A Survey of Direct Methods in Calculus of Variations

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by

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This is to certify that this thesis entitled “A Survey of Direct Methods in Calculus of Variations" submitted towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune, represents work carried out by Roshni Namdeo Patil under the supervision of Prof. Prashanth K. S.

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Abstract

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by Roshni Namdeo Patil

In this thesis, we demonstrate the applications of direct methods of minimization in finite as well as infinite dimensional space. The weak lower semicontinuity (or continuity) of a functional is pivotal for existence of a minimum of a functional. This thesis also develops the method of Lagrange Multipliers which allows extrimization of constrained functionals by presenting Euler-Lagrange equation. This work also gives a general proof of Ekeland's Variational Principle which gives a criterion for existence of a minimizing sequence under certain assumptions.

This thesis introduces Sobolev spaces in one dimension and explores the conditions under which a well-defined integral functional achieves a minimum. The weak lower semicontinuity of the functional in a Sobolev space is necessary for concluding an existence result.

Contents

Introduction

The calculus of variations is a branch of mathematics concerned with solving optimization problems for a function of one or more variables. It is an immense and very active field within which the direct method is a general procedure for constructing a proof of existence of a minimizer (maximizer). The direct methods were introduced by Zaremba and David Hilbert around 1900. Some of its applications include optimal control and minimal surfaces. It can be used to find the path, curve, surface, etc., for which a given function has a stationary value.

An outline of this thesis is as follows. Chapter 1 gives a synopsis of the sufficient conditions for existence of a minimum (or maximum) of a functional defined on a finite dimensional space. It briefly states that a lower semicontinuous, bounded below, coercive functional has a global minimum. The method of Lagrange multipliers is also introduced in case of constrained functionals. Chapter 2 describes the sufficient conditions for existence of a minimum of a functional defined on a reflexive Banach space. It also includes a very general proof of Ekeland's Variational Principle which says that, given a minimizing sequence, it is possible to obtain another sequence close to it which can give more accurate minimum value of a funcional. Chapter 3 talks about some of the most popular problems in calculus of variations. For example, the shortest path and the minimum area of surface of revolution problem. The chapter develops a general method of solving such problems by introducing Euler-Lagrange equations for constrained and unconstrained functionals. Different methods for different types of constraints, like integral side condition and finite side condition have been discussed in this chapter. Also it enhances the beauty of Lagrange multiplier method by giving several physical examples. Finally in chapter 4, beginning with a motivation, Sobolev spaces are defined in one dimension, by considering a simple minimization problem of one dimensional Dirichlet integral in a unit interval. It explores weak convergence in Sobolev spaces followed by the discussion of necessary and sufficient conditions for the integral functional to be sequentially weakly lower semicontinuous. The thesis completes by giving a final existence result and a couple of examples.

Chapter 1

Direct Extremizing Methods in Finite Dimension

1.1 Introduction

Calculus of variations is a field of mathematical analysis that deals with maximizing or minimizing functionals which are collectively called extrema. The analogous problem in calculus is to determine the point(s) in the domain of a function, that yields the minimum or maximum value of the function. In this chapter, we look over the conditions that must be satisfied by a function in order to exhibit an extremum.

1.2 Extrema of Finite Dimensional Functionals

To begin with, let us review some definitions from single variable calculus.

Definition 1.1. A real function f defined on a set X is said to have a global minimum (maximum) at $x^* \in \mathbb{R}$ if

$$f(x^*) \leq (\geq) f(x) \quad \forall x \in \mathbb{R}.$$

Example 1.1.

$$f(x) = |x|$$

From Figure 1.1, it is evident that at $x^* = 0$ the function attains its minimum value. Let us check this with Definition 1.1. $f(0) = 0 \leq |x| = f(x)$. Therefore $f(x^*) \leq f(x) \quad \forall x \in \mathbb{R}$. Thus $f(x)$ indeed has a minimum at $x = 0$.

The above example illustrates that a function need not be differentiable for a minimum to exist.

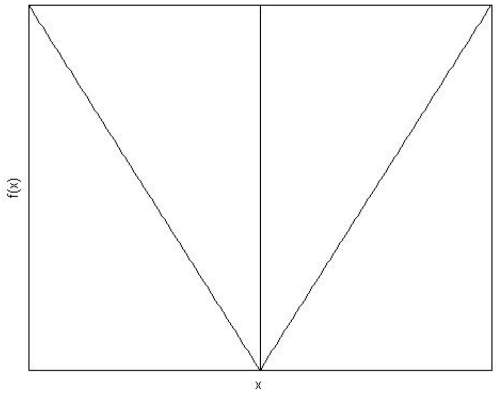

Figure 1.1: $f(x) = |x|$

Example 1.2. Define $g : \mathbb{R} \to \mathbb{R}$ as $g(x) = x^3$. Let $x^* \in \mathbb{R}$ be such that $g(x^*) \leq g(x)\ \forall x \in \mathbb{R}$. That is,

$$(x^*)^3 \leq x^3 \quad \forall x \in \mathbb{R}. \tag{1.1}$$

But note that $(x^* - 1)^3 \leq (x^*)^3$. This implies that (1.1) is not true $\forall x \in \mathbb{R}$. Thus $g(x) = x^3$ does not have a minimum according to Definition 1.1.

Now consider the following example.

Example 1.3.

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Clearly, the minimum of $f(x)$ is 0 and it occurs at all negative x. This example illustrates that it is not necessary for a function to be continuous for a minimum to exist.

Recall that in Definition 1.1 the essential condition for a minimum to exist was $f(x^*) \leq f(x)\ \ \forall x \in \mathbb{R}$. This automatically implies that $f(x)$ should be bounded below. In addition to this, we can impose one more condition on $f(x)$ which will ensure that $f(x)$ exhibits minima.

Suppose $f(x) \to \infty$ as $|x| \to \infty$. Then $f(x)$ can be shown to have a minimum (see Theorem 1.1). A simplest example which will verify this statement is given below.

Example 1.4.

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) = x^2$$

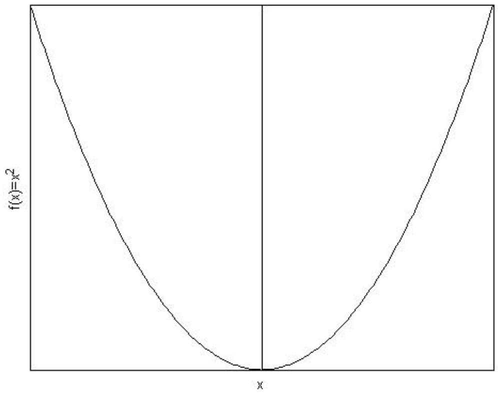

Figure 1.2: $f(x) = x^2$

The above function has a minimum at $x = 0$. The fact that $f(x) \to \infty$ as $|x| \to \infty$ is clearly noticeable from Figure 1.2.

The above argument can be summarized in terms of a theorem as given below.

Theorem 1.1. *If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying:*

(i) f is bounded below,

(ii) if $f(x) \to \infty$ as $|x| \to \infty$.

Then f has a global minimum.

Before proving the theorem, let us look at an example that focuses on the relevance of condition (ii).

Example 1.5.

$$h(x) = e^{-x}$$

From Figure 1.3, it can be seen that $h(x) \to \infty$ as $x \to -\infty$. However, $h(x) \to 0$ as $x \to \infty$. Thus the condition (ii) of Theorem 1.1 is not satisfied. The function $h(x)$ approaches the minimum value of zero at $x = \infty$. Therefore, $h(x)$ does not have a minimum.

Proof of Theorem 1.1. Since f is bounded below, the infimum of f exists. Let $\alpha = \inf f$. Then by definition of infimum there exists a sequence $\{x_\epsilon\} \subset \mathbb{R}$ such that for any $\epsilon > 0$

$$f(x_\epsilon) - \alpha < \epsilon \tag{1.2}$$

But as $\alpha \leq f(x_\epsilon)$, we have

$$\alpha - f(x_\epsilon) \leq 0 < \epsilon. \tag{1.3}$$

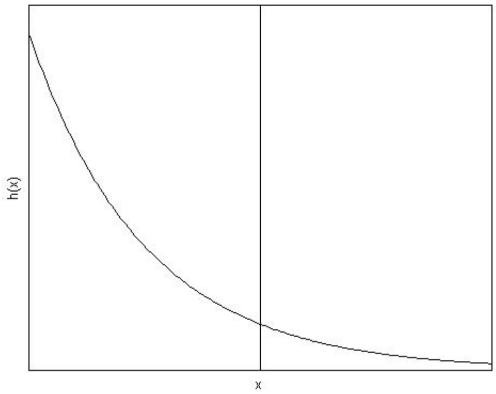

Figure 1.3: $h(x) = e^{-x}$

From (1.2) and (1.3),

$$|f(x_\epsilon) - \alpha| < \epsilon.$$

Now, let $\epsilon = \frac{1}{n}, n \in \mathbb{N}$ to get a sequence $\{x_n\} \subset \mathbb{R}$ such that, $|f(x_n) - \alpha| < \frac{1}{n}$. This implies,

$$f(x_n) \to \alpha \text{ as } n \to \infty. \tag{1.4}$$

Now by condition (ii) $\{x_n\}$ is a bounded sequence. Therefore, by Bolzano-Weierstrass theorem, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Let $x_{n_k} \to x^*$. Since f is continuous, $f(x_{n_k}) \to f(x^*)$. Therefore from (1.4), $\alpha = f(x^*)$. Thus, f attains its infimum at $x = x^*$. In other words, f has a global minimum at $x = x^*$. ■

Remark 1.1. An important ingredient in the above proof is that bounded sequences in $\mathbb{R}$ have convergent subsequences .i.e. $\mathbb{R}$ is locally compact.

It is worthy to note that assumptions (i) and (ii) are not necessary. The following example analyzes this statement.

Example 1.6. $f : \mathbb{R} \to \mathbb{R}$.

$$f(x) = \begin{cases} 0 & x \in (-\infty, -a] \\ (a^2 - x^2)^{\frac{1}{2}} & x \in (-a, a) \\ 0 & x \in [a, \infty). \end{cases}$$

As from Figure 1.4, the above function is bounded below and has a global minimum value of zero. However, $f(x) \nrightarrow \infty$ as $x \to \pm\infty$.

We now make an important relaxation of assumption (ii) in Theorem 1.1 by considering lower semi-continuous functions. The theorem can be shown to hold in this situation as

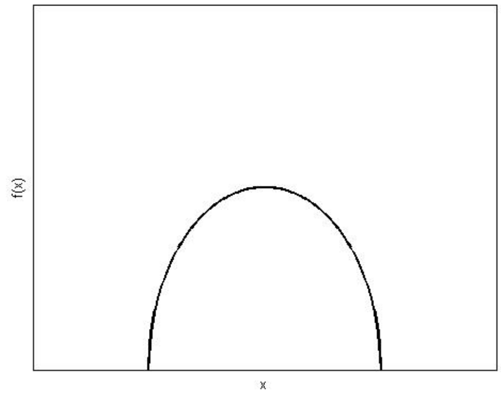

Figure 1.4: $f(x)$

well. Before proving this fact let us review the definition of lower semi-continuous functions.

Definition 1.2 (Lower semi-continuous functions)**.** Let X be a metric space. A function $f : X \to \mathbb{R}$ is lower semi-continuous at x if for every sequence $\{x_n\}$ in X with $x_n \to x$, we have $\liminf_{n\to\infty} f(x_n) \geq f(x)$.

Theorem 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a lower semi-continuous function satisfying:*

(i) f is bounded below,

(ii) if $f(x) \to \infty$ as $|x| \to \infty$.

Then f has a global minimum.

Proof. Let α be the infimum of f. Then by definition of infimum $\exists \{x_n\} \subset \mathbb{R}$ such that $f(x_n) \to \alpha$ as $n \to \infty$.
By condition (ii) $\{x_n\}$ is a bounded sequence. Thus by Bolzano-Weierstrass theorem, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Let $x_{n_k} \to x^*$. Now since f is lower semi-continuous, $\liminf_{k\to\infty} f(x_{n_k}) \geq f(x^*)$. We note that $\liminf_{k\to\infty} f(x_{n_k}) = \alpha$ and hence, $f(x^*) \leq \alpha$. Therefore $f(x^*) = \alpha$. Thus f attains its infimum at x^*. ■

Example 1.7. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases}$$

Consider the sequence $x_n = -\frac{1}{n}$. Then $x_n \to 0$, $\liminf_{n\to\infty} f(x_n) = 0$ and $f(0) = 1$. Hence by definition, f is not lower semi-continuous. Here $\inf_{\mathbb{R}} = 0$ and it is achieved by any negative x.

Example 1.8.

$$f(x) = \begin{cases} x^2 & x < 0, \\ 1 + x^2 & x \geq 0. \end{cases}$$

Consider a sequence $x_n = -\frac{1}{\sqrt{n}}$. Then $x_n \to 0$. $\liminf_{n\to\infty} f(x_n) = \liminf_{n\to\infty} \frac{1}{n} = 0$, while $f(0) = 1$. Therefore, by definition f is not lower semi-continuous. We see that $\lim_{|x|\to\infty} f(x) = \infty$. But $\inf f = 0$ and by the definition of f it is clear that this infimum is never achieved. Therefore, f does not have a global minimum.

Remark 1.2. If a lower semi-continuous function f is defined on a compact interval $[a, b] \subset \mathbb{R}$, we can show the existence of minimum by only assuming condition (i) of Theorem 1.2. This is because any sequence from this interval is bounded and the theorem can be proved by making similar arguments.

One can say that the condition (ii) in Theorem 1.1 is quite strong and need not hold everytime. It is quite interesting that the theorem holds even after weakening condition (ii) as in the following theorem.

Theorem 1.3. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a lower semi-continuous function satisfying:*

(i) f is bounded below,

(ii) $\exists$ a point $x_0 \in \mathbb{R}$ such that

$$f(x_0) < \min\left(\liminf_{x\to\infty} f, \liminf_{x\to-\infty} f\right).$$

Then f has a global minimum.

