Aspects of Linearized Perturbations of Black Holes and Flat Black Strings

A thesis submitted in fulfillment of the requirements

for the degree of Doctor of Philosophy

by

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2019

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH (IISER) PUNE

To Aai and Baba

CERTIFICATE

Certified that the work incorporated in the thesis titled 'Aspects of linearized perturbations of black holes and flat black strings' submitted by Amruta Sadhu was carried out by the candidate. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other University or institution.

Date: 05 July 2019

Suneeta Vardarajan
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DECLARATION

I declare that this thesis written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Acknowledgements

I would like to thank my advisor Dr. Suneeta Vardarajan for her support and guidance during my PhD. She gave me freedom to work on my own and helped me whenever I stumbled. I thank her for the encouragement she gave me to explore my other research interests. I am grateful for her help in not just academic but administrative matters.

I would also like to thank my research advisory committee members Prof. Sukanta Bose, Dr Nabamita Banerjee and Prof Avinash Khare for their valuable comments, support and encouragement throughout my PhD. Attending IUCAA group meetings with Prof. Bose's group has helped me expand horizons of my knowledge.

I could not have done this without the constant friendship of Aparna, Bhagwat and Srikrishna. Thank you for tolerating me for all these years and patiently listening to my rants. Your support has kept me going through all the difficult times. I shall always miss having *chai*s with you.

A big thanks to Mali, Shweta, Sucheta, Shamik for their friendship. Thank you Kajari and Niraja, your unconditional support during my down time has meant a lot to me. I thank Dhanya for being a great labmate. I am also thankful to all the other friends whose name I may have missed.

I would like to thank Manjusha Kaku for the excellent food and support she gave me everyday. I would not have survived without it for all these years. I am grateful to all my family and cousins who have supported me through this time.

Finally, I am eternally indebted to my parents for their love and support. Thank you for all the sacrifices you have made for my sake. I could not have achieved anything without you.

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Synopsis

This thesis presents a study in aspects of linearized perturbations of flat black strings and black holes in D dimensions. Main focus of the thesis is on analysis the non-spherically symmetric perturbations of these objects. We have formulated simplified equations for the non-spherically symmetric scalar and vector perturbations and explored the large D limit of general relativity as an analytical tool to study them. Using these equations, we have studied stability of flat black string and semiclassical stability of black holes in the path integral formulation of Euclidean quantum gravity in D-dimensions. Analyzing classical stability of flat black strings, we proved that the non-spherically symmetric perturbations do not lead to Gregory-Laflamme type instability. As the classical stability of D-dimensional black string is related to semiclassical stability of $(D-1)$-dimensional black hole, this analysis also proves that the Gross-Perry-Yaffe negative mode is the *unique* semiclassically unstable mode of the Schwarzschild-Tangherlini black holes. We have computed, for the first time, quasinormal modes of D-dimensional black strings under non-spherically symmetric perturbations. We have calculated frequencies of $\mathcal{O}(1)$ (called decoupled mode frequencies) and those of order $\mathcal{O}(D)$ (non-decoupled mode frequencies) to various orders in D for vector perturbations of these objects. We have also re-analyzed quasinormal modes of Schwarzschild-Tangherlini black holes in the large D limit, with a different approach from previous works on this topic, by not assuming a $1/D$ expansion of the mode functions. We have studied semiclassical stability of the of D-dimensional Schwarzschild AdS black holes under both non-spherically symmetric and spherically symmetric ($\ell = 0$) perturbations. We have shown in various cases that the non-spherically symmetric perturbations do not lead to instability. In the case of spherically symmetric perturbations, where there is an instability, we have calculated eigenvalue corresponding to the unstable mode to next to leading order in D. We show that

the large black holes are stable but the small black holes are semiclassically unstable. This instability mimics features of thermodynamic (in)stability of (small) large black holes found by Hawking and Page.

Chapter 1

Introduction

Motivated by String theory, the study of black holes in higher dimensions has been of increasing interest. In higher dimensions, along with black holes [1], there exist other black hole like solutions to the Einstein equations that have extended horizons. The simplest such solutions to vacuum Einstein's equation are the 'black string' and 'p-brane' which are obtained by adding a single extra flat dimension and a p-dimensional flat metric respectively to the Schwarzschild black hole solution. Such solutions are shown to arise from low-energy string theory [2]. In this thesis we study aspects of linearized perturbations of flat black strings, Schwarzschild-Tangherlini black holes and Schwarzschild-Anti-de Sitter (SAdS) black holes in D dimensions. We will concentrate on the non-spherically symmetric perturbations of these objects. We will analyze classical stability of black strings and compute its quasinormal modes. We shall also study semiclassical stability of the black holes in the path integral formulation of Euclidean quantum gravity for both non-spherically symmetric and spherically symmetric ($\ell = 0$) perturbations. To this end, we shall employ the large D limit of general relativity as an analytical tool.

Stability of four dimensional black holes under perturbations is well established [3], but some higher dimensional black holes and other objects with extended horizons exhibit instabilities (for eg, instability of ultraspinning black holes [4]. See also [5]). Gregory and Laflamme showed that black strings and p-branes are unstable [6] under spherically symmetric (s-wave) perturbations. They found that for perturbations of the form $e^{(\Omega t - ikz)}$, the frequency Ω is positive for $k < k_{GL} \sim 1/r_0$, where r_0 is the horizon radius where k_{GL} is critical wavelength. In other words, for $k < k_{GL}$ there exist perturbations that are regular on the future horizon and normalizable at spatial infinity but grow in time. For $k > k_{GL}$ there are no such modes and the critical threshold mode $k = k_{GL}$ is time independent. This result was later extended to charged black strings [7] and black strings in AdS spacetime in [8]. The instability vanishes for the extremal p-branes [9]. A thermodynamic argument for the existence of this instability is that the black string has lower entropy than an array of black holes with the same total mass, and must hence be unstable. The link between local thermodynamic instability and classical instability of extended objects is conjectured by Gubser and Mitra in [10], [11]. Further analysis of this link is done in [12], [13], [14], [15]. The endpoint of the Gregory-Laflamme instability has been studied extensively. Gregory and Laflamme argued that the black string will fragment into forming a black hole caged in the extra dimension. During the fragmentation, the curvature at the horizon diverges, forming a naked singularity [16], possibly violating the 'cosmic censorship conjecture' (see also [17]). Horowitz and Maeda argued that the endpoint of such instability will be a non-uniform black string in [18]. In a numerical analysis of a five dimensional black string, Lehner and Pretorius [19] found that the endpoint indeed results in a naked singularity. Sorkin found a critical dimension $D \approx 13.5$ above which the evolution of instability possibly ends in a stable non-uniform black string [20]. The large D limit of this analysis points towards the non-uniform black string being an endpoint [21, 22].

Studies of linearized perturbations of black strings in general D dimensions have been concentrated on the spherically symmetric perturbations. The perturbation equations for non-spherically symmetric perturbations are coupled and it is generally not possible to solve them analytically. From studies of the link between local thermodynamic instability and classical instability of extended objects , non-spherically symmetric perturbations are not expected to cause instabilities. However, because of the difficulty in analyzing the coupled non-spherically symmetric perturbations, there is no fully analytical proof of this even for the simplest flat black string in general dimensions. In this thesis, we formulate equations governing $(D + 1)$-dimensional black string by extending the formalism of gauge invariant variables due to Ishibashi and Kodama [23], [24], [25]. This formalism was developed for studying linearized perturbations of static black hole solutions in D dimensions. The basic idea is to decompose the perturbations and the generators of gauge transformation into scalar, vector and tensor components with respect to the $(D-2)$-sphere and formulate gauge invariant variables in each sector. To extend this formalism to suit our case, using an appropriate gauge choice we write our perturbation equations in terms of the Ishibashi-Kodama variables. We have concentrated on the scalar and vector perturbations. Analysis of tensor perturbations is performed by Kodama in [26], where he has developed we set of completely gauge invariant variable for the black strings. We have obtained sets of coupled equations which we cannot be analytically studied for a general D. Analyzing our equations in the large D limit using the technique of matched asymptotic expansions, we have proved that the non-spherically symmetric perturbations do not lead to instabilities in various cases considered. This work is based on results of [27].

The large D limit was first used by Kol and Sorkin in [28] to study the black string/brane instabilities and Asnin et al in [29] to study the Gross-Perry-Yaffe negative mode. A systematic approach to study general relativity in a $1/D$ expansion as $D \to \infty$ was developed by Emparan, Suzuki and Tanabe in [30]. In the large D limit, general relativity simplifies dramatically. Specifically in the limit $D \to \infty$, the gravitational field of the black hole is strongly localized near the horizon. This localization creates a clear distinction between 'near horizon region' and the region far from horizon in the spacetime and introduces a new length scale r_0/D in the theory where r_0 is the horizon radius. Such a neat separation facilitates the use of matched asymptotic expansion techniques to study properties of black holes and other black objects with extended horizons. Due to the $1/D$ expansion, the equations governing dynamics of the black objects get simplified drastically and can be solved order by order. A striking consequence of the large D limit is seen in the study of black hole quasinormal modes [31], [32], [33]. In this limit there exist two distinct sets of quasinormal modes called decoupled modes with frequency $\sim \mathcal{O}(1)$ and non-decoupled modes with frequency $\sim \mathcal{O}(D)$. The decoupled quasinormal modes, which are localized near the black hole horizon, are a novel feature of the large D limit. These modes were also found in a numerial work by Dias, Hartnett and Santos in [34]. This limit has been used to study various higher dimensional black objects like rotating black holes, black strings etc in [35, 36, 37].

Due to the neat separation between the 'near horizon region' and 'far region' in the large D limit, in [38] Emparan et al proposed an effective theory in a $1/D$ expansion for black holes where the black hole can be effectively identified with a hypersurface Σ_B in the background. They argued that the properties of black holes must be able to be derived from the equations Σ_B satisfies. They derived an effective equation to find such a surface. This effective theory can also be motivated by split between the quasinormal modes of high frequencies $\sim D/r_0$ and those of low frequencies $1/r_0$. The effective theory describes the dynamics of the decoupled sector after the high frequency modes have been intergrated out. Effective theories for describing the dynamics of black string is obtained in [21]. The dynamics of non-uniform black string in effective theory are studied in [39],[40]. Effective theories for charged rotating black holes and black rings are explored in [41, 42, 43, 44, 45, 46]. Relation between effective theory for black branes (to leading order in the 1/D expansion), and hydrodynamic theory of a compressible, viscous fluid, with a conserved particle-number current is found in [47].

A similar $1/D$ effective theory approach named 'a membrane paradigm at large D' was independently proposed by Bhattacharya et al in [48]. The two approaches are shown to be equal in [49]. Bhattacharya et al started with a very simple ansatz for a black hole metric in Kerr schild coordinates that solves the vacuum Einstein's equation at the leading order in 1/D. Upon solving for the perturbative corrections, they observed that the black hole horizon acts like a codimension one 'membrane' embedded in a flat D dimensional background. The membrane is characterized by its shape as embedded in the background and a 'velocity field' defined on it which together satisfy a integrability condition. This integrability condition is viewed as a dynamical equation on the coupled system of the membrane and the velocity. The formalism was extended to subleading orders by Dandekar et al in [50]. A 'charged membrane paradigm, was formulated in [51]. Extension of the membrane paradigm to include the cosmological constant was achieved in [52, 53, 54]. Study of stress tensor and charge current on this membrane was done in [55]. Relation between the membrane paradigm and fluid-gravity correspondence was shown in [56, 57].

The natural extension of the large D limit formalism is to apply it to study the black objects in modified theories of gravity where the equations governing these objects are generally more complicated than those in general relativity. The large D limit has been used to study quasinormal modes of black holes in Gauss-Bonnet gravity by Chen et al [58]. The effective theory formalism has been extended to study various black holes and other extended black objects in the Einstein-Gauss-Bonnet gravity in a series of papers by Chen et al [59, 60, 61, 62]. A membrane paradigm for Einstein-Gauss-Bonnet gravity has been developed in [63]. A large D membrane paradigm formalism for the most general four derivative gravity has been developed in [64].

