mathcal{O})$ is one. An irreducible admissible representation (π, V) of $GL(2, F)$ is called *spherical* (or *unramified*) if it contains a K-fixed vector. A nonzero element of V^K is called a *spherical vector*.

Let $\mathcal{H}$ be the space of locally constant and compactly supported complex-valued functions on $GL(2, F)$. Define $\mathcal{H}_K = \{f \in \mathcal{H} : f(k_1gk_2) = f(g)\, \forall k_1, k_2 \in K\}$. Since $GL(2, F)$ is unimodular one can give $\mathcal{H}$ the structure of an algebra (without unit) under convolution:

$$(f_1 * f_2)(g) = \int_G f_1(gh^{-1}) f_2(h) dh$$

Then $\mathcal{H}_K$ is clearly closed under this convolution and hence forms a subring of $\mathcal{H}$, but it does have identity element ε_K, where

$$\varepsilon_K(g) := \begin{cases} vol(K)^{-1} & \text{if } g \in K; \\ 0 & \text{otherwise.} \end{cases}$$

This $\mathcal{H}_K$ is called the *spherical Hecke algebra*.

Now given any admissible representation (π, V) of GL(2,F), if $f \in \mathcal{H}$, we define an endomorphism $\pi(f)$ of V by

$$\pi(f)v = \int_{GL(2,F)} f(g)\pi(g)v \ dg. \tag{3.3.5}$$

Note that this integral makes sense because f is both locally constant as well as compactly supported (so that it will actually turn out to be a finite sum). Observe that $\pi(f_1 * f_2) = \pi(f_1) \circ \pi(f_2)$. So, π is a representation of the ring $\mathcal{H}$. Hence V is a $\mathcal{H}$ module, in particular one obtains that V^K is a $\mathcal{H}_K$ module.

Proposition 3.3.6. *Let (π, V) be a smooth represenation of $GL(2, F)$. Assume that V is nonzero. Then the following conditions are equivalent.*

1. *π is irreducible*

2. *V is simple as a $\mathcal{H}$-module.*

3. *V^{K_0} is either zero or simple as a $\mathcal{H}_{K_0}$-module for all open subgroups K_0 of G.*

Proof. Refer to Proposition 4.2.3, §4.2, [Bu]. □

Theorem 3.3.7. *A complete set of double coset representatives for $K\backslash GL(2, F)/K$ consists of diagonal matrices*

$$\begin{pmatrix} \varpi^{n_1} & \\ & \varpi^{n_2} \end{pmatrix}$$

where $n_1 \geq n_2$ are integers.

Proof. Refer to Proposition 4.6.2, §4.6, [Bu]. □

Proposition 3.3.8. *The spherical Hecke algebra $\mathcal{H}_K$ is commutative.*

Proof. Refer to Theorem 4.6.1, §4.6, [Bu]. □

Theorem 3.3.9. *Let (π, V) be an irreducible admissible represenation of $GL(2, F)$. Then V^K is at most one-dimensional.*

Proof. By Proposition 3.3.6, V^K(if non-zero) is a finite-dimensional simple $\mathcal{H}_K$ module. But, since $\mathcal{H}_K$ is commutative by Proposition 3.3.8, it follows that V^K can be atmost one-dimensional. □

If k is a non-negative integer, let $T(\mathfrak{p}^k) \in \mathcal{H}_K$ be the charactersitic function of the set of all $g \in Mat_2(\mathcal{O})$ such that the ideal generated by $\det(g)$ in $\mathcal{O}$ is $\mathfrak{p}^k$. Also let, $R(\mathfrak{p}) \in \mathcal{H}_K$ be the charactersitic function of

$$K \begin{pmatrix} \varpi & \\ & \varpi \end{pmatrix} K = K \begin{pmatrix} \varpi & \\ & \varpi \end{pmatrix}.$$

Proposition 3.3.10. $K \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} K = \begin{pmatrix} 1 & \\ & \varpi \end{pmatrix} K \cup \bigcup_{b \bmod \mathfrak{p}} \begin{pmatrix} \varpi & b \\ & 1 \end{pmatrix} K$.

Proof. It follows from easy matrix calculations. □

Proposition 3.3.11. *If $k \geq 1$ we have*

$$T(\mathfrak{p})T(\mathfrak{p}^k) = T(\mathfrak{p}^{k+1}) + qR(\mathfrak{p})T(\mathfrak{p}^{k-1})$$

Proof. Refer to Proposition 4.6.4, §4.6, [Bu]. □

Let χ_1 and χ_2 be two unramified quasicharacters of $F^\times$ (which means $\chi_i|_{\mathcal{O}^\times} = 1$ for $i = 1, 2$) and such that $\chi_1\chi_2^{-1} \neq |x|^{\pm 1}$ and let $(\pi, V) = \mathcal{B}(\chi_1, \chi_2)$. Then (π, V) is an irreducible represenation of $GL(2, F)$. We will show that it is also spherical.

Define $\chi : B(F) \to \mathbb{C}^\times$,

$$\chi \begin{pmatrix} a & b \\ & d \end{pmatrix} = \chi_1(a)\chi_2(d).$$

Then

$$V = \{f : G \to \mathbb{C} \,|\, f(bg) = \delta^{1/2}\chi(b)f(g)\}$$

for $b \in B(F)$ and $g \in GL(2,F)$, where δ is the modular quasicharacter of $B(F)$ defined by

$$\delta \begin{pmatrix} a & b \\ & d \end{pmatrix} = |a||d|^{-1}. \tag{3.3.12}$$

Then for any $\tilde{g} \in GL(2,F)$ and $f \in V$, $(\pi(\tilde{g})f)(g) = f(g\tilde{g})$. Now let $\phi_K \in V$ defined by, $\phi_K(bk) = \delta^{1/2}\chi(b)$. First we have to checck that ϕ_K is well-defined. So let $g = bk = \tilde{b}\tilde{k}$, then $b = \tilde{b}(\tilde{k}k^{-1})$ and hence $\tilde{k}k^{-1} \in K \cap B(F)$ and since χ_1 and χ_2 are unramified, we have $\delta^{1/2}\chi(\tilde{k}k^{-1}) = 1$ and we have $\phi_K(bk) = \phi(\tilde{b}\tilde{k})$. For any $\tilde{k} \in K$, we have

$$(\pi(\tilde{k})\phi_K)(bk) = \phi_K(bk\tilde{k}) = \delta^{1/2}\chi(b) = \phi_K(bk).$$

Hence ϕ_K is a K-fixed vector in V. In such a case, $\mathcal{B}(\chi_1, \chi_2)$ is called a spherical principal series represenation.

