Stability Analysis of Spherically Symmetric Space-Time sourced by Anisotropic Fluids

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by

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under the guidance of

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Certificate

This is to certify that this thesis entitled "The Stability Analysis of Spherically Symmetric Spacetime sourced by Anisotropic Fluids" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by Bhavesh Khamesra at IISER Pune, under the supervision of Dr. Suneeta Vardarajan during the academic year 2013-2014.

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Abstract

In this work, we analyze the stability of spherically symmetric spacetimes sourced by perfect and anisotropic fluids under a class of gravitational perturbations. This has been done in a gauge-invariant formalism developed by Ishibashi and Kodama, where the perturbation equations are written in terms of manifestly gauge-invariant variables. We study axial perturbations of such spacetimes. The perturbation equations reduce to a single master equation and the equations for conservation of the energy momentum tensor. As is well-known, axial perturbations do not excite the fluid perturbations when the source is a perfect fluid. We review this result and then show that when the fluid is anisotropic, this is still true provided the vector defining the direction of anisotropy is constant under the perturbation. Finally we conclude by taking a specific example of a spacetime sourced by an anisotropic fluid and analyzing the axial perturbations of this spacetime.

Contents

Chapter 1

Introduction

Stability analysis under gravitational perturbations has been studied widely in relativity, astrophysics and cosmology. It comes as a useful tool in the study the interior structure of star, its stability and evolution with time. While the study of radial perturbations provides insights about the global stability or instability which may be the cause of stellar collapse, the study of non radial perturbations is used in the resolution of the problem of gravitational waves by non stationary sources [1].

The study of stellar perturbations is of special interest in relativity and astrophysics. A large amount of literature exist which presents and analyze various stellar models and their gravitational collapse situation. Some of most successful stellar models were found with perfect fluids sources. Unlike the dust case, perfect fluids models can predict static spherically symmetric solutions. Such models can be used to model the interior of the neutron stars [2]. Also, perfect fluid models are easier to model due to the high symmetry involved which simplifies the equations to a great extent. Stellar perturbations of stars sourced by perfect fluids have been studied in a great detail, a detailed account can be found in the works of Chandrasekhar [3].

In the course of development of these stellar models, it was realized that static spherically symmetric models can also occur with other fluids. Using an anisotropic fluid with zero radial pressure and non zero angular pressure, equal in both the directions, it was shown in [4] that a static spherically symmetric solution can exist. Such fluid model was also used to analyze the gravitational collapse scenarios in [5] in which a general class of equilibrium solutions were obtained as the result of the gravitational collapse. The issue still remains about their stability. In this work, we explore such spacetime sourced by anisotropic fluid and study about their stability. We consider special anisotropic fluids in which radial pressure differs from angular pressures, which are both equal.

We analyze the stability of these spacetimes under gravitational perturbations. Such an analysis was first done by Regge and Wheeler, who developed the general formalism to analyze the stability of the Schwarzschild spacetime [6]. However, a major problem with such method is the ambiguity in gauge choice. All the observables derived from these theories should be completely independent of the coordinate system the theory is formulated in. A remedy to this problem came from the formalism developed by Gerlach and Sengupta [7], [8], [9] who presented a gauge invariant structure of linearized Einstein equations and Conservation equations for polar and axial perturbations of spherically symmetric spacetime. Mukohyama [10] developed the gauge invariant formalism for maximally symmetric spacetimes including Anti-de Sitter spacetime. This formalism was further developed in a series of papers by Ishibashi and Kodama where they extended the Regge Wheeler master equation to higher dimensions [11].

The study of spacetimes sourced by anisotropic fluids is of special relevance in various spheres. The theoretical interest arises in the study of their stability and to see the degree of variation from the perfect fluid case. It also has applications in astrophysics to model the anisotropies occurring in fluids in the interior of the stars. Such anisotropies cannot be studied with perfect fluid stellar models and need an anisotropic counterpart. Apart from relativity and astrophysics, anisotropic spacetime have been studied in particle physics in the study of the instability associated with anisotropic quark-gluon plasma at weak and strong couplings [12]. There has also been studies to obtain the solutions of anisotropic black branes in the asymptotically Anti-de Sitter spacetime [13].

In this work, we focus on the axial perturbations of a general spherically symmetric stellar model sourced by anisotropic fluids. We present the Einstein equations and the Conservation laws in the gauge invariant form and using these we develop a masters equation for this case. It has earlier been shown that the intricately coupled perturbed Einstein equations can be reduced to simple ordinary differential equations for stationary vacuum black holes [6]. Such a reduction is also possible for perfect fluid stellar models under axial perturbations as the fluid does not excite the axial perturbations [3]. In this work, we show that fluid perturbations, preserving the anisotropy direction, are completely independent of metric perturbations for anisotropic fluids. We further show that equation for axial perturbation in this case reduces to a linear partial differential equation for a function and by employing a suitable ansatz we can reduce it to a simple looking Schrodinger equation. As an example we analyze a specific anisotropic spacetime containing a naked singularity spacetime as (in)stability of naked singularity is a problem of interest in view of cosmic censorship.

In the next chapter we first develop the gauge invariant formalism for a spherically symmetric spacetime. This is continued in chapter 3 with a introduction to fluids in which we give a brief description of fluids, the conservation law and the energy conditions. In chapter 4, we find the master equation for perfect fluid case as a demonstration of the gauge invariant formalism. In chapter 5 we discuss the anisotropic fluid case and obtain the gauge invariant master equation for the perturbations. We discuss an example of naked singularity spacetime sourced by anisotropic fluid.

Chapter 2

Gravitational Perturbations

2.1 Why Study Perturbations

Einstein' s General theory of relativity is one of the most successful theories of gravity. It has made many novel predictions completely replacing its classical counterpart. However, some of these predictions are bizarre and difficult to comprehend. Some of the solutions of Einstein equations are unphysical and violate the laws of causality. This puts the theory in trouble raising questions about its predictive ability.

A possible way out is the stability analysis. It is expected that the unphysical solutions predicted by this theory should not be stable and hence would not exist in real world. A perturbation analysis of such solutions can throw light on their existence, stability and evolution with time. Knowing about the stability of these solutions it will be easier to understand them.

Perturbation analysis has become a useful tool to solve many astrophysical and cosmological problems. It has been used to study the stability of black holes and to understand the emission of gravitational waves from black holes and gravitational collapse scenarios [11]. Perturbation analysis has also been used to study the stellar collapse. Various stellar models have been developed since the Schwarzschild vacuum solution was discovered. However, many of such models lead to prediction of unphysical or unstable solutions which can be ruled out using the perturbation analysis.

Perturbation analysis have also been used in the higher dimensional gravity theories to study the stability of higher dimensional black holes. Valuable information can be extracted from the perturbation analysis about the stability of black holes, gravitational wave emission during the formation of black hole, and quasi-normal model analysis [14]. Perturbation Theory also has many important application in the cosmology. Cosmological Perturbation theory provides a link between the Early Universe models (for eg. Inflationary Universe or Pre-Big-Bang model or Ekpyrotic- type models) and observations from cosmic microwave anisotropies and the spectrum of density fluctuations. [15]

2.2 Perturbation types

Gravitational perturbations can be classified in two categories, radial perturbations and non-radial perturbations. The study of radial perturbations was first initiated by Chandrasekhar in 1964 [16] [17] in order to understand the effect of relativity on the stability conditions. This was later extended by Linblom and Hiscock to fluid models with dissipations and viscosity [18].

The non radial perturbations were first introduced by T. Regge and J.A.Wheeler [6]. They generalized the theory of spherical harmonics to the tensors and based on the parity, they classified the perturbations into two classes:

1.Odd perturbations or axial perturbations: Perturbations with parity $(-1)^{L+1}$,

2.Even perturbations or polar perturbations: Perturbation with parity $(-1)^{L}$

Here L is the angular momentum. Such a classification depends on the metric structure, specifically, it is only valid for spacetime which have spherical symmetry. It has been further extended to general m+n dimensional spacetime where the perturbations can be classified as scalar, vector and tensor perturbations [19].

2.3 Notation and Convention

In this section, we describe the general notation and formalism which will be followed throughout the paper. We use the general method formulated by Ishibashi and Kodama [19] to find the perturbed Einstein equations for general four dimensional spherically symmetric spacetime, and apply it for stability analysis of these spacetimes sourced by perfect and anisotropic fluids in the next sections. We highlight the important theory and results of the formalism in this section.

2.3.1 Background Geometry

Consider a four dimensional spacetime whose metric structure is given as

$$g_{\mu\nu}dx^\mu dx^\nu = g_{ab}(y)dy^a dy^b + \tilde{R}^2(y)\gamma_{ij}(z)dz^i dz^j$$

We assume $g_{ab}(y)$ is the metric defined on the r-t submanifold and $\gamma_{ij}(z)$ is the metric on two sphere S^2. For the latter, we have $\hat{R}_{ij} = K\gamma_{ij}$ for some constant K. The constant K corresponds to the sectional curvature of S^2 and takes values -1 for hyperbolic geometry, 0 for Euclidean geometry and 1 for elliptic geometry.

We adopt the following convention to differentiate tensors associated with these submanifolds.

i) Greek indices $\mu, \nu, \alpha, \beta, \gamma \cdots \rightarrow$ tensors on complete manifold
ii) Latin indices a, b, c, d $\cdots \rightarrow$ tensors on r-t submanifold
iii) Latin indices i, j, k, l $\cdots \rightarrow$ tensors on S^2

We denote covariant derivatives, Connection coefficients and Riemann curvature tensors for the full spacetime, the r-t submanifold and the sphere with metric γ_{ij} as

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu &\rightarrow \nabla_\mu, \Gamma^\alpha_{\beta\gamma}, R_{\mu\nu\alpha\beta} \\ g_{ab}dy^a dy^b &\rightarrow D_a, \bar{\Gamma}^a_{bc}, \bar{R}_{abcd} \\ \gamma_{ij}dz^i dz^j &\rightarrow \hat{D}_i, \hat{\Gamma}^i_{jk}, \hat{R}_{ijkl} \end{aligned}$$

We now give the general formulae for the background spacetime.

