SPINORIAL REPRESENTATIONS OF SYMMETRIC AND ALTERNATING GROUPS

A thesis

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by

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Dedicated to
My Parents & Teachers

Certificate

Certified that the work incorporated in the thesis entitled "*Spinorial Representations of Symmetric and Alternating Groups*", submitted by *Jyotirmoy Ganguly* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Date: February 20, 2019

Dr. Steven Spallone
Thesis Supervisor

Declaration

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Contents

Abstract

Representations of the symmetric group S_n may be regarded as homomorphisms ϕ to the orthogonal group $\mathrm{O}(d, \mathbb{R})$, where d is the degree of ϕ. We give a criterion for whether ϕ lifts to $\mathrm{Spin}(d, \mathbb{R})$ or $\mathrm{Pin}(d, \mathbb{R})$, in terms of the character of ϕ. We give similar criteria for orthogonal representations of the alternating group A_n, and of products of symmetric groups. Using these criteria we count the number of irreducible spinorial representations of the Symmetric groups for some particular cases. Finally we prove that asymptotically most of the irreducible representations of S_n and A_n are spinorial.

1

Introduction

A finite dimensional real representation V of a group G is called orthogonal if for an inner product B on V we have

$$B(v, w) = B(g \cdot v, g \cdot w) \text{ for } v, w \in V.$$

In other words, if $\phi : G \to \mathrm{GL}(V)$ is the representation associated with V then $\phi(G) \subset O(V, B)$. Generally, we drop the notation B and simply write $\mathrm{O}(V)$ or $\mathrm{O}(d)$ to denote the group of orthogonal $d \times d$ matrices, where d denotes the dimension of the vector space V. We know that any representation of S_n is real and orthogonal. Therefore we can consider representations (ϕ, V) of S_n, where V is a finite-dimensional real vector space and ϕ is a homomorphism from S_n to $\mathrm{O}(V)$. If the determinant of ϕ is trivial then we say it is achiral. In this case the image of ϕ lies in $\mathrm{SO}(V)$. It is called chiral otherwise. There is a non-trivial two-fold cover $\mathrm{Pin}(V)$ of $\mathrm{O}(V)$ with covering map $\rho : \mathrm{Pin}(V) \to \mathrm{O}(V)$. We say a representation (ϕ, V) of S_n is spinorial if there exists a homomorphism $\widehat{\phi} : S_n \to \mathrm{Pin}(V)$ such that $\rho \circ \widehat{\phi} = \phi$. Otherwise we say ϕ is aspinorial. In particular we call an achiral representation spinorial if it lifts to $\mathrm{Spin}(V)$, which is a two-fold cover of $\mathrm{SO}(V)$.

The problem of lifting orthogonal representations has been highlighted by Serre [24], Delinge [7] and Prasad-Ramakrishnan [22] (who specifically ask about symmetric groups). The paper [12] gives lifting criteria for representations of reductive connected algebraic groups of characteristic 0 in terms of highest weights. In our thesis we give lifting criteria for representations of symmetric groups, alternating groups and a product of two symmetric groups. Using the criteria we also give the number of irreducible spinorial representations of S_n for some particular cases and show that asymptotically most irreducible representations of S_n and A_n are spinorial.

We in particular prove that one can determine the spinorial representations of S_n from the character values.

Theorem 1.0.1. *A representation (ϕ, V) of $S_n, n \geq 4$ is spinorial if and only if one of the following conditions holds:*

1. $\chi_V(s_1 s_3) \equiv \chi_V(1) \pmod 8$, $\chi_V(s_1) \equiv \chi_V(1) + 2 \pmod 8$. *In this case ϕ is chiral.*
2. $\chi_V(1) \equiv \chi_V(s_1) \equiv \chi_V(s_1 s_3) \pmod 8$. *In this case ϕ is achiral.*

Similarly for Alternating groups we obtain

Theorem 1.0.2. *An orthogonal representation (ϕ, V) of A_n, $n \geq 4$, is spinorial if and only if*

$$\chi_V(1) \equiv \chi_V(s_1 s_3) \pmod 8$$

Let (π_i, V_i) denote a representation of S_i, for $i \in \{1, 2\}$. Let g_i denote the multiplicity of -1 as an eigenvalue of $\pi_i(s_1)$, and f_i the dimension of V_i. We give lifting criteria for the representation $(\pi, V_1 \boxtimes V_2)$ of $S_{n_1} \times S_{n_2}$.

Theorem 1.0.3. *Let V_i be a representation of S_{n_i} for $i \in \{1, 2\}$. The representation $(\pi, V_1 \boxtimes V_2)$ of $S_{n_1} \times S_{n_2}$ is spinorial if and only if $\pi|_{(S_{n_1} \times 1)}$ and $\pi|_{(1 \times S_{n_2})}$ are spinorial and the following condition holds:*

$$g_1 g_2 (1 + f_1 f_2) \equiv 0 \pmod 2.$$

Let BS_n denote a classifying space of S_n and ES_n denote the principal S_n bundle over BS_n. The spinoriality of an achiral representation V of S_n can be detected by the second Stiefel-Whitney class of the associated vector bundle $ES_n \times_{S_n} V$ over BS_n. We work with orthogonal real representations of any finite group G. We take the Stiefel-Whitney classes of a finite- dimensional real representation (ϕ, V) of a group G to be $w_i(\phi) = w_i(EG \times_G V) \in H^i(BG; \mathbb{Z}/2\mathbb{Z})$ as in [19, page 37]. From [9] it follows that for a representation (ϕ, V) of G, we have $w_1(\phi) = \det(\phi)$. The following result gives a lifting criterion for an orthogonal representation (ϕ, V) of a finite group G, with $\det(\phi) = 1$.

The result can be found in [14] in a more general context. We prove it here to make the thesis more self-contained.

Theorem 1.0.4. *Let (ϕ, V) be an orthogonal representation of a finite group G and $w_1(\phi) = 0$. Then ϕ is spinorial if and only if $w_2(\phi) = 0$.*

In the paper [15] the author gives explicit formulas for the character values of irreducible representations of S_n in terms of Young diagrams of the associated partitions. Combining this with the theory of 2-core towers (discussed in Section 2.4) we obtain a characterization of the irreducible spinorial representations of S_n for some particular cases. If we write a number n in the form

$$n = \epsilon + 2^{k_1} + \cdots + 2^{k_r}, \;\; 0 < k_1 < ... < k_r, \;\; \epsilon \in \{0, 1\}, \tag{1.1}$$

then from [17, Corollary 1.3] the number of odd partitions of n is

$$A(n) = 2^{k_1 + \cdots + k_r}.$$

The result can also be found in [18].Determining the higher Stiefel-Whitney classes for S_n We write $s_1(n)$ to denote the number of odd, achiral, spinorial partitions of n.

Theorem 1.0.5. *For $n \geq 4$, we have*

$$s_1(n) = \begin{cases} \frac{1}{8}A(n), & for\ k_2 = k_1 + 1, \\ \frac{1}{4}A(n), & for\ k_2 \geq k_1 + 2, \quad or\ r = 1. \end{cases}$$

Finally we show that asymptotically most irreducible representations of S_n are spinorial. We use the notation $p(n)$ to denote the number of partitions of n.

Theorem 1.0.6. *We have*

$$\lim_{n \to \infty} \frac{\#\{\lambda \vdash n \mid \lambda \text{ is spinorial}\}}{p(n)} = 1.$$

Chapter 2 sets the stage by recalling all basic definitions and notations needed for the rest of the chapters. The first section introduces the Pin group. The rest of the chapter is devoted to the theory of cores and quotients of partitions and related topics. We end this chapter by recalling some relevant results from the paper [3].

Chapter 3 is concerned with the determination of spinorial representations (ϕ, V) of the symmetric groups. We know that S_n is generated by the transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$. Write g_V for the multiplicity of -1 as an eigenvalue of $\phi(s_1)$. For $n \geq 4$, consider the subgroup $C_2 \times C_2$ of S_n generated by s_1 and s_3. Write $\omega : K_4 \to \{\pm 1\}$ for the multiplicative character of $C_2 \times C_2$ taking both s_1 and s_3 to -1. Write h_V for the multiplicity of ω in the restriction of V to $C_2 \times C_2$.

Theorem 1.0.7. *A representation V of S_n, $n \geq 4$, is spinorial if and only if both the following conditions hold:*

1. $g_V \equiv 0$ *or* $3 \pmod 4$,
2. $h_V \equiv g_V \pmod 2$.

We in particular consider the Specht modules V_λ (discussed in Section 2.3). These are irreducible representations of S_n parametrized by partitions λ of n. We give lifting criteria of these representations in terms of the numbers $f_{\lambda/\mu}$ of certain standard skew Young tableaux. In fact we express the g_{V_λ} and h_{V_λ} in terms of numbers of standard skew Young tableaux as follows:

$$g_{V_\lambda} = f_{\lambda/(1,1)}, \quad \text{and} \quad h_{V_\lambda} = f_{\lambda/(2,1,1)} + f_{\lambda/(2,2)} + f_{\lambda/(1^4)}.$$

Chapter 4 investigates the spinorial representations of alternating groups. The alternating group is generated by $u_i = s_1 s_{i+1}$, $1 \leq i \leq n-2$. Write k_V for the multiplicity of -1 as an eigenvalue of $s_1 s_3$. We prove that:

Theorem 1.0.8. *An orthogonal representation (ϕ, V) of A_n for $n \geq 4$, is spinorial if and only if $k_V \equiv 0 \pmod 4$.*

If λ is not a self-conjugate partition then $V_\lambda \mid_{A_n}$ is an irreducible representation of A_n. For a self-conjugate partition λ the representation V_λ decomposes into two irreducible representations $V_\lambda^\pm$.

Theorem 1.0.9. *Suppose $V_\lambda^\pm$ is orthogonal. Then the following statements are equivalent:*

1. *V_λ^+ is spinorial.*
2. *V_λ^- is spinorial.*
3. *$\chi_\lambda(1) \equiv \chi_\lambda(s_1s_3) \pmod{16}$.*

Building on the previous chapters, Chapter 5 explores some corollaries and low dimensional examples. Here we discuss the spinoriality of direct sums, internal tensor products of representations of S_n. For each partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of n, take X_λ to be the set of all ordered partitions of $\{1, 2, \ldots, n\}$ of shape λ,

$$X_\lambda = \{(X_1, \ldots, X_l) \mid X_1 \sqcup \cdots \sqcup X_l = \{1, 2, \ldots, n\}, |X_i| = \lambda_i\}.$$

The action of S_n on $\{1, \ldots, n\}$ gives rise to an action of it on X_λ. Take the vector space $\mathbb{R}[X_\lambda]$ and consider the permutation representation it affords. We obtain lifting criteria in terms of congruence relations of multinomial coefficients. In particular, for $\lambda = (1^n)$, the representation $\mathbb{R}[X_{(1^n)}]$ gives the regular representation of S_n. Our criteria gives the following result.

Theorem 1.0.10. *The regular representation of S_n, $n \geq 4$, is achiral and spinorial.*

Next we explore the spinoriality of the representations of the product of two symmetric groups. Finally, we present in tabular form behaviors of representations of Symmetric and Alternating groups of small sizes.

Chapter 6 adopts a cohomological approach to determine spinoriality of representations of Symmetric groups. Let ϵ denote the sign representation of S_n and ϕ_n denote the standard permutation representation of S_n on $\mathbb{R}^n$, via permutation matrices. Write $e_{\text{cup}} = w_1(\epsilon) \cup w_1(\epsilon)$. From [24, Section 1.5] we obtain that e_{cup} and $w_2(\phi_n)$ generate the group $H^2(S_n, \mathbb{Z}/2\mathbb{Z})$. Using these facts we prove that:

Theorem 1.0.11. *Let (ϕ, V) be any representation of S_n. Then*

$$w_2(\phi) = \left[\frac{g_V}{2}\right] e_{\text{cup}} + \frac{k_V}{2} w_2(\phi_n).$$

Here [.] denotes the greatest integer function.

Chapter 7 is devoted to characterizing the irreducible, spinorial representations of S_n for some particular cases. We write $H(a,b)$ for the hook of the form $(a+1, 1^b)$ and $H^+(a,b)$, $a, b > 0$ for the partition $(a+1, 2, 1^{(b-1)})$. The first section of the chapter recalls the explicit character formulas in terms of contents. Here we also mention the general character formulae in terms of contents given by Michel Lassalle in [15, Theorem 6]. In the next section we focus on the achiral, odd partitions. Here we use the theory of 2-core towers extensively. The results in [15] help us to characterize the odd, achiral, spinorial partitions and count them. Theorem 6.3.2 ensures that for an achiral, spinorial irreducible representation V_λ of S_n, we have $w_1(V_\lambda) = w_2(V_\lambda) = 0$. In fact from [19, Exercise 8.B, page 94] we conclude that $w_3(V_\lambda) = 0$ as well.

In the next two sections we explore the odd, chiral, spinorial partitions of $2^k + \epsilon$. The two theorems stated below summarizes the results.

Theorem 1.0.12. *Let $n \geq 8$ be a power of 2. Then a partition of n is odd, chiral and spinorial if and only if it is a hook of the form $H(a,b)$ with $a > b$ and $b \equiv 3 \pmod 4$. In particular the number of odd, chiral, spinorial partitions of n is $n/8$.*

Theorem 1.0.13. *Let n be of the form $2^k + 1, k \geq 3$. Then a partition of n is odd, chiral and spinorial if and only if it is of the form $H^+(a,b)$ with $b > a$, $b \equiv 0 \pmod 4$ and $v_2(b) \leq k-2$. In particular there are $2^{k-3} - 1$ odd, chiral, spinorial partitions of n.*

The remainder of the chapter investigates the self-conjugate spinorial partitions. We write $v = v_2(f_\lambda)$.

Theorem 1.0.14. *Let λ be a self-conjugate partition of n. If $v \geq 3$ then λ is spinorial. If $v = 2$ then λ is aspinorial. If $v = 1$, then λ is spinorial if and only if $\lambda = H(2^{k-1}, 2^{k-1})$, for some $k \geq 2$.*

Chapter 8 ventures into asymptotic behaviors of the irreducible representations of the symmetric and alternating groups. We prove that

Theorem 1.0.15. *For any fixed non-negative integer m,*

$$\lim_{n\to\infty} \frac{|\{\lambda \vdash n \mid v \leq k_r + m\}|}{p(n)} = 0.$$

where $v = v_2(f_\lambda)$, n has the form as in Equation (1.1).

With similar arguments, we conclude that most irreducible representations of S_n and A_n are spinorial (see 8.0.2 and 8.0.4). For a partition μ such that $|\mu| \leq n$ we obtain a partition $(\mu, 1^{n-|\mu|})$ of n. For example if $\mu = (2,1,1)$ and $n = 6$, we have the partition $(2,1,1,1,1)$ of 6. We also prove that:

Theorem 1.0.16. *For a fixed partition μ and a positive integer b we have*

$$\lim_{n\to\infty} \frac{\#\{\lambda \vdash n \mid \chi_\lambda(\mu, 1^{(n-|\mu|)}) \equiv 0 \pmod{2^b}\}}{p(n)} = 1.$$

2

Preliminaries

This chapter is devoted to the exposition of basic preliminary material which we use extensively throughout the thesis. We begin with a quick review of the Pin group and some of its properties that we use later. Next we define Young tableau associated with a partition λ of n and discuss related concepts. This allows us to discuss Young's natural representation of the Specht modules, which gives irreducible representations of S_n. Finally, we recall some results from the paper [3] which we use later.

2.1 Pin Group

For a real vector space V the tensor algebra $T(V)$ is defined as

$$T(V) = \bigoplus_{i=0}^{\infty} V^{(i)}, \quad \text{where } V^{(i)} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{i\,\text{times}} \text{ and } V^{(0)} = \mathbb{R}.$$

For the multiplicative structure on $T(V)$ see [8, Section 11.5]. Let $Q : V \to \mathbb{R}$ denote a quadratic form. Then we define the Clifford Algebra as

$$C(V, Q) = \frac{T(V)}{\mathfrak{a}},$$

where $\mathfrak{a} \subset T(V)$ denotes the ideal generated by the elements $\{v \otimes v - Q(v) \cdot 1; v \in V\}$. The algebra $C(V, Q)$ has a canonical anti-automorphism $t : C(V, Q) \to C(V, Q)$ defined as

$$t(v_1 \cdots v_r) = v_r \cdots v_1,$$

for $v_i \in V$. Also there is a canonical automorphism $\alpha : C(V,Q) \to C(V,Q)$ given by

$$\alpha(v_1 \cdots v_r) = (-1)^r v_1 \cdots v_r.$$

Using these two maps we define an anti-involution on $C(V,Q)$ as " $*$ " $= \alpha t = t\alpha : C(V,Q) \to C(V,Q)$ as

$$(v_1 \cdots v_r)^* = (-1)^r v_r \cdots v_1.$$

Let $\mathrm{O}(V,Q)$ denote the orthogonal group and $\mathrm{SO}(V,Q)$ denote the special orthogonal group with respect to the quadratic form Q. We define the Pin group as

$$\mathrm{Pin}(V,Q) = \{x \in C(V,Q) \mid x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\},$$

and the homomorphism

$$\rho : \mathrm{Pin}(V,Q) \to \mathrm{O}(V,Q), \quad \rho(x)(v) = \alpha(x) \cdot v \cdot x^*.$$

An important subgroup of $\mathrm{Pin}(V,Q)$ is $\mathrm{Spin}(V,Q)$ defined as

$$\mathrm{Spin}(V,Q) = \rho^{-1}(\mathrm{SO}(V,Q)).$$

From now on we take $V = \mathbb{R}^n$ with the quadratic form $Q : \mathbb{R}^n \to \mathbb{R}, x \mapsto -|x|^2$, i.e. the standard negative definite quadratic form. We write $C_n = C(\mathbb{R}, Q)$. We will use both the notations $\mathrm{Pin}(V)$ and $\mathrm{Pin}(n)$ to denote the group $\mathrm{Pin}(V,Q)$. Similarly, we take the liberty of using the notations $\mathrm{O}(V)$ and $\mathrm{O}(n)$ (resp. $\mathrm{SO}(V)$ and $\mathrm{SO}(n)$) to denote the group $\mathrm{O}(V,Q)$ (resp. $\mathrm{SO}(V,Q)$).

If we consider the standard basis $\{e_1, e_2, ..., e_n\}$ of V, then $e_i \in \mathrm{Pin}(V)$ with

$$\rho(e_i) = \mathrm{diag}(1, 1, \cdots, -1, 1, \cdots, 1),$$

where -1 is at the i-th position. In fact $\rho(u)$ is a reflection when u is a unit vector. We also obtain the relations

1. $e_i^2 = -1 \in \text{Pin}(V)$,
2. $e_i e_j = -e_j e_i$ for $i \neq j$.

As a quick example for $n = 1$ we have $C_1 = \mathbb{C}$ and $\text{Pin}(1) = \mathbb{Z}/4\mathbb{Z}$. More details on the spinor groups can be found in [4, Chapter 1.6].

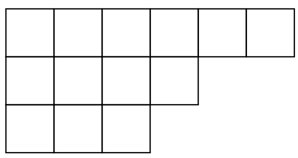
Definition 2.1.1. A finite dimensional real representation (ϕ, V) of a group G is called *orthogonal* if $\phi(G) \subset \text{O}(V)$.

Definition 2.1.2. An orthogonal representation (ϕ, V) of a group G is called *spinorial* if there exists a homomorphism $\widehat{\phi} : G \to \text{Pin}(V)$ such that $\rho \circ \widehat{\phi} = \phi$. So if ϕ is spinorial we obtain the following commutative diagram:

$$
\begin{tikzcd}
 & \text{Pin}(V) \arrow[d, "\rho"] \\
G \arrow[ur, dotted, "\widehat{\phi}"] \arrow[r, "\phi"] & O(V)
\end{tikzcd}
$$

2.2 Young Tableaux

For a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$, we define the associated Young diagram, denoted by $\mathcal{Y}(\lambda)$, as a finite collection of cells arranged in an array of left justified rows such that the i-th row contains λ_i number of cells. Pictorially $\mathcal{Y}(6, 4, 3)$ looks like



The partition λ is called the shape of $\mathcal{Y}(\lambda)$. A Young diagram with its boxes filled in by integers is called a Young tableau. We denote a Young tableau by $t = (t(i, j))$, where $t(i, j)$ denotes the integer in the (i, j)-th cell of the tableau. We are in particular interested in the class of standard Young tableaux.

Definition 2.2.1. A *standard Young tableau* (SYT) of shape λ is a Young diagram of shape λ in which the cells are filled in with the positive integers $\{1, 2, \ldots, n\}$, where $|\lambda| = n$, in such a way that

- the entries increase strictly down each column;

- the entries increase strictly (from left to right) along each row.

For example,

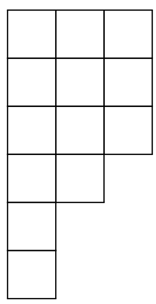
1	2	4	5	6	8
3	7	9	11		
10	12	13			

is a SYT of shape $(6, 4, 3)$. The number of SYT of shape λ is denoted by f_λ.

Definition 2.2.2. The *conjugate partition* of a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ is defined as the partition $\lambda' = (\lambda'_1, \lambda'_2, \cdots, \lambda'_s)$, where λ'_j is the number of parts of λ which are greater than or equal to j:

$$\lambda'_j = |\{1 \leq i \leq l \mid \lambda_i \geq j\}|.$$

The concept of conjugate partitions can be visualized in terms of Young diagrams. The Young diagram of shape λ' is obtained from the Young diagram of shape λ by reflecting it about the principal diagonal. For example, flipping the Young diagram of shape $(6, 4, 3)$ gives



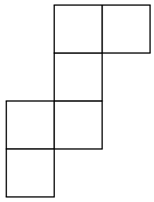
which is the Young diagram of shape $(3, 3, 3, 2, 1, 1)$, the partition conjugate to $(6, 4, 3)$.

Definition 2.2.3. For $\mu \subseteq \lambda$, the *skew Young diagram* of shape λ/μ is the set of cells

$$\lambda/\mu = \{c : c \in \lambda \text{ and } c \notin \mu\}.$$

As for example if $\lambda = (3,2,2,1)$ and $\mu = (1,1)$ then

$\lambda/\mu =$



In a similar fashion as before one can define a standard skew Young tableau.

Definition 2.2.4. A standard skew Young tableau is a skew Young diagram in which the boxes are filled in with positive integers in such a way that the entries increase strictly down each column and along each row from left to right.

The number of standard skew Young tableaux of shape λ/μ is denoted by $f_{\lambda/\mu}$.

Definition 2.2.5. The *content* of a cell $(i,j) \in \mathcal{Y}(\lambda)$ is defined to be $c(i,j) = j - i$. The total content of $\mathcal{Y}(\lambda)$ is defined as

$$C(\lambda) = \sum_{(i,j)\in\mathcal{Y}(\lambda)} (j-i).$$

Here is an example of a Young diagram of $\lambda = (6,4,3)$ with each of its cells filled by its content.

0	1	2	3	4	5
−1	0	1	2		
−2	−1	0			

2.3 Young's Natural Representation

For a Young tableau t of shape λ we define two subgroups of the symmetric group $S_{|\lambda|}$ as

$$R_{\text{tabloids}} = \{g \in S_{|\lambda|} \mid g \text{ preserves each row of } t\},$$

and

$$C_{\text{tabloids}} = \{g \in S_{|\lambda|} \mid g \text{ preserves each column of } t\}.$$

The subgroups R_{tabloids} and C_{tabloids} are called the row stabilizer and column stabilizer of t respectively. Two tableaux t_1 and t_2 of shape λ are called row equivalent, denoted by $t_1 \sim t_2$, if corresponding rows of the two tableaux contain same elements. The equivalence class of a tableau t is given by $\{t\} = R_{\text{tabloids}}t$. Similarly one can define a column equivalence relation on the set of tableaux of shape λ such that the equivalence class of a tableau t becomes $[t] = C_{\text{tabloids}}t$. One can define a column dominance order denoted by '$\rhd$' on the column equivalence classes of the tableaux. For details see [23, page 72]. An element $\sigma \in S_n$ acts on a tableau $t = (t_{i,j})$ of shape λ as $\sigma t = (\sigma(t_{i,j}))$. This induces an action on the set of equivalence classes $\{t\}$ by letting $\sigma\{t\} = \{\sigma t\}$.

Definition 2.3.1. For a Young tableau t, the *associated polytabloid* is

$$e_t = \sum_{\sigma \in C_{\text{tabloids}}} \operatorname{sgn}(\sigma)\sigma\{t\}.$$

Note that $e_t \in \mathbb{R}\{\{t_1\}, \dots, \{t_k\}\}$. Here $\mathbb{R}\{\{t_1\}, \dots, \{t_k\}\}$ denotes the vector space over $\mathbb{R}$ generated by the set $\{\{t_1\}, \dots, \{t_k\}\}$, where $\{t_1\}, \dots, \{t_k\}$ gives a complete list of row equivalent tableaux of shape λ. Next we define the Specht module denoted by V_λ.

Definition 2.3.2. The *Specht module* V_λ is the subspace of $\mathbb{R}\{\{t_1\}, \dots, \{t_k\}\}$ generated by the polytabloids e_t, where t varies over all the tableaux of shape λ.

Theorem 2.3.3. *The set of polytabloids*

$$\{e_t \mid t \textit{ is a standard Young tableau of shape } \lambda\},$$

is linearly independent.

As a result, the set

$$\beta_\lambda = \{e_t \mid t \text{ is a standard Young tableau of shape } \lambda\},$$

is a basis for V_λ. For details about Specht modules we refer the reader to [23, Theorem 2.6.5]. We write $\dim V_\lambda = f_\lambda$.

The representation of $(\phi_\lambda, V_\lambda)$, with respect to the basis β_λ is known as Young's natural representation. Here we indicate how to compute the matrices of the representation.

Since S_n is generated by the transpositions $s_i = (i, i+1)$, for $1 \leq i \leq n-1$, it is enough to compute the matrices for these group elements. We have three cases.

1. If i and $i+1$ are in the same column of t, then
$$\phi_\lambda(s_i)(e_t) = -e_t.$$

2. If i and $i+1$ are in the same row of t, then
$$\phi_\lambda(s_i)(e_t) = e_t \pm \text{ other polytabloids } e_{t'} \text{ such that } [t'] \rhd [t].$$

3. If i and $i+1$ are not in the same row or column of t, then the tableau $t' = s_i t$ is standard and
$$\phi_\lambda(s_i)(e_t) = e_{t'}.$$

The details are provided in the book [23, Section 2.7]. The following example shows the matrices for the representation $V_{(2,1)}$ of S_3. This example is also taken from [23, page 75]. Applying the methods mentioned above yields

$$\phi_\lambda(s_1) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \phi_\lambda(s_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2.4 Core and Quotient of a Partition and 2-core Towers

For $x \in \mathcal{Y}(\lambda)$, let H_x denote the union of the cells in $\mathcal{Y}(\lambda)$ to the right of x with the cells below x, including x itself. If (i,j) denotes the location of x in $\mathcal{Y}(\lambda)$ then the set H_x is called the (i,j)-hook in λ. Write $h_x = |H_x|$ for the "hooklength" of H_x. In the following Young diagram for $\lambda = (6,4,3)$ we have labeled each cell c by its hooklength $h(c)$.

8	7	6	4	2	1
5	4	3	1		
3	2	1			

If $x = (i, j) \in \mathcal{Y}(\lambda)$, then

$$h_x = \lambda_i + \lambda'_j - i - j + 1.$$

The hooklengths of a partition are quite useful. As for example one can quickly compute the dimension f_λ of the representation V_λ with the hooklengths using the following formula, mentioned in [21, Theorem 5.8.3].

$$f_\lambda = \frac{n!}{\prod_{x \in \mathcal{Y}(\lambda)} h_x}. \tag{2.1}$$

The node (i, j) is called the corner of $H_{(i,j)}$. The furthest node to the right of (i, j) in $\mathcal{Y}(\lambda)$, (i, λ_i), is called the hand node of $H_{(i,j)}$. Similarly the furthest node below (i, j), (j, λ'_j) is called the foot node of $H_{(i,j)}$. If h_x is divisible by q, we call it a q-hook.

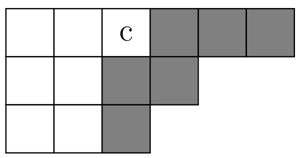
The set

$$\mathcal{R}_{\mathcal{Y}} = \{(i', j') \in \mathcal{Y}(\lambda) \mid (i' + 1, j' + 1) \notin \mathcal{Y}(\lambda)\},$$

is called the rim of $\mathcal{Y}(\lambda)$. Note that one can remove $\mathcal{R}_{\mathcal{Y}}$ from $\mathcal{Y}(\lambda)$ to obtain a new partition. For $x = (i, j) \in \mathcal{Y}(\lambda)$, we put

$$\mathrm{rim}_x = \{(i', j') \in \mathcal{R}_{\mathcal{Y}} \mid i' \geq i, j' \geq j\}.$$

This is called the x-rim hook of $\mathcal{Y}(\lambda)$. For an example we have shaded the c-rim hook of the Young diagram for $\lambda = (6, 4, 3)$, where $c = (1, 3)$.



Note that $h(x) = |\mathrm{rim}_x|$.

For a given partition λ of n and $q \in \mathbb{N}$ we obtain the q-core of λ denoted as $\mathrm{core}_q(\lambda)$ by successively removing all rim-q hooks from $\mathcal{Y}(\lambda)$ until there is no q-hook. This does not depend on the choice of q-hooks at each stage. For details we refer the reader to [20].

The q-quotient of λ is a certain q-tuple of partitions

$$\mathrm{quo}_q(\lambda) = (\lambda_q^{(0)}, \lambda_q^{(1)}, \ldots, \lambda_q^{(q-1)}),$$

such that

$$|\lambda| = |\operatorname{core}_q(\lambda)| + q(|\lambda_q^{(0)}| + |\lambda_q^{(1)}| + \cdots + |\lambda_q^{(q-1)}|).$$

In fact $|\operatorname{quo}_q(\lambda)|$ is the total number of q-hooks to be removed from $\mathcal{Y}(\lambda)$ to obtain $\operatorname{core}_q(\lambda)$. A partition λ can be uniquely recovered from the given pair $(\operatorname{core}_q(\lambda), \operatorname{quo}_q(\lambda))$.

We are in particular interested in the case when $q = 2$. Note that the empty set $\emptyset$ is a 2-core. We call the set of partitions of the form $\{k-1, k-2, k-3, \ldots, 1\}$, $k \in \mathbb{N}$, as "staircase" partitions. Note that the staircase partitions are partitions of triangular numbers, i.e. they are partitions of the numbers $k(k-1)/2$, for $k \in \mathbb{N}$.

