Microlocal Analysis of Certain Imaging Problems

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This is to certify that this thesis entitled "Microlocal Analysis of Certain Imaging Problems" submitted towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune, represents the work carried out by Ashwin T.A.N under the supervision of Dr. Venkateswaran P. Krishnan.

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Dedicated to my friends and family;
If I were a distribution, they would constitute my singular support.

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Abstract

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by Ashwin T.A.N

Microlocal analysis is concerned with the study of propagation of singularities under the action of various operators. In this thesis, we introduce certain techniques from microlocal analysis and apply them to some problems from Synthetic Aperture Radar imaging. Chpater 1 provides a rapid overview of distribution theory and Fourier transforms, including Schwartz kernels and the concept of a wavefront set. In the next chapter, we present (for the most part without proofs) some elements of the theory of pseudodifferential operators. Their significance in imaging stems from the fact that the action of a pseudodifferential operator on a distribution does not introduce any new singularities.

Chapter 3 introduces a more general class of operators called the Fourier integral operators. We show how Fourier Integral operators correspond naturally to certain Lagrangian submanifolds, which leads to the global theory of FIOs. Chapter 3 concludes with a brief discription of classes of distributions associated to two cleanly intersecting Lagrangians (denoted by $I^{p,l}$ where p and l are real numbers).

Finally, in Chapter 4, we consider two of problems from SAR imaging. In the first problem, the transmitter and receiver are combined into one device, and move along a circular trajectory at a constant height above the ground.The scattering operator F is known to be an FIO. The standard technique in imaging problems is to "back-project" the scattered data and thus we wish to understand the composition F^*F. It is a known result that F^*F belongs to an $I^{p,l}$ class. We outline the standard proof, and also give a new proof (Theorem 4.5) that is based on a characterization of $I^{p,l}$ classes due to Greenleaf and Uhlmann. In the second problem, the transmitter and receiver move along a circular trajectory, but separated by a fixed distance at all times. This problem is more complicated, and we present a new result (Theorem 4.6) that under certain restrictions, F^*F belongs to an $I^{p,l}$ class.

Contents

Chapter 1

Distributions and Fourier Transforms

We begin by fixing some notation. If f is a continuous function on an open set $\Omega \subset \mathbb{R}^n$, the *support* of f, denoted by supp f, is defined as the closure (in Ω) of the set $\{x \in \Omega |\, f(x) \neq 0\}$. We let $C^k(\Omega)$ denote the set of all complex valued functions on Ω which have continuous partial derviatives of all orders $\leq k$ (The function itself is included as the 0-th order derivative). $C^\infty(\Omega) = \cap_k C^k(\Omega)$ is the set of all complex valued functions on Ω that have continuous partial derivatives of all orders. Moreover, the set of functions in $C^k(\Omega)$ (resp. $C^\infty(\Omega)$) whose supports are compact subsets of Ω is denoted by $C_c^k(\Omega)$ (resp. $C_c^\infty(\Omega)$). It is easily seen that all these sets are vector spaces.

If $n \in \mathbb{N}$, an n - multi-index is an n - tuple of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$. The length of α is defined as $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If β is another n-multi-index, we define $\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$. We say that $\beta \leq \alpha$ if $\beta_j \leq \alpha_j$ for every j, and in that case we can define $\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$. If $f \in C^k(\Omega)$ and $|\alpha| \leq k$, we denote by $\partial^\alpha f$ the partial derivative $\partial^{|\alpha|} f / \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$. We also set $\alpha! = \alpha_1! \cdots \alpha_n!$, and if $x \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Thus, the formal statement of Taylor's theorem becomes

$$f(x+h) = \sum_{\alpha \geq 0} \frac{\partial^\alpha f(x)}{\alpha!} h^\alpha$$

1.1 Test Functions and Distributions

We call elements of $C_c^\infty(\Omega)$ as *test functions* on Ω. $C_c^\infty(\Omega)$ has the structure of a Fréchet space, which is an example of a topological vector space. Instead of describing the topology of $C_c^\infty(\Omega)$ in detail, we just note when a sequence in $C_c^\infty(\Omega)$ converges to an element in $C_c^\infty(\Omega)$, which will be sufficient for our purposes.

Definition 1.1 ([1], p. 8)**.** *Let Ω be an open subset of $\mathbb{R}^n$. We say that a sequence $\phi_n \in C_c^\infty(\Omega)$ converges to $\phi \in C_c^\infty(\Omega)$ if there is a fixed compact set $K \subset \Omega$ such that supp $\phi_n \subset K$ for every n, and for every multi-index α, $\partial^\alpha \phi_n \to \partial^\alpha \phi$ uniformly.*

Now, if $f \in L^1_{\text{loc}}(\Omega)$, we can define a linear functional (also denoted by f) on $C_c^\infty(\Omega)$ by

$$\langle f, \phi \rangle = \int f\phi \, dx \qquad \forall \phi \in C_c^\infty(\Omega)$$

It can be shown that $\langle f, \cdot \rangle$ is sequentially continuous, and the right hand side of the above equation vanishes for all ϕ if and only if $f = 0$ a.e. Thus, we may view every locally integrable function as a sequentially continuous linear functional on $C_c^\infty(\Omega)$ in a unique way.

Definition 1.2 ([1], p. 7,9)**.** *Let Ω be an open subset of $\mathbb{R}^n$. We say that a linear functional $u : C_c^\infty(\Omega) \to \mathbb{C}$ is a distribution if whenever $\phi_n \to \phi$ in $C_c^\infty(\Omega)$, $\langle u, \phi_n \rangle \to \langle u, \phi \rangle$. Equivalently, u is a distribution if for every compact $K \subset \Omega$, there exists a non-negative integer N_K and $C_K > 0$ such that*

$$|\langle u, \phi \rangle| \le C_K \sum_{|\alpha| \le N_K} \sup |\partial^\alpha \phi| \quad \forall \phi \in C_c^\infty(K)$$

where $C_c^\infty(K) = \{\phi \in C_c^\infty(\Omega) |$ supp $u \subset K\}$. Estimates like the one above are called semi-norm estimates.

Note. If we can take a single $N = N_K$ for all compact $K \subset \Omega$, u is said to be a distribution of finite order and the least such N is called the order of the distribution u. If u is a distribution of order k, it can be extended to a sequentially continuous linear functional on $C_c^k(\Omega)$. (A sequence $\phi_n \to \phi$ in $C_c^k(\Omega)$ iff there is a fixed compact set K containing supp ϕ_n for all n and $\partial^\alpha \phi_n \to \partial^\alpha \phi$ uniformly for all α with $|\alpha| \le k$.)

An example of a distribution not given by a locally integrable function is the Dirac delta distribution, defined by $\langle \delta_{x_0}, \phi \rangle = \phi(x_0)$. The set of all distributions on Ω is denoted by $\mathcal{D}'(\Omega)$. It is equipped with the weak * topology.

Definition 1.3 ([1], p.13)**.** *A sequence $u_n \to u$ in $\mathcal{D}'(\Omega)$ if for every $\phi \in C_c^\infty(\Omega)$, $\langle u_n, \phi \rangle \to \langle u, \phi \rangle$.*

Let $Y \subset X$ be open sets in $\mathbb{R}^n$. Note that there is a canonical inclusion $C_c^\infty(Y) \hookrightarrow C_c^\infty(X)$ (extend $\phi \in C_c^\infty(Y)$ by 0). Thus, if $u \in \mathcal{D}'(X)$, $\langle u, \phi \rangle$ is well defined for every $\phi \in C_c^\infty(Y)$. We define the *restriction* of u to Y by $u|_Y : \phi \mapsto \langle u, \phi \rangle$ for all $\phi \in C_c^\infty(Y)$.

It is easy to see that $u|_Y$ is a distribution on Y. Using the following theorem, one can show that we can recover the distribution u from its restrictions to a family of open sets that covers the whole space.

Theorem 1.1 (Partition of Unity, [1], p. 11)**.** *Let $X \subset \mathbb{R}^n$ be open and let K be a compact subset of X. Let $X_1, \ldots, X_m$ be open subsets of X such that $K \subset X_1 \cup \cdots \cup X_m$. Then there exist functions $\phi_i \in C_c^\infty(X_i)$ $(1 \leq i \leq m)$ such that $0 \leq \phi_i \leq 1$ for every i,*

$$\sum_{i=1}^{m} \phi_i \leq 1 \text{ on } X, \qquad \sum_{i=1}^{m} \phi_i = 1 \text{ on a neighbourhood of } K$$

Corollary 1.1 ([1], p. 12)**.** *Let $X \subset \mathbb{R}^n$ be open and let $X_j \subset X$, $j \in J$ be open subsets such that $X = \cup_{j \in J} X_j$. Suppose that for each $j \in J$, there is a distribution $u_j \in \mathcal{D}'(X_j)$ such that*

$$u_j = u_i \text{ on } X_j \cap X_i, \qquad \forall i, j \in J$$

Then there exists a unique $u \in \mathcal{D}'(X)$ such that $u|_{X_j} = u_j$ for every j.

As an application of this corollary, we can now define the support of a distribution $u \in \mathcal{D}'(X)$. Let X_j be the family of all open subsets of X such that $u|_{X_j} = 0$. Then by the above corollary $u = 0$ on $Y = \cup_j X_j$. Note that Y is the largest open subset of X on which u is 0. The complement of Y is called the support of u.

Definition 1.4. *Let $X \subset \mathbb{R}^n$ be open and $u \in \mathcal{D}'(X)$. Then the support of u is defined as*

$$supp\ u = (\{x \in X |\, u = 0 \text{ on a neighbourhood of } x\})^c$$

Note that if u is a continuous function, the above definition of the support of u coincides with the previous definition. Similarly, we can define the singular support of a distribution $u \in \mathcal{D}'(X)$ as

$$\text{sing supp } u = (\{x \in X |\, u \in C^\infty \text{ on a neighbourhood of } x\})^c$$

The class of distributions with compact support in Ω is denoted by $\mathcal{E}'(\Omega)$. A distribution $u \in \mathcal{D}'(\Omega)$ is in $\mathcal{E}'(\Omega)$ if and only if it can be extended to a sequentially continuous linear functional on $C^\infty(\Omega)$ ([1], p. 34-35).

Definition 1.5 ([1], p. 34)**.** *A sequence $\phi_j \in C^\infty(\Omega)$ is said to converge to $\phi \in C^\infty(\Omega)$ if for every multi-index α, $\partial^\alpha \phi_j \to \partial^\alpha \phi$ uniformly on all compact subsets of Ω.*

We conclude this section by noting that the class of distributions is in a sense *sequentially closed.*

Theorem 1.2 ([1], p. 15)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and let u_j be a sequence in $\mathcal{D}'(\Omega)$ such that $\langle u_j, \phi \rangle$ converges for every $\phi \in C_c^\infty(\Omega)$. Define u on $C_c^\infty(\Omega)$ by*

$$\langle u, \phi \rangle = \lim_{j \to \infty} \langle u_j, \phi \rangle, \qquad \phi \in C_c^\infty(\Omega)$$

Then $u \in \mathcal{D}'(\Omega)$ and $u_j \to u$ in $\mathcal{D}'(\Omega)$.

1.2 Operations on Distributions

Suppose $f \in C^k(\Omega)$ and $\phi \in C_c^\infty(\Omega)$. By integration by parts, we can see that for every multi-index α with $|\alpha| \leq k$,

$$\langle \partial^\alpha f, \phi \rangle = \int (\partial^\alpha f) \phi \, dx = (-1)^{|\alpha|} \int f (\partial^\alpha \phi) \, dx = (-1)^{|\alpha|} \langle f, \partial^\alpha \phi \rangle$$

But the last expression would still make sense for any distribution f. This allows us to extend the notion of a derivative to any distribution u.

Definition 1.6 ([1], p. 17)**.** *If $u \in \mathcal{D}'(\Omega)$ and α is a multi-index, we define $\partial^\alpha u \in \mathcal{D}'(\Omega)$ by $\langle \partial^\alpha u, \phi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle$ for all $\phi \in C_c^\infty(\Omega)$.*

Moreover, it is easy to see that $\partial^\alpha : \mathcal{D}' \to \mathcal{D}'$ is a sequentially continuous linear map. In general, if $\mu : C_c^\infty(Y) \to C_c^\infty(X)$ is a linear map that takes sequences converging to 0 to sequences converging to 0, the transpose ${}^t\mu$ can be extended to a map $\mathcal{D}'(X) \to \mathcal{D}'(Y)$ by setting ([1], p. 29)

$$\langle {}^t\mu u, \phi \rangle = \langle u, \mu \phi \rangle, \qquad u \in \mathcal{D}'(X), \phi \in C_c^\infty(Y)$$

Another such important operation on distributions is multiplication by a smooth function: If $\phi \in C_c^\infty(X)$, the map $\mu : \psi \mapsto \phi\psi$ is sequentially continuous from $C_c^\infty(X) \to C_c^\infty(X)$. The map μ is self-adjoint and if $u \in \mathcal{D}'(X)$, we define $\phi u \in \mathcal{D}'(X)$ by $\langle \phi u, \psi \rangle = \langle u, \phi\psi \rangle$ for all $\psi \in C_c^\infty(X)$.

Consider a polynomial in $\xi \in \mathbb{R}^n$ whose co-effecients are smooth functions of $x \in \mathbb{R}^n$, given by $P(x, \xi) = \sum_{|\alpha| \leq k} a_\alpha(x) \xi^\alpha$. We denote by $P(x, \partial)$ the linear partial differential operator $\sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha$. By the previous discussion it is easy to see that $P(x, \partial) : \mathcal{D}' \to \mathcal{D}'$ is sequentially continuous.

The next operation we consider is the pullback by a diffeomorphism. Let X and Y be open subsets of $\mathbb{R}^n$ and let $f : X \to Y$ be a diffeomorphism. If $u \in C^\infty(Y)$, it pullback is defined by $(f^*u)(x) = u(f(x))$. If $\phi \in C_c^\infty(X)$, the change of variable formula shows that

$$\begin{aligned}\langle f^*u, \phi\rangle &= \int u(f(x))\phi(x)\,dx \\ &= \int u(y)g^*\phi(y)|\det dg(y)|\,dy\end{aligned}$$

where $g = f^{-1}$. Now, if $u \in \mathcal{D}'(Y)$, its pullback $f^*u \in \mathcal{D}'(X)$ is defined by

$$\langle f^*u, \phi\rangle = \langle u(y), g^*\phi(y)|\det dg(y)|\rangle, \qquad \forall\phi \in C_c^\infty(X)$$

Another important operation that we want to extend to distributions is the convolution of two functions.

Definition 1.7 ([2], p. 16)**.** *Let f and g be two continuous functions on $\mathbb{R}^n$, at least one of which has compact support. Then we define the function $f * g$ on $\mathbb{R}^n$ by $(f*g)(x) = \int f(x-y)g(y)\,dy$.*

The following properties of convolution are easily verified.

Proposition 1.1. *Let $f, g, h \in C(\mathbb{R}^n)$, at least two of which have compact support.*

1. $f * g = g * f$.

2. $f * (g * h) = (f * g) * h$.

3. $\tau_h(f * g) = (\tau_h f) * g = f * (\tau_h g)$ *for all* $h \in \mathbb{R}^n$*, where* τ_h *is the translation map* $\tau_h\phi(y) = \phi(y-h)$.

4. *If* $f \in C^j$ *and* $g \in C^k$*, then* $\partial^{\alpha+\beta}(f*g) = (\partial^\alpha f) * (\partial^\beta g)$ *whenever* $|\alpha| \le j$ *and* $|\beta| \le k$.

5. *If f and g both have compact support, supp $f * g \subset$ supp f + supp g*

The 4th property in the above proposition implies that convolution with smooth function leads to a smooth function. Convolutions can be used to approximate a general function (or even a distribution) with smooth functions. Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be such that supp $\rho \subset \{|x| \le 1\}$, $\rho \ge 0$ and $\int \rho\,dx = 1$. Define $\rho_\epsilon(x) = \epsilon^{-n}\rho(x/\epsilon)$. It is clear that each ρ_ϵ is a non-negative function whose integral is equal to 1 and supp $\rho_\epsilon \subset \{|x| \le \epsilon\}$.

Proposition 1.2 ([1], p. 6)**.** *Let $f \in C_c^k(\mathbb{R}^n)$ for some $k \ge 0$. Then $f_\epsilon := f * \rho_\epsilon \in C_c^\infty(\mathbb{R}^n)$ for every ϵ and $f_\epsilon \to f$ in $C_c^k(\mathbb{R}^n)$.*

Let $\tau_x : C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$ be the translation map $\tau_x\phi(y) = \phi(y-x)$. Evidently, this can be extended to $\mathcal{D}'(\mathbb{R}^n)$ by $\langle \tau_x u, \phi\rangle = \langle u, \tau_{-x}\phi\rangle$. Similarly the self-adjoint operation $\phi \mapsto \check{\phi}$ where $\check{\phi}(y) = \phi(-y)$ can also be extended to distributions by $\langle \check{u}, \phi\rangle = \langle u, \check{\phi}\rangle$. The definition of convolution says that

$$(f*g)(x) = \int f(x-y)g(y)\,dy = \langle f, \tau_x \check{g}\rangle \tag{1.1}$$

This immediately suggests an extension: If $f \in \mathcal{D}'(\mathbb{R}^n)$ and $g \in C_c^\infty(\mathbb{R}^n)$ or if $f \in \mathcal{E}'(\mathbb{R}^n)$ and $g \in C^\infty(\mathbb{R}^n)$, we define $f * g$ as a function through the equation 1.1.

Proposition 1.3 ([2], p. 88)**.** *Let f and g be as above. Then $f*g \in C^\infty(\mathbb{R}^n)$ and for any multi-index α,*

$$\partial^\alpha(f*g) = (\partial^\alpha f)*g = f*(\partial^\alpha g)$$

If f and g both have compact support, we still have supp $f*g \subset$ supp f + supp g. Also, if ρ_ϵ is the sequence defined above, then for any distribution $u \in \mathcal{D}'(\mathbb{R}^n)$, $u*\rho_\epsilon$ is a sequence of C^∞ functions that converges to u in $\mathcal{D}'(\mathbb{R}^n)$. Now, let us define the convolution of two distributions at least one of which has compact support. Notice that if $u \in \mathcal{D}'(\mathbb{R}^n)$, the map $\phi \mapsto u*\phi$ is a continuous linear map from $C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ and commutes with translations. In fact, the converse is also true.

Theorem 1.3 ([2], p. 100)**.** *If $\mu : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is a continuous linear map that commutes with translations, there exists a unique $u \in \mathcal{D}'(\mathbb{R}^n)$ such that $\mu\psi = u*\psi$ for all $\psi \in C_c^\infty(\mathbb{R}^n)$.*

Now let u_1, u_2 be two distributions on $\mathbb{R}^n$, at least one of which has compact support. It is easy to see that $u_1*(u_2*\phi)$ is well defined for every $\phi \in C_c^\infty(\mathbb{R}^n)$ and that the map $\phi \mapsto u_1*(u_2*\phi)$ is sequentially continuous. We define u_1*u_2 to be the unique distribution on $\mathbb{R}^n$ such that

$$(u_1*u_2)*\phi = u_1*(u_2*\phi), \qquad \forall \phi \in C_c^\infty(\mathbb{R}^n)$$

Properties 1,2,3 and 5 of Proposition 1.1 continue to hold for any $f \in \mathcal{D}'(\mathbb{R}^n), g \in \mathcal{E}'(\mathbb{R}^n)$. Also, $\partial^\alpha(u_1*u_2) = (\partial^\alpha u_1)*u_2$.