Proposition 3.3.13. *Let ϕ_K be the normalized spherical vector in $\mathcal{B}(\mu_1, \mu_2)$, where μ_1 and μ_2 are unramified quasicharacters of $F^\times$. Let $\beta_1 = \mu_1(\varpi)$ and $\beta_2 = \mu_2(\varpi)$. Then $\pi(T(\mathfrak{p}))\phi_K = \lambda\phi_K$ and $\pi(R(\mathfrak{p}))\phi_K = \rho\phi_K$, where*

$$\lambda = q^{1/2}(\beta_1 + \beta_2), \quad \rho = \beta_1\beta_2.$$

Proof. Since, V^K is a $\mathcal{H}_K$ module, and $\phi_K \in V^K$, hence $\pi(T(\mathfrak{p}))\phi_K \in V^K$ and is a multiple of ϕ_K (since V^K is one dimesnional), so therefore equals $\lambda\phi_K$ for some $\lambda \in \mathbb{C}$; similarly $\pi(R(\mathfrak{p}))\phi_K = \rho\phi_k$ for some $\rho \in \mathbb{C}$, and because $R(\mathfrak{p})$ has an inverse in $\mathcal{H}_K$, we know that $\rho \neq 0$.

$$\begin{aligned}\lambda = (\pi(T(\mathfrak{p}))\phi_K)(1) &= \int_{K\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}K} \phi_K(g)dg \\ &= \sum_{\gamma \in K\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}K/K} \int_K \phi(\gamma k)dk = \sum_{\gamma \in K\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}K/K} \phi_K(\gamma).\end{aligned}$$

Using the representatives from Proposition 3.3.10 this equals

$$(\delta^{1/2}\mu)\begin{pmatrix}1 & \\ & \varpi\end{pmatrix} + q(\delta^{1/2}\mu)\begin{pmatrix}\varpi & b\\ & 1\end{pmatrix} = q^{1/2}(\beta_1+\beta_2),$$

where $\beta_i = \mu_i(\varpi)$. Similarly, $\rho = \beta_1\beta_2$. $\square$

Let us end this section by finding a formula for the spherical Whittaker function. Let $(\pi, V) = \mathcal{B}(\chi_1, \chi_2)$, where χ_1 and χ_2 are unramified. This representation is spherical as well as admits a Whittaker model. The spherical Whittaker function is just the spherical vector in the Whittaker model. More explicitly let ψ be a non-trivial character of F (such that the conductor of ψ is $\mathcal{O}$). Let Λ be the Whittaker functional for (π, V) given by

$$\Lambda(f) = \int_F \phi_K\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\psi(-x)dx,$$

where ϕ_K is the unique spherical vector defined earlier. Then $W_0(g) = \Lambda(\pi(g)\phi_K)$. Let $w_m = W_0\left(\begin{smallmatrix}\varpi^m & \\ & 1\end{smallmatrix}\right)$ and let $w'_m = (\pi(T(\mathfrak{p}))W_0)\left(\begin{smallmatrix}\varpi^m & \\ & 1\end{smallmatrix}\right)$. If $x \in \mathcal{O}$, then $\left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right) \in K$, and

$$\begin{aligned} w_m &= W_0\Big(\begin{pmatrix}\varpi^m & \\ & 1\end{pmatrix}\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\Big)\\ &= W_0\Big(\begin{pmatrix}1 & \varpi^m x\\ & 1\end{pmatrix}\begin{pmatrix}\varpi^m & \\ & 1\end{pmatrix}\Big) = \psi(\varpi^m x)w_m. \end{aligned}$$

So, if $m < 0$ then (because we are chosing the conductor of ψ to be $\mathcal{O}$), we can choose $x \in \mathcal{O}$ such that $\psi(\varpi^m x) \neq 1$ which gives us $w_m = 0$ if $m < 0$. If $m \geq 0$, then we proceed as follows to obtain an expression for w_m:

$$w'_m = \int_{K\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}K} W_0\Big(\begin{pmatrix}\varpi^m & \\ & 1\end{pmatrix}g\Big)dg = \sum_{\gamma\in K\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}K/K}\int_K W_0\Big(\begin{pmatrix}\varpi^m & \\ & 1\end{pmatrix}\gamma\Big)dk.$$

Hence

$$\begin{aligned}
w'_m &= \sum_{\gamma\in K\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}K/K} W_0\Big(\begin{pmatrix}\varpi^m & \\ & 1\end{pmatrix}\gamma\Big)\\
&= W_0\Big(\begin{pmatrix}\varpi^m & \\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ & \varpi\end{pmatrix}\Big) + \sum_{b \bmod \mathfrak{p}} W_0\Big(\begin{pmatrix}\varpi^m & \\ & 1\end{pmatrix}\begin{pmatrix}\varpi & b\\ & 1\end{pmatrix}\Big)\\
&= W_0\begin{pmatrix}\varpi^m & \\ & \varpi\end{pmatrix} + qw_{m+1}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
W_0\begin{pmatrix}\varpi^m & \\ & \varpi\end{pmatrix} &= \int_F \phi_k\Big(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\varpi^m & \\ & \varpi\end{pmatrix}\Big)\psi(-x)dx\\
&= \int_F \phi_k\Big(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\varpi^{m-1} & \\ & 1\end{pmatrix}\begin{pmatrix}\varpi & \\ & \varpi\end{pmatrix}\Big)\psi(-x)dx\\
&= \int_F \phi_k\Big(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\varpi^{m-1} & \\ & 1\end{pmatrix}\Big)\phi_k\begin{pmatrix}\varpi & \\ & \varpi\end{pmatrix}\psi(-x)dx\\
&= \alpha_1\alpha_2 w_{m-1}, \quad (\alpha_i = \chi_i(\varpi)).
\end{aligned}$$

Hence, $w'_m = qw_{m+1} + \alpha_1\alpha_2 w_{m-1}$, but since W_0 is an eigenfunction for $\pi(T(\mathfrak{p}))$ we get $q^{1/2}(\alpha 1 + \alpha_2)w_m = qw_{m+1} + \alpha_1\alpha_2 w_{m-1}$, solving which we get

$$w_m = q^{-m/2}\frac{{\alpha_1}^{m+1} - {\alpha_2}^{m+1}}{\alpha_1 - \alpha_2}w_0. \tag{3.3.14}$$

Definition 3.3.15. *Let π denote an irreducible admissible represenation of $GL(2,F)$ and $\mathcal{W}(\pi)$ be its Whittaker space. Suppose χ is an unitary character of $F^\times$, $g \in GL(2,F)$, $W \in \mathcal{W}(\pi)$ and $s \in \mathbb{C}$. Then the local zeta function attached to (g,χ,W) is defined by the formula*

$$\zeta(g,\chi,W,s) = \int_{F^\times} W\Big(\begin{pmatrix}a & \\ & 1\end{pmatrix}g\Big)\chi(a)|a|^{s-1/2}d^\times a. \tag{3.3.16}$$

Theorem 3.3.17. *1. The integral defining $\zeta(g, \chi, W, s)$ converges for s with sufficiently large real part.*

2. There exists an Euler factor $L(s, \chi \otimes \pi)$ with the property that

$$\frac{\zeta(g, \chi, W, s)}{L(s, \chi \otimes \pi)}$$

is an entire function of s for every g, χ, and W, and such that

$$\zeta(1, \chi, W_0, s) \;=\; L(s, \chi \otimes \pi) \tag{3.3.18}$$

for an appropriate choice of $W_0 \in W(\pi)$

The function $\zeta(g, \chi, W, s)$ possesses an analytic continuation to the whole s-plane and satisfies the functional equation

$$\frac{\zeta(g, \chi, W, s)}{L(s, \chi \otimes \pi)}\varepsilon(s, \chi, \psi, \pi) \;=\; \frac{\zeta(w_1 g, \omega^{-1}\chi^{-1}, W, 1-s)}{L(s, \chi^{-1} \otimes \hat{\pi})} \tag{3.3.19}$$

where ψ is the fixed non-trivial additive character of F, ω is the central character of π, $\varepsilon(s, \chi, \psi, \pi)$ is independent of g and W and $w_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Proof. Refer to Theorem 6.12, §6, [Ge]. □

Using the above theorem, let us calculate the expression for $L(s, \pi)$, when π is a spherical principal series. Suppose that the conductor of the additive character ψ is $\mathcal{O}$ and W_0 is the spherical Whittaker function normalized so that $W_0(1) = 1$. There exists an explicit formula for w_m in terms of α_1 and α_2. Let $m = ord(a)$, and let q be the cardinality of the residue field $\mathcal{O}/(\varpi)$. Then

$$W_0\begin{pmatrix} a & \\ & 1 \end{pmatrix} := \begin{cases} q^{-m/2}\frac{\alpha_1^{m+1}-\alpha_2^{m+1}}{\alpha_1-\alpha_2}, & \text{if } m \geq 0\ , \\ 0 & \text{, otherwise.} \end{cases} \tag{3.3.20}$$

