Erdos-Ko-Rado and Kruskal Katona Theorems for Discrete Structures

April 1, 2014

A thesis submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Program

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April, 2014

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Declaration

This is to certify that this thesis entitled ”Erdos-Ko-Rado and Kruskal Katona Theorems for Discrete Structures” submitted towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune, represents work carried out by Sourajit Basu under the supervision of Dr. Soumen Maity.

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Dedicated to my parents.

Acknowledgements

I would like to thank Dr. Soumen Maity for his incredible patience and understanding while guiding and encouraging me over the course of this thesis. His interactive teaching in the classroom over the course of the last 5 years has influenced me greatly.

I would also like to thank –

Prof. A. Raghuram for his valuable advice during trying times and his belief in me.

Dr. Anupam Kumar Singh for being a wonderful faculty advisor and for his innovative teaching methods.

Dr. Anindya Goswami for showing me the beauty of rigour while addressing fundamental concepts.

Dr. Anisa Chorwadwala for making learning fun while covering oceans of analysis.

Dr. Ayan Mahalonobis for his insightful and thought provoking questions in and outside mathematics.

Dr. Rama Mishra for first introducting me to "real" mathematics.

Dr. Chandrasheel Bhagwat for his contagious energy and encouraging me outside the classroom.

Abstract

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The project we have undertaken concerns extremal combinatorics. Two core concepts in extremal set theory are intersecting families and shadows. A family of subsets of a given set X whose members have size k and pair wise intersect is called an intersecting family. The main results for intersecting families are the Erdos-Ko-Rado and Hilton-Milner theorems, which give an upper bound on the maximum size of intersecting families. Shadow is a property of a family of k-element subsets of a set X. It consists of all $(k-1)$-element subsets of the set X contained in at least one member of the family. The principal result for shadows is the Kruskal-Katona theorem, which gives a lower bound on the size of a shadow. This thesis aims to further understand analogs of Erdos-Ko-Rado, Hilton-Milner and Kruskal-Katona Theorems for other discrete structures such as vector spaces and multisets.

Contents

Chapter 1

Introduction

The aim of this thesis is to study results from the field of extremal combinatorics, specifically extremal set theory questions and their analogs for other discrete structures like multisets and vector spaces. While studying results of extremal set theory, we see the use of certain tools from algebra and probability and also great potential for finding analogous results in other discrete structures.

In general, given a finite set X, extremal combinatorics ask how large or small a family of subsets of X can be if it certain restrictions are satisfied. Two core concepts in extremal set theory are intersecting families and shadows. A family of subsets of a given set X whose members have size k and pair wise intersect is called an intersecting family. The main results for intersecting families are the Erdos-Ko-Rado and Hilton-Milner theorems, which give an upper bound on the maximum size of intersecting families. Shadow is a property of a family of k-element subsets of a set X. It consists of all $(k-1)$-element subsets of the set X contained in at least one member of the family. The principal result for shadows is the Kruskal-Katona theorem, which gives a lower bound on the size of a shadow.

This thesis further aims at understanding of shadows and intersecting families in other structures, primarily multisets and vector spaces. In the latter case, the goals are to study analogs of the Erdos-Ko-Rado, Hilton-Milner, and Kruskal-Katona theorems and understand the techniques to their proofs. In the former case, however, in addition to studying analogs of the Erdos-Ko-Rado and Kruskal-Katona theorems, we explore the special case of a conjecture, which is an extension of the Erdos-Ko-Rado in multisets.

1.1 Outline of thesis

We now present an outline of this thesis. First, we present some results in Section 1.2. Then in 1.3, we introduce some theory about sets, multisets,

vector spaces and hypergraphs.
In Chapter 2, we discuss some literature of combinatorial results from extremal set theory. We then look at analogous results on vector space over finite fields, particularly Erdos-Ko-Rado theorem and Lovszs version of the Kruskal-Katona theorem. We also look at Erdos-Ko-Rado theorem for multisets.
In Chapter 3, we show a construction technique for multisets proving a one way inequality for the Erdos-Ko-Rado theorems extension to t-intersecting k-multisets for the case of $t = 2$.
Finally, in Chapter 4, we present the conclusions and scope for further study.

1.2 Results

The main problem we have worked on is the extension of Erdos-Ko-Rado analog to the t-intersecting k-multisets for the case $t = 2$. Erdos, Ko and Rado proved that given an $[n]$-set, with certain constraints on an integer k, the size of the largest pairwise intersecting k-subset family is bounded above by a function of k and n. Meagher and Purdy [12] extended this notion to pairwise intersecting k-multiset family from an $[m]$-set, where the intersection of two multisets is the multiset containing all elements common to both multisets with repetitions. They showed that an upper bound exits on the size of largest such family which is a function purely of k and m.

1.2.1 t-intersecting k-multisets

A further extension to the notion of pairwise intersecting k-multiset family from an $[m]$-set is that of pairwise t-intersecting k-multisets of an $[m]$-set, where the size of intersection for every pair of multisets is atleast t.
An open problem on this notion is stated as a conjecture in the same paper [12]. The proof to the limit on size and structure of the pairwise intersecting k-multiset family is based on the existence of a graph homomorphism from a Kneser graph $K(n, k)$ to its multiset analog $M(m, k)$. These concepts of Kneser graph and its analog can generalized as: $K(n, k, t)$ be the graph whose vertices are the k-subsets of $[n]$ where two vertices A and B are adjacent if $\mid A \cap B \mid < t$ and $M(m, k, t)$ be the graph whose vertices are the k-multisets of $[m]$ whose vertices C and D are adjacent of $\mid C \cap D \mid < t$. If a bijective homomorphism from $K(n, k, t)$ to $M(m, k, t)$ exists, it could be used to prove

the bound.
We could not establish such a bijective homomorphism. We tried, instead, to argue by principles of construction - starting from any intersecting pair of k-multisets, construct the largest possible family of pairwise intersecting k-multiset family. By using a particular method of construction, which we will describe in detail in later chapter, we could only show a trend in the growth in the size of the family which tended to the bound proposed as a limiting case. It further seemed to agree with the additional condition stated at the end of the conjecture.

1.3 Discrete Structures

We recall the relevant definitions and concepts from the literature related to sets, multisets, vector spaces and hypergraphs.

1.3.1 Sets

Definition 1.
A set is a well defined collection of distinct elements.

Given a set X and an element ϵ, one of the two must be true:
(i) $\epsilon \in X$
(ii) $\epsilon \notin X$

Definition 2.
Given a set X, a subset $Y \subset X$ is a set with every element of Y being an element of X.

Definition 3.
Given two sets X and Y, the union set denoted by $X \cup Y$ is the set of all the elements which are members of either X or Y.

Definition 4.
Given two sets X and Y, the intersection set denoted by $X \cap Y$ is the set of all the elements which are members of both X and Y.

Definition 5.
A set having no element is called a null set and is denoted by ϕ.

Definition 6.
Given two sets X *and* Y*, the cartesian product denoted by* $X \times Y$ *is the set of all ordered pairs* (x, y) *with* $x \in X$ *and* $y \in Y$.

Counting Sets
We will generally be working with an $[n]$- set which we will denote by X.

$$X = \{1, 2, \ldots, n\}$$

If we want to choose a k-subset (a subset with k elements, $k \leq n$)from the set X, the number of distinct subsets is denoted by $\binom{n}{k}$ which is defined as

$$\binom{n}{k} = \frac{n!}{(n-k)!(k)!} \tag{1.1}$$

The following is called Pascal's Identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{1.2}$$

1.3.2 Multisets

Definition 7.
A multiset is defined as a 2-tuple (A, m) *where* A *is some set and* $m : A \rightarrow \mathbb{N}$ *is a function from* A *to the set* $\{1, 2, \ldots, n, \ldots\}$ *of positive natural numbers. The set* A *is called the underlying set of elements. For each* $a \in A$*, the multiplicity of* a *is the number* $m(a)$.

Multiplicity Function

The set indicator function of a subset A of a set X is the function

$$I_A : X \rightarrow \{0, 1\}$$

(1.3)

defined by

$$I_A(x) = \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases}$$

Some important set indicators:

$$I_{A\cup B}(x) = max\{I_A(x), I_B(x)\}$$

(1.4)

$$I_{A\cap B}(x) = min\{I_A(x), I_B(x)\}$$

(1.5)

$$A \subseteq B \Leftrightarrow \forall x, I_A(x) \leq I_B(x)$$

(1.6)

$$I_{A\times B}(x,y) = I_A(x) \cdot I_B(x)$$

(1.7)

$$\mid A \mid = \sum_{x\in X} I_A(x)$$

(1.8)

Allowing the set indicator function to take values {2, 3, ...}, the new function called multiplicity function is defined for a multiset. The concepts of union, intersection, subset, cartesian product and cardinality are defined by the above formulae.