Quasinormal modes are characteristic response of black hole to an perturbation. These modes were first found by Vishveshwara [65] in calculation of scattering of gravitational wave by a Schwarzschild black hole . In the event of the merger of two compact objects, the resulting black hole settles down by emitting gravitational waves that are characterized by its quasinormal mode frequencies. It is possible to find these frequencies from the gravitational wave signal of black hole merger observed by LIGO [66]. In the case of black holes in general relativity, these frequencies are completely characterized by the black hole parameters mass, charge and angular momentum. Quasinormal modes of black holes have been studied extensively in four and higher dimensions. Some excellent reviews are [67, 68, 69]. Quasinormal modes of general D-dimensional Schwarzschild black holes by WKB method in [70, 71, 72, 73]. Highly damped quasinormal modes of D dimensional Schwarzschild black holes were evaluated by Cardoso,Lemos and Yoshida in [74].

In the case of other black objects, study of quasinormal modes of AdS black hole and AdS black brane holds a special importance. This is because in the AdS/CFT correspondence [75], the quasinormal spectra of asymptotically AdS_{D+1} and more general backgrounds correspond to poles of the retarded thermal correlators of dual D-dimensional strongly interacting quantum gauge theories [76, 77, 78]. Quasinormal modes of AdS black branes were computed by Starinets in [79]. A direct evaluation of quasinormal modes of a D dimensional flat black string/brane by analyzing equations for non-spherically symmetric perturbations has not been done yet. In this thesis, we have evaluated these modes for the first time in the large D limit.

As stated before, we have used the techniques of matched asymptotic expansions to study the non-spherically symmetric perturbations of black strings. Using this technique, we have evaluated the vector decoupled quasinormal mode frequencies of black strings. We have also calculated non-decoupled mode frequencies for both scalar and vector perturbations in the large D limit. For evaluating the non-decoupled modes, we have used the Laplace transform method by Nollert and Schmidt [80]. In the Laplace transform method, the quasinormal modes are defined as the poles of Green's function of the perturbation equation.

To understand the effect of large D limit as it pertains to the perturbation equation and its solutions, we have re-analyzed the quasinormal modes of Schwarzschild-Tangherlini black holes in the limit where D is taken to be large. Our analysis differs from previous large D analyses in that we do not a priori assume any particular D-dependence of the mode functions or frequencies that is analytic in D. Solving for the quasinormal mode frequencies, we find the non-decoupled mode frequency to the leading order. In the case of decoupled quasinormal mode frequencies, we find that they are indeed a series in $1/D$ and with our method we can obtain their value to the next to leading order. This work is based on [81].

In Euclidean path integral formulation of quantum gravity [82], a spacetime is semiclassically unstable if there exist a set of perturbations which decrease the action from its classical value. Gross, Perry and Yaffe showed that the Schwarzschild instanton in four dimensions is semiclassically unstable [83]. They found a single nonconformal, spherically symmetric unstable mode called 'negative mode'. [1] The existence of this unique nonconformal negative mode implies that the Schwarzschild solution is a saddle-point of the action rather than a true minimum. In [84], Allen numerically analyzed the unstable mode by considering the Schwarzschild instanton in an isothermal cavity. He found that the negative mode persists till the cavity radius falls below some critical value. Existence of this mode for in higher dimensions is shown in [85]. In [12], Reall showed that the Gregory-Laflamme instability of the black string is related to the Gross-Perry-Yaffe negative mode. Specifically, the eigenvalue corresponding to the negative mode (λ) is related to the critical wavelength k_{GL} as $\lambda = -k_{GL}^2$. Using this correspondence, for static perturbations, our analysis of stability of the $(D+1)$-dimensional black string under the non-spherically symmetric perturbations also serves as analysis of semiclassical stability of corresponding D-dimensional Schwarzschild black hole.

[1] The name comes from the sign of eigenvalue corresponding to the Lichnerowicz laplacian of the black hole.

Thermodynamic properties of black holes in finite isothermal cavities and their relation to the semiclassical stability of the black holes was studied by Whiting and York in [86, 87, 88]. In this setup sign of the second variation of the reduced Euclidean action with respect to the horizon radius r_+ is same as the sign of the heat capacity of the black hole [88]. The reduced Euclidean action here depends on the temperature of the black hole as well as radius and temperature of the cavity walls. A black hole is thermodynamically unstable if it has negative specific heat.

Thermodynamic analysis of SAdS black holes, along the lines of [86], was done by Brown, Creighton and Mann [89]. They studied properties of both $(3+1)$ and $(2+1)$ dimensional black holes. They found that for a suitable choice of boundary temperature, the larger black hole will be thermodynamically stable and the smaller black hole will be unstable. This behaviour is identical to that found by Hawking and Page [90]. Motivated by this result, Prestidge [91] investigated the problem of semi-classical instability of the Schwarzschild-AdS black holes along the lines of Gross, Perry and Yaffe's analysis. He found that the eigenvalue λ undergoes a transition from being positive for large black holes to negative for small black holes. This analysis was done numerically and and there is no analytical formula obtained for the eigenvalue. In this thesis, we have performed this analysis for both non-spherically symmetric and spherically symmetric perturbations in the large D limit. For non-spherically symmetric perturbations we have formulated perturbation equations which for $\Lambda = 0$ reduce to the black string equations. We have showed absence of instabilities in the various cases in the non-spherically symmetric sector. For spherically symmetric perturbations, using the perturbation equation in [91], we have found the eigenvalue up to next to leading order in D.

The plan of this thesis is as follows: In Chapter 2 we will derive the equations for non-spherically symmetric perturbations which we shall analyze in the rest of the thesis. Chapter 3 contains analysis of the black string stability. We will also discuss the technical details of the large D limit in this chapter. Quasinormal modes of black string and Schwarzschild-Tangherlini black holes in the large D limit are discussed in Chapter 4. In Chapter 5 we will analyze semi-classical stability of SAdS black holes in the large D limit. Summary and further discussion of the results is given in Chapter 6.

Chapter 2

Obtaining The Equations

In this chapter we shall obtain equations governing classical stability of the flat black string and semiclassical stability of the Schwarzschild and AdS-Schwarzschild (SAdS) black holes under linearized perturbations. We will concentrate on the non-spherically symmetric perturbations. We will review the notion of semiclassical instability in detail. We shall also discuss the Gregory-Laflamme instability in black string. Then we shall show the relation between classical perturbation of the D dimensional black string and semiclassical perturbations of the $(D-1)$ dimensional Schwarzschild black hole. Our goal in this chapter is to write our equations in terms of the gauge invariant variables introduced by Ishibashi and Kodama (IK) [25, 23, 24]. These variables were introduced to study classical perturbations of black holes. We shall extend their formalism to suit our problem.

2.1 Semiclassical stability of black holes

The partition function Z in the path integral approach of quantum gravity is defined as

$$Z = \int_{\mathcal{M}} D[g] e^{-iI[g]} \tag{2.1}$$

The functional integral is taken over all metrics with some fixed asymptotic behaviour over a manifold $\mathcal{M}$. The action $I[g]$ in the general relativity is given by

$$I = -\frac{1}{16\pi G}\int_{\mathcal{M}}(R-2\Lambda)\sqrt{-g}dx^n - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} K(\pm\sigma)^{1/2}dx^{n-1} \tag{2.2}$$

The second term in the action is a boundary term required to make the variation of the action well defined. The action $I(g)$ is real for Lorentzian metrics and hence the path integral will oscillate and not converge. To get around this issue, the time t is Wick rotated to introduce an imaginary time coordinate $\tau = it$ which makes the metric Euclidean. The action $I[g] = -i\hat{I}[g]$ and the path integral changes to

$$\hat{Z} = \int_{\mathcal{M}} D[g]e^{-\hat{I}[g]} \tag{2.3}$$

In a semiclassical or stationary phase approximation, partition function is evaluated by Taylor expanding the action around the solutions to the classical field equations.

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = 0 \tag{2.4}$$

To expand the action, one considers perturbations around the background metric. The perturbed metric is $\tilde{g}_{ab} = g_{ab} + h_{ab}$. The action is then expanded as,

$$\hat{I}[\tilde{g}] = \hat{I}[g] + \hat{I}_2[h] + \text{higher order terms} \tag{2.5}$$

The first term is action of the classical solution. The next contribution $\hat{I}_2[h]$ comes from the second variation of the action as the first variation of action vanishes for the classical solution. The contribuation quadratic in h_{ab} to the action $\hat{I}_2$ can be written as

$$\hat{I}_2[h] = \frac{1}{32\pi G}\int h^{ab}A_{abcd}h^{cd}(g)^{1/2}d^4x \tag{2.6}$$

where, from [82],

$$
\begin{aligned}
A_{abcd} =& \frac{1}{4} g_{cd} \nabla_a \nabla_b - \frac{1}{4} g_{ac} \nabla_d \nabla_b + \frac{1}{8}(g_{ac} g_{bd} + g_{ab} g_{cd}) \nabla_e \nabla^e + \frac{1}{2} R_{ad} g_{bc} \\
& - \frac{1}{4} R_{ab} g_{cd} + \frac{1}{16} R g_{ab} g_{cd} - \frac{1}{8} R g_{ac} g_{bd} - \frac{1}{8} \Lambda g_{ab} g_{cd} + \frac{1}{4} \Lambda g_{ac} g_{bd} \\
& + (a \leftrightarrow b) + (c \leftrightarrow d) + (a \leftrightarrow b, c \leftrightarrow d)
\end{aligned}
\tag{2.7}
$$

The operator A has a large number of zero eigenvalues arising from invariance of the action under gauge transformations of the metric. To extract contribution from only physical degrees of freedom, a gauge fixing term and associated ghost contributions are added such that the resulting operator has no zero eigenvalues [82]. We are interested in the eigenvalues of this operator. Specifically, we wish to see if this operator has negative eigenvalues.

A spacetime is semiclassically unstable if there exist a set of perturbations which decrease the action from its classical value. Existence of such perturbation mode implies that the classical solution is only a saddle point of the action and not a true extremum. This corresponds to the negative eigenvalues of the operator A_{abcd} (2.7). Even after fixing the gauge and adding the ghost terms, the final form of the operator A_{abcd} is still very complicated. To get only the relevant pieces of the final operator, the perturbation is decomposed in the following manner:

$$
h_{ab} = h_{ab}^{TT} + \frac{1}{4} g_{ab} h + (\nabla_a \xi_b + \nabla_b \xi_a - \frac{1}{2} g_{ab} \nabla_c \xi^c). \tag{2.8}
$$

h_{ab}^{TT} is the transverse traceless part of h_{ab} obeying

$$
\nabla^a h_{ab}^{TT} = 0 \quad g^{ab} h_{ab}^{TT} = 0 \tag{2.9}
$$

The other pieces in 2.8 are a trace part h, and a tracefree vector field ξ_a. The effective action has an extra vector field due to the ghost contribution. These vector fields are further decomposed into their components. Expanding the effective action in terms of these various components, it is shown that the operators acting on h_{ab}^{TT} and the trace h can have negative eigenvalues [83] [82]. But the operator acting on trace piece is gauge dependent and can be made arbitrarily negative by performing conformal transformations on the background metric. By choosing an appropriate gauge, this operator can be made to vanish. Hence the only physical gauge invariant operator that can have negative eigenvalues is the one acting on the transverse-traceless part h^{TT}_{ab} [82, 83]. The operator takes the simple form

$$G_{abcd} = -g_{ac}g_{bd}\nabla_e\nabla^e - 2R_{abcd}. \tag{2.10}$$

Eigenvalues of this operator are determined by the normalizable solutions of the equation

$$G^{ab}_{cd}h^{cd} = \kappa h^{ab}. \tag{2.11}$$

The eigenfunctions h_{ab} are real and transverse-traceless. As stated before, negative eigenvalues of the equation correspond to the unstable solutions. For flat space the eigenvalues are positive definite [83], [92]. Semiclassical stabilty of the four dimensional Schwarzschild black hole instanton was studied by Gross,Perry and Yaffe in [83]. They found existence of a single spherically symmetric, negative mode with eigenvalue $\kappa = -0.19M^2$ showing that the Schwarzschild instanton is unstable, where M is the mass of the corresponding black hole. The existence of a negative mode for the Schwarzschild-AdS black hole instanton was shown by Prestidge in [91]. Prestidge further showed that the negative eigenvalue of the SAdS black hole instanton reduced to that of Schwarzschild instanton in the limit $\Lambda \to 0$. For $\kappa = 0$, the equation (2.11) reduces to the classical perturbations of the background metric. For both Schwarzschild and Schwarzschild AdS black holes classical stability with $\kappa = 0$ has been established by extensive studies [3, 24].