We conclude this section by defining tensor products of distributions. Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be open. If $f \in C(X)$ and $g \in C(Y)$, the tensor product $f \otimes g$ is the function on $X \times Y$ defined by pointwise multiplication: $(f\otimes g)(x,y) = f(x)g(y)$. As a distribution, this is given by

$$\langle f\otimes g, \chi\rangle = \iint f(x)g(y)\chi(x,y)\,dx\,dy \qquad \forall \chi \in C_c^\infty(X\times Y)$$

We want to define tensor products for distributions. If we take $\chi = \phi \otimes \psi$ with $\phi \in C_c^\infty(X)$ and $\psi \in C_c^\infty(Y)$, we get

$$\langle f \otimes g, \phi \otimes \psi \rangle = \langle f, \phi \rangle \langle g, \psi \rangle \tag{1.2}$$

We want our definition of tensor product of distributions to still satisfy this identity.

Theorem 1.4 ([1], p. 44)**.** *The subspace of $C_c^\infty(X \times Y)$ generated by functions of the form $\phi \otimes \psi$, $\phi \in C_c^\infty(X), \psi \in C_c^\infty(Y)$ is dense in $C_c^\infty(X \times Y)$.*

Thus, equation 1.2 already determines the required distribution on a dense subspace of $C_c^\infty(X \times Y)$. The next theorem says that this can be uniquely extended to a distribution on $X \times Y$.

Theorem 1.5 ([1], p. 45)**.** *Let $u \in \mathcal{D}'(X)$ and $v \in \mathcal{D}'(Y)$. Then there exists a unique distribution on $X \times Y$, called the tensor product of u and v and denoted by $u \otimes v$ such that*

$$\langle u \otimes v, \phi \otimes \psi \rangle = \langle u, \phi \rangle \langle v, \psi \rangle, \quad \forall \phi \in C_c^\infty(X), \psi \in C_c^\infty(Y)$$

1.3 Schwartz Kernels

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be open sets. If $k \in \mathcal{D}'(X \times Y)$, we can define a map $\mu_k : C_c^\infty(Y) \to \mathcal{D}'(X)$ by

$$\langle \mu_k(\psi), \phi \rangle = \langle k, \phi \otimes \psi \rangle \qquad \forall \psi \in C_c^\infty(Y), \phi \in C_c^\infty(X) \tag{1.3}$$

or, to use the integral notation,

$$\int_X \mu_k(\psi)(x)\phi(x)\,dx = \int_X \int_Y k(x,y)\psi(y)\phi(x)\,dy\,dx$$

If k is a locally integrable function, this is simply the integral transform

$$\psi \mapsto \mu_k(\psi)(x) = \int_Y k(x,y)\psi(y)\,dy$$

The distribution k is called the *distribution kernel* or *Schwartz kernel* of the map μ_k. The Schwartz kernel theorem says that a very large family of operators $C_c^\infty(Y) \to \mathcal{D}'(X)$ can be respresented in the form 1.3.

Theorem 1.6 (Schwartz kernel theorem, [2], p. 128)**.** *A linear map $\mu : C_c^\infty(Y) \to \mathcal{D}'(X)$ is sequentially continuous if and only if there exists a $k \in \mathcal{D}'(X \times Y)$ such that for all $\phi \in C_c^\infty(X)$ and $\psi \in C_c^\infty(Y)$,*

$$\langle \mu\psi, \phi \rangle = \langle k, \phi \otimes \psi \rangle \tag{1.4}$$

Morevover, the kernel k is uniquely determined by μ.

Remark. If $k \in \mathcal{D}'(X \times Y)$ is a distribution kernel, the associated map from $C_c^\infty(Y)$ to $\mathcal{D}'(X)$ is also usually denoted by k.

Definition 1.8. *If $k \in \mathcal{D}'(X \times Y)$, its transpose ${}^t k \in \mathcal{D}'(Y \times X)$ is defined by*

$$\langle {}^t k, \chi \rangle = \langle k, {}^t \chi \rangle \qquad \forall \chi \in C_c^\infty(Y \times X)$$

where ${}^t\chi(x, y) = \chi(y, x)$.

If k is actually a function, then ${}^t k(y, x) = k(x, y)$ and so the above definition is consistent. Note that the *maps* k and ${}^t k$ are also adjoints of each other: if $\phi \in C_c^\infty(X)$ and $\psi \in C_c^\infty(Y)$, then by definition,

$$\langle {}^t k\phi, \psi \rangle = \langle {}^t k, \psi \otimes \phi \rangle = \langle k, \phi \otimes \psi \rangle = \langle k\psi, \phi \rangle \tag{1.5}$$

Theorem 1.7 ([1], p. 73)**.** *Let $k \in \mathcal{D}'(X \times Y)$. If ${}^t k$ is a continuous linear map of $C_c^\infty(X)$ into $C^\infty(Y)$, then k can be extended to a map $\mathcal{E}'(Y) \to \mathcal{D}'(X)$ that is sequentially continuous in the following sense: if u_j is a sequence in $\mathcal{E}'(Y)$ such that $u_j \to u$ in $\mathcal{D}'(Y)$ and supp u_j are all contained in a fixed compact set K, then $ku_j \to ku$ in $\mathcal{D}'(X)$.*

Remark. Usually, the extension map $\mathcal{E}'(Y) \to \mathcal{D}'(X)$ is also denoted by k

Definition 1.9. *If a Schwartz kernel $k \in \mathcal{D}'(X \times Y)$ is such that both $k : C_c^\infty(Y) \to C^\infty(X)$ and ${}^t k : C_c^\infty(X) \to C^\infty(Y)$ are sequentially continuous linear maps, k is called a regular kernel.*

Corollary 1.2. *If k is a regular kernel, the maps k and ${}^t k$ extend to sequentially continuous linear maps $\mathcal{E}'(Y) \to \mathcal{D}'(X)$ and $\mathcal{E}'(X) \to \mathcal{D}'(Y)$ respectively*

1.4 Fourier Transforms and Tempered Distributions

Let $f \in L^1(\mathbb{R}^n)$. The Fourier transform of f is defined as the function

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\,dx \qquad \forall \xi \in \mathbb{R}^n$$

It is easy to see that $\hat{f}$ is a bounded continuous function with $|\hat{f}(\xi)| \leq ||f||_{L^1}$ for every $\xi \in \mathbb{R}^n$.

Proposition 1.4 ([1], p. 92)**.** *Let $f, g \in L^1(\mathbb{R}^n)$.*

1. $\int f(x)\hat{g}(x)\,dx = \int \hat{f}(\xi)g(\xi)\,d\xi$.

2. *The convolution* $(f * g)(x) = \int f(x-y)g(y)\,dy$ *is defined for a.e.* $x \in \mathbb{R}^n$ *and* $f * g \in L^1(\mathbb{R}^n)$. *Also,* $\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

If $g \in C_c^\infty(\mathbb{R}^n)$, 1 determines the distribution associated to $\hat{f}$ in terms of f. But one can not use this equation to define the Fourier transform for an arbitrary $f \in \mathcal{D}'(\mathbb{R}^n)$, since the Fourier transform does not map C_c^∞ to C_c^∞.

Definition 1.10 ([1], p. 93)**.** *A function* $\phi \in C^\infty(\mathbb{R}^n)$ *is said to be rapidly decreasing if for every pair of multi-indices* α, β,

$$\sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha \phi(x)| < \infty$$

The space of all rapidly decreasing functions on $\mathbb{R}^n$ *is called the Schwartz space on* $\mathbb{R}^n$ *and is denoted by* $\mathcal{S}(\mathbb{R}^n)$.

$\mathcal{S}(\mathbb{R}^n)$ also has a Fréchet space structure. A sequence $\phi_j \in \mathcal{S}(\mathbb{R}^n)$ converges to 0 if for every pair of multi-indices α, β, $\sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha \phi_j(x)| \to 0$ as $j \to \infty$.

It is easy to see that $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ so that we can define the Fourier transform of any rapidly decreasing function. Also, if $\phi \in \mathcal{S}(\mathbb{R}^n)$, $x_j\phi, \partial_j\phi \in \mathcal{S}(\mathbb{R}^n)$ for every j, so that $\mathcal{S}$ is closed under differentiation and multiplication by polynomials. The importance of the class $\mathcal{S}$ is due to the following result ([2], p. 160-161). Let us denote by D the operator $-i\partial$, so that $D_j = -i\partial_j$ and $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$.

Theorem 1.8. *Let* $\phi \in \mathcal{S}(\mathbb{R}^n)$. *Then*

$$\begin{aligned} \mathcal{F}(D^\alpha \phi)(\xi) &= \xi^\alpha \hat{\phi}(\xi) \\ \mathcal{F}(x^\alpha \phi)(\xi) &= (-1)^{|\alpha|} D^\alpha \hat{\phi}(\xi) \end{aligned}$$

and consequently $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *is a continuous linear map. Its inverse, called the Inverse Fourier transform is also continuous and is given by*

$$\mathcal{F}^{-1}\phi(x) = \check{\phi}(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \phi(\xi)\, d\xi$$

Remark. The equation

$$\phi(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \hat{\phi}(\xi)\, d\xi$$

is sometimes called the Fourier Inversion formula.

Note that if P is a polynomial, $\mathcal{F}(P(D)\phi)(\xi) = P(\xi)\hat{\phi}(\xi)$. Thus, if a distribution f extends to a continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$, we may define its Fourier transform.

Definition 1.11. *We define $\mathcal{S}'(\mathbb{R}^n)$ as the space of those distributions on $\mathbb{R}^n$ which extend to sequentially continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$. Elements of $\mathcal{S}'(\mathbb{R}^n)$ are called* tempered distributions. *A sequence $u_j \in \mathcal{S}'(\mathbb{R}^n)$ is said to converge to $u \in \mathcal{S}'(\mathbb{R}^n)$ if $\langle u_j, \phi\rangle \to \langle u, \phi\rangle$ as $j \to \infty$ for every $\phi \in \mathcal{S}(\mathbb{R}^n)$.*

Definition 1.12 ([1], p. 97)**.** *If $u \in \mathcal{S}'(\mathbb{R}^n)$, its Fourier transform $\hat{u}$ is defined by*

$$\langle \hat{u}, \phi\rangle = \langle u, \hat{\phi}\rangle \qquad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$$

It can be verified by simple computations that Theorem 1.8 can be extended to tempered distributions.

Proposition 1.5 ([1], p. 99)**.** *Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then*

$$\begin{aligned} \mathcal{F}(D^\alpha f)(\xi) &= \xi^\alpha \hat{f}(\xi) \\ \mathcal{F}(x^\alpha f)(\xi) &= (-1)^{|\alpha|} D^\alpha \hat{f}(\xi) \end{aligned}$$

Also, the Fourier transform $\mathcal{F}$ is a continuous linear map from $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ and its inverse is also continuous.

We conclude the section by noting that if $u \in \mathcal{E}'(\mathbb{R}^n)$, its Fourier transform is actually a C^∞ function.

Lemma 1.1 ([1], p. 101)**.** *If $u \in \mathcal{E}'(\mathbb{R}^n)$, $\hat{u}$ is a C^∞ function given by $\hat{u}(\xi) = \langle u(x), e^{-ix\cdot\xi}\rangle$.*

Theorem 1.9. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and let $v \in \mathcal{E}'(\mathbb{R}^n)$. Then $u * v \in \mathcal{S}'(\mathbb{R}^n)$ and*

$$\mathcal{F}(u * v)(\xi) = \hat{u}(\xi)\hat{v}(\xi)$$

1.5 The Wavefront Set

Consider a distribution $u \in \mathcal{E}'(\mathbb{R}^n)$. Then we know that its Fourier transform is a smooth function, and $u \in C_c^\infty(\mathbb{R}^n)$ if and only if $\hat{u}(\xi)$ is a rapidly decreasing function of ξ. For a general $u \in \mathcal{E}'(\mathbb{R}^n)$, it is interesting to look at those directions in which $\hat{u}$ is not rapidly decreasing. More specifically, we make the following definition:

Definition 1.13 ([1], p. 145)**.** *We say that $\xi_0 \notin \Sigma(u)$ if there exists a conic neighbourhood $V \ni \xi_0$ such that*

$$|\hat{u}(\xi)| \leq C_N(1 + |\xi|)^{-N} \qquad \forall \xi \in V, N \in \mathbb{N}$$

where $C_1, C_2, \ldots$ are positive constants.

It can be easily seen that $u \in C_c^\infty(\mathbb{R}^n)$ iff $\Sigma(u) = \emptyset$. $\Sigma(u)$ gives us the directions of the singularities of the distribution u. To find the directions of singularities of u *at* x_0, we localize u by multiplying by a cut-off function that is non-zero near x_0.

Definition 1.14 ([1], p. 145)**.** *Let $u \in \mathcal{D}'(\Omega)$. We say that $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus 0)$ is not in the wavefront set of u, denoted by $WF(u)$ if there exists $\phi \in C_c^\infty(\Omega)$ such that $\phi(x_0) \neq 0$ and $\xi_0 \notin \Sigma(\phi u)$.*

The wavefront set gives us the positions as well as the directions of the singularities of a dstribution. The following proposition shows that it is a refinement of the concept of singular support.

Proposition 1.6 ([1], p. 146)**.** *Let $u \in \mathcal{D}'(\Omega)$ and let $\pi : \Omega \times (\mathbb{R}^n \setminus 0) \to \Omega$ be the projection map $(x, \xi) \mapsto x$. Then*

$$sing\ supp\ u = \pi(WF(u))$$

Example. Let $\Omega \ni x_0$ be an open set in $\mathbb{R}^n$. Consider the Dirac delta distribution δ_{x_0} given by $\langle \delta_{x_0}, \phi \rangle = \phi(x_0)$ for $\phi \in C_c^\infty(\Omega)$. It can be shown that $WF(\delta_{x_0}) = \{x_0\} \times (\mathbb{R}^n \setminus 0)$.

We now present some results on how the wavefront set transforms under various operations on distributions.

Let X be an n-dimensional manifold. Consider its cotangent bundle T^*X. If $x_1, x_2, \dots, x_n$ are local coordinates defined on an open set U of X, then we get corresponding local co-ordinates on T^*U by

$$((x_1, x_2, \dots x_n), (\xi_1 dx_1 + \xi_2 dx_2 + \dots + \xi_n dx_n)) \mapsto (x_1, \dots x_n, \xi_1, \dots \xi_n)$$

$(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ are called canonical local co-ordinates on $T^*(X)$. If $(y_1, \dots y_n)$ is another system of local co-ordinates, it can be easily verified that the resulting canonical local cordinates (y, η) are related to (x, ξ) by

$$\eta = {}^t\left(\frac{\partial x}{\partial y}\right)\xi$$

where $\left(\frac{\partial x}{\partial y}\right)$ denotes the usual Jacobian matrix $\left(\frac{\partial x_i}{\partial y_j}\right)_{i,j}$.

Theorem 1.10 ([1], p. 152)**.** *Let X and Y be open subsets of $\mathbb{R}^n$ and let $f : X \to Y$ be a diffeomorphism. If $u \in \mathcal{D}'(Y)$,*

$$WF(f^*u) = \{(x, {}^t df_x \eta) \mid (f(x), \eta) \in WF(u)\}$$

Thus, under diffeomorphisms, the wavefront set transforms like a subset of the cotangent bundle. So, the wavefront set of $u \in \mathcal{D}'(X)$ can be naturally viewed as a subset of $T^*X \setminus 0$. Henceforth, we always regard the wavefront set to be a subset of the cotangent bundle.

The concept of wavefront sets can be used to extend many operations on distributions. For example, if u and v are distributions with disjoint singular supports, we know how to define the product uv. The next theorem shows that this can be done in some cases even if their singular supports are not disjoint.

Theorem 1.11 ([1], p. 153)**.** *Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$ be such that $(x,\xi) \in WF(u)$ implies $(x, -\xi) \notin WF(v)$. Then the product uv can be defined and*

$$WF(uv) \subset WF(u) \cup WF(v) \cup \{(x, \xi+\eta) \,|\, (x,\xi) \in WF(u) \text{ and } (x,\eta) \in WF(v)\}$$

If u, v are compactly supported, then one shows that the integral $\int \hat{u}(\xi-\eta)\hat{v}(\eta)\,d\eta$ is absolutely convergent, and then takes the Inverse Fourier transform of this function to define uv. The definition is extended to general distributions by a partition of unity.

We conclude with some results that relate the Schwartz kernel of an operator to its action on wavefront sets. We first fix some notation. If $C_1 \subset T^*X \times T^*Y$ and $C_2 \subset T^*Y$. We define

$$C_1 \circ C_2 = \{(x,\xi) \in T^*X \,|\, \exists (y,\eta) \text{ such that } (y,\eta) \in C_2, (x,\xi,y,\eta) \in C_1\}$$

i.e., C_1 is viewed as a relation between T^*X and T^*Y and $C_1 \circ C_2$ is the image of C_2 under this relation. If $C_3 \subset T^*Y \times T^*Z$, $C_1 \circ C_3$ is defined as a composition of relations.

$$C_1 \circ C_3 = \{(x,\xi,z,\theta) \,|\, |\exists (y,\eta) \text{ such that } (x,\xi,y,\eta) \in C_1, (y,\eta,z,\theta) \in C_2\}$$

Also, if $A \subset T^*X \times T^*Y$, we define

$$\begin{aligned} A_X &= \{(x,\xi) \,|\, (x,\xi,y,0) \in A \text{ for some } y \in Y\} \\ A_Y &= \{(y,\eta) \,|\, (x,0,y,\eta) \in A \text{ for some } x \in X\} \\ A' &= \{(x,\xi,y,\eta) \,|\, (x,\xi,y,-\eta) \in A\} \end{aligned}$$

Theorem 1.12 ([2], p. 268)**.** *Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be open and let $K \in \mathcal{D}'(X \times Y)$ be a Schwartz kernel. Then K can act upon $u \in \mathcal{E}'(Y)$ to give $Ku \in \mathcal{D}'(X)$, provided $(y,\eta) \in WF(u)$ implies $(x, 0, y, -\eta) \notin WF(K)$ for any x, and we have*

$$WF(Ku) \subset (WF(K))_X \cup WF'(K) \circ WF(u)$$

Theorem 1.13 (Hörmander-Sato Lemma, [2], p. 270)**.** *Let $K_1 \in \mathcal{D}'(X \times Y)$ and $K_2 \in \mathcal{D}'(Y \times Z)$ be Schwartz kernels. Suppose that the projection supp $K_2 \ni (y, z) \mapsto z$ is a proper map and $WF'(K_1)_Y \cap WF(K_2)_Y = \emptyset$. Then we can form the composition of the corresponding operators $K_1 \circ K_2 : C_c^\infty(Z) \to \mathcal{D}'(Z)$ is well defined and*

$$\begin{aligned} WF'(K_1 \circ K_2) \quad &\subset \quad WF'(K_1) \circ WF'(K_2) \cup (WF(K_1)_X \times Z \times \{0\}) \\ &\cup (X \times \{0\} \times WF'(K_2)_Z) \end{aligned}$$

Chapter 2

Pseudodifferential Operators

Consider the linear partial differential operator $P(x,D) = \sum_{|\alpha|\leq k} a_\alpha(x) D^\alpha$ where a_α are C^∞ functions on $\mathbb{R}^n$. If $u \in \mathcal{S}(\mathbb{R}^n)$, we may use the Fourier Inversion formula to write

$$P(x,D)u(x) = \sum_{|\alpha|\leq k} a_\alpha(x) D^\alpha \left(\frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi \right) \tag{2.1}$$

$$= \sum_{|\alpha|\leq k} a_\alpha(x) \left(\frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \hat{u}(\xi) \xi^\alpha\, d\xi \right) \tag{2.2}$$

$$= \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} P(x,\xi) \hat{u}(\xi)\, d\xi \tag{2.3}$$

The polynomial $P(x,\xi)$ is called the *symbol* of the operator $P(x,D)$. Pseudodifferential operators are generalizations of differential operators, in the sense that they are given by expressions of the form 2.3 where $P(x,\xi)$ is allowed to belong to a larger class of functions. We begin by defining this class.