The multiplicity of multiset sum is:

$$I_{A\uplus B}(x) = I_A(x) + I_B(x)$$

(1.9)

The multiplicity of multiset difference is:

$$I_{A\setminus B}(x) = max\{0, I_A(x) - I_B(x)\}$$

$$(1.10)$$

The scalar multiplication of a multiset by a natural number is:

$$I_{n \otimes A}(x) = n \cdot I_A(x) \tag{1.11}$$

Example. Different multiset operations are illustrated below:

$$\{1, 1, 1, 3\} \cap \{1, 1, 2\} = \{1, 1\}$$

$$\{1, 1\} \cup \{1, 2\} = \{1, 1, 2\}$$
$$\{1, 1\} \uplus \{1, 2\} = \{1, 1, 1, 2\}$$
$$\{1, 1\} \subseteq \{1, 1, 1, 2\}$$
$$\mid \{1, 1\} \mid = 2$$
$$\{1, 1\} \times \{1, 2\} = \{(1, 1), (1, 1), (1, 2), (1, 2)\}$$

Counting Multisets

The number of multisets of cardinality k, with elements taken from a finite set of cardinality m, is called the multiset-coefficient or multiset-number and is denoted by

$$\left(\!\!\binom{m}{k}\!\!\right)$$

The value of the multiset coefficient can be written explicitly as:

$$\left(\!\!\binom{m}{k}\!\!\right) = \binom{m+k-1}{k} \tag{1.12}$$

The following is a recurrence relation for counting multisets:

$$\left(\!\!\binom{m}{k}\!\!\right) = \left(\!\!\binom{m}{k-1}\!\!\right) + \left(\!\!\binom{m-1}{k}\!\!\right) \tag{1.13}$$

1.3.3 Vector Space

Definition 8.
A vector space is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms below. The axioms must hold for all $u, v, w \in V$ and for all scalars c and d.

1. $u + v \in V$
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. There is a vector, $\hat{0} \in V$ *such that* $u + \hat{0} = u$
5. For each $u \in V$*, there is a vector* $-u$ *in* V *satisfying* $u + (-u) = 0$
6. $c \cdot u$ *is in* V
7. $c \cdot (u + v) = c \cdot u + c \cdot v$
8. $(c + d) \cdot u = c \cdot u + d \cdot u$
9. $(cd) \cdot u = c(d \cdot u)$
10. $1 \cdot u = u$

Definition 9.
A subspace of a vector space V is a subset H of V that has three properties:

a. The zero vector, $\hat{0}$, of V is in H.
b. For each u and v are in H, $u + v$ is in H.
c. For each u in H and each scalar c, $c \cdot u$ is in H.

Counting Vector Subspaces

We let V denote an n-dimensional vector space over the finite field $GF(q)$. For $k \in \mathbb{Z}^+$, we write $\left[{V \atop k}\right]$ to denote the family of all k-dimensional subspaces of V . For $a, k \in \mathbb{Z}^+$, define the Gaussian binomial coefficient by

$$\left[{a \atop k}\right] := \prod_{0 \le i < k} \frac{q^{a-1} - 1}{q^{k-1} - 1} \tag{1.14}$$

A simple counting argument shows that the size of $\left[{V \atop k}\right]$ is $\left[{n \atop k}\right]$.
If two subspaces of V intersect in the zero subspace, then we say they are

disjoint or that they trivially intersect; otherwise we say the subspaces non-trivially intersect. A family $F \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is called intersecting if any two k-spaces in F non-trivially intersect.

1.3.4 Hypergraphs

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set.

Definition 10.
A hypergraph $H = (E_1, E_2, \ldots, E_m)$ on X is defined to be a family of subsets of X satisfying the following properties:

$$E_i \neq \phi, \forall 1 \leq i \leq m \tag{1.15}$$

$$\bigcup_i E_i = X \tag{1.16}$$

The subsets $E_1, E_2, \ldots, E_m$ are called the edges of the hypergraph and the elements $x_1, x_2, \ldots, x_n$ are called the vertices. Note that the first condition excludes all the empty subsets and the second condition excludes all the isolated vertices from further discussions on hypergraphs.
A hypergraph is also called a set system or a family of sets drawn from the universal set Ω. The difference between a set system and a hypergraph is not well defined and depends on the questions being asked. Hypergraph theory tends to ask questions similar to those of graph theory, such as connectivity and colorability while the theory of set systems tends to ask non graph theoretic questions, such as Sperner theory. We now recall some relevant definitions and concepts from the literature.

Definition 11.
A simple hypergraph (Sperner family) is one in which no edge is a subset of other. If $H = (E_1, E_2, \ldots, E_m)$ is a simple hypergraph, then

$$E_i \subset E_j \implies i = j$$

(1.17)

Definition 12.
The dual of a hypergraph $H = (E_1, E_2, \dots, E_m)$, *on* X *is a hypergraph* $H^* = (X_1, X_2, \dots, X_n)$ *whose vertices* $e_1, e_2, \dots, e_m$ *correspond to the edges of* H *and with edges* $X_1, X_2, \dots, X_n$

$$X_i = \{e_j / x_i \in E_j \ in \ H\}$$

(1.18)

Definition 13.
The order of a hypergraph H *is defined as the number of elements of* X *and is denoted by* $n(H)$.
The number of edges of a hypergraph H is denoted by $m(H)$.

Definition 14.
The rank $r(H)$ *of a hypergraph* H *is*

$$r(H) = \max_j \mid E_j \mid$$

(1.19)

Definition 15.
The anti-rank $s(H)$ *of a hypergraph* H *is*

$$s(H) = \min_j \mid E_j \mid$$

(1.20)

Further, if $r(H) = s(H)$, all the edges have the same cardinality and the hypergraph is said to be uniform.

Definition 16.
Let $J \subset \{1, 2, \dots, m\}$. *Then the family* $H' = (E_j / j \in J)$ *is called the partial hypergraph of* H *generated by* J.

Note that a partial hypergraph contains some of the edges of the hypergraph.

Definition 17.
Let $A \subset X$. *Then the family*

$$H_A = (E_j \cap A, 1 \leq j \leq m, \mid E_j \cap A \mid \neq 0)$$

is called the subhypergraph of H induced by the set A.

Definition 18.
For $x \in X$, the star of x, $H(x)$ is defined as the partial hypergraph formed by edges containing x.

The number of edges of $H(x)$, denoted by $m(H(x))$, is called degree of x.

$$d_H(x) = m(H(x))$$

The maximum degree of the hypergraph H is always denoted as $\Delta(H)$.
Thus,

$$\Delta(H) = \max_{x \in X} d_H(x)$$

Definition 19.
A hypergraph in which all vertices have the same degree is said to be regular.

Also note that $\Delta(H) = r(H*)$, and that the dual of a regular hypergraph is uniform.

Definition 20.
Let r, n be integers, $1 \leq r \leq n$. Then, the r-uniform complete hypergraph on order n (or the r-complete hypergraph) is defined to be a hypergraph, denoted by K_r^n and containing all the r subsets of the set X of cardinality n

Definition 21.
Let H be a simple hypergraph on X of rank r and let $k \leq r$ be an integer. The k-section of the hypergraph, $[H]_k$ is defined to be a hypergraph with edges $F \subset X$ satisfying either $\mid F \mid = k$ and $F \subset E$, for some $E \in H$ or $\mid F \mid < k$ and $F = E$, for some $E \in H$. Note that $[H]_k$ is a simple hypergraph of rank k

Definition 22.
Let $G = (V, E)$ be a graph with vertex set V and edge set E. A graph homomorphism f from $G = (V, E)$ to a graph $G' = (V', E')$ is a mapping $f : V \to V'$, from the vertex set of G to the vertex set of G' such that

$$\{u, v\} \in E \implies \{f(u), f(v)\} \in E'$$

Also, notice that if $f : G \to G'$ is a graph homomorphism from G *to* G', then we have

$$\alpha(G) \geq \alpha(G')$$

where $\alpha(G)$ is the size of the largest independent family of G.