Thus we may break the integral into a sum over $m = 0$ to ∞ to obtain

$$\sum_{m=0}^{\infty} q^{-m/2}\frac{\alpha_1^{m+1} - \alpha_2^{m+1}}{\alpha_1 - \alpha_2} q^{m/2-ms} = (\alpha_1 - \alpha_2)^{-1}\Big(\alpha_1 \sum_{m=0}^{\infty}(\alpha_1 q^{-s})^m - \alpha_2 \sum_{m=0}^{\infty}(\alpha_2 q^{-s})^m\Big).$$

Now summing the geometric series, this equals

$$\begin{aligned}(\alpha_1-\alpha_2)^{-1}(\alpha_1(1-\alpha_1 q^{-s})^{-1}-\alpha_2(1-\alpha_2 q^{-s})^{-1}) &= (\alpha_1-\alpha_2)^{-1}\frac{(\alpha_1-\alpha_2)}{(1-\alpha_1 q^{-s})(1-\alpha_2 q^{-s})} \\ &= (1-\alpha_1 q^{-s})^{-1}(1-\alpha_2 q^{-s})^{-1}\end{aligned}$$

Hence, if $\pi = \mathcal{B}(\chi_1, \chi_2)$ with χ_1, χ_2 unramified quasicharacters, then

$$L(s,\pi) = (1-\chi_1(\varpi)q^{-s})^{-1}(1-\chi_2(\varpi)q^{-s})^{-1} \tag{3.3.21}$$

3.4 Jacquet-Langlands Theory

We recall the definition of restricted direct product. Let Σ be some indexing set and for each ν in Σ, let there be a group G_ν, and for almost all $\nu \in \Sigma$, let there be given a subgroup K_ν of G_ν. Then the restricted direct product of $\{G_\nu\}_\nu$ of $\{K_\nu\}_\nu$ is

$$G = \Big\{(a_\nu)_{\nu\in\Sigma} \in \prod G_\nu \Big| \, a_\nu \in K_\nu \; \textit{for almost all}\, \nu \in \Sigma\Big\}.$$

Let F be a number field and for each ν in Σ, the set of places of F, let F_ν be the completion of F at ν. If ν is non-archimedean, let $\mathcal{O}_\nu$ be the ring of integers in F_ν. The adele ring $\mathbb{A}$ of F is the restricted direct product of F_ν with respect to the $\mathcal{O}_\nu$. The ideles $\mathbb{A}^\times$ are the restricted direct product of the $F_\nu^\times$ with respect to the $\mathcal{O}_\nu^\times$.

Let $\mathbb{A}_f$ be the ring of "finite adeles", i.e., those adeles (a_ν) with $a_\nu = 1$ at every archimedean place ν. Let $F_\infty = \prod_{\nu\in P_\infty} F_\nu$, where P_∞ is the finite set of archimedean places of F. We embed F_∞ in $\mathbb{A}$ by mapping $(a_\nu)_{\nu\in P_\infty}$ to the adele that matches a_ν at every infinite place and that is 1 at every finite place. Then $\mathbb{A} = F_\infty \mathbb{A}_f$ and also $GL(2,\mathbb{A}) = GL(2,F_\infty)GL(2,\mathbb{A}_f)$.

We fix a positive integer N. Let $K_0(N)$ be the following compact subgroup of $GL(2,\mathbb{A}_f)$; $K_0(N) = \prod_{\nu\notin P_\infty} K_0(N)_\nu$, where $K_0(N)_\nu = GL(2,\mathcal{O}_\nu)$ if $p_\nu \nmid N$ (p_ν is the rational prime corresponding to then non-archimedean place ν), while if $p_\nu \mid N$, then $K_0(N)$ is the subgroup of $\mathrm{GL}(2,\mathcal{O}_\nu)$ of the form $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ where $c \equiv 0$ modulo N in the ring $\mathbb{Z}_\nu$ of p-adic integers.

Theorem 3.4.1 (Strong Approximation Theorem)**.**

$$GL(2,\mathbb{A}) = GL(2,\mathbb{Q})GL(2,\mathbb{R})^{+}K_0(N). \tag{3.4.2}$$

Proof. See Theorem 3.3.1, §3.3, [Bu]. □

Let ω denotes a unitary character of $\mathbb{A}^\times/F^\times$. Let $L^2(GL(2,F)\backslash GL(2,\mathbb{A}),\omega)$ be the space of all functions ϕ on $GL(2,\mathbb{A})$ that are measurable with respect to Haar measure and that satisfy

$$\phi\Big(\begin{pmatrix} z & \\ & z \end{pmatrix} g\Big) = \omega(z)\phi(g), z \in \mathbb{A}^\times, \tag{3.4.3}$$

$$\phi(\gamma g) = \phi(g),\ \gamma \in GL(2,F), \tag{3.4.4}$$

$$\int\limits_{Z(\mathbb{A})Gl(2,F)\backslash GL(2,\mathbb{A})} |\phi(g)|^2 dg < \infty. \tag{3.4.5}$$

We say $\phi \in L^2(GL(2,F)\backslash GL(2,\mathbb{A}),\omega)$ is cuspidal if,

$$\int\limits_{\mathbb{A}/F} \phi\Big(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\Big) dx = 0 \tag{3.4.6}$$

for all g.

An automorphic form with central character ω is a function on $GL(2,\mathbb{A})$ satisfying 3.4.3 and 3.4.4, and which is *smooth*, K-$finite$, $\mathcal{Z}$-$finite$ and of moderate growth. We now define these terms.

If F is a number field, then a function f on $GL(2,\mathbb{A})$ is called smooth if for every $g \in GL(2,\mathbb{A})$, there exists a neighbourhood N of g and a smooth function f_g on $GL(2,F_\infty)$ such that for $h \in N$, $f(h) = f_g(h_\infty)$, where we have factored $h = h_\infty h_f$ with $h_\infty \in GL(2,F_\infty)$ and $h_f \in GL(2,\mathbb{A}_f)$. A function f on $GL(2,\mathbb{A})$ is called K-finite if its right translates, by elements of K, span a finite-dimensional dimensional vector space.

If ν is any archimedean place of F, one can define an action of $\mathfrak{gl}(2,F_\nu)$ on the

K-finite vectors by

$$(Xf)(g) = \frac{d}{dt} f(g \exp(tX))|_{t=0}.$$

Here $\mathfrak{gl}(2, F_\nu)$ is the four dimensional Lie algebra whose underlying space is $Mat_2(F_\nu)$, with Lie bracket operation $[X, Y] = XY - YX$. It may be shown that if f is K-finite, then Xf is defined and is also K-finite. This action of $\mathfrak{gl}(2, F_\nu)$ is extended to the universal enveloping algebra $U(\mathfrak{gl}(2, F_\nu))$. The requirement that f be $\mathcal{Z}$-finite means that f lies in a finite-dimensional vector space that is invariant by $\mathcal{Z}$.

To define moderate growth, we need to define a height function $\parallel g \parallel$ on $GL(2, \mathbb{A})$. For any field E, We embed $GL(2, E) \to E^5$ via $g \mapsto (g, det(g)^{-1})$. First we define a local height $\parallel g_\nu \parallel_\nu$ on $GL(2, F_\nu)$ for each place by restricting the height function $(x_1, x_2, x_3, x_4, x_5) \mapsto max_i |x_i|_\nu$ on F_ν^5. We note that $\parallel g_\nu \parallel_\nu \geq 1$ and (if ν is non-archimedean) that $\parallel g_\nu \parallel_\nu = 1$ if $g_\nu \in GL(2, \mathcal{O}_\nu)$. We the define the global height $\parallel g \parallel$ to be the product of local heights. We then say that f is *slowly increasing* if there exists constants C and N such that $f(g) < C \parallel g \parallel^N$ for all $g \in GL(2, \mathbb{A})$.