Proposition 2.4.1. *Any partition λ is a 2-core if and only if it is a staircase partition.*

Proof. If λ is a staircase partition of the form $\{k-1, k-2, k-3, \ldots, 1\}$ for some $k \in \mathbb{N}$, then λ is a 2-core. For the converse let $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ be a 2-core. Then we have $|\lambda_l| = 1$. Otherwise one can remove a partition of shape (2) from the last row of $\mathcal{Y}(\lambda)$. We claim that $|\lambda_i - \lambda_{i+1}| = 1$, for $1 \leq i \leq l-1$. If $|\lambda_i - \lambda_{i+1}| > 1$, then we can remove a domino of shape (2) from the i-th row of $\mathcal{Y}(\lambda)$. If $\lambda_i = \lambda_{i+1}$, let M denote the maximum integer such that $\lambda_i = \lambda_M$. Then we can remove a vertical domino of shape $(1,1)$ from the λ_i-th column of $\mathcal{Y}(\lambda)$. Therefore the λ is of the form $\{k-1, \ldots, 2, 1\}$. □

The 2-core tower of a partition λ, which we denote by '$T_2(\lambda)$' is obtained as follows.

- It has rows numbered $0, 1, 2, ...$ and the ith row has 2^i many nodes. Each node is labeled with a 2-core partition. The 0th row has the partition $\alpha_\phi = \operatorname{core}_2 \lambda$.

- The first row consists of the partitions

  $$\alpha_0, \alpha_1$$

  where , if $\operatorname{quo}_2 \lambda = (\lambda_0, \lambda_1)$,then $\alpha_i = \operatorname{core}_2 \lambda_i$.

- The 2nd row of the tower is

  $$\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}$$

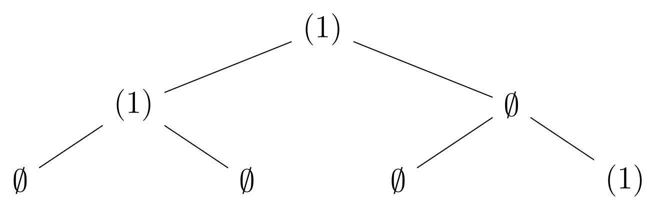
  where, if $\operatorname{quo}_2 \lambda_i = (\lambda_{i0}, \lambda_{i1})$,then $\alpha_{ij} = \operatorname{core}_2 \lambda_{ij}$.

- Recursively, having defined partitions λ_x for a binary sequence x, define the partitions λ_{x0} and λ_{x1} by

$$\mathrm{quo}_2(\lambda_x) = (\lambda_{x0}, \lambda_{x1}), \tag{2.2}$$

  and let $\alpha_{x\epsilon} = \mathrm{core}_2\,\lambda_{x\epsilon}$ for $\epsilon = 0, 1$. The i-th row of the tower consists of the partitions α_x, where x runs over the set of all 2^i binary sequences of length i, listed from left to right in lexicographic order.

A partition is uniquely determined by its 2-core tower, which has non-empty partitions in only finitely many places. For example, $T_2(3, 3, 1)$ looks like:



All other nodes in the tower are labeled by the empty partition.

Let w_i denote the total number of cells in all the nodes in the i-th row of $T_2(\lambda)$. It follows that

$$\sum_i w_i 2^i = n.$$

Let $v = v_2(f_\lambda)$ and let $\nu(n)$ denote the number of 1's in the binary expansion of n, then

$$v = \sum_i w_i - \nu(n).$$

For details on the theory of 2-core towers we refer the reader to [20, Section 6, page 41].

2.5 Macdonald's Theory

We call a partition λ "odd" if f_λ is odd. Otherwise we call it even. From [17] we obtain a nice classification of odd partitions. The following result gives the description of the 2-core towers for odd partitions.

Theorem 2.5.1 (Macdonald). *A partition λ is odd if and only if $T_2(\lambda)$ has at most one nonempty partition in each row, and this partition can only be* (1).

As a result we can count the number of odd partitions for a fixed n. The following result can be found in [17, Corollary 1.3].

Corollary 2.5.2. *The number of odd partitions of n, for n as in 1.1, is given by*

$$A(n) = 2^{k_1+k_2+\cdots+k_r}.$$

Let n, n_1, n_2 be positive integers such that $n_1 + n_2 = n$. The sum is called neat if there is no carry in adding n_1 and n_2 in binary. Otherwise it is called messy. Note that if $n_1 + n_2 = n$ is neat then $A(n) = A(n_1) \cdot A(n_2)$.

Proposition 2.5.3. *Let λ be a partition such that $2^k \leq |\lambda| < 2^{k+1}$, where $k \geq 1$. Then λ is odd if and only if the partition $\mathrm{core}_{2^k}(\lambda)$ is odd and λ has a unique hook of length 2^k.*

For a proof of the proposition we refer the reader to [2, Lemma 1].

2.6 Review of Ayyer-Prasad-Spallone

Here we mention some results from the paper [3] which we use in the thesis.

We know that any representation of S_n is orthogonal. A representation (ϕ, V) of S_n is called achiral if $\det \circ \phi$ is the trivial character of S_n. Otherwise we call ϕ chiral.

The paper [3] gives a characterization of the chiral partitions of S_n in terms of 2-core towers and counts them.

Lemma 2.6.1. *[3, Lemma 9] Let λ be any partition. For each binary sequence x, let λ_x denote the partition obtained recursively from λ by Equation (2.2). Fix $\delta \in \{0, 1\}$. The nodes of $\mathcal{Y}(\lambda_x)$ as x runs over the binary sequences of length i starting with δ, correspond to the nodes of $\mathcal{Y}(\lambda)$ whose hooklengths are multiples of 2^i and hand nodes have content congruent to δ modulo 2.*

Given the 2-core tower of a partition λ we can easily get the corresponding tower for $\mathrm{core}_{2^i}(\lambda)$ for $i \geq 0$ as follows.

Lemma 2.6.2. *[3, Lemma 10] Let λ be any partition. The 2-core tower of $\mathrm{core}_{2^i}(\lambda)$ is obtained by replacing all partitions in rows numbered i and larger by empty partitions in* *the* 2-core *tower of* λ.

Lemma 2.6.3. *[3, Lemma* 12*] Let* λ *be a partition of* n *and* $\alpha = \mathrm{core}_{2^{k_1+1}}(\lambda)$. *Then* λ *is chiral if and only if both the following conditions hold:*

1. α *is a chiral partition of* $2^{k_1} + \epsilon$.

2. *If* μ *is the partition whose 2-core tower is obtained from the* 2-core *tower of* λ *by replacing the partitions appearing in rows numbered* $0, \ldots, k_1$ *by the empty partition, then* $v_2(f_\mu) = 0$.

Note that the second condition of the lemma automatically holds if λ is odd. As a consequence we obtain the following result.

Corollary 2.6.4. *Let* λ *be an odd partition of* n *and* $\alpha = \mathrm{core}_{2^{k_1+1}}(\lambda)$. *Then* λ *is chiral if and only if* α *is a chiral partition of* $2^{k_1} + \epsilon$.

The next result gives a characterization the chiral self-conjugate partitions.

Theorem 2.6.5. *[3, Corollary* 7*] A positive integer* n *admits a self-conjugate chiral partition if and only if* $n = 3$, *or* $n = 2^k + \epsilon$, *for some* $k \geq 2$ *and* $\epsilon \in \{0, 1\}$. *Moreover,* λ *is a self-conjugate chiral partition of* $2^k + \epsilon$ *if and only if* λ *is self-conjugate and* $v_2(f_\lambda) = 1$. *The number of self-conjugate chiral partitions of* $2^k + \epsilon$ *is* 2^{k-2} *for* $k \geq 2$.

3

Spinorial Representations of Symmetric Groups

In this chapter, we determine the spinorial representations of the symmetric groups. In the first section we derive a criterion for spinoriality of any representation of S_n. In the remaining portion of the chapter we discuss the spinoriality of irreducible representations of S_n, known as Specht modules.

3.1 General Case

We know that S_n is generated by the transpositions $s_i = (i, i+1)$, for $1 \leq i \leq n-1$, which satisfy the following relations:

1. ${s_i}^2 = 1, \quad 1 \leq i \leq n-1,$
2. $(s_i s_{i+1})^3 = 1, \quad 1 \leq i \leq n-2,$
3. $[s_i, s_k] = 1, \quad |i-k| > 1.$

For a lift $\widehat{\phi} : S_n \to \mathrm{Pin}(V)$ of ϕ we need to choose elements $h_i = \widehat{\phi}(s_i) \in \mathrm{Pin}(V)$, which satisfy the same relations. Namely

1. ${h_i}^2 = 1, \quad 1 \leq i \leq n-1,$
2. $(h_i h_{i+1})^3 = 1, \quad 1 \leq i \leq n-2,$

3. $[h_i, h_k] = 1, \quad |i-k| > 1.$

We call these the first, second and third lifting conditions.

For the first condition it is enough to verify whether $h_1^2 = 1$, as all the transpositions are conjugate in S_n. Since $\phi(s_1^2) = \mathbb{1}$, the identity matrix, the eigenvalues of the matrix $\phi(s_1)$ are ± 1. Let g_V denote the multiplicity of -1 as an eigenvalue of $\phi(s_1)$. We use the notation $m = g_V$ for convenience. Let $\{e_1, e_2, \ldots, e_m\}$ denote the orthonormal basis for the -1 eigenspace of $\phi(s_1)$. We can extend this basis to obtain an orthonormal basis for the vector space V with respect to which $\phi(s_1)$ takes the diagonal form $A = \mathrm{diag}(\underbrace{-1, -1, \ldots, -1}_{m \text{ times}}, 1, \ldots, 1)$. Note that

$$\begin{aligned}
\mathrm{diag}(-1, -1, \ldots, -1, 1, \ldots, 1) &= \mathrm{diag}(-1, 1, \ldots, 1, 1, \ldots, 1) \\
&\quad \cdots \mathrm{diag}(1, 1, \ldots, -1, 1, \ldots, 1) \\
&= \rho(e_1) \cdots \rho(e_m).
\end{aligned}$$

Therefore we may choose $e_1 \cdot e_2 \cdots e_m$ as a lift of A in $\mathrm{Pin}(n)$. To satisfy the first lifting condition we must have $(e_1 \cdot e_2 \cdots e_m)^2 = 1$.

For each $\phi(s_i) \in \mathrm{O}(V)$, we may choose $\pm h_i \in \mathrm{Pin}(V)$ with $\rho(\pm h_i) = \phi(s_i)$, and the question is whether we may choose signs so that the $\widehat{\phi}(s_i) = \pm h_i$ satisfy these lifting conditions.

Theorem 3.1.1. *The first and second lifting conditions of an orthogonal representation V are satisfied if and only if*

$$g_V \equiv 0 \text{ or } 3 \pmod 4.$$

Proof. We claim that

$$(e_1 e_2 \cdots e_m)^2 = (-1)^{m(m+1)/2}.$$

To see the result we expand $(e_1 e_2 \cdots e_m)^2$ using the following steps:

1. Consider the rightmost e_i and move it $(m-i)$ places towards left using the relation $e_i e_j = -e_j e_i$ for $i \neq j$,

2. Apply the relation $e_i^2 = -1$.

Repeat the process for each $e_i, 1 \leq i \leq m$. Now $(e_1e_2...e_m)^2 = 1$ if and only if $m(m+1)/2$ is even. Again the latter condition holds if and only if $m \equiv 0$ or $3 \pmod 4$.

For the second lifting condition note that

$$\rho((h_ih_{i+1})^3) = \phi((s_is_{i+1})^3) = 1.$$

As $\ker(\rho) = \pm 1$, we obtain $(h_ih_{i+1})^3 = 1$ or -1. If $(h_ih_{i+1})^3 = -1$ then keeping h_i fixed we replace h_{i+1} by $-h_{i+1}$, so that $(-h_ih_{i+1})^3 = 1$. This change in sign does not affect the first condition as $(-h_{i+1})^2 = 1$.

□

Consider the subgroup $H_1 = \langle s_1 \rangle$ of S_n, and the character $\omega_1 : H_1 \to \{\pm 1\}$, given by $\omega_1(s_1) = -1$. For a representation (ϕ, V) of S_n let χ_V denote the character of it. Then we calculate

$$\begin{aligned}(\chi_V \mid_{H_1}, \omega_1) &= \frac{1}{2}(\chi_V(1)\omega_1(1) + \chi_V(s_1)\omega_1(s_1)) \\ &= (\chi_V(1) - \chi_V(s_1))/2.\end{aligned}$$

This gives the dimension of the isotypic component $V(-1)$, i.e. the dimension of the -1 eigenspace of $\phi(s_1)$. In other words, we obtain

$$g_V = \frac{1}{2}(\chi_V(1) - \chi_V(s_1)). \tag{3.1}$$

Now we proceed to deal with the third lifting condition.

Definition 3.1.2. For any element $x \in \mathrm{O}(V)$ the *sharp centralizer* of x, denoted by $Z_{\mathrm{O}(V)}(x)^\sharp$, is $\rho(Z_{\mathrm{Pin}(V)}(y))$, where $y \in \mathrm{Pin}(V)$ such that $\rho(y) = x$.

Since the ambient group is always the orthogonal group $\mathrm{O}(V)$, we denote $Z_{\mathrm{O}(V)}(A)$ by $Z(A)$ and $Z_{\mathrm{O}(V)}(x)^\sharp$ by $Z(x)^\sharp$. We choose $e_1 \cdot e_2 \cdots e_m$ as a lift of A. Let $Z(A)^0$ denote the connected component of $Z(A)$ containing the identity. Here we consider connectedness with respect to the Euclidean topology.

Lemma 3.1.3. *The sharp centralizer $Z(A)^\sharp$ is a closed, normal subgroup of $Z(A)$ with index 2. Moreover we have $Z(A)^0 \subseteq Z(A)^\sharp$.*

Proof. Consider the map $\psi : Z(A) \to \{1, -1\}$ given by $\psi(g) = [\tilde{g}, e_1 e_2 \cdots e_m]$, where $\tilde{g}$ denotes a lift of g in $\mathrm{Pin}(V)$. It is easy to check that ψ is a homomorphism. Observe that $\ker \psi = Z(A)^\sharp$. Simple calculation shows

$$e_1 \cdot e_1 \cdot e_2 \cdots e_m(-e_1) = (-1)^{m+1} e_1 \cdot e_2 \cdots e_m.$$

Therefore $\rho(e_1) \notin Z(A)^\sharp$ for m even. Consequently $\psi(\rho(e_1)) = -1$ when m is even. On the other hand, we have

$$e_{m+1} \cdot e_1 \cdot e_2 \cdots e_m(-e_{m+1}) = (-1)^m e_1 \cdot e_2 \cdots e_m.$$

This tells that if m is odd then $\rho(e_{m+1}) \notin Z(A)^\sharp$ and $\psi(\rho(e_{m+1})) = -1$. Thus we conclude that the map ψ is surjective and $Z(A)^\sharp$ is an index 2 subgroup of $Z(A)$. So $Z(A)^\sharp$ is a normal subgroup of $Z(A)$.

To prove the other part note that $Z(A)^\sharp$ is an index 2 subgroup of $Z(A)$. Therefore we write

$$Z(A) = Z(A)^\sharp \sqcup gZ(A)^\sharp,$$

where $g \in Z(A) \setminus Z(A)^\sharp$. Intersecting both sides with $Z(A)^0$ we obtain

$$Z(A)^0 = (Z(A)^\sharp \cap Z(A)^0) \sqcup (gZ(A)^\sharp \cap Z(A)^0).$$

Note that the first disjoint summand is nonempty as it contains the identity element. Since $Z(A)^0$ is connected it follows that $Z(A)^0 \subseteq Z(A)^\sharp$. $\square$

Lemma 3.1.4. *Two elements $g_1, g_2 \in \mathrm{Pin}(V)$ commute if and only if $\rho(g_2) \in Z(\rho(g_1))^\sharp$.*

Proof. If $g_2 \in Z_{\mathrm{Pin}(V)}(g_1)$, then $\rho(g_2) \in \rho(Z_{\mathrm{Pin}(V)}(g_1)) = Z(\rho(g_1))^\sharp$. For the other way consider $\rho(g_2) \in Z(\rho(g_1))^\sharp$. Since $\ker \rho = \{\pm 1\}$, we have $\pm g_2 \in Z_{\mathrm{Pin}(V)}(g_1)$ proving the claim. $\square$

Recall the third lifting condition which requires $h_j \in Z_{\mathrm{Pin}(V)}(h_i)$, for $|i-j| > 1$. Using the previous lemma we obtain an equivalent criterion which requires $\rho(h_j) \in Z(\rho(h_i))^\sharp$, i.e. $\phi(s_j) \in Z(\phi(s_i))^\sharp$, for $|i-j| > 1$. The next lemma determines the sharp centralizer of $\phi(s_1)$ in $\mathrm{O}(V)$. For convenience we write $N = \dim V$. We also write $\mathrm{O}(n)$ (resp. $\mathrm{SO}(n)$) to denote the real $n \times n$ orthogonal matrix (resp. the real $n \times n$ special orthogonal matrix).

Lemma 3.1.5. *We have*

$$Z(A)^\sharp = \begin{cases} \left[\begin{array}{c|c} \mathrm{O}(m) & 0 \\ \hline 0 & \mathrm{SO}(N-m) \end{array}\right], & \textit{for} \quad m = g_V \textit{ odd}, \\ \\ \left[\begin{array}{c|c} \mathrm{SO}(m) & 0 \\ \hline 0 & \mathrm{O}(N-m) \end{array}\right], & \textit{for} \quad m = g_V \textit{ even}. \end{cases}$$

Proof. Note that

$$Z(A) = \left[\begin{array}{c|c} \mathrm{O}(m) & 0 \\ \hline 0 & \mathrm{O}(N-m) \end{array}\right].$$

The connected component of this group containing identity is

$$Z(A)^0 = \left[\begin{array}{c|c} \mathrm{SO}(m) & 0 \\ \hline 0 & \mathrm{SO}(N-m) \end{array}\right].$$

The sharp centralizer subgroup $Z(A)^\sharp$ is an index 2 subgroup of $Z(A)$ containing $Z(A)^0$. The possible candidates are

1. $\left[\begin{array}{c|c} \mathrm{SO}(m) & 0 \\ \hline 0 & \mathrm{O}(N-m) \end{array}\right].$

2. $\left[\begin{array}{c|c} \mathrm{O}(m) & 0 \\ \hline 0 & \mathrm{SO}(N-m) \end{array}\right].$

3. $C = \left[\begin{array}{c|c} C_1 & 0 \\ \hline 0 & C_2 \end{array}\right]$, such that $\det C_1 = \det C_2$.

Observe that $\rho(e_1) = \mathrm{diag}(-1, 1, \ldots, 1) \in Z(A)^\sharp$, when m is odd. Again $\rho(e_{m+1}) = \mathrm{diag}(1, \ldots, -1, \ldots, 1) \in Z(A)^\sharp$. Therefore we have the result. □

The eigenspace decomposition of V for the operator $\phi(s_1)$ is given by

$$V = V(-1) \oplus V(1),$$

where $V(\pm 1)$ denotes the ± 1 eigenspace of $\phi(s_1)$. So we have

$$V(\pm 1) = \{v \in V \mid \phi(s_1)v = \pm v\}. \tag{3.2}$$

Since s_1 and s_3 commute $\phi(s_3)$ keeps $V(1)$ and $V(-1)$ invariant. Let the eigenspace decomposition of $V(-1)$ for the operator $\phi(s_3)$ be given by

$$V(-1) = V(-1,-1) \oplus V(-1,1),$$

where $V(-1,\pm 1)$ denotes the ± 1 eigenspace of $\phi(s_3)$ restricted to $V(-1)$. So we have

$$V(-1,\pm 1) = \{v \in V \mid \phi(s_1)v = -v, \phi(s_3)v = \pm v\}. \tag{3.3}$$

We write $h_V = \dim V(-1,-1)$.

Theorem 3.1.6. *The third lifting condition of a representation V of S_n holds if and only if*

$$h_V \equiv g_V \pmod 2.$$

Proof. Since all the transpositions in S_n are conjugate, we work with s_1, s_3 only. Consider the basis $\beta_1 = \{v_1, v_2, \ldots, v_m\}$ for $V(-1)$ which can be extended to a basis $\beta = \{v_1, \ldots, v_m, v_{m+1}, v_{m+2}, \ldots, v_N\}$ for V. The matrix form of $\phi(s_1)$ with respect to β looks like

$$\left[\begin{array}{c|c} -I_m & 0 \\ \hline 0 & I_{(N-m)} \end{array}\right].$$

The matrix form of $\phi(s_3)$ with respect to the same basis β looks like

$$\left[\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right] \in \left[\begin{array}{c|c} \mathrm{O}(m) & 0 \\ \hline 0 & \mathrm{O}(N-m) \end{array}\right],$$

where B_1 denotes the matrix for $\phi(s_3)$ with respect to the basis β_1. Whereas B_2 denotes the matrix for $\phi(s_3)$ with respect to the basis $\{v_{m+1}, v_{m+2}, \ldots, v_N\}$. The lemma 3.1.4 ensures that the third lifting condition holds if and only if $\phi(s_3) \in Z(\phi(s_1))^\sharp$. Note that

$$\det(\phi(s_3)) = \det B_1 \cdot \det B_2 = (-1)^{g_V}.$$

If g_V is odd from Theorem 3.1.5 we require $\det B_2 = 1$. Also we have $\det B_1 \cdot \det B_2 = -1$. So the condition becomes $\det B_1 = -1$. As h_V denotes the multiplicity of -1 as an eigenvalue of $\phi(s_3)$ applied on $V(-1)$, we obtain $\det(B_1) = (-1)^{h_v}$. Therefore the third lifting condition holds if and only if h_V is odd. Similarly, if g_V is even, we require $\det B_1 = 1$. It holds if and only if h_V is even. Hence the result follows. □

Combining the theorems 3.1.1 and 3.1.6 we obtain the following result.

Theorem 3.1.7. *A representation (ϕ, V) of S_n, $n \geq 4$, is spinorial if and only if both the following conditions hold:*

1. $g_V \equiv 0$ *or* $3 \pmod 4$*,*
2. $h_V \equiv g_V \pmod 2$*.*

Remark 3.1.8. *For a representation (ϕ, V) of S_n, where $n = 2, 3$ we do not define h_V. In these cases, ϕ is spinorial if and only if $g_V \equiv 0$ or $3 \pmod 4$.*

Consider the subgroup $H_2 = \langle s_1, s_3 \rangle$ of S_n, and the character $\omega_2 : H_2 \to \{\pm 1\}$, given by $\omega_2(s_1) = -1$ and $\omega_2(s_3) = -1$. For a representation (ϕ, V) of S_n let χ_V denote the character of it. As before we calculate

$$\begin{aligned}(\chi_V \mid_{H_2}, \omega_2) &= \frac{1}{4}(\chi_V(1)\omega_2(1) + \chi_V(s_1)\omega_2(s_1) + +\chi_V(s_3)\omega_2(s_3) + \chi_V(s_1 s_3)\omega_2(s_1 s_3)) \\ &= \frac{1}{4}(\chi_V(1) - 2\chi_V(s_1) + \chi_V(s_1 s_3)).\end{aligned}$$

This gives the dimension of the isotypic component $V(-1,-1)$. Thus we have

$$h_V = \frac{1}{4}(\chi_V(1) - 2\chi_V(s_1) + \chi_V(s_1 s_3)). \tag{3.4}$$

In terms of character values we conclude

Corollary 3.1.9. *A representation (ϕ, V) of $S_n, n \geq 4$, is spinorial if and only if one of the following conditions holds:*

1. $\chi_V(s_1 s_3) \equiv \chi_V(1) \pmod 8$, $\chi_V(s_1) \equiv \chi_V(1) + 2 \pmod 8$. *In this case ϕ is chiral.*
2. $\chi_V(1) \equiv \chi_V(s_1) \equiv \chi_V(s_1 s_3) \pmod 8$. *In this case ϕ is achiral.*

Proof. First consider the case when V is achiral and spinorial. Then from Theorem 3.1.7 we obtain $g_V \equiv 0 \pmod 4$ and $h_V \equiv 0 \pmod 2$. Using the expression for g_V in terms of character values as in Equation (3.1) we have

$$\chi_V(1) - \chi_V(s_1) \equiv 0 \pmod 8. \tag{3.5}$$

Again using Equation (3.4) we obtain

$$\chi_V(1) - 2\chi_V(s_1) + \chi_V(s_1s_3) \equiv 0 \pmod 8. \tag{3.6}$$

We rewrite Equation (3.6) as

$$\chi_V(1) - 2\chi_V(s_1) + \chi_V(s_1s_3) = \chi_V(1) - \chi_V(s_1) - (\chi_V(s_1) - \chi_V(s_1s_3)) \equiv 0 \pmod 8.$$

Using Equation (3.5) we conclude $\chi_V(s_1) - \chi_V(s_1s_3) \equiv 0 \pmod 8$. Altogether we obtain $\chi_V(1) \equiv \chi_V(s_1) \equiv \chi_V(s_1s_3) \pmod 8$.

If we start with the identity $\chi_V(1) \equiv \chi_V(s_1) \equiv \chi_V(s_1s_3) \pmod 8$, from the first equation we obtain the condition $g_V \equiv 0 \pmod 4$. It also follows that

$$\begin{aligned}
\chi_V(1) - \chi_V(s_1) - (\chi_V(s_1) - \chi_V(s_1s_3)) &\equiv 0 \pmod 8 \\
\chi_V(1) - 2\chi_V(s_1) + \chi_V(s_1s_3) &\equiv 0 \pmod 8.
\end{aligned}$$

So we have $h_V \equiv 0 \pmod 2$. Therefore V is spinorial.

Now consider V is chiral as well as spinorial. Then from Theorem 3.1.7 we have $g_V \equiv 3 \pmod 4$ and $h_V \equiv 1 \pmod 2$. From Equation (3.1) we have $\chi_V(1) - \chi_V(s_1) \equiv 6 \pmod 8$. This in turn gives $\chi_V(s_1) \equiv \chi_V(1) + 2 \pmod 8$. From Equation (3.4) we also have $\chi_V(1) - 2\chi_V(s_1) + \chi_V(s_1s_3) \equiv 4 \pmod 8$. Write $\chi_V(1) - 2\chi_V(s_1) + \chi_V(s_1s_3) = \chi_V(1) - \chi_V(s_1) - (\chi_V(s_1) - \chi_V(s_1s_3))$, and see that

$$\begin{aligned}
\chi_V(1) - \chi_V(s_1) - (\chi_V(s_1) - \chi_V(s_1s_3)) &\equiv 4 \pmod 8 \\
6 - (\chi_V(s_1) - \chi_V(s_1s_3)) &\equiv 4 \pmod 8 \\
6 - (\chi_V(1) + 2 - \chi_V(s_1s_3)) &\equiv 4 \pmod 8 \\
\chi_V(1) - \chi_V(s_1s_3) &\equiv 0 \pmod 8.
\end{aligned}$$

If the identity $\chi_V(s_1s_3) \equiv \chi_V(1) \pmod 8$, $\chi_V(s_1) \equiv \chi_V(1)+2 \pmod 8$ holds then a similar calculation shows that V is chiral and spinorial. □

The following result shows that the spinoriality of any representation of S_n is detected by its restriction to the subgroup $H_2 = \langle s_1, s_3 \rangle$.

Corollary 3.1.10. *Let (ϕ, V) be a representation of S_n, $n \geq 4$. Then ϕ is spinorial if and only if $\phi\mid_{H_2}$ is spinorial.*

Proof. If ϕ is spinorial then $\phi\mid_{H_2}$ is spinorial. Conversely suppose $\phi\mid_H$ is spinorial. Then certainly the conditions in 3.1.9 are satisfied. □

Corollary 3.1.11. *Any spinorial representation of $S_n, n > 1$, has two lifts.*

Proof. Note that there are two choices for h_1, namely $\pm e_1 \cdots e_{g_V}$. Once we choose h_1 the other h_i's, $2 \leq i \leq n-1$, are fixed by the relation $(h_ih_{i+1})^3 = 1$. Therefore the result follows.

□

3.1.1 Alternative Approach for the Third Lifting Condition

The generators s_i of the symmetric group S_n, $n \geq 4$, in particular satisfy the relations $[s_i, s_j] = 1$, for $|i-j| > 1$. If a representation (ϕ, V) of S_n has a lift $\widetilde{\phi}$, then it preserves the relations. We can rewrite the relation $[s_i, s_j] = 1$ as

$$(s_is_j)^2 = 1. \tag{3.7}$$

Let h'_V denote the multiplicity of -1 as an eigenvalue of $\phi(s_1s_3)$. Following similar arguments as in Lemma 3.1.1, we conclude that the map ϕ preserves the relation 3.7 if and only if $h'_V \equiv 0$ or $3 \pmod 4$.

Now consider the subgroup $H' = \langle s_1s_3 \rangle$ and the character $\omega' : H' \to \{\pm 1\}$ given by $\omega'(s_1s_3) = -1$. Using similar arguments as before we conclude

$$h'_V = \frac{\chi_V(1) - \chi_V(s_1s_3)}{2}. \tag{3.8}$$

Lemma 3.1.12. *The quantity h'_V is always even.*

Proof. Note that ζ_4^2 is conjugate to s_1s_3 in S_n, where ζ_4 is the cycle $(1,2,3,4)$. For any representation (ϕ, V) of S_n the eigenvalues of $\phi(\zeta_4)$ are among $\pm 1, \pm i$. Let h_i (resp. h_{-i}) denote the multiplicity of i (resp. $-i$) as an eigenvalue of $\phi(\zeta_4)$. Since the characters of $\phi(\zeta_4)$ are real we have $h_i = h_{-i}$. The fact $(\pm i)^2 = -1$ implies that $h'_V = h_i + h_{-i} = 2h_i$. Hence the result follows. □

Since h'_V is always even the alternative third lifting condition becomes $h'_V \equiv 0 \pmod 4$. This allows us to formulate an alternative condition for spinorial representations of symmetric groups.

Theorem 3.1.13. *Any representation V of S_n, $n \geq 4$, is spinorial if and only if both the following conditions hold:*

1. $g_V \equiv 0$ *or* $3 \pmod 4$,
2. $h'_V \equiv 0 \pmod 4$.

Proof. We show that these conditions are equivalent to those mentioned in Theorem 3.1.7. Following Equation (3.8) the condition $h'_V \equiv 0 \pmod 4$ can be rewritten as $\chi_V(1) \equiv \chi_V(s_1s_3) \pmod 8$. If ϕ is chiral and spinorial, we have $g_V \equiv 3 \pmod 4$. In terms of characters $\chi_V(s_1) \equiv \chi_V(1) + 2 \pmod 8$. Similarly for ϕ achiral, spinorial we obtain $\chi_V(1) \equiv \chi_V(s_1) \pmod 8$. These are exactly the conditions mentioned in Corollary 3.1.9.

□

3.2 Specht Modules

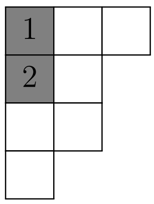
For the representation $V = V_\lambda$, we denote g_V by g_λ and h_V by h_λ. Recall that for $\mu \subset \lambda$, the number of all possible standard skew Young tableaux of shape λ/μ is denoted by $f_{\lambda/\mu}$.

Lemma 3.2.1. *For any irreducible representation V_λ, we have*

$$g_\lambda = f_{\lambda/(1,1)}.$$

Proof. Since any transposition s_k is conjugate with s_1 in S_n, we work specifically with s_1. Recall that g_λ is the multiplicity of -1 as an eigenvalue of $\phi_\lambda(s_1)$. Here we consider the action of $\phi_\lambda(s_1)$ on the basis β_λ (see Section 2.3) to determine g_λ.