Chapter 2

Some Results from Extremal Combinatorics

Let X denote the set $[n] := \{1, ..., n\}$, which is the underlying set we work with. Let 2^X denote the family of all subsets of X. In extremal set theory, we look at maximum or minimum size of a family $F \subseteq 2^X$ that satisfies certain restrictions. We shall, for the most part, restrict ourselves to the family of k-uniform subsets; that is, all sets in the family have size k. We use the symbol $\binom{X}{k}$ to denote the family of all k-subsets of X.

The sections of this chapter fall into three major themes. The first topic is the essential theorems on intersections and shadows viz., Erdos-Ko-Rado, Hilton-Milner and and Kruskal-Katona theorems. Next, we discuss extensions of the above mentioned theorems to vector spaces. Finally, in the last part, we discuss extensions of the Erdos-Ko-Rado theorem to multisets. The aim of this chapter is to give an overview of the literature studied covering the various results from extremal combinatorics. In the next chapter we describe the construction technique to form a intersecting family and the results we obtained while attempting to solve the conjecture on pairwise t-intersecting k-multisets for the special case $t = 2$.

2.1 Erdos-Ko-Rado Theorem

Given a hypergraph H, an intersecting family is defined as the set of edges having non-empty pair wise intersection. A set of edges $A = \{E_1, E_2, \ldots, E_k\}$ is an intersecting family if $E_i \cap E_j \geq 1$ for all $i, j \in \{1, 2, \ldots, k\}$. For any vertex x in a hypergraph H, the star of x, $H(x)$ is an example of an intersecting family. The size of the largest intersecting family of a hypergraph H is always denoted by $\Delta_0(H)$ and satisfies:

$$\Delta_0(H) \geq \max_{x \in X} \mid H(x) \mid = \Delta(H) \tag{2.1}$$

Theorem 2.1.1 [3] *Every hypergraph H of order n, with no repeated edge* *satisfies*

$$\Delta_0(H) \leq 2^{(n-1)} \tag{2.2}$$

Further, every maximal intersecting family of a hypergraph of subsets of an n-set has cardinality $2^{(n-1)}$

Proof. Let A be a maximal intersecting family of subsets of a set X, where $\mid X \mid= n$.
If $B_1 \notin A$, then $\exists\ A_1 \in A$ such that $A_1 \cap B_1 = \phi$ (This follows from the maximality of A, else we could add B_1 to A and get a bigger intersecting family). Thus, we have $A_1 \subset X - B_1$ and hence, $A_1 \cap (X - B_1) \neq \phi$. Again, the maximality of A ensures that $(X - B_1) \in A$.
Further, if $X - B_1 \in A$, then $B_1 \notin A$.
Hence, $B \to X - B$ is a bijection between $P(X) - A$ and A, where $P(X)$ is the power ser of X. Also, the bijection ensures that

$$\mid P(X) - A \mid = \mid A \mid$$

Thus, we have

$$\mid A \mid = \frac{\mid P(X) \mid}{2} = 2^{n-1} \tag{2.3}$$

Lemma 2.1.2 (Greene, Katona, Kleitman) [3] *Let $x_1, x_2, \ldots, x_n$ be points on a circle in that order and let $A = (A_1, A_2, \ldots, A_m)$ be a family of circular intervals of points satisfying the following properties:*

(1) $\mid A_i \mid \leq \frac{n}{2}\ \forall i \in \{1, 2, \ldots, m\}$
(2) $\mid A_i \cap A_j \mid \neq 0\ \forall i, j \in \{1, 2, \ldots, m\}$
(3) $A_i \not\subset A_j\ \forall i \in \{1, 2, \ldots, m\}$.
Then
(4) $m \leq \min_i \mid A_i \mid$ *and*
(5) $\Sigma_i \mid A_i \mid^{-1} \leq 1$
Equality in (5) is attained iff A is a family of circular intervals of cardinality m and each having a point in common.

Proof. Let A_1 be a circular interval of minimum size. Then,
(a) From (2), $\mid A_1 \cap A_i \mid \neq 0\ \forall i \neq 1$.
(b) And, from (1) and (3), all other intervals have only one of their ends

coinciding with an end of A_1.
(c) Also, from (3), the intervals $A_1 \cap A_i$ are all different.
Thus, the number of possible intervals of this form is $\leq (2 \mid A_1 \mid -1)$.
We claim two sets $A_i \cap A_1$ and $A_j \cap A_1$, $i \neq j$, $i \neq 1$ and $j \neq 1$ cannot constitute a partition of A_1. As, if they constitute a partition of A_1, they will have to coincide with opposite sides of A_1. But if they coincide with opposite sides of A_1, then $\mid A_i \cap A_j \mid = 0$, (1) requires ($\mid A_i \mid \leq \frac{n}{2}$) and thus they will violate (2). Hence, A_i and A_j cannot constitute a partition of A_1.
Thus, out of all the total cases, only half of them are possible, i.e,
$m - 1 \leq \mid A_1 \mid -1$.
Hence, we get $m \leq \mid A_1 \mid$, which completes the proof for (4).
Also, we have

$$\sum_{A_i \in A} \mid A_i \mid^{-1} \leq \frac{m}{\mid A_1 \mid} \leq 1 \tag{2.4}$$

which gives us (5).
Also, equality in (5) implies

$$1 = \sum_{1 \leq i \leq m} \mid A_i \mid^{-1} \leq \frac{m}{\mid A_1 \mid} \leq 1 \tag{2.5}$$

So, we have $\mid A_i \mid = \mid A_1 \mid = m$, $1 \leq i \leq m$. Thus, the A_i's are intervals of length whose intital end points are m successive points on a circle.
Conversely, if the A_i satisfy (1), (2), (3) and have lenght m, then obviously we have an equality in (5).

Theorem 2.1.3 (Erdos-Ko-Rado) [3] *Let H be a simple hypergraph of order n and rank $r \leq \frac{n}{2}$. Then*

$$\sum_{E \in H} \binom{n-1}{\mid E \mid -1}^{-1} \leq 1 \tag{2.6}$$

and

$$m(H) \leq \binom{n-1}{r-1}$$

(2.7)

Further, we have equality in (2) when H is a star of K_r^n (and if $r < \frac{n}{2}$).

Proof. Let $X = \{x_1, x_2 \ldots, x_n\}$ be the vertex set of H and for any permutation π of $1, 2, \ldots, n$ denote by H_π the set of edges of H which are circular intervals of the circular sequence $x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_n}, x_{\pi_1}$

Also, for $E \in H$, put

$$\beta(E) =| \{\pi/E \in H_\pi\}$$

(2.8)

Also note that from Lemma 2.1.2.

$$\sum_{E \in H_\pi} \frac{1}{| E |} \leq 1$$

(2.9)

We have then

$$\sum_{E \in H} \frac{\beta(E)}{| E |} = \sum_{E \in H} \sum_{\pi | E \in H_\pi} \frac{1}{| E |} = \sum_{\pi} \sum_{E \in H_\pi} \frac{1}{| E |} \leq n!$$

(2.10)

Let E_0 be an edge of H with $| E_0 |= h$ and let x_0 be an element of E_0. Since E_0 is also an edge of the hypergraph $H' = K_n^h(x_0)$ and from Lemma 2.1.2 we have equality in both the equations above for H'. Thus we have

$$\frac{\beta(E_0)}{| E_0 |} = \frac{1}{m(H')} \sum_{E' \in H'} \frac{\beta(E')}{| E' |} = \frac{n!}{m(H')} = n! \binom{n-1}{| E_0 | -1}^{-1}$$

(2.11)

or,

$$\sum_{E \in H} \binom{n-1}{| E | -1}^{-1} = \frac{1}{n!} \sum_{E \in H} \frac{\beta(E)}{| E |} \leq \frac{n!}{n!} = 1$$

(2.12)

Thus we have the first part of the Erdos-Ko-Rado bound.

For the secong part, note that every $E \in H$ satisfies $| E |\leq r \leq \frac{n}{2}$. Thus, we have

$$m(H) \binom{n-1}{r-1}^{-1} \leq \sum_{E \in H} \binom{n-1}{| E | -1}^{-1} \leq 1$$

$$(2.13)$$

With the previous two results, we have completed the proof.

The intersecting family is further generalized by the concept of t-intersecting family. For a hypergraph $H = (E_1, E_2, \ldots, E_m)$, a t-intersecting family A is a set of edges that intersect in t or more vertices. Thus, if A is a t- intersecting family, then we have

$$\mid E_i \cap E_j \mid \geq t, \forall E_i, E_j \in H \tag{2.14}$$

For example, the family $A = \{(1,2,3,4),(2,3,4,5),(1,2,3,5)\}$ is a 3-intersecting family on $X = \{1,2,3,4,5\}$.
Erdos-Ko-Rado theorem for t-intersecting families is an important result, which is stated below without giving a proof.