Proof. If $\alpha = \inf f$, then by definition of infimum $\exists\{x_n\} \subset \mathbb{R}$ such that $f(x_n) \to \alpha$ as $n \to \infty$. Therefore, $\alpha \leq f(x_0)$.

Claim: $\{x_n\}$ is a bounded sequence.

In order to prove the claim, assume on contrary that $\{x_n\}$ is not a bounded sequence. Then $|x_n| \to \infty$ as $n \to \infty$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. Then from condition (ii),

$$f(x_0) < \liminf_{k\to\infty} f(x_{n_k}) = \alpha.$$

But since $\alpha \leq f(x_0)$, we have,

$$\alpha \leq f(x_0) < \liminf_{k\to\infty} f(x_{n_k}) = \alpha.$$

This contradiction proves that $\{x_n\}$ is a bounded sequence. Then using lower semi-continuity of f, one can prove that f has a global minimum. ■

It is noteworthy that Theorems 1.1, 1.2 and 1.3 are true for a function of more than one variable as well, since Bolzano-Weierstrass theorem holds for each bounded sequence in $\mathbb{R}^n$.

Theorem 1.4. *Suppose a function $f : \mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous and satisfies the following conditions:*

(i) f is bounded below,

(ii) if $f(x) \to \infty$ as $||x|| \to \infty$, where $||\cdot||$ denotes Euclidean norm.

Then f has a global minimum.

Proof. Condition (i) implies that f has an infimum. Let $\alpha = \inf f$, then $f(x_n) \to \alpha$ as $n \to \infty$.
Condition (ii) implies that $\{x_n\} \subset \mathbb{R}^n$ is bounded. Therefore, by Bolzano Weierstrass theorem $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Using this information and the lower semi-continuity of f one can prove as before that there exists $x^* \in \mathbb{R}^n$ such that f attains a minimum at x^*. ■

Similarly, Theorem 1.3 is also true for the functions defined on $\mathbb{R}^n$.

Theorem 1.5. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a lower semi-continuous function satisfying:*

(i) f is bounded below,

(ii) $\exists$ a point $x_0 \in \mathbb{R}$ such that

$$f(x_0) < \min\left(\liminf_{x\to\infty} f, \liminf_{x\to-\infty} f\right).$$

Then f has a global minimum.

1.3 Lagrange Multipliers

In this section, we study minimization of functions subject to certain given constraints. Such problems are known as "Constrained problems". For example, consider a C^1 function $f : \mathbb{R}^n \to \mathbb{R}$. We are interested in finding out the minimum (or maximum) of f on a surface $\{g(x) = 0\} \subset \mathbb{R}^n$. One method of solving such problems is known as Lagrange Multiplier method.

Theorem 1.6 (The method of Lagrange Multipliers [4]). *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be C^1 functions. Let $Z = \{x : g(x) = 0\}$ and assume $\nabla g \neq 0$ on Z. If f has a local minimum (or maximum) on Z at x_0, then there exists a $\lambda \in \mathbb{R}$ such that*

$$\nabla f(x_0) = \lambda \nabla g(x_0).$$

The real number λ is known as Lagrange multiplier.

Proof. Let $\gamma : [-1,1] \to Z$ be a smooth curve such that $\gamma(0) = x_0$. Define a C^1 function $h : \mathbb{R}^n \to \mathbb{R}$, as

$$h(t) = f(\gamma(t)).$$

Therefore,

$$h(0) = f(x_0) = \min_Z f. \tag{1.5}$$

Thus, $h(t) = f(\gamma(t)) \geq f(x_0) \quad \forall t \in \mathbb{R}^n$. By chain rule, $\frac{dh}{dt} = \frac{\partial f}{\partial x_1}\frac{d\gamma_1}{dt} + ... + \frac{\partial f}{\partial x_n}\frac{d\gamma_n}{dt} = \nabla f(x).\frac{d\gamma}{dt}$. From (1.5), $0 = \frac{dh}{dt}(0) = \nabla f(x_0).\frac{d\gamma}{dt}(0)$. However, $\frac{d\gamma}{dt}(0)$ is tangent to Z because it is tangent to the curve γ on Z, which implies that $\nabla f(x_0)$ is in the direction normal to the manifold Z. But we know that $\nabla g(x_0)$ is also normal to Z at x_0. Therefore, there exists a $\lambda \in \mathbb{R}$ such that $\nabla f(x_0) = \lambda \nabla g(x_0)$. This completes the proof of the theorem. ∎

Most problems require extremization of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, whose variables are subject to more than one constraint. We give a general result below.

Theorem 1.7. *Let $f : \mathbb{R}^n \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$, $1 \leq i \leq m < n$ be C^1 functions. Let $Z = \{x : g_i = 0, 1 \leq i \leq m\}$ and assume $\nabla g_i \neq 0$, $\forall\ 1 \leq i \leq m$ on Z. If f has a local minimum (or maximum) on Z at x_0, then there exist constants $c_i \in \mathbb{R}$ such that $\nabla f(x_0) = \sum_{i=1}^{m} c_i \nabla g_i$.*

Example 1.9. Let $g_1 : \mathbb{R}^3 \to \mathbb{R}$ be defined as $g_1(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 1$ and $g_2 : \mathbb{R}^3 \to \mathbb{R}$ be defined as $g_2(x_1, x_2, x_3) = x_3$. Let $Z = \{(x_1, x_2, x_3) : g_1 = 0, g_2 = 0\}$. $\nabla g_1 = (2x_1, 2x_2, 2x_3) \neq 0$ if $x \neq 0$ and $\nabla g_2 = (0,0,1) \neq 0$. We can use Lagrange multiplier method and get the following equation at any maximum (or minimum) x_0 of f on Z.

$$\nabla f(x_0) = (2c_1(x_0)_1,\ 2c_1(x_0)_2,\ 2c_1(x_0)_3 + c_2).$$

Example 1.10. Find the maximum and minimum values of $f(x_1, x_2) = x_1^2 + x_2^2$ subject to $g(x_1, x_2) = x_1^2 + x_2^2 + x_1 x_2 - 9 = 0$.

Solution: Both f and g are C^1 functions. Let $Z = \{x : g(x) = 0\}$. $\nabla g = (2x_1 + x_2, 2x_2 + x_1)$. Note that $\nabla g = 0$ at $(x_1, , x_2) = (0,0)$ but $(0,0) \notin Z$. That is $\nabla g \neq 0$ on Z. Therefore, from Theorem 1.6, if f has a local minimum (or maximum) on Z at (x_1^*, x_2^*) then there exists $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_1^*, x_2^*) = \lambda \nabla g(x_1^*, x_2^*).$$

That is, $(2x_1^*, 2x_2^*) = \lambda(2x_1^* + x_2^*, 2x_2^* + x_1^*)$ or

$$2x_1^* = \lambda(2x_1^* + x_2^*)$$
$$2x_2^* = \lambda(2x_2^* + x_1^*).$$

Solving these equations simultaneously gives $x_2^* = x_1^*$ or $x_2^* = -x_1^*$. Substituting these into $x_1^2 + x_2^2 + x_1x_2 - 9 = 0$, we find that $(x_1^*, x_2^*) = (\sqrt{3}, \sqrt{3})$ or $(-\sqrt{3}, -\sqrt{3})$ or $(3, -3)$ or $(-3, 3)$, giving the minimum and maximum values of f as 6 and 18 respectively.

With these examples we end this chapter with the inference that a continuous (lower semicontinuous) functional, defined on finite dimensional domain, which is bounded below and is coercive attains its minimum. A similar result holds for an upper semicontinuous functional which is bounded above and coercive. In that case the functional attains its maximum in the finite dimensional domain. We will revisit the method of Lagrange Multipliers in chapter 3 where we study the extremization of a functional defined on an infinite dimensional domain.

Chapter 2

Direct Extremization Method in Infinite Dimensions

2.1 Introduction

In preceding chapter we discussed the necessary and sufficient conditions for existence of extremum of a function in finite dimensions. In this chapter we will study a similar approach with certain additional conditions on domain of the functional and the functional itself which assure the existence of its minimum.

Before we proceed, we give a few basic definitions from functional analysis and recall some important theorems.

Definition 2.1 (Weak convergence)**.** Let X be a normed space. We say that a sequence $\{x_n\} \subset X$ converges weakly to $x \in X$ if for every $f \in X'$ we have that $|f(x_n) - f(x)| \to 0$ as $n \to \infty$. Notationally,

$$x_n \xrightarrow{w} x.$$

Definition 2.2 (Weak lower semicontinuity)**.** Let X be a normed space. The functional $f : X \to \mathbb{R}$ is called weak (sequentially) lower semicontinuous at x if for every sequence x_n in X with $x_n \xrightarrow{w} x$, we have $f(x) \le \liminf_{n\to\infty} f(x_n)$.

Theorem 2.1. *Let X be a normed space. Then X is reflexive if and only if every bounded sequence in X has a weak convergent subsequence. (c.f. Theorem 16.5, [1])*

Theorem 2.2 (Cantor Intersection Theorem)**.** *Let X be a metric space. Then X is complete if and only if whenever F_n is a sequence of closed subsets of X such that $\lim_{n\to\infty} diam(F_n) = 0$, there is a point $x \in X$ such that $\bigcap_{n=1}^{\infty} F_n = \{x\}$.*

We now present a general result on the existence of a minimizer of a functional defined on infinite dimensional domain.

Theorem 2.3. *Let X be a reflexive Banach space. If $f : X \to \mathbb{R}$ is a functional satisfying:*

(i) f is bounded below,

(ii) f is coercive,

(iii) f is weakly lower semicontinuous,

then f attains its minimum in X.

Proof. By (i) infimum of f exists. Let $\alpha = \inf f$. Then by definition of infimum there exists a sequence $\{x_n\} \subset X$ such that $f(x_n) \to \alpha$ as $n \to \infty$. Since f is coercive, $\{x_n\}$ is a bounded sequence. Therefore by Theorem 2.1, $\{x_n\}$ has a weak convergent subsequence $\{x_{n_k}\}$, say. Let $x_{n_k} \xrightarrow{w} x^*$. Then, since f is weakly lower semicontinuous,

$$f(x^*) \leq \liminf_{k\to\infty} f(x_{n_k}) = \alpha$$

But $\alpha \leq f(x^*)$. Therefore, $\alpha = f(x^*)$. Thus f attains its infimum at x^*. ∎

Remark 2.1. In the above theorem, reflexivity of the Banach space X is necessary because in an infinite dimensional Banach space a bounded sequence does not necessarily contain a (weakly) convergent subsequence.

Example 2.1. Let Ω be an open subset of $\mathbb{R}^n$. Consider a sequence of functionals $\{f_n\} \subset L^1(\Omega)$. If

(i) $\sup ||f_n||_{L^1} < \infty$ and

(ii) $\{f_n\}$ is uniformly bounded, that is there exists $M \in \mathbb{R}$ such that

$$\int_\Omega |f_n| < M \quad \forall n \in \mathbb{N},$$

then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$f_{n_k} \xrightarrow{w} f \in L^1(\Omega).$$

Therefore uniform boundedness of f_n is necessary for convergence in L^1.

2.2 Ekeland's Variational Principle

It is important to know how far we can relax the conditions on f in Theorem 2.3 and still be certain that the minimum is achieved. For example, we drop the coercivity of f and the fact that X is a normed linear space in Theorem 2.3 and consider a problem of finding minimum of a lower semicontinuous, bounded below functional on a complete metric space. However, lower semicontinuity of f is required instead of weak lower semicontinuity.

The above problem can be posed in terms of the following theorem.

Theorem 2.4 (Ekeland Variational Principle). *Let X be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous functional which is bounded below. Let $\epsilon > 0$ and $x^* \in X$ be given such that*

$$f(x^*) \leq \inf_X f + \frac{\epsilon}{2}.$$

Then given $\lambda > 0$ there exists $x_\lambda \in X$ such that

$$f(x_\lambda) \leq f(x^*) \tag{2.1}$$

$$d(x_\lambda, x^*) \leq \lambda \tag{2.2}$$

$$f(x_\lambda) < f(x) + \frac{\epsilon}{\lambda} d(x, x_\lambda) \ \ \forall x \neq x_\lambda \tag{2.3}$$

Proof. Let us denote $d_\lambda(x, y) = \frac{1}{\lambda} d(x, y)$. Define a binary relation "$\leq$" in X by

$$x \leq y \Leftrightarrow f(x) \leq f(y) - \epsilon d_\lambda(x, y).$$

The relation "$\leq$" defined as above is a partial order since for all x, y and z in X, it is

(i) reflexive: $x \leq x$,
(ii) antisymmetric: if $x \leq y$ and $y \leq x$ then $f(x) \leq f(y) - \epsilon d_\lambda(x, y)$ and $f(y) \leq f(x) - \epsilon d_\lambda(x, y)$ implies $f(x) \leq f(x) - 2\epsilon d_\lambda(x, y)$, that is, $d_\lambda(x, y) = 0$. Therefore, $x = y$, and
(iii) transitive: if $x \leq y$ and $y \leq z$ then $f(x) \leq f(y) - \epsilon d_\lambda(x, y)$ and $f(y) \leq f(z) - \epsilon d_\lambda(y, z)$. Therefore, $f(x) \leq f(z) - \epsilon d_\lambda(y, z) - \epsilon d_\lambda(x, y) \leq f(z) - \epsilon d_\lambda(x, z)$. Hence $x \leq z$.

We now construct the sequences $\{x_n\}$ and $\{S_n\} \subset X$ inductively as follows. Let $x^* = x_1$. Define

$$S_1 = \{x \in X : x \leq x_1\}; \ \ x_2 \in S_1 \ \text{ such that } \ f(x_2) \leq \inf_{S_1} f + \frac{\epsilon}{2^2}.$$

In general define,

$$S_n = \{x \in X : x \leq x_n\}; \ \ x_{n+1} \in S_n \ \text{ such that } \ f(x_{n+1}) \leq \inf_{S_n} f + \frac{\epsilon}{2^{n+1}}.$$

We remark that $S_n \neq \phi$ $\forall n$ since $x_n \in S_n$.