1. Connection Coefficients:

$$\Gamma^a_{bc} = \bar{\Gamma}^a_{bc}, \quad \Gamma^a_{ij} = -\tilde{R}D^a\tilde{R}\gamma_{ij}, \quad \Gamma^i_{bj} = \frac{D_b\tilde{R}}{\tilde{R}}\delta^i_j, \quad \Gamma^i_{jk} = \hat{\Gamma}^i_{jk} \tag{2.1}$$

2. Curvature Tensors:

$$\begin{aligned} R^a_{bcd} = \bar{R}^a_{bcd} \quad R^a_{ibj} &= -\frac{D^a D_b\tilde{R}}{\tilde{R}}g_{ij} \\ R^i_{jkl} = \hat{R}^i_{jkl} &- (D\tilde{R})^2(\delta^i_k\gamma_{jl} - \delta^i_l\gamma_{jk}) \end{aligned} \tag{2.2}$$

3. Ricci Tensors:

$$R_{ab} = \frac{1}{2}\,\bar{R}_{ab} - 2\frac{D_a D_b \tilde{R}}{\tilde{R}}$$

$$R_{ai} = 0$$

$$R_{ij} = \left(-\frac{\Box \tilde{R}}{\tilde{R}} + \frac{K - (D\tilde{R})^2}{\tilde{R}^2} \right) g_{ij}$$

$$R = \bar{R} - 4\frac{\Box \tilde{R}}{\tilde{R}} + 2\frac{K - (D\tilde{R})^2}{\tilde{R}^2}$$

Here $\Box = D^a D_a$ and $\bar{R}$ is the Ricci scalar of g_{ab}. We take our background energy momentum tensor of the form

$$T_{ai} = 0, \quad T^i_j = P\delta^i_j \tag{2.3}$$

where P is the pressure. Using the above quantities, we can write the Einstein equations for the background (κ^2 is the gravitational constant)

$$\bar{G}_{ab} - 2\frac{D_a D_b \tilde{R}}{\tilde{R}} - \left(\frac{K - (D\tilde{R})^2}{\tilde{R}^2} - 2\frac{\Box \tilde{R}}{\tilde{R}} \right) g_{ab} = \kappa^2\, T_{ab} \tag{2.4}$$

$$-\frac{1}{2}\,\bar{R} + \frac{\Box \tilde{R}}{\tilde{R}} = \kappa^2 P \tag{2.5}$$

Though we have specialized to four dimensional spacetime, in general this can be extended to the case when a sherically symmetric spacetime is a warped product of an m dimensional non compact spacetime and an n dimensional sphere.

2.3.2 Decomposition Theorems

The Einstein equations are a set of coupled partial differential equations which are very difficult to solve. To simplify the problem, we classify the perturbation variables based on their tensorial behavior on S^2 into three types. Such a classification leads to decoupling of linearized Einstein equations. Let us first discuss two general theorems of decomposition of vectors and tensors. [19]

1. Hodge Decomposition Theorem: Suppose $(\mathcal{K}^n, \gamma_{ij})$ be a compact Riemann manifold. Any dual vector field on $\mathcal{K}^n$ can be uniquely decomposed as

$$v_i = V_i + \hat{D}_i S \tag{2.6}$$

where $\hat{D}^i V_i = 0$. We refer to V_i and S as vector type and scalar type components of dual vector v_i

2. Suppose ($\mathcal{K}^n$,γ_{ij}) be a compact Riemannian Einstein space, $\hat{R}_{ij} = c\gamma_{ij}$ for some constant c. Any second rank symmetric tensor field t_{ij} can be uniquely decomposed as

$$\begin{aligned} t_{ij} &= t^{(2)}_{ij} + 2\hat{D}_{(i}t^{(1)}_{j)} + t_L\gamma_{ij} + \hat{L}_{ij}t_T \\ \hat{L}_{ij} &:= \hat{D}_i\hat{D}_j - \frac{1}{n}\gamma_{ij}\hat{\triangle} \end{aligned} \tag{2.7}$$

where $\hat{D}^i t^{(2)}_{ij} = 0$, $t^{(2)i}{}_i = 0$, $\hat{D}^i t^{(1)}{}_i = 0$ *and* $t_L = t^m_m/n$. A similar decomposition theorem has been proved by Kodama and Sasaki in which $\mathcal{K}^n$ is considered maximally symmetric [20]. Using these two theorems, we can now write the decomposition of the metric and fluid perturbation variables. Let $\delta g_{\mu\nu} = h_{\mu\nu}$ be the metric perturbation relative to our background spacetime. Its decomposition is given by

$$\begin{aligned} h_{ai} &= \hat{D}_i h_a + h^{(1)}_{ai} \\ h_{ij} &= h^{(2)}_{T\,ij} + 2\hat{D}_{(i}h^{(1)}_{T\,j)} + h_L\gamma_{ij} + \hat{L}_{ij}h^{(0)}_T \end{aligned} \tag{2.8}$$

where $\hat{D}^j h^{(2)}_{T\,ij} = h^{(2)i}_{T}{}_i = 0$, $\hat{D}^i h^{(1)}_{T\,i} = 0$ *and* $\hat{D}^i h^{(1)}_{ai} = 0$. A similar decomposition of the energy momentum tensor perturbation can be written down.

$$\begin{aligned} \delta T_{ai} &= \hat{D}_i \delta T_a + \delta T^{(1)}_{ai} \\ \delta T_{ij} &= \delta T^{(2)}_{T\,ij} + 2\hat{D}_{(i}\delta T^{(1)}_{T\,j)} + \delta T_L\gamma_{ij} + \hat{L}_{ij}\delta T_T \end{aligned} \tag{2.9}$$

here $\hat{D}^j \delta T^{(2)}_{T\,ij} = \delta T^{(2)i}_{T}{}_i = 0$, $\hat{D}^i \delta T^{(1)}_{T\,i} = 0$ *and* $\hat{D}^i \delta T^{(1)}_{ai} = 0$. The perturbed Einstein equations gets decoupled into three types of equations and each type can be dealt individually.

2.3.3 Solving Gauge Ambiguity

One of the common problem in general relativity is the issue of gauge fixing. Since Einstein equations are invariant under gauge transformations generated by an infinitesimal vector field ζ^μ, therefore any perturbation variable should provide the same results as its gauge transformed ones. Clearly having multiple gauge choices creates an ambiguity in representation of perturbation variables. To resolve this ambiguity, we can either fix our gauge choice or work in gauge invariant variables. The two approaches are equivalent. [19]

Let us choose an infinitesimal gauge transformation given by $\delta z^\mu = \zeta^\mu$. The metric perturbation transformation is given by

$$\delta h_{\mu\nu} = -\nabla_\mu \zeta_\nu - \nabla_\nu \zeta_\mu \tag{2.10}$$

For individual components, this can be written in terms of the covariant derivatives on the two submanifolds as

$$\begin{aligned} h_{ab} &\to h_{ab} - D_a\zeta_b - D_b\zeta_a, \\ h_{ai} &\to h_{ai} - \tilde{R}^2 D_a \frac{\zeta_i}{\tilde{R}^2} - \hat{D}_i\zeta_a \\ h_{ij} &\to h_{ij} - 2\hat{D}_{(i}\zeta_{j)} - 2\tilde{R}(D^a\tilde{R})\zeta_a\gamma_{ij} \end{aligned} \tag{2.11}$$

Similarly, the energy momentum tensor perturbation transformation law is given by

$$\delta(\delta T)_{\mu\nu} = -\zeta^\alpha \nabla_\alpha T_{\mu\nu} - T_{\mu\alpha}\nabla_\nu\zeta^\alpha - T_{\nu\alpha}\nabla_\mu\zeta^\alpha \tag{2.12}$$

For individual components, this can be written as

$$\begin{aligned} \delta T_{ab} &\to \delta T_{ab} - \zeta^c D_c T_{ab} - T_{ac}D_b\zeta^c - T_{bc}D_a\zeta^c, \\ \delta T_{ai} &\to \delta T_{ai} - T_{ab}\hat{D}_i\zeta^b - \tilde{R}^2 P D_a\frac{\zeta_i}{\tilde{R}^2} \\ \delta T_{ij} &\to \delta T_{ij} - \zeta^a D_a(\tilde{R}^2 P)\gamma_{ij} - 2P\hat{D}_{(i}\zeta_{j)} \end{aligned} \tag{2.13}$$

Using the previous theorems, we can write the decomposition of the generator ζ^μ. From (2.6), we can see that

$$\zeta_a = T_a, \quad \zeta_i = V_i + \hat{D}_i S$$

where $\hat{D}^i V_i = 0$. In this paper, we focus only on the vector type perturbations. Hence we can write the gauge transformation law for the vector components of the metric perturbation

$$h^{(1)}_{ai} \to h^{(1)}_{ai} - \tilde{R}^2 D_a\left(\frac{V_i}{\tilde{R}^2}\right)$$

$$h^{(1)}_{Ti} \to h^{(1)}_{Ti} - V_i$$

A similar decomposition can be written for the vector components of energy momentum tensor perturbations. From the above transformation laws, we can build the following gauge invariant quantities.

$$F_{ai}^{(1)} := h_{ai}^{(1)} - \tilde{R}^2 D_a \left(\frac{h_{Ti}^{(1)}}{\tilde{R}^2} \right)$$

$$\tau_{ai}^{(1)} := \delta T_{ai}^{(1)} - P h_{ai}^{(1)}$$

$$\tau_{ij}^{(1)} := 2\hat{D}_{(i}\delta T_{Tj)}^{(1)} - 2P\hat{D}_{(i}h_{Tj)}^{(1)}$$

Using these gauge invariant quantities, we can now write the linearized Einstein equations for the perturbed metric. Note that there are no transverse traceless tensor perturbations for S^2 case. We first write the above gauge invariant quantities in terms of vector harmonics as follows. For a more detailed account of vector harmonics $\mathcal{V}_i$ and derivation of the linearized equations refer [19], [21]. For generic modes $m_V := k_v^2 - K = l(l+1) - 2 \neq 0$ [19]

$$F_{ai}^{(1)} = \tilde{R} F_a \mathcal{V}_i, \quad \tau_{ai}^{(1)} = \tilde{R} \tau_a \mathcal{V}_i, \quad \tau_{ij}^{(1)} = \tilde{R}^2 \tau_T \mathcal{V}_{ij}$$

In terms of these quantities we can write the perturbed Einstein Equations as follows:

$$\frac{1}{\tilde{R}^3} D^b \left\{ \tilde{R}^4 \left[D_b \left(\frac{F_a}{\tilde{R}} \right) - D_a \left(\frac{F_b}{\tilde{R}} \right) \right] \right\} - \frac{m_V}{\tilde{R}^2} F_a = -2\kappa^2 \tau_a \tag{2.14}$$

$$\frac{k_V}{\tilde{R}^2} D_a(\tilde{R} F^a) = -\kappa^2 \tau_T \tag{2.15}$$

Chapter 3

Fluids

This chapter gives an introduction to fluids. Section I gives a some example of different types of relativistic fluids and the energy momentum tensors associated with them. Section II describes the conservation laws in relativistic hydrodynamics. Section III discusses various physical energy conditions and their implications. For detailed discussion about the fluids, refer to [22]

3.1 Classification of Fluids

Fluids can be classified on the basis of their physical properties. Fluids display various physical properties like viscosity, density, compressibility, surface tension etc. In general, we have different kinds of fluids like

1. Ideal fluids: Fluids which are not viscous and incompressible.

2. Non Ideal fluids: Fluids which are compressible and viscous.

Let us consider some special types of fluids and their energy momentum tensors

3.2 Stress Energy Tensor

Energy Momentum tensor is a generalization of stress tensor of Newtonian Physics and appears in the Einstein equations as a source of curvature of the spacetime. Let us now describe the energy momentum tensor of some of the special fluids.