Theorem 2.1.4 (Erdos-ko-Rado Theorem for t-intersecting families) [13] *Let $n \geq k \geq t \geq 1$, and let A be a family of k-uniform, t-intersecting subsets of the set $n = \{1, 2, \ldots, n\}$. If $n \geq (k-t+1)(t+1)$, then*

$$\mid A \mid \leq \binom{n-t}{k-t} \tag{2.15}$$

Moreover, if $n > (k-t+1)(t+1)$, then this bound is achieved by a trivially t-intersecting system, that is by a family A containing all the k-subsets of the set n that contain a fixed t-subset from the set n.

2.2 Kruskal-Katona Theorem

The Kruskal-Katona theorem gives a tight lower bound on the size of $r-1$ section of an r- uniform hypergraph.

Theorem 2.2.1 (Kruskal, Katona) [3] *Let H be an r-uniform hypergraph with*

$$m(H) = m = \binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \binom{a_{r-2}}{r-2} + \cdots + \binom{a_s}{s} \tag{2.16}$$

and

$$a_r > a_{r-1} > \cdots > a_s \geq s \geq 1 \tag{2.17}$$

Then,

$$m(H_{r-1}) \geq \binom{a_r}{r-1} + \binom{a_{r-1}}{r-2} + \binom{a_{r-2}}{r-3} + \cdots + \binom{a_s}{s-1} \tag{2.18}$$

The proof of Kruskal-Katona theorem presented here was given by Frankl. This proof requires two lemmas which are stated and proved first before starting with the proof of the theorem. We now prove a lemma that demonstrates that every positive integer m has an r-binomial representation

Lemma 2.2.2 [3] *Let m and r be positive integers. Then there exist integers $a_r, a_{r-1}, \ldots, a_s$ such that*

$$m = \binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \binom{a_{r-2}}{r-2} + \cdots + \binom{a_s}{s} \tag{2.19}$$

and

$$a_r > a_{r-1} > \cdots > a_s \geq s \geq 1 \tag{2.20}$$

Further, the a_i's are uniquely defined by (2.19) and (2.20) and a_r is the largest integer such that

$$m - \binom{a_r}{r} \geq 0$$

Proof. The proof proceeds by induction on r. For any given m, with $r = 1$ the decomposition exists trivially and is unique, as $m = \binom{m}{1}$. We assume that for any $m > 0$, the decomposition exists with $r-1$ and is unique. Let a_r be the largest integer such that $m - \binom{a_r}{r} \geq 0$. Then from our assumption, a decomposition of $m - \binom{a_r}{r}$ with $r - 1$ exists, i.e.

$$m - \binom{a_r}{r} = \binom{a_{r-1}}{r-1} + \binom{a_{r-2}}{r-2} + \cdots + \binom{a_s}{s} \tag{2.21}$$

with

$$a_{r-1} > a_{r-2} > \cdots > a_s \geq s \geq 1 \tag{2.22}$$

We must have $a_r > a_{r-1}$, else we would have

$$m \geq \binom{a_r}{r} + \binom{a_{r-1}}{r-1} \geq \binom{a_r}{r} + \binom{a_r}{r-1} = \binom{a_{r+1}}{r} \tag{2.23}$$

which is not in accordance with our assumption. Hence, the existance of the decomposition is proved.
For proving uniqueness, let us assume two distinct decompositions exist:

$$\begin{aligned} m &= \binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \binom{a_{r-2}}{r-2} + \cdots + \binom{a_s}{s} \\ &= \binom{b_r}{r} + \binom{b_{r-1}}{r-1} + \binom{b_{r-2}}{r-2} + \cdots + \binom{b_s}{s} \end{aligned} \tag{2.24}$$

where the b_i's also satisfy the same equations as a_i's.
Now, observe that

$$m \leq \binom{a_r}{r} + \binom{a_r - 1}{r-1} + \binom{a_r - 2}{r-2} + \cdots + \binom{a_r - r + 1}{1} = \binom{a_r + 1}{r} \tag{2.25}$$

If $a_r < b_r$, then

$$m \leq \binom{a_r + 1}{r} \leq \binom{b_r}{r} \leq m$$

$$(2.26)$$

This implies $m = \binom{a_r+1}{r}$, which violates the definition of r. Thus $a_r = b_r$ and hence, the decomposition is unique. This decomposition of m is also called the r-binomial representation of m.

Lemma 2.2.3 [3] *Let H be an r-uniform hypergraph on $X = \{x_1, x_2, \ldots, x_n\}$. Let $H(x_1)$ be the star of the vertex x_1. Then there exists an r-unifrom hypergraph H' on X with $m(H') = m(H)$, $m([H']_{r-1}) \leq m([H]_{r-1})$ which satisfies*

$$F \in [H' - H'(x_1)]_{r-1} \implies F \cup \{x_1\} \in H' \tag{2.27}$$

Proof. For a vertex $x_j \in X$, $x_j \neq x_1$, put

$$\sigma_{x_j}(E) = \begin{cases} (E - x_j) \cup x_1 & if x_j \in E, x_1 \notin E, (E - x_j) \cup x_1 \notin H \\ E & otherwise \end{cases} \tag{2.28}$$

Also, $\sigma_{x_j}(H) = \{\sigma_{x_j}(E)/E \in H\}$. We claim, $[\sigma_{x_j}(H)]_{r-1} \subset \sigma_{x_j}[H]_{r-1}$.
We have shown that $A \in [\sigma_{x_j}(H)]_{r-1} \implies A \in \sigma_{x_j}[H]_{r-1}$
First, suppose that $A = \sigma_{x_j}(A)$.
If $B \in [\sigma_{x_j}(A)]_{r-1}$, then $B \in [A]_{r-1}$ $(as A = \sigma_{x_j}(A))$. Thus, $A = B \cup \{x_i\}$, for some $i \leq n$. Now, it suffices to prove that $\sigma_{x_j}(B) = B$, as this would imply $B \in \sigma_{x_j}[H]_{r-1}$.
case 1. If $i = j$, $x_j \notin B$, then $\sigma_{x_j}(B) = B$.
case 2. If $i = j$, $x_j \in B$, then $(B - \{x_j\}) \cup \{x_1\} = A - \{x_j\} \in [A]_{r-1}$. Thus, $\sigma_{x_j}(B) = B$.
case 3. If $i = 1$, $x_j \notin B$, then $\sigma_{x_j}(B) = B$.
case 4. If $i \neq 1, i \neq j$, then $\sigma_{x_j}(B) = B$, unless $x_j \in B$ and $x_1 \notin B$. But in this case, $x_j \in A$ and $x_1 \notin A$. Also, we have $\sigma_{x_j}(A) = A$, so we must have $(A - \{x_j\}) \cup \{x_1\} \in H$. Thus $(B - \{x_j\}) \cup \{x_1\} \in [H]_{r-1}$, and so $\sigma_{x_j}(B) = B$.
Next, assume that $A \neq \sigma_{x_j}(A)$. Then $x_j \in A$, $x_1 \notin A$ and $\sigma_{x_j}(A) = (A - \{x_j\}) \cup \{x_1\}$. Now, if $B \in [\sigma_{x_j}(A)]_{r-1}$ and $x_1 \notin B$, then $B* = (B - \{x_1\}) \cup x_j \subset A$ and so $B* \in [H]_{r-1}$. If $B \notin [H]_{r-1}$, then $\sigma_{x_j}(B*) = B$ so that $B \in \sigma_{x_j}[H]_{r-1}$. If finally, $B \in [H]_{r-1}, x_j \notin B$ so that $B = \sigma_{x_j}(B)$.

Thus, the proof is complete.

Now we can move on the proof of Kruskal-Katona Theorem.