For the transverse-traceless perturbations $h_{\mu\nu}$, the eigenvalue equation 2.11 can be written in the terms of Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ as

$$2\delta G_{\mu\nu} + 2\Lambda h_{\mu\nu} = \kappa h_{\mu\nu} \tag{2.12}$$

where $\delta G_{\mu\nu}$ is the first variation of the Einstein tensor.

2.2 Black string and Gregory-Laflamme instability

The (uncharged) black string metric is $D = n + 3$ dimensions, obtained by adding a flat extra dimension (z) to the $n + 2$ dimensional Schwarzschild-Tangherlini metric is

$$g_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_n^2 + dz^2; \tag{2.13}$$

where $f(r) = \left(1 - \frac{b^{n-1}}{r^{n-1}}\right)$ and $r > b$. The horizon is located at $r = b$. The topology of the event horizon is $\mathbb{R} \times \mathbb{S}_n$. A flat black brane of dimension $D = (n + 2 + p)$ is obtained by adding the flat metric corresponding to p extra dimensions.

Gregory and Laflamme [6] analyzed stability of black string/ flat black branes under the spherically symmetric (s-wave) perturbations of the form $h_{MN} = e^{\Omega t}e^{i\lambda z}H_{MN}$. Computationally, they looked for the solutions to the eigenvalue equation

$$(\Delta_L^D + \lambda^2)h_{MN} = 0 \tag{2.14}$$

Δ_L^D is D-dimensional Lichnerowicz operator. Demanding the perturbations to be regular at the horizon and decaying exponentially at infinity, leads to the condition $\Omega > 0$. They numerically integrated the eigenvalue equation and found that there are indeed such solutions for a range of Ω for appropriate values of λ. In their paper, Gregory and Laflamme performed this analysis for $4 \leq D \leq 9$, but the instability has been shown to exist for all values of D [28]. We shall now perform this analysis for the non-spherically symmetric perturbations.

2.3 Non-spherically symmetric perturbations of the black string

We will use capital Roman indices $A, B, ...$ to denote coordinates on the black string. Greek indices $\mu, \nu, ...$ will be used to denote indices only in the Schwarzschild-Tangherlini part of the metric. Coordinates in the black hole part of the metric will be denoted collectively by y.

We consider perturbations of the metric (2.13), with the perturbed metric $\bar{g}_{AB} = g_{AB} +$

$\bar{h}_{AB}$, in linearized perturbation theory. We first make a gauge choice used by Reall [12] that allows us to set $\bar{h}_{Az} = 0$. The only non-zero perturbations left after gauge-fixing are thus the ones with indices in the Schwarzschild part of the metric. Our objective in this section is to establish a connection between stability of the semi-classical perturbations of Schwarzschild-Tangherlini black holes to that of the classical perturbations of the black string. In [12], this gauge choice is used for the spherically symmetric s-wave perturbations where there is indeed an unstable mode in both the cases: the semiclassical Gross-Perry-Yaffe negative mode for Schwarzschild black hole instanton [83] and the Gregory-Laflamme unstable mode for black string [6]. In this gauge, the unstable mode of the black string can be seen as emerging from the semiclassical instability of the lower dimensional black hole and hence will be independent of the z coordinate. The linearized Einstein equation for the black string perturbations is

$$\delta R_{MN} = 0 \tag{2.15}$$

The linearized Ricci tensor δR_{MN} is expressed in terms of the Lichnerowicz Laplacian Δ_L acting on the perturbations as

$$2\delta R_{MN} = \Delta_L \bar{h}_{MN} - \nabla_M \nabla_N \bar{h} + \nabla_M \nabla_S \bar{h}^S_N + \nabla_N \nabla_S \bar{h}^S_M \tag{2.16}$$

$$\Delta_L \bar{h}_{MN} = -\nabla^L \nabla_L \bar{h}_{MN} + R_{ML} \bar{h}^L_N + R_{NL} \bar{h}^L_M - 2R_{MLNS} \bar{h}^{LS}. \tag{2.17}$$

All curvature tensors are those of the black string metric (2.13). $\bar{h} = g^{MN} \bar{h}_{MN}$. For this background metric, the Laplacian acting on symmetric tensors splits in the form

$$\nabla^L \nabla_L = \nabla^\mu \nabla_\mu + \partial_z^2. \tag{2.18}$$

Expanding all the covariant derivatives, and imposing the gauge $\bar{h}_{Mz} = 0$ we see that for the the metric (2.13) the $\delta R_{\mu\nu}$ equation (2.15) reduces to

$$\Delta_L^{Sch} \bar{h}_{\mu\nu} - \nabla_\mu \nabla_\nu \bar{h} + \nabla_\mu \nabla_\sigma \bar{h}^\sigma_\nu + \nabla_\nu \nabla_\sigma \bar{h}^\sigma_\mu = \partial_z^2 \bar{h}_{\mu\nu} \tag{2.19}$$

$\Delta_L^{Sch} \bar{h}_{\mu\nu}$ denotes the Lichnerowicz Laplacian of the Schwarzschild-Tangherlini metric acting on perturbations of the string metric. The other equations with indices on the extra z dimension, $\delta R_{z\mu}$ and δR_{zz} reduce to

$$\partial_z[g^{\sigma\nu}\left(\nabla_\nu \bar{h}_{\mu\sigma} - \nabla_\mu \bar{h}_{\nu\sigma}\right)] = 0 \tag{2.20}$$

$$\partial_z^2\left(g^{\mu\nu}\bar{h}_{\mu\nu}\right) = 0 \tag{2.21}$$

Following Gregory and Laflamme, we choose the ansatz for $\bar{h}_{\mu\nu}(y,z)$

$$\bar{h}_{\mu\nu}(y,z) = e^{i\lambda z} h_{\mu\nu}(y). \tag{2.22}$$

The z-dependence of the string perturbation $\bar{h}_{\mu\nu}$ is in the $e^{i\lambda z}$ term. The perturbation $h_{\mu\nu}(y)$ depends only on the coordinates of the black hole part of the metric (2.13).

(2.19) then becomes

$$\Delta_L^{Sch} h_{\mu\nu} - \nabla_\mu\nabla_\nu h + \nabla_\mu\nabla_\sigma h^\sigma_\nu + \nabla_\nu\nabla_\sigma h^\sigma_\mu = -\lambda^2 h_{\mu\nu} \tag{2.23}$$

Here, $h = g^{\mu\nu}h_{\mu\nu}$. Similarly, the equations (2.20) and (2.21) respectively simplify to

$$\nabla^\sigma h_{\mu\sigma} - \nabla_\mu h = 0 \quad \text{and} \quad h = 0. \tag{2.24}$$

These equations together imply that $h_{\mu\nu}(y)$ is transverse-traceless. Using this fact in (2.23), we get

$$\Delta_L^{Sch} h_{\mu\nu} = -\lambda^2 h_{\mu\nu}. \tag{2.25}$$

Thus, we finally obtain an eigenvalue equation for the Lichnerowicz Laplacian in the Schwarzschild-Tangherlini background. Negative eigenvalues (i.e. real λ) corresponding to normalizable eigentensors are relevant for perturbations of the black string. The normalizability is defined with respect to the volume form of the background. For transverse-traceless $h_{\mu\nu}$, the equation (2.25) is equivalent to

$$\delta G_{\mu\nu} = -\frac{1}{2}\lambda^2 h_{\mu\nu}; \tag{2.26}$$

where $\delta G_{\mu\nu}$ is the first variation of the Einstein tensor evaluated for the transverse traceless perturbation $h_{\mu\nu}$ on the Schwarzschild-Tangherlini background.

For $\lambda = 0$, the equation (2.26) describes the equation for classical perturbations of the Schwarzschild-Tangherlini black holes. Stability of these black holes has been proven by Ishibashi and Kodama [25]. To explore stability of black strings/branes, the non-trivial case to analyze is solutions to (2.25) with $\lambda \neq 0$.

Note that the equation for black string perturbations (2.26) is a special case of the equation (2.12), derived in the previous section for studying the semiclassical perturbations of the SAdS black holes, with cosmological constant Λ set to zero and $\kappa = -\lambda^2$. For $\Lambda = 0$, the equation (2.12) describes the semiclassical (in)stability of the Schwarzschild-Tangherlini black holes. Thus we see that the classical stability of the black string in $D+1$ dimensions is related to the semiclassical stability of the Schwarzschild black holes in D dimensions.

Although we have discussed the classical perturbations of black string with a single flat extra dimension z, these equations can easily be generalized to the case of black branes with p extra flat dimensions. For the black brane, the generalization of the ansatz (2.22) is $\bar{h}_{\mu\nu} = e^{i\lambda_k z^k} h_{\mu\nu}(y)$ where k runs from 1 to p (the number of extra dimensions) and $\lambda^2 = \Sigma_{k=1}^{p} \lambda_k^2$. Equivalently, we could have set the linearized Einstein tensor for the brane to zero and obtained (2.26).

To analyze the perturbation equations, we will adapt a formalism due to Ishibashi and Kodama (IK). This formalism was originally developed for studying classical gravitational perturbations of black holes in higher dimensions [25]. As the linearized Ricci tensor is invariant under a gauge transformation, for classical perturbations of black holes with $\lambda = 0$, (2.26) is invariant. Ishibashi and Kodama introduced manifestly gauge-invariant variables by taking suitable combinations of metric perturbations of the black hole spacetime. The linearized Einstein equation are then written entirely in terms of these variables.

We will use their variables even for $\lambda \neq 0$. Due to the presence of a non-zero right hand side, we will have to take appropriate combinations of the various equations to obtain the eigenvalue equation written entirely in terms of the IK variables. This is done mainly for computational simplicity. Using the IK variables reduces the number of equations that describe the perturbations to a set of five equations. A set of two coupled equations for the vector perturbations and a set of three coupled equations for the scalar perturbations.

The Ishibashi-Kodama variables

We will set up the notation for this thesis by quickly stating the perturbation variables proposed by Ishibashi and Kodama in ([25], [23]) for doing gravitational perturbation theory. This formalism was developed for studying perturbations of black hole metric of the form

$$g_{\mu\nu}dy^{\mu}dy^{\nu} = g_{ab}(x)dx^a dx^b + r^2(x)d\Omega_n^2; \tag{2.27}$$

$$= -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_n^2. \tag{2.28}$$

where

$$f(r) = 1 - \frac{2\Lambda}{n(n+1)}r^2 - \frac{b^{n-1}}{r^{n-1}} \tag{2.29}$$

For Schwarzschild-Tangherlini black holes, $\Lambda = 0$. For the Schwarzschild-AdS black holes, $\Lambda < 0$. Henceforth, we shall denote

$$\sigma^2 = -\frac{2\Lambda}{n(n+1)}. \tag{2.30}$$

For SAdS black holes with negative Λ, $\sigma^2 > 0$ and for Schwarzschild black holes, $\sigma^2 = 0$.

Since, after gauge-fixing, we are working with perturbations of the metric (2.28), we will use the IK variables. Here $g_{ab}(x)$ is the $r-t$ part of the metric and $d\Omega_n^2 = \gamma_{ij}d\tilde{y}^i d\tilde{y}^j$ is the metric of a n-dimensional sphere of unit radius with Ricci tensor given by $\hat{R}_{ij} = (n-1)\gamma_{ij}$. The construction of the gauge invariant variables is actually applicable for a wide range of $(m+n)$ dimensional spacetimes that can be written in the form (2.27), where $d\Omega_n^2$ is any n-dimensional space with constant sectional curvature.