1. Dust: Dust is a collection of particles which are all at rest in some Lorentz frame. In the rest frame, since all the particles are motionless hence their momentum is zero. Therefore the energy momentum tensor takes the form $T^{\mu}_{\nu} = diag(-\rho, 0, 0, 0)$ or in tensor notation

$$T^{\mu\nu} = \rho u^{\mu} u^{\nu}$$

Dust models were one of the initial fluid models studied in the stellar collapse problems in GR. The first such attempt was made by Oppenheimer and Snyder [23]. They found that a spherically symmetric star with dust as the source will always collapse to singularity and no stable stationary states are possible.

2. Perfect Fluids: Perfect fluids are the fluids which do not have viscosity, shear stress or heat conduction properties. They can be completely characterized by the energy density and isotropic pressure in the rest frame. The energy momentum tensor for the perfect fluids in the rest frame of the observer is given by $T^{\mu}_{\nu} = diag(-\rho, p, p, p)$, or in any general frame, is given by the tensor law

$$T^{\mu}_{\nu} = (\rho + p) u^{\mu} u_{\nu} + p \delta^{\mu}_{\nu}$$

Perfect fluid have been widely used as a source in the study of stellar collapse. There is a large amount of literature published in this area due to the wide applications of this model. Recently, in a 1998 article, Skea and Finch [2] analyzed the large number of solutions obtained for the static spherically symmetric perfect fluid stars based on the physical conditions for realistic solutions. This was further extended by Delgaty and Lake [24].

3. Non-Perfect Fluids: Unlike perfect fluids, non perfect fluids are fluids with non zero tangential or shearing stress. These are compressible and viscous in nature. As a result these are difficult to model. It took a considerable time to develop a relativistically consistent theory of such fluids. A brief account of the history of relativistic fluids is presented in [22].

For a general description refer to [25]. The energy momentum tensor for a general fluid is given by [26]:

$$T^{\mu\nu} = \rho(1+\varepsilon) u^{\mu} u^{\nu} + (p - \zeta\theta) \perp^{\mu\nu} - 2\eta\sigma^{\mu\nu} + q^{\mu} u^{\nu} + q^{\nu} u^{\mu}$$

where ρ is the rest mass density, ε is the energy density in the rest frame, p is the pressure, η, ζ are the shear and bulk viscosities, q^μ is the energy flux vector, u^μ is the four velocity and $\perp^{\mu\nu}$ is the projection tensor (spatial) given by

$$\perp^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$$

θ defined as $\theta = \nabla_\mu u^\mu$, describes the convergence or the divergence of world lines of fluid element. $\sigma^{\mu\nu}$ is the traceless, symmetric, spatial shear stress given by

$$\sigma^{\alpha\beta} = \frac{1}{2}(\nabla_\mu u^\alpha \perp^{\mu\beta} + \nabla_\mu u^\beta \perp^{\mu\alpha}) - \frac{1}{3}\theta \perp^{\alpha\beta}$$

In the following chapters, we work with perfect and anisotropic fluid, a special case of real fluids for which the viscosity, shear and dissipation terms are zero. The anisotropic fluid differs from the perfect fluid due to differences in radial and tangential pressures i.e. $p_r \neq p_t$

3.3 Conservation Law

In general relativity, the hydrodynamics equation of non relativistic mechanics is now replaced by a more general law given by

$$\nabla_\mu T^{\nu\mu} = 0 \tag{3.1}$$

As given in the previous chapter, for our case the background metric is written as

$$g_{\mu\nu} dx^\mu dx^\nu = g_{ab}(y) dy^a dy^b + R^2(y)\gamma_{ij}(z) dz^i dz^j$$

where we have denoted radius $\tilde{R}$ as $R$and the energy momentum tensor is assumed to have the form $T_{ai} = 0$ and $T^i_j = P\delta^i_j$ where P is the pressure. Later we will specialize to a definite form of T_{ab}, but for now we can assume it to be of any general form. From the conservation law we have

$$\nabla_\mu T^{a\mu} = 0$$

which gives

$$D_b T^{ab} + 2\frac{D_b R}{R}(T^{ab} - pg^{ab}) = 0 \tag{3.2}$$

while the other equation $\nabla_\mu T^{i\mu} = 0$ is trivially satisfied due to the metric compatibility condition.

Along with the perturbations of the background spacetime, the background fluid is also perturbed. This leads to a change in energy momentum tensor of the fluid by a small perturbation $\tilde{T}_{\mu\nu} = T_{\mu\nu} + \delta T_{\mu\nu}$. Under the linearized approximation, the continuity equation for the perturbed energy momentum tensor can be written as

$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = \nabla_\mu T^{\mu\nu} + \nabla_\mu \delta T^{\mu\nu} + \delta\Gamma^\mu_{\mu\alpha} T^{\alpha\nu} + \delta\Gamma^\nu_{\mu\alpha} T^{\alpha\mu} = 0$$

$$\delta(\nabla_\mu \; T^{\mu\nu}) = \nabla_\mu \delta T^{\mu\nu} + \delta\Gamma^\mu_{\mu\alpha} T^{\alpha\nu} + \delta\Gamma^\nu_{\mu\alpha} T^{\alpha\mu} \tag{3.3}$$

Case 1: $\nu = a$. The above equation becomes

$$\delta\nabla_\mu \; T^{\mu a} = \nabla_\mu \delta T^{\mu a} + \delta\Gamma^\mu_{\mu\alpha} T^{\alpha a} + \delta\Gamma^a_{\mu\alpha} T^{\alpha\mu} \tag{3.4}$$

Consider the first term in the expansion

$$\nabla_\mu \delta T^{\mu a} = \nabla_b \delta T^{ba} + \nabla_i \delta T^{ia} \tag{3.5}$$

It can be easily computed that

$$\delta T^{ab} = g^{ac} g^{bd} \delta T_{cd} - g^{ad} T^{bc} h_{cd} - g^{bc} T^{ad} h_{cd} \tag{3.6}$$

$$\delta T^{ai} = g^{ab} g^{ij} (\delta T_{bj} - P h_{bj}) - T^{ab} g^{ij} h_{jb} \tag{3.7}$$

$$\delta T^{ij} = g^{ik} g^{jl} (\delta T_{kl} - 2P h_{kl}) \tag{3.8}$$

On expanding the two terms in expression (3.5) and using the above equations, we see that both these terms go to zero. (Using the relations $\gamma^{ij}\delta T_{ij} = \gamma^{ij} h_{ij} = 0$ and $\hat{D}^i \delta T_{ai} = 0$). The Christoffels can be treated in the similar manner. Expanding the second term of equation (3.4),

$$\delta\Gamma^\mu_{\mu\alpha} T^{\alpha a} = T^{ab} \frac{g^{\mu\nu}}{2} (-\nabla_\nu h_{\mu b} + \nabla_\mu h_{\nu b} + \nabla_b h_{\mu\nu})$$

To simplify the above expression, we can use the symmetry of the metric. Since the metric is symmetric about μ and ν while the sum of the first two terms are antisymmetric about them, hence their sum vanishes. Since $\nabla_\mu g^{\alpha\beta} = 0$, and for vector perturbations $Tr(\mathbf{h}) = 0$, therefore the last term also vanishes.

$$g^{\mu\nu} \nabla_a h_{\mu\nu} = \nabla_a (g^{\mu\nu} h_{\mu\nu}) = 0$$

This gives,

$$\delta\Gamma^\mu_{\mu\alpha} T^{\alpha\nu} = 0$$

Approaching the same way, using the symmetry of the energy momentum tensor and relations mentioned above we can see that the third term in equation (3.4) also goes to zero. Equation (3.4) thus reduces to

$$\delta(\nabla_\mu \, T^{\mu a}) = 0 \tag{3.9}$$

Case 2: $\nu = i$. Equation (3.3) can be written as

$$\delta(\nabla_\mu \, T^{\mu i}) = \nabla_\mu \delta T^{\mu i} + \delta\Gamma^\mu_{\mu\alpha} T^{\alpha i} + \delta\Gamma^i_{\mu\alpha} T^{\alpha\mu} \tag{3.10}$$

Following the same procedure, lets expand the first term. We have

$$\nabla_\mu \delta T^{\mu i} = \nabla_a \delta T^{ai} + \nabla_j \delta T^{ji}$$

On expanding the covariant derivatives this reduces to

$$\nabla_\mu \delta T^{\mu i} = D_a \delta T^{ai} + \hat{D}_j \delta T^{ji} + \frac{4\partial_a R}{R} \delta T^{ai}$$

δT^{ai} and δT^{ij} can be rewritten using equations (3.7) and (3.8). Using the definition of τ_{ai} from chapter 2 and equation (3.2), we can rewrite the above equation as

$$\nabla_\mu \delta T^{\mu i} = \frac{1}{R^2} g^{ab} g^{ij} D_a (R^2 \tau_{bj}) + g^{ik} g^{jl} \hat{D}_j (\tau_{kl} - P h_{kl}) - T^{ab} g^{ij} D_a h_{jb} - 2P \frac{\partial^a R}{R} g^{ij} h_{aj}$$

Expansion of the second term of equation (3.10) gives

$$\delta\Gamma^\mu_{\mu\alpha} T^{i\alpha} = T^{ij} \frac{g^{\mu\nu}}{2} (-\nabla_\nu h_{\mu j} + \nabla_\mu h_{\nu j} + \nabla_j h_{\mu\nu})$$

Again, using the symmetry property of the metric the first and second term cancels each other. We are then left with last term which again goes to zero as the trace of metric perturbation is zero. Hence

$$\delta\Gamma^\mu_{\mu\alpha} T^{i\alpha} = 0 \tag{3.11}$$

The last term can be similarly simplified to give

$$\delta\Gamma^i_{\mu\alpha} T^{\mu\alpha} = g^{ij} T^{ab} D_a h_{jb} + g^{ij} T^{kl} \hat{D}_k h_{jl} + \frac{2P}{R} (\partial_b R) g^{ab} g^{ij} h_{aj} \tag{3.12}$$

Substituting this in the equation(3.10) and after a little algebra we get

$$\delta(\nabla_\mu T^{i\mu}) = \frac{1}{R^2} g^{ab} g^{ij} D_a (R^2 \tau_{bj}) + \frac{1}{R^4} \gamma^{ik} \gamma^{jl} \hat{D}_j \tau_{kl} \tag{3.13}$$

From chapter 1, we have $\tau_{kl} = R^2 \tau_T \mathcal{V}_{kl}$. Substituting this in the above expression and using the result

$$\hat{D}_j \mathcal{V}^j_k = \frac{m_V}{2k_V} \mathcal{V}_K$$

we have the final equation

$$D_a(R^3 \tau^a) + \frac{m_V}{2k_V} R^2 \tau_T = 0 \tag{3.14}$$

3.4 Energy Conditions

Solving Einstein equations can lead to many possible solutions, some of which may be non-physical and unreasonable. To obtain physically reasonable solutions, we need a physically reasonable energy momentum tensor. We impose some energy conditions on $T^{\mu\nu}$ for this. We give a description of various energy conditions. We have used [27] as a main reference for this section.