2.2.4 Proof to Kruskal-Katona Theorem [3]

Assume that H satisfies

$$F \in [H - H(x_1)]_{r-1} \implies F \cup \{x_1\} \in H \tag{2.28}$$

Also, let

$$H_1 = \{E - \{x_1\}/E \in H(x_1)\}$$

Then,

$$m([H]_{r-1}) \geq m(H_1) + m([H_1]_{r-2}) \tag{2.29}$$

The theorem holds trivially for $r = 1$ and $m = 1$. We proceed by induction on m and r.
Suppose that

$$m(H_1) \geq \binom{a_r - 1}{r - 1} + \cdots + \binom{a_s - 1}{s - 1} \tag{2.30}$$

From the induction hypothesis, for the hypergraph H_1, we get

$$m[H_1]_{r-2} \geq \binom{a_r - 1}{r - 2} + \cdots + \binom{a_s - 1}{s - 2} \tag{2.31}$$

Thus, from (2.29) we get

$$m[H_1]_{r-1} \geq \binom{a_r - 1}{r - 1} + \cdots + \binom{a_s - 1}{s - 1} + \binom{a_r - 1}{r - 2} + \cdots + \binom{a_s - 1}{s - 2} \tag{2.32}$$

or,

$$m[H]_{r-1} \geq \binom{a_r}{r - 1} + \cdots + \binom{a_s}{s - 1} \tag{2.33}$$

Now suppose that

$$m(H_1) < \binom{a_r - 1}{r - 1} + \binom{a_{r-1} - 1}{r - 2} + \cdots + \binom{a_s - 1}{s - 1}$$

$$(2.34)$$

Thus,

$$m(H - H(x_1)) = m(H) - m(H_1) > \binom{a_r}{r} + \dots$$
$$+ \binom{a_s}{s} - \binom{a_r - 1}{r-1} - \binom{a_{r-1} - 1}{r-2} - \dots - \binom{a_s - 1}{s-1} \tag{2.35}$$

or,

$$m(H - H(x_1)) > \binom{a_r - 1}{r} + \dots + \binom{a_s - 1}{s} \tag{2.36}$$

But, we have

$$m(H_1) \geq m(H - H(x_1)) \geq \binom{a_r - 1}{r-1} + \dots + \binom{a_s - 1}{s-1} \tag{2.37}$$

which contradicts (2.34)
This completes our proof.

Corollary 2.2.5 [3] *Let H be an r-uniform hypergraph and let k be an integer with $r > k \geq 2$. If a is the largest integer such that $m(H) \geq \binom{a}{r}$, then*

$$m([H]_k) \geq \binom{a}{k} \tag{2.38}$$

Proof. Let H_1 be a partial hypergraph of H with $m(H_1) = \binom{a}{r}$. From the Theorem (2.2.4),

$$m([H_1]_{r-1}) \geq \binom{a}{r-1} \tag{2.39}$$

Further, let H_2 be a partial hypergraph of $[H_1]_{r-1}$ with $m(H_2) = \binom{a}{r-1}$. From Theorem (2.2.1),

$$m([H_2]_{r-2}) \geq \binom{a}{r-2}$$

$$(2.40)$$

Continuing, we get

$$m([H_{r-k}]_k) \geq \binom{a}{k}$$

$$(2.41)$$

Since $[H_{r-k}]_k \subset [H_k]$, we have

$$m([H_k]) \geq \binom{a}{k}$$

$$(2.42)$$

which completes our proof.

2.3 Hilton-Milner Theorem

Theorem 2.3.1 (Hilton, Milner) [4] *Let $F \subset \binom{X}{k}$ be an intersecting family with $k \geq 2$, $n \geq 2k+1$, and such that there does not exist $x \in X$ such that $F \subset \{f \in \binom{X}{k} : x \in f\}$. We then have*

$$| F | \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

$$(2.43)$$

Equality holds if and only if
(i) $F = \{| f |\} \cup \{| g | \in \binom{X}{k} : x \in g, f \cap g \neq \phi\}$ for some k-subset f and $x \in X \backslash F$.
(ii) $F = \{F \in \binom{X}{3} :| F \cap D | \geq 2\}$ for some 3-subset S if $k = 3$.
We have stated the Hilton-Milner Theorem above without providing the proof.

2.4 Vector Space Analogs

We will now present the Vectos Space analogs of the three primary theorems without stating the proofs (Erdos-Ko-Rado, Kruskal-Katona and Hilton-Milner).

A family $F \subset \left[{V \atop k}\right]$ is called 2-wise intersecting if $\forall\ f_1, f_2 \in F$ we have

$$\bigcap_{i=1}^{2} f_i \neq \phi$$

We define the shadow of F, denoted ∂F, to consist of those $(k-1)$ dimensional subspaces of V, contained in at least one member of F.

We will say that F is an HM-type family if:

$$F = \{W \in \left[{V \atop k}\right] : E \leq W, dim(W \cap U) \geq 1\} \bigcup \left[{E+U \atop k}\right]$$

for some $E \in \left[{V \atop 1}\right]$ and $U \in \left[{V \atop k}\right]$ with $E \nsubseteq U$.
For any family $F \subset \binom{X}{k}$, the covering number $\tau(F)$ is the minimum size of a ser that meets all $f \in F$.

Theorem 2.4.1 (Erdos-Ko-Rado Theorem for Vector Spaces) [5]
Suppose $F \subset \left[{V \atop k}\right]$ is 2-wise intersecting and $n \geq 2k$. Then

$$\mid F \mid \leq \left[{n-1 \atop k-1}\right] \tag{2.44}$$

Theorem 2.4.2 (Kruskal-Katona Theorem due to Lovasz for Vector Spaces) [5]
Let $F \subset \left[{V \atop k}\right]$ and let $y \geq k$ be the real number defined by $\mid F \mid = \left[{y \atop k}\right]$. Then

$$\mid \partial F \mid \geq \left[{y \atop k-1}\right] \tag{2.45}$$

Theorem 2.4.3 (Hilton-Milner Theorem for Vector Spaces) [4]

Theorem. *Suppose $k \geq 3$ and either $q \geq 3$ and $n \geq 2k+1$, or $q = 2$ and $n \geq 2k+2$. For any intersecting family $F \subset {V \brack k}$ with $\tau(F) \geq 2$, we have $\mid F \mid \leq f(n,k,q)$.*

$$f(n,k,q) := {n-1 \brack k-1} - q^{k(k-1)} {n-k-1 \brack k-1} + q^k \tag{2.46}$$

Equality holds if and only if
(i) F is an HM-type family,
(ii) $F = F_3 = \{f \in {V \brack k} : dim(s \cap f) \geq 2\}$ for some $s \in {V \brack 3} if k = 3$

Furthermore, if $k \geq 4$, then there exists an $\epsilon > 0$ (independent of all n, k, q) such that if $\mid F \mid \geq (1-\epsilon) f(n,k,q)$, then F is a subfamily of an HM type family.

If $k = 2$, then a maximal intersecting family F of k-spaces with $\tau(F) > 1$ is the family of all 2-subspaces of a 3-subspace, and the conclusion of the theorem holds.

2.5 Multiset Analogs

We will now present the Multiset analog of the Erdos-Ko-Rado Theorem.

Theorem 2.5.1 (Erdos-Ko-Rado Theorem for Multisets) [12]
Let k, m be positive integers and with $m \geq k+1$. If A is an intersecting collection of multisets of $[m]$, then

$$\mid A \mid \leq \binom{m+k-2}{k-1} \tag{2.47}$$

Moreover, if $m > k+1$, equality in the above equation is achieved if and only if A is a collection of all the k-multisets of $[m]$, each containing a fixed element from $[m]$.

Proof.
The proof of this theorem uses a homomorphism from a Kneser graph to a graph whose vertices are the k-multisets of $[m]$.
A Kneser graph $K(n,k)$, over a set $[n]$ is defined to be a graph whose vertices are all the k-sets of the set $[n]$, denoted by $\binom{[n]}{k}$, and two vertices are adjacent if and only if the k-sets they correspond to are disjoint. We represent by $\alpha(K(n,k))$ the size of largest independent set in $K(n;k)$. Note that an independent set of vertices in $K(n,k)$ is an intersecting k-set system.
We now define a multiset analogue of the Kneser graph. Let k, m be positive integers. Then $M(m,k)$ is defined to be a graph with vertices the k-multisets of the set $[m]$, denoted by $\binom{[m]}{k}$, and two vertices of this graph are adjacent if and only if the multisets they correspond to are disjoint. Thus an independent set in $M(m,k)$ is an intersecting family of k-multisets on the set $[m]$. We denote by $\alpha(M(m,k))$ the size of the maximum intersecting family of $M(m,k)$. Also, the number of vertices in $M(m,k)$ is $\binom{m+k-1}{k}$.
Further, let $n = m+k-1$. Then $K(n,k)$ has the same number of vertices as $M(m,k)$ and $\forall\ B \in \binom{[n]}{k}$, $B \cap [m] \neq \phi$.
For a set $A \subseteq [m]$ of cardinality a where $1 \leq a \leq k$, the number of k-sets, B, from $[n]$ such that $B \cap [m] = A$ will be equal to

$$\binom{n-m}{k-a} = \binom{k-1}{k-a} \tag{2.48}$$

.
Similarly, the number of k-multisets from $[m]$ that contain all of the elements of A and no others will be equal to

$$\left(\!\!\binom{a}{k-a}\!\!\right) = \binom{a+(k-a)-1}{k-a} = \binom{k-1}{k-a} \tag{2.49}$$