We use indices a, b to denote indices from the set (r, t) and indices i, j are coordinates on sphere. Indices μ, ν denote any coordinate in the spacetime with metric (2.28). Covariant derivatives and Ricci tensors on each space are denoted as

$$\begin{aligned} g_{\mu\nu} &\to \nabla_\mu, R_{\mu\nu} \\ g_{ab} &\to D_a, {}^m R_{ab} \\ \gamma_{ij} &\to \hat{D}_i, \hat{R}_{ij}. \end{aligned}$$

We consider perturbations of the metric (2.28) with the perturbed metric denoted by $g^p_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ in linearized perturbation theory. The perturbation $h_{\mu\nu}$ is defined in terms of the original black string perturbation by (2.22). For the semiclassical stability problem, $h_{\mu\nu}$ is just a perturbation on the black hole metric. We first decompose the perturbation $h_{\mu\nu}$ as

$$h_{\mu\nu} dy^\mu dy^\nu = h_{ab} dx^a dx^b + 2h_{ai} dx^a d\tilde{y}^i + h_{ij} d\tilde{y}^i d\tilde{y}^j \tag{2.31}$$

The scalar, vector and tensor components of $h_{\mu\nu}$ are defined as those that are decomposed in terms of scalar, vector and tensor spherical harmonics on the n-sphere, respectively. The components h_{ab} are scalars with respect to transformations on the n-sphere. The other components can be further decomposed as follows:

$$h_{ai} = \hat{D}_i h_a + h^{(1)}_{ai} \tag{2.32}$$

$$h_{ij} = h^{(2)}_{T\ ij} + 2\hat{D}_{(i} h^{(1)}_{T\ j)} + h_L \gamma_{ij} + \hat{L}_{ij} h^{(0)}_T. \tag{2.33}$$

where

$$\hat{D}^j h^{(2)}_{T\ ij} = h^{(2)i}_{T\ \ i} = 0 \tag{2.34}$$

$$\hat{D}^a h^{(1)}_{ai} = 0, \hat{D}^j h^{(1)}_{T\ j} = 0. \tag{2.35}$$

Here $h^{(2)}_{T\ ij}$ is the 'tensor' part, the 'vector' set is $(h^{(1)}_{T\ j}, h^{(1)}_{ai})$, and the 'scalar' set is $(h_{ab}, h_a, h_L, h^{(0)}_T)$. The eigenvalue equations (2.12) decouple for these three classes, which can be studied separately.

The goal is to write combinations of perturbations in each set which are gauge invariant. To do this, we need to decompose the generator of a gauge transformation ξ_μ into its components. Under a gauge transformation generated by any infinitesimal vector ξ_μ, the perturbation transforms as

$$h'_{\mu\nu} = h_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu. \tag{2.36}$$

Decomposition of ξ_μ into a 'vector' part and a gradient of a scalar is written as

$$\xi_a = \xi_a^{(0)}, \xi_i = \xi_i^{(1)} + \hat{D}_i \xi^{(0)}. \tag{2.37}$$

Here $(\xi_a^{(0)}, \xi^{(0)})$ are scalars and $\xi_i^{(1)}$ is a pure vector.

2.3.1 Vector perturbations

We use the notation of [25], [23] to describe the vector perturbations.[1] Consider the ansatz

$$h_{ab} = 0 \qquad h_{ai} = r f_a^{vector} V_i \qquad h_{ij} = 2r^2 H_T^{vector} V_{ij}. \tag{2.38}$$

Here $f_a^{vector}, H_T^{vector}$ are functions of r, t. Vector harmonics V_i and V_{ij} are defined by

$$(\hat{\Delta} + k_v^2) V_i = 0, \hat{D}_i V^i = 0 \tag{2.39}$$

$$V_{ij} = -\frac{1}{2k_v}(\hat{D}_i V_j + \hat{D}_j V_i). \tag{2.40}$$

$k_v^2 = \ell(\ell + n - 1) - 1$ and $\ell = 2, ...$ We denote $\hat{\Delta} = \gamma^{ij} \hat{D}_i \hat{D}_j$.

The generator of gauge transformation only has a non-zero vector component for the vector perturbations.

$$\xi_a = 0 \qquad \xi_i = r L V_i \tag{2.41}$$

[1] Ishibashi and Kodama use the notation f_a and H_T to denote distinct quantities in the vector and scalar case. Here we add a superscript *vector* in the vector case to avoid confusion.

Substituting in (2.36), the functions f_a^{vector} and H_T^{vector} transform as,

$$\delta f_a^{vector} = -rD_a\left(\frac{L}{r}\right), \qquad H_T^{vector} = \frac{k_v}{r}L \tag{2.42}$$

Looking at the transformation, we can formulate a gauge-invariant variable in the class of vector perturbations using the combination

$$F_a = f_a^{vector} + \frac{r}{k_v}D_aH_T^{vector}. \tag{2.43}$$

2.3.2 Scalar perturbations

We shall now construct gauge-invariant variables for scalar perturbations. Again following [23], [25] anstaz for scalar perturbations is chosen as

$$h_{ab} = f_{ab}S \qquad h_{ai} = rf_aS_i \qquad h_{ij} = 2r^2(H_L\gamma_{ij}S + H_TS_{ij}) \tag{2.44}$$

S, S_i and S_{ij} are scalar harmonics satisfying

$$(\hat{\Delta} + k^2)S = 0 \qquad S_i = -\frac{1}{k}\hat{D}_iS \qquad \hat{D}_iS^i = kS$$
$$S_{ij} = \frac{1}{k^2}\hat{D}_i\hat{D}_jS + \frac{1}{n}\gamma_{ij}S \qquad S_i^i = 0$$

$k^2 = \ell(\ell+n-1)$ and $\ell = 0, 1, 2....$ We will not deal with the case $\ell = 0$ which corresponds to spherically symmetric perturbations in this section. The scalar variables constructed by IK are not gauge-invariant for this case. Later we will use a different prescription by [91] to get the eigenvalue equation for this case. The modes with $k^2 = n$ (i.e., $\ell = 1$) are exceptional modes. For these modes, S_{ij} vanishes and thus the construction of gauge-invariant variables is not possible. A detailed prescription to deal with these perturbations is given in [23]. We shall not deal with this mode in this thesis. Therefore, we will consider only $\ell \geq 2$ while discussing the scalar modes.

Gauge invariant variables for scalar perturbations (not defined for $\ell = 0$ and $\ell = 1$)are constructed as follows: First we define

$$X_a = \frac{r}{k}\left(f_a + \frac{r}{k} D_a H_T\right) \tag{2.45}$$

Components of the generator for gauge transformation ξ_μ for the scalar perturbations are

$$\xi_a = T_a S, \quad \xi_i = rLS_i. \tag{2.46}$$

Again substituting in (2.36), transformation of various functions in (2.44) becomes:

$$\begin{array}{ll} \delta f_{ab} = -D_a T_b - D_b T_a & \delta f_a = -rD_a\left(\frac{L}{r}\right) + \frac{k}{r} T_a \\ \delta H_L = -\frac{k}{nr} L - \frac{D^a r}{r} T_a & \delta H_T = \frac{k}{r} L \quad \delta X_a = T_a \end{array}$$

After taking suitable combinations, the gauge invariant variables for scalar class of perturbations in terms of X_a are

$$F_{ab} = f_{ab} + D_a X_b + D_b X_a \tag{2.47}$$

$$F = H_L + \frac{1}{n} H_T + \frac{1}{r} D^a X_a. \tag{2.48}$$

2.3.3 Tensor Perturbations

For the tensor class, IK consider expand the perturbation as

$$h_{ab} = 0, \quad h_{ai} = 0, \quad h_{ij} = 2r^2 H_T T_{ij}. \tag{2.49}$$

The tensor spherical harmonics T_{ij} are defined as

$$(\hat{\Delta} + k_t^2) T_{ij} = 0, \quad T_i^i = 0, \quad \hat{D}^i T_{ij} = 0,$$

where $k_t^2 = \ell(\ell = n - 1) - 2$. As ξ_i has no tensor component, the variable H_T is gauge invariant.

We have not analyzed this class of perturbations in this thesis. Stability of flat black strings under tensor perturbations has been studied by Kodama in [26]. The eigenvalue equation for tensor perturbations is

$$-\Box H_T - \frac{n}{r} DrDH_T + \left(\frac{k_t^2+2}{r^2}\right) H_T = -\lambda^2 H_T \tag{2.50}$$

For $H_T = e^{i\omega t}\tilde{H}_T(r)$, Kodama has shown that the flat black string is stable under tensor perturbations. This analysis has also been done for static perturbations in the context of stability of the Schwarzschild-Tangherlini metric under Ricci flow in [93]. Their results show that the black string is stable under static perturbations. This also proves the semi-classical stability of the Schwarzschild-Tangherlini black holes under tensor perturbations for all n.

In the next two sections, we will consider the equations for the vector and scalar perturbations arising from (2.12), written using IK variables.

2.4 Vector Perturbations

We now write the equations (2.12) in terms of the IK variables F_a defined in the previous section (2.43). We need to write the equations $\delta G_{ai} + \Lambda h_{ai} = -\frac{1}{2}\lambda^2 h_{ai}$ and $\delta G_{ij} + \Lambda h_{ij} = -\frac{1}{2}\lambda^2 h_{ij}$ in terms of the IK variables F_a. To write the variation of the Einstein tensor in terms of these variables, we have used the expressions given in [23], [25].

$$\begin{aligned}
&\frac{1}{r^{n+1}} D^b \left[r^{n+2} \left[D_b\left(\frac{F_a}{r}\right) - D_a\left(\frac{F_b}{r}\right)\right]\right] - \frac{\alpha}{r^2} F_a = \lambda^2 f_a^{vector} \\
&\frac{k_v}{r^n} D_a(r^{n-1}F^a) = \lambda^2 H_T^{vector};
\end{aligned} \tag{2.51}$$

where $\alpha = k_v^2 - (n-1)$.

Although the left hand sides of the two equations are written in terms of F_a, the right hand sides containing the eigenvalue λ are not. Notice that the RHS of the two equations contain f_a and H_T whose combination gives the variable F_a (2.43). We hence combine the equations (2.51) such that the right hand side of the final equation can be written in terms of F_a. The second order differential equations for the variables F_a (i.e., F_r and F_t) is

$$\begin{aligned}\Box F_a - D^b D_a F_b + D_a D^b F_b + n\frac{D^b r D_b F_a}{r} - 2\frac{D^b r D_a F_b}{r} - \frac{\Box r}{r}F_a - n\frac{(Dr)^2}{r^2}F_a \\ -(n-2)\frac{D^b r D_a r}{r^2}F_b + \frac{D^b D_a r}{r}F_b + (n-1)\frac{D_a D^b r}{r}F_b - \frac{\alpha}{r^2}F_a = \lambda^2 F_a.\end{aligned} \tag{2.52}$$

Explicitly evaluating the covariant derivatives, we get a system of coupled equations for components F_r and F_t.

$$f\partial_r^2 F_t - \frac{1}{f}\partial_t^2 F_t + \frac{nf}{r}\partial_r F_t - \left[\frac{nf}{r^2} + \frac{\alpha}{r^2}\right]F_t + \left[f' - \frac{2f}{r}\right]\partial_t F_r = \lambda^2 F_t \tag{2.53}$$

$$\begin{aligned}f\partial_r^2 F_r - \frac{1}{f}\partial_t^2 F_r + \left[2f' + \frac{(n-2)f}{r}\right]\partial_r F_r \\ + \left[f'' + \frac{(n-2)f'}{r} - \frac{2(n-1)f}{r^2} - \frac{\alpha}{r^2}\right]F_r + \frac{f'}{f^2}\partial_t F_t = \lambda^2 F_r\end{aligned} \tag{2.54}$$

The equations (2.53) and (2.54) match with those of Kodama[26]. While we have worked with gauge-fixed variables, in [26], Kodama has constructed gauge-invariant variables in the entire black brane. For the vector perturbations, this scheme leads to a set of three coupled equations which reduce to our set of two coupled equations for special cases. To further simplify the equations, we do a modal decomposition of F_t and F_r.

$$F_t = A(r)e^{i\omega t} \qquad F_r = \frac{B(r)}{f}e^{i\omega t} \tag{2.55}$$

The resulting equations for $A(r)$ and $B(r)$ are:

$$\frac{d^2A}{dr^2} + \frac{n}{r}\frac{dA}{dr} + \left(-\frac{n}{r^2} - \frac{\alpha}{fr^2} - \frac{\lambda^2}{f} + \frac{\omega^2}{f^2}\right)A = \left(\frac{2}{rf} - \frac{f'}{f^2}\right)i\omega B \tag{2.56}$$

$$\frac{d^2B}{dr^2} + \frac{(n-2)}{r}\frac{dB}{dr} + \left(-\frac{2(n-1)}{r^2} - \frac{\alpha}{fr^2} - \frac{\lambda^2}{f} + \frac{\omega^2}{f^2}\right)B = -\frac{f'}{f^2}i\omega A \tag{2.57}$$

We will study these equations to analyze stability and quasinormal modes of flat black strings.