Let us assume the following decomposition of energy momentum tensor:

$$T^{\mu\nu} = \rho e^\mu_0 e^\nu_0 + p_1 e^\mu_1 e^\nu_1 + p_2 e^\mu_2 e^\nu_2 + p_3 e^\mu_3 e^\nu_3 \tag{3.15}$$

where e^μ_σ are functions of the coordinates and form an orthonormal basis ie.

$$g_{\mu\nu} e^\mu_\alpha e^\nu_\beta = \eta_{\alpha\beta} \tag{3.16}$$

where $\eta_{\alpha\beta} = diag(-1, 1, 1, 1)$ is the minkowski metric. Let v^α be any normalized future directed timelike vector. Physically it describes the four velocity of any observer in spacetime.

$$v^\alpha = \gamma(e^\alpha_0 + a e^\alpha_1 + b e^\alpha_2 + c e^\alpha_3) \tag{3.17}$$

where γ is the normalization factor given by $\gamma = (1 - a^2 - b^2 - c^2)^{\frac{1}{2}}$ and a, b, c are arbitrary functions of coordinates. Similarly let us also define a future directed null vector k^α by

$$k^\alpha = e^\alpha_0 + a' e^\alpha_1 + b' e^\alpha_2 + c' e^\alpha_3 \tag{3.18}$$

where again a$'$, b$'$ and c$'$ are functions of coordinates and $a'^2 + b'^2 + c'^2 = 1$.

3.4.1 Weak Energy Condition

It states that for every timelike vector field, the energy density observed by the corresponding observer in spacetime is non negative:

$$\rho = T_{\mu\nu} v^{\mu} v^{\nu} \geq 0$$

Substituting the expressions of $T_{\mu\nu}$ and of v^{μ} from (3.15) and (3.17) and after a little simplification we get

$$\rho + a^2 p_1 + b^2 p_2 + c^2 p_3 \geq 0$$

Using the arbitrariness of a, b and c we get

$$\rho \geq 0, \quad \rho + p_i > 0 \tag{3.19}$$

3.4.2 Null Energy Condition

This is the weakest of all energy conditions. It states that for every future directed null vector k^{α}

$$T_{\mu\nu} k^{\mu} k^{\nu} \geq 0$$

Using equations(3.15) and (3.18) we have

$$\rho + a'^2 p_1 + b'^2 p_2 + c'^2 p_3 \geq 0$$

where a′,b′ and c′ are related as described above. Using this we get

$$\rho + p_i \geq 0 \tag{3.20}$$

3.4.3 Strong Energy Condition

According to strong energy condition,

$$(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) v^{\alpha} v^{\beta} \geq 0$$

Using equation (3.15) and (3.17) and the arbitrary nature of a,b and c we have

$$\rho + p_1 + p_2 + p_3 \geq 0, \quad \rho + p_i \geq 0 \tag{3.21}$$

3.4.4 Dominant Energy Condition

According to dominant energy condition, for any future directed, timelike vector field v^μ, $-T^\mu_\nu v^\nu$ will also be a future directed, timelike or null vector field. In other words, $-T^\mu_\nu v^\nu$ is not spacelike or

$$\rho^2 - a^2 p_1^2 - b^2 p_2^2 - c^2 p_3^2 \geq 0$$

Again, from the arbitrariness of a,b and c we have

$$\rho \geq 0, \quad \rho \geq |p_i| \tag{3.22}$$

In the following sections we will be dealing with the perfect and anisotropic fluids. We can write the energy conditions for these cases:

Case 1: Perfect Fluid

For perfect fluid, we have $p_1 = p_2 = p_3 = p$ and hence, from equation (3.15) and (3.16)

$$\begin{aligned} T^{\mu\nu} &= \rho e_0^\mu e_0^\nu + p(e_1^\mu e_1^\nu + e_2^\mu e_2^\nu + e_3^\mu e_3^\nu) \\ &= (\rho + p) e_0^\mu e_0^\nu + p g^{\mu\nu} \end{aligned} \tag{3.23}$$

Comparing with equation of Energy Momentum Tensor of perfect fluid, we see that $e_0^\mu = u^\mu$. The energy conditions can be written as

1. Weak Energy Condition: $\rho \geq 0,\ \rho + p > 0$
2. Null Energy Condition: $\rho + p \geq 0$
3. Strong Energy Condition: $\rho + 3p \geq 0,\ \rho + p \geq 0$
4. Dominant Energy Condition: $\rho \geq 0,\ \rho \geq |p|$

Case 2: Anisotropic Fluids

For anisotropic fluid, we have $p_1 = p_r$, $p_2 = p_3 = p_t$ and hence, from equation (3.15) and (3.16)

$$\begin{aligned} T^{\mu\nu} &= \rho e_0^\mu e_0^\nu + p_r e_1^\mu e_1^\nu + p_t(e_2^\mu e_2^\nu + e_3^\mu e_3^\nu) \\ &= (\rho + p_t) e_0^\mu e_0^\nu + (p_r - p_t) e_1^\mu e_1^\nu + p_t g^{\mu\nu} \end{aligned} \tag{3.24}$$

Comparing with equation of Energy Momentum Tensor of anisotropic fluid we see that $e_0^\mu = u^\mu$ and $e_1^\mu = s^\mu$. The energy conditions can be written as

1. Weak Energy Condition: $\rho \geq 0, \quad \rho + p_t > 0 \quad \rho + p_r \geq 0$
2. Null Energy Condition: $\rho + p_t \geq 0, \quad \rho + p_r \geq 0$
3. Strong Energy Condition: $\rho + p_r + 2p_t \geq 0, \quad \rho + p_r \geq 0, \quad \rho + p_t \geq 0$
4. Dominant Energy Condition: $\rho \geq 0, \quad \rho \geq |p_r| \quad \rho + p_t \geq 0$

Chapter 4

Perfect Fluids

Perfect fluid is defined as the fluid which has no viscosity, heat conduction or other transport or dissipative processes. Such fluid models have been widely studied in the area of stellar collapse and perturbations. In this section, we use perfect fluid case as an example to demonstrate the formalism described in chapter 1. In the first section we describe the energy momentum tensor of perfect fluid and the background spacetime. We then discuss the Einstein equation for the background and perturbed spacetime. We end with a derivation of master equation. For more detailed analysis and solutions to the master equation, refer to the work of Chandrasekhar and Ferrari [3].

4.1 Background Spacetime

Let us consider a gravitational collapse situation of a spherically symmetric star. The general metric for the interior of the spherically symmetric star is given by

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}dr^2 + R^2 d\Omega^2$$

$d\Omega^2$ represents the metric on 2-sphere. Here ν, ψ and R are functions of the comoving coordinates r and t. The exterior of the star is described in general by Vaidya spacetime.

For a static spacetime, the existence of a timelike killing vector requires all the metric and fluid variables to be functions of only r. Hence, for static metric, we can describe the exterior by Schwarzschild metric given by

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Let us now describe the perfect fluid source [28]. The energy momentum tensor for perfect fluids in the rest frame of the observer is given by

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Here ρ is the energy density of the fluid at rest and p is the isotropic pressure. Dust is a special case of perfect fluid with zero pressure. In general, there is an equation of state which relates the pressure with energy density. For example, radiation is also a perfect fluid with equation of state $\rho = 3p$.

In any general frame, the energy momentum tensor of perfect fluid is given by

$$T^{\beta}_{\alpha} = (\rho + p)u_{\alpha}u^{\beta} + p\delta^{\beta}_{\alpha}$$

where $u^{\alpha} = e^{-\nu}\delta^{\alpha}_{0}$ is the four velocity of the fluid which obeys the relations $u_{\mu}u^{\mu} = -1$. In comoving coordinates $u^{\mu} = \begin{pmatrix} e^{-\nu} \\ 0 \\ 0 \\ 0 \end{pmatrix}$

4.2 Background Einstein equations

We can now find the relation between the metric and the fluid variables using the Einstein equations with cosmological constant taken to be zero. Together with equation of state $p = p(\rho)$, these determine the form of metric and fluid variables. Following [5], in this section we work in the system of units for which $\kappa^2 = 1$. Here prime ($'$) denotes derivative wrt. r and dot($\dot{}$) denotes wrt. t.

$$-2RR''e^{-2\psi} + 2\dot{\psi}R\dot{R}e^{-2\nu} + 2\psi'RR'e^{-2\psi} + 1 + \dot{R}^2e^{-2\nu} - R'^2e^{-2\psi} = \rho R^2 \quad (4.1)$$

$$-2R\ddot{R}e^{-2\nu} + 2\nu'RR'e^{-2\psi} + 2R\dot{\nu}\dot{R}e^{-2\nu} - 1 - \dot{R}^2e^{-2\nu} + R'^2e^{-2\psi} = pR^2 \quad (4.2)$$

$$(R\nu''e^{-2\psi} - R\ddot{\psi}e^{-2\nu} - \ddot{R}e^{-2\nu} + R''e^{-2\psi}) + (\nu'^2e^{-2\psi} - \dot{\psi}^2e^{-2\nu})R+$$
$$(\dot{\nu}\dot{\psi}e^{-2\nu} - \nu'\psi'e^{-2\psi})R + (\nu'R'e^{-2\psi} + \dot{\nu}\dot{R}e^{-2\nu} - \dot{\psi}\dot{R}e^{-2\nu} - \psi'R'e^{-2\psi}) = pR \quad (4.3)$$

$$\nu'\dot{R} + R'\dot{\psi} = \partial_t R' \quad (4.4)$$

where the last equation follows from $R_{10} = 0$. Let us introduce a mass function

$$m = \frac{R}{2}(1 + \dot{R}^2 e^{-2\nu} - R'^2 e^{-2\psi})$$

This is the Misner Sharp mass function which describes the mass enclosed by the spherical star of radius R [29]. We can recast equations (4.1) and (4.2) as

$$\frac{2m'}{R^2 R'} = \rho \tag{4.5}$$

$$\frac{2\dot{m}}{R^2 \dot{R}} = -p \tag{4.6}$$

Since p is non zero, hence for any non static case, the Misner-Sharp mass function and hence the mass of the spherical shell keeps changing with time. Therefore it is not possible to describe the exterior of the star by a Schwarzschild metric and so we use Vaidya spacetime. For static case, ν, ψ and R are functions of only R. The Misner-Sharp function is, thus, constant in time as the time derivative of R will be zero. Hence, for static metric, we can choose our exterior spacetime to be Schwarzschild [5]. Note that for static spacetime, equation (4.4) is trivially satisfied. For static spacetime the Misner-Sharp function becomes:

$$m = \frac{R}{2}(1 - R'^2 e^{-2\psi})$$

Thus for static spacetime, equation (4.5) and (4.6) have the same form with above Misner-Sharp function. Equation (4.3) reduces to

$$\left(\nu'' + \frac{R''}{R}\right) + \nu'(\nu' - \psi') + \left(\nu'\frac{R'}{R} - \psi'\frac{R'}{R}\right) = pe^{2\psi} \tag{4.7}$$

The boundary is chosen at $R = R_b$ such that at $R = R_b$ the pressure goes to zero and for $R < R_b$, pressure is positive. At the boundary, the metric should match the exterior spacetime.