Claim 1: $S_{n+1} \subseteq S_n$ for each $n = 0, 1, ...$

Proof of claim 1: Let $x \in S_{n+1}$ for some fixed n. Then

$$f(x) \leq f(x_{n+1}) - \epsilon d_\lambda(x_{n+1}, x). \tag{2.4}$$

Also, there exists $x_{n+1} \in S_n$. Therefore,

$$f(x_{n+1}) \leq f(x_n) - \epsilon d_\lambda(x_{n+1}, x_n) \tag{2.5}$$

from (2.4) and (2.5),

$$\begin{aligned} f(x) &\leq f(x_n) - \epsilon d_\lambda(x_{n+1}, x_n) - \epsilon d_\lambda(x_{n+1}, x) \\ &\leq f(x_n) - \epsilon d_\lambda(x, x_n). \end{aligned}$$

This implies that $x \leq x_n$. Thus, $x \in S_n$.

Claim 2: Each S_n is closed.

Proof of claim 2: Let $x_j \in S_n$ be such that $x_j \to x \in X$. Now since $x_j \in S_n, f(x_j) \leq f(x_n) - \epsilon d_\lambda(x_j, x_n)$. Taking limits as $j \to \infty$ both sides and using lower semicontinuity of f we have,

$$f(x) \leq \liminf_{j \to \infty} f(x_j) \leq f(x_n) - \epsilon d_\lambda(x, x_n).$$

This implies that $x \in S_n$. This is true for all $n \in \mathbb{N}$. Hence each S_n is closed.

Claim 3: $\lim_{n \to \infty} diam(S_n) = 0$.

Proof of claim 3: Fix $n \in \mathbb{N}$ and let $x \in S_n$. Therefore,

$$f(x) \leq f(x_n) - \epsilon d_\lambda(x, x_n). \tag{2.6}$$

By claim 1, $x \in S_{n-1}$ and there exists $x_n \in S_{n-1}$ such that

$$f(x_n) \leq \inf_{S_{n-1}} f + \frac{\epsilon}{2^n} \leq f(x) + \frac{\epsilon}{2^n}. \tag{2.7}$$

Now, from (2.6) and (2.7) we have,

$$d_\lambda(x, x_n) \leq 2^{-n}. \tag{2.8}$$

Thus $d(x, x_n) \to 0$ as $n \to \infty$.

We may now apply the Cantor Intersection theorem. Let $\{x_\lambda\} = \bigcap_{n=1}^{\infty} S_n$. Then $x_\lambda \in S_1$. Therefore $x_\lambda \leq x_1 = x^*$. We have, $f(x_\lambda) \leq f(x^*) - \epsilon d_\lambda(x_\lambda, x^*) \leq f(x^*)$, thus proving (2.1). Now let $x \neq x_\lambda$. We claim that $x \not\leq x_\lambda$ because otherwise $x \in S_n \;\; \forall n$ by construction of $S_n's$. That is, $x \in \bigcap_{n=1}^{\infty} S_n$; which is a contradiction. So $x \nleq x_\lambda$ implies that

$$f(x_\lambda) < f(x) + \frac{\epsilon}{\lambda} d(x, x_\lambda)) \ \ \forall x \neq x_\lambda.$$

Hence (2.3) is true.

Finally to prove (2.2) we use the triangle inequality;

$$d_\lambda(x^*, x_n) \leq \sum_{j=1}^{n-1} d_\lambda(x_j, x_{j+1}) \leq \sum_{j=1}^{n-1} 2^{-j}$$

where the last inequality follows from (2.8). Taking limit as $n \to \infty$ gives $d_\lambda(x^*, x_\lambda) \leq 1$, thus proving (2.2). ∎

In conclusion, given any minimizer $x^* \in X$ of a functional defined on a complete metric space, one can find another minimizing sequence x_λ such that $f(x_\lambda) \leq f(x^*)$ as long as x_λ and x^* are close enough. (2.3) tells us that derivative of f goes to $'0'$ whenever distance between x_λ and x^* is very less; which is the criterion for finding critical points.

We terminate this chapter by giving an example that exploits application of Ekeland's Variational Principle.

Example 2.2. Consider a sequence of triangle functions $f(x)$ as shown in Fig. 2.1, where the height of triangles is $\frac{1}{n}, n \in \mathbb{N}$ and the length of base of each triangle is twice its height so that the slope remains ± 1. Consider a sequence $\{x_n\} \subset \mathbb{R}$ such that $x_n \to \infty$ as $n \to \infty$, but as mentioned above, $|f'(x_n)| = 1 \forall n \in \mathbb{R}$. Hence f does not achieve a minimum.

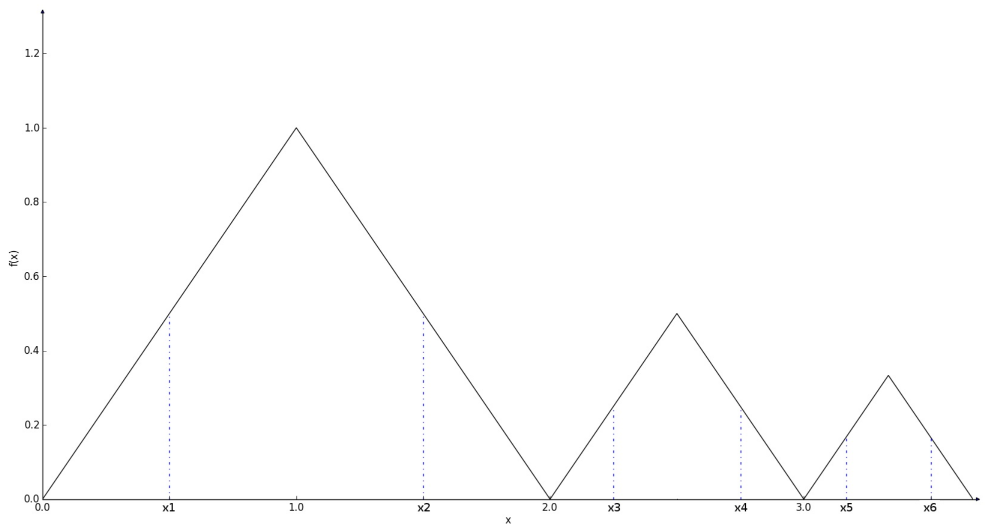

Figure 2.1: Triangular wave

Chapter 3

Euler Lagrange Equation at an Extremal in Infinite Dimensional Case

3.1 Introduction

So far we studied the necessary and sufficient conditions for existence of extremum of a function in finite and infinite dimensions. In this chapter, we delve into some problems of calculus of variations in infinite dimensions. We refer the reader to [3] for more details.

Consider points P and Q in $x-y$ plane (See figure 3.1). We intend to find a shortest path between these two points. One can also ask which curve between P and Q will generate the surface of revolution of smallest area when revolved about $x-$axis or which curve gives shortest time of descent.

We now examine each of these problems thoroughly.

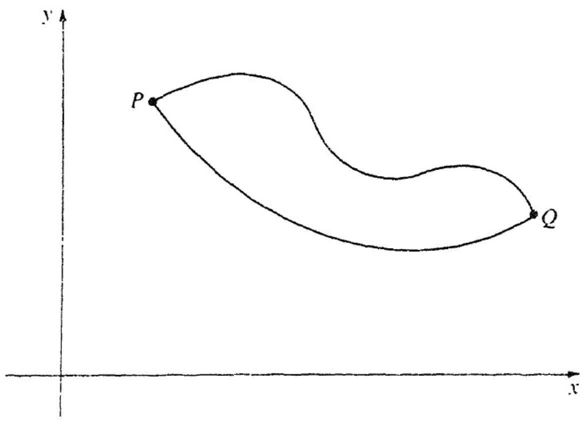

Figure 3.1: Paths between two points

Example 3.1. Consider a curve joining two points A and B as shown in figure 3.2. In order to find shortest path between A and B, we must minimize the arclength of the curves between them. For infinitesimal ds ,

$$\text{Arclength of the curve} = I_1 := \int_a^b \sqrt{1+(y')^2}\, dx. \tag{3.1}$$

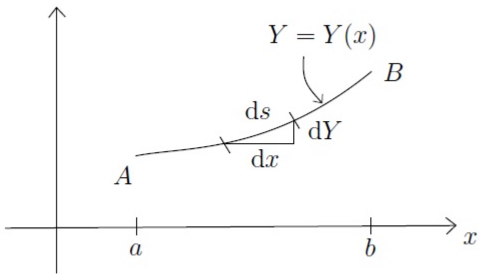

Figure 3.2: A simple curve

Example 3.2. Another problem could be finding a curve generating a surface of minimim area after revolution about x—axis. From figure 3.3, it is evident that area of surface of revolution of the curve from A to B about x—axis is

$$I_2 := \int_a^b 2\pi y\, ds = \int_a^b 2\pi y \sqrt{1+(y')^2}\, dx \tag{3.2}$$

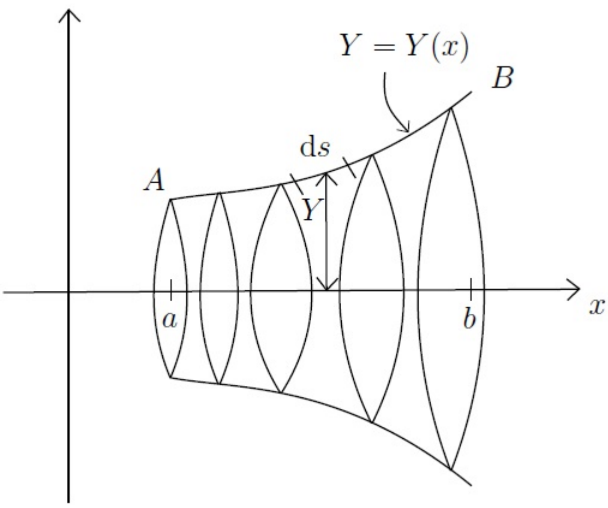

Figure 3.3: A volume of revolution, of the curve $Y(x)$, around x—axis.

Example 3.3 (Time of descent)**.** Another interesting problem could be finding out the path that gives quickest time of descent from P to Q (See figure 3.4). For a freely falling body, the equation of motion is $\frac{d^2y}{dt^2} = g$. This implies that

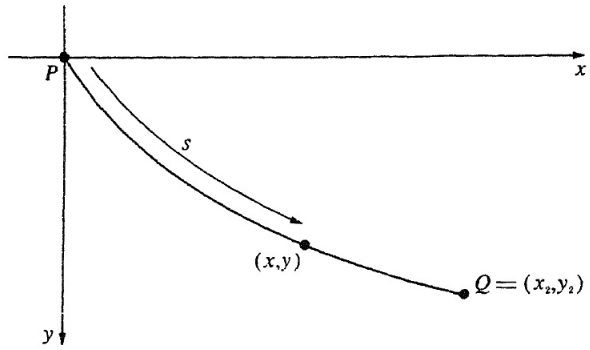

Figure 3.4:

$$v(t) = \frac{dy}{dt} = gt + v_0, \tag{3.3}$$

$$y(t) = \frac{gt^2}{2} + v_0 t + y_0, \tag{3.4}$$

where v_0 and y_0 are initial velocity and initial position respectively. If the body falls from rest starting at $y = 0$ then $v_0 = y_0 = 0$. Therefore, (3.3) and (3.4) gives $v = gt$ and $y = \frac{gt^2}{2}$. That is,

$$v = \sqrt{2gy} \tag{3.5}$$

For our convenience we shift the point P to origin. But we know that $v = \frac{ds}{dt}$. Therefore from (3.5),

$$\text{the total time of descent} = I_3 := \int \frac{ds}{v} = \int_0^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}} dx \tag{3.6}$$

We pose a general problem and it can then be seen that all the three problems stated above are special cases of this general problem.

Let P and Q be the points with coordinates (x_1, y_1) and (x_2, y_2) respectively. Consider a family of curves

$$y = y(x) \tag{3.7}$$

with P and Q as end points. Then we wish to find the function in this family which minimizes the integral

$$I := \int_{x_1}^{x_2} f(x, y, y')\, dx. \tag{3.8}$$

Integrals of the form as above are called functionals. Thus, in order to find the curve of shortest length, the minimum area of surface of revolution and the shortest time of descent, we minimize I_1, I_2 and I_3 given by (3.1), (3.2), and (3.6) respectively.

The choice of function (3.7) seems deliberately naive and we must ask the question what type of functions are to be allowed and what conditions the function $f(x, y, y')$ in (3.8) must satisfy. Well, we always assume that $f(x, y, y')$ has continuous partial derivatives of the second order with respect to x, y and y' so that the integral in (3.8) is well defined. Thus, it suffices to assume that $y'(x)$ is continuous. Once and for all, we assume that the functions $y(x)$ having continuous second order derivatives and satisfying the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$ are allowed. We will call such functions as "admissible".

3.2 Euler-Lagrange Equation for Free Functionals

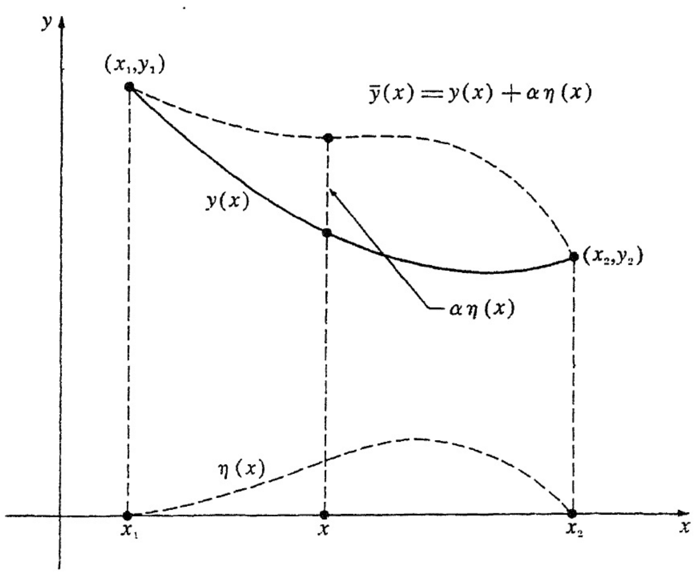

Figure 3.5:

In this section, we study the conditions satisfied by a functional having an extremum in an admissible domain consisting of smooth functions. Our aim is to find the admissible function y that minimizes the integral (3.8) locally. Since y gives minimum value to I, any perturbation in y must increase the value of I. We use this fact to find a differential equation for y as follows.