Definition 2.1 ([3], p. 1)**.** *Let Ω be an open set in $\mathbb{R}^n$ and $m \in \mathbb{R}$. We define $S^m(\Omega \times \mathbb{R}^N)$ to be the set of all functions $P \in C^\infty(\Omega \times \mathbb{R}^N)$ which satisfy the following estimates: given any compact set $K \subset \Omega$ and multi-indices α and β, there exists a constant $c = c(K,\alpha,\beta) > 0$ such that*

$$\sup_{x\in K} |\partial_x^\alpha \partial_\xi^\beta P(x,\xi)| \leq c(1+|\xi|)^{m-|\beta|}$$

for all $\xi \in \mathbb{R}^N$. Elements of $S^m(\Omega \times \mathbb{R}^N)$ are called symbols *of order m on $\Omega \times \mathbb{R}^N$.*

Note that if $m < m'$, $S^m(\Omega \times \mathbb{R}^N) \subset S^{m'}(\Omega \times \mathbb{R}^N)$. So it is natural to define $S^\infty(\Omega \times \mathbb{R}^N) = \bigcup_{m\in\mathbb{R}} S^m(\Omega \times \mathbb{R}^N)$ and $S^{-\infty}(\Omega \times \mathbb{R}^N) = \bigcap_{m\in\mathbb{R}} S^m(\Omega \times \mathbb{R}^N)$.

Remark. If $P \in S^{m_1}$ and $Q \in S^{m_2}$ it is easy to see that $P + Q \in S^{\max(m_1,m_2)}$, and an application of Leibniz formula will show that $PQ \in S^{m_1+m_2}$. Further, if $P \in S^m$, it follows immediately from the definition that $\partial_x^\alpha \partial_\xi^\beta P(x,\xi) \in S^{m-|\beta|}$.

Remark. If $N = \dim \Omega$, we simply write $S^m(\Omega)$ for $S^m(\Omega \times \mathbb{R}^N)$.

Definition 2.2. *Suppose $P(x,\xi) \in S^m(\Omega)$. Then we define the operator $P(x,D)$ by*

$$P(x,D)u(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} P(x,\xi)\hat{u}(\xi)\, d\xi \tag{2.4}$$

$P(x,D)$ is called the pseudodifferential operator *(ΨDO for short) associated to the symbol $P(x,\xi)$. If $P(x,\xi)$ is of order m, then we say $P(x,D)$ is also of order m. The set of all pseudodifferential operators of order m on Ω is denoted by $\Psi^m(\Omega)$. As before, we also set $\Psi^\infty(\Omega) = \bigcup_{m\in\mathbb{R}} \Psi^m(\Omega)$ and $\Psi^{-\infty}(\Omega) = \bigcap_{m\in\mathbb{R}} \Psi^m(\Omega)$.*

Consider $P(x,\xi) = \sum_{|\alpha|\le k} a_\alpha(x)\xi^\alpha$ where a_α are C^∞ functions on Ω. It is easy to see that this is a symbol of order k on Ω. Further, by equation 2.3, the pseudodifferential operator associated to it is precisely $P(x,D) = \sum_{|\alpha|\le k} a_\alpha(x) D^\alpha$. Thus pseudodifferential operators are generalizations of differential operators.

Note that the integral in 2.4 converges whenever $u \in \mathcal{S}(\mathbb{R}^n)$, as $P(x,\cdot)$ has polynomial growth. However, it is more natural to view $P(x,D)$ as acting on functions on Ω and we usually restrict the domain of $P(x,D)$ to $C_c^\infty(\Omega)$. The following lemma is easily proved by differentiating under the integral.

Lemma 2.1. *Let $P(x,D)$ be a ΨDO on Ω. Then $P(x,D) : C_c^\infty(\Omega) \to C^\infty(\Omega)$ is a continuous linear map*

2.1 Kernels of Pseudodifferential Operators

Consider a pseudodifferential operator $P(x,D)$ on Ω. If $u, v \in C_c^\infty(\Omega)$, we have

$$\begin{aligned} \langle P(x,D)u(x), v(x)\rangle &= \frac{1}{(2\pi)^n} \iint e^{ix\cdot\xi} P(x,\xi)\hat{u}(\xi)v(x)\, d\xi\, dx \\ &= \frac{1}{(2\pi)^n} \iiint e^{i(x-y)\cdot\xi} P(x,\xi)u(y)v(x)\, dy\, d\xi\, dx \end{aligned}$$

Let $\check{P}_2$ denote the Inverse Fourier transform of $P(x,\xi)$ with respect to ξ (in the sense of distributions). Then the above equation reduces to

$$\langle P(x,D)u(x), v(x)\rangle = \int \check{P}_2(x, x-y)u(y)v(x)\, dy\, dx$$

From this, we can easily see that the Schwartz kernel of $P(x,D)$ is given by $K(x,y) = \check{P}_2(x, x-y)$. Also, its transpose ${}^tK(x,y) = \check{P}_2(y, y-x)$ gives rise to the map

$$
{}^tP(x,D)u(x) = \frac{1}{(2\pi)^n} \iint e^{i(y-x)\cdot\xi} P(y,\xi)u(y)\,dy\,d\xi \tag{2.5}
$$

which, as we can easily see, is again a continuous linear map from $C_c^\infty(\Omega)$ to $C^\infty(\Omega)$. Thus, K is a regular kernel, and by Corollary 1.2, both $P(x,D)$ and ${}^tP(x,D)$ extend to sequentially continuous linear maps from $\mathcal{E}'(\Omega)$ to $\mathcal{D}'(\Omega)$. In fact, we can say more.

Definition 2.3 ([4], p. 11). *A regular Schwartz kernel $k(x,y) \in \mathcal{D}'(\Omega \times \Omega)$ is said to be* very regular *if it is a C^∞ function outside the diagonal $\Delta = \{(x,x) | x \in \Omega\}$.*

We will see that the kernel of any ΨDO is very regular. An important property of very regular kernels is the so-called pseudolocal property.

Definition 2.4. *An operator $T : \mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$ is said to be pseudolocal if sing supp $Tu \subset$ sing supp u for every $u \in \mathcal{E}'(\Omega)$.*

Theorem 2.1 ([4]). *If a Schwartz kernel $K(x,y) \in \mathcal{D}'(\Omega \times \Omega)$ is very regular, then the associated map $K : \mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$ is pseudolocal.*

Theorem 2.2 ([5], p. 273). *Let $P \in S^m(\Omega)$ and let $K(x,y)$ be the kernel of the ΨDO associated to P.*

1. *The function $f_\alpha(x,z) = z^\alpha \check{P}_2(x,z)$ is in $C^j(\Omega \times \mathbb{R}^n)$ for all multi-indices α with $|\alpha| > m+n+j$. If A is a compact subset of Ω, f_α and all its derivatives of order $\leq j$ are bounded on $A \times \mathbb{R}^n$.*

2. *If $|\alpha| > m+n+j$, $(x-y)^\alpha K(x,y) \in C^j(\Omega \times \Omega)$. In particular, $K(x,y)$ is C^∞ on $\Omega \times \Omega \setminus \Delta_\Omega$.*

The basic idea of the proof is that $z^\alpha \check{P}_2(x,z)$ is the inverse Fourier transform of $D_\xi^\alpha P(x,\xi)$ (up to a scalar multiple). Since differentiation in the ξ variables reduces the order of $P(x,\xi)$, for $|\alpha|$ sufficiently high, the Inverse Fourier transform can be interpreted in the classical sense as an integral. So, 1 follows by arguments involving differentiating under the integral. Now 2 follows from 1 since $K(x,y) = \check{P}_2(x, x-y)$.

Corollary 2.1. *Every pseudodifferential operator is pseudolocal*

Corollary 2.2. *If $P \in S^{-\infty}(\Omega)$, then $P(x,D)$ maps $\mathcal{E}'(\Omega) \to C^\infty(\Omega)$.*

Proof. Theorem 2.2 implies that the kernel K of P is in $C^\infty(\Omega \times \Omega)$. Let $u \in \mathcal{E}'(\Omega)$. Approximating u by a sequence of C_c^∞ functions, we get that Pu is the smooth function given by $Pu(x) = \langle u(y), K(x,y) \rangle$. □

Remark. Operators that map $\mathcal{E}'(\Omega) \to C^\infty(\Omega)$ are called *smoothing* operators. We have shown that every $P \in \Psi^{-\infty}(\Omega)$ is smoothing.

Note that since a pseudodifferential operator maps $C_c^\infty(\Omega) \to C^\infty(\Omega)$ and $\mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$, it does not in general make sense to compose two ΨDOs. However, this can be easily overcome by adding a simple condition on the kernels of the ΨDOs.

Definition 2.5 ([3], p. 28)**.** *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be open sets, and let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projection maps onto the first and second co-ordinates respectively. We say that a closed set $W \subset X \times Y$ is* proper *if for all compact subsets $K \subset X, K' \subset Y$, the sets $\pi_1^{-1}(K) \cap W$ and $\pi_2^{-1}(K') \cap W$ are also compact.*

Definition 2.6. *Let $T : C_c^\infty(Y) \to C^\infty(X)$ be a continuous linear map with Schwartz kernel K. T is said to be properly supported if supp K is a proper subset of $X \times Y$.*

Proposition 2.1 ([5], p. 276)**.** *Let $T : C_c^\infty Y \to C^\infty(X)$ be properly supported. Then T maps $C_c^\infty(Y) \to C_c^\infty(X)$ and $\mathcal{E}'(Y) \to \mathcal{E}'(X)$. Furthermore, T can be extended to continuous linear maps $C^\infty(Y) \to C^\infty(X)$ and $\mathcal{D}'(Y) \to \mathcal{D}'(X)$.*

Thus, it makes sense to speak of $S \circ T$ if at least one of them is compactly supported. It can also be shown that if both S and T are compactly supported, then so is $S \circ T$.

2.2 Action on Sobolev Spaces

Let Ω be an open set in $\mathbb{R}^n$ and let $s \in \mathbb{R}$. The Sobolev space $H_c^s(\Omega)$ is defined by

$$H_c^s(\Omega) = \left\{ u \in \mathcal{E}'(\Omega) \middle| \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi < \infty \right\}$$

with norm defined by $||u||_{H^s} = (\int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi)^{1/2}$. It is a known fact that $\mathcal{E}'(\Omega) = \cup_{s \in \mathbb{R}} H_c^s(\Omega)$. The following Theorem says that if $P \in \Psi^m(\Omega)$, the action of P on u reduces the order of regularity (measured by s) by at most m.

Theorem 2.3 ([5], p. 295)**.** *If $P \in \Psi^m(\Omega)$, then P is a continuous linear map from $H_c^s(\Omega) \to H_{loc}^{s-m}(\Omega)$ for all $s \in \mathbb{R}$. That is, given any $\phi \in C_c^\infty(\Omega)$, there exists $C_{s,\phi} > 0$ such that*

$$||\phi P u||_{H^{s-m}} \leq C_{s,\phi} ||u||_{H^s} \qquad \forall u \in H_c^s(\Omega)$$

2.3 The Symbolic Calculus

Suppose m_j is a sequence of real numbers strictly decreasing to $-\infty$ and $P_j \in S^{m_j}(\Omega \times \mathbb{R}^N)$ for all $j \geq 0$. Since the orders are decreasing, the terms of the series are in a sense getting smaller and smaller. Though the series $\sum_0^\infty P_j$ need not converge, the following notion of *asymptotic sum* is very useful: We say that the formal sum $\sum_0^\infty P_j$ is an *asymptotic expansion* of $P \in S^{m_0}(\Omega \times \mathbb{R}^N)$, and write $P \sim \sum_0^\infty P_j$ if

$$P - \sum_{j=0}^{k-1} P_j \in S^{m_k}(\Omega \times \mathbb{R}^N) \qquad \forall k > 0$$

Proposition 2.2 ([3], p. 8)**.** *Let m_j be a sequence of real numbers strictly decreasing to $-\infty$ and let $P_j \in S^{m_j}(\Omega \times \mathbb{R}^N)$. Then there exists $P \in S^{m_0}(\Omega \times \mathbb{R}^N)$ such that $P \sim \sum_0^\infty P_j$. Furthermore, if Q is another such symbol, $P - Q \in S^{-\infty}(\Omega \times \mathbb{R}^N)$.*

Remark. Operators in $\Psi^{-\infty}(\Omega)$, and smoothing operators in general, are regarded as in a sense *negligible*. Thus, while considering an operator $P \in \Psi^m(\Omega)$, we are generally only interested in its equivalence class in $\Psi^m(\Omega)/\Psi^{-\infty}(\Omega)$.

Definition 2.7 ([4], p. 32)**.** *A symbol $P \in S^m(\Omega)$ is said to be* classical *if P admits an asymptotic expansion of the form*

$$P(x,\xi) \sim \sum_{j=0}^{\infty} \chi(\xi) P_j(x,\xi) \tag{2.6}$$

Here χ is a smooth function such that $\chi(\xi) = 0$ for $|\xi| \leq 1/2$ and $\chi(\xi) = 1$ for $|\xi| \geq 1$, and $P_j(x,\xi) \in C^\infty(\Omega \times (\mathbb{R}^n \setminus 0))$ is positively homogeneous of degree $m - j$, that is,

$$P_j(x, t\xi) = t^{m-j} P(x,\xi) \qquad \forall x \in \Omega,\ \xi \in \mathbb{R}^n \setminus 0,\ \text{and } t > 0$$

It can be checked that $\chi(\xi)P_j(x,\xi) \in S^{m-j}(\Omega)$. ΨDOs associated to classical symbols are called *classical pseudodifferential operators.* Note that all linear partial differential operators are classical ΨDOs, since any polynomial in ξ with C^∞ coefficients in x is classical.

Definition 2.8. *If P is a classical symbol with an asymptotic expansion as in equation 2.6, $P_0(x,\xi)$ is called the principal symbol of P.*

We now move on to computing the transposes, adjoints and compositions of ΨDOs. Recall that if $P(x,D)$ is a ΨDO, its transpose is given by (equation 2.5)

$${}^tP(x,D)u(x) = \frac{1}{(2\pi)^n} \iint e^{i(x-y)\cdot\xi} P(y,-\xi) u(y)\, dy\, d\xi$$

This indicates that the class of ΨDOs can be profitably extended by allowing the symbols to depend on both x and y.

Definition 2.9. *Let $X \subset \mathbb{R}^p$ and $Y \subset \mathbb{R}^n$ be open sets and let m be a real number. We denote by $S^m(X \times Y \times \mathbb{R}^N)$ the set of all $a \in C^\infty(X \times Y \times \mathbb{R}^N)$ that satisfy the following estimates: given a compact subset $K \subset X \times Y$ and multi-indices α, β, γ, there exist positive constants $C = C_{K,\alpha,\beta,\gamma}$ such that*

$$\sup_{(x,y)\in K} |\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi)| \leq C(1 + |\xi|)^{s-|\gamma|} \qquad \forall \xi \in \mathbb{R}^N$$

If Ω is an open set in $\mathbb{R}^n$, we denote $A^m(\Omega) = S^m(\Omega \times \Omega \times \mathbb{R}^n)$ and their elements are called amplitudes *of order m.*

Given $a \in A^m(\Omega)$, we define $P_a : C_c^\infty(\Omega) \to C^\infty(\Omega)$ by

$$P_a u(x) = \frac{1}{(2\pi)^n} \iint e^{i(x-y)\cdot\xi} a(x, y, \xi) u(y)\, dy\, d\xi$$

Integration must be carried out in the indicated order. The kernel K of P_a is given by $K(x, y) = \check{a}_3(x, y, x - y)$, where $\check{a}_3$ denotes the Inverse Fourier transform of a in the third variable.

Proposition 2.3 ([5], p. 285)**.** *If $a \in A^m(\Omega)$, there exists $b \in A^m(\Omega)$ such that P_b is properly supported and $P_a - P_b$ is a smoothing operator.*

Recall that if an operator $T : C_c^\infty(Y) \to C^\infty(X)$ is properly supported, it can be extended to a continuous map from $C^\infty(Y) \to C^\infty(X)$. The next theorem shows that the class of properly supported P_a's exactly coincides with the class of properly supported ΨDO's.

Theorem 2.4 ([5], p. 286)**.** *Let $a \in A^m(\Omega)$ be such that P_a is properly supported. Define*

$$P(x, \xi) = e^{-ix\cdot\xi} P_a(e^{ix\cdot\xi}) \tag{2.7}$$

Then $P \in S^m(\Omega)$ and $P(x, D) = P_a$. Furthermore, we have the asymptotic expansion

$$P(x, \xi) \sim \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha a(x, y, \xi)|_{y=x}$$

Corollary 2.3. *If $a \in A^m(\Omega)$, there exists a properly supported $Q \in \Psi^m(\Omega)$ such that $P_a - Q$ is a smoothing operator. Moreover, if $a \in S^m(\Omega)$, there exists a properly supported $Q \in \Psi^m(\Omega)$ such that $a(x, D) - Q \in \Psi^{-\infty}(\Omega)$.*

Remark. In general, different symbols can give rise to the same ΨDO. However if $P \in \Psi^m$ is properly supported, equation 2.7 gives us a canonical choice for the symbol of P. We denote it by σ_P.

If S and T are linear maps from $C_c^\infty(\Omega)$ to $C^\infty(\Omega)$, we say that S is the adjoint of T and write $S = T^*$ if $\langle Tu, \overline{v}\rangle = \langle u, \overline{Sv}\rangle$. T^* has distribution kernel $K^*(x,y) = \overline{K(y,x)}$. It follows that if T is properly supported then so are ${}^tT$ and T^*. Also, if $P \in S^m(\Omega)$, equation 2.5 shows that ${}^tP(x,D) = P_a$ where $a(x,y,\xi) = P(y,-\xi)$ and a similar computation shows that $P(x,D)^* = P_b$ where $b(x,y,\xi) = \overline{P(y,\xi)}$. Moreover, by an application of the previous theorem, we can conclude the following:

Theorem 2.5 ([5], p. 291)**.** *If $P \in \Psi^m(\Omega)$ is properly supported, then ${}^tP, P^* \in \Psi^m(\Omega)$ and*

$$\begin{aligned}\sigma_{{}^tP}(x,\xi) &\sim \sum_{|\alpha|\geq 0} \frac{(-1)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha D_x^\alpha \sigma_P(x,-\xi),\\ \sigma_{P^*}(x,\xi) &\sim \sum_{|\alpha|\geq 0} \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha \overline{\sigma_P(x,\xi)}\end{aligned}$$

Theorem 2.6 ([5], p. 291)**.** *If $P \in \Psi^m(\Omega)$ and $Q \in \Psi^{m'}(\Omega)$ are properly supported, then $QP := Q \circ P \in \Psi^{m+m'}(\Omega)$. Moreover, $QP = P_a$ where $a(x,y,\xi) = \sigma_Q(x,\xi)\sigma_{{}^tP}(y,-\xi)$ and*

$$\sigma_{QP}(x,\xi) = \sum_{|\alpha|\geq 0} \frac{1}{\alpha!}\partial_\xi^\alpha \sigma_Q(x,\xi)\cdot D_x^\alpha \sigma_P(x,\xi)$$

2.4 Propagation of Singularities

We have seen that pseudodifferential operators are pseudolocal: If $P \in \Psi^\infty(\Omega)$ and $u \in \mathcal{E}'(\Omega)$, sing supp $Pu \subset$ sing supp u. This result can be further refined .