Theorem 3.4.7 (Tensor Product Theorem)**.** *Let (π, V) be an irreducible representation of $GL(n, \mathbb{A})$, then there exists for each archimedean place ν of F an irreducible (g_∞, K_ν)-module (π_ν, V_ν) and for each non-archimedean place ν there exists an irreducible admissible representation (π_ν, V_ν) of $GL(n, F_\nu)$ such that for almost ν, V_ν contains a K_ν fixed vector ξ_ν^0 such that π is the restricted tensor product of π_ν.*

Proof. See Theorem 3.3.3, §3.3, [Bu]. □

Theorem 3.4.8 (Existence of Whittaker model for Automorphic Representation)**.** *Let F be a number field, $\mathbb{A}$ be its adele ring and let (π, V) be a cuspidal automorphic representation of $GL(2, F)$, so $V \subset A_0(GL(2, F)\backslash GL(2, \mathbb{A}), \omega)$ where ω is a character of $\mathbb{A}^\times / F^\times$. If $\phi \in V$ and $g \in GL(2, \mathbb{A})$, let*

$$W_\phi(g) = \int_{\mathbb{A}/F} \phi\Big(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\Big) \psi(-x) dx \tag{3.4.9}$$

(where ψ is a non-trivial additive character of $\mathbb{A}/F$). Then the space W of functions W_ϕ is a $Whittaker$ model for π. We have the $Fourier$ expansion

$$\phi(g) = \sum_{\alpha \in F^\times} W_\phi\Big(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\Big). \tag{3.4.10}$$

Proof. Define

$$F_\phi(x) = \phi\Big(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g \Big) \; for \; x \in \mathbb{A}.$$

Clearly F_ϕ is continuous, $\because \phi$ is automorphic we have, $\phi(\gamma g) = \phi(g)$ $\forall \gamma \in GL(2, F)$. Choose $\gamma = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, where $a \in F^\times$. Then

$$F_\phi(x+a) = \phi\Big(\begin{pmatrix} 1 & x+a \\ & 1 \end{pmatrix} g \Big) = \phi\Big(\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g \Big) = \phi\Big(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g \Big) = F_\phi(x).$$

$\therefore$ $F_\phi(x + a) = F_\phi(x)$, $\forall a \in F$. Hence F_ϕ can be regarded as a continuous function on the compact group $\mathbb{A}/F$. Therefore one can expand F_ϕ as a fourier series in terms of characters of $\mathbb{A}/F$, but we know chracters of $\mathbb{A}/F$ are of the form ψ_a taking x to $\psi(ax)$ for $a \in F$. So let

$$\phi\Big(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g \Big) = \sum_{\alpha \in F} c(\alpha)\psi(\alpha x), \tag{3.4.11}$$

where,

$$c(\alpha) = \int_{\mathbb{A}/F} \phi\Big(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g \Big) \psi(-\alpha x) dx.$$

If $\alpha = 0$ then

$$c(0) = \int_{\mathbb{A}/F} \phi\Big(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g \Big) dx = 0 \; (\because \phi \; is \;\; cuspidal)$$

So we may restrict our summation to $\alpha \in F^\times$. If $\alpha \neq 0$, then because ϕ is automorphic

$$\begin{aligned} c(\alpha) = \int_{\mathbb{A}/F} \phi\Big(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g \Big) \psi(-\alpha x) dx &= \int_{\mathbb{A}/F} \phi\Big(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g \Big) \psi(-\alpha x) dx \\ &= \int_{\mathbb{A}/F} \phi\Big(\begin{pmatrix} \alpha & \alpha x \\ & 1 \end{pmatrix} g \Big) \psi(-\alpha x) dx. \end{aligned}$$

Hence

$$\begin{aligned}
c(\alpha) &= \int\limits_{\mathbb{A}/F} \phi\Big(\begin{pmatrix} 1 & \alpha x \\ & 1 \end{pmatrix}\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\Big)\psi(-\alpha x)dx \\
&= \int\limits_{\mathbb{A}/F} \phi\Big(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\Big)\psi(-x)dx \\
&= W_\phi\Big(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\Big).
\end{aligned}$$

Hence evaluating $F_\phi(x)$ at $x = 0$ using the Fourier expansion we obtain the equality

$$\phi(g) = \sum_{\alpha \in F^\times} W_\phi\Big(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\Big).$$

Now

$$\begin{aligned}
W_\phi\Big(\begin{pmatrix} 1 & \tilde{x} \\ & 1 \end{pmatrix} g\Big) &= \int\limits_{\mathbb{A}/F} \phi\Big(\begin{pmatrix} 1 & \tilde{x}+x \\ & 1 \end{pmatrix} g\Big)\psi(\tilde{x})\psi(-\tilde{x}-x)dx \\
&= \psi(\tilde{x})W_\phi(g).
\end{aligned}$$

$\because$ ϕ is of moderate growth so is W_ϕ.

$$\begin{aligned}
W_{\pi(\tilde{g})\phi}(g) &= \int\limits_{\mathbb{A}/F} \pi(\tilde{g})\phi\Big(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\Big)\psi(x)dx \\
&= \int\limits_{\mathbb{A}/F} \phi\Big(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\tilde{g}\Big)\psi(-x)dx \\
&= W_\phi(g\tilde{g}) = \pi(\tilde{g})W_\phi
\end{aligned}$$

From the definition it is obvious that $\phi \longmapsto W_\phi$ is injective as well as from eq. (3.4.10) it is clear that it is onto. Hence the space $\mathcal{W}$ of functions W_ϕ comprises a Whittaker model for π. $\square$

Theorem 3.4.12. *Suppose $\pi = \bigotimes \pi_\nu$ has central character ω. If π occurs in $L_0^2(GL(2,F)\backslash GL(2,\mathbb{A}),\omega)$, then $L(s,\chi\otimes\pi)$ satisfies the following properties for every character χ of $\mathbb{A}^\times/F^\times$*

1. *$L(s,\chi\otimes\pi)$ extends to an entire function bounded in vertical strips.*

2. *$L(s,\chi\otimes\pi)$ satisfies the functional equation*

$$L(s,\chi\otimes\pi)=\varepsilon(s,\chi,\pi)L(1-s,\chi^{-1}\otimes\tilde{\pi}) \tag{3.4.13}$$

   where $\tilde{\pi}(g)=\omega^{-1}(g)\pi(g)$

Proof. Suppose first that π occurs in L_0 and that its represenation space is V. For each ν choose a function $W_\nu(g_\nu)$ in $\mathcal{W}(\pi_\nu)$ such that $W_\nu=W_\nu^0$ for almost all ν. Then

$$W(g)=\prod_\nu W_\nu(g_\nu)$$

certainly belongs to $\mathcal{W}(\pi)$ and

$$\phi(g)=\sum_{\alpha\in F^\times} W\Big(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}g\Big) \tag{3.4.14}$$

belongs to the space of K-finite functions in V.