4.3 Perturbed Einstein Equations and Master Equation

Using the gauge invariant Einstein equations derived in chapter 1, we can now write down the perturbed Einstein equations for stars sourced by perfect fluid. Let us rewrite the perturbed Einstein equations derived in chapter 1

$$\frac{1}{R^3}D^b\left[R^4\left(D_b\left(\frac{F_a}{R}\right)-D_a\left(\frac{F_b}{R}\right)\right)\right]-\frac{m_V}{R^2}F_a=-2\kappa^2\tau_a \tag{4.8}$$

$$\frac{k_V}{R^2}D_a(RF^a)=-\kappa^2\tau_T \tag{4.9}$$

For perfect fluid case, $R\tau_a=(p+\rho)\alpha u_a$ and $\tau_T=0$. Thus, for $a=t$ we can rewrite equation (4.8) as

$$\begin{aligned}&Re^{-2\psi}\left[\partial_r\Big(\partial_r\Big(\frac{F_t}{R}\Big)-\partial_t\Big(\frac{F_r}{R}\Big)\Big)-(\nu'+\psi')\Big(\partial_r\Big(\frac{F_t}{R}\Big)-\partial_t\Big(\frac{F_r}{R}\Big)\right]\\&+4R'e^{-2\psi}\left[\partial_r\left(\frac{F_t}{R}\right)-\partial_t\left(\frac{F_r}{R}\right)\right]-\frac{m_v}{R^2}F_t=\frac{2\kappa^2}{R}\Big[(p+\rho)\tilde{\alpha}e^{\nu}\Big]\end{aligned} \tag{4.10}$$

For $a=r$ equation (4.8) becomes

$$R\left[\partial_t\left(\partial_t\left(\frac{F_r}{R}\right)-\partial_r\left(\frac{F_t}{R}\right)\right)\right]-R(\dot\nu+\dot\psi-4\dot R)\left[\partial_t\left(\frac{F_r}{R}\right)-\partial_r\left(\frac{F_t}{R}\right)\right]+\frac{m_V}{R^2}F_re^{2\nu}=0 \tag{4.11}$$

Similarly we can write the equation (4.9) as

$$\begin{aligned}&-R^2\left[e^{-2\nu}\partial_t\left(\frac{F_t}{R}\right)-e^{-2\psi}\partial_r\left(\frac{F_r}{R}\right)\right]-2R\left[\frac{F_t}{R}\dot Re^{-2\nu}-\frac{F_r}{R}R'e^{-2\psi}\right]\\&+R^2\left[(\dot\nu-\dot\psi)\frac{F_t}{R}e^{-2\nu}-(\psi'-\nu')\frac{F_r}{R}e^{-2\psi}\right]=0\end{aligned} \tag{4.12}$$

For static metric, equation (4.10) remains same while equation (4.11) becomes

$$Re^{-2\nu}\left[\partial_t\Big(\partial_t\left(\frac{F_r}{R}\right)-\partial_r\left(\frac{F_t}{R}\right)\Big)\right]+\frac{m_V}{R^2}F_r=0 \tag{4.13}$$

and equation(4.12) becomes

$$R\left[e^{-2\nu}\partial_t\left(\frac{F_t}{R}\right)-e^{-2\psi}\partial_r\left(\frac{F_r}{R}\right)\right]-\frac{2F_r}{R}R'e^{-2\psi}+R(\psi'-\nu')\frac{F_r}{R}e^{-2\psi}=0 \tag{4.14}$$

Let us now derive the master equation for perfect fluids. Consider the perturbed conservation law derived in section 3.3 (equation (3.14))

$$g^{ab}D_a(R^3\tau_b)+\frac{m_V}{2k_V}R^2\tau_T=0 \tag{4.15}$$

From this expression, we can write τ_T in terms of τ_a and substitute in expression (4.9). We get

$$D_a(RF^a-\frac{2\kappa^2}{m_V}R^3\tau^a)=0$$

We know that divergence of a curl is always zero. Hence we can rewrite the expression in the bracket as a curl.

$$\epsilon^{ab}D_b\tilde{\Omega}=RF^a-\frac{2\kappa^2}{m_V}R^3\tau^a \tag{4.16}$$

Substituting the value of F^a in equation (4.8), we get the master equation:

$$R^2D_a\left(\frac{1}{R^2}D^a\tilde{\Omega}\right)-\frac{m_V}{R^2}\tilde{\Omega}=-\frac{2\kappa^2}{m_V}R^2\epsilon^{ab}D_a(R\tau_b) \tag{4.17}$$

Further, for static metric, we can consider the perturbations of specific frequency due to the existence of a timelike killing vector field. Hence let us assume $\tilde{\Omega}=e^{\lambda t}\Omega(r)$ and $\tilde{\alpha}=e^{\lambda t}\alpha(r)$ [6]. Substituting these expressions in above equation we get a radial equation for our static metric given by

$$\begin{gathered}\Omega''+\left(\nu'-\psi'-\frac{2R'}{R}\right)\Omega'-\left[\frac{m_V}{R^2}e^{2\psi}+\lambda^2e^{2(\psi-\nu)}\right]\Omega= \\ \frac{-2\kappa^2}{m_V}R^2e^{2\psi+\nu}\Big[\alpha(p'+\rho')+(p+\rho)(\alpha'+\nu'\alpha)\Big]\end{gathered} \tag{4.18}$$

We can convert this to Schrodinger type equation with a new set of coordinate defined as $dr^*=R^2e^{\psi-\nu}dr$. Under this coordinate change, above equation becomes

$$\frac{d^2\Omega}{dr^{*2}} - \left[\frac{m_V}{R^2}e^{2\nu} + \lambda^2\right]\frac{\Omega}{R^4} = \frac{-2\kappa^2}{m_V}e^{2\nu+\psi}\left[\alpha\left(\frac{dp}{dr^*} + \frac{d\rho}{dr^*}\right) + (p+\rho)\left(\frac{d\alpha}{dr^*} + \alpha\frac{d\nu}{dr^*}\right)\right] \tag{4.19}$$

Further from the perturbed conservation law we have

$$(p+\rho)\lambda\alpha e^{\lambda t+\nu} = 0$$

which implies that $\alpha = 0$. Hence RHS of equation (4.19) goes to zero showing that fluid perturbations do not excite axial perturbations and hence metric perturbations remain decoupled from fluid perturbations. For a more detailed analysis, refer to the work of Chandrasekhar and Ferrari [3] Reference used: [19]

Chapter 5

Anisotropic Fluids

5.1 Energy Momentum Tensor

In this chapter, we consider the axial perturbation of stars sourced by anisotropic fluids. The energy momentum tensor for anisotropic fluids in comoving coordinates is given by ([30])

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p_r & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & p_t \end{pmatrix}$$

where ρ is the energy density of the fluid, p_r is the radial(normal) pressure and p_t is the tangential pressure, all dependent on t and r in general. In the limit $p_r \to p_t$ we recover the perfect fluid case.The tensor equation for the above energy momentum tensor is

$$T^{\mu}_{\nu} = (\rho + p_t)u_{\nu}u^{\mu} + (p_r - p_t)s_{\nu}s^{\mu} + p_t\delta^{\mu}_{\nu} \tag{5.1}$$

where $u^{\mu} = e^{-\nu}\delta^{\mu}_{0}$ is the four velocity of the fluid and $s^{\mu} = e^{\psi}\delta^{\mu}_{0}$ is the unit vector in the radial direction. As can be verified, these quantities satisfy the relations $u^{\mu}u_{\mu} = -1$, $s_{\mu}s^{\mu} = 1$ and $s_{\mu}u^{\mu} = 0$

5.2 Energy Conditions

In this section we discuss the energy conditions for the Perturbed Energy Momentum Tensor. Following section 3.4 we can write the decomposition of perturbed energy momentum tensor as

$$\tilde{T}^{\alpha\beta} = \tilde{\rho}\tilde{e}_0^\alpha\tilde{e}_0^\beta + \tilde{p}_1\tilde{e}_1^\alpha\tilde{e}_1^\beta + \tilde{p}_2\tilde{e}_2^\alpha\tilde{e}_2^\beta + \tilde{p}_3\tilde{e}_3^\alpha\tilde{e}_3^\beta \tag{5.2}$$

where vectors $\tilde{e}_\mu^\alpha$ form an orthonormal basis i.e.

$$\tilde{g}_{\alpha\beta}\tilde{e}_\mu^\alpha\tilde{e}_\nu^\beta = \eta_{\mu\nu}$$

These basis vectors are functions of coordinates and $\eta_{\mu\nu}$ is the minkowski metric. The inverse metric is given by

$$\tilde{g}^{\alpha\beta} = \eta^{\mu\nu}\tilde{e}_\mu^\alpha\tilde{e}_\nu^\beta$$

The perturbed energy momentum tensor can also be written as the sum of background and the perturbation.

$$\tilde{T}^{\alpha\beta} = T^{\alpha\beta} + \delta T^{\alpha\beta}$$

Following section 3.4, from equation (3.15) and (3.16) we can also write the background energy momentum tensor in the following form:

$$T^{\alpha\beta} = (\rho + p_t)\hat{e}_0^\alpha\hat{e}_0^\beta + p_t g^{\alpha\beta} + (p_r - p_t)\hat{e}_1^\alpha\hat{e}_1^\beta$$

By comparison with equation(5.1) we see that $u^\mu = \hat{e}_0^\alpha$ and $s^\mu = \hat{e}_1^\alpha$. The perturbation of the above gives

$$\delta T_{\alpha\beta} = 2(\rho + p_t)\hat{e}_{(0\alpha}\delta\hat{e}_{0\beta)} + p_t h_{\alpha\beta} + 2(p_r - p_t)\hat{e}_{(1\alpha}\hat{e}_{1\beta)}$$

$$\tilde{T}_{\alpha\beta} = (\rho + p_t)[\hat{e}_{0\alpha}\hat{e}_{0\beta} + 2\hat{e}_{(0\alpha}\delta\hat{e}_{0\beta)}] + p_t\tilde{g}_{\alpha\beta} + (p_r - p_t)(\hat{e}_{1\alpha}\hat{e}_{1\beta} + 2\hat{e}_{(1\alpha}\hat{\delta}e_{1\beta)})$$

where $\delta u_\alpha = \delta\hat{e}_{0\alpha}$ and $\delta s_\alpha = \delta\hat{e}_{1\alpha}$. Proceeding in the same way as section 3.3, we define a future directed timelike and null vector field.

$$\tilde{v}^\alpha = \gamma(\tilde{e}_0^\alpha + a\tilde{e}_1^\alpha + b\tilde{e}_2^\alpha + c\tilde{e}_3^\alpha)$$

$$\tilde{k}^\alpha = \gamma(\tilde{e}_0^\alpha + a'\tilde{e}_1^\alpha + b'\tilde{e}_2^\alpha + c'\tilde{e}_3^\alpha)$$

We can now write down the energy conditions

1. Weak Energy Condition:

$$\tilde{T}_{\alpha\beta}\tilde{v}^\alpha\tilde{v}^\beta \geq 0$$

This can be simplified to

$$\tilde{T}_{\alpha\beta}\tilde{v}^{\alpha}\tilde{v}^{\beta} = \gamma^2[\rho + p_r a^2 + p_t(b^2 + c^2)] \geq 0$$

Again, using the arbitrariness of a, b and c we can find the conditions on energy density and pressure.

i. Let $a = b = c = 0$ which gives $\rho \geq 0$

ii. Let $b = c = 0$. Thus we have $\rho + p_r > \rho + p_r a^2 \geq 0$.

iii. Let $a = c = 0$.Thus we have $\rho + p_t > \rho + p_t b^2 \geq 0$.