1. We obtain the eigenvalue -1 only when 1 and 2 are in the first column of the tableau. As we have $\phi_\lambda(s_1)e_t = -e_t$. The only possible SYT with 1 and 2 in the same column are like:



   Observe that 1 always occurs in the $(1,1)$-th cell of a standard young tableau and 2 can occur in two places, either below it or to the right of it.

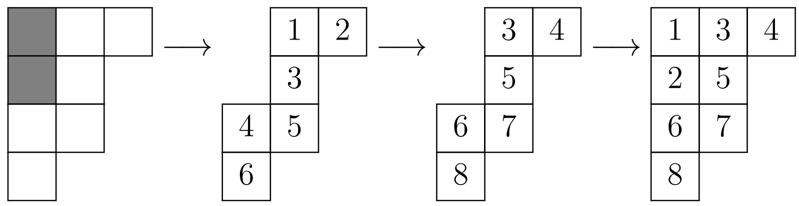
2. If 1 and 2 are in the first row then

$$(1,2)e_t = e_t \pm \text{other polytabloids } t' \text{ such that } [t'] \rhd [t].$$

   Therefore for these basis elements we obtain 1 in the diagonal positions and some non-zero entries in the up diagonal positions of the matrix, $\phi_\lambda(s_1)$.

3. There will be no case where 1 and 2 occur in a different row and a different column.

So the image matrix of $\phi_\lambda(s_1)$ is upper triangular where the diagonal entries as ± 1. The multiplicity of the eigenvalue -1 is equal to the number of SYT with 1 and 2 in the same column. We claim that the number is equal to $f_{\lambda/(1,1)}$. The justification for this is as follows:



We can fill up the standard skew tableau of shape $\lambda/(1,1)$ with the numbers $\{1, 2, ..., n-2\}$. Then we can insert the two cells in the top left corner filled up with the numbers 1 and 2 and then add 2 to all the remaining cells. As a result we obtain a standard Young tableaux filled up with the numbers $\{1, 2, ..., n\}$ such that 1 and 2 are in the same column. Therefore $g_\lambda = f_{\lambda/(1,1)}$.

$\square$

Lemma 3.2.2. *For $n \geq 4$,*

$$h_\lambda = f_{\lambda/(1,1,1,1)} + f_{\lambda/(2,2)} + f_{\lambda/(2,1,1)}.$$

Proof. Recall that $h_\lambda = \dim V_\lambda(-1,-1)$. We again use Young's natural representation (see Section 2.3) to determine h_λ in terms of the number of standard skew Young tableaux. Here we first take a basis β of $V_\lambda(-1)$. The set β consists of the polytabloids e_t such that 1 and 2 lie in the first column of t as in the proof of 3.2.1. We consider the action of $\phi_\lambda(s_3)$ on β.

1. Let 3 and 4 lie in the same column of t. Then we obtain $\phi_\lambda(s_3)e_t = -e_t$. The possible tableaux t such that $e_t \in \beta$ with 3 and 4 lying in same column are as below.

$$\begin{ytableau} *(gray)1 & & \\ *(gray)2 & \\ *(gray)3 & \\ *(gray)4 \end{ytableau} \quad , \quad \begin{ytableau} *(gray)1 & *(gray)3 & \\ *(gray)2 & *(gray)4 \\ & \\ \\ \end{ytableau}$$

Following similar arguments as in the previous lemma we conclude that the total number of such SYT will be $f_{\lambda/(1,1,1,1)} + f_{\lambda/(2,2)}$.

2. Consider the subset β_1 of β such that β_1 contains the elements e_t where t has 3 and 4 in a different row and a different column. The possible tableaux t such that $e_t \in \beta_1$ are as follows.

$$\begin{ytableau} *(gray)1 & *(gray)4 & \\ *(gray)2 & \\ *(gray)3 & \\ \\ \end{ytableau} \quad , \quad \begin{ytableau} *(gray)1 & *(gray)3 & \\ *(gray)2 & \\ *(gray)4 & \\ \\ \end{ytableau}$$

The total number of SYT of this kind is $2f_{\lambda/(2,1,1)}$. So we have $|\beta_1| = 2f_{\lambda/(2,1,1)}$. For these tableaux we have $\phi_\lambda(s_3)e_t = e'_t$, where t' is obtained from t by switching the positions of 3 and 4. Without loss of generality, we assume that β_1 looks like $\{t_1, s_3t_1, t_2, s_3t_2, \ldots, t_{|\beta_1|/2}, s_3t_{|\beta_1|/2}\}$. The matrix of $\phi_\lambda(s_3)$ with respect to the basis β_1 contains $f_{\lambda/(2,1,1)}$ many blocks of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore the multiplicity of -1 as an eigenvalue of the matrix of $\phi_\lambda(s_3)$ with respect to the basis β_1 is $f_{\lambda/(2,1,1)}$.

The set of polytabloids e_t where t varies over the set of SYT mentioned in cases 1 and 2 form a basis for $V_\lambda(-1,-1)$. As a result we obtain

$$h_\lambda = f_{\lambda/(1,1,1,1)} + f_{\lambda/(2,2)} + f_{\lambda/(2,1,1)}.$$

□

Next we deduce an expression for h_λ in terms of character values. The expression $\chi_\lambda(\mu)$ denotes the character value of the representation V_λ at the conjugacy class with cycle type μ.

Theorem 3.2.3. *Given a representation* $(\phi_\lambda, V_\lambda)$ *of* S_n *for* $n \geq 4$*, we have*

$$h_\lambda \equiv \frac{f_\lambda - \chi_\lambda(\zeta_4)}{2} \pmod 2,$$

where $\zeta_4 = (1,2,3,4)$.

Proof. We start with the character table of S_4.

Table 3.1: Character Table of S_4.

x	1	s_1	s_1s_2	s_1s_3	ζ_4
$\lvert C_x\rvert$	1	6	8	3	6
$\chi_{(4)}$	1	1	1	1	1
$\chi_{(1^4)}$	1	-1	1	1	-1
$\chi_{(2,2)}$	2	0	-1	2	0
$\chi_{(3,1)}$	3	1	0	-1	-1
$\chi_{(2,1^2)}$	3	-1	0	-1	1

From [25, Exercise 7.62, page 469], we have if $\lambda \vdash n$ and $\mu \vdash k \leq n$, then

$$\chi_\lambda(\mu 1^{(n-k)}) = \sum_{\nu \vdash k} f_{\lambda/\nu}\chi_\nu(\mu).$$

Taking $k = 4$ and $\mu = (2,2)$, we have

$$\begin{aligned}\chi_\lambda(s_1s_3) &= f_{\lambda/(4)}\chi_{(4)}(s_1s_3) + f_{\lambda/(3,1)}\chi_{(3,1)}(s_1s_3) + f_{\lambda/(2,2)}\chi_{(2,2)}(s_1s_3)\\ &\quad + f_{\lambda/(2,1^2)}\chi_{(2,1^2)}(s_1s_3) + f_{\lambda/(1^4)}\chi_{(1^4)}(s_1s_3)\\ &= f_{\lambda/(4)} - f_{\lambda/(3,1)} + 2f_{\lambda/(2,2)} - f_{\lambda/(2,1^2)} + f_{\lambda/(1^4)}.\end{aligned}$$

Again taking $k = 4$ and $\mu = (4)$, we obtain

$$\begin{aligned}\chi_\lambda(\zeta_4) &= f_{\lambda/(4)}\chi_{(4)}(\zeta_4) + f_{\lambda/(3,1)}\chi_{(3,1)}(\zeta_4) + f_{\lambda/(2,2)}\chi_{(2,2)}(\zeta_4)\\ &\quad + f_{\lambda/(2,1^2)}\chi_{(2,1^2)}(\zeta_4) + f_{\lambda/(1^4)}\chi_{(1^4)}(\zeta_4)\\ &= f_{\lambda/(4)} - f_{\lambda/(3,1)} + 0.f_{\lambda/(2,2)} + f_{\lambda/(2,1^2)} - f_{\lambda/(1^4)}.\end{aligned}$$

Putting these values we calculate

$$\chi_\lambda(s_1s_3) - \chi_\lambda(\zeta_4) = 2f_{\lambda/(1^4)} + 2f_{\lambda/(2,2)} - 2f_{\lambda/(2,1^2)}.$$

Therefore

$$\frac{\chi_\lambda(s_1s_3) - \chi_\lambda(\zeta_4)}{2} \equiv f_{\lambda/(1^4)} + f_{\lambda/(2,2)} + f_{\lambda/(2,1^2)} \pmod 2.$$

So we obtain

$$h_\lambda \equiv \frac{\chi_\lambda(s_1s_3) - \chi_\lambda(\zeta_4)}{2} \pmod 2, \tag{3.9}$$

Following Section 3.1.1 we write h'_{V_λ} to denote the multiplicity of -1 as an eigenvalue of $\phi_\lambda(s_1s_3)$. As $(s_1s_3)^2 = 1$, the matrix $\phi_\lambda(s_1s_3)$ is similar to the matrix $\mathrm{diag}(\underbrace{-1,-1,\ldots,-1}_{h'_{V_\lambda}\text{ times}},1,\ldots,1)$. So one calculates

$$\begin{aligned}\chi_\lambda(s_1s_3) &= f_\lambda - h'_{V_\lambda} - h'_{V_\lambda}\\ &= f_\lambda - 2h'_{V_\lambda}.\end{aligned}$$

From Theorem 3.1.12 we obtain h'_{V_λ} is even. Therefore $\chi_\lambda(s_1s_3) \equiv f_\lambda \pmod 4$. Using this relation and Equation 3.9 we conclude

$$h_\lambda \equiv \frac{f_\lambda - \chi_\lambda(\zeta_4)}{2} \pmod 2.$$

□

4

Alternating Groups

In this chapter we determine the spinorial representations of the alternating groups. In the first section we derive a criterion for spinoriality of any orthogonal representation of A_n. In the remaining portion of the chapter, we discuss the spinoriality of irreducible representations of A_n.

4.1 Spinorial Representations of Alternating Groups

From [6, page 66] we obtain that A_n is generated by the elements

$$u_i = s_1 s_{i+1}, \quad (i = 1, 2, \ldots, n-2),$$

which satisfy the relations:

$$\begin{gathered} u_1^3 = u_j^2 = (u_{j-1}u_j)^3 = 1, \quad (2 \leq j \leq n-2), \\ (u_i u_j)^2 = 1, \quad (1 \leq i < j-1, j \leq n-2). \end{gathered}$$

For an orthogonal representation (ϕ, V) of A_n let k_V denote the multiplicity of -1 as an eigenvalue of $\phi(s_1 s_3)$.

Proposition 4.1.1. *For an orthogonal representation (ϕ, V) of A_n, $n \geq 4$, the quantity k_V is even.*

Proof. For a suitable choice of basis of V we have $\phi(u_2) = (\underbrace{-1, -1, \ldots, -1}_{k_V \text{ times}}, 1, \ldots, 1)$. Therefore $\det(\phi(u_2)) = (-1)^{k_V}$. Consider the map $\det \circ \phi(u_2) : A_n \to \{\pm 1\}$. Since the group $\{\pm 1\}$ is commutative, there exists a unique map $f : A_n/[A_n, A_n] \to \{\pm 1\}$ such that $\det \circ \phi(u_2) = f \circ q$, where $q : A_n \to A_n/[A_n, A_n]$ denotes the quotient map. In other words we obtain the following commutative diagram.

$$\begin{tikzcd}
A_n \arrow[rr, "\det \circ \phi(u_2)"] \arrow[dr, "q"'] & & \{\pm 1\} \\
& A_n/[A_n, A_n] \arrow[ur, "f"'] &
\end{tikzcd}$$

We know that

$$[A_n, A_n] = \begin{cases} A_n, & \text{for } n \geq 5 \\ V_4, & \text{for } n = 4, \end{cases}$$

where V_4 denotes the Klein's four group. If $n \neq 4$, clearly $\det(\phi(u_2)) = 1$. Therefore k_V is even. For $n = 4$, we have $A_n/[A_n, A_n] \cong C_3$, where C_3 denotes the cyclic group of order 3. Therefore in this case f denotes the unique map which takes all the elements to the identity. As a result we obtain $\det(\phi(u_2)) = 1$, proving the claim that k_V is even. □

Theorem 4.1.2. *An orthogonal representation (ϕ, V) of A_n, $n \geq 4$, is spinorial if and only if $k_V \equiv 0 \pmod 4$.*

Proof. Suppose the representation ϕ is spinorial. Write l_i for a lift of u_i. Then the elements l_i must satisfy similar relations as u_i, namely:

$$l_j^2 = 1,\ 1 \leq j \leq n-2, \quad (l_i l_j)^2 = 1,\ 1 \leq i < j-1,\ j \leq n-2, \tag{4.1}$$

$$l_1^3 = (l_{j-1} l_j)^3 = 1, \quad 1 \leq j \leq n-2. \tag{4.2}$$

Note that $u_2 = s_1 s_3$. Since $u_2^2 = 1$, for a suitable choice of basis of V we have $\phi(u_2) =$

$(\underbrace{-1,-1,\ldots,-1}_{k_V \text{ times}},1,\ldots,1)$. So we may take $l_2 = e_1 \cdot e_2 \cdots e_{k_V}$, where $e_i \in \mathrm{Pin}(V)$ (see Section 2.1). The relation $l_2^2 = 1$ holds if $k_V \equiv 0$ or $3 \pmod 4$. Since k_V is even (see Proposition 4.1.1), we must have $k_V \equiv 0 \pmod 4$.

For the converse take $k_V \equiv 0 \pmod 4$. Note that the odd permutation $s_1 \in S_n$ commutes with u_2. Therefore from [21, Exercise 4.6.10] we conclude that all the elements of cycle type $(2,2)$ in A_n lie in the same conjugacy class. Note that all the elements $u_j, (2 \le j \le n-2)$ and $u_i u_j, (1 \le i < j-1, j \le n-2)$ have cycle type $(2,2)$. Therefore they are all conjugate to u_2 in A_n.

Therefore to check the relations 4.1 it is enough to check whether $l_2^2 = 1$. We can take $l_2 = e_1 \cdot e_2 \cdots e_{k_V}$, where $e_i \in \mathrm{Pin}(V)$. Then following a similar argument as in Proposition 3.1.1 we conclude that $l_2^2 = 1$ as we have $k_V \equiv 0 \pmod 4$. To check the relations 4.2 note that if $(l_i l_{i+1})^3 = -1$, then keeping l_i fixed we replace l_{i+1} with $-l_{i+1}$. Similar adjustment fixes the relation $l_1^3 = 1$, for a suitable choice of l_1. So ϕ is spinorial. □

As an immediate consequence we obtain the lifting criterion in terms of the character values.

Corollary 4.1.3. *An orthogonal representation (ϕ, V) of $A_n, n \ge 4$ is spinorial if and only if*

$$\chi_V(1) \equiv \chi_V(s_1 s_3) \pmod 8.$$

Proof. Consider the subgroup $H_3 = \langle s_1 s_3 \rangle$ of A_n. Take the character $\omega_3 : H_3 \to C_2$ given by $\omega_3(s_1 s_3) = -1$. For an orthogonal representation (ϕ, V) of A_n, we calculate

$$\begin{aligned} (\chi\,|_{H_3}, \omega_3) &= \frac{1}{2} \cdot (\chi_V(1) \cdot \omega_3(1) - \chi_V(s_1 s_3) \cdot \omega_3(s_1 s_3)) \\ k_V &= \frac{1}{2} \cdot (\chi_V(1) - \chi_V(s_1 s_3)). \end{aligned}$$

From the previous theorem we obtain ϕ is spinorial if and only if $k_V \equiv 0 \pmod 4$. So ϕ is spinorial if and only if $\chi_V(1) - \chi_V(s_1 s_3) \equiv 0 \pmod 8$. □

Corollary 4.1.4. *An orthogonal representation (ϕ, V) of S_n, $n \ge 4$, is spinorial if and only if $\phi\,|_{H_3}$ is spinorial, where $H_3 = \langle s_1 s_3 \rangle$.*

The corollary follows easily from the previous result.

Corollary 4.1.5. *Let (ϕ, V) be a spinorial representation of $A_n, n > 2$. Then ϕ has a unique lift.*

Proof. Recall from [6, page 66] that A_n is generated by the elements u_i. Since ϕ is spinorial, let $l_i \in \mathrm{Pin}(V)$ denote the lift of u_i. The relation $l_1^3 = 1$ leaves one possible choice for l_1. The other l_i's are fixed by the relations $(l_i l_{i+1})^3 = 1$. So there exists a unique lift for ϕ. □

4.2 Specht Modules

From [21, section 4.6.2] we obtain for each partition λ, the restriction of the irreducible representation V_λ of S_n to A_n is

1. an irreducible representation of A_n if $\lambda \neq \lambda'$.

2. a sum of two non-isomorphic irreducible representations V_λ^+ and V_λ^- of A_n if $\lambda = \lambda'$. The representation V_λ^- is a twist of V_λ^+ by conjugation by an odd permutation in S_n. In other words if $\phi_\lambda^\pm$ denotes the representations $V_\lambda^\pm$ then we have $\phi_\lambda^-(g) = \phi_\lambda^-(x^{-1}gx)$, where x denotes an odd element in S_n.

We study these two cases separately.

4.2.1 Case of Non Self-Conjugate Partitions

First we consider the cases when $\lambda \neq \lambda'$. Here the restriction of V_λ to A_n gives an irreducible representation. For simplicity we denote k_{V_λ} by k_λ. Here we obtain an expression for k_λ in terms of the number of standard skew Young tableaux.

Theorem 4.2.1. *For the irreducible representation $V_\lambda \mid_{A_n}$, $n \geq 4$,*

$$k_\lambda = 2(f_{\lambda/(2,1,1)} + f_{\lambda/(3,1)}).$$

Proof. We denote the representation $V_\lambda \mid_{A_n}$ by ϕ_λ. Following Section 2.3 we record the action of $\phi_\lambda(s_1 s_3)$ on the elements of the basis β_λ of V_λ. The elements of β_λ can be categorized into 6 different types depending on the position of $1, 2, 3$ and 4 in the corresponding SYT. Note that $\phi_\lambda(s_1 s_3) = \phi_\lambda(s_1)\phi_\lambda(s_3)$.

1. If $1, 2$ are in the same column and $3, 4$ are in the same column of t, then

$$\phi_\lambda(s_1)\phi_\lambda(s_3)e_t = \phi_\lambda(s_1)e_t = e_t.$$

2. If $1, 2$ are in the same column and $3, 4$ are in the same row of t, then

$$\phi_\lambda(s_3)e_t = e_t \pm \text{other polytabloids } e_{t'},$$

   where $[t'] \trianglerighteq [t]$. Applying $\phi_\lambda(s_1)$ to it we obtain

$$\phi_\lambda(s_1)(e_t \pm \text{other polytabloids } e_{t'}) = -e_t \pm \text{other polytabloids } e_{t''},$$

   where $[t''] \trianglerighteq [t]$. In this case 3 and 4 are not in the same column as that of $1, 2$. So we obtain an upper triangular matrix.

3. If $1, 2$ are in the same column and $3, 4$ are in a different row and different column of t, then

$$\phi_\lambda(s_1)\phi_\lambda(s_3)e_t = \phi_\lambda(s_1)e_{t'} = -e_{t'},$$

   where t' is the tableau with the positions of 3 and 4 interchanged.

4. If $1, 2$ are in the same row and $3, 4$ are in the same column of t, then

$$\phi_\lambda(s_1)\phi_\lambda(s_3)e_t = \phi_\lambda(s_1)(-e_t) = -e_t \pm \text{other polytabloids } e_{t'},$$

   where $[t'] \trianglerighteq [t]$.

5. If $1, 2$ are in the same row and $3, 4$ are in the same row of t, then

$$\phi_\lambda(s_3)e_t = e_t \pm \text{other polytabloids } e_{t'},$$

   where $[t'] \trianglerighteq [t]$. Again we have

$$\phi_\lambda(s_1)(e_t \pm \text{other polytabloids } e_{t'}) = (e_t \pm \text{other polytabloids } e_{t''}),$$

where $[t''] \trianglerighteq [t]$.

6. If $1, 2$ are in the same column and $3, 4$ are in a different row and different column of t, then

$$\phi_\lambda(s_1)\phi_\lambda(s_3)e_t = \phi_\lambda(s_1)e_{t'} = e_{t'} \pm \text{other polytabloids } e_{t''},$$

where $[t''] \trianglerighteq [t']$ and $t' = s_3 \cdot e_t$.

Using these observations we determine k_λ in terms of standard skew Young tableaux. Let k_i denote the multiplicity of -1 as an eigenvalue of $\phi_\lambda(s_1 s_3)$ acting on the basis elements of type i, $1 \leq i \leq 6$. So we have $k_\lambda = \sum_{i=1}^{6} k_i$.

- From cases 2 and 4 we obtain -1 in the diagonal positions of $\phi_\lambda(s_1 s_3)$. We have $k_2 = f_{\lambda/(3,1)}$ and $k_4 = f_{\lambda/(2,1,1)}$.
- The number of polytabloids of type 3 is $2f_{\lambda/(2,1,1)}$. The polytabloids of type 3 can be paired up so that the matrix of $\phi_\lambda(s_1 s_3)$ on these basis elements consists $f_{\lambda/(2,1,1)}$ many blocks of the form $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Therefore $k_3 = f_{\lambda/(2,1,1)}$.
- The number of polytabloids of type 6 is $2f_{\lambda/(3,1)}$. The polytabloids of type 6 can be paired up so that the matrix of $\phi_\lambda(s_1 s_3)$ on these basis elements consists $f_{\lambda/(3,1)}$ many blocks of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore we have $k_6 = f_{\lambda/(3,1)}$.
- For the basis elements of type 1 and 5 we obtain 1 in the matrix of $\phi_\lambda(s_1 s_3)$ and some non-zero up-diagonal entries. So we have $k_1 = k_2 = 0$.

Therefore we obtain $k_\lambda = \sum_{i=1}^{6} k_i = 2(f_{\lambda/(2,1,1)} + f_{\lambda/(3,1)})$. □

Theorem 4.2.2. *For any partition $\lambda \vdash n$, we have*

$$f_{\lambda/(2,1,1)} + f_{\lambda/(3,1)} \equiv \frac{\chi_\lambda(s_1) - \chi_\lambda(\zeta_4)}{2} \pmod 2.$$

Proof. Using [25, Exercise 7.62, page 469] we deduce

$$\begin{aligned}\chi_\lambda(s_1) &= f_{\lambda/(4)}\chi_{(4)}(s_1) + f_{\lambda/(3,1)}\chi_{(3,1)}(s_1) + f_{\lambda/(2,2)}\chi_{(2,2)}(s_1) \\ &\quad + f_{\lambda/(2,1,1)}\chi_{(2,1,1)}(s_1) + f_{\lambda/(1^4)}\chi_{(1^4)}(s_1) \\ &= f_{\lambda/(4)} + f_{\lambda/(3,1)} - f_{\lambda/(2,1,1)} - f_{\lambda/(1^4)}\end{aligned}$$

A similar calculation gives

$$\chi_\lambda(\zeta_4) = f_{\lambda/(4)} - f_{\lambda/(3,1)} + f_{\lambda/(2,1,1)} - f_{\lambda/(1^4)}.$$

Thus one gets

$$f_{\lambda/(2,1,1)} + f_{\lambda/(3,1)} \equiv \frac{\chi_\lambda(s_1) - \chi_\lambda(\zeta_4)}{2} \pmod 2.$$

□

Corollary 4.2.3. *Consider a non self-conjugate partition λ such that $n = |\lambda| \geq 4$. Then $V_\lambda \mid_{A_n}$ is spinorial if and only if*

$$\chi_\lambda(s_1) \equiv \chi_\lambda(\zeta_4) \pmod 4. \tag{4.3}$$

Proof. From Theorem 4.1.2 we have $V_\lambda \mid_{A_n}$ is spinorial if and only if $k_\lambda \equiv 0 \pmod 4$. Combining the results from Theorems 4.2.1 and 4.2.2 we obtain $k_\lambda \equiv \chi_\lambda(s_1) - \chi_\lambda(\zeta_4) \pmod 4$. Therefore for $k_\lambda \equiv 0 \pmod 4$ we require $\chi_\lambda(s_1) \equiv \chi_\lambda(\zeta_4) \pmod 4$. □

4.2.2 Case of Self-Conjugate Partitions

Theorem 4.2.4. *For self-conjugate partitions λ, the representation $V_\lambda \mid_{A_n}$ is spinorial.*

Proof. From [21, Theorem 4.4.2] we obtain $V_{\lambda'} \cong V_\lambda \otimes \epsilon$, where ϵ denotes the sign representation of S_n. Consequently we have

$$\chi_{\lambda'}(\mu) = \epsilon(\mu)\chi_\lambda(\mu), \tag{4.4}$$

where μ denote the cycle type of a conjugacy class in S_n. Since $\epsilon(s_1) = \epsilon(\zeta_4) = -1$, from Equation (4.4) we conclude $\chi_\lambda(s_1) = \chi_\lambda(\zeta_4) = 0$. Therefore the condition in Corollary 4.2.3 holds proving our claim. □

Let $\mathrm{SP}(n)$ denote the set of self-conjugate partitions of n and $\mathrm{DOP}(n)$ denote the set of partitions of n with distinct odd parts. We obtain a bijection (see [21, Lemma 4.6.16]) between these two sets

$$\theta : \mathrm{DOP}(n) \to \mathrm{SP}(n).$$

For $\mu \in \mathrm{DOP}(n)$ such that $\mu = (2m_1 + 1, 2m_2 + 1, \ldots, 2m_r + 1)$, define $\epsilon_\mu = (-1)^{\sum m_i}$. Let w_μ denote an element with cycle type μ and C_{w_μ} denote the conjugacy class in S_n containing all permutations with cycle type μ. From [21, Section 5.12] we obtain that C_{w_μ} splits into two conjugacy classes in A_n of equal cardinality. If w_μ^+ lies in one of these classes, then $w_\mu^- = \nu w_\mu^+ \nu^{-1}$ lies in the other for any odd permutation ν. We denote the corresponding conjugacy classes by $C_{w_\mu^+}$ and $C_{w_\mu^-}$. In particular let $C_{w_\mu}^\lambda$ denote the conjugacy class in A_n such that $\theta(\mu) = \lambda$. As mentioned in the beginning of Section 4.2, for $\lambda = \lambda'$ we have

$$V_\lambda \mid_{A_n} = V_\lambda^+ \oplus V_\lambda^-.$$

We denote the character of $V_\lambda^\pm$ by $\chi_\lambda^\pm$. From [21, Section 5.12], we obtain if $w \notin C_{w_\mu}^\lambda$, then

$$\chi_\lambda^+(w) = \chi_\lambda^-(w) = \chi_\lambda(w)/2, \tag{4.5}$$

where χ_λ denotes the character of the representation V_λ of S_n. For $w_\mu^\pm \in C_{w_\mu}^\lambda$ we have

$$\chi_\lambda^\pm(w_\mu^+) = \frac{1}{2}\left(\epsilon_\mu \pm \sqrt{\frac{\epsilon_\mu n!}{c_\mu}}\right) \quad \text{and} \quad \chi_\lambda^\pm(w_\mu^-) = \chi_\lambda^\mp(w_\mu^+). \tag{4.6}$$

From [21, Exercise 5.12.1] we obtain

$$\chi_\lambda(w_\mu) = \chi_\lambda^+(w_\mu) + \chi_\lambda^-(w_\mu) = \epsilon_\mu, \tag{4.7}$$

where χ_λ denote the character of the representation V_λ of S_n.

Lemma 4.2.5. *For $g \in A_n$, let β and β' denote the cycle types of g and g^2 respectively. Then $\beta \in \mathrm{DOP}(n)$ if and only if $\beta' \in \mathrm{DOP}(n)$. In this case in fact we have $\beta = \beta'$.*

Proof. Let $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$ denote the cycle type of g. Let β_i, $1 \leq i \leq r$ are all odd and distinct. Suppose $\sigma_i = (a_1, a_2, \ldots, a_{\beta_i})$ denotes the cycle of length β_i. Then $\sigma^2 = (a_1, a_3, a_5, \ldots, a_{\beta_i}, a_2, a_4, \ldots, a_{\beta_i - 1})$ is also a cycle of length β_i. As a result g^2 will have cycle type β.

For the converse let β' contains distinct odd parts. If one of the β_i is even then taking the cycle of length β_i, we calculate

$$(x_1, x_2, \ldots, x_{\beta_i})^2 = (x_1, x_3, \ldots, x_{\beta_i - 1})(x_2, x_4, \ldots, x_{\beta_i}),$$

where both the cycles are of length $\beta_i/2$. This violates the assumption that β' contains distinct parts. Also if $\beta_i = \beta_j$, where both are odd, then previous calculations confirm that β' also contain two odd parts of equal length. Therefore β must contain all distinct odd parts. From the previous part we conclude that $\beta' = \beta$. □

From Lemma 4.2.5 it follows that if $g \in C^{\lambda}_{w_\mu}$ then $g^2 \in C^{\lambda}_{w_\mu}$. Since $C^{\lambda}_{w_\mu}$ splits in two equal parts of the same cardinality we obtain

$$\sum_{g \in C^{\lambda}_{w_\mu}} \frac{\chi_\lambda(g^2)}{2} = \frac{|C^{\lambda}_{w_\mu}|}{2} \left(\chi^{+}_{\lambda}(w^{+}_{\mu}) + \chi^{+}_{\lambda}(w^{-}_{\mu}) \right). \tag{4.8}$$

This rest of this section discusses spinoriality of the representations $V^{\pm}_{\lambda}$. For that we first need to ensure whether they are orthogonal. Following [4, Proposition 6.8] we obtain that V_λ is orthogonal if and only if $\frac{1}{|A_n|} \sum_{g \in A_n} \chi^{+}_{\lambda}(g^2) = 1$. For a finite dimensional representation V of a finite group G the quantity $\frac{1}{|G|} \sum_{g \in G} \chi_V(g^2)$ is known as its Frobenius-Schur indicator, where χ_V denotes the character of V. As representations of S_n are orthogonal it follows that

$$\frac{1}{|S_n|} \sum_{g \in S_n} \chi_\lambda(g^2) = 1. \tag{4.9}$$

The following theorem determines when $V^{\pm}_{\lambda}$ are orthogonal. This result can be found in [5]. Here we prove it in our own way.