Next consider the Mellin transform

$$\zeta(g,W,\chi,s)=\int\limits_{\mathbb{A}^\times/F^\times}\phi\Big(\begin{pmatrix}x & \\ & 1\end{pmatrix}g\Big)\chi(x)|x|^{s-1/2}d^\times x \tag{3.4.15}$$

Now $\phi\Big(\begin{pmatrix}x & \\ & 1\end{pmatrix}g\Big)=\phi_g\begin{pmatrix}x & \\ & 1\end{pmatrix}$(where $\phi_g=\pi(g)\phi$) and since $\phi_g\in V$, it is rapidly decreasing as $|x|\to\infty$; which means for any $N>0$ there exists constant B_N such that

$$\phi_g\begin{pmatrix}x & \\ & 1\end{pmatrix}<B_N|x|^{-N}$$

for $|x|$ sufficiently large. Also we claim that $\phi_g\begin{pmatrix}x & \\ & 1\end{pmatrix}$ is rapidly decreasing as

$|x| \to 0$, which means for any $N > 0$ there exists a constant B'_N such that

$$\phi_g \begin{pmatrix} x & \\ & 1 \end{pmatrix} < B'_N |x|^N$$

when $|x|$ is sufficiently small. This can be seen in the following way: since ϕ_g is automorphic

$$\begin{aligned}\phi_g \begin{pmatrix} x & \\ & 1 \end{pmatrix} = \phi_g \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} x & \\ & 1 \end{pmatrix} &= \Big(\pi \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \phi_g\Big) \begin{pmatrix} 1 & \\ & x \end{pmatrix} \\ &= \omega(x) \Big(\pi \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \phi_g\Big) \begin{pmatrix} x^{-1} & \\ & 1 \end{pmatrix}\end{aligned}$$

so the rapid decrease of $\phi_g \begin{pmatrix} x & \\ & 1 \end{pmatrix}$ as $|x| \to 0$ follows from the rapid decrease of

$$\Big(\pi \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \phi_g\Big) \begin{pmatrix} x^{-1} & \\ & 1 \end{pmatrix}$$

as $|x| \to \infty$ which is already established. Now because $\phi\Big(\begin{pmatrix} x & \\ & 1 \end{pmatrix} g\Big)$ is rapidly decreasing as $|x| \to \infty$ or 0. The integral in eq. (3.4.15) converges for all values of s and represents an entire function bounded in vertical strips. But $w\Big(\begin{pmatrix} x & \\ & 1 \end{pmatrix} g\Big)$ is also rapidly decreasing at ∞. Hence

$$\int_{\mathbb{A}^\times} W\Big(\begin{pmatrix} x & \\ & 1 \end{pmatrix} g\Big) \chi(x) |x|^{s-1/2} d^\times x = \prod_\nu \zeta(g_\nu, \chi_\nu, w_\nu, s)$$

converges for $Re(s)$ sufficiently large and in this range equals $\zeta(g, \chi, W, s)$. (Here $\zeta(g_\nu, \chi_\nu, W_\nu, s)$ is the local zeta function defined in 3.3.16). Now suppose W_v is chosen such that

$$\zeta(g_\nu, \chi_\nu, W_\nu, s) = L(s, \chi_\nu \otimes \pi_\nu).$$

With this choice of $W(g) = \prod W_\nu(g_\nu)$, we have for $Re(s)$ sufficiently large,

$$\zeta(g, \chi, W, s) = \prod_\nu L(s, \chi_\nu \otimes \pi_\nu) = L(s, \chi \otimes \pi).$$

Consequently $L(s, \chi \otimes \pi)$ satisfies condition 1 of the theorem since $\zeta(g, \chi, W, s)$ does. To verify (2) first observe that

$$\begin{aligned}
\zeta(g, W, \chi, s) &= \int_{\mathbb{A}^\times/F^\times} \phi\Big(\begin{pmatrix} x & \\ & 1 \end{pmatrix} g\Big)\chi(x)|x|^{s-1/2}d^\times x \\
&= \int_{\mathbb{A}^\times/F^\times} \phi\Big(w_1 \begin{pmatrix} x & \\ & 1 \end{pmatrix} g\Big)\chi(x)|x|^{s-1/2}d^\times x \ \Big(w_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\Big) \\
&= \int_{\mathbb{A}^\times/F^\times} \phi\Big(\begin{pmatrix} 1 & \\ & x \end{pmatrix} w_1 g\Big)\chi(x)|x|^{s-1/2}d^\times x.
\end{aligned}$$

Now substitute x^{-1} for x and use the invariance of ϕ under the central quasicharacter to obtain

$$\int_{\mathbb{A}^\times/F^\times} \phi\Big(\begin{pmatrix} x & \\ & 1 \end{pmatrix} w_1 g\Big)(\chi\omega)^{-1}(x)|x|^{-s+1/2}d^\times x$$

So we get

$$\zeta(g, W, \chi, s) = \zeta(w_1 g, W, \omega^{-1}\chi^{-1}, 1-s)$$

Consequently by the local functional equations,

$$\begin{aligned}
\frac{\zeta(w_1 g, W, \omega^{-1}\chi^{-1}, 1-s)}{L(s, \chi^{-1} \otimes \hat{\pi})} &= \prod_\nu \frac{\zeta(w_1 g_\nu, W_\nu, \omega_\nu^{-1}\chi_\nu^{-1}, 1-s)}{L(s, \chi_\nu^{-1} \otimes \hat{\pi_\nu})} \\
&= \prod_\nu \varepsilon(s, \chi_\nu, \psi_\nu, \pi_\nu)\frac{\zeta(g_\nu, W_\nu, \omega_\nu\chi_\nu, s)}{L(s, \chi_\nu \otimes \pi_\nu)} \\
&= \varepsilon(s, \chi, \psi, \pi)\frac{\zeta(g, W, \chi, s)}{L(s, \chi \otimes \pi)} \\
&= \varepsilon(s, \chi, \psi, \pi)\frac{\zeta(w_1 g, W, \omega^{-1}\chi^{-1}, 1-s)}{L(s, \chi \otimes \pi)}
\end{aligned}$$

and from this the functional equation follows immediately. It can be seen that the global ε-factor is independent of choice of ψ and hence we may write $\varepsilon(s, \chi, \pi)$ for $\varepsilon(s, \chi, \psi, \pi)$. $\square$

3.5 Automorphic forms as functions on Adele group

Let χ be a primitive Dirichlet character mod N. Let ω be the adelization of χ which means $\omega = \prod_\nu \omega_\nu$ is a character of $\mathbb{A}^\times/\mathbb{Q}^\times$ of finite order and if p is a rational prime not dividing N, if ν is the place of $\mathbb{Q}$ corresponding to p and $\varpi_\nu \in \mathcal{O}_\nu$ be a generator of the maximal ideal in $\mathcal{O}_\nu$, then $\chi(p) = \omega_\nu(\varpi_\nu)$. It is also known that ω_∞ is trivial on $\mathbb{R}_+^\times$ and if ν is a non-archimedean place not dividing N, then ϖ_ν is unramified (that is, trivial on $\mathcal{O}_\nu^\times$) and if ν is a non-archimedean place dividing N, then ω_ν is trivial on the subgroup of $\mathcal{O}_\nu^\times$ consisting of the elements congruent to identity modulo N.

In general, if d is an integer prime to N, then

$$\chi(d) = \prod_{\nu \notin S_f(N)} \omega_\nu(d_\nu).$$

Now because $\omega = \prod_\nu \omega_\nu$ is trivial on $\mathbb{Q}^\times$, we get if $(d, N) = 1$ then

$$\chi(d) = \prod_{\nu \in S_f(N)} \omega_\nu^{-1}(d_\nu). \tag{3.5.1}$$

We define a character λ of $K_0(N)$ by

$$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \prod_{\nu \in S_f(N)} \omega_\nu(d_\nu) \tag{3.5.2}$$

Let $f \in S_k(N, \chi)$. If $g \in GL(2, \mathbb{A})$, then strong approximation theorem can be used to write g $= \gamma g_\infty k_0$, where $\gamma \in GL(2, \mathbb{Q})$, $g_\infty \in GL(2, \mathbb{R})^+$ and $k_0 \in K_0(N)$. Define a function ϕ_f on $GL(2, \mathbb{A})$ by

$$\phi_f(g) = F(g_\infty)\lambda(k_0) \tag{3.5.3}$$

where $F(g_\infty) = (f|_k g_\infty)(i)$.