2.5 Scalar Perturbations

In this section we write eigenvalue equations $2\delta G_{\mu\nu} + 2\Lambda h_{\mu\nu} = -\lambda^2 h_{\mu\nu}$ in terms of the IK variables for scalar class of perturbations. To achieve this, as in the case of vector perturbations, we first use [23] to express $2\delta G_{\mu\nu} + 2\Lambda h_{\mu\nu}$ in terms of F_{ab} and F. As the indices a, b take the values (r, t), we have four IK variables $(F_{rr}, F_{rt}, F_{tt}, F)$. Again, as in the vector case, the right hand side of the eigenvalue equations cannot expressed in terms of either F_{ab} of F. To get the eigenvalue equations completely in terms of the IK variables, we will have to explicitly evaluate the eigenvalue equations for components of $\delta G_{\mu\nu}$ and take their combinations. Rather than getting a set of equations for F_{ab} and F, to simplify the equations further, we construct three new functions from their components. Following IK [24] the three functions are;

$$W = r^{n-2}(F^t_t - 2F) \qquad Y = r^{n-2}(F^r_r - 2F) \qquad Z = r^{n-2}F^r_t. \tag{2.58}$$

The goal is to take appropriate combinations of the eigenvalue equations for $\delta G_{\mu\nu}$ so that they can be expressed entirely in terms of just three perturbation variables W, Y and Z. We first need to invert the relations (2.58) to obtain F^a_a and F in terms of W, Y and Z. To accomplish this, we use the traceless part of the equation $2\delta G_{ij} + 2\Lambda h_{ij} = -\lambda^2 h_{ij}$. The equation, when written partly in terms of the new variables, becomes

$$W + Y + 2nF = 2\lambda^2 \frac{r^2}{k^2} H_T. \tag{2.59}$$

We can now write F in terms of W, Y and H_T using this relation. For IK, in the case $\lambda = 0$, F can be completely expressed in terms of W and Y without the extra H_T factor.

Subsequently using (2.59) and (2.58) we get:

$$F = -\frac{W+Y}{2nr^{n-2}} + \frac{\lambda^2}{nk^2}(r^2 H_T) \tag{2.60a}$$
$$F_t^r = \frac{Z}{r^{n-2}} \tag{2.60b}$$
$$F_t^t = \frac{W(n-1)-Y}{2nr^{n-2}} + \frac{2\lambda^2}{nk^2}(r^2 H_T) \tag{2.60c}$$
$$F_t^t = \frac{Y(n-1)-W}{2nr^{n-2}} + \frac{2\lambda^2}{nk^2}(r^2 H_T) \tag{2.60d}$$

Our choice of variables is motivated by those of Ishibashi and Kodama who studied the linearized Einstein equation where $\lambda = 0$. Their variables therefore correspond to (2.58) with λ is zero i.e. without the extra H_T terms. Hence their expressions of F_a^a, F can be solely written in terms of W, Y and Z.

Substituting our new variables in the eigenvalue equations, we obtain six equations (2.61)-(2.66). Due to the H_T factors in (2.60), these equations have terms containing derivatives of H_T in addition to components of $h_{\mu\nu}$.

Equation for δG_{ti} :

$$\partial_t W + \partial_r Z = \lambda r^{n-2}\left[X_t + \frac{1}{k^2}\partial_t(r^2 H_T)\right] \tag{2.61}$$

Equation for δG_{ri} :

$$\partial_r Y + \frac{f'}{2f}Y - \frac{f'}{2f}W - \frac{1}{f^2}\partial_t^2 Z = \lambda r^{n-2}\left[X_r + \frac{1}{k^2}\partial_r(r^2 H_T)\right] \tag{2.62}$$

Equation for δG_t^r :

$$\begin{aligned}&\left[\frac{k^2}{r^2} - f'' - \frac{nf'}{r} + 2(n+1)\sigma^2\right] Z + f\partial_t\partial_r Y + \left(\frac{2f}{r} - \frac{f'}{2}\right)\partial_t Y\\ &+ f\partial_t\partial_r W - \left(\frac{(n-2)f}{r} + \frac{f'}{2}\right)\partial_t W = -\lambda r^{n-2} f_t^r\\ &- \frac{2\lambda}{nk^2}\left[n\partial_t\partial_r(r^2 H_T) - \frac{nf'}{r}\partial_t(r^2 H_T)\right]\end{aligned} \tag{2.63}$$

Equation for δG^r_r :

$$\begin{aligned}
&\frac{1}{f}\partial_t^2 W - \frac{f'}{2}\partial_r W + \frac{1}{f}\partial_t^2 Y - \left(\frac{f'}{2} + \frac{nf}{r}\right)\partial_r Y \\
&+ \left[\frac{n-1}{r^2}(f-1) - \frac{2(n+1)}{n}\sigma^2 + \frac{(n+2)f'}{2r} + \frac{f''}{n}\right] W \\
&+ \left[\frac{1-f}{r^2} + \frac{2(n^2-1)}{n}\sigma^2 - \frac{3n-2}{2r}f' - \frac{n-1}{n}f'' + \frac{k^2-nK}{r^2}\right] Y \\
&+ \frac{2n}{rf}\partial_t Z = -\lambda r^{n-2} f^r_r \\
&- \frac{2\lambda}{nk^2}\left[-\frac{nk^2}{r^2}(H_T r^2) - \frac{n}{f}\partial_t^2(r^2 H_T) + \frac{nf'}{2}\partial_r(r^2 H_T) + \frac{n^2 f}{r}\partial_r(r^2 H_T) + 2\Lambda(r^2 H_T)\right]
\end{aligned} \tag{2.64}$$

Equation for δG^i_i :

$$\begin{aligned}
&\frac{1}{2f}\partial_t^2 W + \frac{f'}{4}\partial_r W - \frac{f}{2}\partial_r^2 Y - \left(\frac{3f'}{4} + \frac{f}{r}\right)\partial_r Y \\
&+ \left[\frac{(n-1)(n-2)(f-1)}{2nr^2} + \frac{(6n-4-n^2)f'}{4nr} + \frac{f''}{2n}\right] W \\
&+ \left[\frac{(n-1)(n-2)(f-1)}{2nr^2} + \frac{(-n^2+2n-4)f'}{4nr} - \frac{(n-1)f''}{2n}\right] Y \\
&+ \left(\frac{1}{rf} - \frac{f'}{2f^2}\right)\partial_t Z + \frac{1}{f}\partial_t\partial_r Z = -\lambda r^{n-2} H_L + \frac{\lambda}{nk^2}\left[\frac{(n-1)k^2}{r^2}(H_T r^2)\right. \\
&\left. - nf\partial_r^2(r^2 H_T) + \frac{n}{f}\partial_t^2(r^2 H_T) - nf'\partial_r(r^2 H_T) - \frac{n(n-1)f}{r}\partial_r(r^2 H_T) - 2\Lambda(r^2 H_T)\right]
\end{aligned} \tag{2.65}$$

Equation for δG^t_t :

$$\begin{aligned}
&- f\partial_r^2 W + \left(\frac{n-4}{r}f - \frac{f'}{2}\right)\partial_r W - f\partial_r^2 Y - \left(\frac{f'}{2} + \frac{4f}{r}\right)\partial_r Y \\
&- \left[\frac{n-1}{r^2} - \frac{(2n-3)f}{r^2} - \frac{2(n^2-1)}{n}\sigma^2 + \frac{n-2}{2r}f' + \frac{n-1}{n}f'' - \frac{k^2}{r^2}\right] W \\
&- \left[\frac{n-1}{r^2} + \frac{2(n+1)}{n}\sigma^2 - \frac{n-3}{r^2}f + \frac{(n-2)f'}{2r} - \frac{f''}{n}\right] Y = -\lambda r^{n-2} f^t_t \\
&- \frac{2\lambda}{nk^2}\left[-\frac{nk^2}{r^2}(H_T r^2) + nf\partial_r^2(r^2 H_T) + \frac{nf'}{2}\partial_r(r^2 H_T) + \frac{n^2 f}{r}\partial_r(r^2 H_T) + 2\Lambda(r^2 H_T)\right]
\end{aligned} \tag{2.66}$$

Recall that σ^2 and Λ are related by (2.30). Our goal is to get the final equations completely in terms of W, Y and Z by taking suitable combinations of the eigenvalue equations, in analogy with the work of Ishibashi and Kodama, and despite the extra H_T factors present in our expressions. Expanding the variables F_{ab} and F in (2.58), we get

$$\frac{W}{r^{n-2}} = f_t^t - \frac{2}{f}\partial_t X_t + \left(f' - \frac{2f}{r}\right) X_r - 2H_L - \frac{2H_T}{n} \tag{2.67}$$

$$\frac{Y}{r^{n-2}} = f_r^r + 2f\partial_r X_r + \left(f' - \frac{2f}{r}\right) X_r - 2H_L - \frac{2H_T}{n} \tag{2.68}$$

$$\frac{Z}{r^{n-2}} = f_t^r + f\partial_r X_t + f\partial_t X_r - f' X_t \tag{2.69}$$

Looking at the expressions above, the final W, Y and Z equations are obtained as follows: Looking at the expression of Z (2.69), we see that adding (2.63), and derivatives of (2.61) and (2.62) with appropriate coefficients will give the right-hand side of the resulting equation in terms of the Z variable.

$$\begin{aligned}&\partial_r^2 Z - \frac{1}{f^2}\partial_t^2 Z - \left(\frac{(n-2)}{r} + \frac{f'}{f}\right)\partial_r Z - \\ &\left[\frac{k^2}{fr^2} - \frac{f''}{f} - \frac{nf'}{fr} + \frac{2(n+1)}{f}\sigma^2\right] Z - \left(\frac{2}{r} - \frac{f'}{f}\right)\partial_t Y - \frac{f'}{f}\partial_t W = \frac{\lambda^2}{f} Z\end{aligned} \tag{2.70}$$

The equation for Y (2.68)) is obtained by adding (2.64), the derivative of (2.62) and (2.65).

$$\begin{aligned}&\partial_r^2 Y - \frac{1}{f^2}\partial_t^2 Y - \left(\frac{n}{r} - \frac{f'}{f}\right)\partial_r Y - \\ &\left[-\frac{2(n-1)}{nr^2 f} + \frac{-n^2+2n-2}{nr^2} + \frac{(2-n)}{nrf}f' - \frac{f''}{f} + \frac{f'^2}{2f^2} + \frac{k^2}{r^2 f} + \frac{2(n^2-1)}{nf}\sigma^2\right] Y - \\ &\left[-\frac{2(n-1)}{nr^2 f} + \frac{2n-2}{nr^2} + \frac{2-n}{nrf}f' + \frac{f''}{f} - \frac{f'^2}{2f^2} + \frac{2(n+1)}{nf}\sigma^2\right] W + \frac{2f'}{f^3}\partial_t Z = \frac{\lambda^2}{f} Y\end{aligned} \tag{2.71}$$

To obtain the equation for W (2.67), we add (2.66), the derivative of (2.61), (2.65) and

(2.62).

$$
\begin{aligned}
&\partial_r^2 W - \frac{1}{f^2}\partial_t^2 W - \left(\frac{(n-4)}{r} - \frac{f'}{f}\right)\partial_r W - \\
&\left[-\frac{2(n-1)}{nr^2 f} + \frac{n^2-2}{nr^2} + \frac{(2-3n)}{nrf}f' - \frac{f''}{f} + \frac{f'^2}{2f^2} + \frac{k^2}{r^2 f} + \frac{2(n^2-1)}{nf}\sigma^2\right]W - \\
&\left[-\frac{2(n-1)}{nr^2 f} - \frac{2}{nr^2} + \frac{2+n}{nrf}f' + \frac{f''}{f} - \frac{f'^2}{2f^2} + \frac{2(n+1)}{nf}\sigma^2\right]Y - \left(\frac{2f'}{f^3} - \frac{4}{rf^2}\right)\partial_t Z = \frac{\lambda^2}{f}W
\end{aligned}
\tag{2.72}
$$

We see that the extra H_T terms automatically vanish and the equations are coupled and final equations are only in terms of W, Y and Z. Furthermore, in the static limit, the Z equation decouples and we get coupled equations for (W, Y).