Theorem 4.2.6. *The representations $V^{\pm}_{\lambda}$ are orthogonal if and only if $\epsilon_\mu = 1$.*

Proof. We write $V_\lambda|_{A_n}$ for V_λ without any ambiguity to avoid complex notations. Taking dual on both sides of the equation

$$V_\lambda = V_\lambda^+ \oplus V_\lambda^-,$$

we obtain

$$V_\lambda^\vee = V_\lambda^{+\vee} \oplus V_\lambda^{+\vee}.$$

So either the representations $V_\lambda^\pm$ are self-dual or they are dual of each other. A self-dual representation always has real character values. If $\epsilon_\mu = -1$, then we deduce

$$\overline{\chi_\lambda^\pm(w_\mu^+)} = \chi_\lambda^\mp(w_\mu^+) \neq \chi_\lambda^\pm(w_\mu^+),$$

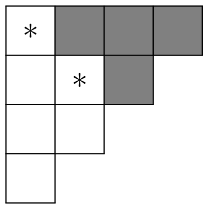
where $\overline{a}$ denotes the complex conjugation of a. Clearly the representations are not self-dual. Since an orthogonal representation is always self-dual in this case $V_\lambda^\pm$ are not orthogonal. For $\epsilon_\mu = 1$, then using Equations 4.5 and 4.6 we deduce $\overline{\chi_\lambda^\pm(w_\mu^+)} = \chi_\lambda^\pm(w_\mu^+)$, and $\overline{\chi_\lambda^\pm(w_\mu^-)} = \chi_\lambda^\mp(w_\mu^-)$. Therefore the representations are self-dual. Now it remains to determine whether the representations are orthogonal. We work with V_λ^+ as similar calculations work for V_λ^-. Expanding out the Frobenius-Schur indicator for the representation V_λ^+ we obtain

$$\begin{aligned}
\frac{1}{|A_n|}\sum_{g\in A_n}\chi_\lambda^+(g^2) &= \frac{2}{n!}\sum_{g\notin C^\lambda_{w_\mu}}\chi_\lambda^+(g^2) + \frac{2}{n!}\sum_{g\in C^\lambda_{w_\mu}}\chi_\lambda^+(g^2)\\
&= \frac{2}{n!}\sum_{g\notin C^\lambda_{w_\mu}}\frac{\chi_\lambda(g^2)}{2} + \frac{2}{n!}\frac{|C^\lambda_{w_\mu}|}{2}\left(\chi_\lambda^+(w_\mu^+) + \chi_\lambda^+(w_\mu^-)\right) \quad \text{(use Equations 4.5 and 4.8)}\\
&= \frac{1}{n!}\sum_{g\notin C^\lambda_{w_\mu}}\chi_\lambda(g^2)\\
&+ \frac{|C^\lambda_{w_\mu}|}{n!}\left(\frac{1}{2}\left(\epsilon_\mu + \sqrt{\frac{\epsilon_\mu n!}{c_\mu}}\right) + \frac{1}{2}\left(\epsilon_\mu - \sqrt{\frac{\epsilon_\mu n!}{c_\mu}}\right)\right) \quad \text{(use Equation (4.6))}\\
&= \frac{1}{n!}\sum_{g\notin C^\lambda_{w_\mu}}\chi_\lambda(g^2) + \frac{|C^\lambda_{w_\mu}|}{n!}\\
&= \frac{1}{|S_n|}\sum_{g\in S_n}\chi_\lambda(g^2) \quad \text{(use Equation (4.7))}\\
&= 1 \quad \text{(use Equation (4.9))}.
\end{aligned}$$

So we conclude that the representation V_λ^+ is orthogonal. Similar calculations prove that the representation V_λ^- is orthogonal.

If $V^\pm$ is orthogonal, then it is self-dual. Therefore we must have $\epsilon_\mu = 1$. □

Remark 4.2.7. *Note that the quantity $\sum m_i$ is equal to the number of up diagonal cells in the Young tableau of shape $\theta(\mu) = \lambda$. As an example, for $\lambda = (4,3,2,1)$, we have shaded the up diagonal cells and put stars in the diagonal ones. In this case $\mu = (7,3)$, so that we have $m_1 = 3$ and $m_2 = 1$. As a result we obtain $\epsilon_\mu = (-1)^{(3+1)} = 1$.*



Theorem 4.2.8. *Suppose $V_\lambda^\pm$ is orthogonal. Then the following statements are equivalent:*

1. *V_λ^+ is spinorial.*
2. *V_λ^- is spinorial.*
3. *$\chi_\lambda(1) \equiv \chi_\lambda(s_1 s_3) \pmod{16}$.*

Proof. We write π (resp. $\pi^\pm$) to denote the representation V_λ (resp. $V_\lambda^\pm$).

(1) $\iff$ (2) : We know that for any $g \in A_n$, $\pi^-(g) = \pi^+(x^{-1}gx)$, where x is an odd permutation in S_n. Now suppose π^+ is spinorial. So there exists a homomorphism $\widehat{\pi^+}$ such that $\widehat{\pi^+} \circ \rho = \pi^+$. Then we define a lift $\widehat{\pi^-}$ for π^- as $\widehat{\pi^-}(g) = \widehat{\pi^+}(x^{-1}gx)$. To verify this we see that

$$\begin{aligned}\rho \circ \widehat{\pi^-}(g) &= \rho \circ \widehat{\pi^+}(x^{-1}gx)\\ &= \pi^+(x^{-1}gx)\\ &= \pi^-(g)\end{aligned}$$

Similar argument shows that π^+ is spinorial if π^- is spinorial.

(1) $\iff$ (3) : Let $\chi_\lambda^\pm(\mu)$ denote the character values of $V_\lambda^\pm$ on the conjugacy class with cycle type μ in A_n. Then the fact

$$V_\lambda\,|_{A_n} = V_\lambda^+ \oplus V_\lambda^-,$$

gives

$$\chi_\lambda(\mu) = \chi^+_\lambda(\mu) + \chi^-_\lambda(\mu). \tag{4.10}$$

Since $\dim V^+_\lambda = \dim V^-_\lambda$ we have $\chi^+_\lambda(1) = \chi_\lambda(1)/2$. Note that the cycle type of s_1s_3 is $(2,2)$, which does not contain distinct odd parts. Therefore from [21, Theorem 4.6.13] it follows that for an odd permutation $x \in S_n$, $x^{-1}s_1s_3x \in C_{A_n}(s_1s_3)$, where $C_{A_n}(s_1s_3)$ denotes the conjugacy class as s_1s_3 in A_n. It follows that $\chi^+_\lambda(s_1s_3) = \chi^-_\lambda(s_1s_3)$. Using Equation (4.10) we conclude that $\chi^+_\lambda(s_1s_3) = \chi_\lambda(s_1s_3)/2$. If V^+_λ is spinorial then according to Corollary 4.1.3 we have $\chi^+_\lambda(1) \equiv \chi^+_\lambda(s_1s_3) \pmod 8$. Therefore from the above discussion it follows that $\chi_\lambda(1) \equiv \chi_\lambda(s_1s_3) \pmod{16}$.

□

5

Some Corollaries and Examples

In this chapter we discuss spinoriality of direct sum and internal tensor product of representations of S_n. We also discuss the spinoriality of permutation representations of S_n. Next we give lifting criteria for representations of $S_{n_1} \times S_{n_2}$. We conclude the chapter with some examples.

5.1 Direct Sum

Lemma 5.1.1. *Consider representations of V_i of S_n. If $V = \bigoplus_i V_i$, then $g_V = \sum_i g_{V_i}$ and $h_V = \sum_i h_{V_i}$.*

Proof. We know that for $V = \bigoplus\limits_i V_i$,

$$\chi_V(\mu) = \sum_i \chi_{V_i}(\mu), \tag{5.1}$$

where $\chi_V(\mu)$ (resp. $\chi_{V_i}(\mu)$) denotes the character value of the representation V (resp.

V_i) at the conjugacy class μ. Using Equation (3.1) we calculate

$$\begin{aligned} g_V &= \frac{1}{2} \cdot (\chi_V(1) - \chi_V(s_1)) \\ &= \frac{1}{2} \cdot (\sum_i \chi_{V_i}(1) - \sum_i \chi_{V_i}(s_1)) \\ &= \sum_i \left(\frac{1}{2} \cdot (\chi_{V_i}(1) - \chi_{V_i}(s_1)) \right) \\ &= \sum_i g_{V_i}. \end{aligned}$$

We obtain an expression for h_V in terms of character values as mentioned in Equation (3.4),

$$h_V = \frac{1}{4} \cdot (\chi_V(1) - 2\chi_V(s_1) + \chi_V(s_1 s_3)).$$

Here applying Equation (5.1) we easily obtain the required result. □

Corollary 5.1.2. *Let (ϕ, V) be a representation of S_n. Then the following statements are true.*

1. *If V is chiral then the representation $V^{\oplus 4}$ is spinorial.*

2. *If V is achiral then the representation $V \oplus V$ is spinorial.*

The corollary follows easily from Lemma 5.1.1 and Theorem 3.1.7.

Let ϵ denote the sign representation of S_n.

Theorem 5.1.3. *Let V be a chiral representation of S_n. Then V is spinorial if and only if $V \oplus \epsilon$ is spinorial.*

Proof. We write $V' = V \oplus \epsilon$. Note that V' is achiral. From Theorem 5.1.1 we obtain $g_{V'} = g_V + 1$ and $h_{V'} = h_V + 1$. Now if V' is spinorial we have $g_{V'} \equiv 0 \pmod 4$ and $h_{V'}$ is even. This gives $g_V \equiv 3 \pmod 4$ and h_V is odd. Therefore V is spinorial.

On the other hand if V is spinorial then $g_V \equiv 3 \pmod 4$ and h_V is odd. Then we obtain that $g_{V'} \equiv 0 \pmod 4$ and $h_{V'}$ is even. Therefore V' is spinorial. □

5.2 Internal Tensor Product

For any two representations V and W of a group G, we can consider the representation $V \otimes W$ of G. The action of G on $V \otimes W$ is given by

$$g \cdot (v \otimes w) = g \cdot v \otimes g \cdot w, \quad g \in G, \ v \otimes w \in V \otimes W.$$

If $\chi_{V \otimes W}(\mu)$ denotes the character of $V \otimes W$ at the conjugacy class with cycle type μ then we have

$$\chi_{V \otimes W}(\mu) = \chi_V(\mu) \cdot \chi_W(\mu), \tag{5.2}$$

where $\chi_V(\mu)$ (resp. $\chi_W(\mu)$) denotes the character value of V (resp. W) at the conjugacy class with cycle type μ.

Theorem 5.2.1. *Let (ϕ, V) be any representation of S_n. Take the representation $V_1 = V \otimes V$. Then we have the following results.*

- *If ϕ is achiral then V_1 is always spinorial.*
- *If ϕ is chiral then V_1 is spinorial if and only if $\dim V$ is odd.*

Proof. Let f_V denote the dimension of V. From Equation (3.1) we have $g_V = \frac{1}{2} \cdot (\chi_V(1) - \chi_V(s_1))$. This gives

$$\chi_V(s_1) = \chi_V(1) - 2g_V. \tag{5.3}$$

Then we calculate

$$\begin{aligned}
g_{V_1} &= \frac{1}{2} \cdot (\chi_{V_1}(1) - \chi_{V_1}(s_1)) \\
&= \frac{1}{2} \cdot ((\chi_V(1))^2 - (\chi_V(s_1))^2) \\
&= \frac{1}{2} \cdot (\chi_V(1) - \chi_V(s_1)) \cdot (\chi_V(1) + \chi_V(s_1)) \\
&= g_V \cdot (\chi_V(1) + \chi_V(s_1)) \\
&= 2g_V \cdot (f_V - g_V) \quad \text{(Use Euation 5.3 and put } \chi_V(1) = f_V\text{)}.
\end{aligned}$$

So g_{V_1} is always even. This implies V_1 is always achiral. From Equation (3.4) we have

$h_V = \frac{1}{4} \cdot (\chi_V(1) - 2\chi_V(s_1) + \chi_V(s_1s_3))$. This gives

$$\chi_V(s_1s_3) = 4h_V - \chi_V(1) - 2\chi_V(s_1). \tag{5.4}$$

As before we calculate

$$\begin{aligned}
h_{V_1} &= \frac{1}{4} \cdot ((\chi_V(1))^2 - 2(\chi_V(s_1))^2 + (\chi_V(s_1s_3))^2) \\
&= \frac{1}{4} \cdot ((\chi_V(1))^2 - (\chi_V(s_1))^2 + (\chi_V(s_1s_3))^2 - (\chi_V(s_1))^2) \\
&= \frac{1}{4} \cdot (\chi_V(1) - \chi_V(s_1))(\chi_V(1) + \chi_V(s_1)) \\
&+ \frac{1}{4} \cdot (\chi_V(s_1s_3) - \chi_V(s_1))(\chi_V(s_1s_3) + \chi_V(s_1)) \\
&= g_V(f_V - g_V) + (2h_V - g_V)(2h_V - 3g_V + f_V) \quad \text{(Use Equations 5.4, 3.1 and put } \chi_V(1) = f_V\text{)} \\
&= 4h_V^2 - 8h_Vg_V + 2g_V^2 + 2h_Vf_V \\
&= 2h_V(f_V - 2g_V + h_V) + 2(g_V - h_V)^2.
\end{aligned}$$

This shows that h_{V_1} is always even. Therefore V_1 is spinorial if and only if

$$g_{V_1} = 2g_V \cdot (f_V - g_V) \equiv 0 \pmod 4. \tag{5.5}$$

If V is achiral the condition 5.5 holds. Therefore for V achiral V_1 is always spinorial. If V is chiral, i.e. g_V is odd, then V_1 is spinorial if and only if $f_V \equiv g_V \pmod 2$. □

5.3 Permutation Representations

For each partition $\lambda = (\lambda_1, \dots, \lambda_l)$ of n let

$$X_\lambda = \{(X_1, \dots, X_l) \mid X_1 \sqcup \cdots \sqcup X_l = \{1, \dots, n\}, |X_i| = \lambda_i\},$$

denote the set of all ordered partitions of $\{1, \dots, n\}$ of shape λ. The action of S_n on $\{1, \dots, n\}$ gives rise to an action of it on X_λ. Let $\mathbb{R}[X_\lambda]$ denote the vector space of all $\mathbb{R}$-valued functions on X_λ. We define the permutation representation $(\beta_{X_\lambda}, \mathbb{R}[X_\lambda])$ of S_n

as

$$\beta_{X_\lambda}(g)f(x) = f(g^{-1}\cdot x) \text{ for all } g \in S_n, f \in \mathbb{R}[X_\lambda].$$

For $V = \mathbb{R}[X_\lambda]$ we have

$$g_V = \frac{\chi_V(1) - \chi_V(s_1)}{2}.$$

It is well known that the character of a permutation representation is equal to the number of fixed points. The dimension of the vector space $V = \mathbb{R}[X_\lambda]$ is

$$\chi_V(1) = \binom{n}{\lambda_1, \ldots, \lambda_l}.$$

We have

$$\chi_V(s_1) = |\{x \in X_\lambda \mid s_1 \cdot x = x\}|.$$

Now an element $x \in X_\lambda$ is fixed by s_1 if and only if 1 and 2 lie in the same part X_i, for some $i \in \{1, 2, \ldots, l\}$. Therefore we have

$$\chi_V(s_1) = \sum_{|\lambda_i| \geq 2} \binom{n-2}{\lambda_1, \ldots, \lambda_i - 2, \ldots, \lambda_l}.$$

Putting all these values we conclude,

$$g_V = \frac{1}{2}\left(\binom{n}{\lambda_1, \ldots, \lambda_l} - \sum_{|\lambda_i| \geq 2} \binom{n-2}{\lambda_1, \ldots, \lambda_i - 2, \ldots, \lambda_l}\right). \tag{5.6}$$

The quantity g_V can also be obtained as follows: The orbits of s_1 in X_λ are of cardinality 1 or 2. We call an orbit of size 2 as a doubleton orbit. A doubleton orbit yields -1 as an eigenvalue for $\beta_{X_\lambda}(s_1)$. Therefore g_V equals the number of doubleton orbits. From [3, Lemma 17] we conclude that

$$g_V = \sum_{1 \leq i < j \leq l} \binom{n-2}{\lambda_1, \ldots, \lambda_i - 1, \ldots, \lambda_j - 1, \ldots, \lambda_l}.$$

Similarly we obtain a formulation for h_V in terms of multinomial coefficients. From

Equation (3.4) we obtain

$$h_V = \frac{1}{4}(\chi_V(1) - 2\chi_V(s_1) + \chi_V(s_1 s_3)).$$

Note that

$$\chi_V(s_1 s_3) = |\{x \in X_\lambda \mid s_1 s_3 \cdot x = x\}|.$$

Now an element $x \in X_\lambda$ is fixed by $s_1 s_3$ if and only if either the elements $\{1, 2, 3, 4\}$ lie in the same part X_i or $\{1, 2\}$ lie in X_i and $\{3, 4\}$ lie some other part X_j. Therefore we obtain

$$\chi_V(s_1 s_3) = \sum_{\substack{1 \le i < j \le l \\ |\lambda_i| \ge 2, |\lambda_j| \ge 2}} \binom{n-4}{\lambda_1, \dots, \lambda_i - 2, \dots, \lambda_j - 2, \dots, \lambda_l} + \sum_{|\lambda_k| \ge 4} \binom{n-4}{\lambda_1, \dots, \lambda_i - 4, \dots, \lambda_l}.$$

Putting this value we obtain:

$$\begin{aligned} h_V = \frac{1}{4}\Bigg(&\binom{n}{\lambda_1, \dots, \lambda_l} - 2 \sum_{|\lambda_k| \ge 2} \binom{n-2}{\lambda_1, \dots, \lambda_k - 2, \dots, \lambda_l} \\ &+ \sum_{\substack{1 \le i < j \le l \\ |\lambda_i| \ge 2, |\lambda_j| \ge 2}} \binom{n-4}{\lambda_1, \dots, \lambda_i - 2, \dots, \lambda_j - 2, \dots, \lambda_l} + \sum_{|\lambda_k| \ge 4} \binom{n-4}{\lambda_1, \dots, \lambda_i - 4, \dots, \lambda_l}\Bigg). \end{aligned}$$

We know that for $\lambda = (1^n)$, the representation $\mathbb{C}[X_{(1^n)}]$ is isomorphic to the regular representation of S_n.

Theorem 5.3.1. *The regular representation of S_n, $n \ge 4$, is achiral and spinorial.*

Proof. Note that $\dim(\mathbb{R}[X_{(1^n)}]) = n!$. From Equation (5.6) we have $g_{(\mathbb{R}[X_{(1^n)}])} = \frac{n!}{2}$. For $n \ge 4$, we obtain $g_{(\mathbb{R}[X_{(1^n)}])} \equiv 0 \pmod 4$. Again from the expression above for h_V in terms of multinomial coefficients we obtain $h_{(\mathbb{R}[X_{(1^n)}])} = \frac{n!}{4} \equiv 0 \pmod 2$, for $n \ge 4$. Therefore from 3.1.7 we conclude that the regular representation of S_n, $n \ge 4$, is achiral and spinorial. □

5.4 Product of Symmetric Groups

Consider the representations (π_i, V_i) of S_{n_i} for $i \in \{1, 2\}$. In this section, we discuss the spinoriality of the representation $(\pi, V_1 \boxtimes V_2)$ of $S_{n_1} \times S_{n_2}$. The action of $S_{n_1} \times S_{n_2}$ on $V_1 \boxtimes V_2$ is given by

$$(x_1, x_2) \cdot (v_1 \otimes v_2) = (x_1 \cdot v_1) \otimes (x_2 \cdot v_2), \quad \text{for } x_i \in S_{n_i}, v_i \in V_i.$$

Let f_i, g_i, h_i denote the dimensions of $V_i, V_i(-1), V_i(-1,-1)$ respectively. Let $\mathrm{O}(n)$ (resp. $\mathrm{SO}(n)$) denote the group of $n \times n$ real orthogonal (resp. special orthogonal) matrices.

Theorem 5.4.1. *Let V_i be a representation of S_{n_i} for $i \in \{1, 2\}$. The representation $(\pi, V_1 \boxtimes V_2)$ of $S_{n_1} \times S_{n_2}$ is spinorial if and only if the representations $\pi|_{(S_{n_1} \times 1)}$ and $\pi|_{(1 \times S_{n_2})}$ are spinorial and the following condition holds:*

$$g_1 g_2 (1 + f_1 f_2) \equiv 0 \pmod 2.$$

Proof. If the representation π is spinorial, then both the representations $\pi|_{S_{n_1} \times 1}$ and $\pi|_{1 \times S_{n_2}}$ are spinorial. One computes $\pi|_{S_{n_1} \times 1} \cong \pi_1^{\oplus f_2}$ and $\pi|_{1 \times S_{n_2}} \cong \pi_2^{\oplus f_1}$. So following Lemma 5.1.1 we write down the necessary and sufficient conditions.

1. The representation $\pi_1^{\oplus f_2}$ is spinorial if and only if $g_1 f_2 \equiv 0 \text{ or } 3 \pmod 4$, and $h_1 f_2 \equiv g_1 f_2 \pmod 2$.

2. The representation $\pi_2^{\oplus f_1}$ is spinorial if and only if $g_2 f_1 \equiv 0 \text{ or } 3 \pmod 4$, and $h_2 f_1 \equiv g_2 f_1 \pmod 2$.

Now the elements $(s_i, 1) \in S_{n_1} \times 1$ and $(1, s_j) \in 1 \times S_{n_2}$ commute. Since all transpositions are conjugate in S_n we work with the elements $(s_1, 1)$ and $(1, s_2)$ for convenience. From Lemma 3.1.4 we conclude that the lifts of these two elements commute if and only if $\pi(1, s_2) \in Z(\pi(s_1, 1))^\sharp$. Consider a basis $\beta_{g_1} = \{v_1, v_2, \ldots, v_{g_1}\}$ for $V_1(-1)$ which can be extended to obtain a basis $\beta_1 = \{v_1, v_2, \ldots, v_{f_1}\}$ for V_1. Also consider a basis $\beta_{g_2} = \{w_1, w_2, \ldots, w_{g_2}\}$ for $V_2(-1)$ which can be extended to obtain a basis $\beta_2 = \{w_1, w_2, \ldots, w_{f_2}\}$ for V_2 . Then a basis of $V_1(-1) \boxtimes V_2$ is $\beta_{g_1 2} = \{v_1 \otimes w_1, v_1 \otimes w_2, \ldots, v_{g_1} \otimes w_{f_2}\}$. With respect to the basis $\beta = \{v_1 \otimes w_1, v_1 \otimes w_2, \ldots, v_{f_1} \otimes w_{f_2}\}$ for

$V_1 \boxtimes V_2$ the matrices for $\pi(s_1, 1)$ and $\pi(1, s_j)$ look like

$$\begin{pmatrix} -I_{g_1 f_2} & \\ & I_{(f_1 f_2 - g_1 f_2)} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix} \quad \text{respectively,}$$

where $B_1 \in \mathrm{O}(g_1 f_2)$ and $B_2 \in \mathrm{O}(f_1 f_2 - g_1 f_1)$. Notice that B_1 denotes the matrix of $\pi(1, s_2)$ with respect to the basis $\beta_{g_1 2}$. Following same argument as in Lemma 3.1.5 we obtain

$$Z(\pi(g_1, 1))^\sharp = \begin{cases} \left[\begin{array}{c|c} \mathrm{O}(g_1 f_2) & 0 \\ \hline 0 & \mathrm{SO}(f_1 f_2 - g_1 f_2) \end{array}\right], & \text{for } g_1 f_2 \text{ odd}, \\ \\ \left[\begin{array}{c|c} \mathrm{SO}(g_1 f_2) & 0 \\ \hline 0 & \mathrm{O}(f_1 f_2 - g_1 f_2) \end{array}\right], & \text{for } g_1 f_2 \text{ even}. \end{cases}$$

Therefore we require

- $\det B_1 = 1$, for $g_1 f_2$ even.
- $\det B_2 = 1$, for $g_1 f_2$ odd.

Consider the subspace

$$V_1 \boxtimes V_2(-1, -1) = \{v \in V_1 \boxtimes V_2 \mid \pi(s_1, 1)v = -v, \pi(1, s_2)v = -v\},$$

of $V_1 \boxtimes V_2$. A basis for $V_1 \boxtimes V_2(-1, -1)$ is $\beta_{g_1 g_2} = \{v_1 \otimes w_{j_1}, v_1 \otimes w_1 \ldots, v_1 \otimes w_{g_2}, \ldots, v_{g_1} \otimes w_{g_2}\}$. So $\dim(V_1 \boxtimes V_2(-1, -1)) = g_1 g_2$. We have $\det(B_1) = (-1)^{g_1 g_2}$. For $g_1 f_2$ is even, $\det(B_1) = 1$ if and only if $g_1 g_2 \equiv 0 \pmod 2$.

For $g_1 f_2$ odd, it remains to check whether $\det B_2 = 1$. Consider the subspace

$$V_1 \boxtimes V_2(1, -1) = \{v \in V_1 \boxtimes V_2 \mid \pi(s_1, 1)v = v, \pi(1, s_2)v = -v\},$$

of $V_1 \boxtimes V_2$. The set $\beta_3 = \{v_i \otimes w_j \mid g_1 + 1 \leq i \leq f_1, 1 \leq j \leq f_2\}$ is a basis for $V_1 \boxtimes V_2$. Now B_2 denotes the matrix of $\pi(1, s_2)$ with respect to the basis β_3. We have $\dim(V_1 \boxtimes V_2(1, -1)) = (f_1 - g_1)g_2$, so that $\det(B_2) = (-1)^{(f_1 - g_1)g_2}$. Hence for $g_1 f_2$ odd,

$\det(B_2) = 1$ if and only if $(f_1 - g_1)f_2g_2 \equiv 0 \pmod 2$. As a summary we write down the required conditions.

1. $(f_1 - g_1)g_2 \equiv 0 \pmod 2$, if g_1f_2 is odd.
2. $g_1g_2 \equiv 0 \pmod 2$, if g_1f_2 is even.

This two conditions can be combined in the following way.

$$\begin{aligned} g_1g_2(1 + g_1f_2) + (f_1 - g_1)g_2g_1f_2 &\equiv 0 \pmod 2 \\ g_1g_2 + g_1^2g_2f_2 + g_1g_2f_1f_2 - g_1^2g_2f_2 &\equiv 0 \pmod 2 \\ g_1g_2(1 + f_1f_2) &\equiv 0 \pmod 2. \end{aligned}$$

$\square$

5.5 Examples

For $n = 2$, there are only two representations of $S_2 = C_2$, namely the trivial and the sign representation. The trivial representation of S_n given by the partition (n), is always spinorial. On the other hand the partition (1^n) gives the sign representation of S_n, denoted by ϵ.

Proposition 5.5.1. *The sign representation of S_n is aspinorial.*

Proof. Consider the sign representation $\epsilon : S_n \to \{\pm 1\}$. Note that $\mathrm{O}(1) = \{\pm 1\}$. Taking the negative definite quadratic form we obtain $\mathrm{Pin}(1) = C_4$, the cyclic group of order 4. If we take g as the generator of C_4, the elements of the group are $1, g, g^2, g^3$. Since $\ker(\rho) = \{\pm 1\}$ we must have $\rho(g) = -1$. Suppose there exists a lift $\widehat{\epsilon} : S_n \to C_4$ such that the following diagram is commutative.

$$\begin{tikzcd} & C_4 \arrow[d, "\rho"] \\ S_n \arrow[ur, dotted, "\widehat{\epsilon}"] \arrow[r, "\epsilon"] & \mathrm{O}(1) \end{tikzcd}$$

Consider the element $s_1 \in S_n$. Note that $\epsilon(s_1) = -1$. The diagram commutes iff $\rho \circ \widehat{\epsilon}(s_1) = -1$. So $\widehat{\epsilon}(s_1)$ is either g or g^3, both are of order 4. For any homomorphism $\widehat{\epsilon} : S_n \to C_4$ this is not possible. $\square$

Remark 5.5.2. *For an alternative approach note that $g_\epsilon = -1$. So from Theorem 3.1.9 we conclude that ϵ is aspinorial.*

Let ϕ_n denote the standard permutation representation of S_n via permutation matrices.

Proposition 5.5.3. *The representation ϕ_n of S_n is aspinorial.*

Proof. We calculate

$$\phi_n(s_1) = \begin{pmatrix} R_{\pi/2} & 0 \\ 0 & I_{n-2} \end{pmatrix},$$

where $R_{\pi/2}$ denotes the reflection matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore we have $g_{\phi_n} = 1$, which denotes the multiplicity of -1 as an eigenvalue of $\phi_n(s_1)$. Thus from Theorem 3.1.7 we conclude that ϕ_n is aspinorial. □

Consider the irreducible representation $V_{(2,1)}$ of S_3. From Section 2.3 we obtain the image of s_1 under the representation $V_{(2,1)}$. Note that we can conjugate it to the diagonal form

$$\phi(s_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So $g_{V_{(2,1)}} = 1$. Therefore from Theorem 3.1.7 we have $V_{(2,1)}$ is aspinorial.

Here we present tables exhibiting the nature of the representations of S_n, for $1 \leq n \leq 6$. Here we use all the methods developed in the previous chapters.

Table 5.1: Table exhibiting the nature of the representations of S_n, $n = 2, 3$.

λ	Chirality of V_λ	Spinoriality of V_λ	Spinoriality of $V_\lambda\|_{A_n}$	Orthogonality of $V_\lambda^\pm$	Spinoriality of $V_\lambda^\pm$
For $n = 2$					
(2)	achiral	spinorial	spinorial	-	-
(1^2)	chiral	aspinorial	spinorial	-	-
For $n = 3$					
(3)	achiral	spinorial	spinorial	-	-
$(2, 1)$	chiral	aspinorial	spinorial	not orthogonal	-
(1^3)	chiral	aspinorial	spinorial	-	-

Table 5.2: Table exhibiting the nature of the representations of S_n, $4 \leq n \leq 6$.

λ	Chirality of V_λ	Spinoriality of V_λ	Spinoriality of $V_\lambda\|_{A_n}$	Orthogonality of $V_\lambda^\pm$	Spinoriality of $V_\lambda^\pm$
For $n = 4$					
(4)	achiral	spinorial	spinorial	-	-
$(3,1)$	chiral	aspinorial	aspinorial	-	-
$(2,2)$	chiral	aspinorial	spinorial	not orthog-onal	-
$(2,1^2)$	achiral	aspinorial	aspinorial	-	-
(1^4)	chiral	aspinorial	spinorial	-	-
For $n = 5$					
(5)	achiral	spinorial	spinorial	-	-
$(4,1)$	chiral	aspinorial	aspinorial	-	-
$(3,2)$	achiral	aspinorial	aspinorial	-	-
$(3,1^2)$	chiral	spinorial	spinorial	orthogonal	aspinorial
$(2^2,1)$	chiral	aspinorial	aspinorial	-	-
$(2,1^3)$	chiral	aspinorial	aspinorial	-	-
(1^5)	chiral	aspinorial	spinorial	-	-
For $n = 6$					
(6)	achiral	spinorial	spinorial	-	-
$(5,1)$	chiral	aspinorial	aspinorial	-	-
$(4,2)$	chiral	aspinorial	spinorial	-	-
$(4,1^2)$	achiral	aspinorial	aspinorial	-	-
(3^2)	achiral	aspinorial	aspinorial	-	-
$(3,2,1)$	achiral	spinorial	spinorial	orthogonal	spinorial
$(3,1^3)$	achiral	aspinorial	aspinorial	-	-
(2^3)	chiral	aspinorial	aspinorial	-	-
$(2^2,1^2)$	achiral	aspinorial	spinorial	-	-
$(2,1^4)$	achiral	aspinorial	aspinorial	-	-
(1^6)	chiral	aspinorial	spinorial	-	-

Table 5.3: Table exhibiting the nature of the representations of S_n, S_n, $3 \leq n \leq 10$ corresponding to the self-conjugate partitions.