Hence there exists a bijection, $f : \binom{[n]}{k} \to \left(\!\binom{m}{k}\!\right)$, such that for any $B \in$

$\binom{[n]}{k}$, the set of distinct elements in $f(B)$ will be equal to $B \cap [m]$.
If $A, B \in \binom{[n]}{k}$ are two adjacent vertices in the Kneser graph, then $(A \cap [m]) \bigcap (B \cap [m]) = \phi$ and hence $f(A) \cap f(B) = \phi$. Therefore $f(A)$ is adjacent to $f(B)$ if A is adjacent to B and so the bijection $f : \binom{[n]}{k} \to \left(\!\binom{m}{k}\!\right)$ is a graph homomorphism. In fact, $K(n,k)$ is isomorphic to a spanning subgraph of $M(m,k)$. Thus,

$$\alpha(M(m.k)) \leq \alpha(K(n.k))$$

. (2.50)

From the Erdos-Ko-Rado Theorem, we have that if $n \geq 2k$,

$$\alpha(K(n.k)) = \binom{n-1}{k-1}$$

Thus, for $m \geq k+1$,

$$\alpha(M(m.k)) \leq \binom{n-1}{k-1} = \binom{m+k-2}{k-1}$$

. (2.51)

An intersecting collection of k-multisets from $[m]$ consisting of all k-multisets containing a fixed element from $[m]$ will have size $\binom{m+(k-1)-1}{k-1} = \binom{m+k-2}{k-1}$. Therefore,

$$\alpha(M(m.k)) = \binom{m+k-2}{k-1} \tag{2.52}$$

which gives the upper bound on A.
To prove the uniqueness statement in the theorem, let $m > k+1$ and let A be an intersecting multiset system of size $\binom{m+k-2}{k-1}$. With the homomorphism defined above, the pre-image of A will be an independent set in $K(n,k)$ of size $\binom{n-1}{k-1}$. Since $m > k+1$ and $n = m+k-1$, it follows that

$n > 2k$ so, by the Erdos-Ko-Rado theorem, $f^{-1}(A)$ will be a collection of all the k-subsets of $[n]$ that contain a fixed element from $[n]$. If the fixed element, x, is an element of $[m]$, then it follows from the definition of f that every multiset in A will contain x. Thus A will be a collection of all the k-multisets from $[m]$ that contain a fixed element from $[m]$ as required. If $x \notin [m]$, then $f^{-1}(A)$ will include the sets $A = \{1, m+1, ..., n\}$ and $B = \{2, m+1, ..., n\}$ since $m > k+1$ implies that $m > 2$. But $f(A) \cap f(B) = \phi$ which contradicts our assumption that A is an intersecting collection of multisets. Therefore, when $m > k+1$, if A is an intersecting collection of multisets of the maximum possible size, then A is the collections of all k-multisets containing a fixed element from $[m]$.

The case when $m = k+1$ is analogous to the case when $n = 2k$ in the Erdos-Ko-Rado Theorem. The size of the largest possible intersecting collection is equal to $\binom{m+k-2}{k-1}$ but collections attaining this bound are not limited to those having a common element in all k-multisets.

Chapter 3

t-intersecting k-multisets : size and structure for the special case $t = 2$

In this chapter, we consider an open problem stated in [12] by Meagher and Purdy.

Conjecture. Let k, m and t be positive integers with $t \leq k$ and $m \leq t(k-t)+2$. If A is a collection of t-intersecting k-multisets of $[m]$, then

$$|A| \leq \binom{m+k-t-1}{k-t} \tag{3.1}$$

Moreover, if $m > t(k-t)+2$, equality holds if and only if A is a collection of all the k-multisets from $[m]$ that contain a fixed t-multiset from $[m]$.

We will restrict ourselves to the case of $t = 2$, that is, where A is a collection of 2-intersecting k-multisets of $[m]$.
2-intersecting k-multisets are the collection of of multisets which are pairwise intersecting and the size of the intersection for every pair is atleast 2.
The problem thus reduces to:
Let k, m be positive integers with $2 \leq k$ and $m \geq 2k-2$. If A is a collection of 2-intersecting k-multisets of $[m]$, then

$$|A| \leq \binom{m+k-3}{k-2} \tag{3.2}$$

Moreover, if $m > 2k-2$, equality holds if and only if A is a collection of all the k-multisets from $[m]$ that contain a fixed 2-multiset from $[m]$.

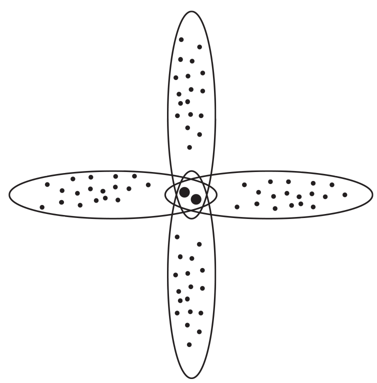
I. Collection of all the k-multisets from $[m]$ that contain a fixed 2-multiset from $[m]$

If A is a collection of all the k-multisets from $[m]$ that contain a fixed 2-multiset from $[m]$, then we fix any 2-multiset from $[m]$. The k-multiset can be formed by filling in the remaining $k-2$ places by any $(k-2)$-multiset from $[m]$. So, the size of such an intersecting family would be same as the number of distinct $(k-2)$-multisets which can be generated from $[m]$, which is

$$|A| = \left(\!\!\binom{m}{k-2}\!\!\right)$$
$$= \binom{m+k-3}{k-2}$$

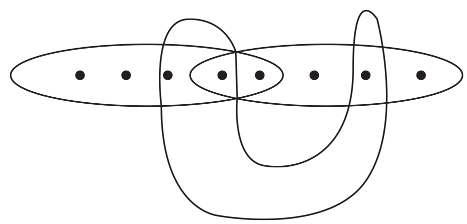
II. **A method of constructing an intersecting family.**

The intersecting family described in (I) looks like:

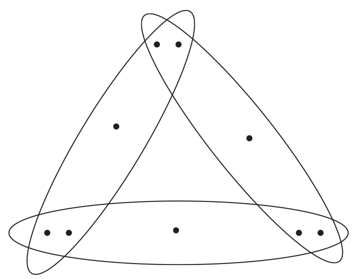


a)
i.e. if $\{A_i\} \in A$, $i \in \alpha$, then $\bigcap A_i =$ fixed 2-multiset for any $\alpha\prime \in \alpha$

However, 2-intersecting families can also look like:



b)



c)

Construction:
We know that for a fixed k, $m \geq 2k-2$.
Take $m = 2k-2$.

Consider two k-sets (not multisets) such that their union set is $[m]$. These two sets will have two elements in common (by Pigeonhole principle). Let these sets be γ^* and δ^*.

In (I) we have seen the case if all the k-multisets have the same 2-multiset in common.

We now want to construct an intersecting family such that we account for intersecting families of the form described in figures (b) and (c).