Let us check that this is well defined. We must show that if $g_\infty, \tilde{g}_\infty \in GL(2, \mathbb{R})^+$, $\gamma \in GL(2, \mathbb{Q})$ and $k_0 \in K_0$ such that $g_\infty = \gamma \tilde{g}_\infty k_0$, then

$$F(g_\infty) = F(\tilde{g}_\infty)\lambda(k_0).$$

Write $\gamma = \gamma_\infty \gamma_f$, where $\gamma_\infty \in GL(2, \mathbb{R})$ and $\gamma_f \in GL(2, \mathbb{A}_f)$. Evidently $g_\infty = \gamma_\infty \tilde{g}_\infty$ and $\gamma_f = k_0^{-1}$. The first relation implies that γ_∞ has positive determinant and the second that it is in $\Gamma_0(N)$. Thus we have

$$F(g_\infty) = \chi(d_\infty) F(\tilde{g}_\infty), \quad \gamma_\infty = \begin{pmatrix} a_\infty & b_\infty \\ c_\infty & d_\infty \end{pmatrix},$$

so what we require is that $\lambda(k_0) = \chi(d_\infty)$, and because $k_0 = \gamma_f^{-1}$, this follows from (3.5.3).

The function ϕ defined by (3.5.4) is an automorphic form with a central quasicharacter ω. It must be checked that if $z \in \mathbb{A}^\times$, then

$$\phi\Big(\begin{pmatrix} z & \\ & z \end{pmatrix} g \Big) = \omega(z)\phi(g) \tag{3.5.4}$$

But

$$\mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}_+^\times \prod_{\nu \ non-archimedean} \mathcal{O}_\nu^\times$$

So using the above equation it is sufficient to check (3.5.4) for individual case, namely $z \in \mathbb{Q}^\times$, $z \in \mathbb{R}_+^\times$, or when $z \in \mathcal{O}_\nu^\times$ for ν non-archimedean which can be done easily.

Theorem 3.5.5. *Suppose that f is an eigenfunction of all the Hecke operators T_p when $p \nmid N$.Then ϕ lies in an irreducible subspace of $L_0^2(GL(2,F)\backslash GL(2,\mathbb{A}), \omega)$.*

Proof. See Theorem 3.6.1, §3.6, [Bu]. □

3.6 Hecke L-function versus Jacquet-Langlands L-function

Now suppose $\pi = \bigotimes \pi_p$ is the representation of $GL(2, \mathbb{A}_\mathbb{Q})$ generated by the cusp form $f(z)$ in $S_k(SL(2,\mathbb{Z}))$ and let $\mathfrak{p} = p\mathbb{Z}_p$

Proposition 3.6.1. $p^{\frac{k}{2}-1}\pi(T(\mathfrak{p}))(\phi_f) = \phi_{T(p)f}$

Proof. See Lemma 3.7, §3, [Ge]. □

Hence the eigenvalue for $T(p)$ on $f(z)$ is:

$$a_p = p^{\frac{k-1}{2}}(p^{s_1} + p^{-s_1}).$$

(Assuming $\pi_p \equiv \mathcal{B}(|x|_p^{s_1}, |x|_p^{-s_1})$.)

By Jacquet-Langlands' Theory,

$$L(s,\pi) = \prod_{p\leq\infty} L(s,\pi_p) = (2\boldsymbol{\pi})^{-s-\frac{k-1}{2}}\Gamma(s+\frac{k-1}{2})\prod_{p<\infty}(1-p^{s_1}p^{-s})^{-1}(1-p^{-s_1}p^{-s})^{-1} \tag{3.6.2}$$

$$= (2\boldsymbol{\pi})^{-s-\frac{k-1}{2}}\Gamma(s+\frac{k-1}{2})\prod_{p<\infty}(1-p^{-s-\frac{k-1}{2}}a_p + p^{-2s})^{-1}. \tag{3.6.3}$$

Hence the L-function $L(s,\pi)$ which agrees with Hecke's L-function

$$\Lambda(s',f) = (2\boldsymbol{\pi})^{-s'}\Gamma(s')L(s',f)$$

with $s' = s + \frac{k-1}{2}$. So we have

Theorem 3.6.4. *Let f be a holomorphic cusp form of weight k on $SL(2,\mathbb{Z})$ and let $\pi = \bigotimes \pi_p$ be the representation of $GL(2,\mathbb{A}_{\mathbb{Q}})$ generated by f. Furthermore let χ be the primitive Dirichlet character which corresponds to a Hecke character $\tilde{\chi}$. Then*

$$L(s,\tilde{\chi}\otimes\pi) \ = \Lambda(s';f,\chi) \tag{3.6.5}$$

where $s' = s + \frac{k-1}{2}$.

Chapter 4

p-adic L-function for modular forms

4.1 Construction of the non-archimedean measure

Let $\varphi(z) = \sum_{n=1}^{\infty} b_n e^{2\pi i n z}$ be a holomorphic cusp form of weight $w+2$ for $SL(2,\mathbb{Z})$. We assume that φ is a normalized Hecke eigenform, i.e., $b_1 = 1$ and $\varphi|T(n) = b_n\varphi$ for all $n \geq 1$,where $T(n)$ is the n^{th} Hecke operator.

For any Dirichlet character χ we define the twisted L-function

$$L(s;\varphi,\chi) = \sum_{n=1}^{\infty} \chi(n) b_n n^{-s} = \frac{(2\pi)^s}{\Gamma(s)} \int_0^{\infty} \varphi_\chi(iz) z^{s-1} dz \tag{4.1.1}$$

where $\varphi_\chi(z) = \sum_{n=1}^{\infty} \chi(n) b_n e^{2\pi i n z}$.

Let p be an odd prime and χ be a primitive Dirichlet character modulo p^m, $m \geq 0$. We define the Gauss sum attached to χ as follows

$$G(\chi) = \sum_{b \; mod \; p^m} \chi(b) e^{2\pi i \frac{b}{p^m}}. \tag{4.1.2}$$

We will be using the following property of $G(\chi)$

$$G(\chi)G(\chi^{-1}) = \chi(-1)p^m.$$

For a proof of this see, [Mi] chapter 3, section 1.

Furthermore, we denote by ρ a root of the equation $x^2 - b_p x + p^{w+1} = 0$.

Theorem 4.1.3 (Manin)**.** *There exist numbers $\omega^\pm \in \mathbb{C}$ such that all the values*

$$\frac{1}{\omega^\pm} \int_0^{i\infty} \varphi_\chi(y) y^t dy \tag{4.1.4}$$

are algebraic integers (for $t = 0, 1, \dots, w$) if the subscript on $\omega^\pm$ is same as in the formula $(-1)^t \chi(-1) = \pm 1$. Moreover, these numbers are rational over the field $\mathbb{Q}(\chi(\mathbb{Z}), G(\chi), b_1, b_2, \dots)$.

Proof. Refer to Theorem 1.2, [Ma1]. $\square$

Definition 4.1.5. $Q_k(x) = \int_0^{i\infty} \varphi(z + x) z^k dz$, $x \in \mathbb{Q}$, $k \geq 0$.

We list some of the properties of $Q_k(x)$.