Let $\eta(x)$ be a C^2 function such that $\eta(x_1) = 0 = \eta(x_2)$ (See figure 3.5). If α is a small parameter then let

$$\bar{y}(x) = y(x) + \alpha\eta(x) \tag{3.9}$$

With $\eta(x)$ fixed, we substitute $\bar{y}(x) = y(x) + \alpha\eta(x)$ and $\bar{y'}(x) = y'(x) + \alpha\eta'(x)$ in (3.8). We have,

$$p(\alpha) := \int_{x_1}^{x_2} f(x, \bar{y}, \bar{y}')\, dx = \int_{x_1}^{x_2} f(x,\ y(x) + \alpha\eta(x),\ y'(x) + \alpha\eta'(x))\, dx.$$

Note that when $\alpha = 0$, (3.9) yields $\bar{y}(x) = y(x)$. Therefore, $p(0) = I$ since $y(x)$ is the minimum of I by our hypothesis. That is, $p(\alpha)$ must have a minimum value at $\alpha = 0$. Thus, $p'(0) = 0$ and

$$p'(\alpha) = \int_{x_1}^{x_2} \frac{\partial}{\partial \alpha} f(x, \bar{y}, \bar{y}')\, dx.$$

By chain rule,

$$p'(\alpha) = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial \bar{y}} \eta(x) + \frac{\partial f}{\partial \bar{y}'} \eta'(x) \right) dx,$$

$p'(0) = 0$ implies

$$0 = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x) \right) dx$$

By using integration by parts formula we get,

$$\int_{x_1}^{x_2} \eta(x) \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] dx = 0.$$

This integral must vanish for all $\eta(x)$. Therefore, we conclude,

$$\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial y} = 0. \tag{3.10}$$

This is called "Euler's equation". To summarize, we can say that if y is an admissible function that minimizes the integral in (3.8), then y satisfies Euler's equation in (3.10). Here, one can ask the question that if an admissible function y satisfies equation (3.10)

then does y minimize I? Well, not necessarily. This is because the condition $p'(0) = 0$ can indicate a maximum or point of inflection as well.
In order to interpret the importance of Euler's equation we emphasize on the fact that the partial derivatives $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial y'}$ are computed by treating x, y and y' as independent variables. $\frac{\partial f}{\partial y'}$ is however a function of x explicitly and also implicitly through y and y'. Therefore,

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = f_{y'x} = f_{y'y'}y'' + f_{y'y}y' + f_{y'x}.$$

Therefore the Euler's equation becomes

$$f_{y'y'}y'' + f_{y'y}y' + (f_{y'x} - f_y) = 0. \tag{3.11}$$

This is a second order differential equation unless $f_{y'y'} = 0$ and is usually very difficult to solve, but fortunately there are a few cases when it can be solved explicitly.

CASE A: If f is a function of y' alone then, $f(x, y, y') = f(y')$. In this case (3.11) reduces to

$$f_{y'y'}y'' = 0.$$

If $f_{y'y'} \neq 0$, the above equation yields $y(x) = c_1 x + c_2$. So the extremals are all straight lines.

CASE B: If f is a function of x and y' then, $f(x, y, y') = f(x, y')$. In this case (3.10) becomes

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0. \tag{3.12}$$

That is, $\frac{\partial f}{\partial y'} = c_3$. Solving this will give us the extremizing curve $y(x)$.

CASE C: If f is a function of y and y' then, $f(x, y, y') = f(y, y')$. Note that

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}y' - f\right) = y'\left[\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y}\right] - \frac{\partial f}{\partial x}$$

Since $\frac{\partial f}{\partial x}(y, y') = 0$, using (3.10) we have,

$$\frac{\partial f}{\partial y'} y' - f = c_4. \tag{3.13}$$

Let us now use Euler's equation to solve the three problems mentioned in the beginning of this chapter.

Example 3.4. In order to minimize the arclength we must minimize the integral $I_1 = \int_a^b \sqrt{1 + (y')^2}\, dx$. Therefore the function $f(y') = \sqrt{1 + (y')^2}$ falls under CASE A. Hence the functions that minimize I_1 are the straight lines $y(x) = c_1 x + c_2$. The unknown constants c_1 and c_2 can be obtained by applying boundary conditions at the two end points A and B (see figure 3.2).

Example 3.5. In order to minimize the area of surface of revolution about x—axis, we must minimize the integral $I_2 = \int_a^b 2\pi y \sqrt{1 + (y')^2}\, dx$. Therefore the function $f(y, y') = 2\pi y \sqrt{1 + (y')^2}$ falls under CASE C. Therefore from (3.13), we have

$$\frac{y (y')^2}{\sqrt{1 + (y')^2}} - y \sqrt{1 + (y')^2} = c_5.$$

On simplification, the above equation reduces to $c_5 y' = \sqrt{y^2 + c_5^2}$. In order to solve for y, we use the method of separation of variables,

$$x = c_5 \int \frac{dy}{\sqrt{y^2 - c_5^2}}.$$

To solve this integration, we substitute $y = c_5 \cosh t$ so that $dy = c_5 \sinh t\, dt$ and $x = c_5 t + c_6 = c_5 \cosh^{-1}\left(\frac{y}{c_5}\right) + c_6$, that is,

$$y(x) = c_5 \cosh\left(\frac{x - c_6}{c_5}\right).$$

Thus the minimum area of surface of revolution can be obtained by revolving this extremal $y(x)$ about x—axis.

Example 3.6. In order to find the shortest time of descent, we minimize the integral $I_3 = \int_0^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}} dx$. Again the function $f(y, y') = \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}$ does not depend on x and hence the function falls under CASE C. Solving (3.13) for this f gives,

$$\frac{(y')^2}{\sqrt{y(1_{(}y')^2)}} - \frac{\sqrt{1+(y')^2}}{\sqrt{y}} = c.$$

On simplification we obtain, $y\left[1+\left(\frac{dy}{dx}\right)^2\right] = c$ or $dx = \left(\frac{y}{c-y}\right)^{1/2} dy$.

In order to solve this differential equation we introduce a new variable ϕ as below,

$$\left(\frac{y}{c-y}\right)^{1/2} = \tan\phi$$

so that $y = c\sin^2\phi$, $dy = 2c\sin\phi\cos\phi\, d\phi$ and $dx = c(1-\cos 2\phi)d\phi$.

Integrating the above equation is now easy and we have, $x = \frac{c}{2}(2\phi - \sin 2\phi) + c_1$. If the curve has to pass through origin, we must have $x = y = 0$, that is, $x = \phi = 0$. This implies that $c_1 = 0$. Hence $x = \frac{c}{2}(2\phi - \sin 2\phi)$ and $y = c\sin^2\phi = \frac{c}{2}(1-\cos 2\phi)$. Let $a = \frac{c}{2}$ and $\theta = 2\phi$. Substituting these new variables, we finally obtain the following equations.

$$x = a(\theta - \sin\theta) \quad and \quad y = a(1-\cos\theta)$$

which are the equations of a cycloid generated by a circle of radius a rolling under the x—axis (See figure 3.6), where a is chosen so that the first inverted arch passes through the point (x_2, y_2) in Figure 3.4. Hence the shortest time of descent curve is a cycloid connecting A and B.

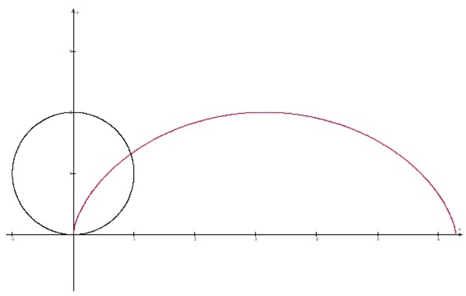

Figure 3.6: Cycloid curve shown with its generating circle

Similar results can be concluded for integrals depending on two or more unknown functions. For example, consider the problem of finding the functions $y(x)$ and $z(x)$ satisfying the boundary conditions $y(x_1) = y_1$, $y(x_2) = y_2$, $z(x_1) = z_1$, $z(x_2) = z_2$ so that the integral

$$I = \int_{x_1}^{x_2} f(x, y, z, y', z')\, dx, \tag{3.14}$$

is minimized. As earlier, we introduce C^2 functions $\eta_1(x)$ and $\eta_2(x)$ vanishing at the end points. We define the neighboring functions as

$$\bar{y}(x) = y(x) + \alpha_1 \eta_1(x), \quad \bar{z}(x) = z(x) + \alpha_2 \eta_2(x).$$

Then consider the following integral depending on α,

$$p(\alpha_1, \alpha_2) = \int_{x_1}^{x_2} f\left(x,\ y + \alpha_1\eta_1,\ z + \alpha_2\eta_2,\ y' + \alpha_1\eta_1',\ z' + \alpha_2\eta_2'\right)\, dx$$

Since $y(x)$ and $z(x)$ give minimum value to I, we must have $p'(0,0) = 0$. Therefore,

$$0 = p'(0,0) = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta_1 + \frac{\partial f}{\partial z}\eta_2 + \frac{\partial f}{\partial y'}\eta_1' + \frac{\partial f}{\partial z'}\eta_2' \right) dx.$$

Eliminating η_1' and η_2' using integration by parts,

$$\int_{x_1}^{x_2} \left\{ \eta_1(x)\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] + \eta_2(x)\left[\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right)\right] \right\} dx = 0.$$

Once again we are lead to Euler's equations

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0, \quad \frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) - \frac{\partial f}{\partial z} = 0. \tag{3.15}$$

In other words, if $y(x)$ and $z(x)$ are the functions minimizing the integral (3.14), then the function f satisfy the Euler's equations (3.15). Similar to the one variable method, different cases can be considered for solving (3.15).
In a similar fashion, results for functions involving more than two independent variables can be obtained.

Some of the very important results of physics are obtained by using the techniques given in this chapter. For example, the famous Hamilton's Principle can be proved by using the method explained in this section.

Example 3.7 (Hamilton's Principle)**.** Consider a particle of mass m moving under the influence of force given by

$$\mathbf{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}.$$

Then we know that the potential energy V of the particle is such that $-\frac{\partial V}{\partial x} = F_1$, $-\frac{\partial V}{\partial y} = F_2$, $-\frac{\partial V}{\partial z} = F_3$. Let the position of the particle at time t be $\mathbf{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ so that the velocity of the particle is $\mathbf{v}(t) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$ and $T = \frac{m\mathbf{v}^2}{2}$ is the kinetic energy of the system.

The Lagrangian L is defined as the kinetic energy, T of the system minus the potential energy, V. That is,

$$L = T - V = \frac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] - V(x,y,z). \tag{3.16}$$

Our aim is to find out the path followed by the particle under the force field $\vec{F}$ in moving from point P_1 at time t_1 to point P_2 at time t_2.

The action, A is by definition, integral over time of the Lagrangian. Therefore,

$$A = \int_{t_1}^{t_2} (T - V)\, dt.$$

Therefore, in this case, the integrand is a function of the form $f(x,y,z,\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt})$. According to Hamilton's principle, the system described between two specified states at two specified times is a stationary point of the action functional. Hence the Euler's equation (3.10) must be satisfied for $f = L$ given by (3.16). We have,

$$m\frac{d^2x}{dt^2} + \frac{\partial V}{\partial x} = 0,\ m\frac{d^2y}{dt^2} + \frac{\partial V}{\partial y} = 0,\ m\frac{d^2z}{dt^2} + \frac{\partial V}{\partial z} = 0.$$

This can be written as,

$$m\frac{d^2\mathbf{r}}{dt^2} = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k} = \mathbf{F}.$$

This is precisely Newton's second law of motion. Thus we conclude that Newton's law is a necessary condition for the action of the particle to have a stationary value.

Example 3.8. Let us determine the equation of the geodesic on a right circular cylinder of radius $'r_0'$, by using the method mentioned in this section. In cylindrical coordinates (r,θ,z) the infinitesimal arclength is given by $(ds)^2 = (dr)^2 + (r\,d\theta)^2 + (dz)^2$. Since $r = r_0$ is a constant, $dr = 0$. Therefore we have, $\frac{ds}{d\theta} = \sqrt{r_0^2 + \left(\frac{dz}{d\theta}\right)^2}$. On integration we obtain,

$$s = \int_{\theta_1}^{\theta_2} \sqrt{{r_0}^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta.$$

In order to find the geodesic we minimize the above integral. Let $f := \sqrt{{r_0}^2 + \left(\frac{dz}{d\theta}\right)^2} d\theta$. Here, f is independent of z. Thus f falls under CASE B. Therefore from (3.12), $\frac{d}{d\theta}\left(\frac{\partial f}{\partial z'}\right) = 0$. This implies that $\frac{\partial f}{\partial z'} = \text{constant} = k$.

$$\frac{\partial f}{\partial z'} = \frac{z'}{\sqrt{r_0^2 + z'^2}} = k.$$

On simplification we have, if $0 < k < 1$,

$$\frac{dz}{d\theta} = \frac{k}{\sqrt{1-k^2}} r_0.$$

Integrating, we obtain $z(\theta) = \frac{k}{\sqrt{1-k^2}} r_0 \theta + c_1$. Thus the equation of the geodesic on the given cylinder is a circular helix $z = k^* \theta + c_1$ where $k^* = \frac{k}{\sqrt{1-k^2}} r_0$.

3.3 Euler-Lagrange Equation for Constrained Functionals

Isoperimetric problems are one of the oldest problems considered by ancient Greeks. It all started with the problem of finding a curve, among all the curves of a given length, that encloses a maximum area. So in isoperimetric problems, the basic idea is to maximize a particular integral given a side condition.

We now show different methods of obtaining Euler-Lagrange equations for functionals with constraints $i.e$, subjected to side conditions.