The sets γ^* and δ^* defined above have the property:

$$\gamma^* \cap \delta^* = \{2 - set\} = \{t_1, t_2\}$$

We now construct two sub-families Θ and Λ as:

Table 1: Intersecting sub-families Θ and Λ

Θ	Λ
Θ is a collection of k-multisets from $[m]$ and satisfies the following properties: (i) Fix element t_1 in all k-multisets of Θ.	Λ is a collection of k-multisets from $[m]$ and satisfies the following properties: (i)Fix element t_2 in all k-multisets of Λ.
(ii) Element t_2 does not belong to any k-multiset of Θ	(ii) Element t_1 does not belong to any k-multiset of Λ
(iii) From the remaining $(k-2)$ elements of γ^* (i.e. $\gamma^* \backslash \{t_1,t_2\}$) and the $(k-2)$ elements of δ^* (i.e. $\delta^* \setminus \{t_1, t_2\}$), take the Cartesian Product of the two sets to obtain $(k-2)^2$ ordered pairs with one element from γ^* and one element from δ^*. From these $(k-2)^2$ elements choose any one and call it $\{p_1,\, p_2\}$ with $p_1 \in \gamma^*$ and $p_2 \in \delta^*$. Assign p_1, p_2 to all elements to Θ.	(iii) From the remaining $(k-2)$ elements of γ^* (i.e. $\gamma^* \backslash \{t_1,t_2\}$) and the $(k-2)$ elements of δ^* (i.e. $\delta^* \setminus \{t_1, t_2\}$), take the Cartesian Product of the two sets to obtain $(k-2)^2$ ordered pairs with one element from γ^* and one element from δ^*. From these $(k-2)^2$ elements choose any one and call it $\{p_1,\, p_2\}$ with $p_1 \in \gamma^*$ and $p_2 \in \delta^*$. Assign p_1, p_2 to all elements to Λ.
Thus, Θ contains k-multisets which look like $\{t_1,\, p_1,\, p_2,\, \ldots (k-3)$-blank spaces $\ldots\}$	Thus, Λ contains k-multisets which look like $\{t_2,\, p_1,\, p_2,\, \ldots (k-3)$-blank spaces $\ldots\}$

Let us note some of the properties of the classes Θ and Λ that we have constructed:

i) If $\theta \in \Theta$ and $\lambda \in \Lambda$, then

$$\begin{gathered}
\mid \theta \cap \gamma^* \mid \geq 2 \; \forall \; \theta \in \Theta \\
\mid \lambda \cap \gamma^* \mid \geq 2 \; \forall \; \lambda \in \Lambda \\
\mid \theta \cap \delta^* \mid \geq 2 \; \forall \; \theta \in \Theta \\
\mid \lambda \cap \delta^* \mid \geq 2 \; \forall \; \lambda \in \Lambda \\
\mid \theta \cap \lambda \mid \geq 2 \; \forall \; \theta \in \Theta \text{ and } \lambda \in \Lambda
\end{gathered}$$

So γ^*, δ^*, Θ and Λ together satisfy the properties of a 2-intersecting k-multiset family.

ii)

$$\mid \Theta \mid = \mid \Lambda \mid = \left(\!\!\binom{m-1}{k-3}\!\!\right) \tag{3.3}$$

Since t_1 does not belong to Λ and t_2 does not belong to Θ, we have only $m-1$ elements to choose from in each case. From these $m-1$ elements, we choose any $(k-3)$-multiset to complete Θ and Λ as k-multisets.

We now construct two sub-families Γ and Δ as:

Table 2: Intersecting sub-families Γ and Δ

Γ	Δ
Γ is a collection of k-multisets from $[m]$ and satisfies the following properties:	Δ is a collection of k-multisets from $[m]$ and satisfies the following properties:
(i) t_1 and t_2 belong to every multiset of Γ	(i) t_1 and t_2 belong to every multiset of Δ
(ii) $p_1 \in$ every multiset of Γ and $p_2 \notin$ any multiset of Γ	(ii) $p_2 \in$ every multiset of Δ and $p_1 \notin$ to any multiset of Δ
Thus, Γ contains k-multisets which look like $\{p_1, t_1, t_2, \ldots (k-3)\text{-blank spaces} \ldots\}$	Thus, Δ contains k-multisets which look like $\{p_2, t_1, t_2, \ldots (k-3)\text{-blank spaces} \ldots\}$

Let us note some of the properties of the classes Γ and Δ that we have constructed:

i) If $\gamma \in \Gamma$ and $\delta \in \Delta$, then

$$
\begin{gathered}
\mid \gamma \cap \gamma^* \mid \geq 2 \ \forall \ \gamma \in \Gamma \\
\mid \delta \cap \gamma^* \mid \geq 2 \ \forall \ \delta \in \Delta \\
\mid \gamma \cap \delta^* \mid \geq 2 \ \forall \ \gamma \in \Gamma \\
\mid \delta \cap \delta^* \mid \geq 2 \ \forall \ \delta \in \Delta \\
\mid \gamma \cap \theta \mid \geq 2 \ \forall \ \gamma \in \Gamma, \, \theta \in \Theta \\
\mid \delta \cap \theta \mid \geq 2 \ \forall \ \delta \in \Delta, \, \theta \in \Theta \\
\mid \gamma \cap \lambda \mid \geq 2 \ \forall \ \gamma \in \Gamma, \, \lambda \in \Lambda \\
\mid \delta \cap \lambda \mid \geq 2 \ \forall \ \delta \in \Delta, \, \lambda \in \Lambda \\
\mid \delta \cap \gamma \mid \geq 2 \ \forall \ \delta \in \Delta, \, \gamma \in \Gamma
\end{gathered}
$$

So γ^*, δ^*, Γ, Δ, Θ and Λ together satisfy the properties of a 2-intersecting k-multiset family.

ii)

$$\mid \Gamma \mid = \mid \Delta \mid = \left(\!\!\binom{m-1}{k-3}\!\!\right)$$

$$(3.4)$$

Since t_1 does not belong to Λ and t_2 does not belong to Θ, we have only $m-1$ elements to choose from in each case. From these $m-1$ elements, we choose any $(k-3)$-multiset to complete Θ and Λ as k-multisets.

Finally, we construct a family Ψ as follows:
Ψ is the collection of k-multisets such that the elements $\{t_1, t_2, p_1, p_2\}$ are fixed in every element $\psi \in \Psi$.
We can see that Ψ along with γ^*, δ^*, Γ, Δ, Θ and Λ satisfy all the properties of a 2-intersecting k-multiset family.

$$|\Psi| = \left(\!\!\binom{m}{k-4}\!\!\right) \tag{3.5}$$

Ψ contains k-multisests which look like
$(t_1, t_2, p_1, p_2, \ldots (k-4) - blankspaces \ldots)$.
The remaining k-4 spaces can be filled in any way from $[m]$.
Further notice that Ψ , Γ, Δ, Θ and Λ are disjoint, and γ^* and δ^* are, each, special cases of Γ and Δ respectively.

III. Combinatorial comparison of size of construction

$$|\Theta| = |\Lambda| = |\Gamma| = |\Delta| = \left(\!\!\binom{m-1}{k-3}\!\!\right) = \binom{m+k-5}{k-3}$$

$$|\Psi| = \left(\!\!\binom{m}{k-4}\!\!\right) = \binom{m+k-5}{k-4}$$

We have established that Ψ , Γ, Δ, Θ and Λ are disjoint (by construction). So,

$$|\Theta| + |\Lambda| + |\Gamma| + |\Delta| = 4\left(\!\!\binom{m-1}{k-3}\!\!\right) = 4\binom{m+k-5}{k-4} \tag{3.6}$$

and,

$$| \Theta | + | \Lambda | + | \Gamma | + | \Delta | + | \Psi | = 4 \left(\!\!\binom{m-1}{k-3}\!\!\right) + \left(\!\!\binom{m}{k-4}\!\!\right)$$

$$= (4 + \frac{k-3}{m-1})\binom{m+k-5}{k-4} \tag{3.7}$$

From (I),

$$| A | = \left(\!\!\binom{m}{k-2}\!\!\right) = \binom{m+k-3}{k-2}$$

Now, consider the following inequality:

$$\eta\binom{m+k-5}{k-3} \leq \binom{m+k-3}{k-2} \tag{3.8}$$

or,

$$\eta\frac{(m+k-5)!}{(k-3)!(m-2)!} \leq \frac{(m+k-3)!}{(k-2)!(m-1)!}$$

or,

$$\eta \leq \frac{(m+k-3)(m+k-4)}{(k-2)(m-1)}$$

We had started with a restriction on k as $m = 2k$ -2

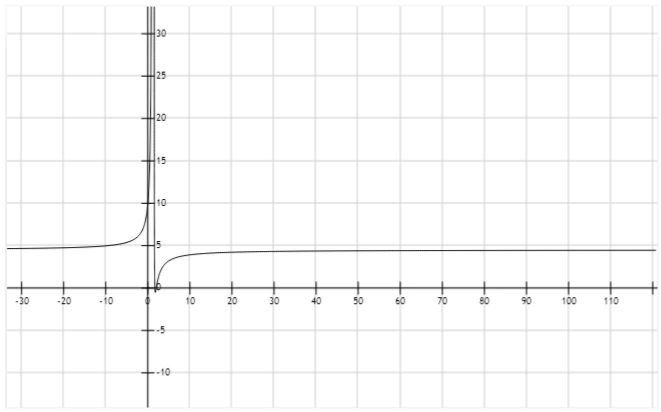
So,

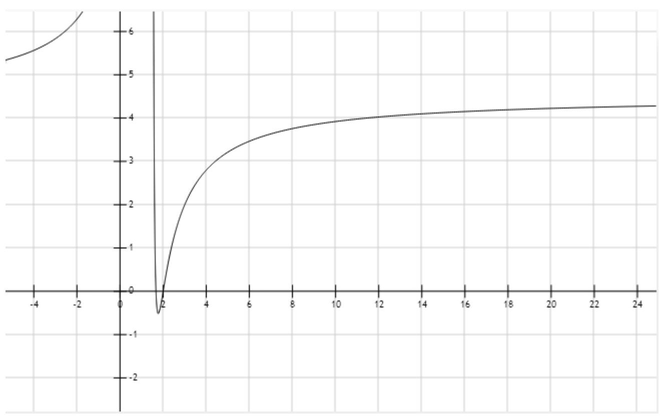
$$\eta \leq \frac{(3k-5)(3k-6)}{(2k-3)(k-1)} \tag{3.9}$$