λ	f_λ	Chirality of V_λ	Spinoriality of V_λ	Orthogonality of $V_\lambda^\pm$	Spinoriality of $V_\lambda^\pm$
$(2,1)$	2	chiral	aspinorial	not orthog-onal	-
$(2,2)$	2	chiral	aspinorial	not orthog-onal	-
$(3,1,1)$	6	chiral	spinorial	orthogonal	aspinorial
$(3,2,1)$	16	chiral	aspinorial	orthogonal	spinorial
$(4,1^3)$	20	achiral	aspinorial	not orthog-onal	-
$(4,2,1^2)$	90	chiral	aspinorial	not orthog-onal	-
$(3,3,2)$	42	chiral	aspinorial	not orthog-onal	-
$(5,1^4)$	70	chiral	spinorial	orthogonal	spinorial
(3^3)	42	chiral	aspinorial	not orthog-onal	-
$(5,2,1^3)$	448	achiral	spinorial	orthogonal	spinorial
$(4,3,2,1)$	768	achiral	spinorial	orthogonal	spinorial

Table 5.4: Table exhibiting the nature of the permutation representations of S_n, $2 \leq n \leq 5$.

λ	Dimension of $\mathbb{R}[X_\lambda]$	$g_{(\mathbb{R}[X_\lambda])}$	$h_{(\mathbb{R}[X_\lambda])}$	Chirality of $\mathbb{R}[X_\lambda]$	Spinoriality of $\mathbb{R}[X_\lambda]$
For $n = 2$					
(2)	1	0	-	achiral	spinorial
(1^2)	2	1	-	chiral	aspinorial
For $n = 3$					
(3)	1	0	-	achiral	spinorial
$(2, 1)$	3	1	-	chiral	aspinorial
(1^3)	6	3	-	chiral	spinorial
For $n = 4$					
(4)	1	0	0	achiral	spinorial
$(3, 1)$	4	1	0	chiral	aspinorial
$(2, 2)$	6	2	1	achiral	aspinorial
$(2, 1^2)$	12	5	2	chiral	aspinorial
(1^4)	24	12	6	achiral	spinorial
For $n = 5$					
(5)	1	0	0	achiral	spinorial
$(4, 1)$	5	1	0	chiral	aspinorial
$(3, 2)$	10	3	1	chiral	spinorial
$(3, 1^2)$	20	7	2	chiral	aspinorial
$(2^2, 1)$	30	12	5	achiral	aspinorial
$(2, 1^3)$	60	27	12	chiral	aspinorial
(1^5)	120	60	30	achiral	spinorial

Using Young's rule as mentioned in [21, Theorem 3.3.1] we obtain the complete decomposition of the permutation representations $\mathbb{R}[X_\lambda]$ in irreducible representations , as λ varies over partitions of 3.

$$\begin{aligned}
\mathbb{R}[X_{(3)}] &= V_{(3)} \\
\mathbb{R}[X_{(2,1)}] &= V_{(3)} \oplus V_{(2,1)} \\
\mathbb{R}[X_{(1,1,1)}] &= V_{(3)} \oplus V_{(2,1)}^{\oplus 2} \oplus V_{(1,1,1)}.
\end{aligned}$$

Similarly for partitions of 4, we obtain:

$$\begin{aligned}
\mathbb{R}[X_{(4)}] &= V_{(4)} \\
\mathbb{R}[X_{(3,1)}] &= V_{(4)} \oplus V_{(3,1)} \\
\mathbb{R}[X_{(2,2)}] &= V_{(4)} \oplus V_{(3,1)} \oplus V_{(2,2)} \\
\mathbb{R}[X_{(2,1,1)}] &= V_{(4)} \oplus V_{(3,1)}^{\oplus 2} \oplus V_{(2,2)} \oplus V_{(2,1,1)} \\
\mathbb{R}[X_{(1,1,1,1)}] &= V_{(4)} \oplus V_{(3,1)}^{\oplus 3} \oplus V_{(2,2)}^{\oplus 2} \oplus V_{(2,1,1)}^{\oplus 3} \oplus V_{(1,1,1,1)}.
\end{aligned}$$

Here we record the complete decomposition of the permutation representations $\mathbb{C}[X_\lambda]$, as λ varies over all the partitions of 5.

$$\begin{aligned}
\mathbb{R}[X_{(5)}] &= V_{(5)} \\
\mathbb{R}[X_{(4,1)}] &= V_{(5)} \oplus V_{(4,1)} \\
\mathbb{R}[X_{(3,2)}] &= V_{(5)} \oplus V_{(4,1)} \oplus V_{(3,2)} \\
\mathbb{R}[X_{(3,1,1)}] &= V_{(5)} \oplus V_{(4,1)}^{\oplus 2} \oplus V_{(3,2)} \oplus V_{(3,1,1)} \\
\mathbb{R}[X_{(2,2,1)}] &= V_{(5)} \oplus V_{(4,1)}^{\oplus 2} \oplus V_{(3,2)}^{\oplus 2} \oplus V_{(3,1,1)} \oplus V_{(2,2,1)} \\
\mathbb{R}[X_{(2,1,1,1)}] &= V_{(5)} \oplus V_{(4,1)}^{\oplus 3} \oplus V_{(3,2)}^{\oplus 3} \oplus V_{(3,1,1)}^{\oplus 3} \oplus V_{(2,2,1)}^{\oplus 2} \oplus V_{(2,1,1,1)} \\
\mathbb{R}[X_{(1^5)}] &= V_{(5)} \oplus V_{(4,1)}^{\oplus 4} \oplus V_{(3,2)}^{\oplus 5} \oplus V_{(3,1,1)}^{\oplus 6} \oplus V_{(2,2,1)}^{\oplus 5} \oplus V_{(2,1,1,1)}^{\oplus 4} \oplus V_{(1^5)}.
\end{aligned}$$

Therefore one can verify the results in 5.4 by using Lemma 5.1.1.

6

Relation with Stiefel-Whitney Classes

This chapter reviews the connection between the spinoriality of representations of finite groups and the Stiefel-Whitney classes. The results 6.2.4 and 6.3.2 are already available in the literature for a more general context. See for example [9], [14]. We prove these results for finite groups to make the thesis more self-contained. We give lifting criteria for the representations of S_n in terms of first and second Stiefel-Whitney classes. Let ϵ denote the sign representation of S_n and ϕ_n denote the standard permutation representation of S_n on $\mathbb{R}^n$, via permutation matrices. Write $e_{\text{cup}} = w_1(\epsilon) \cup w_1(\epsilon)$. From [24, Section 1.5] we obtain that e_{cup} and $w_2(\phi_n)$ generate the group $H^2(S_n, \mathbb{Z}/2\mathbb{Z})$. We showed that for any representation (ϕ, V) of S_n, $w_2(\phi) = [\frac{g_V}{2}]e_{\text{cup}} + \frac{k_V}{2} w_2(\phi_n)$, where $[.]$ denotes the greatest integer function.

6.1 Definition and Notation

Definition 6.1.1. A *real vector bundle* $\xi = (E, \pi, B)$ consists of topological spaces E and B, called the total space and base space respectively, with a projection map $\pi : E \to B$ so that the following properties hold:

1. For each $b \in B$, the set $\pi^{-1}(b)$ has a real vector space structure.

2. For each $b \in B$, there exists a neighborhood $U \in B$ and a homeomorphism

$$h : U \times \mathbb{R}^n \to \pi^{-1}(U),$$

so that, for each $b \in U$, the correspondence $x \to h(b,x)$ defines an isomorphism between the vector spaces $\mathbb{R}^n$ and $\pi^{-1}(b)$. In this way we obtain an open cover $\{U_i\}$ of B and the pair (U_i, h_i) is called a bundle chart.

If $U_i \cap U_j \neq \emptyset$, then we obtain transition maps $t_{ij} : U_i \cap U_j \to \mathrm{GL}(n, \mathbb{R})$ given by $h_j \circ h_i^{-1}(b,v) = (b, t_{ij}(v))$, for $b \in U_i \cap U_j$ and $v \in \mathbb{R}$.

Definition 6.1.2. Let E be a vector bundle as above and let B_1 be an arbitrary topological space. Given a map $f : B_1 \to B$ we have a bundle f^*E, called the *pullback bundle* of E by the map f defined as

$$f^*(E) = \{(b,e) \in B_1 \times E \mid f(b) = \pi(e)\}.$$

The projection map $\pi_1 : f^*E \to B_1$ is defined as $\pi_1(b,e) = b$. Therefore we have the following commutative diagram.

$$\begin{array}{ccc}
f^*E & \xrightarrow{\hat{f}} & E \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\
B_1 & \xrightarrow{f} & B
\end{array}$$

For any finite group G there exists a contractible free right G space EG and the quotient space EG $/G$ is called a *classifying space* of G, denoted by BG. This classifying space BG is well defined up to homotopy equivalence. In fact, we have

$$H^m_{\mathrm{Gr}}(G, \mathbb{Z}/2\mathbb{Z}) \cong H^m_{\mathrm{Top}}(\mathrm{BG}, \mathbb{Z}/2\mathbb{Z}) \quad \text{for } m \geq 0. \tag{6.1}$$

Here $H^m_{\mathrm{Gr}}(.)$ denotes the m-th cohomology group and $H^m_{\mathrm{Top}}(.)$ denotes the m-th singular cohomology group with coefficients in $\mathbb{Z}/2\mathbb{Z}$. The elements of $H^2_{\mathrm{Gr}}(G, \mathbb{Z}/2\mathbb{Z})$ correspond to the group extensions of G by $\mathbb{Z}/2\mathbb{Z}$. For a Lie group G we in particular consider the subgroup $H^2_{\text{cont.}}(G, \mathbb{Z}/2\mathbb{Z})$ of $H^2_{\text{Gr}}(G, \mathbb{Z}/2\mathbb{Z})$. The elements of this cohomology group correspond to the group extensions of G by $\mathbb{Z}/2\mathbb{Z}$ in the category of Lie groups. For a Lie group G in fact we have

$$H^2_{\text{cont.}}(G, \mathbb{Z}/2\mathbb{Z}) \cong H^2_{\text{Top}}(\text{BG}, \mathbb{Z}/2\mathbb{Z}).$$

See [1] for reference.

We know that the space EG over BG also gives the universal principal G bundle with the continuous map $\pi : \text{EG} \to \text{BG}$. For details about principal bundles, we refer the reader to [11, Sections 10, 11].

Let $\{U_i, h_i\}$ denote the local trivializations for this bundle, i.e. $\{(U_i)\}$ forms an open cover for BG and $h_i : \pi^{-1}(U_i) \to U_i \times G$ are homeomorphisms. If $p_1 : U_i \times G \to U_i$ and $p_2 : U_i \times G \to G$ denote the projections to the first and second components respectively. Then we have $h_i(u' \cdot g) = (p_1(u'), p_2(u') \cdot g)$ for $u' \in \pi^{-1}(U_i)$. In fact we have the following commutative diagram.

$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[rr, "h_i"] \arrow[dr, "\pi"'] & & U_i \times G \arrow[dl, "\pi_1"] \\
& U_i &
\end{tikzcd}$$

For $U_i \cap U_j \neq \emptyset$ we obtain the transition map

$$\psi_{ij} = h_j \circ h_i^{-1} : U_i \cap U_j \times G \to U_i \cap U_j \times G,$$

such that $\psi_{ij}(p, g) = (p, f_{ij}(p)g)$, where $p \in U_i \cap U_j$ and $f_{ij} : U_i \cap U_j \to G$ is a continuous map.

For a finite dimensional representation (ϕ, V) of G, consider the right G action on the set EG $\times V$ as

$$(\xi, v) \cdot g = (\xi \cdot g, \phi(g^{-1}) \cdot v), \ \xi \in \text{EG}, v \in V.$$

The orbit space of this action consists of the equivalence classes $[\xi, v]$ such that $[\xi.g, v] = [\xi, \phi(g)v]$. This orbit space is denoted by $\text{EG} \times_G V$ and is called the associated vector bundle over BG of rank n and fiber V. The transition maps are given by $\tilde{\psi}_{ij} : (U_i \cap U_j) \times$

$V \to (U_i \cap U_j) \times V$, such that

$$\tilde{\psi}_{ij}(p, v) = (p, \phi(f_{ij}(p))v), \quad p \in U_i \cap U_j.$$

In general for a left G-space F, the relations $(e, y) \cdot g = (e \cdot g, g^{-1}y)$, for $e \in \mathrm{EG}, y \in F$ and $g \in G$ defines a right G structure on the set $\mathrm{EG} \times F$. The orbit space of this action consists of the equivalence classes $[e, y]$ such that $[e \cdot g, y] = [e, g^{-1} \cdot y]$. This orbit space is denoted by $\mathrm{EG} \times_G F$ and is called the associated fiber bundle over BG with fiber F.

Definition 6.1.3. [13, page 38] Let G be a subgroup of $\mathrm{GL}(V)$. Then the *structure group* of a vector bundle with fiber V is G, if there exists an atlas of bundle charts (see [13, page 2]) such that all the transition maps have their values in G.

In a similar fashion, we define the structure group of a principal G bundle.

Definition 6.1.4. [13, page 61] A subgroup H of G is called the *structure group* of a principal G bundle if there exists an atlas of bundle charts such that all the transition maps have their values in H.

For a real vector bundle ξ there corresponds a sequence of cohomology classes $w_i(\xi) \in H^i(B(\xi), \mathbb{Z}/2\mathbb{Z})$, for $i \geq 0$, called Stiefel-Whitney classes. For an axiomatic definition see [19, page 37].

Definition 6.1.5. The *Stiefel-Whitney classes* of a finite dimensional real representation (ϕ, V) of a group G are defined as $w_i(\phi) = w_i(\mathrm{EG} \times_G V) \in H^i(\mathrm{BG}; \mathbb{Z}/2\mathbb{Z})$.

Definition 6.1.6. [19, page 96] An *orientation* for a vector bundle ξ is a function which assigns an orientation to each fiber F of ξ, subject to the following local compatibility condition. For every point b in the base space, there should exist a local trivialisation (U, h), with $b \in U$ and $h : U \times \mathbb{R}^n \to \pi^{-1}(U)$, so that for each fiber $F = \pi^{-1}(b)$ over U the homomorphism $x \to h(b, x)$ from $\mathbb{R}^n$ to F is orientation preserving.

In particular a vector bundle with fiber V is orientable if for the associated transition function $t_{ij} : V \to V$, we have $\det(t_{ij}) > 0$.

6.2 Determinant of Representations and w_1

From [9] we obtain that for a representation (ϕ, V) of G, we have

$$w_1(\phi) = \det(\phi).$$

Definition 6.2.1. A real orthogonal representation (ϕ, V) is called *achiral* if $\det(\phi(g)) = 1$, for all $g \in G$.

So ϕ is achiral if and only if $w_1(\phi) = 0$.

Definition 6.2.2. [11, Definition 2.1, page 74] For a closed subgroup H of G, let $\xi = (X, p, B)$ be a principal G-bundle and $\eta = (Y, q, B)$ a principal H-bundle. Let $f : Y \to f(Y) \subset X$ be a homeomorphism onto a closed subset $f(Y)$ such that $f(y \cdot h) = f(y) \cdot h$, for $y \in Y$ and $h \in H$. Then η is called a *restriction* of ξ.

Theorem 6.2.3. *The tructure group of the universal principal G-bundle* EG *over* BG *cannot be reduced to a proper subgroup H of G.*

Proof. Take a proper subgroup H of G. Consider the universal principal H-bundle EH over BH. Note that H has a left action on G. Then we can think of the associate G bundle $\mathrm{EH} \times_H G$ over BH. Then there will be a classifying map $f_H : \mathrm{BH} \to \mathrm{BG}$ such that $\mathrm{EH} \times_H G$ is the pullback of EG by f_H.

Consider the universal principal G-bundle $(\mathrm{EG}, \mathrm{BG})$. Suppose there is a H-bundle η such that η is a restriction of EG. From [11, Theorem 4.1, page 76] it follows that this implies that the structure group of the principal G-bundle EG over BG can be reduced to H. Now we apply the theorem [11, Theorem 5.1, page 77] for $X = \mathrm{EG}$ and $B = \mathrm{BG}$. Then we obtain a classifying map $g : \mathrm{BG} \to \mathrm{BH}$ such that η is isomorphic to $g^*(\mathrm{EH})$ and $f_H \circ g$ is homotopic to the identity map on BG. We write in notation

$$f_H \circ g \sim \mathbb{1}. \tag{6.2}$$

$$\begin{tikzcd} \mathrm{BG} \arrow[rr, "\mathbb{1}"] \arrow[dr, "g"'] & & \mathrm{BG} \\ & \mathrm{BH} \arrow[ur, "f_H"'] & \end{tikzcd}$$

Here $\mathbb{1}$ denotes the classifying map for $(\mathrm{EG}, \mathrm{BG})$ itself as the universal principal G-bundle. We know that for any group G, $\pi_1(\mathrm{BG}) = G$, where $\pi_1(\mathrm{BG})$ denotes the fundamental group of the space BG. We have the maps $f_H^* : H \to G$, $g^* : G \to H$ and $\mathbb{1}^* : G \to G$ at the fundamental group level. Note that $\mathbb{1}^*$ denotes the identity map from G to itself. From Equation (6.2) we obtain $f_H^* \circ g^* = \mathbb{1}$. Therefore we have $g^* : G \to H$ is injective and $f_H^* : H \to G$ is surjective. Since G is a finite group and H is a proper subgroup of G, it follows that $G = H$. $\square$

Theorem 6.2.4. *Let (ϕ, V) be a real orthogonal representation of a finite group G. Then $\det(\phi)(g) > 0$, for all $g \in G$, if and only if the associated vector bundle $\mathrm{EG} \times_G V$ is orientable.*

Proof. Consider a real, orthogonal representation (ϕ, V) of G we have $\det(\phi(g)) > 0$, for all $g \in G$. In particular we obtain $\det(\phi(f_{i,j}(p))) > 0$, for $p \in U_i \cap U_j$. So the vector bundle $\mathrm{EG} \times_G V$ is orientable.

For the converse let the vector bundle $\mathrm{EG} \times_G V$ be orientable. Then we have $\det(\phi(f_{ij}(p))) > 0$ for all i, j and $p \in U_i \cap U_j$. Consider the group $G_+ = \{g \in G \mid \det(\phi(g)) > 0\}$. Note that for any two elements $g, h \in G_+$ we have $\det(\phi(gh^{-1})) = \det(\phi(g)) \cdot \det(\phi(h^{-1})) > 0$. So G_+ is a subgroup of G. If G_+ is a proper subgroup of G, then by Definition 6.1.4 the structure group of the universal principal G bundle EG over BG becomes G_+. But we proved in Theorem 6.2.3 that this cannot hold. So the only possibility is $G = G_+$. As a result we have $\phi(g) > 0$ for all $g \in G$ proving ϕ is achiral.

$\square$

6.3 Spinoriality of Representations and w_2

By the standard representation of $\mathrm{SO}(n)$ we mean the natural inclusion of $\mathrm{SO}(n)$ in $\mathrm{GL}_n(\mathbb{R})$. From [16, Appendix B, page 381] we obtain $\mathrm{BSO}(n)$ is the Grassmannian of oriented n planes in $\mathbb{R}^\infty$. The vector bundle $\mathrm{ESO}(n) \times_{\mathrm{SO}(n)} V$, where V denotes the standard representation of $\mathrm{SO}(n)$, is the universal oriented n-plane bundle over $\mathrm{BSO}(n)$. Note that the oriented frame bundle of $\mathrm{ESO}(n) \times_{\mathrm{SO}(n)} V$ is $\mathrm{ESO}(n)$. Write $E_1 = \mathrm{EG} \times_G V$ and $E_2 = \mathrm{ESO}(n) \times_{\mathrm{SO}(n)} V$.

Lemma 6.3.1. *Let (ϕ, V) denote an achiral representation of S_n. Then there exists a map $\phi_B : \mathrm{BG} \to \mathrm{BSO}(n)$ such that E_1 is the pullback of the bundle E_2 by ϕ_B.*

Proof. From Milnor's construction (see [11, Section 4.11]) we obtain that any element of EG looks like

$$\langle x, t\rangle = (t_0 x_0, t_1 x_1, \ldots, t_k x_k, \ldots),$$

where $x_i \in G$, $t_i \in [0,1]$, such that only a finite number of $t_i \neq 0$ and $\sum_i t_i = 1$. The right G action on EG is given by $\langle x, t\rangle \cdot y = \langle xy, t\rangle$, for $y \in G$. Similarly we obtain $\mathrm{ESO}(n)$. Now we define a map $m_\phi : \mathrm{EG} \to \mathrm{ESO}(n)$ as

$$m_\phi\langle x, t\rangle = (t_0\phi(x_0), t_1\phi(x_1), \ldots, t_k\phi(x_k), \ldots).$$

The map is well-defined as we have $m_\phi(\langle x, t\rangle \cdot y) = m_\phi(\langle x, t\rangle) \cdot \phi(y)$. Since $\mathrm{BG} = \mathrm{EG}/G$ and $\mathrm{BSO}(n) = \mathrm{ESO}(n)/\mathrm{SO}(n)$, m_ϕ induces a map $\phi_B : \mathrm{BG} \to \mathrm{BSO}(n)$, such that $\phi_B(eG) = m_\phi(e)\,\mathrm{SO}(n)$, where $e \in \mathrm{EG}$ and $m_\phi(e) \in \mathrm{ESO}(n)$. The map ϕ_B is well-defined. If $e_1 G = e_2 G$, then $e_1 = e_2 g$, for some $g \in G$. Applying m_ϕ to both sides of the equation we obtain

$$m_\phi(e_1) = m_\phi(e_2 g) = m_\phi(e_2)\phi(g).$$

Therefore it follows that $\phi_B(e_1 g) = \phi_B(e_2 g)$. Now we define a map $m_V : E_1 \to E_2$, such that $m_V(e, v) = (m_\phi(e), v)$, where $e \in \mathrm{EG}$ and $v \in V$. The map m_V is well-defined as

$$m_V(e \cdot g, g^{-1} \cdot v) = (m_\phi(e) \cdot \phi(g), \phi(g)^{-1} \cdot v).$$

Let $\pi_1 : E_1 \to \mathrm{BG}$ and $\pi_2 : E_2 \to \mathrm{BSO}(n)$ denote the vector bundle maps to the corresponding base spaces. We have $\pi_1(e, v) = eG$, for $e \in \mathrm{EG}$ and $v \in V$. Similarly

$\pi_2(e', v) = e' \operatorname{SO}(n)$, for $e' \in \operatorname{ESO}(n)$ and $v \in V$. Note that

$$\begin{aligned} \phi_B(\pi_1(e,v)) &= \phi_B(eG) \\ &= m_\phi(e) \operatorname{SO}(n) \\ &= \pi_2(m_\phi(e), v). \end{aligned}$$

This shows that the following diagram is commutative.

$$\begin{tikzcd} \mathrm{EG} \times_G V \arrow[r, "m_v"] \arrow[d] & \operatorname{ESO}(n) \times_{SO(n)} V \arrow[d] \\ \mathrm{BG} \arrow[r, "\phi_B"] & \operatorname{BSO}(n) \end{tikzcd}$$

From the definition it follows that m_V induces a surjective map $\hat{m_V} : V \to V$, at the fiber level. So $\hat{m_V}$ is also injective. Therefore $m_v : E_1 \to E_2$ gives a bundle map. From [19, Lemma 3.1] we conclude that E_1 is isomorphic to the pullback of E_2 by ϕ_B, denoted as $\phi_B^*(E_2)$. □

The following result can be found in [14] in a more general context.

Theorem 6.3.2. *Let (ϕ, V) be an orthogonal representation of a finite group G and $w_1(\phi) = 0$. Then ϕ is spinorial if and only if $w_2(\phi) = 0$.*

Proof. The fact $w_1(\phi) = 0$ indicates that $\phi(g) \in \operatorname{SO}(n)$ for all $g \in G$. In other words, ϕ is achiral. From Lemma 6.3.1 we conclude that there exists a map $\phi_B : \mathrm{BG} \to \operatorname{BSO}(n)$ such that i.e E_1 is isomorphic the pullback of E_2 by the map ϕ_B. We denote the pullback as $E_1 = \phi_B^*(E_2)$. From [16, page 81] we obtain

$$H^2_{\text{cont.}}(\operatorname{SO}(n), \mathbb{Z}/2\mathbb{Z}) \cong H^2(\operatorname{BSO}(n), \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \tag{6.3}$$

and $w_2(E_2) \in H^2(\operatorname{BSO}(n), \mathbb{Z}/2\mathbb{Z})$ is the non zero element. This element represents the class of the non-trivial extension $\operatorname{Spin}(n)$ of $\operatorname{SO}(n)$ in $H^2_{\text{cont.}}(\operatorname{SO}(n), \mathbb{Z}/2\mathbb{Z})$. Let G' denote the pullback $\operatorname{Spin}(n) \times_{\operatorname{SO}(n)} G = \{(e,g) \in \operatorname{Spin}(n) \times G \mid \rho(e) = \phi(g)\}$. From [27, Exercise

6.6.4] we conclude that $\phi_B^*(w_2(E_2))$ is the class of the extension G'. From Axiom 2 mentioned in [19, page 37] we have

$$\phi_B^*(w_2(E_2)) = w_2(E_1) = w_2(\phi). \tag{6.4}$$

From Equation (6.1) we have $H^2_{\text{Top}}(\text{BG}, \mathbb{Z}/2\mathbb{Z}) \cong H^2_{\text{cont.}}(G, \mathbb{Z}/2\mathbb{Z})$. Therefore we have the map

$$\phi_B^* : H^2_{\text{cont.}}(\text{SO}(n), \mathbb{Z}/2\mathbb{Z}) \to H^2(G, \mathbb{Z}/2\mathbb{Z}).$$

If $w_2(\phi) = 0$, then G' is the trivial double cover of G. In other words for the short exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to G' \to G \to 1,$$

there exists a section $s : G \to G'$. Let Pr_{Spin} be the projection map from G' to $\text{Spin}(n)$. Then we can define the lift map $\widehat{\phi} : G \to \text{Spin}(n)$ as $\widehat{\phi} = \text{Pr}_{\text{Spin}} \circ s$.

For the converse take ϕ to be spinorial. Then there exists a lift $\widehat{\phi}$ such that $\rho \circ \widehat{\phi} = \phi$. This gives $\widehat{\phi}^* \circ \rho^* = \phi_B^*$, i.e. the following diagram commutes.

$$\begin{tikzcd}
 & H^2_{\text{cont.}}(\text{Spin}(n), \mathbb{Z}/2\mathbb{Z}) \arrow[dl, "\widehat{\phi}^*"'] \\
H^2(G, \mathbb{Z}/2\mathbb{Z}) & H^2_{\text{cont.}}(\text{SO}(n), \mathbb{Z}/2\mathbb{Z}) \arrow[l, "\phi_B^*"'] \arrow[u, "\rho^*"']
\end{tikzcd}$$

Since $\text{Spin}(n)$ is simply-connected we have $H^2_{\text{cont.}}(\text{Spin}(n), \mathbb{Z}/2\mathbb{Z}) = 0$. From Equation (6.4) we have $w_2(\phi) = \phi_B^*(w_2(E_2))$. Following the commutativity of the diagram we deduce

$$w_2(\phi) = \phi_B^*(w_2(E_2)) = \widehat{\phi}^* \circ \rho^*(w_2(E_2)) = \widehat{\phi}^*(0) = 0.$$

Hence the condition is necessary and sufficient. □

Corollary 6.3.3. *Any orthogonal representation of a finite group of odd order is achiral and spinorial.*

Proof. From [8, page 807, Corollary 29] it follows that if $|G|$ is odd then $H^m(G, \mathbb{Z}/2\mathbb{Z}) = 0$ for all $m \geq 1$. The fact $H^1(G, \mathbb{Z}/2\mathbb{Z}) = 0$ tells that $w_1(\phi) = 0$, i.e. ϕ is achiral. Since

$H^2(G, \mathbb{Z}/2\mathbb{Z}) = 0$, we have $w_2(\phi) = 0$. So the result follows from Theorem 6.3.2. $\square$

Corollary 6.3.4. *Let (ϕ, V) be a representation of $S_n, n \geq 4$. Then ϕ is spinorial if and only if $w_2(\phi) + w_1(\phi) \cup w_1(\phi) = 0$.*

Proof. If V is achiral then we have $w_1(\phi) = 0$. Therefore the condition follows from Theorem 6.3.2. If V is chiral then consider the representation (ϕ', V'), where $V' = V \oplus \epsilon$ and ϵ denotes the sign representation of S_n. From Theorem 5.1.3 we obtain that V is spinorial if and only if V' is spinorial. Since ϕ' is achiral, from Theorem 6.3.2 we conclude that ϕ' is spinorial if and only if $w_2(\phi') = 0$. We calculate

$$\begin{aligned} w_2(V') &= w_2(V \oplus \epsilon) \\ &= w_2(V) + w_2(\epsilon) + w_1(V) \cup w_1(\epsilon) \\ &= w_2(V) + w_1(V) \cup w_1(V). \end{aligned}$$

For the last equality we used the facts that $w_2(\epsilon) = 0$, since the sign representation has dimension 1 and $w_1(\epsilon) = w_1(V) \in H^1(S_n, \mathbb{Z}/2\mathbb{Z})$ is the only non zero element. Therefore ϕ is spinorial if and only if $w_2(\phi) + w_1(\phi) \cup w_1(\phi) = 0$. $\square$

6.4 Expression of w_2 in Terms of Character Values

Let ϵ denote the sign representation of S_n and ϕ_n denote the standard permutation representation of S_n on $\mathbb{R}^n$, via permutation matrices. We have $H^1(S_n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, for $n \geq 2$, and $H^2(S_n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, for $n \geq 4$. For reference see [24, Section 1.5]. The only non-zero element in $H^1(S_n, \mathbb{Z}/2\mathbb{Z})$ is $w_1(\epsilon)$. Since $w_2(\epsilon) = 0$, we obtain

$$w_2(\epsilon \oplus \epsilon) = w_1(\epsilon) \cup w_1(\epsilon) \in H^2(S_n, \mathbb{Z}/2\mathbb{Z}).$$

Write $e_{\text{cup}} = w_1(\epsilon) \cup w_1(\epsilon)$. From [24, Section 1.5] we also obtain the following facts.

- The elements e_{cup} and $w_2(\phi_n)$ generate $H^2(S_n, \mathbb{Z}/2\mathbb{Z})$.
- The only non-zero element in $H^2(A_n) = \mathbb{Z}/2\mathbb{Z}$ is $w_2(\phi_n)$.

Note that for a group G, the m-th cohomology group $H^m(G, \mathbb{Z}/2\mathbb{Z})$ is a vector space over the finite field $\mathbb{Z}/2\mathbb{Z}$ (see [8, Proposition 20, page 801]). Consider the subgroups $C_2 = \langle s_1 \rangle$ and A_n of S_n. We obtain the maps $i_1^* : H^2(S_n, \mathbb{Z}/2\mathbb{Z}) \to H^2(C_2, \mathbb{Z}/2\mathbb{Z})$, and $i_2^* : H^2(S_n, \mathbb{Z}/2\mathbb{Z}) \to H^2(A_n, \mathbb{Z}/2\mathbb{Z})$.

Proposition 6.4.1. *The map*

$$i^* : H^2(S_n, \mathbb{Z}/2\mathbb{Z}) \to H^2(C_2, \mathbb{Z}/2\mathbb{Z}) \oplus H^2(A_n, \mathbb{Z}/2\mathbb{Z}),$$

given by $i^*(\alpha) = i_1^*(\alpha) \oplus i_2^*(\alpha)$, for $\alpha \in H^2(S_n, \mathbb{Z}/2\mathbb{Z})$, is bijective for $n \geq 4$.