Theorem 2.7 ([5], p.307)**.** *If $P \in \Psi^\infty(\Omega)$ and $u \in \mathcal{E}'(\Omega)$,*

$$WF(Pu) \subset WF(u)$$

Remark. This property of pseudodifferential operators is called microlocality

There is a class of pseudodifferential operators for which the reverse inclusion also holds.

Definition 2.10 ([5], p. 297)**.** *A symbol $P \in S^m(\Omega)$ and its corresponding operator $P(x,D) \in \Psi^m(\Omega)$ are said to be* elliptic of order m *if for every compact set $K \subset \Omega$, there exist positive constants c_K, r_K such that*

$$|P(x,\xi)| \geq c_K|\xi|^m \qquad \forall x \in K, |\xi| \geq r_K$$

Remark. If $P(x, \xi)$ is a classical symbol with principal symbol $P_0(x, \xi)$, P is elliptic iff $P_0(x, \xi) \neq 0$ for $\xi \neq 0$.

Elliptic ΨDOs are invertible in $\Psi^{\infty}/\Psi^{-\infty}$.

Theorem 2.8 ([5], p. 298)**.** *If $P \in \Psi^m(\Omega)$ is elliptic, there exists a $Q \in \Psi^{-m}(\Omega)$ such that $QP - I \in \Psi^{-\infty}(\Omega)$ and $PQ - I \in \Psi^{-\infty}(\Omega)$. Here I is the identity operator. Q is called a* parametrix *for P.*

Corollary 2.4 (The Elliptic Regularity Theorem)**.** *If P is an elliptic ΨDO, $WF(Pu) = WF(u)$.*

Proof. By Theorem 2.8, there exists a ΨDO Q such that $QP - I \in \Psi^{-\infty}$. This implies that $WF(u) = WF(QPu) \subset WF(Pu)$ since Q is ΨDO. This along with Theorem 2.7 implies that $WF(Pu) = WF(u)$. $\square$

Example. If a distribution u satisfies the Cauchy-Riemann equations (in the sense of distributions) in $\Omega \subset \mathbb{R}^2$, the above theorem implies that u must in fact be an analytic function in Ω.

Chapter 3

Fourier Integral Operators

Fourier Integral Operators are extensions of ΨDOs, in the sense that they are given by expressions of the form

$$Au(x) = \int e^{i\phi(x,y,\xi)} a(x,y,\xi) u(y)\, dy\, d\xi$$

where $a(x,y,\xi) \in S^\infty$ and the function ϕ satisfies certain properties.

Definition 3.1 ([3], p. 9)**.** *Let Ω be an open set in $\mathbb{R}^n$. A function $\phi \in C^\infty(\Omega \times (\mathbb{R}^N \setminus \{0\}))$ is called a* phase function *if for all $(x,\xi) \in \Omega \times (\mathbb{R}^N \setminus \{0\})$,*

1. *Im $\phi(x,\xi) \geq 0$*
2. *$\phi(x,\lambda\xi) = \lambda\phi(x,\xi)$ for all $\lambda > 0$*
3. *$\nabla_{x,\xi}\phi(x,\xi) \neq 0$*

where $\nabla_{x,\xi}$ denotes the operator $(\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{\xi_1}, \ldots \partial_{\xi_N})$.

Note. From now on, we will denote $\mathbb{R}^N \setminus \{0\}$ by $\dot{\mathbb{R}}^N$.

For example, $\phi = (x-y)\cdot\xi \in C^\infty(\Omega \times \Omega \times \mathbb{R}^n)$ is a phase function, and for this ϕ, A is nothing but the pseudodifferential operator P_a (up to a scalar multiple). Note that the kernel of A is given by

$$A(x,y) = \int e^{i\phi(x,y,\xi)} a(x,y,\xi)\, d\xi$$

which, clearly, need not be absolutely convergent. In the case of ΨDOs, such integrals were interpreted as Inverse Fourier transforms of tempered distributions. We begin by interpreting the above integral in the general case of ϕ being a phase function.

3.1 Oscillatory Integrals

Suppose $\phi \in C^\infty(\Omega \times \dot{\mathbb{R}}^N)$ is a phase function and $a(x,\xi) \in S^m(\Omega \times \mathbb{R}^N)$. Then if $m < -N$, it makes sense to define

$$I(a,\phi)(x) = \int e^{i\phi(x,\xi)} a(x,\xi)\, d\xi \tag{3.1}$$

In fact, if $m + k < -N$, $k \in \mathbb{N}$ we may differentiate under the integral k times and see that $I(a,\phi) \in C^k(\Omega)$. Note that due to the positive homogeneity condition on ϕ, $\partial_x^\alpha \phi(x,\xi) \in O(1+|\xi|)$ for any multi-index α.

There is a way of extending this definition to interpret $I(a,\phi)$ as a distribution for *any* $a \in S^\infty(\Omega \times \mathbb{R}^N)$.

Lemma 3.1 ([7], p. 89)**.** *Let $\phi \in C^\infty(\Omega \times \dot{\mathbb{R}}^N)$ be a phase function. Then there exist $a_j \in S^0(\Omega \times \mathbb{R}^N)$ and $b_j, c \in S^{-1}(\Omega \times \mathbb{R}^N)$ such that the differential operator*

$$L = \sum_{j=1}^N a_j(x,\xi)\frac{\partial}{\partial \xi_j} + \sum_{j=1}^n b_j(x,\xi)\frac{\partial}{\partial x_j} + c(x,\xi)$$

satisfies ${}^tL(e^{i\phi}) = e^{i\phi}$.

Proof. The function

$$\Phi(x,\xi) = \sum_{j=1}^n \left|\frac{\partial \phi}{\partial x_j}\right|^2 + |\xi|^2 \sum_{j=1}^N \left|\frac{\partial \phi}{\partial \xi_j}\right|^2$$

is $\neq 0$ for $\xi \neq 0$ and is positively homogeneous of degree 2 with respect to ξ. Let $\chi(\xi) \in C_c^\infty(\mathbb{R}^N)$ be a cutoff function that is equal to 1 in a neighbourhood of 0, and define

$$\begin{aligned} {}^tL &= \frac{(1-\chi(\xi))}{i\Phi(x,\xi)}\left(\sum_{j=1}^N |\xi|^2 \overline{\frac{\partial\phi}{\partial\xi_j}}\frac{\partial}{\partial\xi_j} + \sum_{j=1}^n \overline{\frac{\partial\phi}{\partial x_j}}\frac{\partial}{\partial x_j}\right) + \chi(\xi) \\ &= \sum a_j' \frac{\partial}{\partial \xi_j} + \sum b_j' \frac{\partial}{\partial x_j} + c' \end{aligned}$$

Then we have ${}^tL(e^{i\phi}) = e^{i\phi}$ and $a_j' \in S^0, b_j' \in S^{-1}, c' \in S^{-\infty}$. Note that $L = {}^t({}^tL) = a_j \sum \frac{\partial}{\partial \xi_j} + b_j \sum \frac{\partial}{\partial x_j} + c$ where

$$a_j = -a_j', \quad b_j = -b_j', \quad c = c' - \sum \partial a_j'/\partial \xi_j - \sum \partial b_j'/\partial x_j \in S^{-1}$$

Thus, L has all the required properties. $\square$

If $m < -N$, and $u \in C_c^\infty(\Omega)$, we have

$$\langle I(a,\phi), u\rangle = \iint e^{i\phi(x,\xi)} a(x,\xi) u(x)\, d\xi\, dx \tag{3.2}$$

$$= \iint ({}^tL)^k (e^{i\phi}) a(x,\xi) u(x)\, d\xi\, dx \tag{3.3}$$

$$= \iint e^{i\phi} L^k (a(x,\xi) u(x))\, d\xi\, dx \tag{3.4}$$

Now, it can be easily verified that if $a \in S^m$, $L^k(au) \in S^{m-k}$, so that for any value of m, the last integral converges absolutely if we take k large enough. Thus, for any $a \in S^\infty$, 3.4 defines $I(a,\phi)$ as a distribution on Ω. By convention, we write

$$I(a,\phi)(x) = \int e^{i\phi(x,\xi)} a(x,\xi)\, d\xi$$

no matter what the order of a, with the understanding that $I(a,\phi)$ is to be interpreted as a distribution as in equation 3.4. Note that $I(a,\phi)$ is a distribution of order $\leq k$ if $m - k + N < 0$.

Definition 3.2 ([3], p. 12). *Let $a \in S^\infty(\Omega \times \mathbb{R}^N)$ and let $\phi \in C^\infty(\Omega \times \dot{\mathbb{R}}^N)$ be a phase function. Then, $I(a,\phi)$, as defined above, is called the* oscillatory integral *or the* Fourier integral distribution *associated to the symbol a and phase function ϕ.*

Note. This definition is independent of L. In fact, This is the unique way of extending the definition of $I(a,\phi)$ such that the map $S^m \ni a \mapsto I(a,\phi) \in \mathcal{D}'(\Omega)$ is continuous for every $m \in \mathbb{R}$. For details see [3], Theorem 1.11 or [7], Proposition 1.2.2.

Definition 3.3. *Let $\phi \in C^\infty(\Omega \times \dot{\mathbb{R}}^N)$ be a phase function. We define the* critical set *of ϕ as $C_\phi = \{(x,\xi) \in \Omega \times \dot{\mathbb{R}}^N \,|\, d_\xi \phi(x,\xi) = 0\}$.*

Note that when $\phi = (x-y)\cdot\xi \in C^\infty(\Omega\times\Omega\times\mathbb{R}^n)$ as in the kernels of ΨDOs, $C_\phi = \Delta$, the diagonal set in $\Omega \times \Omega$. The singularities of $I(a,\phi)$ are determined by the behavior of a near C_ϕ.

Proposition 3.1 ([3], p. 12). *Let $\phi \in C^\infty(\Omega \times \dot{\mathbb{R}}^N)$ be a phase function. If $a \in S^m(\Omega \times \mathbb{R}^N)$ vanishes in a conical neighbourhood of C_ϕ, then $I(a,\phi) \in C^\infty(\Omega)$.*

Corollary 3.1. *Let $I(a,\phi)$ be defined as above and let π be the projection map $\Omega \times \mathbb{R}^N \ni (x,\xi) \mapsto x$. Then*

$$\text{sing supp } I(a,\phi) \subset \pi(C_\phi)$$

Remark. Let $V \subset \Omega \times \mathbb{R}^N$ is a conic open set. If $\phi \in C^\infty(V)$ satisifies conditions 1-3 in Definition 3.1, it is called a phase function in V. Moreover, we can define $I(a, \phi)$ as above for all $a \in S^m(\Omega \times \mathbb{R}^n)$ whose support is contained in a conic open subset of V, and the obvious analogues of the above results continue to hold.

A Fourier integral operator is an operator whose Schwartz kernel is a Fourier integral distribution:

Definition 3.4 (Preliminary). *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be open sets. Suppose $a \in S^\infty(X \times Y \times \mathbb{R}^N)$ and $\phi \in C^\infty(X \times Y \times \dot{\mathbb{R}}^N)$ is a phase function. Let $K(x,y) = I(a,\phi)(x,y)$ be the Fourier integral distribution as defined above. We may now define $A : C_c^\infty(Y) \to \mathcal{D}'(X)$ by*

$$\langle Au(x), v(x)\rangle = \langle K(x,y), v(x) \otimes u(y)\rangle \qquad u \in C_c^\infty(Y), v \in C_c^\infty(X) \tag{3.5}$$

Operators such as A are called Fourier integral operators.

We will make some modifications to this definition later.

Theorem 3.1 ([3], p. 13). *Let A be defined as in equation 3.5.*

1. *If for fixed x, $(y,\xi) \mapsto \phi(x,y,\xi)$ is a phase function, then $A : C_c^\infty(Y) \to C^\infty(X)$ is continuous.*

2. *If for fixed y, $(x,\xi) \mapsto \phi(x,y,\xi)$ is a phase function, then ${}^tA : C_c^\infty(X) \to C^\infty(Y)$ is continuous and A has a continuous extension $\mathcal{E}'(Y) \to \mathcal{D}'(X)$.*

If both 1 and 2 hold, $K(x,y)$ is evidently a regular kernel.

3.2 The Method of Stationary Phase

Let Ω be an open set in $\mathbb{R}^n$ and let $u \in C_c^\infty(\Omega)$. Let $\phi \in C^\infty(\Omega)$ be real valued. We want to investigate the behaviour of integrals of the form

$$I(\lambda) = \int e^{i\lambda\phi(x)} u(x)\, dx$$

as $\lambda \to \infty$. Note that if $\phi(x) = x \cdot \xi$ for some $\xi \in \mathbb{R}^n$, $I(\lambda)$ is rapidly decreasing as $\lambda \to \infty$ (as the Fourier transform maps $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$). The same is in fact true for a larger class of functions ϕ.

Lemma 3.2 ([3])**.** *Let $I(\lambda)$ be defined as above, and suppose that $\nabla\phi \neq 0$ on supp u. Then for every $N \in \mathbb{N}$ there exists a constant C_N such that*

$$|I(\lambda)| \leq \left(C_N \sum_{|\alpha|\leq N} \sup |\partial^\alpha u| \right) \lambda^{-N}$$

Proof. Consider the operator

$${}^tL = \frac{1}{i\lambda|\nabla\phi|^2} \sum \frac{\partial\phi}{\partial x_j} \frac{\partial}{\partial x_j}$$

Clearly, ${}^tL(e^{i\lambda\phi}) = e^{i\lambda\phi}$. Then, by integration by parts, we have for every $N \in \mathbb{N}$,

$$\begin{aligned} I(\lambda) &= \int ({}^tL)^N (e^{i\lambda\phi(x)}) u(x)\, dx \\ &= \int e^{i\lambda\phi} L^N(u(x))\, dx \end{aligned}$$

from which the result immediately follows. □

This indicates that the asymptotic behaviour of $I(\lambda)$ is determined by the behavior of ϕ, u near points x where $\nabla\phi(x) = 0$. Such points are called critical points of ϕ.

Definition 3.5. *Let $\phi \in C^\infty(\Omega)$. We say that ϕ has a* non-degenerate critical point *at $x_0 \in \Omega$ if $\nabla\phi(x_0) = 0$ and $\det D^2\phi(x_0) \neq 0$. Here $D^2\phi$ denotes the Hessian matrix $(\frac{\partial^2\phi}{\partial x_i \partial x_j})_{i,j}$.*

If ϕ has only non-degenerate critical points, we have the following result regarding the asymptotic behaviour of $I(\lambda)$.

Theorem 3.2 (Stationary Phase asymptotics, [3], p. 22)**.** *Let ϕ be a real-valued C^∞ function on Ω such that ϕ has a non-degerate critical point at $x_0 \in \Omega$ and $\nabla\phi \neq 0$ everywhere else. Let $I(\lambda)$ be as defined above. Then there exist differential operators P_{2k} of order $\leq 2k$ such that for every compact $K \subset \Omega$ and $N \in \mathbb{N}$, we have*

$$\left| I(\lambda) - \left(\sum_{k=0}^{N-1} (P_{2k}u)(x_0) \lambda^{-k-n/2} \right) e^{i\lambda\phi(x_0)} \right| \leq C_{K,N} \lambda^{-N-n/2} \sum_{|\alpha|\leq N} \sup |\partial^\alpha u|$$

for all $u \in C^\infty(K)$, $\lambda \geq 1$. Furthermore, we have

$$P_0 = \frac{(2\pi)^{n/2} e^{i\frac{\pi}{4}(\mathrm{sgn} D^2\phi(x_0))}}{|\det D^2\phi(x_0)|^{1/2}} \in \mathbb{C}$$

Note. For a regular symmetric matrix Q with r positive and $n-r$ negative eigenvalues, sgn $Q := r - (n-r)$.

3.3 Symplectic Geometry of the Cotangent Bundle

In this section we record some facts about the cotangent bundle T^*X, where X is an open subset of $\mathbb{R}^n$. T^*X has a global parametrization

$$((x_1, x_2, \ldots, x_n), \xi_1 dx_1 + \xi_2 dx_2 + \ldots + \xi_n dx_n) \mapsto (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$$

This is sometimes called the canonical parametrization of T^*X. We define the *canonical 1-form* on T^*X by

$$\omega = \sum_{j=1}^{n} \xi_j dx_j$$

and the *canonical 2-form* σ on T^*X by $\sigma = d\omega$, that is,

$$\sigma = \sum_{j=1}^{n} d\xi_j \wedge dx_j$$

Suppose $\rho \in T^*X$. Consider $t = \sum_{j=1}^{n} t_{x_j} \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} s_{\xi_j} \frac{\partial}{\partial \xi_j} \in T_\rho(T^*X)$ and $s = \sum_{j=1}^{n} s_{x_j} \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} s_{\xi_j} \frac{\partial}{\partial \xi_j} \in T_\rho(T^*X)$. Then

$$\sigma_\rho(t, s) = \sum_{j=1}^{n} t_{\xi_j} s_{x_j} - t_{x_j} s_{\xi_j}$$

from which we can see that σ_ρ is a non-degenerate bilinear form. T^*X along with the canonical 2-form σ is an example of what is called a symplectic manifold.

Definition 3.6 ([3], p. 60)**.** *A submanifold $\Lambda \subset T^*X$ is called a Lagrangian submanifold if $\dim \Lambda = \dim X$ and $\sigma|_\Lambda = 0$, that is, for every $\rho \in \Lambda$ and $s, t \in T_\rho \Lambda$ we have $\sigma_\rho(s, t) = 0$.*

Definition 3.7 ([3], p. 100)**.** *Let X and Y be open subsets of $\mathbb{R}^n$ and let σ_X and σ_Y represent the canonical 2-forms on T^*X and T^*Y respectively. A C^∞ map $\chi : T^*X \to T^*Y$ is called a canonical transformation if $\chi^*\sigma_Y = \sigma_X$.*

Moreover, χ is called a *homogeneous* canonical transformation if for $\lambda > 0$, we have $\chi(x, \xi) = (y, \eta) \Rightarrow \chi(x, \lambda\xi) = (y, \lambda\eta)$. The following proposition gives a useful alternative characterization of canonical transformations.

Proposition 3.2. *A C^∞ map $\chi : T^*X \to T^*Y$ is a canonical transformation if and only if the graph of χ, $\Gamma_\chi = \{(\rho, \chi(\rho)) | \, \rho \in T^*X\}$ is a Lagrangian submanifold of $T^*X \times T^*Y$ with respect to the 2-form $\sigma_X - \sigma_Y$, i.e, Γ_χ is a 2n-dimensional submanifold of $T^*X \times T^*Y$ and $(\sigma_X - \sigma_Y)|_{\Gamma_\chi} = 0$.*

3.4 The Global Theory of Fourier Integral Distributions

Let $V \subset X \times \dot{\mathbb{R}}^N$ be a conic open set.