2. Null Energy Condition:

$$\tilde{T}_{\alpha\beta}\tilde{k}^{\alpha}\tilde{k}^{\beta} \geq 0$$

This gives

$$\gamma^2[\rho + p_r a'^2 + p_t(b'^2 + c'^2)] \geq 0$$

Using the arbitrariness of b and c we can see that $\rho + p_r \geq 0$ and $\rho + p_t \geq 0$

3. Strong Energy Condition:

$$(\tilde{T}_{\alpha\beta} - \frac{1}{2}\tilde{T}\tilde{g}_{\alpha\beta})\tilde{v}^{\alpha}\tilde{v}^{\beta} \geq 0$$

The second term in the above expression reduces to trace of perturbed energy momentum tensor (v is a timelike normalized vector field). Substituting the expressions of $\tilde{T}_{\alpha\beta}$ and $\tilde{v}^{\alpha}$ from above and after a little simplification

$$2\gamma^2[(p_t+\rho)(1+2u_\alpha\delta u^\alpha+2u^\alpha\delta u_\alpha)+(p_r-p_t)\tilde{a}^2(1+2s_\alpha\delta s^\alpha+2s^\alpha\delta s_\alpha)-p_t(1-\tilde{a}^2-\tilde{b}^2-\tilde{c}^2)] \geq -\tilde{T}^\alpha_\alpha$$

Since we are only considering the axial perturbations hence the perturbation term do not contribute to the trace of perturbed energy momentum tensor. Therefore,

$$\tilde{T}^\alpha_\alpha = T^\alpha_\alpha$$

Thus the strong energy condition yields:

$$2\gamma^2[\rho + a^2 p_r + p_t(b^2 + c^2)] \geq \rho - p_r - 2p_t$$

Choosing $a = b = c = 0$ we get $\rho + p_r + 2p_t > 0$

5.3 Conservation Law

Let us consider the gravitational collapse situation similar to the perfect fluid case. The form of the metric of stellar interior is given by

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}dR^2 + R^2 d\Omega^2$$

where ν, ψ and R are functions of the comoving coordinates r and t and $d\Omega^2$ represents the metric on a 2-sphere. The exterior of the star is described by the Vaidya spacetime. For the above metric, we find the conservation law for anisotropic fluid to obtain relations between energy density, radial and tangential pressure:

$$\dot{\rho} + (p_r + \rho)\dot{\psi} + (p_t + \rho)\frac{2\dot{R}}{R} = 0 \tag{5.3}$$

$$p_r' + (p_r + \rho)\nu' + (p_r - p_t)\frac{2R'}{R} = 0 \tag{5.4}$$

For a static metric metric, pressure and density will also be a function of only r. As a result, first equation (5.3) is trivially satisfied.

To find the functional form of ν, ψ and R, we need to solve the Einstein equations. For the above metric the Einstein equations are (assuming $\kappa^2 = 1$ for this section):

$$\frac{2m'}{R^2R'} = \rho \tag{5.5}$$

$$\frac{2\dot{m}}{R^2\dot{R}} = -p_r \tag{5.6}$$

$$(\nu''e^{-2\psi} - \ddot{\psi}e^{-2\nu} - \frac{\ddot{R}}{R}e^{-2\nu} + \frac{R''}{R}e^{-2\psi})R^2 + (\nu'^2e^{-2\psi} - \dot{\psi}^2e^{-2\nu})R^2 + (\dot{\nu}\dot{\psi}e^{-2\nu} - \nu'\psi'e^{-2\psi})R^2 + (\nu'R'e^{-2\psi} + \dot{\nu}\dot{R}e^{-2\nu} - \dot{\psi}\dot{R}e^{-2\nu} - \psi'R'e^{-2\psi})R = p_tR^2 \tag{5.7}$$

where m is the Misner Sharp mass function as described in the previous chapter.

$$m = \frac{R}{2}(1 + \dot{R}^2e^{-2\nu} - R'^2e^{-2\psi})$$

For $p_r = 0$, the Misner-Sharp function does not change with time and hence, only a function of radial coordinate. In this case we can describe the exterior of the star by the Schwarzschild metric.

$$ds^2 = -(1 - \frac{2GM}{R})dt^2 + (1 - \frac{2GM}{R})dr^2 + R^2 d\Omega^2$$

However, for any $p_r \neq 0$, the Misner-Sharp function changes with time and hence, we cannot describe the exterior by Schwarzschild metric. In such a case, we have to choose more general Vaidya metric to describe the stellar exterior [5]. For the static case the time derivatives will be zero and equations become

$$2\psi'\frac{R'}{R}e^{-2\psi} + \frac{1}{R^2} - \frac{R'^2}{R^2}e^{-2\psi} = \rho \tag{5.8}$$

$$2\nu' R' e^{-2\psi} - 1 + R'^2 e^{-2\psi} = p_r R^2 \tag{5.9}$$

$$(\nu'' e^{-2\psi} + \frac{R''}{R}e^{-2\psi}) + \nu'^2 e^{-2\psi} - \nu'\psi' e^{-2\psi} + (\nu'\frac{R'}{R}e^{-2\psi} - \psi'\frac{R'}{R}e^{-2\psi}) = p_t \tag{5.10}$$

The above equations can be solved to obtain the functions ν, ψ and R. For static metric, the exterior spacetime will be Schwarzschild. For detailed solutions refer [30]. To consider the axial perturbations let us first consider the conservation law for the perturbed energy momentum tensor. The perturbation of energy momentum tensor can be written as

$$\delta T_{\mu\nu} = 2(\rho + p_t)u_{(\mu}\delta u_{\nu)} + 2(p_r - p_t)s_{(\mu}\delta s_{\nu)} + p_t h_{\mu\nu}$$

Let $\delta u_i = \tilde{\alpha}\mathcal{V}_i$ and $\delta s_i = \tilde{\beta}\mathcal{V}_i$. $\delta u_a = \delta s_a = 0$ because these will behave like scalars on S^2. For static metric we can assume the time dependence of alpha and beta of exponential order i.e. $\tilde{\alpha}(r,t) = \alpha(r)e^{\lambda t}$ and $\tilde{\beta}(r,t) = \beta(r)e^{\lambda t}$. Substituting these expressions into equation (3.14) the perturbed conservation law becomes

$$(p_t + \rho)\lambda\alpha e^{\psi-\nu} + (p_r - p_t)\beta' + (p_r' - p_t')\,\beta + (p_r - p_t)\beta\left(\nu' + \frac{2R'}{R}\right) = 0 \tag{5.11}$$

This gives us a relation between alpha and beta. Also it can also be seen that metric perturbations are completely decoupled from fluid perturbations. However, unlike perfect fluid case, fluid perturbations are not zero and also anisotropic in nature, as is evident from the presence of factor beta.

The anisotropic nature of fluid changes the behavior of the fluid which leads to fluid excitation by axial perturbations. As can be seen, in the absence of anisotropy in the background fluids, $p_r = 0$ and therefore $\alpha = 0$ which is consistent with the perfect fluid analysis.

5.4 Perturbed Einstein Equations

We had derived the general gauge invariant form of perturbed Einstein Equation. Using these equations, we can now write the perturbed Einstein equation for anisotropic fluids. For convenience of readers, we rewrite the equations (2.14) and (2.15)

$$\frac{1}{R^3}D^b\Bigg[R^4\left(D_b\left(\frac{F_a}{R}\right)-D_a\left(\frac{F_b}{R}\right)\right)\Bigg]-\frac{m_V}{R^2}F_a=-2\kappa^2\tau_a \tag{5.12}$$

$$\frac{k_V}{R^2}D_a(RF^a)=-\kappa^2\tau_T \tag{5.13}$$

For anisotropic case, $R\tau_a=(p_t+\rho)\alpha u_a+(p_r-p_t)\beta s_a$ and $\tau_T=0$. Thus, for $a=t$ we can rewrite equation (5.12) as

$$\begin{aligned}&Re^{-2\psi}\Bigg[\partial_r\Big(\partial_r\Big(\frac{F_t}{R}\Big)-\partial_t\Big(\frac{F_r}{R}\Big)\Big)-(\nu'+\psi')\Big(\partial_r\left(\frac{F_t}{R}\right)-\partial_t\left(\frac{F_r}{R}\right)\Big)\Bigg]\\&+4R'e^{-2\psi}\Bigg[\partial_r\left(\frac{F_t}{R}\right)-\partial_t\left(\frac{F_r}{R}\right)\Bigg]-\frac{m_v}{R^2}F_t=\frac{2\kappa^2}{R}\Big[(p_t+\rho)\tilde{\alpha}e^{\nu}\Big]\end{aligned} \tag{5.14}$$

For $a=r$ equation (5.12) becomes

$$\begin{aligned}&-Re^{-2\nu}\left[\partial_t\left(\partial_t\left(\frac{F_r}{R}\right)-\partial_r\left(\frac{F_t}{R}\right)\right)\right]+R(\dot{\nu}+\dot{\psi})e^{-2\nu}\left[\partial_t\left(\frac{F_r}{R}\right)-\partial_r\left(\frac{F_t}{R}\right)\right]\\&-4R\dot{R}e^{-2\nu}\left[\partial_t\left(\frac{F_r}{R}\right)-\partial_r\left(\frac{F_t}{R}\right)\right]-\frac{m_V}{R^2}F_r=-\frac{2\kappa^2}{R}(p_r-p_t)\tilde{\beta}e^{\psi}\end{aligned} \tag{5.15}$$

Similarly we can write the equation (5.13) as

$$\begin{aligned}&-R^2\left[e^{-2\nu}\partial_t\left(\frac{F_t}{R}\right)-e^{-2\psi}\partial_r\left(\frac{F_r}{R}\right)\right]-2R\left[\frac{F_t}{R}\dot{R}e^{-2\nu}-\frac{F_r}{R}R'e^{-2\psi}\right]\\&+R^2\Bigg[(\dot{\nu}-\dot{\psi})\frac{F_t}{R}e^{-2\nu}-(\psi'-\nu')\frac{F_r}{R}e^{-2\psi}\Bigg]=0\end{aligned} \tag{5.16}$$

For static metric,s equation (5.14) remains same while equation (5.15) becomes

$$Re^{-2\nu}\Bigg[\partial_t\Bigg(\partial_t\left(\frac{F_r}{R}\right)-\partial_r\left(\frac{F_t}{R}\right)\Bigg)\Bigg]+\frac{m_V}{R^2}F_r=\frac{2\kappa^2}{R}(p_r-p_t)\tilde{\beta}e^{\psi} \tag{5.17}$$

and equation(5.16) becomes

$$-R^2\left[e^{-2\nu}\partial_t\left(\frac{F_t}{R}\right)-e^{-2\psi}\partial_r\left(\frac{F_r}{R}\right)\right]+2R\frac{F_r}{R}R'e^{-2\psi}-R^2(\psi'-\nu')\frac{F_r}{R}e^{-2\psi}=0 \tag{5.18}$$