1. $Q_k(x)$ is well defined.
2. $Q_k(x + 1) = Q_k(x)$ for all x.
3. $\sum_{r=0}^{p-1} Q_k(\frac{x+r}{p}) = b_p p^{-k} Q_k(x) - p^{w-2k} Q_k(px)$ for all x.
4. $Q_k(x) + Q_k(-x) = \begin{cases} 2Re Q_k(x) & \text{if } n \equiv 1 \pmod 2 \\ 2iIm Q_k(x) & \text{if } n \equiv 0 \pmod 2). \end{cases}$

For the proofs, refer to [Ma2].

Let K be a finite extension of the field $\mathbb{Q}_p$ containing the values of the character χ, ρ, $G(\chi)$ and all b_n. Let $\mathcal{O}$ be the ring of integers in K, and let $\mathfrak{m}$ be the maximal ideal in $\mathcal{O}$. Write $\chi = \chi_0 \chi_1$, where χ_0 is the tame component and χ_1 is the wild component, i.e., the conductor of χ_0 is relatively prime to p, and the conductor of χ_1 is a p-power. Let $t = \chi_1(1 + p) - 1$, then $t \in \mathfrak{m}$, in such a case denote χ_1 by $\chi_{(t)}$.

Definition 4.1.6. $\mu_k(a + (p^m)) = \rho^{-m} p^{mk} Q_k(\frac{a}{p^m}) - \rho^{-(m+1)} p^{mk-k+w} Q_k(\frac{a}{p^{m-1}})$.

Theorem 4.1.7. *μ_k is a K-valued distribution on $\mathbb{Z}_p^\times$ and if ρ is a p-adic unit, then $\mu_k(U)$ is bounded for compact-open subsets U of $\mathbb{Z}_p^\times$ and hence μ_k is a K-valued measure on $\mathbb{Z}_p^\times$.*

Proof. To see that μ_k is a distribution, we must check that

$$\mu_k(a+(p^m)) = \sum_{r=0}^{p-1} \mu_k(a+rp^m+(p^{m+1})).$$

So, let's simplify the right hand side and for this we will be using property (1), (3) of $Q_k(x)$.

$$\begin{aligned}
\sum_{r=0}^{p-1} \mu_k(a+rp^m+(p^{m+1})) &= \sum_{r=0}^{p-1} \rho^{-(m+1)} p^{(m+1)k} Q_k\Big(\frac{a+rp^m}{p^{m+1}}\Big) \\
&\quad - \sum_{r=0}^{p-1} \rho^{-(m+2)} p^{(m+1)k-k+w} Q_k\Big(\frac{a+rp^m}{p^m}\Big) \\
&= \rho^{-(m+1)} p^{mk}\Big(b_p p^{-k} Q_k\Big(\frac{a}{p^m}\Big) - p^{w-2k} Q_k\Big(\frac{a}{p^{m-1}}\Big)\Big) \\
&\quad - \rho^{-(m+2)} p^{(m+1)k-k+(w+1)} Q_k\Big(\frac{a}{pp^m}\Big) \\
&= \rho^{-(m+2)} p^{mk} (\rho b_p - p^{w+1}) Q_k\Big(\frac{a}{p^m}\Big) \\
&\quad - \rho^{-(m+1)} p^{mk-k+w} Q_k\Big(\frac{a}{p^{m-1}}\Big) \\
&= \rho^{-m} p^{mk} Q_k\Big(\frac{a}{p^m}\Big) - \rho^{-(m+1)} p^{mk-k+w} Q_k\Big(\frac{a}{p^{m-1}}\Big) \\
&= \mu_k(a+(p^m)).
\end{aligned}$$

For a proof of boundedness of μ_k when ρ is a p-adic unit we refer to [Ma1]. □

Now define

$$Q_k^+(x) = \frac{i}{\omega^+} Im\, Q_k(x), \;\; Q_k^-(x) = \frac{1}{\omega^-} Re\, Q_k(x),$$

From Theorem 1.3 and properties of $Q_k(x)$ it follows immediately that for $0 \le k \le w$, $Q_k^{\pm}(x)$ are algebraic integers over the field $\mathbb{Q}(b_1, b_2, \dots)$. Therefore, we can construct two K-valued measures $\mu_k^{\pm}$ as:

$$\mu_k^{\pm}(a+(p^m)) = \rho^{-m} p^{mk} Q_k^{\pm}\Big(\frac{a}{p^m}\Big) - \rho^{-(m+1)} p^{mk-k+w} Q_k^{\pm}\Big(\frac{a}{p^{m-1}}\Big). \tag{4.1.8}$$

Lemma 4.1.9. $\varphi_\chi(z) = \frac{G(\chi)}{p^m} \sum_{a \bmod p^m} \chi^{-1}(-a)\varphi(z+\frac{a}{p^m})$.

Proof.

$$\begin{aligned}
\frac{G(\chi)}{p^m}\sum_{a \bmod p^m}\chi^{-1}(-a)\varphi\Big(z+\frac{a}{p^m}\Big) &= \frac{G(\chi)}{p^m}\sum_{a \bmod p^m}\chi^{-1}(-a)\sum_{n=0}^{\infty}b_n e^{2\pi inz}e^{2\pi in\frac{a}{p^m}}\\
&= \frac{G(\chi)}{p^m}\sum_{n=0}^{\infty}b_n e^{2\pi inz}\sum_{a \bmod p^m}\chi^{-1}(-a)e^{2\pi in\frac{a}{p^m}}\\
&= \frac{G(\chi)}{p^m}\sum_{n=0}^{\infty}\chi(n)b_n e^{2\pi inz}\sum_{a \bmod p^m}\chi^{-1}(-na)e^{2\pi in\frac{a}{p^m}}\\
&= \frac{G(\chi)}{p^m}\sum_{n=0}^{\infty}\chi(n)b_n e^{2\pi inz}\sum_{\tilde{a} \bmod p^m}\chi^{-1}(-\tilde{a})e^{2\pi i\frac{\tilde{a}}{p^m}}\\
&= \frac{G(\chi)}{p^m}\varphi_\chi(z)G(\chi^\star)\ \ (\textit{where}\ \ \chi^\star(a)=\chi^{-1}(-a))\\
&= \varphi_\chi(z).
\end{aligned}$$

□

Theorem 4.1.10. *For all $k=0,\dots,w$, in case ρ is a p-adic unit, we have*

$$\frac{p^m}{G(\chi)}\frac{1}{\omega^\pm}\int_0^{i\infty}\varphi_\chi(z)z^k dz \;=\; \frac{\rho^m}{p^{mk}}\int_{\mathbb{Z}_p^\times}\chi^{-1}(-a)d\mu_k^\pm(a), \tag{4.1.11}$$

where the superscripts on $\omega^\pm$ and $\mu_k^\pm$ are taken as in the formula $(-1)^k\chi(-1)=\pm 1$.