Definition 3.8 ([3], p. 119). *A real valued phase function $\phi \in C^\infty(V)$ is said to be* non-degenerate *if whenever $d_\xi \phi(x,\xi) = 0$, the vectors $\nabla_{x,\xi}(\frac{\partial \phi}{\partial \xi_i})$, $1 \leq i \leq N$ are linearly independent at (x,ξ).*

If ϕ is a non-degenerate phase function, the Submersion Level Set Theorem implies that $C_\phi = \{(x,\xi) \in V |\, d_\xi \phi(x,\xi) = 0\}$ is an n-dimensional submanifold of $X \times \dot{\mathbb{R}}^N$. Henceforth, we only consider non-degenerate phase functions. Now consider the map

$$j_\phi : C_\phi \ni (x,\xi) \mapsto (x, d_x\phi(x,\xi)) \in T^*X \setminus 0$$

Lemma 3.3. *dj_ϕ is injective at every point of C_ϕ*

Since every immersion is locally an embedding, we can shrink V such that $j : C_\phi \to j_\phi(C_\phi)$ is a smooth diffeomorphism. In fact we have

Proposition 3.3 ([3], p. 119). *$\Lambda_\phi := j_\phi(C_\phi)$ is a conic Lagrangian submanifold of $T^*X \setminus 0$*

Corollary 3.1 can now be extended as follows:

Proposition 3.4 ([7], p. 123). *Let $V \subset \Omega \times \dot{\mathbb{R}}^N$ be a conic open set and let $\phi(x,\xi)$ be a non-degenerate phase function in V. If $a \in S^m(X \times \mathbb{R}^N)$ is such that its support is contained in a conic open subset of V , then*

$$WF(I(a,\phi)) \subset \Lambda_\phi$$

Proposition 3.5 ([3], p. 120). *Let $\Lambda \subset T^*X \setminus 0$ be a conic Lagrangian submanifold and let $(x_0,\xi_0) \in \Lambda$. Then there exists a non-degenerate phase function ϕ such that $\Lambda = \Lambda_\phi$ in a neighbourhood of (x_0,ξ_0).*

Theorem 3.3 ([3], p. 121). *Let $V_1 \subset X \times \dot{\mathbb{R}}^{N_1}$ and $V_2 \subset X \times \dot{\mathbb{R}}^{N_2}$ be conic open sets and let $\phi_1 \in C^\infty(V_1)$ and $\phi_2 \in C^\infty(V_2)$ be non-degenerate phase functions. Assume also that j_{ϕ_i} maps $(x_0,\theta_i) \in C_{\phi_i}$ to (x_0,ξ_0) and that $\Lambda_{\phi_1} = \Lambda_{\phi_2}$. Then for every conic neighbourhood $U_2 \subset V_2$ of (x_0,θ_2), there exists a conic neighbourhood $U_1 \subset V_1$ of (x_0,θ_1), such that for every $a_1 \in S^{m+n/4-N_1/2}(X \times \mathbb{R}^{N_1})$ that has its support in U_1, there exists $a_2 \in S^{m+n/4-N_2/2}(X \times \mathbb{R}^{N_2})$ with support in U_2 such that $I(a_1,\phi_1) = I(a_2,\phi_2)$ modulo C^∞.*

Proposition 3.5 and Theorem 3.3 now lead to the following "global" definition of Fourier Integral distributions.

Definition 3.9 ([3], p. 122). *Let $\Lambda \subset T^*X \setminus 0$ be a conic Lagrangian submanifold and let $m \in \mathbb{R}$. Then we define $I^m(X, \Lambda)$ as the set of all $u \in \mathcal{D}'(X)$ such that*

1. *$WF(u) \subset \Lambda$.*

2. *If $(x_0, \xi_0) \in \Lambda$ and if $\phi \in C^\infty(V)$ is a non-degenerate phase function, with $V \subset X \times \dot{\mathbb{R}}^N$ an open cone, such that $\Lambda_\phi = \Lambda$ in a neighbourhood of (x_0, ξ_0), there exists $a \in S^{m+n/4-N/2}(X \times \mathbb{R}^N)$ with support in a cone $\subset\subset V$ such that $u = I(a, \phi)$ modulo C^∞.*

Definition 3.10 ([3], p. 126). *$C \subset T^*(X \times Y) \setminus 0 \simeq (T^*X \setminus 0) \times (T^*Y \setminus 0)$ is called a canonical relation if it is a Lagrangian submanifold for the symplectic form $\sigma_X - \sigma_Y$.*

Define

$$C' = \{(x, \xi; y, -\eta) | (x, \xi; y, \eta) \in C\}$$

Then C is a canonical relation iff C' is a Lagrangian manifold with respect to the standard symplectic form $\sigma_X + \sigma_Y$. Note that the graph of any canonical transformation $\chi : T^*X \to T^*Y$ is a canonical relation. The class of Fourier Integral operators of order m associated to C is by definition the set of those operators whose distribution kernels $K(x, y) \in I^m(X \times Y, C')$. Also, by Theorem 1.12, it follows that

Proposition 3.6. *Let $C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ be a canonical relation and let $K(x, y) \in I^m(X \times Y, C')$. Let A be the operator associated to the Schwartz kernel K. Then*

$$WF(Au) \subset C \circ WF(u) \qquad \forall u \in \mathcal{D}'(Y)$$

We conclude this section by computing the canonical relation of the adjoint of an FIO. Let A be an FIO associated to the canonical relation $C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$. If A is locally given by

$$Au(x) = \iint e^{i\phi(x,y,\xi)} a(x, y, \xi) u(y)\, dy\, d\xi \qquad u \in C_c^\infty(Y)$$

then C is locally given by $(x, d_x\phi, y, -d_y\phi)$. Now

$$A^*v(y) = \iint e^{-i\phi(x,y,\xi)} \overline{a(x, y, \xi)} v(x)\, dx\, d\xi \qquad v \in C_c^\infty(X)$$

so that its canonical relation is locally given by $(y, -d_y\phi, x, d_x\phi)$. It follows that$A^*$ is also an FIO with canonical relation

$$C^t := \{(y, \eta, x, \xi) |\, (x, \xi, y, \eta) \in C\}$$

3.5 Composition of Fourier Integral Operators

Let $C_1 \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ and $C_2 \subset (T^*Y \setminus 0) \times (T^*Z \setminus 0)$ be two canonical relations and let $F_1 \in I^m(X \times Y, C_1'), F_2 \in I^{m'}(Y \times Z, C_2')$. The Hörmander-Sato lemma (Theorem 1.13) shows that the wavefront set of the kernel of $F_1 \circ F_2$ satisfies

$$WF'(K_{F_1 \circ F_2}) \subset C_1 \circ C_2$$

We are interested in finding out when the composition of two FIO's is again an FIO.

Definition 3.11 ([8], p. 490). *Two submanifolds M and N of a smooth manifold X are said to intersect transversally if*

$$T_pN + T_pM = T_pX \qquad \forall p \in M \cap N$$

If M and N intersect transversally, $M \cap N$ will be a submanifold with codim $M \cap N =$ codim $M+$ codim N.

Theorem 3.4 (Hörmander [7], p. 178). *Let $C_1 \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ and $C_2 \subset (T^*Y \setminus 0) \times (T^*Z \setminus 0)$ be conic canonical relations such that*

1. *$C_1 \times C_2$ and $\Delta' := (T^*X \setminus 0) \times \Delta_{(T^*Y \setminus 0)} \times (T^*Z \setminus 0)$ intersect transversally.*
2. *The natural projection $C_1 \times C_2 \cap \Delta' \to T^*(X \times Z) \setminus 0$ is injective and proper.*

Then $C_1 \circ C_2 \in (T^*X \setminus 0) \times (T^*Z \setminus 0)$ is a conic canonical relation. If $A_1 \in I^{m_1}(X \times Y, C_1')$ and $A_2 \in I^{m_2}(X \times Y, C_2')$ are properly supported Fourier Integral Operators, then $A_1 \circ A_2 \in I^{m_1+m_2}(X \times Z, (C_1 \circ C_2)')$

Duistermaat and Guillemin [10] and Weinstein [11] extended this result to the case of clean intersection.

Definition 3.12 ([8], p. 490). *Two submanifolds M and N of a manifold X are said to intersect* cleanly *if $M \cap N$ is also a submanifold and*

$$T_p(M \cap N) = T_pM \cap T_pN \qquad \forall p \in M \cap N$$

Transverse intersection is a special case of clean intersection. It can be shown that if M and N intersect cleanly, codim $M+$ codim $N =$ codim $M \cap N + e$, where e is a non-negative integer, called the excess of the intersection. The clean intersection is transverse if and only if $e = 0$.

Theorem 3.5 (Duistermaat and Guillemin [10], Weinstein [11], Ref: [9], p. 21). *Let C_1 and C_2 be conic canonical relations as before. Suppose*

1. *$C_1 \times C_2$ and $\Delta' := (T^*X \setminus 0) \times \Delta_{(T^*Y\setminus 0)} \times (T^*Z \setminus 0)$ intersect cleanly with excess e.*

2. *The projection $C_1 \times C_2 \cap \Delta' \to T^*(X \times Z) \setminus 0$ is injective and proper.*

Then $C_1 \circ C_2 \in (T^*X \setminus 0) \times (T^*Z \setminus 0)$ is a conic canonical relation. If $A_1 \in I^{m_1}(X \times Y, C_1')$ and $A_2 \in I^{m_2}(X \times Y, C_2')$ are properly supported Fourier Integral Operators, then $A_1 \circ A_2 \in I^{m_1+m_2+e/2}(X \times Z, (C_1 \circ C_2)')$

Let $C \subset (T^*X \setminus 0) \times T^*Y \setminus 0)$ be a canonical relation and let π_L and π_R denote the canonical projections of C into T^*X and T^*Y respectively. If either one of π_L and π_R is a local diffeomorphism, then it can be shown that so is the other and C is what is called a local canonical graph [18].

Definition 3.13 ([13], p. 462)**.** *A homogeneous (conic) canonical relation $C \subset (T^*X \setminus 0) \times T^*Y \setminus 0)$ is called a local canonical graph if every $(x, \xi, y, \eta) \in C$ has a neighbourhood of the form $V_1 \times V_2$ where V_1, V_2 are conic open sets in $T^*X \setminus 0 \ni (x, \xi)$ and $T^*Y \setminus 0 \ni (y, \eta)$ respectively, such that $C \cap V_1 \times V_2$ is the graph of a canonical transformation from V_1 to V_2.*

If either one of $C_1 \subset T^*X \times T^*Y$ and $C_2 \subset T^*Y \times T^*Z$ is a local canonical graph, it can be shown that the transverse intersection condition holds ([13], p. 464). In the next chapter, we consider an FIO F with associated canonical relation C, where both π_L and π_R have singularities, and the clean intersection condition does not hold. We will show that the Schwartz kernel of F^*F is in a class of distributions associated to two cleanly intersecting Lagrangians introduced by Guillemin and Uhlmann [14]. Before we define this class, we note that any two pairs of cleanly intersecting Lagrangians are equivalent.

Theorem 3.6 ([14])**.** *Let $\Lambda_1, \Lambda_2 \subset T^*X$ and $\Lambda_3, \Lambda_4 \subset T^*Y$ be two pairs of Lagrangians cleanly intersecting in codimension k. Then for every $\rho_1 \in \Lambda_1 \cap \Lambda_2$ and $\rho_2 \in \Lambda_3 \cap \Lambda_4$, there exists a canonical transformation $\chi : T^*X \to T^*Y$ and neighbourhoods $T^*X \supset V_1 \ni \rho_1$ and $T^*Y \supset V_2 \ni \rho_2$ such that $\chi(V_1) = V_2$, $\chi(\rho_1) = \rho_2$, $\chi(V_1 \cap \Lambda_1) = V_2 \cap \Lambda_3$ and $\chi(V_1 \cap \Lambda_2) = V_2 \cap \Lambda_4$.*

Let $\widetilde{\Lambda_1} = T_0^*\mathbb{R}^n = \{(x, \xi) | x = 0\}$ and let $\Lambda_2 = \{(x, \xi) | x'' = \xi' = 0\}$ where we write $x' = (x_1, \ldots, x_k)$ and $x'' = (x_{k+1}, \ldots, x_n)$ and similarly for ξ.

Definition 3.14. *We define $S^{p,l}(m, n, k)$ to be the set of all $a(z, \xi, \sigma) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^k)$ such that for every compact $K \subset \mathbb{R}^m$ and multi-indices $\alpha \in \mathbb{Z}_+^m, \beta \in \mathbb{Z}_+^n, \gamma \in \mathbb{Z}_+^k$, there is a positive constant $C_{K,\alpha,\beta}$ such that*

$$|\partial_z^\alpha \partial_\xi^\beta \partial_\sigma^\gamma a(z, \xi, \sigma)| \leq c_{K,\alpha,\beta}(1 + |\xi|)^{p-|\beta|}(1 + |\sigma|)^{l-|\gamma|} \quad \forall (z, \xi, \sigma) \in K \times \mathbb{R}^n \times \mathbb{R}^k$$

Set $m = n + k$ and let $z = (x, s)$ denote an element in $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^k$ and let ξ, σ be dual co-ordinates to x, s.

Definition 3.15 ([14]). *We define $I^{p,l}(\mathbb{R}^n, \widetilde{\Lambda_1}, \widetilde{\Lambda_2})$ to be the set of all distributions u such that $u = u_1 + u_2$ with $u_1 \in C_c^\infty$ and u_2 given by the oscillatory integral*

$$u_2 = \int e^{i((x'-s)\cdot\xi' + x''\cdot\xi'' + s\cdot\sigma)} a(x, \xi, \sigma)\, d\xi\, d\sigma\, ds$$

where $a \in S^{p',l'}(n + k, n, k)$ where $p' = p - n/4 + k/2$ and $l' = l - k/2$.

Using $\widetilde{\Lambda_1}, \widetilde{\Lambda_2}$ as a model case, we define $I^{p,l}$ classes for any pair of cleanly intersecting Lagrangians.

Definition 3.16 ([14]). *Let Λ_1 and Λ_2 be a pair of Lagrangian submanifolds in $T^*X \setminus 0$ cleanly intersecting in codimension k. Then we say $u \in I^{p,l}(X, \Lambda_1, \Lambda_2)$ if $u = u_1 + u_2 + \sum v_i$ where $u_1 \in I^{p+l}(\Lambda_1 \setminus \Lambda_2), u_2 \in I^p(\Lambda_2 \setminus \Lambda_1)$, the sum $\sum v_i$ is locally finite and $v_i = F w_i$, where F is a zero-order FIO associated to a conic canonical transformation $\chi : T^*\mathbb{R}^n \setminus 0 \to T^*X \setminus 0$ and $w_i \in I^{p,l}(\mathbb{R}^n, \widetilde{\Lambda_1}, \widetilde{\Lambda_2})$.*

Chapter 4

Applications to Some Imaging Problems

In this chapter, we will apply some of the concepts and tools developed so far to some problems from Synthetic Aperture Radar (SAR) imaging. In SAR imaging, a region on the surface of the earth is illuminated by electromagnetic waves sent from a moving platform (such as an airplane or a satellite). The waves scatter off the terrain and these backscattered waves are measured by a receiver, which is then used to image the surface. Under certain linearizing approximations, the operator F that relates the ground reflectivity function and the scattering data is a Fourier Integral Operator [16]. The conventional method of recovering the image is to "back-project" the scattered data, and thus we wish to understand the operator F^*F.

In the first problem we consider, the transmitter and receiver are colocated (called monostatic SAR), and move along a circular trajectory at a constant height above the ground. We will first outline a proof originally due to Nolan and Cheney ([12]) and Felea ([18]) that the Schwartz kernel of F^*F belongs to an $I^{p,l}$ class. The proof is based on the fact that the canonical relation $\mathcal{C}$ of F is what is called a two sided fold, and a result of Melrose and Taylor [17] that such two-sided folds can be locally put into a relatively simple form. In the next section, we give an alternative proof which does not use this result of Melrose and Taylor. Finally, in section 3, we consider the more complicated problem where the transmitter and receiver move along a circular trajectory, but separated (bistatic SAR) by a fixed distance at all times.

4.1 The Monostatic Case

Let the earth's surface be modelled as the x-y plane in $\mathbb{R}^3$. Suppose the radar moves along a circular path of radius R, centred at $(0,0,h)$, at a constant height h above the ground. We will use the model from [12]: the operator that maps the ground reflectivity function $f(x,y)$ to scattering data $Ff(\alpha,t)$ is an FIO given by

$$Ff(\alpha,t) = \int e^{-i\omega(t-2\sqrt{(x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2})}a(\alpha,t,x,y,\omega)f(x,y)\,dx\,dy\,d\omega$$

where $a \in S^m(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R})$. Here $Ff(\alpha,t)$ measures the scattered waves received at the point $(R\cos\alpha, R\sin\alpha, h)$ at time t. Consider the function

$$\phi(\alpha,t,x,y,\omega) = -\omega(t-2\sqrt{(x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2})$$

Note that ϕ is real valued, positively homogeneous of degree 1 in ω and $\nabla\phi \neq 0$ on $\mathbb{R}^2\times\mathbb{R}^2\times(\mathbb{R}\setminus 0)$. So ϕ is a phase function. Also it is easy to see that $\nabla\partial_\omega\phi \neq 0$, so that ϕ is also non-degenerate.

We begin by analyzing the canonical relation $\mathcal{C}$ of F. By definition, $\mathcal{C} \subset T^*\mathbb{R}^2 \times T^*\mathbb{R}^2$ is given by

$$\mathcal{C} = \left\{\left(\alpha,t,\frac{\partial\phi}{\partial\alpha},\frac{\partial\phi}{\partial t},x,y,-\frac{\partial\phi}{\partial x},-\frac{\partial\phi}{\partial y}\right)\middle|\frac{\partial\phi}{\partial\omega}=0,\left(\frac{\partial\phi}{\partial\alpha},\frac{\partial\phi}{\partial t}\right)\neq 0,\left(-\frac{\partial\phi}{\partial x},-\frac{\partial\phi}{\partial y}\right)\neq 0\right\}$$

Note that

$$\frac{\partial\phi}{\partial\omega}=0 \Longrightarrow t = 2\sqrt{(x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2}$$

We will parametrize $\mathcal{C}$ as follows: Define

$$G(x,y,\alpha,\omega) = (G_1, G_2, \ldots, G_8)$$

where

$$\begin{aligned}
G_1 &= \alpha\\
G_2 &= t = 2\sqrt{(x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2}\\
G_3 &= \frac{\partial\phi}{\partial\alpha} = -\frac{2\omega R(y\cos\alpha - x\sin\alpha)}{\sqrt{(x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2}}\\
G_4 &= \frac{\partial\phi}{\partial t} = -\omega\\
G_5 &= x\\
G_6 &= y\\
G_7 &= -\frac{\partial\phi}{\partial x} = -\frac{2\omega(x-R\cos\alpha)}{\sqrt{(x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2}}\\
G_8 &= -\frac{\partial\phi}{\partial y} = -\frac{2\omega(y-R\sin\alpha)}{\sqrt{(x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2}}
\end{aligned}$$

Since $(G_1, G_4, G_5, G_6) = (\alpha, -\omega, x, y)$, G is clearly an injective immersion. Let us remove points of the form $(x, y) = (R\cos\alpha, R\sin\alpha)$ from the domain space (x, y, α, ω). Then G is a paramterization of $\mathcal{C}$.

Definition 4.1 ([15], p. 109-111)**.** *Let M and N be smooth manifolds of the same dimension. Suppose $f : M \to N$ is a smooth map such that*

1. *f has full rank everywhere except on a submanifold $\Sigma \subset M$ where it drops rank by 1.*

2. *The determinant of the Jacobian of f vanishes to exactly first order on Σ.*

3. *For every $p \in \Sigma$*
$$T_p\Sigma \cap \ker(df(p)) = \{0\}$$

then we say that f is a fold.