For the inequality to hold, we investigate the function of k, $f(k)$, defined as.

$$f(k) = \frac{(3k-5)(3k-6)}{(2k-3)(k-1)}$$

This is an increasing function for $k \geq 3$





Taking limiting condition of $k \to \infty$ on both sides of equation, we get

$$\eta \leq \frac{9}{2} \tag{3.10}$$

Note that if we take $m > 2k - 2$ the upper bound on η also increases.

Now we go back to equation (3.6) and (3.7). Remember that γ^* and δ^* are special cases of Γ and Δ.
So,

$$\mid \Theta \mid + \mid \Lambda \mid + \mid \Gamma \mid + \mid \Delta \mid + \mid \Psi \mid = 4 \left(\binom{m-1}{k-3} \right)$$

$$= (4 + \frac{k-3}{m-1})\binom{m+k-5}{k-4} \tag{3.11}$$

Take

$$\eta' = 4 + \frac{k-3}{m-1} \tag{3.12}$$

and once again use the constraint $m = 2k - 2$
We get

$$\eta' = 4 + \frac{k-3}{2k-3} \tag{3.13}$$

Taking limiting condition of $k \to \infty$ on both sides of equation, we get

$$\eta' = 4 + \frac{1}{2}$$

or

$$\eta' = \frac{9}{2} \tag{3.14}$$

We see that the size of the intersecting family defined by $\Gamma \cup \Delta \cup \Theta \cup \Lambda \cup \Psi$ is limited by the same as the size given in the conjecture.
If we take $m > 2k - 2$, then a strict inequality holds, which seems to agree with the additional condition of *"moreover, if $m > t(k-t)+2$, equality holds if and only if A is a collection of all the k-multisets from $[m]$ that contain a fixed t-multiset from $[m]$"*.

IV. Example

Let $k = 5$, $m = 2k - 2 = 8$ and $t = 2$ and

$$[m] = \{1, 2, 3, 4, 5, 6, 7, 8\}$$
$$\gamma^* = \{1, 2, 3, 4, 5\}$$
$$\delta^* = \{4, 5, 6, 7, 8\}.$$

Note that $t_1 = 4$, $t_2 = 5$

$$| A |= \binom{8+5-3}{5-2} = \binom{10}{3} = 120$$

So, $\gamma^* \backslash \{4,5\} \times \delta^* \backslash \{4,5\} = \{(1,6),(1,7),(1,8),(2,6),(2,7),(2,8),(3,6),(3,7),(3,8)\}$
We choose $p_1 = 1$ and $p_2 = 6$ and generate Θ, Λ, Γ and Δ

$$| \Theta |=| \Lambda |=| \Gamma |=| \Delta |= \left(\!\!\binom{7}{2}\!\!\right) = \binom{8}{2} = 28$$

$\Theta = \{$(4,1,6,1,1), (4,1,6,2,2), (4,1,6,3,3), (4,1,6,4,4), (4,1,6,6,6), (4,1,6,7,7), (4,1,6,1,2), (4,1,6,1,3), (4,1,6,1,4), (4,1,6,1,6), (4,1,6,1,7), (4,1,6,2,3), (4,1,6,2,4), (4,1,6,2,6), (4,1,6,2,7), (4,1,6,3,4), (4,1,6,3,6), (4,1,6,3,7), (4,1,6,4,6), (4,1,6,4,7), (4,1,6,6,7), (4,1,6,8,8), (4,1,6,1,8), (4,1,6,2,8), (4,1,6,3,8), (4,1,6,4,8), (4,1,6,6,8), (4,1,6,7,8)$\}$

$\Lambda = \{$(5,1,6,1,1), (5,1,6,2,2), (5,1,6,3,3), (5,1,6,5,5), (5,1,6,6,6), (5,1,6,7,7), (5,1,6,1,2), (5,1,6,1,3), (5,1,6,1,5), (5,1,6,1,6), (5,1,6,1,7), (5,1,6,2,3), (5,1,6,2,5), (5,1,6,2,6), (5,1,6,2,7), (5,1,6,3,5), (5,1,6,3,6), (5,1,6,3,7), (5,1,6,5,6), (5,1,6,5,7), (5,1,6,6,7), (5,1,6,8,8), (5,1,6,1,8), (5,1,6,2,8), (5,1,6,3,8), (5,1,6,5,8), (5,1,6,6,8), (5,1,6,7,8)$\}$

$\Gamma = \{$(1,4,5,1,1), (1,4,5,1,2), (1,4,5,1,3), (1,4,5,1,4), (1,4,5,1,5), (1,4,5,1,7), (1,4,5,1,8), (1,4,5,2,2), (1,4,5,2,3), (1,4,5,2,4), (1,4,5,2,5), (1,4,5,2,7), (1,4,5,2,8), (1,4,5,3,3), (1,4,5,3,4), (1,4,5,3,5), (1,4,5,3,7), (1,4,5,3,8), (1,4,5,4,4), (1,4,5,4,5), (1,4,5,4,7), (1,4,5,4,8), (1,4,5,5,5), (1,4,5,5,7), (1,4,5,5,8), (1,4,5,7,7), (1,4,5,7,8), (1,4,5,8,8)$\}$

$\Delta = \{$(4,5,6,2,2), (4,5,6,2,3), (4,5,6,2,4), (4,5,6,2,5), (4,5,6,2,6), (4,5,6,2,7), (4,5,6,2,8), (4,5,6,3,3), (4,5,6,3,4), (4,5,6,3,5), (4,5,6,3,6), (4,5,6,3,7), (4,5,6,3,8), (4,5,6,4,4), (4,5,6,4,5), (4,5,6,4,6), (4,5,6,4,7), (4,5,6,4,8), (4,5,6,5,5), (4,5,6,5,6), (4,5,6,5,7), (4,5,6,5,8), (4,5,6,6,6), (4,5,6,6,7), (4,5,6,6,7), (4,5,6,7,7), (4,5,6,7,8), (4,5,6,8,8)$\}$

so,

$$| \Theta | + | \Lambda | + | \Gamma | + | \Delta |= 4\left(\!\!\binom{7}{2}\!\!\right) = 4\binom{8}{2} = 112$$

We now generate Ψ:

$\Psi = \{(1,4,6,5,1), (1,4,6,5,2), (1,4,6,5,3), (1,4,6,5,4), (1,4,6,5,5), (1,4,6,5,6), (1,4,6,5,7), (1,4,6,5,8)\}$

$$|\Psi| = 8$$

We see that,

$$|\Theta| + |\Lambda| + |\Gamma| + |\Delta| + |\Psi| = 120 = |A|$$

i.e., the construction has yielded the desired intersecting family whose size is bounded by the bound proposed in the conjecture.

Chapter 4

Conclusion

This thesis aims to further understanding of shadows and intersecting families in sets and other discrete structures.

In Chapter 2, we present Erdos-Ko-Rado and Kruskal-Katona theorems for sets, vector spaces and multisets.

In Chapter 3, we give a technique to construct a collection of 2-intersecting k-multisets of $[m]$. We see that the 2-intersecting family we constructed has a size which tends to the bound proposed in the conjecture. Further, it meets the bound only in the case of $m = 2k - 2$ and if $m > 2k - 2$ then, it is strictly below the bound. It would be interesting to prove that the size of 2-intersecting k-multiset family is at most $\binom{m+k-3}{k-2}$.

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