I. Lagrange Multipliers : Consider a function $z = f(x, y)$. Our aim is to find the points (x, y) that yields stationary values for $z = f(x, y)$ subject to the side condition $g(x, y)$=0. Let us assume that x is an independent variable while y depends on x. We have,

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial x} = 0. \tag{3.17}$$

Since z is a function of x alone by our assumption, $\frac{dz}{dx} = 0$ for z to be stationary. That is,

$$0 = \frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x}. \tag{3.18}$$

Using (3.17), we have

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\frac{\partial g/\partial x}{\partial g/\partial y} = 0, \tag{3.19}$$

$$g(x, y) = 0. \tag{3.20}$$

Solving (3.19) and (3.20) simultaneously, gives the required points (x, y).
Although easy, this method has certain drawbacks. It might not always be possible to write y as a function of x; or vice versa. Therefore, alternatively we can use the following method which is practically more advantageous. We begin by defining a function $F(x, y, \lambda)$ as follows.

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y),$$

where x, y and λ are independent of each other. Therefore,

$$\left.\begin{aligned} \tfrac{\partial F}{\partial x} &= \tfrac{\partial f}{\partial x} + \lambda \tfrac{\partial g}{\partial x} = 0 \\ \tfrac{\partial F}{\partial y} &= \tfrac{\partial f}{\partial y} + \lambda \tfrac{\partial g}{\partial y} = 0 \\ \tfrac{\partial F}{\partial \lambda} &= g(x, y) = 0. \end{aligned}\right\} \tag{3.21}$$

Eliminating λ from first two of these equations we have,

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\,\frac{\partial g/\partial x}{\partial g/\partial y} = 0$$

and

$$g(x, y) = 0.$$

Notice that the motive behind introducing a new parameter λ is to reduce the number of equations. That is, we include the side condition in $F(x, y, \lambda)$ at the cost of a single parameter λ and then examine the conditions satisfied by F instead of f. It is interesting to note that we arrive at the same set of equations (3.19) and (3.20). But the benefit of using this method is that we need not consider y as a function of x or vice versa. This method is known as method of Lagrange multipliers and the parameter λ is called "Lagrange Multiplier". This method can also be used for problems involving functions of more than two variables with several side conditions.

II. Integral Side Condition: We now consider the problem with a side condition involving integral. The problem is to find a function $y(x)$ that gives a stationary value to the integral

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx$$

with the side condition $J = \int_{x_1}^{x_2} g(x, y, y')\, dx = c$; given the boundary values $y(x_1) = y_1$ and $y(x_2) = y_2$.

As before, the idea is to perturb $y(x)$ which is the actual stationary function. However, here we cannot use a neighboring function of the form $\bar{y}(x) = y(x) + \alpha\eta(x)$ because this will not maintain the integral J at a constant value c. Hence we use a two-parameter family of functions $\bar{y}(x) = y(x) + \alpha_1\eta_1(x) + \alpha_2\eta_2(x)$ where α_1 and α_2 are not independent, but are constrained by

$$q(\alpha_1, \alpha_2) := \int_{x_1}^{x_2} g(x, \bar{y}, \bar{y}')\, dx = c, \tag{3.22}$$

where η_i is a C^2 function vanishing at the end-points, for $i = 1, 2$.

Notice that $\bar{y}'(x) = y(x)$ at $\alpha_1 = \alpha_2 = 0$. Therefore,

$$p(\alpha_1, \alpha_2) := \int_{x_1}^{x_2} f(x, \bar{y}, \bar{y}')\, dx \tag{3.23}$$

has stationary value at $\alpha_1 = \alpha_2 = 0$. As in the earlier method, we introduce a Lagrange multiplier as follows. Define a function $K(\alpha_1, \alpha_2, \lambda)$ as

$$K(\alpha_1, \alpha_2, \lambda) = p(\alpha_1.\alpha_2) + \lambda q(\alpha_1, \alpha_2) = \int_{x_1}^{x_2} \big(f(x, \bar{y}, \bar{y}') + \lambda g(x, \bar{y}, \bar{y}')\big)\, dx.$$

Therefore,

$$K(\alpha_1, \alpha_2, \lambda) = \int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')\, dx \tag{3.24}$$

where $F = f + \lambda g$. Now, differentiating (3.24) with respect to α_i under integration sign, we get,

$$\frac{\partial K}{\partial \alpha_i} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \bar{y}}\eta_i(x) + \frac{\partial F}{\partial \bar{y}'}\eta_i'(x)\right) dx \quad \text{for } i = 1, 2.$$

Note that $\frac{\partial K}{\partial \alpha_1} = \frac{\partial K}{\partial \alpha_2} = 0$ when $\alpha_1 = \alpha_2 = 0$ in regard of (3.22) and (3.23). Hence,

$$0 = \frac{\partial K}{\partial \alpha_i}|_{\alpha_i=0} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \eta_i(x) + \frac{\partial F}{\partial y'} \eta_i'(x) \right) dx \quad \text{for } i = 1, 2.$$

Integrating the second term in parenthesis by parts and applying the fact that η_i vanishes at the endpoints x_1 and x_2, the above equation is reduced to:

$$\int_{x_1}^{x_2} \eta_i(x) \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right] dx = 0.$$

One more time, we arrive at the Euler's equation

$$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = 0.$$

This is a second order differential equation. We are therefore going to end up with two constants of integration and the parameter λ. To find these three unknown quantities, we have two boundary conditions and a side condition $J = c$.

The integrals involving more than two variables are handled in the same manner. For example if

$$I = \int_{x_1}^{x_2} f(x, y, z, y', z')\, dx \;\text{ and }\; J = \int_{x_1}^{x_2} g(x, y, z, y', z')\, dx,$$

then the stationary functions $y(x)$ and $z(x)$ must satisfy Euler's equations,

$$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = 0, \quad \frac{d}{dx} \left(\frac{\partial F}{\partial z'} \right) - \frac{\partial F}{\partial z} = 0,$$

where $F = f + \lambda g$.

Example 3.9. Suppose a curve is expressed parametrically by $x = x(t), y = y(t)$, then as seen in the previous section, the length of the curve will be

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2}\, dt \tag{3.25}$$

when t is traversed in anti-clockwise direction from t_1 to t_2.

To find the area A enclosed by the curve, we use Green's theorem:

Theorem 3.1 (Green's Theorem). *Let C be a positively oriented, piecewise smooth, simple, closed curve and let D be the region enclosed by the curve. If P and Q have continuous first* *order partial derivatives on D then,*

$$\int\limits_C P\,dx + Q\,dy = \iint\limits_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA.$$

We know that $A = \iint\limits_D dA$. That is $Q_x - P_y = 1$ in Green's theorem. If we let $P = \frac{y}{2}$ and $Q = \frac{x}{2}$, we have

$$A = \frac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) dt. \tag{3.26}$$

Therefore, the problem is to maximize (3.26) with respect to the side condition (3.25), where L is a given constant.

Example 3.10. We solve the problem mentioned in previous example by using the method of Lagrange Multipliers. Suppose that we want to find out the curve $y(x)$ that will enclose maximum area between the $x-$ axis and its graph, given the constraint that the curve has $(-a, 0)$ and $(a, 0)$ as end points and that the arclength of the curve is L. That is, we have to maximize the integral $I = \int\limits_{-a}^{a} y(x)\,dx$ with side condition $J = \int\limits_{-a}^{a} \sqrt{1+y'^2}\,dx = L$.

Let $F(y, y') := y + \lambda\sqrt{1+y'^2}$. Therefore by using results obtained in previous section, we conclude that F satisfies (3.13). Hence $\frac{\partial F}{\partial y'}y' - F = c_1$. Substituting value of F and simplifying,

$$\frac{dy}{dx} = \frac{\sqrt{\lambda^2 - (y+c_1)^2}}{y+c_1}.$$

Integrating gives, $-\sqrt{\lambda^2-(y+c_1)^2} = x + c_2$ or $(x+c_2)^2 + (y+c_1)^2 = \lambda^2$, which is an equation of a circle of radius λ centered at (c_2, c_1). In order to find the constants c_1, c_2 and λ we have the conditions $y(-a) = 0 = y(a)$ and the given side condition $J = L$.

Example 3.11. In this example given a constant $k \neq 0$, we try to find out the extremum of the integral $I(y) = \int_0^1 y^2 dx$ subject to the constraints

$$J(y') = \int_0^1 (y')^2\,dx = k^2 \;\text{ and }\; y(0) = 0 = y(1) \tag{3.27}$$

Let $F = y^2 + \lambda(y')^2$. Then F should satisfy Euler's equation. Therefore, $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$. On substituting for F and simplifying,

$$y'' - \frac{1}{\lambda}y = 0 \tag{3.28}$$

Notice that, at this stage we do not have any information about λ except that there can be

three cases:
(1) if $\frac{1}{\lambda} > 0$, the general solution of (3.28) is $y(x) = Ae^{\frac{1}{\lambda}x} + Be^{-\frac{1}{\lambda}x}$. After imposing the boundary conditions (3.27), we get $A = B = 0$.

(2) if $\frac{1}{\lambda} = 0$, $y'' = 0$. that is, $y(x) = ax + b$. After imposing boundary conditions (3.27), we get $a = b = 0$.
So in case (1) and (2) above we have $\int_0^1 y'^2 = 0$ and hence the constraint is not satisfied. So we need to assume $\frac{1}{\lambda} < 0$.

(3) if $\frac{1}{\lambda} < 0$, then let $\frac{1}{\lambda} = -\gamma^2$. The general solution of (3.28) in this case is $y(x) = C\cos\gamma x + D\sin\gamma x$. Imposing the boundary conditions (3.27) we get, $C = 0$ and $\gamma = n\pi$, so that the solutions of (3.28) are given by

$$y_n(x) = D\sin n\pi x. \quad \text{for } n = 1, 2, 3, \ldots.$$

Then $y = \sum_{n=1}^{\infty} c_n y_n$ is also the solution provided $\sum_{n=1}^{\infty} {c_n}^2 < \infty$. Now, to find out the maximum value of I we use the obtained solution,

$$\begin{aligned} I(y) &= \int_0^1 y^2\,dx \\ &= \int_0^1 \left(\sum_{n=1}^{\infty} c_n \sin(n\pi x)\right)\left(\sum_{m=1}^{\infty} c_m \sin(m\pi x)\right) dx \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_n c_m \int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx \\ &= \sum_{n=1}^{\infty} c_n^2 \frac{1}{2}. \end{aligned}$$

Now, $y' = \sum_{n=1}^{\infty} c_n n\pi\cos(n\pi x)$. Also,

$$\begin{aligned} k^2 = J(y') &= \int_0^1 (y')^2\,dx \\ &= \int_0^1 \left(\sum_{n=1}^{\infty} c_n n\pi\cos(n\pi x)\right)\left(\sum_{m=1}^{\infty} c_m m\pi\cos(m\pi x)\right) dx \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_n c_m mn\pi^2 \int_0^1 \cos(n\pi x)\cos(m\pi x)\,dx \\ &= \sum_{n=1}^{\infty} c_n^2 n^2\pi^2 \frac{1}{2}. \end{aligned}$$

Therefore, $c_1^2\pi^2\frac{1}{2} + \sum\limits_{n=2}^{\infty} c_n^2 n^2\pi^2\frac{1}{2} = k^2$ or $\frac{c_1^2}{2} = \frac{k^2}{\pi^2} - \sum\limits_{n=2}^{\infty} c_n^2 n^2\frac{1}{2}$ and

$$\begin{aligned} I(y) &= \sum_{n=1}^{\infty} c_n^2\frac{1}{2} \\ &= \frac{c_1^2}{2} + \sum_{n=2}^{\infty} c_n^2\frac{1}{2} \\ &= \frac{k^2}{\pi^2} - \sum_{n=2}^{\infty} c_n^2 n^2\frac{1}{2} + \sum_{n=2}^{\infty} c_n^2\frac{1}{2} \\ &= \frac{k^2}{\pi^2} - \sum_{n=2}^{\infty} c_n^2\frac{1}{2}(n^2-1). \end{aligned}$$

Therefore $I(y)$ achieves the maximum value $\frac{k^2}{\pi^2}$ only when $c_n = 0$, $n = 2, 3, 4, \ldots$.

III. Finite Side Condition: Let us now consider a problem of finding the parametrized functions $x(t), y(t), z(t)$ that gives a stationary value to the integral of the form

$$I = \int_{t_1}^{t_2} f(\dot{x}, \dot{y}, \dot{z})\, dt, \tag{3.29}$$

where t is the independent variable; subject to the side condition of the type $G(x, y, z) = 0$.

Let us assume that the curve lies on the surface where $G_z \neq 0$. Then we can write z as a function of x and y. Let $z = g(x, y)$. Then,

$$\dot{z} = \frac{\partial g}{\partial x}\dot{x} + \frac{\partial g}{\partial y}\dot{y}. \tag{3.30}$$

Inserting this in (3.29) we have,

$$I = \int_{t_1}^{t_2} f\left(\dot{x}, \dot{y}, \frac{\partial g}{\partial x}\dot{x} + \frac{\partial g}{\partial y}\dot{y}\right) dt.$$

From (3.15) we know that the Euler's equations in this case are

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}} + \frac{\partial f}{\partial \dot{z}}\frac{\partial g}{\partial x}\right) - \frac{\partial f}{\partial \dot{z}}\frac{\partial \dot{z}}{\partial x} = 0, \\ \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}} + \frac{\partial f}{\partial \dot{z}}\frac{\partial g}{\partial y}\right) - \frac{\partial f}{\partial \dot{z}}\frac{\partial \dot{z}}{\partial y} = 0. \end{aligned}\right\} \tag{3.31}$$

Also, from (3.30) we have $\frac{\partial \dot{z}}{\partial x} = \frac{d}{dt}\left(\frac{\partial g}{\partial x}\right)$ and $\frac{\partial \dot{z}}{\partial y} = \frac{d}{dt}\left(\frac{\partial g}{\partial y}\right)$. Substituting this in (3.31) we get,

$$\left.\begin{aligned}\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}} + \frac{\partial f}{\partial \dot{z}}\frac{\partial g}{\partial x}\right) - \frac{\partial f}{\partial \dot{z}}\frac{d}{dt}\left(\frac{\partial g}{\partial x}\right) = 0,\\ \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}} + \frac{\partial f}{\partial \dot{z}}\frac{\partial g}{\partial y}\right) - \frac{\partial f}{\partial \dot{z}}\frac{d}{dt}\left(\frac{\partial g}{\partial y}\right) = 0.\end{aligned}\right\}$$

Differentiating the terms in parenthesis with respect to t, we get,

$$\left.\begin{aligned}\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) + \frac{\partial g}{\partial x}\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{z}}\right) = 0,\\ \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right) + \frac{\partial g}{\partial y}\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{z}}\right) = 0.\end{aligned}\right\} \tag{3.32}$$

Let $\lambda(t)$ be a function such that

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{z}}\right) = \lambda(t)G_z. \tag{3.33}$$

Notice that $G(x,y,z) = G(x,y,g(x,y)) = 0$. This implies

$$\left.\begin{aligned}\frac{\partial g}{\partial x} = -\frac{G_x}{G_z},\\ \frac{\partial g}{\partial y} = -\frac{G_y}{G_z}.\end{aligned}\right\} \tag{3.34}$$

From (3.32), (3.33) and (3.34) we get,

$$\left.\begin{aligned}\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) - \lambda(t)G_x = 0,\\ \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right) - \lambda(t)G_y = 0.\end{aligned}\right\} \tag{3.35}$$

Eliminating $\lambda(t)$ from (3.33) and (3.35) we arrive at the following equation valid in the set $\{G_x, G_y, G_z \neq 0\}$

$$\frac{\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)}{G_x} = \frac{\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right)}{G_y} = \frac{\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{z}}\right)}{G_z}. \tag{3.36}$$

(3.36) together with the side condition $G(x,y,z) = 0$ gives the desired functions $x(t), y(t)$ and $z(t)$ that stationarize I.