We have thus successfully reduced the six eigenvalue equations for the scalar perturbations to three coupled second order partial differential equations for W, Y and Z. Our later computations are made simpler by the further change of variables:

$$
\hat{\psi} = \frac{f^{1/2}}{r^{(n-4)/2}}W \qquad \hat{\phi} = \frac{f^{1/2}}{r^{n/2}}Y \qquad \hat{\eta} = \frac{1}{r^{(n-2)/2}f^{1/2}}Z; \tag{2.73}
$$

We do a modal decomposition $\hat{\psi}(r,t) = e^{i\omega t}\psi(r)$ for all three variables. Finally, the three coupled perturbation equations are:

$$
\begin{aligned}
&-\frac{d^2\psi}{dr^2} + \left[\frac{n^3-2n^2+8n-8}{4nr^2} + \frac{f'^2}{4f^2} - \frac{(n^2+2n-4)}{2n}\frac{f'}{fr} - \frac{f''}{2f}\right. \\
&\left.+\frac{2(n^2-1)\sigma^2}{nf} - \frac{2(n-1)}{nr^2 f} + \frac{k^2}{fr^2} + \frac{\lambda^2}{f} - \frac{\omega^2}{f^2}\right]\psi = \\
&\left[\frac{4}{f} - \frac{2f'r}{f^2}\right](i\omega)\eta + \left[\frac{2(n-1)}{nf} + \frac{2}{n} - \frac{n+2}{n}\frac{rf'}{f} - \frac{r^2 f''}{f} + \frac{f'^2 r^2}{2f^2} + \frac{2(n+1)}{n}\frac{\sigma^2 r^2}{f}\right]\phi
\end{aligned}
\tag{2.74}
$$

$$
\begin{aligned}
&-\frac{d^2\phi}{dr^2} + \left[\frac{n^3-2n^2+8n-8}{4nr^2} + \frac{f'^2}{4f^2} - \frac{(n^2+2n-4)}{2n}\frac{f'}{fr} - \frac{f''}{2f}\right. \\
&\left.+\frac{2(n^2-1)\sigma^2}{nf} - \frac{2(n-1)}{nr^2 f} + \frac{k^2}{fr^2} + \frac{\lambda^2}{f} - \frac{\omega^2}{f^2}\right]\phi = \\
&\frac{2f'}{f^2 r}\eta(i\omega) + \left[\frac{2(n-1)}{nr^4 f} - \frac{2(n-1)}{nr^4} - \frac{2-n}{nr^3}\frac{f'}{f} - \frac{f''}{r^2 f} + \frac{f'^2}{2f^2 r^2} + \frac{2(n+1)}{n}\frac{\sigma^2}{r^2 f}\right]\psi
\end{aligned}
\tag{2.75}
$$

$$-\frac{d^2\eta}{dr^2}+\left[\frac{n^2-2n}{4r^2}-\frac{(n+2)f'}{2rf}+\frac{3f'^2}{4f^2}-\frac{3f''}{2f}+\frac{k^2}{fr^2}-\frac{\omega^2}{f^2}+\frac{\lambda^2}{f}+\frac{2(n+1)\sigma^2}{f}\right]\eta$$
$$=\left[\frac{f'}{f}-\frac{2}{r}\right]\frac{r(i\omega)}{f}\phi-\frac{f'}{f^2}\frac{(i\omega)}{r}\psi \qquad (2.76)$$

We will use these equations for ψ, ϕ and η for analysis of the scalar perturbations. These equations are still intricately coupled and it is not possible to decouple them without making a large n approximation. In the rest of the thesis, we shall study the vector and scalar perturbation equations obtained in this chapter. For $\sigma = 0$, these equations describe classical perturbations on black string. We will analyze stability and quasinormal modes of black strings. For static perturbations with non-zero σ, the equations describe semi-classical perturbations of SAdS black holes.

2.6 Summary

In this chapter, we analyze the non-spherically symmetric perturbations of the black string/flat black brane. The perturbations are decomposed in terms of the scalar, vector and tensor spherical harmonics on the n-sphere part of the brane metric. By an appropriate choice of gauge, and by generalizing perturbation variables introduced by Ishibashi and Kodama for black hole perturbation theory, we have rewritten the brane perturbation equations in a vastly simplified form. The tensor perturbations which reduce to an ODE for a single function have already been discussed in [26]. It is the vector and scalar perturbations that have eluded analysis before. In our formulation, the vector perturbations reduce to a system of two coupled ODEs. The scalar perturbations reduce to three coupled ODEs. We show that these equations also describe the semiclassical perturbations of black holes in the path integral formulation of Euclidean quantum gravity.

Chapter 3

Black Strings: Stability

We will be analyzing stability of the black string in this chapter. We will investigate stability under non-spherically symmetric perturbations to find whether there are unstable modes like the spherically symmetric Gregory-Laflamme mode. We will prove that there are no such unstable modes for the black strings in the cases considered in the large D limit[1]. Furthermore, classical stability of $(D+1)$ dimensional black string is related to semiclassical stability of corresponding D dimensional black hole. Using this, we show that there are no non-spherically symmetric unstable mode which implies that the spherically symmetric Gross-Perry-Yaffe negative mode is the unique unstable mode for the Schwarzschild-Tangherlini black holes. Stability of the black string under tensor perturbations has been proved by Kodama [26]. We shall analyze equations obtained for non-spherically symmetric perturbations in Chapter (2) in the large D limit for our investigation. We will first study the vector perturbations in detail and then analyze the scalar perturbations.

3.1 Large D limit

The perturbation equations we obtained in the previous chapter are coupled and cannot be fully solved analytically. To study these equations, we will use the large D limit as an analytical tool. This limit has first been used by Kol and others [28] to analyze spherically

[1]Non-spherically symmetric perturbations of black string were studied numerically in [94]. However the analysis has incorrect equations.

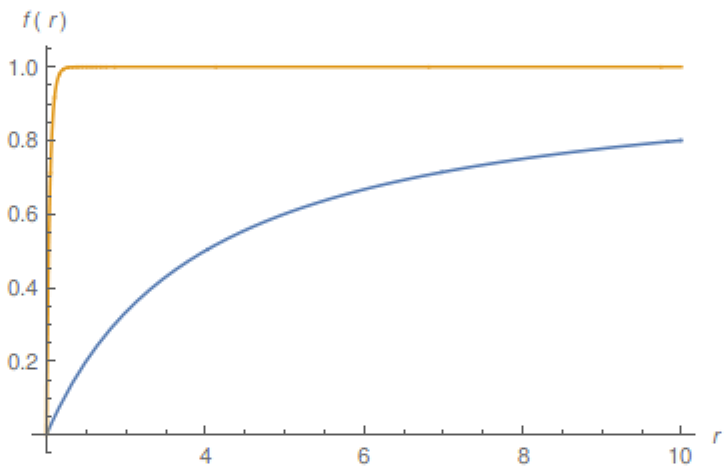

Figure 3.1: $f(r)$ for Schwarzschild black holes in $D = 4$ (blue) and $D = 52$ (yellow). The horizon is at $r = 2$.

symmetric perturbations of the black string. This limit was extensively studied by Emparan et al to analyze linearized perturbations of various black objects. To understand the effect of the large D limit, let us look at a D-dimensional black hole.

The Schwarzschild-Tangherlini black hole metric in $D = n + 2$ dimensions is

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_n^2 \tag{3.1}$$

with

$$f = \left(1 - \frac{b^{n-1}}{r^{n-1}}\right). \tag{3.2}$$

The horizon radius is b and $d\Omega_n^2$ is metric of n-sphere with unit radius. In this thesis the horizon radius b is independent of D. This implies that for a fixed b, the metric remains finite as $D \to \infty$. In the following discussion of the large D limit, we have interchanged $D \to \infty$ and $n \to \infty$. While it does not affect our discussion of the features of the large D limit, particularly the leading behaviour. In the large n limit we can see that this function increases steeply near the horizon and becomes constant away from the horizon.

In the above figure we have plotted the function $f(r)$ for $D = 4$ and $D = 52$. We can see that for $D = 52$, $f(r)$ increases steeply near the horizon ($b = 2$ in the figure). The function $f(r)$ shows a step-like behaviour, where after a steep increase, it becomes constant. We can see that the effect of large D is visible even at a finite $D \approx 50$. In fact the step-like behaviour starts showing even for $D \approx 30$.

In the large n limit, gradient of $f(r)$ near the horizon becomes very large as

$$f'(r)|_b \sim \frac{n-1}{b} \sim \frac{n}{b}. \tag{3.3}$$

In the limit $n \to \infty$, the gravitational potential vanishes far outside the horizon, but there is a small region around the horizon on the scale of b/n where the black hole exerts its gravitational influence. The gravitational influence is appreciable around in the region

$$r - b \leq \frac{b}{n} + \mathcal{O}(n^{-2}) \tag{3.4}$$

The function $f = \left(1 - \frac{b^{n-1}}{r^{n-1}}\right)$ which appears in the background metric is an increasing function and $f(r) \to 1$ as $r \to \infty$ as shown in figure (3.1). As the gradient of f is large near the horizon, in the large n limit, this function increases steeply from zero in the interval $b < r < b + \frac{b}{n}$. Outside this region, the function$f(r)$ is almost constant for $r > \frac{b}{n}$. This creates two distinct regions in the black hole spacetime. The two distinct regions called a near region and far region are defined as follows:

$$\begin{aligned} \text{Near region} \quad & r - b \ll b \\ \text{Far region} \quad & r - b \gg \frac{b}{n-1} \end{aligned}$$

The definition of the far region as is decided by the sphere of influence of gravitational field around the horizon in the large n limit. The function $f(r)$ is almost constant in the far region. These definitions of the two regions allow for the existence of an overlap region in $\frac{b}{n-1} \ll r - b \ll b$.

Existence of such distinct regions facilitates the use of the method of matched asymptotic expansions to analyze perturbation equations where $f(r)$ and its derivatives appear in the coefficient functions. We solve our equations in the two distinct regions separately and then extend the solutions in the two regions to the overlap region and match the two solutions. This is not different from the established strategy for solving the black hole perturbation equations very near the horizon and at infinity and trying to match the solutions in the interior region in four dimensions. However, the simplified structure of the distinct regions and existence of a small overlap region simplifies the process and leads to more concrete matching procedures compared to that in four dimensions where there is no overlap between near horizon region and infinity.

Solving the equations in the near region becomes easier if we define a new coordinate

$$R = \left(\frac{r}{b}\right)^{n-1}. \tag{3.5}$$

In term of this coordinate, the near and far regions are

$$\begin{aligned} \text{Near region} \quad & \ln R \ll n-1 \\ \text{Far region} \quad & \ln R \gg 1 \end{aligned}$$

Using the near-region approximation, r can be expanded in terms of R as

$$r \sim b\left[1 + \frac{\ln R}{n-1} + \frac{(\ln R)^2}{2(n-1)^2} + \ldots\right] \tag{3.6}$$

This expression is a consequence of the near region definition $r - b << b$.

In the coming chapters, we shall use the large n limit as an analytical tool to solve equations obtained in the previous chapter. We shall mainly use the technique of matched asymptotic expansions to analyze the questions regarding stability of the black string and the semi-classical stability of SAdS black holes. Motivated by the expansion of the radial coordinate r in terms of R written as a series in $1/n$, we can further simplify the analysis by assuming a $1/n$ expansion of the perturbations and various parameters like ω, λ present in our equations while working in the near region approximation.

$$h_{\mu\nu} = h^{(0)}_{\mu\nu} + \frac{h^{(1)}_{\mu\nu}}{n} + \frac{h^{(2)}_{\mu\nu}}{n^2} + \ldots \tag{3.7}$$

In this expansion, $h^{(i)}_{\mu\nu}$s are independent of n. Of course, the perturbation $h_{\mu\nu}$ may have an overall n-dependence. The far region solution, where no such approximation is made, when extended to the overlap solution gets expanded as a series in $1/n$, giving further validation to the use of $1/n$ expansion in the near region. This expansion then facilitates us to solve our equations and perform matching order by order in n.

We shall now proceed with our analysis of black string instability for non-spherically symmetric perturbations. The problem of spherically symmetric perturbations in the large D limit has been analyzed by Kol and others in [29] and using matched asymptotic expansions by Emparan et al in [30]. They studied the Gregory Laflamme instability in the large D limit and calculated the value of the unstable mode.