Proof. Note that $H^2(A_n, \mathbb{Z}/2\mathbb{Z}) = H^2(C_2, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Therefore for $n \geq 4$, we have the map $i^* : \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. So i^* is a linear operator on the vector space $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Consider the following cases.

- The restriction of $\pi_1 = \epsilon \oplus \epsilon$ to C_2 is aspinorial but its restriction to A_n is spinorial. Therefore $i_1^*(w_2(\epsilon \oplus \epsilon)) \neq 0$, whereas $i_2^*(\epsilon \oplus \epsilon) = 0$.
- The restrictions of $\pi_2 = \phi_n \oplus \epsilon$ to both C_2 and A_n are aspinorial. Therefore $i_1^*(w_2(\phi_n \oplus \epsilon)) \neq 0$, whereas $i_2^*(\phi_n \oplus \epsilon) \neq 0$.
- The restriction of $\pi_3 = \pi_1 \oplus \pi_2$ to C_2 is spinorial but its restriction to A_n is aspinorial. Therefore $i_1^*(w_2(\pi_1 \oplus \pi_2)) = 0$, whereas $i_2^*(\pi_1 \oplus \pi_2) \neq 0$.

This shows that the map i^* is surjective. Since i^* is a map between vector spaces of the same dimension it follows that i^* is bijective. □

Theorem 6.4.2. *Let (ϕ, V) be an achiral representation of S_n. Then*

$$w_2(\phi) = \frac{g_V}{2} e_{\text{cup}} + \frac{k_V}{2} w_2(\phi_n).$$

Equivalently the representation ϕ is spinorial if and only if its restrictions to both $C_2 = \langle s_1 \rangle$ and A_n are spinorial.

Proof. From [24, Section 1.5] it follows that

$$w_2(\phi) = c_1 e_{\text{cup}} + c_2 w_2(\phi_n),$$

where $c_i \in \mathbb{Z}/2\mathbb{Z}$ are scalars. Theorem 3.1.1 says that ϕ is spinorial if and only if $g_V \equiv 0$

(mod 4). Since for an achiral representation V, g_V is even we choose $c_1 = g_V/2$. Note that if ϕ is spinorial $g_V/2 \equiv 0 \pmod 2$. From Theorem 4.1.2 we obtain $V\mid_{A_n}$ is spinorial if and only if $k_V \equiv 0 \pmod 4$. Since k_V is always even we take $c_2 = k_V/2$. Again if ϕ is spinorial $k_V/2 \equiv 0 \pmod 2$. So we have

$$w_2(\phi) = \frac{g_V}{2} e_{\text{cup}} + \frac{k_V}{2} w_2(\phi_n).$$

Note that since i^* is bijective, $i^*(w_2(\phi)) = 0$ if and only if $i_1^*(w_2(\phi)) = 0$ and $i_2^*(w_2(\phi)) = 0$. This ensures that ϕ is spinorial if and only if $\phi\mid_{A_n}$ and ϕ_{C_2} are spinorial. □

Theorem 6.4.3. *Let (ϕ, V) be a chiral representation of S_n. Then*

$$w_2(\phi) = \frac{g_V - 1}{2} e_{\text{cup}} + \frac{k_V}{2} w_2(\phi_n).$$

Equivalently ϕ is spinorial if and only if $w_2(\pi) = e_{\text{cup}}$.

Proof. Take $V' = V \oplus \epsilon$. Since V' is achiral, from the previous theorem we obtain

$$w_2(\phi') = w_2(\phi) + w_1(\phi) \cup w_1(\epsilon)$$
$$w_2(\phi) = w_2(\phi') - e_{\text{cup}}.$$

Since k_V denotes the multiplicity of -1 as an eigenvalue of $s_1 s_3$ we have $k_{V'} = k_V$. From the previous calculation we obtain

$$\begin{aligned} w_2(\phi) &= \frac{g_{V'}}{2} e_{\text{cup}} + \frac{k_V}{2} w_2(\phi_n) - e_{\text{cup}} \\ &= \frac{g_V + 1}{2} e_{\text{cup}} + \frac{k_V}{2} w_2(\phi_n) - e_{\text{cup}} \\ &= \frac{g_V - 1}{2} e_{\text{cup}} + \frac{k_V}{2} w_2(\phi_n). \end{aligned}$$

Let for a chiral representation (ϕ, V) of S_n, $w_2(\phi) = e_{\text{cup}}$. Then we have $k_V/2 \equiv 0 \pmod 4$, i.e. $k_V \equiv 0 \pmod 4$. Note that $k_V = h'_V$. Also $\frac{g_V - 1}{2} \equiv 0 \pmod 2$, i.e. $g_V \equiv 3 \pmod 4$. Then from Theorem 3.1.13 it follows that ϕ is spinorial.

On the other hand, if ϕ is spinorial then $\phi\mid_{A_n}$ is spinorial. As a result, we have $k_V/2 \equiv 0 \pmod 2$. Also $g_V \equiv 3 \pmod 4$. Therefore we calculate $w_2(\phi) = e_{\text{cup}}$. □

Combining the last two results we obtain

Theorem 6.4.4. *Let (ϕ, V) be any representation of S_n. Then*

$$w_2(\phi) = \left[\frac{g_V}{2}\right] e_{\text{cup}} + \frac{k_V}{2} w_2(\phi_n).$$

7

Characterizing Spinorial Partitions

In general it is difficult to enumerate the irreducible spinorial representations of S_n. In this chapter we characterize the spinorial partitions for some particular cases. The first section of this chapter sets the stage by recalling some works of Michel Lassalle. In the paper [15] he gives explicit formula for the characters of irreducible representations of symmetric groups in terms of power sum symmetric polynomials with variables as the contents of the associated partitions. Throughout the chapter we use the notation "v" for $v_2(f_\lambda)$.

7.1 Lassalle's Character Formulas

Recall that the content of the (i,j)th cell in a Young diagram is given by $(j-i)$. In general for any positive integer m, we define

$$C_m(\lambda) = \sum_{(i,j)\in \mathcal{Y}(\lambda)} (j-i)^m. \tag{7.1}$$

Note that $C_1(\lambda) = C(\lambda)$, as defined in 2.2.5.

From [15] we obtain the expressions for character values in terms of contents. Take the functions $c_r^\lambda(x)$ as defined in [15, Section 4.4, page 393]. Let $s(p,i)$ denote the Stirling number of the first kind which counts the number of ways to permute a list of p numbers in i cycles. Let for any positive integer m, $(n)_m$ denote the lowering factorial

$$(n)_m = n(n-1)\cdots(n-m+1).$$

Theorem 7.1.1. *[15, Theorem 4] For $\mu = (p, 1^{n-p})$ we have*

$$(n)_p \chi_\lambda(\mu) = f_\lambda \sum_{i\geq 2} s(p+1,i) c_i^\lambda(p).$$

In particular for $p = 2$ (see [15, page 395]), we obtain

$$\chi_\lambda(s_1) = \frac{f_\lambda}{\binom{n}{2}} C(\lambda). \tag{7.2}$$

Similarly putting $p = q = 2$,in Theorem [15, Theorem 5, page 396] we obtain

$$\chi_\lambda(s_1 s_3) = \frac{f_\lambda}{6\binom{n}{4}} \cdot (C(\lambda)^2 - 3C_2(\lambda) - n + n^2). \tag{7.3}$$

[1] Using Equation (7.2) in the expression of g_V in Equation (3.1) we deduce

$$g_\lambda = \frac{f_\lambda \cdot \left(\binom{n}{2} - C(\lambda)\right)}{2\binom{n}{2}}. \tag{7.4}$$

From [15, Section 5, page 395], we also obtain

$$\chi_\lambda(\zeta_4) = \frac{f_\lambda}{6\binom{n}{4}} \cdot (C_3(\lambda) - (2n-3)C(\lambda)), \tag{7.5}$$

where ζ_4 denotes the cycle type $(1,2,3,4)$.

Using the value of $\chi_\lambda(\zeta_4)$ (as in Equation (7.5)) in Theorem 3.2.3 we obtain

$$h_\lambda \equiv \frac{f_\lambda \cdot \left(6\binom{n}{4} - (C_3(\lambda) - (2n-3)C(\lambda))\right)}{12\binom{n}{4}} \pmod 2. \tag{7.6}$$

[1] A list of these formulas is available at the site http://igm.univ-mlv.fr/~lassalle/resucarac.

We use Equation (7.6) to count the number of irreducible spinorial partitions of S_n for some particular cases. In general we have the character value for the representation V_λ of S_n at a conjugacy class of a given cycle type.

Let $\rho = (\rho_1, \dots, \rho_r)$ be a partition with weight $|\rho| \leq n$. Let $M^{(r)}$ denote the set of upper triangular $r \times r$ matrices with non-negative integers, and 0 on the diagonal. For any $1 \leq i < j \leq r$, let $\epsilon_{ij} \in \{0, 2\}$, and define $\theta_{ij} = 1$ if $\epsilon_{ij} = 0$ and $\theta_{ij} = \rho_i \rho_j$ otherwise.

Theorem 7.1.2. *[15, Theorem 6] For $\mu = (\rho_1, \rho_2, \dots, \rho_r, 1^{(n-|\rho|)})$ we have*

$$(n)_{|\rho|} \chi_\lambda(\mu) = f_\lambda \sum_{\epsilon \in \{0,2\}^{r(r-1)/2}} \sum_{(i_1, i_2, \dots, i_r) \in N^r} A^{\epsilon}_{i_1, i_2, \dots, i_r}(\rho_1, \rho_2, \dots, \rho_r) \prod_{k=1}^{r} c^{\lambda}_{i_k}(\rho_k), \tag{7.7}$$

where

$$\begin{aligned} A^{\epsilon}_{i_1, i_2, \dots, i_r}(\rho_1, \rho_2, \dots, \rho_r) = & \sum_{a, b \in M^{(r)}} \left(\prod_{1 \leq i, j \leq r} \theta_{ij} \binom{a_{ij} + 1}{b_{ij} + 1} \frac{\rho_i (-\rho_i)^{b_{ij}} + \rho_j^{b_{ij}+1}}{\rho_i + \rho_j} \right) \\ & \times \prod_{k=1}^{r} s \left(\rho_k + 1, i_k + \sum_{l<k} (a_{lk} + \epsilon_{lk}) - \sum_{l>k} (a_{lk} - b_{lk}) \right), \end{aligned}$$

and the convention that the sum on a_{ij}, b_{ij} is restricted to $a_{ij} = b_{ij} = 0$ when $\epsilon_{ij} = 0$.

In short for $\mu = (\rho_1, \rho_2, \dots, \rho_r, 1^{(n-|\rho|)})$, where $|\rho| = \sum_{i=1}^{r} |\rho_i|$, we have

$$(n)_{|\rho|} \chi_\lambda(\mu) = f_\lambda \cdot \widehat{A}(\mu), \tag{7.8}$$

where $\widehat{A}(\mu) \in \mathbb{Z}$. For details we refer the reader to [15, Theorem 6].

7.2 Case of Achiral, Odd Partitions

Throughout this section we consider

$$n = \epsilon + 2^{k_1} + \cdots + 2^{k_r}, \; 0 < k_1 < \cdots < k_r, \; \epsilon \in \{0, 1\}. \tag{7.9}$$

Theorem 6.3.2 ensures that for an achiral, spinorial irreducible representation V_λ of S_n, we have $w_1(V_\lambda) = w_2(V_\lambda) = 0$. In fact from [19, Exercise 8.B, page 94] we conclude that $w_i(V_\lambda) = 0$, for $1 \leq i \leq 3$. Let $\mu_i = \mathrm{core}_{2^{k_i}}(\lambda)$. Since λ is odd the partition $\mu_i = \mathrm{core}_{2^{k_i}}(\lambda)$ is also odd. As $2^{k_i} \leq |\mu_i| < 2^{k_i+1}$, from 2.5.3 we obtain that there is a unique hook H_{k_i} of size 2^{k_i} in μ_i for $1 \leq i \leq r$. Let c_i denote the foot node content (recall Definition 2.2.5) of the hook of H_{k_i}.

Theorem 7.2.1. *Let λ be an odd, achiral partition of n.*

1. *Suppose $k_1 \geq 2$ and $k_2 = k_1 + 1$. Then λ is spinorial if and only if $c_1 \equiv \epsilon \pmod 4$ and $c_2 \equiv \epsilon \pmod 2$.*

2. *Suppose $k_1 \geq 2$ and $k_2 > k_1 + 1$. Then λ is spinorial if and only if $c_1 \equiv \epsilon \pmod 4$.*

3. *Suppose $k_1 = 1$ and $k_2 = 2$. Then λ is spinorial if and only if $c_1 = \epsilon$ and $c_2 \equiv 2 + \epsilon \pmod 4$.*

4. *Suppose $k_1 = 1$ and $k_2 \geq 3$. Then λ is spinorial if and only if $c_1 = \epsilon$ and either $c_2 \equiv 2 + \epsilon \pmod 4$ or $c_2 \equiv 1 - \epsilon \pmod 4$.*

Proof. As λ is achiral, the first lifting condition requires $g_\lambda \equiv 0 \pmod 4$. Note that $v_2\binom{n}{2} = k_1 - 1$. Since λ is odd, from Equation (7.4) we obtain an equivalent first lifting condition as

$$v_2\left(\binom{n}{2} - C(\lambda)\right) \geq k_1 + 2. \tag{7.10}$$

We have $2^{k_r} \leq n < 2^{k_r+1}$. So from Proposition 2.5.3 it follows that there exist a unique hook H_{k_r} of length 2^{k_r}. Removing the rim-hook R_{k_r} from $\mathcal{Y}(\lambda)$ we obtain the partition $\mu_{2^{k_r}} = \mathrm{core}_{2^{k_r}}(\lambda)$ such that $|\mu_{2^{k_r}}| = \epsilon + 2^{k_1} + 2^{k_2} + \cdots + 2^{k_{r-1}}$. Following Proposition 2.5.3 we conclude that $\mu_{2^{k_r}}$ contains a unique hook of size $2^{k_{r-1}}$. In this way we obtain that $\mu_{2^{k_i}}$ contains a unique hook H_{k_i} of size 2^{k_i}, for $1 \leq i \leq r$. Let R_{k_i} denote the corresponding rim-hook. If c_i denotes the content of the foot node of R_{k_i} then the contents of the other nodes of the rim are $c_i + 1, c_i + 2, \ldots, c_i + 2^{k_i} - 1$. As a result we obtain

$$\begin{aligned} C(R_i) &= 2^{k_i} c_i + \binom{2^{k_i}}{2} \\ &= 2^{k_i} c_i + 2^{k_i - 1}(2^{k_i} - 1). \end{aligned}$$

The union of all these rim-hooks R_{k_i} misses at most one cell with content 0. Therefore we have $C(\lambda) = \sum_i C(R_i)$. Note that $C(R_i) \equiv 0 \pmod{2^{k_i-1}}$. Therefore $C(\lambda) \equiv C(\mathrm{core}_{2^{k_1+3}})(\lambda) \pmod{2^{k_1+2}}$.

We calculate

$$C(\lambda) \equiv 2^{k_1}c_1 + 2^{k_2}c_2 + 2^{2k_1-1} + 2^{2k_2-1} - (2^{k_1-1} + 2^{k_2-1} + 2^{k_3-1}) \pmod{2^{k_1+1}}. \tag{7.11}$$

We have

$$\begin{aligned}\binom{n}{2} &= (2^{k_1-1} + 2^{k_2-1} + \cdots + 2^{k_r-1})(2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} - 1 + 2\epsilon) \\ &\equiv 2^{2k_1-1} + 2^{2k_2-1} - (2^{k_1-1} + 2^{k_2-1} + 2^{k_3-1}) + \epsilon(2^{k_1} + 2^{k_2}) \pmod{2^{k_1+1}}.\end{aligned}$$

Altogether we have

$$\binom{n}{2} - C(\lambda) \equiv 2^{k_1}(\epsilon + \epsilon 2^{k_2-k_1} - c_1 - 2^{k_2-k_1}c_2) \pmod{2^{k_1+2}}.$$

Therefore the condition 7.10 requires

$$(\epsilon + \epsilon 2^{k_2-k_1} - c_1 - 2^{k_2-k_1}c_2) \equiv 0 \pmod 4. \tag{7.12}$$

For $\epsilon = 0$, the condition 7.12 holds for the following cases:

1. if $k_2 = k_1 + 1$ then one of the following should hold.
   - $c_1 \equiv 0 \pmod 4$ and c_2 is even,
   - $c_1 \equiv 2 \pmod 4$ and c_2 is odd.
2. if $k_2 \geq k_1 + 2$, then we only require $c_1 \equiv 0 \pmod 4$.

For $\epsilon = 1$, the condition 7.12 holds for the following cases:

1. if $k_2 = k_1 + 1$ then one of the following should hold.
   - $c_1 \equiv 3 \pmod 4$ and c_2 is even,
   - $c_1 \equiv 1 \pmod 4$ and c_2 is odd.

2. if $k_2 \geq k_1 + 2$, then we only require $c_1 \equiv 1 \pmod 4$.

Since λ is achiral the third lifting condition requires $h_\lambda \equiv 0 \pmod 2$. Recall that we have the congruence equality

$$h_\lambda \equiv \frac{f_\lambda(6\binom{n}{4} - (C_3(\lambda) - (2n-3)C(\lambda)))}{12\binom{n}{4}} \pmod 2. \tag{7.13}$$

Note that

$$v_2\left(12\binom{n}{4}\right) = \begin{cases} k_1, & \text{for } k_1 > 1, \\ k_2, & \text{for } k_1 = 1. \end{cases}$$

Therefore we require

$$v_2\left(6\binom{n}{4} - C_3(\lambda) + (2n-3)C(\lambda)\right) \geq \begin{cases} k_1 + 1, & \text{for } k_1 > 1, \\ k_2 + 1, & \text{for } k_1 = 1. \end{cases}$$

We first consider the case when $k_1 > 1$. For a rim-hook R_i of length 2^i we get

$$\begin{aligned} C_3(R_i) &= \sum_{k=0}^{2^i-1} (c_i - k)^3 \\ &= 2^i c_i^3 + \sum_k 3c_i^2 k + \sum_k 3c_i k^2 + \sum_k k^3 \\ &= 2^i c_i^3 + 2^{2i-2}(2^i-1)^2 + c_i 2^{i-1}(2^i-1)(2^{i+1}-1) + 3c_i^2 2^{i-1}(2^i-1) \\ &= 2^i c_i^3 + 2^{2i-2}(2^i-1)^2 + 2^{i-1}(2^i-1)c_i(2^{i+1}-1+3c_i). \end{aligned}$$

Note that the term $2^{i-1}(2^i-1)c_i(2^{i+1}-1+3c_i)$ is always even. Also $v_2(2^{2i-2}(2^i-1)^2) \geq i$, for $i > 1$. Therefore we have $C_3(R_i) \equiv 0 \pmod{2^i}$. The union of the rim-hooks R_{k_i} misses at most one cell with content 0. Therefore we obtain $C_3(\lambda) = \sum_i C_3(R_i)$. Observe that

$$C_3(\lambda) \equiv C_3(\mathrm{core}_{2^{k_1+1}}(\lambda)) \pmod{2^{k_1+1}}.$$

So we obtain

$$C_3(\lambda) \equiv C_3(R_{k_1}) \equiv 2^{k_1-1}(2c_1^3 + 2^{k_1-1} + c_1 - 3c_1^2) \pmod{2^{k_1+1}}. \tag{7.14}$$

Also we have $C(\lambda) \equiv 2^{k_1-1}(2c_1 - 1 - 2^{k_2-k_1}) \pmod{2^{k_1+1}}$ and $2n \equiv 2\epsilon \pmod{2^{k_1+1}}$. Therefore we obtain

$$(2n-3)C(\lambda) \equiv 2^{k_1-1}(-2\epsilon - 6c_1 - 32^{k_1} + 3 + 32^{k_2-k_1}) \pmod{2^{k_1+1}}. \tag{7.15}$$

We calculate

$$\begin{aligned} 6\binom{n}{4} &= \frac{n(n-1)(n-2)(n-3)}{4} \\ &= (2^{k_1-1} + \cdots + 2^{k_r-1})(2^{k_1} + \cdots + 2^{k_r} - 1 + 2\epsilon) \\ &\quad (2^{k_1-1} + \cdots + 2^{k_r-1} - 1)(2^{k_1} + \cdots + 2^{k_r} - 3 + 2\epsilon). \end{aligned}$$

Therefore we obtain

$$6\binom{n}{4} \equiv 2^{k_1-1}(3\cdot 2^{k_1-1} - 3 - 3\cdot 2^{k_2-k_1}) \pmod{2^{k_1+1}}. \tag{7.16}$$

Using the congruence relations in Equations (7.14), (7.15), 7.16 we deduce

$$\begin{aligned} 6\binom{n}{4} - C_3(\lambda) + (2n-3)C(\lambda) &\equiv 2^{k_1-1}(3\cdot 2^{k_1-1} - 3 - 3^{k_2-k_1} - 2c_1^3 - 2^{k_1-1} - c_1 + 3c_1^2) \\ &\quad + 2^{k_1-1}(-2\epsilon - 6c_1 - 3\cdot 2^{k_1} + 3 + 3\cdot 2^{k_2-k_1}) \pmod{2^{k_1+1}} \\ &\equiv 2^{k_1-1}(c_1(3c_1 - 7 - 2c_1^2) - 2\epsilon) \pmod{2^{k_1+1}}. \end{aligned}$$

So for $h_\lambda \equiv 0 \pmod 2$, we need

$$v_2(c_1(3c_1 - 7 - 2c_1^2) - 2\epsilon) \geq 2. \tag{7.17}$$

Here we write down all the possible cases where the conditions 7.12 and 7.17 hold.

1. Let $\epsilon = 0$. Then for $k_2 = k_1 + 1$, we require $c_1 \equiv 0 \pmod 4$ and c_2 is even. For $k_2 > k_1 + 1$ the condition on c_1 is sufficient.

2. Let $\epsilon = 0$. Then for $k_2 = k_1 + 1$, we require $c_1 \equiv 1 \pmod 4$ and c_2 is odd. For $k_2 > k_1 + 1$ the condition on c_1 is sufficient.

First consider the cases where $k_2 = k_1 + 1$. For $\epsilon = 0$, the condition 7.17 holds if and only if $c_1 \equiv 0 \pmod 4$ and c_2 is even. For $\epsilon = 1$, the condition holds if and only if $c_1 \equiv 1 \pmod 4$ and c_2 is odd. If $k_2 > k_1 + 1$, the conditions on c_1 will suffice for both the cases.

Now we consider the case for $k_1 = 1$. Recall that for this case we require

$$v_2\left(6\binom{n}{4} - C_3(\lambda) + (2n-3)C(\lambda)\right) \geq k_2 + 1. \tag{7.18}$$

From similar calculation as above for $k_1 = 1$, we obtain

$$\begin{aligned} 6\binom{n}{4} - C_3(\lambda) + (2n-3)C(\lambda) \equiv{} & 2^{k_2-1}(-2 - 2\epsilon + c_2 - 2c_2^3 + 3c_2^2) \\ & + 4c_1\epsilon + 2\epsilon - c_1 - 2c_1^3 - 3c_1^2 \pmod{2^{k_2+1}}. \end{aligned}$$

For $\epsilon = 0$ from the first condition we get either $c_1 \equiv 0 \pmod 4$ or $c_1 \equiv 2 \pmod 4$. As $k_1 = 1$, the foot node content of a hook of size 2 is either 0 or 1. Therefore $c_1 = 0$. Putting $\epsilon = c_1 = 0$ the condition becomes $v_2(-2 + c_2 - 2c_2^3 + 3c_2^2) \geq 2$. This holds if and only if $c_2 \equiv 2 \pmod 4$ or $c_2 \equiv 1 \pmod 4$. Here we write down characterizations of spinorial partitions for different cases.

1. If $k_2 = 2$, then we require $c_1 = 0$ and $c_2 \equiv 2 \pmod 4$.

2. If $k_3 \geq 3$, then we require $c_1 = 0$ and $c_2 \equiv 2 \pmod 4$ or $c_2 \equiv 1 \pmod 4$.

Similarly for $\epsilon = 1$, we obtain $c_1 = 1$ and $c_2 \equiv 0 \pmod 4$ or $c_2 \equiv 3 \pmod 4$. Again we list down all possible cases.

1. If $k_2 = 2$, then we require $c_1 = 1$ and $c_2 \equiv 3 \pmod 4$.

2. If $k_2 \geq 3$, then we require $c_1 = 1$ and $c_2 \equiv 3 \pmod 4$ or $c_2 \equiv 0 \pmod 4$.

□

Corollary 7.2.2. *Let λ is an odd, achiral partition of n. Then λ is spinorial if and only if* $\mathrm{core}_{2^{k_2+1}}(\lambda)$ *is spinorial.*

The corollary follows directly from the previous theorem.

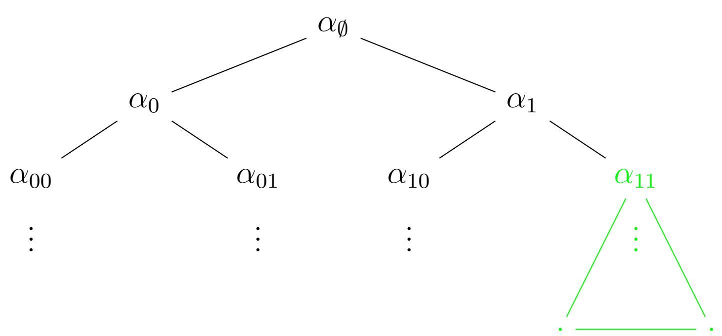
From Section 2.5 we obtain a nice description of odd partitions in terms of 2-core towers. It will be nice if we can provide a description of odd, achiral, spinorial partitions in terms of 2-core towers. Here we state a conjecture based on some observations. For an odd partition λ let x_{k_i} denote the binary sequence of length k_i denoting the position of the unique non-zero cell in the k_i-th row. We write $\bar{\epsilon} = 1 - \epsilon$.

Conjecture 7.2.3. *Let λ be an odd, achiral partition of n such that $\mathrm{core}_4(\lambda) = \emptyset$.*

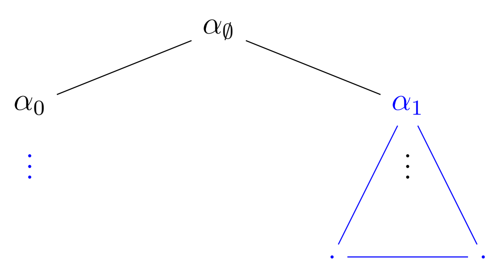
1. *Suppose $k_1 \geq 2$, and $k_2 = k_1 + 1$. Then λ is spinorial if and only if x_{k_1} begins with $\bar{\epsilon}\,\bar{\epsilon}$ and x_{k_2} begins with $\bar{\epsilon}$.*

2. *Suppose $k_1 \geq 2$, and $k_2 > k_1 + 1$. Then λ is spinorial if and only if x_{k_1} begins with $\bar{\epsilon}\,\bar{\epsilon}$.*

We present some of the cases pictorially for better understanding. Consider an odd, achiral, spinorial partition of n with $\mathrm{core}_4(\lambda) = \emptyset$. From 3rd row on wards the tower can be divided into four sub towers with upper vertex $\alpha_{\epsilon\delta}$, where $\epsilon, \delta \in \{0, 1\}$. If $k_1 \geq 2$ and $\epsilon = 0$, then the unique non-zero element in the k_1-st row will lie in the sub tower with upper vertex α_{11}. This particular sub tower is labeled as green in the following figure.



With the same restriction we obtain that the unique non-zero element in the k_2-nd row will lie in the sub tower with upper vertex α_1, labeled as blue.



The number of odd partitions of n is given by $A(n) = 2^{k_1+\cdots+k_r}$. (see Section 2.5 for reference.) Let $s_1(n)$ denote the number of irreducible, spinorial representations of S_n with odd dimension and trivial determinant. Using the previous theorem we determine $s_1(n)$ in the following result.

Theorem 7.2.4. *For $n \geq 4$, we have*

$$s_1(n) = \begin{cases} \frac{1}{8}A(n), & for\ k_2 = k_1 + 1, \\ \frac{1}{4}A(n), & for\ k_2 \geq k_1 + 2,\ \ or\ r = 1. \end{cases}$$

Proof. Let λ be an odd, achiral and spinorial partition of n. Write $\mu_i = \mathrm{core}_{2^{k_i+1}}(\lambda)$. Note that μ_i is an odd partition of $\epsilon + 2^{k_1} + \cdots + 2^{k_i}$. We write $n_1 = \epsilon + 2^{k_1} + 2^{k_2}$ and $n_2 = n - n_1 = 2^{k_3} + \cdots + 2^{k_r}$. Since $\mathrm{core}_{2^{k_1+1}}(\mu_2) = \mu_1$, from Corollary 2.6.4 it follows that μ_2 is an achiral partition of n_1. We first prove the result for μ_2.

The Young diagram $\mathcal{Y}(\mu_2)$ contains two hooks $H(a_i, b_i)$ of size 2^{k_i}, for $i \in \{1, 2\}$. Let d_2 denote the corner cell of the hooks $H(a_2, b_2)$. Also let c_i denote the foot node content of $H(a_i, b_i)$.

Take $k_1 \geq 2$. Note that μ_1 is a partition of $2^{k_1} + \epsilon$. The first lifting condition requires

$$c_1 \equiv \epsilon \pmod 4. \tag{7.19}$$

If $\epsilon = 0$ we have $\mu_1 = H(a_1, b_1)$ such that $a + b + 1 = 2^{k_1}$ and $c_1 = -b_1$. There are 2^{k_1} possible odd hooks $H(a_1, b_1)$, for $0 \leq b_1 \leq 2^{k_1} - 1$. So there will be 2^{k_1-2} possible hooks satisfying $c_1 = -b_1 \equiv 0 \pmod 4$.

If $\epsilon = 1$, then we obtain μ_1 is a partition of $2^{k_1} + 1$ of the form $H^+(a_1, b_1) = (a_1 + 1, 2, 1^{(b_1-1)})$. As $1 \leq a_1 \leq 2^{k_1} - 2$, the set of foot-node contents of the hook $H(a_1, b_1)$ of length 2^{k_1} is

$$S = \{1, -1, -2, \ldots, -2^{k_2} + 2, -2^{k_1}\}.$$

So we have $\#\{c_1 \mid c_1 \equiv x \pmod 4\} = 2^{k_1-2}$, where $x = \{0, 1, 2, 3\}$. Therefore there are 2^{k_1-2} cases such that $c_1 \equiv 1 \pmod 4$.