It is worthy to note that if we adopt Lagrange multiplier method in this case, by defining a function, $F(x,y,z,\dot{x},\dot{y},\dot{z},\lambda(t)) = f(\dot{x},\dot{y},\dot{z}) + \lambda(t)G(x,y,z)$, then equations (3.33) and (3.35) are nothing but Euler's equations for F. The procedure is same except that the Lagrange multiplier here is a function of the independent variable t instead of being a constant.

Let us look at an example that emphasizes on the convenience of using the above method.

Example 3.12. Consider a problem of finding a geodesic on a sphere of radius a, $x^2+y^2+z^2=a^2$. Therefore we have,

$$f(\dot{x},\dot{y},\dot{z})=\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2},\ \ G(x,y,z)=x^2+y^2+z^2-a^2$$

Therefore, from (3.36) we have,

$$\frac{f\ddot{x}-\dot{x}\dot{f}}{2xf^2}=\frac{f\ddot{y}-\dot{y}\dot{f}}{2yf^2}=\frac{f\ddot{z}-\dot{z}\dot{f}}{2zf^2}.$$

on simplification, we obtain

$$\frac{x\ddot{y}-y\ddot{x}}{x\dot{y}-y\dot{x}}=\frac{y\ddot{z}-z\ddot{y}}{y\dot{z}-z\dot{y}}=\frac{\dot{f}}{f}$$

This implies,

$$\frac{d/dt\,(x\dot{y}-y\dot{x})}{x\dot{y}-y\dot{x}}=\frac{d/dt\,(y\dot{z}-z\dot{y})}{y\dot{z}-z\dot{y}}.$$

Therefore, $x\dot{y}-y\dot{x}=c_1\,(y\dot{z}-z\dot{y})$ or

$$\frac{\dot{x}+c_1\dot{z}}{x+c_1z}=\frac{\dot{y}}{y}.$$

This yields $x+c_1z=c_2y$, which is an equation of a plane passing through origin. Hence the geodesics on sphere are arcs of great circles.

One can also prove that the total energy of a system is constant as shown in the following example.

Example 3.13. Consider a constrained system of n particles. Let there be k constraints, $G_j(x_1,y_1,z_1,........,x_n,y_n,z_n)=0,\ j=1,2,........,k$. The total number of degrees of freedom in this case is $m=3n-k$. We can therefore, express the $3n$ numbers x_i,y_i and z_i $(i=1,2,...,n)$ into m numbers. Let these new m coordinates be denoted by generalized coordinates $q_1,......,q_m$.

Now, consider an i^{th} particle of mass m_i having the kinetic energy

$$T=\frac{1}{2}\sum_{i=1}^{n}m_i\left[\left(\frac{dx_i}{dt}\right)^2+\left(\frac{dy_i}{dt}\right)^2+\left(\frac{dz_i}{dt}\right)^2\right];$$

In generalized coordinates we have,

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \left[\left(\sum_{j=1}^{m} \frac{\partial x_i}{\partial q_j}\dot{q}_j \right)^2 + \left(\sum_{j=1}^{m} \frac{\partial y_i}{\partial q_j}\dot{q}_j \right)^2 + \left(\sum_{j=1}^{m} \frac{\partial z_i}{\partial q_j}\dot{q}_j \right)^2 \right] \tag{3.37}$$

We also note that T is a homogeneous function of degree 2 in q_j since

$$T(\lambda q_1, \lambda q_2, ..., \lambda q_m) = \lambda^2 T(q_1, ..., q_m) \text{ for any} \lambda \in \mathbb{R}.$$

Under the assumption that the potential energy is a function of q_j alone ($j = 1, 2, ..., k$), we have $L = T - V$ as a function of the following form.

$$L = L(q_1, q_2, ..., q_m, \dot{q_1}, \dot{q_2}, ..., \dot{q_m}).$$

From Hamilton's principle, $\int_{t_1}^{t_2} L$ is minimized. Therefore, Euler's equation at the minimum of this integral is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \quad j = 1, 2, ..., m. \tag{3.38}$$

These are popularly known as Lagrange's equations.

We take note of the following identity,

$$\frac{d}{dt}\left[\sum_{j=1}^{m} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right] = \sum_{j=1}^{m} \dot{q}_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right] - \frac{\partial L}{\partial t}$$

where $\frac{dL}{dt} = \sum_{j=1}^{m} \frac{\partial L}{\partial q_j}\frac{dq_j}{dt} + \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt}$.

Since L satisfies (3.38) we have,

$$\sum \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = E \tag{3.39}$$

where E is a constant. Notice that $\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$. Therefore since T is homogeneous of degree 2,

$$\sum_{j=1}^{m} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = \sum_{j=1}^{m} \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T.$$

Thus, from (3.39) we get, $2T - L = E$, that is, $2T - (T - V) = E$ or $T + V = E$, which states that the total energy is constant.

Chapter 4

Sobolev Spaces in One Dimension

In this chapter we shall give a framework for direct minimization of functionals defined on infinite dimensional space. In the first section we consider a simple problem of minimizing a Dirichlet integral on an interval in $\mathbb{R}$ followed by introduction of Sobolev spaces. In the following section we apply the direct minimizing methods to the integrals defined on Sobolev spaces of one dimension. For further details, refer to [2].

To begin with, we quickly review a few basic definitions that we will be referring to in this chapter.

Definition 4.1. [5] Let I be an open interval in $\mathbb{R}$. Then we say that $u \in L^p(I)$ has a weak derivative $v \in L^p(I)$ if

$$\int_I u\phi'\, dx = -\int_I v\phi\, dx \quad \forall \phi \in C_c^\infty(I).$$

Definition 4.2. Given a subset $A \subset X$ of a space X, the sequential closure $[A]_{\text{seq}}$ is the set $[A]_{\text{seq}} = \{x \in X : a_n \to x, a_n \in A\}$. That is, $[A]_{\text{seq}}$, the set of all points $x \in X$ for which there is a sequence in A that converges to x.

4.1 Sobolev Spaces

In this section we introduce the notion of Sobolev spaces by solving the problem of minimizing the one dimensional Dirichlet integral in $(0,1)$

$$\mathcal{D}(u) = \int_0^1 |u'(x)|^2\, dx$$

in the class $K(\alpha,\beta) := \{u \in C^1([0,1]) : u(0) = \alpha, u(1) = \beta\},\ \alpha,\beta \in \mathbb{R}$, where u' denotes weak derivative of u.

By definition of $K(\alpha,\beta)$, it is clear that

$$0 \leq \inf_{u\in K(\alpha,\beta)} \mathcal{D}(u) < +\infty. \tag{4.1}$$

Let $\{u_k\} \subset K(\alpha,\beta)$ be a minimizing sequence for $\mathcal{D}(u)$. Then, $\mathcal{D}(u_k) \to \inf\limits_{K(\alpha,\beta)} \mathcal{D}$. Thus by definition of infimum, $\mathcal{D}(u_k) \leq \inf\limits_{K(\alpha,\beta)} \mathcal{D}+1$ for all k. That is,

$$\int_0^1 |u_k'|^2\,dx \leq \inf_{K(\alpha,\beta)} \mathcal{D}+1, \quad k \geq 1. \tag{4.2}$$

By fundamental theorem of calculus, we know that for all $x, y \in [0,1]$,

$$|u_k(x)-u_k(y)| \leq \int_y^x |u_k'(t)|\,dt.$$

Applying Hölder inequality on the right hand side of the above inequality gives

$$|u_k(x)-u_k(y)| \leq \left(\int_0^1 |u_k'(t)|^2\,dt\right)^{\frac{1}{2}} |x-y|^{\frac{1}{2}} \quad \forall x,y \in [0,1]. \tag{4.3}$$

Taking $y=0$,

$$|u_k(x)|-|u_k(0)| \;\leq\; |u_k(x)-u_k(0)| \;\leq\; \left(\int_0^1 |u_k'(t)|^2\,dt\right)^{\frac{1}{2}} |x|^{\frac{1}{2}}$$

Thus we have,

$$|u_k(x)| \;\leq\; |u_k(0)| + \left(\int_0^1 |u_k'(t)|^2\,dt\right)^{\frac{1}{2}} \tag{4.4}$$

From (4.3) and (4.4), we can say that $\{u_k\}$ is a bounded sequence in $C^{0,\frac{1}{2}}([0,1])$ with the help of (4.2) and (4.1). In fact, the functions u_k are uniformly bounded and equicontinuous. The Arzelà-Ascoli theorem thus implies that $\{u_k\}$ contains a uniformly convergent subsequence. Calling this subsequence again as $\{u_k\}$, we have,

$$u_k \to u \quad \text{uniformly in } [0,1].$$

By (4.2) we note that the L^2-norms of u_k' are bounded. Now, since $L^2(0,1)$ is a reflexive space, every bounded sequence in $L^2(0,1)$ will have a weak convergent subsequence. Hence u_k' converges weakly in $L^2(0,1)$ to a function, say v. Thus,

$$\int_0^1 u_k'\phi \, dx \to \int_0^1 v\phi \, dx \quad \forall \phi \in L^2(0,1). \tag{4.5}$$

Since the set of all smooth compactly supported functions is dense in $L^p([0,1])$, the above convergence also holds for all $\phi \in C_c^\infty(0,1)$. Moreover, as u_k' is a weak derivative of u_k, by definition we have,

$$\int_0^1 u_k'\phi \, dx = -\int_0^1 u_k\phi' \, dx.$$

Under the limit $k \to \infty$ and using (4.5) we conclude that

$$\int_0^1 u\phi' \, dx = -\int_0^1 v\phi \, dx \quad \forall \phi \in C_c^\infty(0,1)$$

The above equality says that v is a weak derivative of u, which we will denote by u' instead of v.

In summary, we define the following convergence, named as $\tau-$convergence on $K(\alpha,\beta)$.

Definition 4.3. A sequence of functions $u_k \in K(\alpha,\beta)$ with bounded Dirichlet's integral is said to $\tau-$converge to u if

$$\begin{cases} u_k \to u & \text{uniformly in } [0,1] \\ u_k' \to u' & \text{weakly in } L^2[0,1] \end{cases}$$

where u' denotes the weak derivative of u.

By the above definition it is clear that $K(\alpha,\beta)$ is not closed with respect to $\tau-$convergence. The following example illustrates this.

Example 4.1. Consider a sequence of functions $\{u_n\}$ defined on $(-1,1)$ by

$$u_n(x) := 1 - \left(1 + \frac{1}{n}\right)^{\frac{1}{2}} + \left(x^2 + \frac{1}{n}\right)^{\frac{1}{2}}$$

in the class $K(1,1) = \{u \in C^1([-1,1]) : u(-1) = 1 = u(1)\}$. It is clear that $u_n \to u = |x|$ as $n \to \infty$. We now claim that $u_n \xrightarrow{\tau} u$ as $n \to \infty$.
Define $G : (-1,1) \to \mathbb{R}$ by

$$G(x) := 1 - \left(1 + \frac{1}{n}\right)^{\frac{1}{2}} + \left((x+1)^2 + \frac{1}{n}\right)^{\frac{1}{2}}$$

Then $|u_n(x)| \leq G(x)$ for all $x \in (-1,1)$. Therefore by using Dominated convergence theorem we have,

$$\lim_{n\to\infty} \int_{-1}^{1} u_n \varphi' \, dx = \int_{-1}^{1} u\varphi' \, dx \quad \forall \varphi \in C_c^\infty(-1,1).$$

Hence the claim follows. But $u(x) \notin K(1,1)$. Thus $K(1,1)$ is not closed.

We introduce a completion $K_{(\tau)}(\alpha,\beta)$, as a sequential closure of $K(\alpha,\beta)$ with respect to the $\tau-$convergence.