Theorem 4.1 ([12], [18])**.** *The canonical relation $\mathcal{C}$ of F is a two-sided fold, that is, both the canonical left and right projections, $\pi_L : \mathcal{C} \to T^*\mathbb{R}^2$ that maps (y, η, x, ξ) to (y, η) and $\pi_R : \mathcal{C} \to T^*\mathbb{R}^2$ that maps (y, η, x, ξ) to (x, ξ) are folds.*

Proof. Step 1: Let us first consider $\pi_L : (x, y, \alpha, \omega) \mapsto (G_1, G_2, G_3, G_4)$. We have

$$(d\pi_L)(x, y, \alpha, \omega) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{\partial G_2}{\partial x} & \frac{\partial G_2}{\partial y} & * & * \\ \frac{\partial G_3}{\partial x} & \frac{\partial G_3}{\partial y} & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So that
$$\det(d\pi_L)(x, y, \alpha, \omega) = \det \begin{pmatrix} \frac{\partial G_2}{\partial x} & \frac{\partial G_2}{\partial y} \\ \frac{\partial G_3}{\partial x} & \frac{\partial G_3}{\partial y} \end{pmatrix}$$

Now,

$$\begin{aligned}
\frac{\partial G_2}{\partial x} &= \frac{2(x - R\cos\alpha)}{\sqrt{(x - R\cos\alpha)^2 + (y - R\sin\alpha)^2 + h^2}} \\
\frac{\partial G_2}{\partial y} &= \frac{2(y - R\sin\alpha)}{\sqrt{(x - R\cos\alpha)^2 + (y - R\sin\alpha)^2 + h^2}} \\
\frac{\partial G_3}{\partial x} &= -\frac{4\omega R(((x - R\cos\alpha)^2 + (y - R\sin\alpha)^2 + h^2)(-\sin\alpha) - (y\cos\alpha - x\sin\alpha)(x - R\cos\alpha))}{((x - R\cos\alpha)^2 + (y - R\sin\alpha)^2 + h^2)^{3/2}} \\
\frac{\partial G_3}{\partial x} &= -\frac{4\omega R(((x - R\cos\alpha)^2 + (y - R\sin\alpha)^2 + h^2)(\cos\alpha) - (y\cos\alpha - x\sin\alpha)(x - R\cos\alpha))}{((x - R\cos\alpha)^2 + (y - R\sin\alpha)^2 + h^2)^{3/2}}
\end{aligned}$$

We finally get

$$K(x,y,\alpha,\omega) := \det(d\pi_L)(x,y,\alpha,\omega) = -\frac{4\omega R(x\cos\alpha + y\sin\alpha - R)}{(x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2}$$

Let Σ be the submanifold $x\cos\alpha + y\sin\alpha = R$. $\det(d\pi_L) = 0$ iff $x\cos\alpha+y\sin\alpha-R=0$ so that π_L has full rank everywhere except on Σ. It is also easy to see that π_L drops rank exactly by 1 on Σ. We also have

$$\begin{aligned}
\frac{\partial K}{\partial x} &= -\frac{4\omega R(((x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2)\cos\alpha - 2(x\cos\alpha+y\sin\alpha-R)(x-R\cos\alpha))}{((x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2)^2} \\
&= \frac{-4\omega R}{(x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2}\cos\alpha \text{ on } \Sigma \\
\frac{\partial K}{\partial y} &= -\frac{4\omega R(((x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2)\sin\alpha - 2(x\cos\alpha+y\sin\alpha-R)(y-R\sin\alpha))}{((x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2)^2} \\
&= \frac{-4\omega R}{(x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2}\sin\alpha \text{ on } \Sigma
\end{aligned}$$

Since ω and R are both non-zero, $(\frac{\partial K}{\partial x}, \frac{\partial K}{\partial y})$ is of the form $(A\cos\alpha, A\sin\alpha)$ with $A \neq 0$. So, G vanishes to exactly first order on Σ.

Now suppose $p = (x,y,\alpha,\omega) \in \Sigma$. The tangent space of Σ at p is given by

$$T_p\Sigma = \ker(\cos\alpha, \sin\alpha, y\cos\alpha - x\sin\alpha, 0)$$

that is, all tangent vectors orthogonal to the gradient vector at p of the function $x\cos\alpha + y\sin\alpha - R$.

$$\ker(d\pi_L)(p) = \left\{ (a,b,0,0) \middle| (a,b) \in \ker \begin{pmatrix} \frac{\partial G_2}{\partial x}(p) & \frac{\partial G_2}{\partial y}(p) \\ \frac{\partial G_3}{\partial x}(p) & \frac{\partial G_3}{\partial y}(p) \end{pmatrix} \right\}$$

Suppose $(a,b,0,0) \in T_p\Sigma \cap \ker(d\pi_L)(p)$. Then $a\cos\alpha + b\sin\alpha = 0$ which implies that $(a,b) = (-t\sin\alpha, t\cos\alpha)$ for some t. Next, note that

$$\ker \begin{pmatrix} \frac{\partial G_2}{\partial x}(p) & \frac{\partial G_2}{\partial y}(p) \\ \frac{\partial G_3}{\partial x}(p) & \frac{\partial G_3}{\partial y}(p) \end{pmatrix} = \ker \begin{pmatrix} x-R\cos\alpha & y-R\sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

(the second matrix was obtained from the first by elementary row operations) and thus

$$\begin{aligned}
\begin{pmatrix} x-R\cos\alpha & y-R\sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} -t\sin\alpha \\ t\cos\alpha \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\Longrightarrow t\cos^2\alpha + t\sin^2\alpha &= 0 \\
\Longrightarrow t &= 0
\end{aligned}$$

which proves that $T_p\Sigma \cap \ker(d\pi_L)(p) = \{0\}$ for every $p \in \Sigma$. Thus, π_L has a fold singularity along Σ.

Step 2: Next, consider $\pi_R : (x, y, \alpha, \omega) \mapsto (G_5, G_6, G_7, G_8)$. We have

$$(d\pi_R)(x,y,\alpha,\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & \frac{\partial G_7}{\partial \alpha} & \frac{\partial G_7}{\partial \omega} \\ * & * & \frac{\partial G_8}{\partial \alpha} & \frac{\partial G_8}{\partial \omega} \end{pmatrix}$$

Now,

$$\begin{aligned} \frac{\partial G_7}{\partial \alpha} &= -\frac{2\omega R((y - R\sin\alpha)(x\cos\alpha + y\sin\alpha - R) + h^2\sin\alpha)}{((x - R\cos\alpha)^2 + ((y - R\sin\alpha)^2 + h^2)^{3/2}} \\ \frac{\partial G_7}{\partial \omega} &= -\frac{2(x - R\cos\alpha)}{\sqrt{(x - R\cos\alpha)^2 + (y - R\sin\alpha)^2 + h^2}} \\ \frac{\partial G_8}{\partial \alpha} &= \frac{2\omega R((x - R\cos\alpha)(x\cos\alpha + y\sin\alpha - R) + h^2\cos\alpha)}{((x - R\cos\alpha)^2 + ((y - R\sin\alpha)^2 + h^2)^{3/2}} \\ \frac{\partial G_8}{\partial \omega} &= -\frac{2(y - R\sin\alpha)}{\sqrt{(x - R\cos\alpha)^2 + (y - R\sin\alpha)^2 + h^2}} \\ \implies \det(d\pi_R)(x,y,\alpha,\omega) &= \frac{4\omega R(x\cos\alpha + y\sin\alpha - R)}{(x - R\cos\alpha)^2 + (y - R\sin\alpha)^2 + h^2} \\ &= -\det(d\pi_L)(x,y,\alpha,\omega) \end{aligned}$$

Thus, we see that π_R has full rank everywhere except on Σ where it drops rank by 1. Further, since $\det(d\pi_L) = -\det(d\pi_R)$, by repeating the same arguments made for π_L, we can conclude that $\det(d\pi_R)$ vanishes to exactly first order on Σ.

Now consider $p = (x, y, \alpha, \omega) \in \Sigma$. As before,

$$T_p\Sigma = \ker(\cos\alpha, \sin\alpha, y\cos\alpha - x\sin\alpha, 0)$$

and

$$\ker(d\pi_R)(p) = \left\{ (0,0,c,d) \middle| (c,d) \in \ker \begin{pmatrix} \frac{\partial G_7}{\partial \alpha}(p) & \frac{\partial G_7}{\partial \omega}(p) \\ \frac{\partial G_8}{\partial \alpha}(p) & \frac{\partial G_8}{\partial \omega}(p) \end{pmatrix} \right\}$$

Now suppose $(0, 0, c, d) \in T_p\Sigma \cap \ker(d\pi_R)(p)$. Then $c(y\cos\alpha - x\sin\alpha) = 0$. For fixed α, $y\cos\alpha - x\sin\alpha = 0$ is the line through the origin in the x-y plane making an angle α with positive x-axis. It's intersection with Σ is the single point $(R\cos\alpha, R\sin\alpha)$ which is not in the domain. So clearly $c = 0$. Next,

$$(0,0,0,d) \in \ker \begin{pmatrix} \frac{\partial G_7}{\partial \alpha}(p) & \frac{\partial G_7}{\partial \omega}(p) \\ \frac{\partial G_8}{\partial \alpha}(p) & \frac{\partial G_8}{\partial \omega}(p) \end{pmatrix}$$

implies that $(x - R\cos\alpha)d = 0$ and $(y - R\sin\alpha)d = 0$ which implies $d = 0$. Thus,

$$T_p\Sigma \cap \ker(\pi_R)_*(p) = \{0\}$$

So π_R has a fold singularity at p. Thus, we can conclude that $\mathcal{C}$ is a two-sided fold. $\square$

Now, by appealing to the following theorem of Felea, we can conclude that F^*F is in an $I^{p,l}$ class.

Theorem 4.2 (Felea [18])**.** *Let X and Y be open subsets of $\mathbb{R}^n$ and let $C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ be a two-sided fold. If G is a properly supported FIO of order m associated to C, $G^*G \in I^{2m,0}(\Delta_{T^*Y}, \widetilde{C})$ where $\widetilde{C} \subset (T^*Y \setminus 0) \times (T^*Y \setminus 0)$ is another two-sided fold that cleanly intersects Δ_{T^*Y}.*

The main ingredient of the proof is the following result of Melrose and Taylor.

Theorem 4.3 (Melrose and Taylor [17])**.** *Let X and Y be open subsets of $\mathbb{R}^n$ and $C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ be a canonical relation. Suppose the canonical left and right projections π_L and π_R have fold type singularities at $p = (x, \xi, y, \eta)$. Then there exist conic neighbourhoods U of (x, ξ) and V of (y, η), and homogeneous canonical transformations $\chi_1 : U \to T^*\mathbb{R}^n$ and $\chi_2 : V \to T^*\mathbb{R}^2$ such that $(\chi_1 \times \chi_2)(C \cap (U \times V)) \subset C_0$ near $\xi_2 \neq 0$, where $C_0 = N^*\{x_2 - y_2 = (x_1 - y_1)^3; x_i = y_i, 3 \leq i \leq n\}$.*

4.2 An Alternative Proof

Let F^* denote the L^2 adjoint of F. Then

$$F^*g(x,y) = \int e^{-i\phi}\overline{a}(x,y,\alpha,t,\omega)g(\alpha,t)\,d\omega\,d\alpha\,dt$$

From this, we can easily see that the Schwartz kernel of F^*F is given by

$$\begin{aligned} K_{F^*F}(x,y,x',y') &= \int e^{i\omega(t-2\sqrt{(x'-R\cos\alpha)^2+(y'-R\sin\alpha)^2+h^2})-i\theta(t-2\sqrt{(x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2})} \\ &\quad \overline{a}a d\omega\,d\theta\,dt\,d\alpha \end{aligned}$$

Define

$$\begin{aligned} \Phi &= (\omega-\theta)(t - 2\sqrt{(x-R\cos\alpha)^2 + (y-R\sin\alpha)^2 + h^2}) \\ &\quad -2\omega(\sqrt{(x'-R\cos\alpha)^2 + (y'-R\sin\alpha)^2 + h^2} - \sqrt{(x-R\cos\alpha)^2 + (y-R\sin\alpha)^2 + h^2}) \end{aligned}$$

We first perform integration in θ, t. Set $b(x,y,x',y',\alpha,\omega) =$

$$\int e^{i\Phi}\overline{a}a\,d\theta\,dt$$

Now put $\theta = \lambda\omega$. The Phase function Φ is now homogeneous of order 1 in ω. Now apply stationary phase asymptotics in terms of powers of ω. $\partial_\lambda \Phi = 0$ implies $t = 2\sqrt{(x-R\cos\alpha)^2 + (y-R\sin\alpha)^2 + h^2}$ and $\partial_t \Phi = 0$ implies $\lambda = 1$. Now we get the simplified expression

$$\begin{aligned} K_{F^*F}(x,y,x',y') &= \int e^{-2i\omega(\sqrt{(x'-R\cos\alpha)^2+(y'-R\sin\alpha)^2+h^2}-\sqrt{(x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2})} \\ &\quad \widetilde{a}(x,y,x',y',\alpha,\omega)d\alpha\, d\omega \end{aligned}$$

plus lower order terms. Here,

$$\begin{aligned} \widetilde{a} &= \overline{a}(x',y',\alpha,\omega,2\sqrt{(x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2}) \times \\ &\quad a(x,y,\alpha,\omega,2\sqrt{(x-R\cos\alpha)^2+(y-R\sin\alpha)^2+h^2}) \end{aligned}$$

The Hormander-Sato lemma states that $WF(K_{F^*F}) \subset \mathcal{C}^t \circ \mathcal{C}$ where $\mathcal{C}^t$ is given by

$$\mathcal{C}^t = \left\{ \left(x, y, -\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y}, \alpha, t, \frac{\partial\phi}{\partial\alpha}, \frac{\partial\phi}{\partial t} \right) \middle| \frac{\partial\phi}{\partial\omega} = 0 \right\}$$

This implies that

$$\mathcal{C}^t \circ \mathcal{C} = \left\{ \left(\begin{array}{llll} x_1, & y_1, & -\frac{\partial\phi}{\partial x}(\alpha,t,x_1,y_1,\omega), & -\frac{\partial\phi}{\partial y}(\alpha,t,x_1,y_1,\omega); \\ x_2, & y_2, & -\frac{\partial\phi}{\partial x}(\alpha,t,x_2,y_2,\omega), & -\frac{\partial\phi}{\partial y}(\alpha,t,x_2,y_2,\omega) \end{array} \right) \right|$$

$$\left. \begin{array}{ll} t & = 2\sqrt{(x_1-R\cos\alpha)^2+(y_1-R\sin\alpha)^2+h^2} \\ & = 2\sqrt{(x_2-R\cos\alpha)^2+(y_2-R\sin\alpha)^2+h^2} \\ & \quad y_2\cos\alpha - x_2\sin\alpha = y_1\cos\alpha - x_1\sin\alpha \end{array} \right\}$$

The conditions defining $\mathcal{C}^t \circ \mathcal{C}$ in the above equation mean the following. For a fixed α and ω, (x_1, y_1) and (x_2, y_2) lie on the same circle centred at $(R\cos\alpha, R\sin\alpha)$ and, if the two points are different, the chord joining them makes an angle α with the positive x-axis. So this means that either $(x_1, y_1) = (x_2, y_2)$ or

$$\begin{aligned} x_2 &= x_1 - 2(x_1\cos\alpha + y_1\sin\alpha - R)\cos\alpha \\ y_2 &= y_1 - 2(x_1\cos\alpha + y_1\sin\alpha - R)\sin\alpha \end{aligned}$$

So, $\mathcal{C}^t \circ \mathcal{C} \subset \Delta \cup \widetilde{\mathcal{C}}$ where $\widetilde{\mathcal{C}}$ corresponds to the part of $\mathcal{C}^t \circ \mathcal{C}$ where the above two equaions hold, and Δ(diagonal) corresponds to the part where $(x_1, y_1) = (x_2, y_2)$. We will show that $\widetilde{\mathcal{C}}$ is also a two-sided fold. We first parametrize $\widetilde{\mathcal{C}}$ as follows:

$$H(x, y, \alpha, \omega) = (H_1, H_2, \ldots, H_8)$$

where

$$
\begin{aligned}
H_1 &= x - 2(x\cos\alpha + y\sin\alpha - R)\cos\alpha \\
H_2 &= y - 2(x\cos\alpha + y\sin\alpha - R)\sin\alpha \\
H_3 &= -\frac{2\omega(x - 2(x\cos\alpha + y\sin\alpha - R)\cos\alpha - R\cos\alpha)}{\sqrt{(x - R\cos\alpha)^2 + (y - R\sin\alpha)^2 + h^2}} \\
H_4 &= -\frac{2\omega(y - 2(x\cos\alpha + y\sin\alpha - R)\sin\alpha - R\sin\alpha)}{\sqrt{(x - R\cos\alpha)^2 + (y - R\sin\alpha)^2 + h^2}} \\
H_5 &= x \\
H_6 &= y \\
H_7 &= -\frac{2\omega(x - R\cos\alpha)}{\sqrt{(x - R\cos\alpha)^2 + (y - R\sin\alpha)^2 + h^2}} \\
H_8 &= -\frac{2\omega(y - R\sin\alpha)}{\sqrt{(x - R\cos\alpha)^2 + (y - R\sin\alpha)^2 + h^2}}
\end{aligned}
$$

We have

$$
dH(x, y, \alpha, \omega) = \begin{pmatrix} * & * & a & 0 \\ * & * & b & 0 \\ * & * & * & * \\ * & * & * & * \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & * & c \\ * & * & * & d \end{pmatrix}
$$

where

$$
\begin{aligned}
a &= \frac{\partial H_1}{\partial \alpha} = -2(x\cos\alpha + y\sin\alpha - R)(-\sin\alpha) - 2(y\cos\alpha - x\sin\alpha)(\cos\alpha) \\
b &= \frac{\partial H_2}{\partial \alpha} = -2(x\cos\alpha + y\sin\alpha - R)(\cos\alpha) - 2(y\cos\alpha - x\sin\alpha)(\sin\alpha) \\
c &= \frac{\partial H_7}{\partial \omega} = -\frac{2(x - R\cos\alpha)}{\sqrt{(x - R\cos\alpha)^2 + (y - R\sin\alpha)^2 + h^2}} \\
d &= \frac{\partial H_8}{\partial \omega} = -\frac{2(y - R\sin\alpha)}{\sqrt{(x - R\cos\alpha)^2 + (y - R\sin\alpha)^2 + h^2}}
\end{aligned}
$$

Since $(x, y) \neq (R\cos\alpha, R\sin\alpha)$, the vector $(c, d) \neq 0$. Also,

$$
\begin{pmatrix} b \\ a \end{pmatrix} = -2 \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x\cos\alpha + y\sin\alpha - R \\ y\cos\alpha - x\sin\alpha \end{pmatrix}
$$

which is a rotation matrix operating on a vector that is non-zero whenever $(x, y) \neq (R\cos\alpha, R\sin\alpha)$. So at least one of a and b is non-zero. This proves that rank $dH = 4$ and thus H is an immersion.

Next we show that H is injective. Suppose $(H_1, H_2, \ldots, H_8)$ are given. Then we know x,y,$x' = x - 2(x\cos\alpha + y\sin\alpha - R)\cos\alpha$ and $y' = y - 2(x\cos\alpha + y\sin\alpha - R)\sin\alpha$. Suppose that $(x, y) \neq (x', y')$. Then the (unique) perpendicular bisector of the line segment joining (x, y) and (x', y') gives us the line $x\cos\alpha + y\sin\alpha - R = 0$, and from this, we can determine α uniquely. If $(x, y) = (x', y')$, both points are on the tangent and drawing a line through this point in the direction $(H_7, H_8) \neq 0$ gives us the line $x\cos\alpha + y\sin\alpha - R = 0$ and again α can be uniquely determined. Finally using the formula for H_7 or H_8 (at least one of which is non-zero), we can determine ω. This proves that H is injective.