The third lifting condition requires $c_2 \equiv \epsilon \pmod 2$, if $k_2 = k_1 + 1$. The hand node content of the hook $H(a_2, b_2)$ is $c_2 + 2^{k_2} - 1$. From Lemma 2.6.1 it follows that for the spinorial partitions the non zero cell in the k_2-nd row of $T_2(\lambda)$ occupies the positions whose binary representations start with $1 - \epsilon$. So there are 2^{k_2-1} possibilities. Altogether we obtain

$$s_1(n_1) = \begin{cases} \frac{1}{8}A(n_1), & for\ k_2 = k_1 + 1, \\ \frac{1}{4}A(n_1), & for\ k_2 \geq k_1 + 2, \text{ or } r = 1. \end{cases}$$

For $k_1 = 1$, we have $\mu_2 = \text{core}_{2^{k_2+1}}(\lambda)$ is an odd partition of $\epsilon + 2 + 2^{k_2}$. Then $\mathcal{Y}(\mu_2)$ contains two hooks $H(a_i, b_i)$ of size 2^{k_i}, for $i \in \{1, 2\}$.

If $\epsilon = 0$, the first lifting condition requires $c_1 = 0$. Therefore we have $\nu = \text{core}_4(\lambda) = (2)$. This gives

$$\text{quo}_2(\nu) = (\emptyset, (1)). \tag{7.20}$$

For the hook $H(a_2, b_2)$ of size 2^{k_2} we have

$$a_2 + b_2 + 1 = 2^{k_2}. \tag{7.21}$$

We record all the partitions μ_2 which satisfy the conditions 7.20 and 7.21 for $0 \leq a_2 \leq 2^{k_2} - 1$. Since μ_2 contains a unique hook of size 2^{k_2}, we mention the corner cell and foot node content of it.

1. If $a_2 = 0$, then we obtain $\mu_2 = (2, 1^{(2^{k_2})})$, so that $d_2 = (2, 1)$. Therefore we calculate $c_2 = (1 - (2^{k_2} + 1)) = -2^{k_2}$.

2. If $a_2 = 1$, then we obtain $\mu_2 = (2, 2, 1^{(2^{k_2} - 2)})$, so that $d_2 = (2, 1)$. Therefore we calculate $c_2 = 1 - 2^{k_2}$.

3. For $2 \leq a_2 \leq 2^{k_2} - 2$, we obtain $\mu_2 = (a_2 + 1, 3, 1^{(b_2 - 1)})$, so that $d_2 = (1, 1)$. As a result we have $c_2 = (1 - (b_2 + 1)) = -b_2$.

4. For $a_2 = 2^{k_2} - 1$, we obtain $\mu_2 = (2 + 2^{k_2})$, so that $d_2 = (1, 3)$. As a result we have $c_2 = 3 - 1 = 2$.

From Equation (7.21) we obtain $b_2 = 2^{k_2} - a_2 - 1$. Therefore we have the set of foot node contents as

$$S = \{-2^{k_2}, -2^{k_2} + 1, -2^{k_2} + 3, -2^{k_2} + 4, \ldots, 1, 2\}.$$

Note that S does not contain the numbers 0 and $-2^{k_2} + 2$. Since $k_2 \geq 2$ we conclude

$$\#\{c_2 \mid c_2 \equiv x \pmod 4\} = 2^{k_2 - 2},$$

for $x \in \{0,1,2,3\}$. Now the second lifting condition requires $c_2 \equiv 2 \pmod 4$, if $k_2 = 2$. So there are 2^{k_2-2} possibilities. If $k_2 \geq 3$, we require $c_2 \equiv 2 \pmod 4$ or $c_2 \equiv 1 \pmod 4$. Therefore there are 2^{k_2-1} options.

For $\epsilon = 1$, the first condition requires $c_1 = 1$. Since $\mathrm{core}_2(\lambda) = 1$ we have

$$\nu' = \mathrm{core}_4(\lambda) = (3). \tag{7.22}$$

This gives $\mathrm{quo}_2(\nu') = ((1), \emptyset)$.

Again we record all the partitions which satisfy the conditions 7.21 and 7.22 for $0 \leq a_2 \leq 2^{k_2} - 1$. Since μ_2 contains a unique hook of size 2^{k_2}, we mention its corner cell and foot node content of it.

1. If $a_2 = 0$, then we obtain $\mu_2 = (3, 1^{(2^{k_2})})$, so that $d_2 = (1,2)$. Therefore we obtain $c_2 = (1 - (2^{k_2} + 1)) = -2^{k_2}$.

2. If $a_2 = 1, 2$, then we obtain $\mu_2 = (3, a_2+1, 1^{(2^{k_2}-a_2-1)})$, so that $d_2 = (1,2)$. Therefore we calculate $c_2 = 1 - 2^{k_2}, c_2 = (1 - (2^{k_2} - 1)) = 2 - 2^{k_2}$ for $a_2 = 1, 2$ respectively.

3. For $3 \leq a_2 \leq 2^{k_2} - 2$, we obtain $\mu_2 = (a_2 + 1, 4, 1^{(b_2-1)})$, so that $d_2 = (1,1)$. As a result we have $c_2 = (1 - (b_2 + 1)) = -b_2$.

4. For $a_2 = 2^{k_2} - 1$, we obtain $\mu_2 = (3 + 2^{k_2})$, so that $d_2 = (1,4)$. As a result we have $c_2 = 4 - 1 = 3$.

From Equation (7.21) we obtain $b_2 = 2^{k_2} - a_2 - 1$. Therefore the set of foot node contents is

$$S_1 = \{-2^{k_2}, -2^{k_2} + 1, -2^{k_2} + 2, -2^{k_2} + 4, \ldots, 2, 1, 3\}.$$

Note that S does not contain the numbers 0 and $-2^{k_2} + 3$. Since $k_2 \geq 2$ we conclude

$$\#\{c_2 \mid c_2 \equiv x \pmod 4\} = 2^{k_2-2},$$

for $x \in \{0,1,2,3\}$. Now the second lifting condition requires $c_2 \equiv 3 \pmod 4$, if $k_2 = 2$. So there are 2^{k_2-2} possibilities. If $k_2 \geq 3$, we require $c_2 \equiv 3 \pmod 4$ or $c_2 \equiv 0 \pmod 4$. Therefore there are 2^{k_2-1} possibilities.

Therefore we obtain the number of odd, achiral, spinorial partitions of n_1 as:

$$s_1(n_1) = \begin{cases} \frac{1}{8}A(n_1), & for\ k_2 = k_1 + 1, \\ \frac{1}{4}A(n_1), & for\ k_2 \geq k_1 + 2, \text{ or } r = 1. \end{cases}$$

From Corollary 7.2.2 it follows that λ is spinorial if and only if μ_2 is spinorial. Note that the unique non-zero entry in the k_i-th row of $T_2(\lambda)$,for $i \geq k_3$ can occur at any one of the 2^{k_i} possible places. Therefore we have

$$s_1(n) = \begin{cases} \frac{1}{8}A(n), & for\ k_2 = k_1 + 1, \\ \frac{1}{4}A(n), & for\ k_2 \geq k_1 + 2, \text{ or } r = 1. \end{cases}$$

□

Corollary 7.2.5. *Let $n \geq 2$ be a power of 2. Then an odd, achiral partition of n is spinorial if and only if it is a hook of the form $H(a, b)$ such that $b \equiv 0 \pmod 4$.*

The corollary follows directly from Theorem 7.2.1.

7.3 Case of Odd Partitions of 2^k

The problem of enumerating the chiral odd spinorial partitions seems dif and only ificult in general. In this and the following section we solve this problem in the special case when n is of the form $2^k + \epsilon$. When $\epsilon = 0$, a partition of n is odd precisely when it is a hook. For any integer n, we write $\mathrm{Od}(n) = \frac{n}{2^{v_2(n)}}$.

Theorem 7.3.1. *let $n \geq 8$ be a power of 2. Then a partition of n is odd, chiral and spinorial if and only if it is a hook of the form $H(a, b)$ with $a > b$ and $b \equiv 3 \pmod 4$. In particular the number of odd, chiral, spinorial partitions of n is $n/8$.*

Proof. For a chiral partition λ the first lifting condition is $g_\lambda \equiv 3 \pmod 4$. An equivalent condition is $g_\lambda + 1 \equiv 0 \pmod 4$. We calculate

$$g_\lambda + 1 = \frac{f_\lambda\left(\binom{n}{2} - C(\lambda)\right) + 2\binom{n}{2}}{2\binom{n}{2}}.$$

In terms of 2-adic valuation the first lifting condition requires

$$v_2\left(f_\lambda\left(\binom{n}{2}-C(\lambda)\right)+2\binom{n}{2}\right)\geq k+2. \tag{7.23}$$

The fact $v=v_2(f_\lambda)=0$ ensures that λ is a hook of length 2^k. Let us denote the hook by $H(a,b)=(a+1,1^b)$, such that $a+b+1=2^k$.The foot node content of the hook is $-b$. Then the contents of the others cells in the hook are $-b+1,-b+2,\ldots,-b+2^k-1$. Adding all this up we obtain $C(\lambda)=-2^k\cdot b+\binom{n}{2}$. Also $f_\lambda=\binom{a+b}{b}$. Using these values we derive

$$\begin{aligned} f_\lambda\left(\binom{n}{2}-C(\lambda)\right)+2\binom{n}{2} &= \binom{a+b}{b}\cdot 2^k\cdot b+2\binom{n}{2} \\ &= \binom{a+b}{b}\cdot 2^k\cdot b+2^k(2^k-1) \\ &= 2^k\left(2^k-1+\binom{a+b}{b}\cdot b\right). \end{aligned}$$

So we need to check whether $v_2(1-b\cdot\binom{a+b}{b})\geq 2$. This holds if and only if $b\binom{a+b}{b}\equiv 1 \pmod 4$. Note that the condition implies that b is odd. Since $\lambda=H(a,b)$ we have $a+b=2^k-1$. So it follows that a is even. From Lemma 7.3.3 below we obtain $b\binom{a+b}{b}\equiv b(-1)^{\min(a,b)} \pmod 4$. Therefore the first lifting condition requires

$$b(-1)^{\min(a,b)}\equiv 1 \pmod 4. \tag{7.24}$$

1. If $a>b$, the condition 7.24 becomes $b(-1)^b\equiv 1 \pmod 4$. Therefore we require

$$b\equiv 3 \pmod 4 \quad \text{if } a>b. \tag{7.25}$$

   In this case we have $1\leq b\leq 2^{k-1}-1$. So there are 2^{k-3} possibilities for b.

2. If $a<b$, the condition 7.24 becomes $b(-1)^a\equiv 1 \pmod 4$. Therefore we require

$$b\equiv 1 \pmod 4 \quad \text{if } a<b. \tag{7.26}$$

   In this case we have $2^{k-1}+1\leq b<2^k$. So there are 2^{k-3} possibilities for b.

The third lifting condition for an achiral partition λ is $h_\lambda \equiv 1 \pmod 2$. Here we use the congruence equality of h_λ.

$$h_\lambda \equiv \frac{f_\lambda(6\binom{n}{4} - (C_3(\lambda) - (2n-3)C(\lambda)))}{12\binom{n}{4}} \pmod 2.$$

For $n = 2^k$, we have

$$12\binom{n}{4} = 2^k(2^k-1)(2^{k-1}-1)(2^k-3).$$

So for $k \geq 3$, we have $v_2\left(12\binom{n}{4}\right) = k$. Since f_λ is odd, for h_λ odd we require

$$v_2\left(6\binom{n}{4} - (C_3(\lambda) - (2n-3)C(\lambda))\right) = 1. \tag{7.27}$$

For $\lambda = H(a, b)$, we compute

$$\begin{aligned}
C_3(\lambda) &= (-b)^3 + (-b+1)^3 + (-b+2)^3 + \cdots + (-b+n-1)^3 \\
&= -b^3 + (1^3 + 2^3 + \cdots + (n-1)^3) - 3b(1^2 + 2^2 + \cdots + (n-1)^2) \\
&+ 3b^2(1 + 2 + \cdots + n - 1) \\
&= -nb^3 + \left(\frac{n(n-1)}{2}\right)^2 - 3b\left(\frac{n(n-1)(2n-1)}{6}\right) + 3b^2\frac{n(n-1)}{2} \\
&= -nb^3 + \frac{n^4 + n^2 - 2n^3}{4} - \frac{b}{2}\cdot(2n^3 - 3n^2 + n) + \frac{3}{2}\cdot b^2(n^2 - n).
\end{aligned}$$

Since $n = 2^k$, $k \geq 3$ then we have

$$C_3(\lambda) \equiv -nb^3 - \frac{1}{2}bn - \frac{3}{2}b^2n \pmod{2^{k+1}}. \tag{7.28}$$

We compute

$$\begin{aligned}
6\binom{n}{4} &= 6 \cdot \frac{n(n-1)(n-2)(n-3)}{24} \\
&= \frac{n(n-1)(n-2)(n-3)}{4} \\
&= \frac{n^4 - 6n^3 + 11n^2 - 6n}{4}.
\end{aligned}$$

Since $n = 2^k$, we conclude that

$$6\binom{n}{4} \equiv -\frac{3}{2}n \pmod{2^{k+1}}. \tag{7.29}$$

Again we have

$$\begin{aligned}(2n-3)C(\lambda) &= (2n-3)\cdot(-nb + \frac{1}{2}\cdot(n^2-n))\\ &= -2n^2b + (n^3-n^2) + 3nb - \frac{3}{2}\cdot(n^2-n),\end{aligned}$$

which gives

$$(2n-3)C(\lambda) \equiv 3nb + \frac{3}{2}n \pmod{2^{k+1}}. \tag{7.30}$$

Using Equations (7.28), (7.29), 7.30 we obtain

$$\begin{aligned}6\binom{n}{4} - (C_3(\lambda) - (2n-3)C(\lambda)) &= -\frac{3}{2}n + nb^3 + \frac{1}{2}bn + \frac{3}{2}b^2n + 3nb + \frac{3}{2}n\\ &= \frac{7}{2}bn + \frac{3}{2}b^2n + nb^3\\ &\equiv \frac{n}{2}\cdot b(7+3b+2b^2) \pmod{2^{k+1}}.\end{aligned}$$

Therefore the condition 7.27 holds if and only if

$$b(7+3b+2b^2) \equiv 2 \pmod{4}. \tag{7.31}$$

1. If $a > b$, the condition 7.25 holds if and only if $b \equiv 3 \pmod{4}$. For such b we obtain $b(7+3b+2b^2) \equiv 2 \pmod{4}$. So the condition 7.31 holds. As a result the partitions $\lambda = H(a,b)$, for $a > b$ and $b \equiv 3 \pmod{4}$ are spinorial. In this case the foot node content of $H(a,b)$ is $-b \equiv 1 \pmod{4}$.

2. If $a < b$, the condition 7.25 holds if $b \equiv 1 \pmod{4}$. But then the condition 7.31 does not hold. So the partitions $\lambda = H(a,b)$, for $a < b$ and $b \equiv 1 \pmod{4}$ are aspinorial. In this case the foot node content of $H(a,b)$ is $-b \equiv 3 \pmod{4}$.

Therefore we arrive at the conclusion that an odd, chiral partition $\lambda = H(a,b)$ is spinorial if and only if

$$a > b \quad \text{and} \quad b \equiv 3 \pmod{4}.$$

Note that the conditions $a > b$ and $a + b + 1 = 2^k$ implies $1 \leq b \leq 2^{k-1} - 1$. This gives

$$\#\{b \mid 1 \leq b \leq 2^{k-1} - 1,\ b \equiv 3 \pmod 4\} = 2^{k-3}.$$

So the number of odd, chiral, spinorial partitions of 2^k, for $k \geq 3$, is 2^{k-3}. $\square$

Lemma 7.3.2. *For $n = 2^k$, where $k \geq 2$, we have*

$$Od(n!) \equiv 3 \pmod 4.$$

Proof. Any odd number is congruent to 1 or -1 mod 4. Here we count the number of odd numbers in $n!$ which are -1 mod 4.

The odd numbers occurring in $n!$ are

$$1, 3, 5, 7, \ldots, 2^k - 1.$$

So there are even number of terms which are -1 mod 4. (in fact there are 2^{k-2} of them). Now consider the numbers with 2-valuation 1. The odd parts of them will be the numbers

$$1, 3, 5, \ldots, 2^{k-1} - 1.$$

There are 2^{k-2} such numbers and 2^{k-3} of them are -1 mod 4.

Similarly in the cases with 2-valuation less than or equal to 2^{k-3}, the product of the odd parts are 1 mod 4. There are only two numbers with 2-valuation 2^{k-2}, namely 2^{k-2}, $3 \cdot 2^{k-2}$. Then the product of there odd parts will be -1 mod 4. Finally, there is one number with 2-valuation 2^{k-1} and one number with 2-valuation 2^k. Hence the result follows.

$\square$

Lemma 7.3.3. *If $a + b + 1 = 2^k, k \geq 1$, then*

$$\binom{a+b}{b} \equiv (-1)^{\min(a,b)} \pmod 4. \tag{7.32}$$

Proof. For any number n of the form

$$n = 2^{k_0} + 2^{k_1} + \cdots + 2^{k_r},\ 0 \le k_0 < k_1 < \cdots < k_r,$$

we define $\mathrm{bin}(n) = \{k_0, k_1, \ldots, k_r\}$. If $a + b + 1 = 2^k$, then $\mathrm{bin}(a) \subseteq \mathrm{bin}(2^k - 1)$. So the binomial coefficient $\binom{a+b}{a}$ is odd. Without loss of generality we consider $a > b$. Therefore we have $1 \le b \le 2^{k-1} - 1$. Now we calculate

$$\begin{aligned}\binom{a+b}{b} &= \frac{(2^k-1)(2^k-2)\cdots(2^k-b)}{1\cdot 2\cdots b}\\ &= \frac{2^k-1}{1}\cdot\frac{2^k-2}{2}\cdots\frac{2^k-b}{b}.\end{aligned}$$

We prove the claim for $k \ge 2$ first. For any odd number $d \le b$,

$$\frac{2^k-d}{d} \equiv -1 \pmod 4.$$

For any even number $e \le b$ write $e = 2^i \cdot e_1$, where $e_1 = \mathrm{Od}(e)$ and $i < k-1$. If $k - i = 1$, then $e = 2^{k-1}\cdot e_1 \ge 2^{k-1}$ which violates our assumption. So

$$\frac{2^k-e}{e} = \frac{2^{k-i}-e_1}{e_1} \equiv -1 \pmod 4.$$

Since there are altogether b terms and $a < b$, we have

$$\binom{a+b}{b} \equiv (-1)^b \pmod 4.$$

For $k = 1$, we have $a = 1, b = 0$. So the result follows.

□

Here is an interesting consequence of Lemma 7.3.3, although we don't use it anywhere else. For any $n \ge 1$ we denote by $A_1(n)$ (resp. $A_3(n)$) the number of partitions of n such that $f_\lambda \equiv 1 \pmod 4$ (resp. $f_\lambda \equiv 3 \pmod 4$).

Theorem 7.3.4. *If $n \ge 2$ is a power of 2, we have $A_1(n) = A_3(n) = n/2$.*

Proof. The odd partitions of n, where n is a power of 2, are of the form $H(a, b) =$

$(a+1, 1^b)$. So that $a+b+1 = 2^k$, for some $k \geq 1$. From Lemma 7.3.3 we have

$$f_{H(a,b)} = \binom{a+b}{b} \equiv (-1)^{\min(a,b)} \pmod 4. \tag{7.33}$$

When $b < a$, we obtain $0 \leq b \leq 2^{k-1} - 1$. Now using 7.33 we deduce

$$f_{H(a,b)} \equiv \begin{cases} 1, & \text{for } b \text{ even}, \\ -1, & \text{for } b \text{ odd}. \end{cases}$$

So for half of the cases $f_{H(a,b)} \equiv 1 \pmod 4$. On the other hand when $a < b$, we have $0 \leq a \leq 2^{k-1} - 1$. Similar argument shows that again for the half of the cases $f_{H(a,b)} \equiv 1 \pmod 4$. So the result follows.

□

7.4 Case of Odd Partitions of $2^k + 1$

Write $H^+(a,b)$, $a, b > 0$, to denote the partition $(a+1, 2, 1^{(b-1)})$ of $2^k + 1$, so that $a + b + 1 = 2^k$. Note that the relation shows that a and b are of dif and only iferent parity.

Theorem 7.4.1. *Let n be of the form $2^k + 1, k \geq 3$. Then a partition of n is odd, chiral and spinorial if and only if it is of the form $H^+(a,b)$ with $b > a$, $b \equiv 0 \pmod 4$ and $v_2(b) \leq k-2$. In particular there are $2^{k-3} - 1$ odd, chiral, spinorial partitions of n.*

Proof. Let λ be an odd partition of $2^k + 1$. If λ is also chiral the first lifting condition requires $g_\lambda \equiv 3 \pmod 4$. An equivalent condition is $g_\lambda + 1 \equiv 0 \pmod 4$. Therefore as in Equation (7.23) we obtain the condition

$$v_2\left(f_\lambda\left(\binom{n}{2} - C(\lambda)\right) + 2\binom{n}{2}\right) \geq k+2. \tag{7.34}$$

From Lemma 2.5.3 it follows that λ contains a unique hook of size 2^k. Then the possible forms of λ are

$$H^+(a,b), \text{ for } a, b > 0, \quad (2^k + 1), \quad (1^{(2^k+1)}).$$

Let λ be of the form $H^+(a,b)$, $a,b>0$. For convenience in calculation we take the foot node content of the unique hook of size 2^k as $c+1$, for $c \in \mathbb{Z}$. Note that $c+1=-b$. Then the contents of the other nodes in the corresponding rim-hook are $c+2,\ldots,c+2^k$. Observe that the rim-hook only misses the cell $(2,2)$, which has content 0. Therefore we calculate

$$C(\lambda) = 2^k c + \binom{n}{2}. \tag{7.35}$$

Then one calculates

$$\begin{aligned} f_\lambda\left(\binom{n}{2} - C(\lambda)\right) + 2\binom{n}{2} &= -f_{H^+(a,b)}2^k \cdot c + 2^k(2^k+1) \\ &= 2^k(2^k+1-f_{H^+(a,b)} \cdot c). \end{aligned}$$

Therefore the condition in Equation (7.34) boils down to $v_2(1-f_{H^+(a,b)} \cdot c) \geq 2$. This holds if and only if $c \cdot f_{H^+(a,b)} \equiv 1 \pmod 4$. Putting $c=-(b+1)$ we obtain the condition as

$$(b+1) \cdot f_{H^+(a,b)} \equiv 3 \pmod 4. \tag{7.36}$$

The condition requires b to be even. We now consider dif and only iferent possibilities. Note that $a+b+1=2^k$.

1. If $b \equiv 2 \pmod 4$, then $a \equiv 1 \pmod 4$. From Lemma 7.4.2 below we conclude

$$(b+1) \cdot f_{H^+(a,b)} \equiv \begin{cases} (-1)^{\min(a,b)+1} \pmod 4, & \text{for } k \geq 3, \\ (-1)^{\min(a,b)} \pmod 4. & \text{for } k=2, \end{cases}$$

   Here we have $i=v_2(b)=1$. Following the assumption in the theorem we ignore the case for $k=2$. Now for the condition 7.36 we require $b<a$.

2. If $b \equiv 0 \pmod 4$, then $a \equiv 3 \pmod 4$. Then from Lemma 7.4.2 below we obtain

$$(b+1) \cdot f_{H^+(a,b)} \equiv \begin{cases} (-1)^{\min(a,b)} \pmod 4, & \text{for } k \geq i+2, \\ (-1)^{\min(a,b)+1} \pmod 4, & \text{for } k=i+1, \end{cases}$$

where $i = v_2(b) \geq 2$. For the condition 7.36 we need

$$a < b, \text{ for } k \geq i+2, \tag{7.37}$$

$$b < a, \text{ for } k = i+1. \tag{7.38}$$

We have

$$h_\lambda \equiv \frac{f_\lambda(6\binom{n}{4} - (C_3(\lambda) - (2n-3)C(\lambda)))}{12\binom{n}{4}} \pmod 2.$$

For $n = 2^k, k \geq 3$, we obtain $v_2\left(12\binom{n}{4}\right) = k$. Since f_λ is odd, for h_λ to be odd we require

$$v_2\left(6\binom{n}{4} - (C_3(\lambda) - (2n-3)C(\lambda))\right) = k. \tag{7.39}$$

We calculate

$$\begin{aligned} C_3(\lambda) &= \sum_{i=1}^{2^k}(c+i)^3 \\ &= 2^k c^3 + 2^{2k-2}(2^k+1)^2 + 3c^2 \cdot 2^{k-1}(2^k+1) + c2^{k-1}(2^k+1)(2^{k+1}+1) \\ &= 2^k c^3 + 2^{2k-2}(2^k+1)^2 + 2^{k-1}(2^k+1)c(3c+1+2^{k+1}). \end{aligned}$$

As a result we obtain

$$C_3(\lambda) \equiv 2^k \cdot c^3 + 3c^2 \cdot 2^{k-1} + c \cdot 2^{k-1} \pmod{2^{k+1}}. \tag{7.40}$$

Similarly we calculate $C(\lambda) = 2^k c + 2^{k-1}(2^k+1) \equiv 2^k c + 2^{k-1} \pmod{2^{k+1}}$. Therefore we obtain

$$(2n-3)C(\lambda) \equiv -2^k c - 2^{k-1} \pmod{2^{k+1}}. \tag{7.41}$$

Also for $k \geq 3$, we have

$$6\binom{2^k+1}{4} \equiv 2^{k-1} \pmod{2^{k+1}}. \tag{7.42}$$

Putting all the congruence relations in Equations (7.40), (7.41), (7.42) we derive

$$\begin{aligned} 6\binom{n}{4} - (C_3(\lambda) - (2n-3)C(\lambda)) &\equiv 2^{k-1} - 2^k \cdot c^3 - 3c^2 \cdot 2^{k-1} - c \cdot 2^{k-1} - 2^k \cdot c - 2^{k-1} \pmod{2^{k+1}} \\ &\equiv 2^{k-1}\left(-2c^3 - 3c^2 - 3c\right) \pmod{2^{k+1}}. \end{aligned}$$

Therefore the condition 7.39 holds if $(-2c^3 - 3c^2 - 3c) \equiv 2 \pmod 4$. Putting $c = -(b+1)$ we derive

$$\begin{aligned} -2c^3 - 3c^2 - 3c &= 3(b+1) + 2(b+1)^2 - 3(b+1)^2 \\ &= 2 + 3b + 3b^2 + 2b^3 \\ &= 2 + b(3 + 3b + 2b^2). \end{aligned}$$

So for the condition 7.39 we require

$$b(3 + 3b + 2b^2) \equiv 0 \pmod 4. \tag{7.43}$$

From the first lifting condition we have b is even. Elementary calculation shows that the condition 7.43 holds if and only if $b \equiv 0 \pmod 4$. From 7.37 we obtain that if $i = k-1$, then we need $b < a$. Since $1 \le b \le 2^k - 2$, the only possibility for b is 2^{k-1}. But then $a = 2^k - 2^{k-1} - 1 = 2^{k-1} - 1 < b$. So for $b = 2^{k-1}$, $H^+(a,b)$ is not spinorial. For $i \le k-2$, we require $a < b$. So an odd, chiral partition λ of 2^k+1, $k \ge 3$, of the form $H^+(a,b)$ is spinorial if and only if

$$b \equiv 0 \pmod 4 \quad \text{and} \quad a < b, \quad i \le k-2.$$

Since $a + b + 1 = 2^k$, the conditions $a < b$ and $i \le k-2$ gives $2^{k-1} + 1 \le b \le 2^k - 2$. So we have

$$\#\{b \mid 2^{k-1} + 1 \le b \le 2^k - 2, b \equiv 0 \pmod 4\} = 2^{k-3} - 1.$$

So there are $2^{k-3} - 1$ odd, chiral, spinorial partitions of the form $H^+(a,b)$.

The other two possibilities for λ are (2^k+1) and $(1^{(2^k+1)})$. The partition (2^k+1) corresponds to the trivial representation which is achiral. The partition $(1^{(2^k+1)})$ corresponds to the sign representation of S_n which is aspinorial (see Proposition 5.5.1). Hence the result follows.

□

Lemma 7.4.2. *For $k \geq 2$, $a, b > 0$, we have*

$$f_{H^+(a,b)} \equiv \begin{cases} (-1)^{\min(a,b)} \pmod 4, & \text{for } i \leq k-2, \\ (-1)^{\min(a,b)+1} \pmod 4, & \text{for } i = k-1, \end{cases}$$

where $i = v_2(a)$ if a is even, or $i = v_2(b)$ if b is even.

We illustrate the result with a couple of examples. Consider the partition $H^+(6,1) = (7,2)$. Here $a = 6$ and $b = 1$. Since a is even, we have $i = v_2(6) = 1$. Since $a+b+2 = 9 = 2^3+1$, we have $k = 3$. So we have $i = k-2$. Therefore from Theorem 7.4.2 we obtain

$$f_{H^+(6,1)} \equiv (-1)^{\min(6,1)} \equiv -1 \pmod 4.$$

Here we draw the Young diagram $\mathcal{Y}(7,2)$ with each of its nodes filled by its hooklength.

8	7	5	4	3	2	1
2	1					

Using hooklength formula (see Equation (2.1) for reference) we obtain $f_{(7,2)} = 27 \equiv 3 \pmod 4$. So it matches with the result in the theorem.

Next take the partition $H^+(3,4) = (4,2,1^3)$. The Young diagram $\mathcal{Y}(4,2,1^3)$ looks like as below.

8	4	2	1
5	1		
3			
2			
1			

Here the cells of the Young diagram are filled with their hooklengths. Here we have $i = v_2(4) = 2$. Since $H^+(3,4)$ is a partition of 9, we have $k = 3$, so that $i = k-1$. Therefore from Theorem 7.4.2 we calculate

$$f_{H^+(3,4)} \equiv (-1)^{\min(3,4)+1} \equiv 1 \pmod 4.$$

From the hooklength formula we derive the same conclusion, i.e. $f_{(4,2,1^3)} = 189 \equiv 1 \pmod 4$.

Now we prove the theorem.

Proof. If λ is an odd partition of 2^k+1 then from Lemma 2.5.3 it contains a unique hook of length 2^k. Therefore λ must be of the form $H^+(a,b) = (a+1, 2, 1^{(b-1)})$, where $a+b+1 = 2^k$. From the hooklength formula we calculate

$$\begin{aligned} f_{H^+(a,b)} &= \frac{(a+b+2)!}{(a-1)!(b-1)!(a+b+1)(a+1)(b+1)} \\ &= \frac{ab(a+b+2)(a+b)!}{(a+1)(b+1)a!b!} \\ &= \frac{ab(2^k+1)}{(a+1)(b+1)}\binom{a+b}{b}. \end{aligned}$$

Here we consider dif and only iferent possible cases. We use the relation $a+b+1 = 2^k$, repeatedly to draw conclusions.