We now define the Dirichlet integral of $u \in K_{(\tau)}(\alpha,\beta)$ as Lebesgue integral of $|u'|^2$ by,

$$\mathcal{D}_{(\tau)}(u) := \int_0^1 |u'|^2 \, dx.$$

We claim that if $\{u_k\}$ is a minimizing sequence in $K_{(\tau)}(\alpha,\beta)$ then $\mathcal{D}_{(\tau)}$ is lower semi-continuous on $K_{(\tau)}$. To see this, we use the following inequality for any two functions $u_1, u_2 \in L^2(0,1)$,

$$|u_1|^2 - |u_2|^2 \geq 2u_2(u_1 - u_2).$$

Since $\{u_k\}$ is a minimizing sequence in $K_{(\tau)}(\alpha,\beta)$, by the previous discussion, there exists $u \in K_{(\tau)}(\alpha,\beta)$ such that u_k $\tau-$converges to u. Using this we have,

$$\int_0^1 |u_k'|^2\,dx - \int_0^1 |u'|^2\,dx \geq 2\int_0^1 u'(u_k' - u')\,dx.$$

Passing through the limit as $k \to \infty$ and using the fact that $u_k' \to u'$ weakly in $L^2[0,1]$ we arrive at the following inequality.

$$\liminf_{k\to\infty} \int_0^1 |u_k'|^2\,dx \geq \int_0^1 |u'|^2\,dx.$$

That is, $\mathcal{D}_{(\tau)}(u) \leq \liminf\limits_{k\to\infty} \mathcal{D}_{(\tau)}(u_k)$. Therefore, $\mathcal{D}_{(\tau)}$ is sequentially lower semicontinuous on $K_{(\tau)}$ in context of definition 4.3. Hence by using the direct method of minimization, Theorem 2.3 of chapter 2, we conclude that $\mathcal{D}_{(\tau)}$ achieves minimum in $K_{(\tau)}(\alpha,\beta)$.

As of now, we do not know what precisely the class $K_{(\tau)}(\alpha,\beta)$ is! We now analyze the nature of functions in $K_{(\tau)}(\alpha,\beta)$ and give a proper definition.

Let $Y = C^\infty([0,1])$ equipped with the norm,

$$||u||_{H^{1,2}(0,1)} := \left(\int_0^1 |u|^2\,dx + \int_0^1 |u'|^2\,dx\right)^{\frac{1}{2}}$$

induced by the scalar product

$$< u, v >_{H^{1,2}(0,1)} := \int_0^1 (uv + u'v')\,dx.$$

Let $H^{1,2}(0,1)$ denote completion of Y with respect to this inner product. Clearly $H^{1,2}(0,1)$ is a Hilbert space with the norm

$$||u||_{H^{1,2}(0,1)} = \left(< u, u >_{H^{1,2}(0,1)}\right)^{\frac{1}{2}}.$$

In addition, from (4.3) we know that

$$H^{1,2}(0,1) \subset C^{0,\frac{1}{2}}(0,1).$$

In fact one can show that $H^{1,2}(0,1) = \{u \in C^{0,\frac{1}{2}}(0,1) : u' \in L^2(0,1)\}$. Equivalently, $H^{1,2}(0,1) = \{u \in L^2(0,1) : u' \in L^2(0,1)\}$.

We now give a formal definition of Sobolev spaces for every real number $p \geq 1$.

Let $I \subset \mathbb{R}$ be an open interval. Let $C^\infty(\bar{I})$ be given the norm

$$||u||_{H^{1,p}(I)} := \left(\int_0^1 \left(|u|^p \; dx + |u'|^p \right) \; dx \right)^{\frac{1}{p}}. \tag{4.6}$$

Definition 4.4. The completion of $C^\infty(\bar{I})$ with respect to norm in (4.6) is denoted by $H^{1,p}(I)$. The spaces $H^{1,p}(I), 1 \leq p < \infty$ are called Sobolev spaces. We denote by $H_0^{1,p}(I)$ the completion of $C_0^\infty(I)$ with respect to the norm in (4.6).

It can be shown that the space $H^{1,\infty}(I) = \{u \in L^\infty[0,1] : u' \in L^\infty[0,1]\}$ endowed with the norm $||u||_{H^{1,\infty}(I)} = ||u||_{L^\infty(I)} + ||u'||_{L^\infty(I)}$ is exactly the space of Lipschitz functions on $[0,1]$ endowed with the norm $||u||_{Lip(I)} = ||u||_{L^\infty(I)} + \sup\limits_{\substack{x \neq y \\ x,y \in I}} \frac{|u(x)-u(y)|}{|x-y|}$.

4.2 Semicontinuity and Existence Results

In this section, we will apply direct methods of extremization for functionals $\mathcal{F}(u)$ defined on Sobolev spaces. We first state the necessary and sufficient conditions for the semicontinuity of the functional followed by a general result of existence of a minimizer.

Let I be a bounded open interval in $\mathbb{R}$. We define the Sobolev space $H^{1,p}(I,\mathbb{R}^n)$ as

$$H^{1,p}(I,\mathbb{R}^n) = \{u = \left(u^1, ..., u^N\right) : u^i \in H^{1,p}(I), i = 1, ..., N\}.$$

We define

$$C_c^\infty(I,\mathbb{R}^n) = \{u = \left(u^1, ..., u^N\right) : u^i \in C_c^\infty(I), i = 1, ..., N\}.$$

We recall the τ-convergence defined in previous section (see definition 4.3). We can show that a sequence of functions $\{u_k\}$ converges weakly to some u in $H^{1,p}(I,\mathbb{R}^n)$, $p \geq 1$, if and only if u_k $\tau-$converges to u.

4.2.1 A lower Semicontinuity Theorem

Let $F(x,u,p) : I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function. We shall consider the problem of minimizing the following functional

$$\mathcal{F}(u) = \int_I F(x, u(x), u'(x))\, dx, \quad u \in H^{1,p}(I). \tag{4.7}$$

Definition 4.5. We say that $\mathcal{F}$ is lower semicomtinuous with respect to the uniform convergence of equi-Lipschitzian function if

$$\mathcal{F}(u) \leq \liminf_{k\to\infty} \mathcal{F}(u_k)$$

for all u_k with $\sup_k |u_k|_{Lip} < +\infty$ and $u_k \to u$ uniformly.

Remark 4.1. If $\mathcal{F}(\cdot, I)$ is weakly lower semicontinuous then for any interval $(a, b) \subset (0, 1)$, $\mathcal{F}(\cdot, (a, b))$ is also weakly lower semicontinuous. This can be shown by extending any $u \in H^{1,m}(a, b)$ by constants on either side.

One can now ask, under which conditions on the integrand $F(x, u, p)$ is the functional $\mathcal{F}(u)$ weakly sequentially lower semicontinuous in $H^{1,p}(I),\ p \geq 1$. The following theorem gives an insight to an answer to this question.

Theorem 4.1. *If $\mathcal{F}(u)$ is weakly sequentially lower semicontinuous with respect to the uniform convergence of equi-Lipschitzian function, then for any $(x_0, u_0, p_0) \in I \times \mathbb{R}^N \times \mathbb{R}^N$ we have,*

$$\int_I F(x_0, u_0, p_0 + \varphi'(x))\, dx \geq F(x_0, u_0, p_0)\ meas(I) \quad \forall \varphi \in C_c^\infty(I, \mathbb{R}^n). \tag{4.8}$$

In particular, the integrand $F(x, u, p)$ is convex in p for any $(x, u) \in I \times \mathbb{R}^N$.

Proof. We first consider the simpler case where $F = F(p)$. Without loss of generality assume that $I = (0, 1)$. Given a $\varphi \in C_c^\infty(I, \mathbb{R}^n)$, we extend it periodically to $\mathbb{R}$ and define a function φ_ν, $\nu \geq 1$ by

$$\varphi_\nu(x) := \frac{1}{\nu}\varphi(\nu x).$$

We also define, for $(u_0, p_0) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$\begin{aligned} v(x) :=& u_0 + p_0 x, \\ u_\nu(x) :=& v(x) + \varphi_\nu(x). \end{aligned}$$

Claim: The sequence $\{u_\nu\}$ has equibounded Lipschitz constants.
To prove this, consider x, y in I. Then $\sup\limits_{x\neq y} \frac{|u_\nu(x)-u_\nu(y)|}{|x-y|} \leq |p_0| + \frac{1}{\nu} \sup\limits_{x\neq y} \frac{|\varphi(\nu x)-\varphi(\nu y)|}{|x-y|} = |p_0| + \frac{1}{\nu}|\varphi'(\xi)|$ for some $\xi = \xi(x,y)$ lying between x and y. Hence $|u_\nu - v|_{Lip(I)} \leq |p_0| + \frac{1}{\nu}|\varphi'|_{L^\infty(I)}$. Clearly $u_\nu \to v$ uniformly as $\nu \to \infty$. Thus by our assumption $\mathcal{F}(v) \leq \liminf\limits_{\nu\to\infty} \mathcal{F}(u_\nu)$. That is,

$$F(p_0)\, meas(I) = \int_I F(p_0)\, dx \leq \liminf_{\nu\to\infty} \int_I F(p_0 + \varphi'(\nu x))\, dx.$$

If we now change the variable $\nu x = y$ and use the periodicity of φ, we have

$$F(p_0)\, meas(I) \leq \liminf_{\nu\to\infty} \frac{1}{\nu} \int_0^\nu F(p_0 + \varphi'(y))\, dy = \int_0^1 F(p_0 + \varphi'(y))\, dy.$$

We have thus proved (4.8) for the case $F = F(p)$.
To prove (4.8) for general case $F = F(x, u, u')$, we again assume I to be a unit interval $(0,1)$ and extend φ periodically to $\mathbb{R}$. For $x_0 \in I$, consider the interval $R = (x_0, x_0 + h)$ where h is so small that $R \subset I$ and define the following,

$$\begin{aligned}
\varphi_\nu(x) &:= \frac{h}{\nu}\varphi\left(\frac{\nu}{h}(x - x_0)\right),\\
v(x) &:= u_0 + p_0(x - x_0),\\
u_\nu(x) &:= v(x) + \varphi_\nu(x).
\end{aligned}$$

As before, we can show that the sequence $\{u_\nu\}$ has equibounded Lipschitz constants and converges to v uniformly. We now divide the interval R into subintervals I_i, each of size $\frac{h}{\nu}$ as follows.
Define $I_i := (x_i, x_{i+1})$, $x_i = x_0 + i\frac{h}{\nu}$, $i = 0, ..., \nu - 1$. Since φ is a periodic function with period 1, φ_ν is a periodic function with period $\frac{h}{\nu}$. Thus we have,

$$\varphi_\nu(x) = \varphi_\nu(x \pm i\frac{h}{\nu}) \quad i = 0, ..., \nu - 1.$$

On simplification we obtain, $\varphi_\nu(x) = \frac{h}{\nu}\varphi\left(\frac{\nu}{h}(x - x_i)\right)$ so that

$$\varphi'_\nu(x) = \varphi'\left(\frac{\nu}{h}(x - x_i)\right). \tag{4.9}$$

Using the above data we have,

$$\begin{aligned}
\int_{I_i} F(x,\, u_\nu(x),\, u'_\nu(x))\, dx &= \int_{x_i}^{x_{i+1}} F\left(x,\, u_\nu,\, p_0 + \varphi'_\nu\right)\, dx \\
&= \int_{x_i}^{x_i+\frac{h}{\nu}} F\left(x,\, u_\nu,\, p_0 + \varphi'\left(\frac{\nu}{h}(x - x_i)\right)\right)\, dx
\end{aligned}$$

where the last equality follows from (4.9). Now changing the variable, $\frac{\nu}{h}(x - x_i) = y$, the above equation becomes

$$\int_{I_i} F(x,\, u_\nu(x),\, u'_\nu(x))\, dx = \frac{h}{\nu}\int_{I} F\left(x_i + \frac{h}{\nu}y,\, u_\nu\left(x_i + \frac{h}{\nu}y\right),\, p_0 + \varphi'(y)\right)\, dy.$$

Therefore we have,

$$\begin{aligned}
\int_{R} F(x,\, u_\nu(x),\, u'_\nu(x))\, dx &= \sum_{i=0}^{\nu-1}\int_{I_i} F(x,\, u_\nu(x),\, u'_\nu(x))\, dx \\
&= \sum_{i=0}^{\nu-1}\frac{h}{\nu}\int_{I} F\left(x_i + \frac{h}{\nu}y,\, u_\nu\left(x_i + \frac{h}{\nu}y\right),\, p_0 + \varphi'(y)\right)\, dy \\
&= \sum_{i=0}^{\nu-1}\frac{h}{\nu}\int_{I} F\left(x_0 + i\frac{h}{\nu} + \frac{h}{\nu}y,\, u_\nu\left(x_0 + i\frac{h}{\nu} + \frac{h}{\nu}y\right),\, p_0 + \varphi'(y)\right)\, dy.
\end{aligned}$$

Taking limit as $\nu \to \infty$ both sides we get,

$$\lim_{\nu\to\infty}\int_{x_0}^{x_0+h} F(x,\, u_\nu(x),\, u'_\nu(x))\, dx = \int_{x_0}^{x_0+h}\left(\int_{I} F\left(x,\, v(x),\, p_0 + \varphi'(y)\right)\, dy\right)\, dx.$$

From our assumption, we also have, $\liminf\limits_{\nu\to\infty}\int_R F(x,\, u_\nu(x),\, u'_\nu(x))\, dx \geq \int_R F(x,\, v(x),\, v'(x))\, dx$. That is,

$$\int_{x_0}^{x_0+h} dx \int_I F\left(x,\ v(x),\ p_0 + \varphi'(y)\right) dy \geq \int_{x_0}^{x_0+h} F(x,\ v(x),\ v'(x))\, dx.$$

Dividing by h and taking limit as $h \to 0$ both sides we get,

$$\int_I F\left(x_0, v(x_0), p_0 + \varphi'(y)\right) dy \geq F(x_0, v(x_0), v'(x_0)).$$

Substituting the values of $v(x_0)$ and $v'(x_0)$ in the above inequality we arrive at equation (4.8).