Proposition 4.1. *$\widetilde{\mathcal{C}}$ is a two-sided fold.*

Proof. Let π'_L and π'_R denote the canonical left and right projections of $\widetilde{\mathcal{C}}$. We have

$$\pi'_R = \left(x, y, -\frac{2\omega(x - R\cos\alpha)}{\sqrt{(x - R\cos\alpha)^2 + (y - R\sin\alpha)^2 + h^2}}, -\frac{2\omega(y - R\sin\alpha)}{\sqrt{(x - R\cos\alpha)^2 + (y - R\sin\alpha)^2 + h^2}} \right)$$

But this is exactly the same as π_R, the canonical right projection of $\mathcal{C}$. So we can immediately conclude that π'_R has a fold singularity along the submanifold Σ given by $x\cos\alpha + y\sin\alpha - R = 0$ and has full rank everywhere else.

Now consider the map

$$T : (x, y, \alpha, \omega) \mapsto (x - 2(x\cos\alpha + y\sin\alpha - R)\cos\alpha, y - 2(x\cos\alpha + y\sin\alpha - R)\sin\alpha, \alpha, \omega)$$

The left projection π'_L is simply the composition $\pi'_R \circ T$. T has the following two properties:

1. $T|_\Sigma$ is the identity map on Σ.

2. T is a diffeomorphism with

$$\begin{aligned} dT(x, y, \alpha, \omega) &= \begin{pmatrix} -\cos 2\alpha & -\sin 2\alpha & * & * \\ -\sin 2\alpha & \cos 2\alpha & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \det(dT(x, y, \alpha, \omega)) &= -1 \end{aligned}$$

Since $(d\pi'_L) = (d\pi'_R) \circ dT$, we can conclude that π'_L has full rank everywhere except on the submanifold Σ where it drops rank by 1. It also follows that the determinant of $(d\pi_L)$ vanishes to exactly first order on Σ.

Finally, suppose $p \in \Sigma$. Since T is the identity map of Σ, $dT(p)$ is the identity map on $T_p\Sigma$ and thus, $T_p\Sigma \cap \ker(d\pi'_L)(p) = T_p\Sigma \cap \ker(d\pi'_R)(p) = \{0\}$. So, π'_L also has a fold singularity along Σ. This proves that $\widetilde{\mathcal{C}}$ is a two-sided fold. $\square$

For the sake of convenience we will henceforth change notation as follows:

$$\begin{aligned} (x, y) &\mapsto x = (x_1, x_2) \\ (x', y') &\mapsto y = (y_1, y_2) \end{aligned}$$

The final step is to show that F^*F is in $I^{p.l}(\Delta, \widetilde{\mathcal{C}})$. The proof will depend on the following theorem, which gives a sufficient condition on a distribution u to be in an $I^{p,l}$ class.

Theorem 4.4 (Greenleaf and Uhlmann [19]). *Let Λ_0 and Λ_1 be two cleanly intersecting Lagrangian submanifolds of $T^*X \times T^*Y$. If $u \in \mathcal{D}'(X \times Y)$, then $u \in I^{p,l}(\Lambda_0, \Lambda_1)$ if there is an $s_o \in \mathbb{R}$ such that, for all first order classical pseudodifferential operators with principal symbols σ_{P_i} vanishing on $\Lambda_0 \cup \Lambda_1$, we have $P_1 P_2 \cdots P_r u \in H^{s_0}_{\text{loc}}$.*

Note that s_0 is independent of r. To use this result, we need to show that Δ and $\widetilde{\mathcal{C}}$ intersect cleanly, and find the ideal of smooth functions vanishing on $\Delta \cup \widetilde{\mathcal{C}}$. We will follow the procedure found in [18]. Suppose M and N are codimension n submanifolds that intersect in codimension k, and there exist smooth functions f_i, g_j such that

$$\begin{aligned} M &= \{f_1 = f_2 = \cdots f_k = f_{k+1} = \cdots f_n = 0\} \text{ and} \\ N &= \{g_1 = g_2 = \cdots g_k = f_{k+1} = \cdots f_n = 0\} \end{aligned}$$

Then, M and N intersect cleanly if $\{\nabla f_i | 1 \leq i \leq n\}$ and $\{\nabla g_i, \nabla f_{k+j} | 1 \leq i \leq k, 1 \leq j \leq n-k\}$ are linearly independent [18]. Also, the following proposition is an easy consequence of the implicit function theorem and the fundamental theorem of calculus.

Proposition 4.2 (Felea [18]). *The ideal of smooth functions vanishing on $M \cup N$ is generated by $\{f_{k+j} | 1 \leq j \leq n-k\}$ and $\{f_i g_j | 1 \leq i, j \leq k\}$.*

We claim that

$$\begin{aligned} \Delta &= \{(y, \eta, x, \xi) \in T^*\mathbb{R}^2 \times T^*\mathbb{R}^2 | f_1 = f_2 = f_3 = f_4 = 0\} \text{ and} \\ \widetilde{\mathcal{C}} &= \{(y, \eta, x, \xi) \in T^*\mathbb{R}^2 \times T^*\mathbb{R}^2 | f_1 = f_2 = f_3 = g_4 = 0\} \end{aligned}$$

where

$$\begin{aligned}
f_1 &= \theta_1\varphi_1 + \theta_2\varphi_2 \\
f_2 &= \theta_1 w_1 + \theta_2 w_2 \\
f_3 &= w_1\theta_2 - w_2\theta_1 - z_1\varphi_2 + z_2\varphi_1 \\
f_4 &= (w_1\theta_2 - w_2\theta_1 - z_1\varphi_2 + z_2\varphi_1)^2 - 4R^2(\varphi_1^2 + \varphi_2^2) \\
g_4 &= (z_1\theta_2 - z_2\theta_1 - w_1\varphi_2 + w_2\varphi_1)^2 - 4R^2(\theta_1^2 + \theta_2^2)
\end{aligned}$$

where we have used the change of variables

$$\begin{aligned}
z &= y + x \\
w &= y - x \\
\theta &= \xi + \eta \\
\varphi &= \xi - \eta
\end{aligned}$$

First of all, note that on Δ, $w = \varphi = 0$, so that we immediately have $f_1 = f_2 = f_3 = f_4 = 0$. Conversely, if $f_1 = f_2 = f_3 = f_4 = 0$, we have

$$\begin{aligned}
f_4 = 0 &\Rightarrow \varphi_1^2 + \varphi_2^2 = 0 \Rightarrow \varphi = 0, \text{ also} \\
f_2 = 0 &\Rightarrow \theta_1 w_1 + \theta_2 w_2 = 0 \\
f_3 = 0 &\Rightarrow w_1\theta_2 - w_2\theta_1 = 0 \\
&\Rightarrow w = 0
\end{aligned}$$

This proves that $\Delta = \{f_1 = f_2 = f_3 = f_4 = 0\}$. Next, it can be easily verified using the parametrization H that every point of $\widetilde{\mathcal{C}}$ satisfies $f_1 = f_2 = f_3 = g_4 = 0$. Conversely, suppose $(y, \eta, x, \xi) \in \widetilde{\mathcal{C}}$. Then if $x \neq y$, the perpendicular bisector of the line joining x and y is tangent (at say the point P) to the circle of radius R with center at the origin. If $x = y$, they must lie on a tangent to the same circle. Further, ξ and η must be of the same magnitude and point in the direction from x to P and y to P respectively. It can be proved that we must have $f_1 = f_2 = f_3 = f_4 = 0$. We skip the calculations. Also, it can be easily verified that $\{\nabla f_i | 1 \leq i \leq 4\}$ and $\{\nabla g_4, \nabla f_i | 1 \leq i \leq 3\}$ are linearly independent. Thus, Δ and $\widetilde{\mathcal{C}}$ intersect cleanly, and the ideal of smooth functions vanishing on $\Delta \cup \widetilde{\mathcal{C}}$ is generated by $J = \{f_1, f_2, f_3, f_4 g_4\}$.

Theorem 4.5. *There exist real numbers p and l such that $F^*F \in I^{p,l}(\Delta, \widetilde{\mathcal{C}})$*

Proof. The proof will use the iterated regularity condition of Theorem 4.4. It is enough now to check that $P_1 P_2 \cdots P_r K_{F^*F} \in H^{s_0}_{\text{loc}}$ where the P_i are first order classical ΨDOs with principal symbols being multiples of functions in J. The idea of the proof is from [18]. We will use the following facts for the proof:

(a) ([6], p. 105) If P is a ΨDO with principal symbol p and $u(x) = \int e^{i\psi(x,\theta)}a(x,\theta)d\theta$ then $Pu(x) = \int e^{i\psi(x,\theta)}a(x,\theta)p(x, d_x\psi)d\theta$ plus lower order terms.

(b) If $a \in S^r$ then $u(x) = \int_{\mathbb{R}^N} e^{i\psi(x,\theta)}a(x,\theta)d\theta \in H^{s_0}_{\mathrm{loc}}$ for some $s_0 = s_0(m, N, n) \in \mathbb{R}$ where n is the dimension of the variable x.

Let P_1 be a classical pseudodifferential operator with principal symbol $p_1 = c_1 f_1$ where c_1 is a homogeneous function of degree -1 in (ξ, η) variables (so that P_1 is of first order). By (a),

$$P_1 K_{F^*F} = \int e^{-i\Phi(x,y,\alpha,\omega)}\widetilde{a}(x,y,\alpha,\omega)p_1(x, -\partial_x\Phi, y, \partial_y\Phi)d\alpha\, d\omega$$

where

$$\Phi = 2\omega(\sqrt{(y_1 - R\cos\alpha)^2 + (y_2 - R\sin\alpha)^2 + h^2} - \sqrt{(x_1 - R\cos\alpha)^2 + (x_2 - R\sin\alpha)^2 + h^2})$$

Let us define

$$\begin{aligned} X &= \sqrt{(x_1 - R\cos\alpha)^2 + (x_2 - R\sin\alpha)^2 + h^2} \\ Y &= \sqrt{(y_1 - R\cos\alpha)^2 + (y_2 - R\sin\alpha)^2 + h^2} \end{aligned}$$

then clearly,

$$\begin{aligned} \Phi &= 2\omega(Y - X) \\ \partial_\omega\Phi &= 2(Y - X) \\ \partial_\alpha\Phi &= 2\omega R\left(\frac{y_1\sin\alpha - y_2\cos\alpha}{Y} - \frac{x_1\sin\alpha - x_2\cos\alpha}{X}\right) \end{aligned}$$

$$\begin{aligned} -\frac{\partial\Phi}{\partial x_1} &= \frac{2\omega(x_1 - R\cos\alpha)}{X} \quad ; \quad -\frac{\partial\Phi}{\partial x_2} = \frac{2\omega(x_2 - R\sin\alpha)}{X} \\ \frac{\partial\Phi}{\partial y_1} &= \frac{2\omega(y_1 - R\cos\alpha)}{X} \quad ; \quad \frac{\partial\Phi}{\partial y_2} = \frac{2\omega(y_2 - R\sin\alpha)}{X} \end{aligned}$$

Now,

$$\begin{aligned}
f_1(y,\eta,x,\xi) &= \theta_1\varphi_1+\theta_2\varphi_2\\
&= \xi_1^2+\xi_2^2-\eta_1^2-\eta_2^2 \text{ so that}\\
f_1(y,\partial_y\Phi,x,-\partial_x\Phi) &= 4\omega^2\left[\frac{(x_1-R\cos\alpha)^2+(x_2-R\sin\alpha)^2}{X^2}-\frac{(y_1-R\cos\alpha)^2+(y_2-R\sin\alpha)^2}{Y^2}\right]\\
&= 4\omega^2\left[1-\frac{h^2}{X^2}-1+\frac{h^2}{Y^2}\right]\\
&= \frac{4\omega^2h^2}{X^2Y^2}(X+Y)(X-Y)\\
&= \frac{-2\omega^2h^2}{X^2Y^2}\partial_\omega\Phi
\end{aligned}$$

So $P_1K_{F^*F}$ becomes

$$P_1K_{F^*F}=\int e^{-i\Phi}\widetilde{a}(x,y,\alpha,\omega)\frac{(-2c_1\omega^2h^2(X+Y))}{X^2Y^2}\partial_\omega\Phi d\alpha\, d\omega$$

By integration by parts, this integral becomes

$$i\int e^{-i\Phi}\partial_\omega\left(\widetilde{a}(x,y,\alpha,\omega)\frac{2c_1\omega^2h^2(X+Y)}{X^2Y^2}\right)d\alpha\, d\omega$$

Now since $\widetilde{a}$ is a symbol of order $2m$ and c_1 is homogeneous in ω of degree -1, it is easy to see that $\partial_\omega\left(\widetilde{a}\frac{(2c_1\omega^2h^2(X+Y))}{X^2Y^2}\right)$ is again a symbol of order $2m$. Thus by (b), we conclude that $P_1K_{F^*F}\in H^{s_0}_{\mathrm{loc}}$ for some $s_0\in\mathbb{R}$.

Next, suppose P_2 is a classical ΨDO with principal symbol c_2f_2 where c_2 is a smooth function homogeneous of degree 0. Now, $f_2=\theta_1w_1+\theta_2w_2=(y_1-x_1)(\xi_1+\eta_1)+(y_2-x_2)(\xi_2+\eta_2)$. Thus we have

$$\begin{aligned}
f_2(y,\partial_y\Phi,x,-\partial_x\Phi) &= 2\omega\left[\frac{(y_1-x_1)(x_1-R\cos\alpha)+(y_2-x_2)(x_2-R\sin\alpha)}{X}\right.\\
&\qquad\left.+\frac{(y_1-x_1)(y_1-R\cos\alpha)+(y_2-x_2)(y_2-R\sin\alpha)}{Y}\right]\\
&= 2\omega\left[-\frac{(x_1-R\cos\alpha)^2+(x_2-R\sin\alpha)^2}{X}+\frac{(y_1-R\cos\alpha)^2+(y_2-R\sin\alpha)^2}{Y}\right.\\
&\qquad\left.+((y_1-R\cos\alpha)(x_1-R\cos\alpha)+(y_2-R\sin\alpha)(x_2-R\sin\alpha))\left(\frac{1}{X}-\frac{1}{Y}\right)\right]\\
&= 2\omega\left[Y-X+((y_1-R\cos\alpha)(x_1-R\cos\alpha)+(y_2-R\sin\alpha)(x_2-R\sin\alpha)+h^2)\left(\frac{1}{X}-\frac{1}{Y}\right)\right]\\
&= \omega\left(1+\frac{(y_1-R\cos\alpha)(x_1-R\cos\alpha)+(y_2-R\sin\alpha)(x_2-R\sin\alpha)+h^2}{XY}\right)\partial_\omega\Phi\\
&= \omega k(x,y,\alpha)\partial_\omega\Phi
\end{aligned}$$

Again applying (a) we get

$$\begin{aligned} P_2 K_{F^*F} &= \int e^{-i\Phi(x,y,\alpha,\omega)} \widetilde{a}(x,y,\alpha,\omega) p_2(x, -\partial_x \Phi, y, \partial_y \Phi) d\alpha\, d\omega \\ &= \int e^{-i\Phi} \widetilde{a} c_2 \omega k \partial_\omega \Phi d\alpha\, d\omega \\ &= -i \int e^{-i\Phi} \partial_\omega (\widetilde{a} c_2 \omega k) d\alpha\, d\omega \text{ by integration by parts} \end{aligned}$$

Since $\widetilde{a}$ is a symbol of order $2m$ and c_2 and k are independent of ω, we conclude that $\partial_\omega(\widetilde{a} c_2 \omega h)$ is again a symbol of order $2m$ and hence by (b), $P_2 K_{F^*F} \in H^{s_0}_{\mathrm{loc}}$.

Next let P_3 be a classical ΨDO with principal symbol $c_3 f_3$ where c_3 is a homogeneous smooth function of degree 0. Since $f_3 = w_1\theta_2 - w_2\theta_1 - z_1\varphi_2 + z_2\varphi_1 = 2(y_1\eta_2 - y_2\eta_1 - x_1\xi_2 + x_2\xi_1)$, we get

$$\begin{aligned} f_3(y, \partial_y\Phi, x, -\partial_x\Phi) &= 4\omega\left[\frac{y_1(y_2 - R\sin\alpha) - y_2(y_1 - R\cos\alpha)}{Y} - \frac{x_1(x_2 - R\sin\alpha) - x_2(x_1 - R\cos\alpha)}{X}\right] \\ &= -4\omega R\left[\frac{y_1\sin\alpha - y_2\cos\alpha}{Y} - \frac{x_1\sin\alpha - x_2\cos\alpha}{X}\right] \\ &= -2\partial_\alpha\Phi \end{aligned}$$

Thus, applying (a) we get

$$\begin{aligned} P_3 K_{F^*F} &= \int e^{-i\Phi} \widetilde{a}(-2c_3)\partial_\alpha\Phi\, d\alpha\, d\omega \\ &= 2i \int e^{-i\Phi} \partial_\alpha(\widetilde{a} c_3)\, d\alpha\, d\omega \end{aligned}$$

by integration by parts. Since $\partial_\alpha(\widetilde{a} c_3)$ is a symbol of order $2m$, we again get that $P_3 K_{F^*F} \in H^{s_0}_{\mathrm{loc}}$.