We now find the master equation for the anisotropic case. Following the same methodology, as in perfect fluid case, it can be seen that we obtain the same Master equation (equation (4.17)). Simplifying this for the static background metric we get

$$\begin{aligned}&\Omega''+\left(\nu'-\psi'-\frac{2R'}{R}\right)\Omega'-\left[\frac{m_V}{R^2}e^{2\psi}+\lambda^2e^{2(\psi-\nu)}\right]\Omega=\\&\frac{-2\kappa^2}{m_V}R^2e^{2\psi}\left[e^{\nu}\Big(\alpha(p'+\rho')+(p+\rho)(\alpha'+\nu'\alpha)\Big)+e^{\psi}(p_r-p_t)\lambda\beta\right]\end{aligned} \tag{5.19}$$

We make a coordinate change $r^*=\int R^2e^{\psi-\nu}dr$ in the above equation.

$$\begin{aligned}\frac{d^2\Omega}{dr^{*2}}-\left[\frac{m_V}{R^2}e^{2\nu}+\lambda^2\right]\frac{\Omega}{R^4}=\frac{-2\kappa^2}{m_V}e^{2\nu+\psi}\Bigg[&\left(\frac{dp}{dr^*}+\frac{d\rho}{dr^*}\right)\alpha\\&+(p+\rho)\left(\frac{d\alpha}{dr^*}+\alpha\frac{d\nu}{dr^*}\right)+\lambda\beta(p_r-p_t)\Bigg]\end{aligned} \tag{5.20}$$

We see that we get a set of coupled ordinary differential equations (5.11) and (5.20) for the perturbation variables. Further, the metric perturbations though shown to be independent of fluid perturbations, are intricately coupled. To solve the above equations we need a constraint condition relating alpha and beta.

Let us look at the special case for which perturbations of the anisotropic term (s^μ) vanishes. Then from equation (5.11),

$$(p_t+\rho)\lambda\alpha e^{\psi-\nu}=0$$

which gives $\alpha=0$. Thus, for the isotropic perturbations, the fluid and metric perturbations completely decouple and fluid perturbations are not excited by the axial perturbations as seen from the above. In this work we study this case further for a naked singularity spacetime. The general case of non zero anisotropic perturbations will be part of later work.

Chapter 6

Example of an anisotropic spacetime and its axial perturbations

In this chapter, we consider a naked singularity spacetime sourced by anisotropic fluid as an example. There are many examples of anistropic spacetimes without such singularities, but a good motivation for choosing this is to investigate, if as expected, naked singularities are unstable. Our formalism will be the first step in such an investigation. In the first section we describe the background spacetime . In the second section we obtain the master equation and convert it to Schrodinger form. We end this chapter with a discussion of the nature of the solution and the stability of the spacetime.

6.1 Background Spacetime

In a recent study [5] of gravitational collapse of anisotropic fluid stellar models, it has been found that equilibrium configurations can arise as a result of such a collapse beginning from regular initial conditions and the resulting static spacetime can be regular or have naked singularities. We use one of these naked singularity spacetimes as our background. For a detailed discussion about this spacetime refer [5]

We choose a Misner Sharp mass function(for static case) defined in the previous chapter of the form $F(r) = M_o r^3$ and $R = cr^3$. Then solution of the Einstein equations gives the functional form of energy density and pressure.

$$\rho = \frac{M_o}{R^2} \tag{6.1}$$

$$p_t = \frac{M_o^2}{4R^2(1-M_o)} \tag{6.2}$$

where ρ is the energy density of the fluid after collapse and p_t is the corresponding tangential pressure. We work with the fluid for which $p_r = 0$. For $R \to 0$ both tangential pressure and energy density diverge at center showing that $R = 0$ is a curvature singularity. The resulting spacetime consists of an interior metric inside a spherical ball of fluid matched to a Schwarzschild exterior. When $M_o < 1$ we do not have a horizon and this is favoured by reasonable assumptions (e.g., energy conditions etc.) on the fluid. In that case the above singularity becomes a naked singularity. The spacetime is then given by

$$ds^2 = -(1-M_o)\left(\frac{R}{R_b}\right)^{\frac{M_o}{1-M_o}} dt^2 + \frac{dR^2}{1-M_o} + R^2 d\Omega^2 \tag{6.3}$$

where $M_o = 2M_{Total}/R_b$ and $R < R_b$. As in the previous chapters, R_b is the physical radius of the boundary of the final equilibrium state and M_{Total} is total mass at the boundary. This spacetime is a final state of collapse and hence static. Thus, exterior metric (for $R > R_b$) is given by the Schwarzschild metric which can be written as

$$ds^2 = -\left(1 - \frac{M_o R_b}{R}\right) dt^2 + \left(1 - \frac{M_o R_b}{R}\right)^{-1} dR^2 + R^2 d\Omega^2$$

It can be seen that equation (6.3) reduces to the Schwarzschild metric for $R \to R_b$ and thus it is continuous at the boundary.

6.2 Master Equation

We can now use the formalism developed in previous chapters to find the master equation for our background spacetime. The non zero Christoffels of the Lorentzian metric are following:

$$\Gamma^0_{10} = \Gamma^0_{01} = \left(\frac{M_o}{1-M_o}\right)\frac{1}{2R}$$

$$\Gamma^1_{00} = \frac{M_o(1-M_o)}{2R}\left(\frac{R}{R_b}\right)^{\frac{M_o}{1-M_o}}$$

The master equation in covariant form is(equation (4.17)):

$$R^2 D_a\left(\frac{1}{R^2}D^a\tilde{\Omega}\right) - \frac{m_V}{R^2}\tilde{\Omega} = -\frac{2\kappa^2}{m_V}R^2\epsilon^{ab}D_a(R\tau_b)$$

Let $\tilde{\Omega}(R,t) = \Omega(R)e^{\lambda t}$ and let prime denote the derivative w.r.t R. The master equation for the naked singularity spacetime is

$$(1-M_o)\,\Omega'' + \left[\frac{M_o}{2R} - \frac{2(1-M_o)}{R}\right]\Omega' - \left[\frac{\lambda^2}{1-M_o}\left(\frac{R_b}{R}\right)^{\frac{M_o}{1-M_o}} + \frac{m_V}{R^2}\right]\Omega =$$
$$\frac{2\kappa^2}{m_V}R^2\left[\frac{\lambda M_0^2}{4R^2(1-M_0)^{\frac{5}{2}}}\beta - (1-M_o)^{\frac{1}{2}}\left(\frac{R}{R_b}\right)^{\frac{M_o}{2(1-M_o)}}\left[(p_t+\rho)\frac{\alpha M_o}{2R(1-M_o)} + \partial_r\left((p_t+\rho)\alpha\right)\right]\right] \tag{6.4}$$

We consider special perturbations in which the anisotropy direction is not perturbed i.e. $\beta = 0$. Again from the conservation equation it can be seen that for this case $\alpha = 0$. Then the RHS of the master equation vanishes and it reduces to

$$(1-M_o)\,\Omega'' + \left[\frac{M_o}{2R} - \frac{2(1-M_o)}{R}\right]\Omega' - \left[\frac{\lambda^2}{1-M_o}\left(\frac{R_b}{R}\right)^{\frac{M_o}{1-M_o}} + \frac{m_V}{R^2}\right]\Omega = 0 \tag{6.5}$$

This is the master equation for our naked singularity spacetime . This is only valid for $R < R_b$. For $R > R_b$, we look at the Schwarzschild master equation:

$$\left(1 - \frac{M_oR_b}{R}\right)\Omega'' + \left(\frac{3M_oR_b}{R} - 2\right)\frac{\Omega'}{R} - \left[\left(1 - \frac{M_oR_b}{R}\right)^{-1}\lambda^2 + \frac{m_V}{R^2}\right]\Omega = 0 \tag{6.6}$$

6.2.1 Regge Wheeler Equation

In 1957, Regge and Wheeler used the gravitational perturbation method to show the stability of the Schwarzschild singularity. They obtained a Schrodinger type equation for axial perturbations also know as Regge Wheeler equation. For our case the exterior metric is Schwarzschild. We here show that master equation derived for Schwarzschild metric effectively reduces to the Regge Wheeler equation. The master equation for the exterior case is given by equation(6.6). We change our perturbation variable from Ω to Φ defined by $\Omega = R\Phi$. We can write the Master equation for Schrodinger case($R > R_b$) in terms of perturbation variable Φ

$$\left(1 - \frac{M_oR_b}{R}\right)\Phi'' + \frac{M_oR_b}{R^2}\Phi' - \left[\left(1 - \frac{M_oR_b}{R}\right)^{-1}\lambda^2 + \frac{l(l+1)}{R^2} - \frac{3M_oR_b}{R^3}\right]\Phi = 0 \tag{6.7}$$

This is a second order ODE of the form $y''+P(x)y'+Q(x)y=0$. This can be converted to Sturm Liouville form by multiplying it with an integrating factor. We change the variable from R to $d\tilde{R}=(1-\frac{M_oR_b}{R})^{-1}dR$. Replacing R from above equation we get the Regge Wheeler equation:

$$\frac{d^2}{d\tilde{R}^2}(\Phi)-\left[\lambda^2+\left(1-\frac{M_oR_b}{R}\right)\left(\frac{l(l+1)}{R^2}-\frac{3M_oR_b}{R^3}\right)\right]\Phi=0 \tag{6.8}$$

which can again be written as

$$\left(-\frac{d^2}{d\tilde{R}^2}+V_2(R)\right)\Phi=0$$

where the potential is given by

$$V_2(R)=\left[\lambda^2+\left(1-\frac{M_oR_b}{R}\right)\left(\frac{l(l+1)}{R^2}-\frac{3M_oR_b}{R^3}\right)\right]$$

6.2.2 Equation for Ω in Schrodinger form

Let us now use the formalism developed in section 5.4. We rewrite the master equation for the interior and the exterior case to obtain a Schrodinger type equation. The master equation for the interior is given by equation (6.5) while for the exterior its given by equation (6.6). We use a change of variable from R to $\tilde{r}$ as introduced in section 5.4. For interior($R<R_b$),

$$d\tilde{r}=\frac{R^2}{1-M_o}\left(\frac{R_b}{R}\right)^{\frac{M_o}{2(1-M_o)}}dR$$

which on integration gives

$$R=(k\tilde{r})^{\frac{2(1-M_o)}{6-7M_o}};\ k=\frac{6-7M_o}{2}R_b^{-\frac{M_o}{2(1-M_o)}}$$

while for the exterior ($R>R_b$) its given by

$$d\tilde{r}=R^2\left(1-\frac{M_oR_b}{R}\right)^{-1}dR$$

which can be integrated to give

$$\tilde{r}=\frac{R^3}{3}+\frac{M_oR_bR^2}{2}+M_o^2R_b^2R+M_o^3R_b^3ln(R-M_oR_b)+C$$

where C is a constant of integation. It can be seen that this coordinate change is continuous at R_b and at $R=R_b$, $d\tilde{r}=\frac{R_b^2}{1-M_o}dR$. With the following coordinate change the master equations takes the Schrodinger equation form. In general it can be represented by

$$-\frac{d^2\Omega}{d\tilde{r}^2} + V(R)\Omega = 0 \tag{6.9}$$

where potential term V for interior is given by

$$\begin{aligned} V(R) &= V_1 \qquad R < R_b \\ &= V_2 \qquad R \geq R_b \end{aligned} \tag{6.10}$$

where

$$V_1 = \frac{\lambda^2}{R^4} + \frac{m_V(1-M_o)}{R^6}\left(\frac{R}{R_b}\right)^{\frac{M_o}{1-M_o}} \tag{6.11}$$

$$V_2 = \frac{\lambda^2}{R^4} + \frac{m_V}{R^6}\left(1 - \frac{M_o R_b}{R}\right) \tag{6.12}$$

These potential can be expressed in terms of $\tilde{r}$ using the transformations given above.