3.2 Vector perturbations

As we are looking at the stability of flat black string (2.13), we have $f(r) = \left(1 - \frac{b^{n-1}}{r^{n-1}}\right)$. For the vector perturbations, we analyze the equations (3.9) and (3.10). Recall, in terms of the original perturbation variables F_t and F_r, A and B are

$$F_t = A(r)e^{i\omega t} \qquad F_r = \frac{B(r)}{f}e^{i\omega t}. \tag{3.8}$$

For the flat black string case,

$$\frac{d^2A}{dr^2} + \frac{n}{r}\frac{dA}{dr} + \left(-\frac{n}{r^2} - \frac{\alpha}{fr^2} - \frac{\lambda^2}{f} + \frac{\omega^2}{f^2}\right)A = \left(\frac{2}{rf} - \frac{(n-1)b^{n-1}}{f^2r^n}\right)i\omega B \tag{3.9}$$

$$\frac{d^2B}{dr^2} + \frac{(n-2)}{r}\frac{dB}{dr} + \left(-\frac{2(n-1)}{r^2} - \frac{\alpha}{fr^2} - \frac{\lambda^2}{f} + \frac{\omega^2}{f^2}\right)B = -\frac{(n-1)b^{n-1}}{f^2r^n}i\omega A \tag{3.10}$$

For discussions on stability, we need to investigate if there are normalizable solutions to the set of coupled equations that are regular at the horizon, with $\omega = -i\Omega$ and Ω real. That is, we want to see if there are solutions to the coupled set of equations that are normalizable at the spatial boundaries and grow in time. The Gregory-Laflamme unstable mode for the spherically symmetric perturbation is of this kind. We will show that there are no such solutions, indicating the stability of the black string under vector perturbations.

As stated before, we shall analyze these equations using the large n limit. The strategy is to look at the coupled equations (3.9) and (3.10) in the near and far regions. We see that the equations decouple in both these regions in this limit and can be solved. Then we shall select solutions that are normalizable at appropriate boundaries in each region. Next we extend the solutions to overlap region by taking far limit of the near region solution and near limit of the far region solution. We then see if the solutions can be matched in the overlap region.

3.2.1 The equations in the near region

In this section, we analyze (3.9) and (3.10) in the near region defined by $r - b << b$. We substitute $i\omega = \Omega$ in the two equations in order to study black string stability. Among the terms that have similar form in these equations, we only keep pieces which are of leading order in n. We first assume ℓ, λ and ω to be at least of order n. As $k_v^2 = \ell(\ell + n - 1)$, $k_v^2 \sim \mathcal{O}(n^2)$. We use the notation $k_v^2/n^2 = \hat{k}_v^2$, $\lambda^2/n^2 = \hat{\lambda}^2$, $i\omega = \Omega$ and $\Omega^2/n^2 = \hat{\Omega}^2$. This makes the hatted quantities of $\mathcal{O}(1)$. We shall comment on the cases where these quantities are of lower orders in n later. For $\hat{k}_v \sim \mathcal{O}(1)$, $\alpha = k_v^2 - (n-1)$ can be replaced by k_v^2 for large n. We then rewrite the equations in terms of the near region coordinate $R = (\frac{r}{b})^{n-1}$. To write functions of r in terms of the variable R in the equations, we use the approximate relation (3.6) which is valid in the near region. We also implicitly assume an expansion of A and B as

$$A = \sum_{i \geq 0} \frac{A_i}{n^i} \qquad B = \sum_{i \geq 0} \frac{B_i}{n^i} \tag{3.11}$$

Thus, in the near region, large n approximation, the equations obeyed by A and B are

$$\begin{aligned} &\frac{d^2 A}{dR^2} + \frac{2}{R}\frac{dA}{dR} - \left[\frac{\hat{k}_v^2}{R(R-1)} + \frac{\hat{\lambda}^2 b^2}{R(R-1)} + \frac{\hat{\Omega}^2 b^2}{(R-1)^2}\right] A \\ &\qquad\qquad\qquad\qquad = -\frac{\hat{\Omega} b}{R(R-1)^2} B + \frac{2\hat{\Omega} b}{nR(R-1)} B. \\ &\frac{d^2 B}{dR^2} + \frac{2}{R}\frac{dB}{dR} - \left[\frac{\hat{k}_v^2}{R(R-1)} + \frac{\hat{\lambda}^2 b^2}{R(R-1)} + \frac{\hat{\Omega}^2 b^2}{(R-1)^2}\right] B = -\frac{\hat{\Omega} b}{R(R-1)^2} A. \end{aligned} \tag{3.12}$$

Let us look at the right-hand side of the first equation in (3.12) contains a term of the form $\frac{2\hat{\Omega}b}{nR(R-1)}B$. this term appears to be sub-leading in n in comparison to a similar term $\frac{\hat{k}_v^2+\hat{\lambda}^2b^2}{R(R-1)}A$ on the left-hand side. This is true only if the overall leading order behaviour in n of A and B in (3.11) is similar. This is the likely scenario in such systems of coupled equations. In such a case, the sub-leading term on the right can be dropped. If on the other hand, A is sub-leading in n in comparison with B, the term must be retained. We will analyze both cases. We will also comment on the possibility of B being sub-leading in comparison with A.

Case 1: Leading order behaviour in n of A and B is similar.

We can take $A = A_0 + A_1/n + ...$ and $B = B_0 + B_1/n + ...$, where $A_0, B_0 \neq 0$. With foreknowledge of the results, we will drop the subscripts of the leading terms in n in A and B for simplicity. In this case, in the large n limit, (3.12) reduces to

$$\frac{d^2A}{dR^2} + \frac{2}{R}\frac{dA}{dR} - \left[\frac{\hat{k}_v^2}{R(R-1)} + \frac{\hat{\lambda}^2b^2}{R(R-1)} + \frac{\hat{\Omega}^2b^2}{(R-1)^2}\right]A = -\frac{\hat{\Omega}b}{R(R-1)^2}B. \tag{3.13a}$$

$$\frac{d^2B}{dR^2} + \frac{2}{R}\frac{dB}{dR} - \left[\frac{\hat{k}_v^2}{R(R-1)} + \frac{\hat{\lambda}^2b^2}{R(R-1)} + \frac{\hat{\Omega}^2b^2}{(R-1)^2}\right]B = -\frac{\hat{\Omega}b}{R(R-1)^2}A. \tag{3.13b}$$

It is clear from the form of (3.13a) that a simple sum and difference of the two equations decouples them. We define

$$\xi = (R-1)^{-\hat{\Omega}b}(A+B) \qquad \zeta = (R-1)^{\hat{\Omega}b}(A-B)$$

The equation obeyed by ξ is

$$R(1-R)\frac{d^2\xi}{dR^2} + \left[2 - (2\hat{\Omega}b+2)R\right]\frac{d\xi}{dR} - [\hat{\Omega}b - (\hat{k}_v^2 + \hat{\lambda}^2b^2)]\xi = 0. \tag{3.14}$$

The solutions of this equation for $2\hat{\Omega}b$ not an integer are given in terms of hypergeometric functions.

$$\xi = C_1F(p,q,2\hat{\Omega}b;1-R) + C_2(R-1)^{1-2\hat{\Omega}b}F(2-p,2-q,2-2\hat{\Omega}b;1-R); \tag{3.15}$$

where

$$p = \frac{1}{2}\left[1 + 2\hat{\Omega}b + \sqrt{1 + 4\hat{\Omega}^2 b^2 + 4(\hat{k}_v^2 + \hat{\lambda}^2 b^2)}\right] \tag{3.16}$$

$$q = \frac{1}{2}\left[1 + 2\hat{\Omega}b - \sqrt{1 + 4\hat{\Omega}^2 b^2 + 4(\hat{k}_v^2 + \hat{\lambda}^2 b^2)}\right] \tag{3.17}$$

Normalizability at the horizon requires that A, B to be, at the very least, finite at the horizon. For $\hat{\Omega}b > 1$, this implies $C_2 = 0$. For $\hat{\Omega}b < 1$, both linearly independent solutions for $(A+B)$ approach zero as $R \to 1$. However we need finiteness of the original perturbation variables F_t and F_r at the horizon, which are related to A, B by (3.8). This requires $\hat{\Omega}b > 1$. Henceforth, we shall assume this. We impose the boundary condition $C_2 = 0$. The solution for $(A+B)$ is

$$(A + B) = (R-1)^{\hat{\Omega}b} C_1 F(p, q, 2\hat{\Omega}b; 1 - R) \tag{3.18}$$

The equation and general solution for ζ can be obtained by replacing $\hat{\Omega}b$ by $-\hat{\Omega}b$ in (3.14) and (3.15) respectively. Since $(A - B) = (R-1)^{-\hat{\Omega}b}\zeta$, the solution $(A - B)$ that is regular at the horizon is now given by $C_1 = 0$.

For the special case $\hat{\Omega}b = N$ a positive integer, and $p \neq 1, 2, ..., 2N - 1$, the general solution for ξ is now

$$\xi = C_1 F(p, q, 2\hat{\Omega}b; 1 - R) + C_2 \ln(R-1) F(p, q, 2\hat{\Omega}b; 1 - R).$$

Finiteness of the perturbation at the horizon implies $C_2 = 0$. If p is one of the integers $1, 2, ..., 2N - 1$, then the general solution for ξ is given by (3.15). As the finite solution in all cases is the same for all cases, we shall not explicitly discuss these cases further.

Case 2: A is sub-leading in n in comparison to B (or vice-versa).

If A is sub-leading in n, then in the expansion (3.11), where $A = A_0 + A_1/n + ...$, and $B = B_0 + B_1/n + ...$), we set $A_0 = 0$. The first equation in equations (3.12) then implies $B_0 = 0$. This brings us back to **Case 1**. The case $B_0 = 0$ where B is sub leading to A will lead us back to conclusion $A_0 = 0$ after a similar analysis. This implies that the two variables A and B can only have similar overall n dependence.

3.2.2 The far region equations

The far region is defined by $r \gg b + \frac{b}{n}$. In this limit $(b^{n-1}/r^{n-1}) \sim e^{-n \ln r}$ is a small quantity for large n and large r. Hence we can approximate $f(r) \sim 1$ in the far region. We can neglect terms that have f' (or f''), which fall off at least as b^{n-1}/r^n, in (3.9) and (3.10) as they are negligible compared to other terms that fall off as $1/r^2$. We further use the large n approximation to retain only the leading n parts in the remaining terms. As in the near region analysis, we have assumed $k_v^2, \Omega^2 = -\omega^2$ and λ^2 are of order n^2. We will later comment on the case when these quantities are of lower order in n.

Let us analyze the term on the right-hand side of (3.10) $-\frac{(n-1)b^{n-1}}{r^n f^2}\Omega A$ in detail. In this approximation, the dominant decaying terms retained on the left-hand side are of the form $\frac{1}{r^2}B$. For the two sides to be comparable, we need at least $A \sim r^{n-2}B$. But in this case, the right-hand side of the equation for A (3.9) could be neglected in this limit. If this is not the case, then we can neglect the right-hand side of the equation for B (3.10) in this limit. In either case, one of the equations will have the right-hand side zero. This decoupled equation can be solved and the solution can be substituted in the other equation as a source term. We also note that in either situation, for the stability analysis, we additionally require normalizability of both sets of perturbations. In the following analysis, we will assume that the right-hand side of equation (3.10) can be neglected. The other case - neglecting the right-hand side of the equation (3.9) is almost identical computationally. This is due to the fact that in the large n approximation, for the set of parameters considered, the left-hand side of (3.10) is identical to that of (3.9) with the replacement of A by B. The only difference between the two cases is in the type of source term in each of the equations.

Neglecting the right-hand side of (3.10), we have, in the large n far region,

$$\frac{d^2A}{dr^2} + \frac{n}{r}\frac{dA}{dr} + \left(-\frac{k_v^2}{r^2} - \lambda^2 - \Omega^2\right) A = \left(\frac{2}{r}\right)\Omega B \tag{3.19}$$

$$\frac{d^2B}{dr^2} + \frac{n}{r}\frac{dB}{dr} + \left(-\frac{k_v^2}{r^2} - \lambda^2 - \Omega^2\right) B = 0 \tag{3.20}$$

The general solution for B is given in terms of modified Bessel functions of order $\nu =$

$\sqrt{\frac{(n-1)^2}{4}+k_v^2}$ as

$$B = r^{\frac{1-n}{2}}[D_1 I_\nu(\sqrt{\lambda^2+\Omega^2}r) + D_2 K_\nu(\sqrt{\lambda^2+\Omega^2}r)] \tag{3.21}$$

We note that for $k_v, \lambda, \Omega \sim \mathcal{O}(n)$, the large n limit implies the limit of large order and large argument for the modified Bessel functions. We rewrite $\sqrt{\lambda^2+\Omega^2}r = \nu z$ so that $I_\nu(\sqrt{\lambda^2+\Omega^2}r) = I_\nu(\nu z)$. In terms of $z = \frac{\sqrt{\lambda^2+\Omega^2}}{\nu}r$, we use standard expansions for large order and large argument for the modified Bessel functions. As $r \to \infty$, $I_\nu(\nu z) \sim e^{\nu z} \to \infty$ whereas $K_\nu(\nu z) \sim e^{-\nu z} \to 0$. Normalizability at infinity thus implies in (3.21) that $D_1 = 0$. The far region solution is

$$B = D_2 r^{\frac{1-n}{2}} K_\nu(\sqrt{\lambda^2+\Omega^2}r) \tag{3.22}$$

We now need to match the solutions in the near region and the far region in the overlap region $\frac{b}{n-1} \ll r - b \ll b$.