Proof. Let's first evaluate the archimedean integral

$$\begin{aligned}
\frac{p^m}{G(\chi)}\int_0^{i\infty}\varphi_\chi(z)z^k dz &= \int_0^{i\infty}\Big(\sum_{a \bmod p^m}\chi^{-1}(-a)\varphi\Big(z+\frac{a}{p^m}\Big)z^k\Big)dz\\
&= \sum_{a \bmod p^m}\chi^{-1}(-a)\int_0^{i\infty}\varphi\Big(z+\frac{a}{p^m}\Big)z^k dz\\
&= \sum_{a \bmod p^m}\chi^{-1}(-a)Q_k\Big(\frac{a}{p^m}\Big)
\end{aligned}$$

$$= \frac{1}{2} \sum_{a \bmod p^m} \Big(\chi^{-1}(-a) Q_k\Big(\frac{a}{p^m}\Big) + \chi^{-1}(a) Q_k\Big(\frac{-a}{p^m}\Big)\Big)$$
$$= \omega^{\pm} \sum_{a \bmod p^m} \chi^{-1}(-a) Q_k^{\pm}\Big(\frac{a}{p^m}\Big).$$

Let's now calculate the non-archimedean integral, because χ is constant on cosets modulo p^m,the integral will actually be a finite sum, i.e.,

$$\frac{\rho^m}{p^{mk}} \int_{\mathbb{Z}_p^\times} \chi^{-1}(-a) d\mu_k^{\pm}(a) = \frac{\rho^m}{p^{mk}} \Big(\sum_{a \bmod p^m} \chi^{-1}(-a) \mu_k^{\pm}(a + (p^m))\Big)$$
$$= \sum_{a \bmod p^m} \chi^{-1}(-a) \mathbb{Q}_k^{\pm}\Big(\frac{a}{p^m}\Big) - \frac{p^{w-k}}{\rho} \sum_{a \bmod p^m} \chi^{-1}(-a) \mathbb{Q}_k^{\pm}\Big(\frac{a}{p^{m-1}}\Big).$$

claim: $\sum_{a \bmod p^m} \chi^{-1}(-a) \mathbb{Q}_k^{\pm}\Big(\frac{a}{p^{m-1}}\Big) = 0.$

This is because one can split the sum on a mod p^m into residue classes mod p^{m-1} and since $Q_k^{\pm}$ has period 1, it is constant on residue classes modulo p^{m-1}, and since χ is a primitive character mod p^m, so is $\chi^\star$ defined by $\chi^\star(a) = \chi^{-1}(-a)$. Hence,

$$\sum_{\substack{\tilde{a} \equiv a \bmod p^{m-1} \\ \tilde{a} \bmod p^m}} \chi^\star(\tilde{a}) = 0.$$

So,

$$\frac{\rho^m}{p^{mk}} \int_{\mathbb{Z}_p^\times} \chi^{-1}(-a) d\mu_k^{\pm}(a) = \sum_{a \bmod p^m} \chi^{-1}(-a) \mathbb{Q}_k^{\pm}\Big(\frac{a}{p^m}\Big). \tag{4.1.12}$$

Now the theorem is evident from the above calculations. □

4.2 p-adic interpolation of critical values of $L(s,\Delta)$

The Ramanujan Δ-function is defined as

$$\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n,\ q = e^{2\pi i z},\ Im(z) > 0$$

It is, up to scalar multiples, the unique weight 12 holomorphic cusp form for $SL(2,\mathbb{Z})$. Since $S_{12}(SL(2,\mathbb{Z}))$ is one dimensional, hence by the theory of Hecke operators, the

Fourier coefficients $\tau(n)$ are multiplicative and hence the L-function of Δ, defined initially as a Dirichlet series, admits an Euler product:

$$L(s,\Delta) := \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p (1-\tau(p)p^{-s} + p^{11-2s})^{-1},\ Re(s) \gg 0.$$

The reciprocal of the Euler factor at p may be factored as:

$$1-\tau(p)p^{-s} + p^{11-2s} = (1-\alpha_p p^{-s})(1-\beta_p p^{-s}),$$

with, $\alpha_p + \beta_p = \tau(p)$ and $\alpha_p\beta_p = p^{11}$.

Now let $\pi(\Delta)$ denote the cuspidal automorphic representation of $GL(2,\mathbb{A})$ attached to Δ, where $\mathbb{A}$ is the adele ring of $\mathbb{Q}$. Hence,

$$\pi(\Delta) = \bigotimes_{\nu \leq \infty} \pi_\nu(\Delta),$$

where for $\nu = p < \infty$,

$$\pi_\nu(\Delta) \equiv \mathcal{B}(|x|_p^{s_p}, |x|_p^{-s_p})$$

such that $\tau(p) = p^{\frac{11}{2}}(p^{s_p} + p^{-s_p})$.

Then from Chapter 2, we know the relation between the Jacquet-langlands L-function and the classical L-function of Δ. Namely,

$$L(s,\pi(\Delta)\otimes\chi) = (2\boldsymbol{\pi})^{(-s-\frac{11}{2})}\Gamma(s+\frac{11}{2})L(s+\frac{11}{2},\Delta\otimes\chi), \tag{4.2.1}$$

where χ is a primitive Dirichlet charcater.

Now fix an odd prime p, such that one of the roots ρ of the equation $x^2 - \tau(p) + p^{11}$ is a p-adic unit. Then, the p-adic measure associated with Δ is given by

$$\mu_k(a+(p^m)) = \rho^{-m}p^{mk}Q_k\Big(\frac{a}{p^m}\Big) - \rho^{-(m+1)}p^{mk-k+w}Q_k\Big(\frac{a}{p^{m-1}}\Big).$$

Then using 4.1.1 and Theorem 4.1.10, we obatin, for a primitive Dirichlet character χ mod p^m

$$\frac{i^{k+1}p^m}{G(\chi)}\frac{\Gamma(k+1)L(k+1,\Delta\otimes\chi)}{\omega^{\pm}(2\boldsymbol{\pi})^{k+1}} = \frac{\rho^m}{p^{mk}}\int_{\mathbb{Z}_p^\times} \chi^{-1}(-a)d\mu_k^{\pm}(a), \tag{4.2.2}$$

where the superscripts on $\omega^{\pm}$ and $\mu_k^{\pm}$ are taken as in the formula $(-1)^k\chi(-1) = \pm 1$. This equation is valid for $k \in \{1, 2, \ldots, 11\}$. Now putting $k = 5$ and using eq.(4.2.1) along with (4.2.2), we obtain the following

Theorem 4.2.3.

$$\frac{L(\frac{1}{2}, \pi(\Delta) \otimes \chi)}{\omega^{\pm}\ G(\chi)} = -\frac{\rho^m}{p^{6m}} \int_{\mathbb{Z}_p^{\times}} \chi^{-1}(-a) d\tilde{\mu}^{\pm}(a) \tag{4.2.4}$$

where, χ is a primitive Dirichlet character mod p^m, and $\tilde{\mu}^{\pm}(a + (p^n)) = \mu_5^{\pm}(a + (p^n))$ and where the superscripts on $\omega^{\pm}$ and $\tilde{\mu}^{\pm}$ are taken as in the formula $-\chi(-1) = \pm 1$.

Theorem 4.2.5. *Let p be an odd prime. Let μ be a bounded K-valued measure on $\mathbb{Z}_p$. Then there exists a unique power series $A = A_\mu \in K[[T]]$ convergent for all $t \in \mathfrak{m}$, such that for any primitive Dirichlet character χ of conductor p^m we have*

$$\int_{\mathbb{Z}_p^{\times}} \chi\ d\mu = A_\mu(t) \tag{4.2.6}$$

where $t = \chi(1 + p) - 1$.

Proof. See theorem 8.7, [Ma1]. See also the discussion in §2.3 . □

Now we want to apply the above theorem under the hypothesis of Theorem 4.2.3, since in that case χ is a character mod p^m, so is $\chi^\star$ ($\chi^\star(a) = \chi^{-1}(-a)$). Let the power series associated to the measure $\mu_5^{\pm}$ via Theorem 4.2.5 be $A^{\pm}(T)$. Then by comparing eq. (4.2.4) and (4.2.6) we obtain the following

Theorem 4.2.7.

$$\frac{L(\frac{1}{2}, \pi(\Delta) \otimes \chi)}{\omega^{\pm}\ G(\chi)} = -\frac{\rho^m}{p^{6m}} A^{\pm}(\chi^\star(1 + p) - 1) \tag{4.2.8}$$

where, $\chi^\star(a) = \chi^{-1}(-a)$ and the superscripts on $\omega^{\pm}$ and $A^{\pm}$ are taken as in the formula $-\chi(-1) = \pm 1$.

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