6.3 Nature of the Solution

To solve the master equation, we need to impose some boundary conditions. We choose the boundary conditions that solutions vanish for $R \to 0$ and are continuous at infinity. Let's consider the Schrodinger type equation for the interior. This is a Schrodinger type equation with zero eigen value. Let Ψ be any general solution of Ω. Then it will satisfy this equation.

$$-\frac{d^2\Psi(\tilde{r})}{d\tilde{r}^2} + V(\tilde{r})\Psi(\tilde{r}) = 0$$

Outgoing solution : We now look at realistic class of solutions. The Schrodinger equation for $R < R_b$ is

$$\frac{d^2}{d\tilde{r}^2}(\Omega) - \left[\frac{\lambda^2}{(k\tilde{r})^{\frac{8(1-M_o)}{6-7M_o}}} + \frac{m_V(1-M_o)}{R_b^{\frac{M_o}{1-M_o}}}\frac{1}{(k\tilde{r})^2}\right]\Omega = 0$$

where $k = \frac{6-7M_o}{2}R_b^{-\frac{M_o}{2(1-M_o)}}$. Let Ψ be a solution of this equation. Then, this can be rewritten as

$$-\frac{d^2\Psi(\tilde{r})}{d\tilde{r}^2} + \left(\frac{K_1}{\tilde{r}^f} + \frac{K_2}{\tilde{r}^2}\right)\Psi(\tilde{r}) = 0$$

where $f = \frac{8(1-M_o)}{6-7M_o}$, $K_1 = \frac{\lambda^2}{k^f}$ and $K_2 = \frac{m_V(1-M_o)}{k^2} R_b^{\frac{-M_o}{1-M_o}}$

Let us look at the behavior of this equation at the boundaries. The boundary is defined by $R = 0$. From the transformation we can see that the corresponding boundary in the new coordinate is

1. $\tilde{r} \to 0$ if $6 - 7M_o > 0$
2. $\tilde{r} \to \infty$ if $6 - 7M_o < 0$

Consider the first condition. In the limit of $\tilde{r}$ tending to zero, both $\tilde{r}^{-2}$ or $\tilde{r}^{-f}$ tend to infinity but one of them will dominate over the other.

Case I: If $0 < M_o < \frac{2}{3}$, $\tilde{r}^{-2}$ dominates.
Case II: If $\frac{2}{3} < M_o < \frac{6}{7}$, $\tilde{r}^{-f}$ dominates

For the second condition, it can be seen that $\tilde{r}^{-f}$ dominates as f becomes negative for this case and $\tilde{r} \to \infty$.

Case 1: $0 < M_o < \frac{2}{3}$

In this region, $\tilde{r}^{-2}$ dominates and hence near $\tilde{r} \to 0$ boundary our Schrodinger type equation becomes:

$$-\frac{d^2\Psi(\tilde{r})}{d\tilde{r}^2} + \frac{K_2}{\tilde{r}^2}\Psi(\tilde{r}) = 0$$

Ansatz: $\Psi(\tilde{r}) = \tilde{r}^p$. Plugging this in the above equation, we get two roots of p:

$$\begin{aligned} p_1 &= \frac{1}{2} + \frac{1}{2}(1 + 4K_2)^{\frac{1}{2}} > 0 \\ p_2 &= \frac{1}{2} - \frac{1}{2}(1 + 4K_2)^{\frac{1}{2}} < 0 \end{aligned} \tag{6.13}$$

The general solution is $\Psi(\tilde{r}) = c_1\tilde{r}^{p_1} + c_2\tilde{r}^{p_2}$. For $r \to 0$, $r^{-|p_2|} \to \infty$ hence $c_2 = 0$.

Case 2: $\frac{2}{3} < M_o < 1$

For this case, the first term of the potential $\tilde{r}^{-f}$ dominates. Hence the equation becomes

$$\frac{d^2\Psi(\tilde{r})}{d\tilde{r}^2} - \frac{K_1}{\tilde{r}^f}\Psi(\tilde{r}) = 0$$

We can obtain solutions to this equation using Frobenius method if f is a rational number. However if f is an irrational number, Frobenius method does not apply and we need to find some alternate way to obtain the general solution to this equation. We solve this equation for $p < q$. The other case can be solved in a similar manner.

Ansatz: $\Psi = \sum_{n=0}^{\infty} a_n \tilde{r}^{\frac{n}{q}}$. Substituting in the Schrodinger equation we have,

$$\sum_{n=-2q}^{-p-1} \left(\frac{n}{q}+2\right)\left(\frac{n}{q}+1\right) a_{n+2q}\tilde{r}^{\frac{n}{q}} + \sum_{m=-p}^{\infty} \left[\left(\frac{m}{q}+2\right)\left(\frac{m}{q}+1\right) a_{m+2q} - K_1 a_{m+p}\right] \tilde{r}^{\frac{m}{q}} = 0$$

This implies that

$$\left(\frac{n}{q}+2\right)\left(\frac{n}{q}+1\right) a_{n+2q} = 0 \qquad \forall\, n \in (-2q, -p-1)$$

Thus, we have either a_0 or a_q as arbitrary depending on the choice of n and rest a_n are zero where m is from 1 to $2q - (p+1)$. We can choose a_o as the independent parameter by keeping $n = -2q$. Recurrence relation is

$$a_{m+2q} = \frac{K_1 a_{m+p}}{(\frac{m}{q}+2)(\frac{m}{q}+1)}; \qquad m = -p, -p+1, ...$$

Let us consider an example. Let $M_o = \frac{2}{5}$. This gives $p = 3$ and $q = 2$. Substituting in the above equations we get either a_o or a_2 as arbitrary. We choose $n = -4$, so $a_2 = 0$. The recurrence relation gives:

$$a_{m+4} = \frac{K_1 a_{m+3}}{(\frac{m}{2}+2)(\frac{m}{2}+1)}; \qquad m = -3, -2, -1...$$

Ratio test:

$$\lim_{q\to\infty} |\frac{a_{q+1}}{a_q}| = \lim_{q\to\infty} \frac{4K_1}{q^2-1} \to 0$$

Hence we have a converging series with radius of convergence as infinite.

$$\Psi = a_0(1 - 4K_1\tilde{r}^{\frac{1}{2}})$$

For r approaching zero, this solution converges to a_o while the derivative diverges to negative infinity.

Let us now look at the solutions for the exterior case. The exterior of the star is given by Schwarzschild metric for which the perturbation can be described by equation (6.17) and (6.20). For $R \to \infty$, the master equation (6.5) becomes

$$\Omega'' - \frac{2\Omega'}{R} + \omega^2\Omega = 0$$

where we have assumed $\lambda = i\omega$ since we are now look for wave like solutions. To obtain solution for this case we change our variable from R to x defined as $R = \frac{1}{x}$. The above equation becomes

$$\Omega'' + \frac{4\Omega'}{x} + \frac{\omega^2}{x^4}\Omega = 0$$

where the prime(') now denotes derivative wrt. x. This is a standard differential equation solution of the form

$$xy'' + ay' + bx^n y = 0$$

general solution of which are given by

$$y = x^{\frac{1-a}{2}} \left[C_1 J_\nu \left(\frac{2\sqrt{b}}{n+1} x^{\frac{n+1}{2}} \right) + C_2 Y_\nu \left(\frac{2\sqrt{b}}{n+1} x^{\frac{n+1}{2}} \right) \right]$$

where $n \neq -1$, $b \neq 0$, $\nu = \frac{|1-a|}{n+1}$. In our case $n = -3$, $a = 4$, $b = \omega^2$ and $\nu = \frac{-3}{2}$. Thus solution for our case becomes $\Omega = x^{\frac{-3}{2}} \left[c_1 J_{\frac{-3}{2}} \left(\frac{-\omega}{x} \right) + c_2 Y_{\frac{-3}{2}} \left(\frac{-\omega}{x} \right) \right]$ For $R \to \infty$, this solution reduces to

$$\Omega = \frac{1}{x} \left[c_1 \sin \frac{\omega}{x} + c_2 \cos \frac{\omega}{x} \right]$$

where c_1 and c_2 are some constants. In terms of R, the solution is

$$\Omega = R \left[c_1 \sin \omega R + c_2 \cos \omega R \right]$$

In terms of Regge Wheeler master variable Φ, the solution becomes

$$\Phi = \left[c_1 \sin \omega R + c_2 \cos \omega R \right]$$

which matches with the solution given by the Regge Wheeler [6]. Thus we have wave solutions.

Conclusions

In this work, we explored the stability issues of spherically symmetric spacetimes sourced by anisotropic fluid. The initial problem is set up by studying the behavior of axial perturbations. To remove the gauge ambiguity, this problem is developed in a gauge invariant form. We then derive the master equation governing the behavior of perturbations, thus reducing the coupled Einstein equations into a single ordinary differential equation. We have further shown that for special cases, this differential equation can be reduced to a Schrodinger type equation.

We find that fluid perturbations are independent of the metric perturbations. However unlike the perfect fluid case, the fluid perturbations remain anisotropic in nature and hence can be non zero in general. These will thus contribute in the master equation. However, if we assume perturbations which do not change the direction of anisotropy, then fluid perturbations vanish showing that anisotropic fluids do not then excite axial perturbations. We have also analyzed a static solutions(naked singularity) as a result of anisotropic fluid collapse where the fluid has zero radial pressure. We compute the nature of the solutions close to the naked singularity and asymptotically at infinity.

This project can be expanded into various directions. The first extension to this work would be to complete the stability analysis of spacetime example dealt in chapter 6 by obtaining approximate solutions in the interior and the exterior and matching the two solutions at the fluid boundary. The questions of interest in this problem are (in)stability of naked singularity as well as quasi-normal modes of this spacetime. The question is whether the spectrum of these modes would be different from that of the star or Schwarzschild black hole. Another direction would be to repeat the analysis of this thesis to polar perturbation of anisotropic fluids. Lastly we would like to consider perturbing the direction of anisotropy as well (i.e. $\beta \neq 0$) which would lead to shear viscosity terms in energy momentum tensor.

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