1. If $a \equiv 0 \pmod 4$, then $b \equiv 3 \pmod 4$. Write $a = 2^i \operatorname{Od}(a)$, where $i \geq 2$. Putting $b+1 = 2^k - a$ we calculate

$$\begin{aligned} f_{H^+(a,b)} &= \frac{2^i \cdot \operatorname{Od}(a)(2^k-a-1)(2^k+1)}{(a+1)\cdot \operatorname{Od}(2^k-a)}\binom{a+b}{b} \\ &\equiv \frac{\operatorname{Od}(a)(-1)}{(2^{k-i}-\operatorname{Od}(a))}(-1)^{\min(a,b)} \pmod 4. \end{aligned}$$

   Here we used the facts that $a+1 \equiv 1 \pmod 4$. If $k-i \geq 2$, then $\frac{\operatorname{Od}(a)}{(2^{k-i}-\operatorname{Od}(a))} \equiv -1 \pmod 4$. If $k-i = 1$, then $\frac{\operatorname{Od}(a)}{(2^{k-i}-\operatorname{Od}(a))} \equiv 1 \pmod 4$. This gives

$$f_{H^+(a,b)} \equiv \begin{cases} (-1)^{\min(a,b)} \pmod 4, & \text{for } k-i \geq 2, \\ (-1)^{\min(a,b)+1} \pmod 4, & \text{for } k-i = 1. \end{cases}$$

2. If $a \equiv 2 \pmod 4$, then $b \equiv 1 \pmod 4$. Write $a = 2 \cdot \operatorname{Od}(a)$.

$$f_{H^+(a,b)} \equiv \frac{\operatorname{Od}(a)}{(-1)(2^{k-1}-\operatorname{Od}(a))}(-1)^{\min(a,b)} \pmod 4.$$

   Here we used the fact that $a+1 \equiv -1 \pmod 4$. If $k-1 \geq 2$, i.e $k \geq 3$, then $\frac{\operatorname{Od}(a)}{(2^{k-1}-\operatorname{Od}(a))} \equiv -1 \pmod 4$. If $k-1 = 1$, i.e. $k = 2$, then $\frac{\operatorname{Od}(a)}{(2^{k-1}-\operatorname{Od}(a))} \equiv 1 \pmod 4$. This gives

$$f_{H^+(a,b)} \equiv \begin{cases} (-1)^{\min(a,b)} \pmod 4, \text{ for } k \geq 3, \\ (-1)^{\min(a,b)+1} \pmod 4, \text{ for } k = 2. \end{cases}$$

3. If $a \equiv 3 \pmod 4$, then $b \equiv 0 \pmod 4$. Write $b = 2^i \cdot \mathrm{Od}(b)$, where $i \geq 2$. Putting $a + 1 = 2^k - b$ we calculate

$$f_{H^+(a,b)} \equiv \frac{\mathrm{Od}(b)(-1)}{(2^{k-i} - \mathrm{Od}(b))}(-1)^{\min(a,b)} \pmod 4.$$

Here we used the facts that $b+1 \equiv 1 \pmod 4$. If $k - i \geq 2$, then $\frac{\mathrm{Od}(b)}{(2^{k-i} - \mathrm{Od}(b))} \equiv -1 \pmod 4$. If $k - i = 1$, then $\frac{\mathrm{Od}(b)}{(2^{k-i} - \mathrm{Od}(b))} \equiv 1 \pmod 4$. This gives

$$f_{H^+(a,b)} \equiv \begin{cases} (-1)^{\min(a,b)} \pmod 4, \text{ for } k - i \geq 2, \\ (-1)^{\min(a,b)+1} \pmod 4, \text{ for } k - i = 1. \end{cases}$$

4. If $a \equiv 1 \pmod 4$, then $b \equiv 2 \pmod 4$. Taking $b = 2 \cdot \mathrm{Od}(b)$ and $a + 1 = 2^k - b$, and following a similar argument as before we obtain

$$f_{H^+(a,b)} \equiv \begin{cases} (-1)^{\min(a,b)} \pmod 4, \text{ for } k \geq 3, \\ (-1)^{\min(a,b)+1} \pmod 4, \text{ for } k = 2. \end{cases}$$

$\square$

As a summary of the counting results we present a Venn diagram showing the number of odd, spinorial, chiral partitions of $n = 2^k + \epsilon$.

- The number of odd partitions of n is $A(n) = 2^k$ as obtained from [17].
- The total number of chiral partitions of n, $b(n)$ can be obtained from the paper [3].
- From the same paper it follows that the number of odd, chiral partitions is $A(n)/2 = 2^{k-1}$. As a result the number of odd, achiral partitions of n is $A(n)/2 = 2^{k-1}$.
- The portion labeled as $s_1(n)$ denotes the set of odd, achiral, spinorial representations of n. From Section 7.2 we have $s_1(n) = 2^{k-2}$.
- The portion labeled as $s_2(n)$ denotes the set of odd, chiral, spinorial representations of n. From Sections 7.3 and 7.4 we obtain $s_2(n) = 2^{k-3}$, if $n = 2^k$ and $s_2(n) = 2^{k-3} - 1$, if $n = 2^k + 1$.
- The portion labeled as $s_3(n)$ denotes the set of even, chiral, spinorial partitions of n. The counting for this portion is not known.
- The portion labeled as $s_4(n)$ denotes the set of even, achiral, spinorial partitions of n.The counting for this portion is not known.

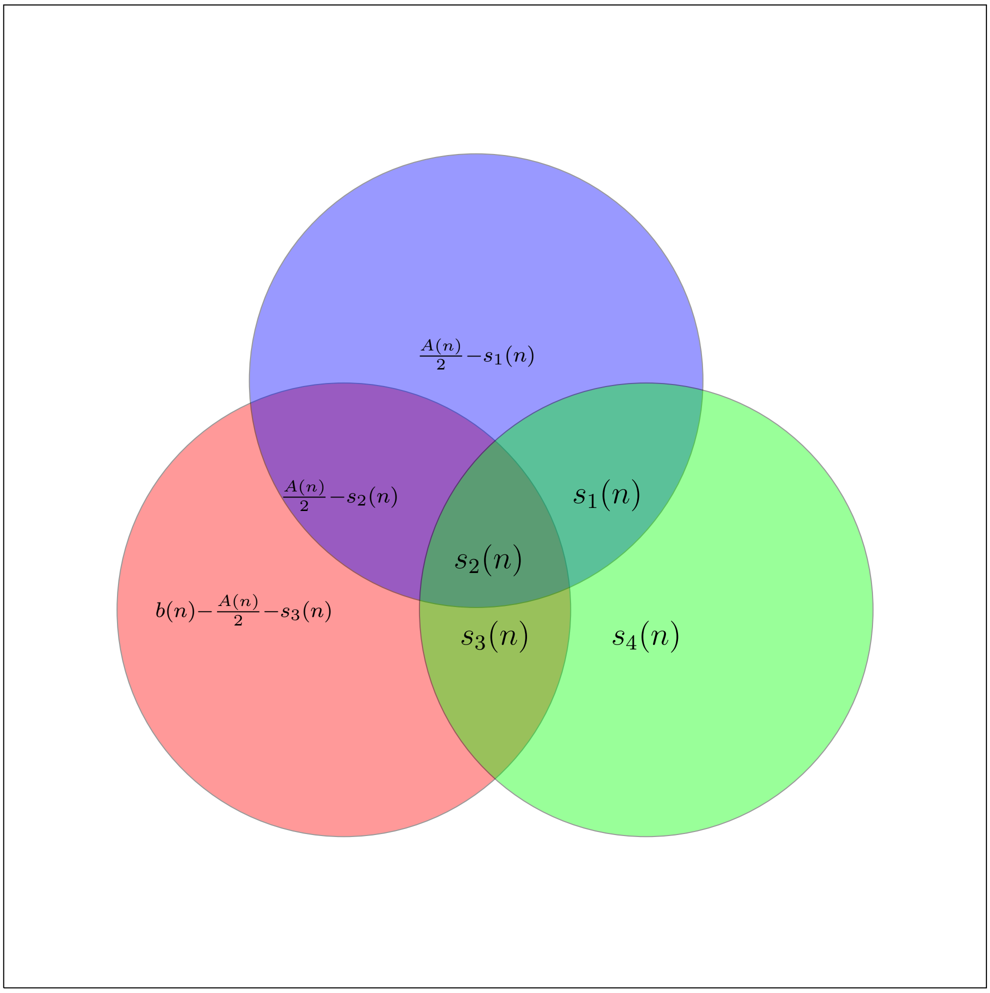


Figure 7.1: Venn diagram showing the number of odd, chiral and spinorial partitions of $n = 2^k + \epsilon$. The circle labeled in blue denotes the odd partitions of n. The circle labeled in red denotes the chiral partitions of n, whereas the circle labeled in green denotes the spinorial partitions of n.

7.5 Case of Self-Conjugate Partitions

For a self-conjugate partition λ, f_λ is even unless $\lambda = (1)$. We write $v = v_2(f_\lambda)$. Let λ' denote the conjugate of a partition λ. Then from [21, Theorem 4.4.2] we obtain $V_{\lambda'} \cong V_\lambda \otimes \epsilon$. Consequently we have

$$\chi_{\lambda'}(\mu) = \epsilon(\mu)\chi_\lambda(\mu). \tag{7.44}$$

Theorem 7.5.1. *Let λ be a self-conjugate partition of n. If $v \geq 3$ then λ is spinorial. If $v = 2$ then λ is aspinorial.*

Proof. Note that if μ denotes an odd cycle type then from Equation (7.44) we have $\chi_\lambda(\mu) = 0$. This yields

$$\chi_\lambda(s_1) = \chi_\lambda(\zeta_4) = 0. \tag{7.45}$$

From lemma 3.1.12 we obtain $\chi_\lambda(s_1 s_3) \equiv f_\lambda \pmod 4$. Putting these values in Theorem 3.2.3 and Equation (3.1), we deduce

$$g_\lambda = f_\lambda/2 \quad \text{and} \quad h_\lambda \equiv f_\lambda/2 \pmod 2. \tag{7.46}$$

For a partition λ if $v \geq 3$ then from 3.1.7 we conclude that λ is always achiral and spinorial. But if $v = 2$, then λ is achiral and aspinorial as $g_\lambda \equiv 2 \pmod 4$. $\square$

Recall from Section 2.4 that the hook with the corner cell (i, j) in $\mathcal{Y}(\lambda)$ is denoted by $H_{(i,j)}$. We write $|H_{(i,j)}| = h_{(i,j)}$.

Lemma 7.5.2. *Let λ be a self-conjugate partition of $2^k + \epsilon$. Suppose $\mathcal{Y}(\lambda)$ contains two hooks of size 2^{k-1}, H_a and H_b, where a, b denotes the corner cells of the hooks. Then a and b occupy the positions $(1, 2)$ and $(2, 1)$.*

Proof. Let (i, j) and (i', j') denote two nodes in $\mathcal{Y}(\lambda)$, such that $(i, j) < (i', j')$ with respect to lexicographic order, i.e, either $i < i'$, or if $i = i'$, then $j < j'$. Then it is easy to check that $h_{(i,j)} > h_{(i',j')}$. Neither a nor b occupy the $(1, 1)$th position. This is because if a occurs in the $(1, 1)$-th cell then $a < b$ with respect to the lexicographic order. So we obtain $h_a > h_b$, which is a contradiction. Suppose a does not occupy either of the positions $(1, 2)$ or $(2, 1)$. Then we have $h_{1,2} \geq h_a + 1$. Since λ is self-conjugate we have

$h_{2,1} \geq h_a + 1$. Note that $H_{1,2}$ and $H_{2,1}$ intersect in exactly one cell $(2,2)$. Also $H_{1,2} \cup H_{2,1}$ does not contain the $(1,1)$th cell. Then we have

$$|\lambda| \geq h_{1,2} + h_{2,1} - 1 + 1 \geq 2h_a + 2 = 2^k + 2,$$

which can't be true. As a result a occupies one of the positions $(1,2)$ or $(2,1)$. The fact $h_{(1,2)} = h_{(2,1)}$ ensures that b occupies the other position.

□

Theorem 7.5.3. *Let λ be a self-conjugate partition with $v = 1$. Then λ is spinorial if and only if $\lambda = H(2^{k-1}, 2^{k-1})$, for some $k \geq 2$.*

Proof. If $\lambda = H(2^{k-1}, 2^{k-1})$ then from the hook-length formula we calculate

$$f_\lambda = \frac{(2^k+1)!}{(2^k+1)\cdot(2^{k-1})!\cdot(2^{k-1})!} = \frac{2^k!}{(2^{k-1}!)^2}.$$

The fact $v = 1$ ensures that $g_\lambda = f_\lambda/2$ is odd. Note that $\mathrm{Od}((2^{k-1}!)^2) \equiv 1 \pmod 4$. From Theorem 7.3.2 we conclude that $g_\lambda \equiv 3 \pmod 4$ for $k \geq 2$. Hence λ is spinorial.

For the converse take λ to be spinorial. If $v = 1$ then from 7.46 we have both h_λ and g_λ are odd. From 3.1.7 it follows that λ is spinorial if and only if $g_\lambda \equiv 3 \pmod 4$. Since λ is chiral, from Theorem 2.6.5 we conclude that $n = 3$, or $n = 2^k + \epsilon$ for some $k \geq 2$ and $\epsilon \in \{0, 1\}$. Following [3, Theorem 5], we obtain the 2-core tower of λ as

$$w_i(\lambda) = \begin{cases} 2 & \text{if } i = k-1, \\ 1 & \text{if } i = 0 \text{ and } \epsilon = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus λ contains two hooks H_a and H_b of length 2^{k-1}, where a and b denote the corner cells of the hooks. Lemma 7.5.2 ensures that a and b occupy the positions $(1,2)$ and $(2,1)$. Also H_a and H_b intersect at most at one cell $(2,2)$. Since $(1,1) \notin H_a \cup H_b$, we conclude that

$$|\{(H_a \cup H_b) \cup (1,1)\}| \geq 2^k - 1 + 1 = 2^k.$$

If $n = 2^k + 1$, then at most one cell, namely $(3,3)$, remains out of this set. Consequently there can be at most three diagonal cells in $\mathcal{Y}(\lambda)$. Following the discussions in the section 4.2.2 we obtain a partition $\mu \in$ DOP such that $\theta(\mu) = \lambda$. Since $\lambda \vdash 2^k$ is obtained from μ by folding (see [21, lemma 4.6.16]) we conclude that the parts of μ denote the hook lengths of the diagonal cells of $\mathcal{Y}(\lambda)$. Therefore μ have at most three parts. Here we list down dif and only iferent possibilities.

1. If the two hooks H_a and H_b intersect in the cell $(2,2)$, then we have either $\mu = (2x+1, 2y+1, 1)$ or $\mu = (2x+1)$.

2. If the two hooks H_a and H_b do not intersect, then we obtain $\mu = (2^{k+1})$. In this case we have $\lambda = (2^{k-1}+1, 1^{(2^k-1)})$.

Let $h_{i,j}$ denote the hook-length of the (i,j)-th node in $\mathcal{Y}(\lambda)$. Since λ is self-conjugate we have $h_{i,j} = h_{j,i}$, for $i \neq j$. Therefore we obtain $\mathrm{Od}(h_{i,j} \cdot h_{j,i}) = \mathrm{Od}((h_{i,j})^2) \equiv 1 \pmod 4$. If H_λ denote product of all hook lengths of $\mathcal{Y}(\lambda)$, then we have

$$\mathrm{Od}(H_\lambda) \equiv \prod_{x \in D(\mathcal{Y}(\lambda))} h_x \pmod 4, \tag{7.47}$$

where $D(\mathcal{Y}(\lambda))$ denotes the diagonal cells in $\mathcal{Y}(\lambda)$. As $v = 1$, from Equation (7.46) we obtain

$$g_\lambda = \frac{\mathrm{Od}(n!)}{\mathrm{Od}(H_\lambda)}. \tag{7.48}$$

Using Lemma 7.3.2 and 7.46 we deduce that the condition $g_\lambda \equiv 3 \pmod{}$ holds if and only if

$$\mathrm{Od}(H_\lambda) \equiv 1 \pmod 4 \tag{7.49}$$

Now we study dif and only iferent cases one by one.

- Let the two hooks H_a and H_b intersect in the cell $(2,2)$ and $\mu = (2x+1, 2y+1, 1)$. Since $|\mu|$ is odd, it must be a partition of 2^k+1. The Young diagram $\mathcal{Y}(\lambda)$ for the corresponding partition λ will contain three diagonal nodes, namely (i,i) for $1 \leq i \leq 3$. Since $2x+1$ and $2y+1$ denote the hook-lengths of the $(1,1)$-th and the $(2,2)$-th nodes in $\mathcal{Y}(\lambda)$ respectively. We have

$$|H_{(1,1)} \cup H_{(2,2)}| = 2x+1+2y+1 = 2^k.$$

Note that the set $H_{(1,1)} \cup H_{(2,2)}$ misses only one node in $\mathcal{Y}(\lambda)$, namely $(3,3)$. Since $k \geq 2$ the condition on x, y becomes $2x + 2y + 2 \equiv 0 \pmod 4$. In other words

$$x + y + 1 \equiv 0 \pmod 2. \tag{7.50}$$

Note that the condition 7.50 holds if and only if x and y are of dif and only iferent parity. Without loss of generality we assume that x is even and y is odd. Take $x = 2p$ and $y = 2q + 1$. Then the possible hook-lengths of the diagonal nodes are $4p + 1, 4q + 3$, and 1. From Equation (7.47) we deduce that

$$\mathrm{Od}(H_\lambda) \equiv (4p + 1) \cdot (4q + 3) \cdot 1 \equiv 3 \pmod 4.$$

Therefore in this case the condition 7.49 does not hold.

- Let the two hooks H_a and H_b intersect in the cell $(2,2)$ and $\mu = (2x + 1, 1)$. Note that in this case μ is a partition of 2^k. So we have $2x + 1 + 1 = 2^k$. Since $k \geq 2$, we conclude $2x + 1 \equiv 3 \pmod 4$. The Young diagram $\mathcal{Y}(\lambda)$ for the corresponding partition λ will contain two diagonal nodes, namely (i, i) for $1 \leq i \leq 2$. Note that the hook-lengths of the two diagonal nodes are $2x+1$ and 1. From Equation (7.47) we deduce that

  $$\mathrm{Od}(H_\lambda) \equiv (2x + 1) \cdot 1 \equiv 3 \pmod 4.$$

  So in this case the condition 7.49 does not hold.

Therefore we conclude that the two hooks H_a and H_b do not intersect and $\mu = (2^k + 1)$. So the corresponding partition λ is a hook of the form $(2^{k-1} + 1, 1^{(2^{k-1})})$.

$\square$

8

Asymptotic Results

This chapter investigates the asymptotic nature of the number of irreducible spinorial partitions of n. It turns out that the growth of the number of irreducible aspinorial partitions of n is much slower than that of the partition function. We prove similar result for irreducible spinorial representations of A_n. Finally we show that the character values of the irreducible representations of the symmetric groups are mostly divisible by high powers of 2. Throughout this chapter we assume

$$n = \epsilon + 2^{k_1} + \cdots + 2^{k_r}, 0 < k_1 < \ldots < k_r,\ \epsilon \in \{0, 1\}.$$

We also write $v = v_2(f_\lambda)$.

Theorem 8.0.1. *For any fixed non-negative integer m,*

$$\lim_{n\to\infty} \frac{\#\{\lambda \vdash n \mid v \leq k_r + m\}}{p(n)} = 0.$$

Proof. Note that

$$r \leq k_r \leq \log_2 n < k_r + 1. \tag{8.1}$$

Recall from Section 2.4 that the total number of cells of partitions in $T_2(\lambda)$ is $w = v + \nu(n)$. For $v \leq m + k_r$ we obtain

$$w \leq m + k_r + r + \epsilon \leq m + 2\log_2 n + \epsilon.$$

Since $n < 2^{k_r+1}$, the nodes of $T_2(\lambda)$ after the k_r-th row will remain unoccupied. If N denotes the total number of nodes up to the k_r-th row, then

$$N = 1 + 2 + 2^2 + \cdots + 2^{k_r} = 2^{k_r+1} - 1.$$

Then using Inequality 8.1 we obtain

$$N = 2^{k_r+1} - 1 \leq 2^{\log_2 n+1} = 2n.$$

There are $\binom{w+N-1}{N-1}$ many ways to distribute w cells in N nodes. Note that for a fixed number of cells assigned to a node there can be at most one 2-core with that many cells. This follows from the fact that any 2-core partition is a staircase partition. (See Proposition 2.4.1). The total number of partitions with $v \leq m + k_r$ is bounded above by the quantity

$$\sum_{v=0}^{m+k_r} \binom{v+r+\epsilon+N-1}{N-1}. \tag{8.2}$$

For $v \leq m + k_r$, we deduce

$$\binom{v+r+\epsilon+N-1}{N-1} \leq \binom{m+k_r+r+\epsilon+N-1}{N-1} = \binom{m+k_r+r+\epsilon+N-1}{m+k_r+r+\epsilon}.$$

Also note that
$$\binom{m+k_r+r+\epsilon+N-1}{m+k_r+r+\epsilon} \leq \binom{m+k_r+r+N}{m+k_r+r+1}.$$

Putting these bounds in the expression in Equation (8.2), we obtain

$$(m + \log_2 n + 1)\binom{m+k_r+r+N}{m+k_r+r+1}. \tag{8.3}$$

Now,

$$\begin{aligned}
\binom{m+r+k_r+N}{m+k_r+r+1} &= \frac{(m+r+k_r+N)\cdot(m+r+k_r+N-1)\cdots N}{(m+k_r+r+1)!} \\
&\leq (k_r+r+m+N)\cdots N \\
&\leq (k_r+r+m+N)^{k_r+r+m+1} \\
&\leq (2\log_2 n + m + 2n)^{2\log_2 n+m+1} \quad \text{(Use Inequality 8.1)}.
\end{aligned}$$

This gives an upper bound for the expression in Equation (8.2) as:

$$\begin{aligned}(m+\log_2 n+1)\binom{m+k_r+r+N}{m+k_r+r+1} &\leq (m+\log_2 n+1)(2\log_2 n+m+2n)^{2\log_2 n+m+1}\\ &\leq (2\log_2 n+m+2n)^{2\log_2 n+m+2}.\end{aligned}$$

The last inequality follows from the fact that $(m+\log_2 n+1) \leq (2\log_2 n+m+2n)$. We write

$$(2\log_2 n+m+2n)^{2\log_2 n+m+2} = \exp\left((2\log_2 n+m+2)\log(2\log_2 n+m+2n)\right). \quad (8.4)$$

So we have

$$\#\{\lambda \vdash n \mid v \leq k_r+m\} \leq \exp\left((2\log_2 n+m+2)\log(2\log_2 n+m+2n)\right).$$

According to Hardy-Ramanujan [10], as $n \to \infty$,

$$p(n) \sim \frac{1}{4n\sqrt{3}}\exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

Certainly the theorem follows from this.

$\square$

Recall that a partition λ is spinorial when the associated irreducible representation V_λ is spinorial.

Theorem 8.0.2. *We have*

$$\lim_{n\to\infty}\frac{\#\{\lambda \vdash n \mid \lambda \text{ is spinorial}\}}{p(n)} = 1.$$

Proof. Following Equation (7.2) we compute

$$v_2(\chi_\lambda(s_1)) = v + v_2(C(\lambda)) - (k_1 - 1). \quad (8.5)$$

As a result if $v \geq k_1 + 2$ then we obtain $\chi_\lambda(s_1) \equiv 0 \pmod 8$. Similarly Equation (7.3) gives

$$v_2(\chi_\lambda(s_1s_3)) = v + v_2(C(\lambda)^2 - 3C_2(\lambda) - n + n^2) - v_2\left(6\binom{n}{4}\right). \tag{8.6}$$

Recall that

$$v_2\left(6\binom{n}{4}\right) = \begin{cases} k_1 - 1, & \text{for } k_1 > 1, \\ k_2 - 1, & \text{for } k_1 = 1. \end{cases}$$

From Equation (8.6) it follows that if $v \geq k_2 + 2$, then

$$\chi_\lambda(s_1) \equiv \chi_\lambda(s_1s_3) \equiv 0 \pmod 8.$$

Since $k_2 \leq k_r$, putting $m = 2$ in Theorem 8.0.1 we conclude

$$\lim_{n\to\infty} \frac{\#\{\lambda \vdash n \mid v \leq k_2 + 2\}}{p(n)} = 0.$$

This shows that for most of the partitions

$$\chi_\lambda(s_1) \equiv \chi_\lambda(s_1s_3) \equiv 0 \pmod 8.$$

Therefore using Corollary 3.1.9 we obtain the required result.

□

Remark 8.0.3. *Theorem 8.0.1 suggests that most of the irreducible partitions V_λ are even dimensional. The fact that for most of the partitions $\chi_\lambda(s_1) \equiv 0 \pmod 8$ implies that most of the irreducible representations of S_n are achiral. These two observations can be also found in [3, Section 5]. Altogether with Theorem 8.0.2 we obtain that most of the irreducible representations of S_n are even dimensional, achiral and spinorial.*

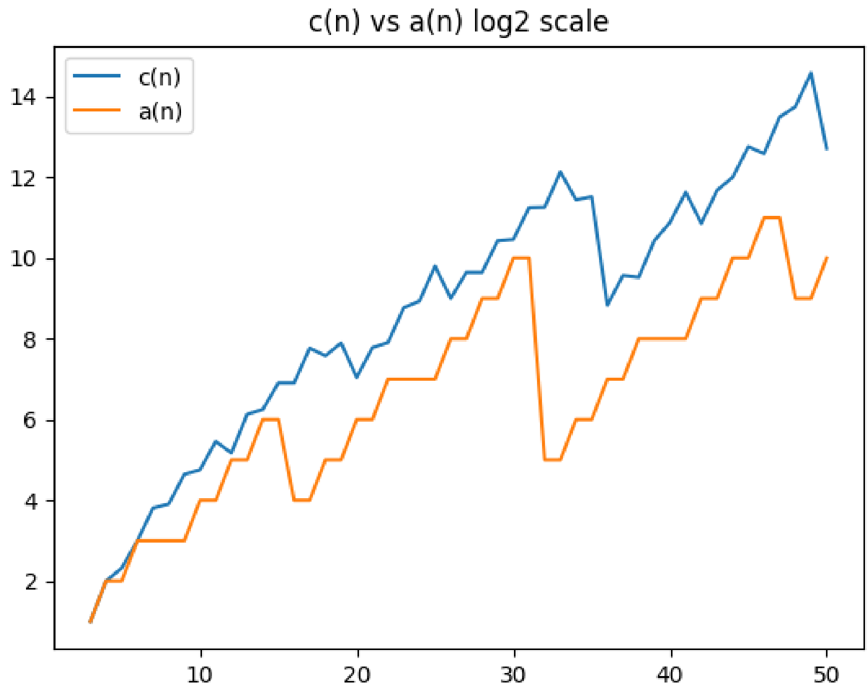


Figure 8.1: A $\log 2$ scale plot showing the number of aspinirial partitions $c(n)$ vs. the number of odd partitions $a(n)$ for $4 \le n \le 50$.

Here is a $\log_2$ scale plot showing the number of aspinorial partitions $c(n)$, drawn in blue and the number of odd partitions $a(n)$, drawn in orange. The figure shows that $c(n)$ follows a similar pattern as $a(n)$ which illustrates our conclusion that aspinorial partitions are rare.

Theorem 8.0.4. *We have*

$$\lim_{n\to\infty} \frac{\#\{\textit{Irreducible spinorial representations of } A_n\}}{\#\{\textit{Irreducible representations of } A_n\}} = 1.$$

Proof. We know that if λ is not self-conjugate then $V_\lambda \mid_{A_n}$ is an irreducible representation of A_n. On the other hand if λ is self-conjugate then $V_\lambda \mid_{A_n}$ decomposes as a sum of two irreducible representations $V_\lambda^{\pm}$. Let $s(n)$ denote the number of self-conjugate partitions of n. Then we obtain

$$\begin{aligned}\#\{\text{Irreducible spinorial representations of } A_n\} = \#\{\lambda \in p(n) \setminus s(n) : V_\lambda\mid_{A_n} \text{ is spinorial}\} \\ + 2\#\{\lambda \in s(n) : V_\lambda^\pm \text{ is spinorial}\}.\end{aligned}$$

Let us denote $a(n) = \#\{\text{Irreducible representations of } A_n\}$. Using [21, Theorem 4.6.7] one calculates $a(n) = 2s(n) + (p(n) - s(n))/2$. Simplifying the expression we obtain $a(n) = (3s(n) + p(n))/2$. From [28, section 5.1] we obtain

$$\frac{s(n)}{p(n)} \sim (6n)^{1/4} e^{-\frac{c\sqrt{n}}{2}} \text{ as } n \to \infty,$$

where $c = 2\sqrt{\pi^2/6}$. Therefore we conclude

$$\lim_{n\to\infty} \frac{\#\{\lambda \in s(n) \mid V_\lambda^\pm \text{ is spinorial}\}}{a(n)} = 0.$$

Note that if V_λ is a spinorial representation of S_n then $V_n\mid_{A_n}$ is a spinorial representation of A_n. So from Theorem 8.0.2 it follows that

$$\lim_{n\to\infty} \frac{\#\{\lambda \in p(n) \setminus s(n) : V_\lambda\mid_{A_n} \text{ is spinorial}\}}{a(n)} = 1.$$

□

Using the same line of argument we prove that the character values of the irreducible representations of the symmetric groups are mostly divisible by high powers of 2. For a partition μ such that $|\mu| \leq n$ we obtain a partition $(\mu, 1^{n-|\mu|})$ of n. For example if $\mu = (3, 2)$ and $n = 7$, we have the partition $(3, 2, 1, 1)$ of 7.

Theorem 8.0.5. *For a fixed partition μ and a positive integer b we have*

$$\lim_{n\to\infty} \frac{\#\{\lambda \vdash n \mid \chi_\lambda(\mu, 1^{(n-|\mu|)}) \equiv 0 \pmod{2^b}\}}{p(n)} = 1.$$

Proof. From Equation (7.8) we obtain

$$v_2(\chi_\lambda(\mu)) = v_2(f_\lambda) + v_2(\widehat{A}(\mu)) - v_2((n)_{|\rho|}), \tag{8.7}$$

where $(n)_{|\rho|} = n(n-1)\cdots(n-|\rho|+1)$. Note that $(n)_{|\rho|} = |\rho|!\binom{n}{|\rho|}$. Recall that for any integers n, a, where $a \leq n$, we have

- $v_2(n!) = n - \nu(n)$.
- $v_2\binom{n}{a} = \nu(a) + \nu(n-a) - \nu(n)$.

Here $\nu(n)$ denotes the number of 1's appearing in the binary expansion of n. Then we calculate

$$\begin{aligned}
v_2((n)_{|\rho|}) &= v_2\left(|\rho|!\binom{n}{|\rho|}\right) \\
&= v_2(|\rho|!) + v_2\left(\binom{n}{|\rho|}\right) \\
&= |\rho| - \nu(|\rho|) + \nu(|\rho|) + \nu(n-|\rho|) - \nu(n) \\
&= |\rho| + \nu(n-|\rho|) - \nu(n) \\
&\leq |\rho| + k_r.
\end{aligned}$$

Using this inequality in Equation (8.7) we obtain

$$v_2(\chi_\lambda(\mu, 1^{(n-|\mu|)})) \geq v_2(f_\lambda) + v_2(\widehat{A}(\mu)) - (|\rho| + k_r).$$

So if $v = v_2(f_\lambda) \geq |\rho| + k_r + b$, then $v_2(\chi_\lambda(\mu, 1^{(n-|\mu|)})) \geq b$. Now taking $m \geq |\rho| + b$ in Theorem 8.0.1 to obtain

$$\lim_{n\to\infty} \frac{\#\{\lambda \vdash n \mid v \leq k_r + |\rho| + b\}}{p(n)} = 0.$$

Hence the result follows.

$\square$

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