Finally, we consider a classical pseudodifferential operator P_4 with principal symbol $c_4 f_4 g_4$, where c_4 is homogeneous of degree -3. Let us first analyze f_4 and g_4. We have

$$\begin{aligned} f_4(y,\eta,x,\xi) &= (w_1\theta_2 - w_2\theta_1 - z_1\varphi_2 + z_2\varphi_1)^2 - 4R^2(\varphi_1^2 + \varphi_2^2) \\ &= 4(y_1\eta_2 - y_2\eta_1 - x_1\xi_2 + x_2\xi_1)^2 - 4R^2((\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2) \end{aligned}$$

so that

$$\begin{aligned} f_4(y, \partial_y\Phi, x, -\partial_x\Phi) &= 16\omega^2 R^2\Bigg[\left(\frac{y_1\sin\alpha - y_2\cos\alpha}{Y} - \frac{x_1\sin\alpha - x_2\cos\alpha}{X}\right)^2 \\ &\quad - \left(\frac{x_1 - R\cos\alpha}{X} - \frac{y_1 - R\cos\alpha}{Y}\right)^2 - \left(\frac{x_2 - R\sin\alpha}{X} - \frac{y_2 - R\sin\alpha}{Y}\right)^2\Bigg] \\ &= -16\omega^2 R^2\left(\frac{y_1\cos\alpha + y_2\sin\alpha - R}{Y} - \frac{x_1\cos\alpha + x_2\sin\alpha - R}{X}\right)^2 \end{aligned}$$

Similarly,

$$\begin{aligned}
g_4(y,\eta,x,\xi) &= (z_1\theta_2 - z_2\theta_1 - w_1\varphi_2 + w_2\varphi_1)^2 - 4R^2(\theta_1^2+\theta_2^2) \\
&= 4(y_1\eta_2 - y_2\eta_1 + x_1\xi_2 - x_2\xi_1)^2 - 4R^2((\xi_1+\eta_1)^2 + (\xi_2+\eta_2)^2)
\end{aligned}$$

which implies

$$\begin{aligned}
g_4(y,\partial_y\Phi,x,-\partial_x\Phi) &= 16\omega^2R^2\Bigg[\left(\frac{y_1\sin\alpha - y_2\cos\alpha}{Y} + \frac{x_1\sin\alpha - x_2\cos\alpha}{X}\right)^2 \\
&\quad - \left(\frac{x_1 - R\cos\alpha}{X} + \frac{y_1 - R\cos\alpha}{Y}\right)^2 - \left(\frac{x_2 - R\sin\alpha}{X} + \frac{y_2 - R\sin\alpha}{Y}\right)^2\Bigg] \\
&= -16\omega^2R^2\left(\frac{y_1\cos\alpha + y_2\sin\alpha - R}{Y} + \frac{x_1\cos\alpha + x_2\sin\alpha - R}{X}\right)^2
\end{aligned}$$

Thus, multiplying f_4 and g_4, we get

$$\left[16\omega^2R^2\left\{\left(\frac{y_1\cos\alpha + y_2\sin\alpha - R}{Y}\right)^2 - \left(\frac{x_1\cos\alpha + x_2\sin\alpha - R}{X}\right)^2\right\}\right]^2$$

Now, notice that

$$\begin{aligned}
\left(\frac{y_1\cos\alpha + y_2\sin\alpha - R}{Y}\right)^2 &= \frac{(y_1 - R\cos\alpha)^2 + (y_2 - R\sin\alpha)^2 - (y_1\sin\alpha - y_2\cos\alpha)^2}{Y^2} \\
&= 1 - \frac{h^2}{Y^2} - \left(\frac{y_1\sin\alpha - y_2\cos\alpha}{Y}\right)^2 \quad \text{and, similarly} \\
\left(\frac{x_1\cos\alpha + x_2\sin\alpha - R}{X}\right)^2 &= \frac{(x_1 - R\cos\alpha)^2 + (x_2 - R\sin\alpha)^2 - (x_1\sin\alpha - x_2\cos\alpha)^2}{X^2} \\
&= 1 - \frac{h^2}{X^2} - \left(\frac{x_1\sin\alpha - x_2\cos\alpha}{X}\right)^2
\end{aligned}$$

So that

$$\left(\frac{y_1\cos\alpha + y_2\sin\alpha - R}{Y}\right)^2 - \left(\frac{x_1\cos\alpha + x_2\sin\alpha - R}{X}\right)^2$$

$$\begin{aligned}
&= -\left(\frac{y_1\sin\alpha - y_2\cos\alpha}{Y} - \frac{x_1\sin\alpha - x_2\cos\alpha}{X}\right)\left(\frac{y_1\sin\alpha - y_2\cos\alpha}{Y} + \frac{x_1\sin\alpha - x_2\cos\alpha}{X}\right) \\
&\quad + h^2\left(\frac{1}{X^2} - \frac{1}{Y^2}\right) \\
&= \frac{h^2(X+Y)}{2X^2Y^2}\partial_\omega\Phi - \frac{1}{2\omega R}\left(\frac{y_1\sin\alpha - y_2\cos\alpha}{Y} + \frac{x_1\sin\alpha - x_2\cos\alpha}{X}\right)\partial_\alpha\Phi
\end{aligned}$$

Substituting this in the expression for f_4g_4 we get,

$$f_4(y,\partial_y\Phi,x,-\partial_x\Phi)g_4(y,\partial_y\Phi,x,-\partial_x\Phi) = \omega^4k_1(x,y,\alpha)\partial_\omega\Phi + \omega^3k_2(x,y,\alpha)\partial_\alpha\Phi$$

Thus, (a) now implies that

$$\begin{aligned} P_4 K_{F^*F} &= \int e^{-i\Phi}\widetilde{a}c_4 f_4(y,\partial_y\Phi,x,-\partial_x\Phi)g_4(y,\partial_y\Phi,x,-\partial_x\Phi)\,d\alpha\,d\omega \\ &= \int e^{-i\Phi}\widetilde{a}c_4\omega^4 k_1\partial_\omega\Phi\,d\alpha\,d\omega + \int e^{-i\Phi}\widetilde{a}c_4\omega^3 k_2\partial_\alpha\Phi\,d\alpha\,d\omega \\ &= -i\int e^{-i\Phi}\partial_\omega(\widetilde{a}c_4\omega^4 k_1)\,d\alpha\,d\omega - i\int e^{-i\Phi}\partial_\alpha(\widetilde{a}c_4\omega^3 k_2)\,d\alpha\,d\omega \end{aligned}$$

by integration by parts. Note that $\widetilde{a}$ is a symbol of order $2m$, c_4 is homogeneous of degree -3 in ω and k_1, k_2 are independent of ω. Thus, $\partial_\omega(\widetilde{a}c_4\omega^4 k_1)$ and $\partial_\alpha(\widetilde{a}c_4\omega^3 k_2)$ are both symbols of order $2m$. Thus, applying (b), we conclude that $P_4 K_{F^*F} \in H^{s_0}_{\text{loc}}$.

Note that in the above proof, we have only used the fact that $\widetilde{a}$ is a symbol of order $2m$, and have shown that for $i = 1,2,3,4$, $P_i K_{F^*F} = \int e^{-i\Phi}\widetilde{b}\,d\alpha\,d\omega$ where $\widetilde{b}$ is again a symbol of order $2m$. Thus, it is easy to see that the iterated regularity condition of Theorem 4.4 is satisfied and $F^*F \in I^{p,l}(\Delta, \widetilde{\mathcal{C}})$ for some p and l. □

4.3 The Bistatic Case

Suppose now that the transmitter and receiver still move along a circular path, but are separated by a fixed angle 2α at all times, i.e., at tims s, the trasmitter is at $(\cos(s-\alpha), \sin(s-\alpha))$ and the receiver is at $(\cos(s+\alpha), \sin(s+\alpha))$. Note that we have taken $R = 1$ and $h = 0$ for simplicity. Using the model from [12], the scattering operator F is an FIO given by

$$Ff(s,t) = \int e^{-i\omega(t-\sqrt{(x-\cos(s-\alpha))^2+(y-\sin(s-\alpha))^2}-\sqrt{(x-\cos(s+\alpha))^2+(y-\sin(s+\alpha))^2})}af(x,y)dxdyd\omega$$

where a is of order m. Here $Ff(s,t)$ measures the scattered waves received at the point $(\cos(s+\alpha), \sin(s+\alpha))$ at time t. Set

$$A = \sqrt{(x-\cos(s-\alpha))^2 + (y-\sin(s-\alpha))^2}, \quad B = \sqrt{(x-\cos(s+\alpha))^2 + (y-\sin(s+\alpha))^2}$$

It can be easily verified that

$$\phi = -\omega(t - A - B)$$

is a non-degenerate phase function. Now, let us consider the canonical relation $\mathcal{C}$ of F. By definition,

$$\mathcal{C} = \left\{ \left(\alpha, t, \frac{\partial\phi}{\partial s}, \frac{\partial\phi}{\partial t}, x, y, -\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y}\right) \middle| \frac{\partial\phi}{\partial\omega} = 0, \left(\frac{\partial\phi}{\partial s}, \frac{\partial\phi}{\partial t}\right) \neq 0, \left(-\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y}\right) \neq 0 \right\}$$

Note that

$$\frac{\partial\phi}{\partial\omega} = 0 \Rightarrow t = A + B$$

We parametrize $\mathcal{C}$ as follows:

$$G(x, y, s, \omega) = (G_1, G_2, \cdots, G_8)$$

where

$$\begin{aligned}
G_1 &= s \\
G_2 &= t = A + B \\
G_3 &= \frac{\partial\phi}{\partial s} = \omega\Bigg[\frac{(x - \cos(s-\alpha))\sin(s-\alpha) - (y - \sin(s-\alpha)\cos(s-\alpha)}{A} \\
&\quad + \frac{(x - \cos(s+\alpha))\sin(s+\alpha) - (y - \sin(s+\alpha)\cos(s+\alpha)}{B}\Bigg] \\
G_4 &= \frac{\partial\phi}{\partial t} = -\omega \\
G_5 &= x \\
G_6 &= y \\
G_7 &= -\frac{\partial\phi}{\partial x} = -\omega\left[\frac{x - \cos(s-\alpha)}{A} + \frac{x - \cos(s+\alpha)}{B}\right] \\
G_8 &= -\frac{\partial\phi}{\partial y} = -\omega\left[\frac{y - \sin(s-\alpha)}{A} + \frac{y - \sin(s+\alpha)}{B}\right]
\end{aligned}$$

Since $(G_1, G_4, G_5, G_6) = (s, -\omega, x, y)$, G is an injective immersion. If we remove points (x, y, s, ω) such that $\omega = 0$ or

$$\left(\frac{x - \cos(s-\alpha)}{A} + \frac{(x - \cos(s+\alpha)}{B}\right) = 0 \quad \text{and} \quad \left(\frac{y - \sin(s-\alpha)}{A} + \frac{y - \sin(s+\alpha)}{B}\right) = 0$$

from the domain of G, then G is a parametrization of $\mathcal{C}$.

Let $a = \cos\alpha$ and $b = \sin\alpha$. Let us restrict the range of F to only those (s, t) such that $t > 2a/b$. With this restriction,we can show that $\mathcal{C}$ is a two-sided fold. Note that when $\alpha = 0$, $2a/b = 0$, so that this generalizes Theorem 4.1.

Theorem 4.6. *$\mathcal{C}$, if restricted as described above, is a two-sided fold*

Proof. Step 1: Let us first consider the canonical left projection

$$\pi_L(x, y, s, \omega) = (G_1, G_2, G_3, G_4)$$

We have

$$d\pi_L = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{x-\cos(s-\alpha)}{A} + \frac{x-\cos(s+\alpha)}{B} & \frac{y-\sin(s-\alpha)}{A} + \frac{y-\sin(s+\alpha)}{B} & * & * \\ \omega\left(\frac{(y-\sin(s-\alpha))C}{A^3} + \frac{(y-\sin(s+\alpha))D}{B^3}\right) & -\omega\left(\frac{(x-\cos(s-\alpha))C}{A^3} + \frac{(x-\cos(s+\alpha))D}{B^3}\right) & * & * \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

where

$$C = x\cos(s-\alpha) + y\sin(s-\alpha) - 1 \qquad D = x\cos(s+\alpha) + y\sin(s+\alpha) - 1$$

so that $\det(d\pi_L)(x,y,s,\omega) = \omega\frac{CB^2+DA^2}{A^2B^2}\left[1 + \frac{(x-\cos(s-\alpha))(x-\cos(s+\alpha))+(y-\sin(s-\alpha))(y-\sin(s+\alpha))}{AB}\right]$.

Second term vanishes iff $\left(\frac{x-\cos(s-\alpha)}{A}, \frac{y-\sin(s-\alpha)}{A}\right) = -\left(\frac{x-\cos(s+\alpha)}{B}, \frac{y-\sin(s+\alpha)}{B}\right)$. But this means that both phase variables of π_R are 0 and this is excluded. So $\det(d\pi_L)$ vanishes on the submanfold Σ given by $CB^2 + DA^2 = 0$. Also, if $d\pi_L = 0$, both the phase variables of π_R are 0, so that $\det(d\pi_L)$ drops rank exactly by 1 on Σ.

Next, suppose $p = (x,y,s,\omega) \in \Sigma$. Let

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and define $f(x,y) = (bx-1)[y^2 + a^2 + (b-x)^2] + 2a^2y^2$. Then it can be verified that $CB^2 + DA^2 = f(x', y')$. We have,

$$\begin{aligned} \left.\frac{\partial f}{\partial x}\right|_{(x'(p),y'(p))} &= b[y'^2 + a^2 + (b-x')^2] + 2(x'-b)(bx'-1) \\ \left.\frac{\partial f}{\partial y}\right|_{(x'(p),y'(p))} &= 2(bx'-1)y' + 4a^2y' \end{aligned}$$

Now, $\partial_y f = 0$ implies $y' = 0$ or $x' = \frac{1-2a^2}{b}$. Suppose $y' = 0$. Then $f(x', y') = 0$ implies $x' = 1/b$ which implies $\partial_x f \neq 0$. Next, suppose $x' = \frac{1-2a^2}{b}$. Then $f(x', y') = -2a^2\left[a^2 + \left(b - \frac{1-2a^2}{b}\right)^2\right] \neq 0$. So, $\nabla_{x',y'} f(x', y') \neq 0$. This proves that $\det(d\pi_L)$ vanishes to exactly first order on Σ.

Next, we claim that $T_p\Sigma \cap (\ker(d\pi_L)(p)) = \{0\}$. Let $v = (v_1, v_2, v_3, v_4) \in T_p\Sigma \cap (\ker(d\pi_L)(p))$. Clearly, $v_3 = v_4 = 0$. As before, we define

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Now, $v \in \ker(d\pi_L)(p)$ implies

$$\left(\frac{x-\cos(s-\alpha)}{A} + \frac{x-\cos(s+\alpha)}{B}, \frac{y-\sin(s-\alpha)}{A} + \frac{y-\sin(s+\alpha)}{B}\right) \cdot (v_1, v_2) = 0$$

$$\Rightarrow \left(\frac{x'-\cos\alpha}{A'} + \frac{x'-\cos\alpha}{B'}, \frac{y'+\sin\alpha}{A'} + \frac{y'-\sin\alpha}{B'}\right) \cdot (v_1', v_2') = 0 \tag{4.1}$$

where

$$A' = \sqrt{(x'-b)^2 + (y'+a)^2} \qquad B' = \sqrt{(x'-b)^2 + (y'-a)^2}$$

Next, $\nabla(CB^2 + DA^2) \cdot \overline{v} = 0$ implies

$$\frac{\partial f}{\partial x'}(x'(p), y'(p))v_1' + \frac{\partial f}{\partial y'}(x'(p), y'(p))v_2' = 0 \tag{4.2}$$

If we prove that $\left(\frac{x'-\cos\alpha}{A'} + \frac{x'-\cos\alpha}{B'}, \frac{y'+\sin\alpha}{A'} + \frac{y'-\sin\alpha}{B'}\right)$ and $\left(\frac{\partial f}{\partial x'}, \frac{\partial f}{\partial y'}\right)$ are not scalar multiples of each other, we must have $(v_1', v_2') = 0$ and we are done.

Consider the curve $(bx-1)(y^2 + a^2 + (b-x)^2) + 2a^2y^2 = 0$. This curve has the following properties:

- The maximum value of x on the curve is $1/b$ and this is attained at $(1/b, 0)$.
- x increases monotonically as a function of y when $y \in (-\infty, 0)$, decreases monotonically when $y \in (0, \infty)$, and $x \geq b$ iff $y \in [-a, a]$.
- touches the unit circle tangentially at (b,a) and (b,-a).
- has a vertical asymptote $x = \frac{1-2a^2}{b}$ as $y \to \infty$ and $y \to -\infty$.

Since $t > 2a/b$, we consider only larger ellipses $A' + B' = c$ where $c > 2a/b$, i.e., ellipses with foci (b, a) and $(b, -a)$ and semi-minor axis $> 1/b - b$. It can be easily verified that the right half of this ellipse can not intersect the above curve. So, it follows that $x' < b$ and $\frac{x'-b}{A'} + \frac{x'-b}{B'} < 0$. Also $\frac{\partial f}{\partial x}(x', y') > 0$. We also have

$$\begin{aligned}
\frac{y'+\sin\alpha}{A'} + \frac{y'-\sin\alpha}{B'} < 0, &\qquad \frac{\partial f}{\partial y}(x', y') < 0 \text{ when } y' < -a \\
\frac{y'+\sin\alpha}{A'} + \frac{y'-\sin\alpha}{B'} > 0, &\qquad \frac{\partial f}{\partial y}(x', y') > 0 \text{ when } y' > a
\end{aligned}$$

from which it follows that $\left(\frac{x'-\cos\alpha}{A'} + \frac{x'-\cos\alpha}{B'}, \frac{y'+\sin\alpha}{A'} + \frac{y'-\sin\alpha}{B'}\right)$ and $\left(\frac{\partial f}{\partial x'}, \frac{\partial f}{\partial y'}\right)$ can not be scalar multiples of each other. This proves that π_L is a fold.

Step 2: Consider $\pi_R(x, y, s, \omega) = (G_5, G_6, G_7, G_8)$. We have

$$d\pi_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & -\omega\left(\frac{(y-\sin(s-\alpha))C}{A^3} + \frac{(y-\sin(s+\alpha))D}{B^3}\right) & -\left(\frac{x-\cos(s-\alpha)}{A} + \frac{x-\cos(s+\alpha)}{B}\right) \\ * & * & \omega\left(\frac{(x-\cos(s-\alpha))C}{A^3} + \frac{(x-\cos(s+\alpha))D}{B^3}\right) & -\left(\frac{y-\sin(s-\alpha)}{A} + \frac{y-\sin(s+\alpha)}{B}\right) \end{pmatrix}$$

Note that $\det(d\pi_R)(x, y, s, \omega) = -\det(d\pi_L)(x, y, s, \omega)$. Thus repeating the same arguments as in the case of $d\pi_L$, we can conclude that π_R drops rank by 1 on Σ and $\det(d\pi_R)$ vanishes to exactly first order on Σ. Finally, suppose $p = (x, y, s, \omega) \in \Sigma$ and let $v = (v_1, v_2, v_3, v_4) \in T_p\Sigma \cap (\ker(d\pi_R)(p))$. Clearly, $v_1 = v_2 = 0$. Let (x', y') be defined as before. Since $v \in T_p\Sigma$, we must have

$$\frac{\partial}{\partial s}(CB^2 + DA^2)v_3 = 0$$

Now, $\frac{\partial}{\partial s}(CB^2 + DA^2)(p) =$

$$\begin{aligned} & (y\cos(s-\alpha) - x\sin(s-\alpha))[(x - 2\cos(s+\alpha))^2 + (y - 2\sin(s+\alpha)^2 - 1] \\ & +(y\cos(s+\alpha) - x\sin(s+\alpha))[((x - 2\cos(s-\alpha))^2 + (y - 2\sin(s-\alpha)^2 - 1] \\ = \; & [y'b + x'a][(x' - 2b)^2 + (y' - 2a)^2 - 1] + [y'b - x'a][(x' - 2b)^2 + (y' + 2a)^2 - 1] \end{aligned}$$

As before, since we only consider large ellipses $A' + B' = c$ where $c > 2a/b$, we must have $0 < x' < b$ and $|y'| > a$, and each of the terms within the square brackets in the above expression is positive. Thus, $\frac{\partial}{\partial s}(CB^2 + DA^2) > 0$ which means that $v_3 = 0$. Finally, since $v \in \ker(d\pi_R)(p)$, we get

$$\begin{aligned} -\left(\frac{x - \cos(s-\alpha)}{A} + \frac{x - \cos(s+\alpha)}{B}\right)v_4 &= 0 \\ -\left(\frac{y - \sin(s-\alpha)}{A} + \frac{y - \sin(s+\alpha)}{B}\right)v_4 &= 0 \end{aligned}$$

and since both the phase variables can't vanish, $v_4 = 0$ and thus π_R is also a fold. This completes the proof. □

Finally, by an application of Theorem 4.2 we conclude that

Corollary 4.1. *If F is the FIO restricted as described above, $F^*F \in I^{2m,0}(\Delta, \widetilde{C})$ where $\Delta \subset T^*\mathbb{R}^2 \times T^*\mathbb{R}^2$ is the diagonal Lagrangian and $\widetilde{C}$ is a two-sided fold that cleanly intersects Δ.*

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