We now show that $F(x,u,p)$ is convex in p. Once again we assume that $I = (0,1)$. Set $p = \lambda p_1 + (1-\lambda)p_2,\ \lambda \in (0,1);\ p_1, p_2 \in \mathbb{R}^N$. Define a function $\tilde{\varphi} : (0,1) \to \mathbb{R}^N$ such that

$$\tilde{\varphi}'(x) = \begin{cases} p_1 & \text{if } x \in (0,\lambda) \\ p_2 & \text{if } x \in (\lambda, 1). \end{cases}$$

By fundamental theorem of calculus, $\tilde{\varphi}(1) - \tilde{\varphi}(0) = \int_0^1 \tilde{\varphi}'(t)\, dt$. This implies that,

$$\tilde{\varphi}(1) = \tilde{\varphi}(0) + \int_0^\lambda p_1\, dt + \int_\lambda^1 p_2\, dt = \tilde{\varphi}(0) + p. \tag{4.10}$$

Now define $\varphi(x) := \tilde{\varphi}(x) - \tilde{\varphi}(0) - px$ so that with the help of (4.10) we have $\varphi(0) = 0 = \varphi(1)$. Applying (4.8) now yields

$$\begin{aligned} F(x_0, u_0, p) &\leq \int_0^1 F\left(x_0, u_0, p + \varphi'\right) dx \\ &= \int_0^1 F\left(x_0, u_0, \tilde{\varphi}'\right) dx \\ &= \int_0^\lambda F(x_0, u_0, p_1)\, dx + \int_\lambda^1 F(x_0, u_0, p_2)\, dx \\ &= \lambda F(x_0, u_0, p_1) + (1-\lambda) F(x_0, u_0, p_2). \end{aligned}$$

This shows the convexity of F in p. ∎

Remark 4.2. The above theorem holds in the situation of $H^{1,1}$ spaces. That is, if $u_k \xrightarrow{w} u$ in $H^{1,1}$ then $\mathcal{F}(u) \leq \liminf\limits_{k\to\infty} \mathcal{F}(u_k)$ holds. The proof is done by modifying the above proof which we will not give here.

The next theorem investigates the sufficient conditions for weak sequential lower semicontinuity of $\mathcal{F}(u)$.

Theorem 4.2. *Let I be a bounded open interval in $\mathbb{R}$ and let $F(x,u,p)$ be a function satisfying the following conditions:*

(i) *F and F_p are continuous in (x,u,p);*

(ii) *F is non-negative or bounded from below by L^1 function;*

(iii) *F is convex in p.*

Then the functional $\mathcal{F}(u)$ in (4.7) is sequentially weakly lower semicontinuous in $H^{1,m}(I,\mathbb{R}^N)$ for all $m \geq 1$, i.e. if $u_k \xrightarrow{w} u$ in $H^{1,m}(I,\mathbb{R}^N)$ then

$$\mathcal{F}(u) \leq \liminf_{k\to\infty} \mathcal{F}(u_k). \tag{4.11}$$

Proof. It is sufficient to prove the theorem for $m = 1$, since $u_k \xrightarrow{w} u$ in $H^{1,m}(I,\mathbb{R}^N)$, $m > 1$ implies that $u_k \xrightarrow{w} u$ in $H^{1,1}(I,\mathbb{R}^N)$, since I is a bounded interval in $\mathbb{R}$.
Let $\{u_k\}$ be a sequence such that $u_k \xrightarrow{w} u$ in $H^{1,1}(I,\mathbb{R}^N)$. Then $u_k \to u$ uniformly on $\bar{I}$. Let $\liminf\limits_{k\to\infty} \mathcal{F}(u_k)$ be finite. If not (4.11) is trivially true. We also note that $\mathcal{F}(u)$ is finite for any $u \in H^{1,m}(I,\mathbb{R}^N)$. Then for any $\epsilon > 0$ we can find a compact subset $K \subset I$ such that $meas(I \setminus K) < \epsilon$ and the following are true:

(a) $u_k \to u$ uniformly in K by Egorov's theorem,

(b) u and u' are continuous in K by Lusin's theorem,

(c) $\int_K F(x,u,u')\,dx \geq \int_I F(x,u,u')\,dx - \epsilon$ by Lebesgue's absolute continuity theorem.

From condition (iii) we have,

$$\begin{aligned}
\mathcal{F}(u_k) = \int_I F(x,u_k,u_k')\,dx &\geq \int_K F(x,u_k,u_k')\,dx \\
&\geq \int_K F_p(x,u_k,u')(u_k'-u')\,dx + \int_K F(x,u_k,u')\,dx \\
&= \int_K F(x,u_k,u')\,dx + \int_K F_p(x,u,u')(u_k'-u')\,dx \\
&\quad + \int_K \left[F_p(x,u_k,u') - F_p(x,u,u')\right](u_k'-u')\,dx.
\end{aligned}$$

Now, from (b) we know that u and u' are continuous on K, therefore $F_p(x,u,u')$ is bounded on K. Hence $\int_K F_p(x,u,u')(u'_k - u')\,dx \to 0$ as $k \to \infty$. Also, we note that $(u'_k - u')$ are equibounded in $L^1(I)$ since $\{u'_k\}$ is a weakly convergent sequence and hence weakly sequentially compact. As F_p is continuous, from (a) we have, $F_p(x,u_k,u') - F_p(x,u,u') \to 0$ uniformly on K as $k \to \infty$. Thus, $\int_K \big[F_p(x,u_k,u') - F_p(x,u,u')\big](u'_k - u')\,dx \to 0$ as $k \to \infty$. Therefore using (ii) we conclude that

$$\liminf_{k\to\infty} \int_I F(x,u_k,u'_k)\,dx \geq \liminf_{k\to\infty} \int_K F(x,u_k,u'_k)\,dx \geq \int_K F(x,u,u')\,dx \geq \int_I F(x,u,u')\,dx - \epsilon.$$

Since the above inequality holds for all ϵ, we obtain (4.11). ■

4.2.2 Existence results in Sobolev spaces

Before proving the main result, we state important definitions and theorems.

Definition 4.6. Let I be an open interval in $\mathbb{R}$. A family of functions f_n is said to be equi-absolutely continuous on I if for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$ such that

$$\sum_{i=1}^{N} |f_n(\beta_i) - f_n(\alpha_i)| \leq \epsilon$$

for all finite systems of nonoverlapping intervals $[\alpha_i, \beta_i]$, $i = 1, \ldots, N$ in I with $\sum_{i=1}^{N}(\beta_i - \alpha_i) < \delta$ and for all $n \in \mathbb{N}$.

Definition 4.7. Let $F(x,u,p)$ be a function defined on $I \times \mathbb{R}^N \times \mathbb{R}^N$, $N \geq 1$, where I is a bounded open interval in $\mathbb{R}$. We say that $F(x,u,p)$ has superlinear growth in p at infinity if there exists a function $\Theta(p) : \mathbb{R}^N \to \mathbb{R}$ such that

$$\begin{cases} F(x,u,p) \geq \Theta(p) & \forall\ x,\ u,\ p, \\ \frac{\Theta(p)}{|p|} \to \infty & \text{as } |p| \to \infty. \end{cases}$$

We recall the following two well known theorems.

Theorem 4.3. *Let Ω be a bounded open interval in $\mathbb{R}$. Let $\mathcal{C}$ be a subset of $L^1(\Omega)$. Then the following are equivalent:*

(i) *the functions u in $\mathcal{C}$ are equibounded in $L^1(\Omega)$ and the set functions*

$$E \to \int_E |u|\, dx, \quad E \subset \Omega,\ u \in \mathcal{C},$$

are equiabsolutely continuous;

(ii) *there exists a function* $\Theta : (0, +\infty) \to \mathbb{R}$ *such that*

$$\lim_{t\to\infty} \frac{\Theta(t)}{t} = \infty,$$
$$\sup_{u\in\mathcal{C}} \int_\Omega \Theta(|u|)\, dx < \infty.$$

Theorem 4.4. *Let* $\{u_k\}$ *be a sequence in* $H^{1,1}(a,b)$, $a, b \in \mathbb{R}$. *Suppose that*

(i) $\sup_k ||u_k||_{1,1} < \infty$,

(ii) *the set functions* $E \to \int_E |Du_k|\, dx, \quad E \subset (a,b)$ *are equiabsolutely continuous.*

Then there exists a subsequence $\{u_{k_i}\}$ *such that* $u_{k_i} \xrightarrow{w} u$ *in* $H^{1,1}(a,b)$.

We can now prove the following existence theorem.

Theorem 4.5. *Let* I *be a bounded open interval in* $\mathbb{R}$. *Suppose that the function* $F(x,u,p)$ *satisfies the following conditions:*

(i) $F(x,u,p)$ *and* $F_p(x,u,p)$ *are continuous in* (x,u,p);

(ii) $F(x,u,p)$ *is convex in* p;

(iii) $F(x,u,p)$ *has a superlinear growth in* p.

Then there exists a minimizer of

$$\mathcal{F}(u) := \int_I F(x,u,u')\, dx$$

in the class $\mathcal{C}(\alpha,\beta) := \{u \in H^{1,1}\big((a,b),\mathbb{R}^N\big) : u(a) = \alpha, u(b) = \beta\}$, *where* α, β *are fixed vectors in* $\mathbb{R}^N$.

Proof. Let $\{u_k\}$ be a minimizing sequence of $\mathcal{F}(u)$ in $\mathcal{C}(\alpha,\beta)$. Then $\mathcal{F}(u_k) \to \inf_{w\in\mathcal{C}} \mathcal{F}(w)$. Also, for large $k \geq 1$,

$$\inf_{w\in\mathcal{C}} \mathcal{F}(w) + 1 \geq \mathcal{F}(u_k) = \int_I F(x,u_k,u_k')\, dx \geq \int_I \Theta(u_k'(x))\, dx$$

for some function Θ as in Definition 4.7. Thus the integrals $\int_I \Theta(u_k'(x))\,dx$ are equibounded. From Theorem 4.3 we infer that the sequence $\{u_k\}$ is equibounded in $H^{1,1}((a,b),\mathbb{R}^N)$ and the set functions $E \to \int_E |u_k'|\,dx,\ E \subset (a,b),\ u_k \in \mathcal{C}$ are equiabsolutely continuous. Therefore by Theorem 4.4, there exists a subsequence denoted by $\{u_k\}$ such that $u_k \xrightarrow{w} u$ in $H^{1,1}((a,b),\mathbb{R}^N)$. Hence applying Theorem 4.2 gives,

$$\mathcal{F}(u) \leq \liminf_{k\to\infty} \mathcal{F}(u_k).$$

At once we have,

$$\inf_{w\in\mathbb{C}} \mathcal{F}(w) = \liminf_{k\to\infty} \mathcal{F}(u_k) \geq \mathcal{F}(u) \geq \inf_{w\in\mathbb{C}} \mathcal{F}(w).$$

This implies that $\mathcal{F}(u) = \inf_{w\in\mathbb{C}} \mathcal{F}(w)$. Therefore $\mathcal{F}$ attains a minimum in the class $\mathcal{C}(\alpha,\beta)$.

■

We remark that the convexity condition in the above theorem cannot be eliminated. The following example demonstrates this.

Example 4.2. Consider the functional

$$\mathcal{F}(u) = \int_0^1 \left[(1-|u'|^2)^2 + u^2\right]\,dx.$$

The integrand $F(u,p) = \left[(1-|p|^2)^2 + u^2\right]$ is not convex in p. We claim that the minimum problem

$$\min\{\mathcal{F}(u) : u \in H^{1,1}(0,1),\ u(0) = u(1) = 0\}$$

has no solution. To see this, we define a function $\varphi(x)$ by $\varphi(x) := \frac{1}{2} - |x - \frac{1}{2}|$ on $[0,1]$ and extend it periodically to $\mathbb{R}$ as in Theorem 4.1. The functions u_h defined as follows

$$u_h(x) := \frac{1}{h}\varphi(hx), \quad (h \in \mathbb{N})$$

are in $H^{1,1}(0,1)$ and $u_h(0) = 0 = u_h(1)$. Also, $|u_h'(x)| = 1$ almost everywhere. Therefore, since φ is periodic,

$$\mathcal{F}(u_h) = \frac{1}{h^2} \int_0^1 \varphi^2(hx)\, dx$$

By using the change of variable $hx = y$, we have $\frac{1}{h^3} \int_0^h \varphi^2(y)\, dy = \frac{1}{h^2} \int_0^1 \varphi^2(y)\, dy$. Therefore $\mathcal{F}(u_h) \to 0$ as $h \to \infty$. But there is no function $u \in H^{1,1}(0,1)$ satisfying $\mathcal{F}(u) = 0$ because this would mean that $\int_0^1 u^2\, dx = 0$ $i.e.$ $u = 0$. But $\mathcal{F}(0) = 1$. Hence the minimum problem does not have any solution.

We end this chapter with an example which shows that condition (iii) in Theorem 4.5 cannot be dropped.

Example 4.3. Consider for every $\alpha \in \mathbb{R}$ and $p > 1$ the functional

$$\mathcal{F}_{\alpha,p}(u) := \int_0^1 x^\alpha\, |u'|^p\, dx$$

defined for every $u \in H^{1,1}(0,1)$, and the following minimum problem

$$\min\{\mathcal{F}_{\alpha,p}(u) : u \in H^{1,1}(0,1),\ u(0) = a,\ u(1) = b\} \tag{4.12}$$

where $a, b \in \mathbb{R}$ with $a \neq b$.

It is easy to see that the integrand $x^\alpha\, |u'|^p$ does not have a superlinear growth when $\alpha > 0$. For instance consider a case when $\alpha = 2 = p$ and $a \neq b$.
Taking a sequence $u_h(x) = a + (b-a)\frac{\arctan(hx)}{\arctan h}$,

$$u_h'(x) = \frac{(b-a)}{\arctan h}\,\frac{h}{1 + h^2 x^2}.$$

Therefore,

$$\mathcal{F}_{2,2}(u_h) = \frac{(b-a)^2}{2h\arctan^2 h}\left[\arctan h - \frac{h}{1+h^2}\right] \to 0 \text{ as } h \to \infty.$$

But there is no u in $H^{1,1}(0,1)$ satisfying $\mathcal{F}_{2,2}(u) = 0$ because this would imply that $u'(x) = 0$ almost everywhere, $i.e$, $u(0) = u(1)$.
However, when $\alpha < 0$, all three conditions of existence Theorem 4.5 are satisfied and problem (4.12) exhibits a solution.

In order to apply direct methods to functionals $\mathcal{F}(u)$, its semicontinuity is the key point. Hence, we explored the necessary and sufficient conditions for sequential weak lower semicontinuity of the functionals $\mathcal{F}$. We conclude that convexity of the integrand $F(x,u,p)$ plays a crucial role in proving existence of a minimum of $\mathcal{F}(u)$ under suitable coerciveness assumptions; where F has to be a non-negative continuous function defined on $I \times \mathbb{R}^N \times \mathbb{R}^N$ with F_p continuous in (x,u,p).

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