3.2.3 Matching of Solutions

In order to match the two solutions, we extend normalizable solution in both the regions to the overlap region and try to match the solutions. We will first extend the far solution to overlap region. We use the unified asymptotic expansion for large order and large argument of the modified Bessel functions in B. The form of (3.22) in the overlap region is then obtained by changing variables from r to R in (3.22) using the approximate formula (3.6) valid in the overlap region.

Let us denote $\kappa = \sqrt{\lambda^2+\Omega^2}$. For simplicity of calculation, we define a new coordinate $z = \kappa r/\nu$. Here

$$\nu = \sqrt{\frac{n^2+1}{4}+k^2} \approx \frac{n}{2}\sqrt{1+4\hat{k}^2} \tag{3.23}$$

The far region solution (3.22) can be now written as

$$B = D_2 \left(\frac{\kappa z}{\nu}\right)^{\frac{1-n}{2}} K_\nu(\nu z)$$

The uniform asymptotic expansion for modified Bessel function with large order and large argument is,

$$K_\nu(\nu z) = \sqrt{\frac{\pi}{2\nu}} \frac{1}{(1+z^2)^{1/4}} e^{-\nu\eta} \left[1 + \sum_{m=1}^{\infty} (-1)^m \frac{u_m(\tilde{t})}{\nu^m}\right] \tag{3.24}$$

where

$$\eta = \sqrt{1+z^2} + \ln\left[\frac{z}{1+\sqrt{1+z^2}}\right] \qquad\qquad \tilde{t} = \frac{1}{\sqrt{1+z^2}}$$

and $u_m(\tilde{t})$ are polynomials in $\tilde{t}$. As we only want leading order solution, we are only consider terms to highest order in n in the expansion. We ignore the polynomial terms as they are divided by ν. Substituting for η we get, up to a constant,

$$K_\nu(\nu z) \approx \frac{1}{z^\nu}\left(1+\sqrt{1+z^2}\right) \frac{1}{(1+z^2)^{1/4}} \exp\left[-\nu\sqrt{1+z^2}\right]. \tag{3.25}$$

To extend $K_\nu(\nu z)$ in the overlap region, we write this expression in terms of R. Here, $R = \frac{r^{n-1}}{b^{n-1}}$. To expand the expression in orders of n, we use the following definition of r in terms of R, valid in the overlap region,

$$r = b\left[1 + \frac{\ln R}{n-1}\right].$$

Let us look at each term in (3.25) individually. For z^ν, the term is directly proportional to r^n. Hence we use the definition $r^n = b^n R$ for large n.

$$\frac{1}{z^\nu} = \left(\frac{\nu}{\kappa}\right)^\nu \frac{1}{r^\nu} = \left(\frac{\nu}{\kappa}\right)^\nu b^{-\nu} R^{-\frac{\nu}{n}} = \left(\frac{\nu}{\kappa b}\right)^\nu R^{-\frac{\sqrt{1+4\hat{k}^2}}{2}} \tag{3.26}$$

The next term becomes

$$\begin{aligned}\left[1+\sqrt{1+z^2}\right]^\nu &= \exp\left[\nu \ln\left\{1 + \left(1 + \frac{\kappa^2 b^2}{\nu^2}\left((1+2\frac{\ln R}{n}\right)\right)^{1/2}\right\}\right] \\ &= \left(1+\sqrt{1+\frac{\kappa^2 b^2}{\nu^2}}\right)^\nu \exp\left[\frac{\kappa^2 b^2}{\nu n\sqrt{1+\frac{\kappa^2 b^2}{\nu^2}}\left(1+\sqrt{1+\frac{\kappa^2 b^2}{\nu^2}}\right)} \ln R\right]. \end{aligned} \tag{3.27}$$

κ and ν are of order n. Hence the constant multiplying $\ln R$ is of $\mathcal{O}(1)$. Similarly substituting for z we get,

$$\exp\left[-\nu\sqrt{1+z^2}\right] = \left(1+\frac{\kappa^2 b^2}{\nu^2}\right)^{-\nu/2} \exp\left[\frac{-\kappa^2 b^2}{n\nu\sqrt{1+\frac{\kappa^2 b^2}{\nu^2}}}\ln R\right] \tag{3.28}$$

The coefficient of $\ln R$ is of order 1 in this term. The remaining term in (3.25) becomes,

$$\frac{\sqrt{z}}{(1+z^2)^{1/4}} = \left(1+\frac{\nu^2}{\kappa^2 b^2}\right)^{-1/4} \exp\left[\frac{\nu^2}{2n\kappa^2 b^2}\left(1+\frac{\nu^2}{\kappa^2 b^2}\right)^{-1}\ln R\right] \tag{3.29}$$

Here, the constant multiplying $\ln R$ is of order $1/n$. Therefore, when substituted in (3.25), this term becomes sub-leading compared to the other terms in expansion. As we are only interested in terms that are leading order in n, we can ignore this term in the final expression. Substituting all the expressions in terms of R back in (3.25), we get the following expression for $K_\nu(\nu z)$. In this, we have absorbed all the constants in D_0.

$$K_\nu(\nu z) = (const.) R^{-\frac{\sqrt{1+4\hat{\Omega}^2 b^2+4(\hat{k}^2+\hat{\lambda}^2 b^2)}}{2}} \tag{3.30}$$

The factor $r^{\frac{1-n}{2}}$ in B can be written as $\sim R^{-\frac{1}{2}}$. Finally, the far region solution when extended to the overlap region is

$$B = D_1 R^{-\frac{1}{2}-\frac{\sqrt{1+4\hat{\Omega}^2 b^2+4(\hat{k}_v^2+\hat{\lambda}^2 b^2)}}{2}} \tag{3.31}$$

Similarly, the uniform asymptotic expansion for the modified Bessel function of first kind $I_\nu(\nu z)$ for large order and large argument is

$$I_\nu(\nu z) = \frac{1}{\sqrt{2\pi\nu}}\frac{1}{(1+z^2)^{1/4}} e^{\nu\eta}\left[1+\sum_{m=1}^{\infty}\frac{u_m(\tilde{t})}{\nu^m}\right] \tag{3.32}$$

Leading order expression for $I_\nu(\nu z)$ in the overlap region in the large n approximation can be obtained by replacing $(-\nu)$ by ν in the expansion of $K_\nu(\nu z)$. The final expression for

I_ν is

$$I_\nu(\nu z) = (const.)R^{\frac{\sqrt{1+4\hat{\Omega}^2 b^2+4(\hat{k}^2+\hat{\lambda}^2 b^2)}}{2}} \tag{3.33}$$

In order to solve for A, we need to substitute the expression for B as the source term in the right-hand side of the first equation in (3.19). Solving for A involves complicated integrals involving modified Bessel functions.

$$\frac{d^2A}{dr^2} + \frac{n}{r}\frac{dA}{dr} + \left(-\frac{k_v^2}{r^2} - \lambda^2 - \Omega^2\right) A = \left(\frac{2}{r}\right) \Omega B. \tag{3.34}$$

Looking at the form of B (3.22), let us take $A = r^{-(n-1)/2}S$. Then the equation (3.34) becomes a modified Bessel equation with source terms. This is

$$\frac{d^2S}{dz^2} + \frac{1}{z}\frac{dS}{dz} - [1 + \frac{\nu^2}{z^2}]S = \frac{2\hat{\Omega}}{\sqrt{\hat{\lambda}^2 + \hat{\Omega}^2 z}} K_\nu(\nu z). \tag{3.35}$$

The Wronskian of the two linearly independent solutions to the homogeneous equation $W[I_\nu(\nu z), K_\nu(\nu z)] = \frac{1}{z}$. Using the method of variation of parameters, we write the solution to (3.35). This takes the form (replacing $dz = \frac{\sqrt{\lambda^2+\Omega^2}}{\nu} dr$)

$$S = -\frac{2\hat{\Omega}}{\nu} I_\nu(\nu z) \int (K_\nu(\nu z))^2 \; dr \; + \; \frac{2\hat{\Omega}}{\nu} K_\nu(\nu z) \int K_\nu(\nu z) I_\nu(\nu z) \; dr. \tag{3.36}$$

We cannot solve these integrals exactly. But as we only need the form of solution A at infinity and in the overlap region, we shall look at these integrals in those limits. Inserting the asymptotic expansions for modified Bessel equations for large argument and order, we note that as $r \to \infty$, $S \to 0$ exponentially as $e^{-\nu z}$. As the particular solution is decaying as $r \to \infty$, the normalizable solution at this boundary can be obtained by choosing the constants in the general solution appropriately.

In the overlap region, we can change variables from r to R using (3.6) and $K_\nu(\nu z)I_\nu(\nu z) \sim [1 + (\frac{\lambda^2+\Omega^2}{\nu^2})b^2]^{-1/2}$. We have $dr \sim dR/R$. From the expressions of $K_\nu(\nu z)$ (3.30 and $I_\nu(\nu z)$ (3.33) obtained above, we observe that in the large n approximation in the overlap region, $K_\nu(\nu z) = cR^{-d}$ and $I_\nu(\nu z) = \tilde{c}R^d$ where c is a constant and $d = \frac{\sqrt{1+4\hat{\Omega}^2 b^2+4(\hat{k}_v^2+\hat{\lambda}^2 b^2)}}{2}$. Using this, we evaluate (3.36) in the overlap region to obtain $S = (const.)R^{-d}$, and $A =$

$r^{-(n-1)/2}S = (const.)R^{-1/2-d}$.

Thus A has the same behaviour as B as $r \to \infty$. Further, in the overlap region, we have the same power law behaviour,

$$A = (const.)R^{-\frac{1}{2}-\frac{\sqrt{1+4\hat{\Omega}^2b^2+4(\hat{k}_v^2+\hat{\lambda}^2b^2)}}{2}}. \tag{3.37}$$

Thus $A+B$ and $A-B$ have the same behaviour.

We now need to extend the near region solution to overlap region. In order to do this, we use the transformation properties of hypergeometric functions relating functions of argument $(1-R)$ to those of argument $1/R$. We will first consider the variable $A+B$ (3.18).

$$\begin{aligned} A+B &= (R-1)^{\hat{\Omega}b}C_1 F(p,q,2\hat{\Omega}b;1-R) \\ &= (R-1)^{\hat{\Omega}b}C_1\big[\tilde{c_1}R^{-p}F(p,p-1,p-q+1;1/R) \\ &\qquad\qquad + \tilde{c_2}R^{-q}F(q,q-1,q-p+1;1/R)\big]; \end{aligned}$$

The constants $\tilde{c_1}, \tilde{c_2}$ depend on p, q (3.16). To extend the solution to the overlap region, we approximate $(R-1) \approx R$ as we are sufficiently far from horizon. We also take the limit $R \to \infty$ in the hypergeometric function. After this extension, explicitly putting the values of $\tilde{c_1}$ and $\tilde{c_2}$, we get

$$\begin{aligned} A+B = C_1 &\frac{\Gamma(p+q-c+1)\Gamma(q-p)}{\Gamma(q)\Gamma(q-c+1)} R^{-\frac{1}{2}-\frac{\sqrt{1+4\hat{\Omega}^2b^2+4(\hat{k}_v^2+\hat{\lambda}^2b^2)}}{2}} + \\ &C_1\frac{\Gamma(p+q-c+1)\Gamma(p-q)}{\Gamma(p)\Gamma(p-c+1)} R^{-\frac{1}{2}+\frac{\sqrt{1+4\hat{\Omega}^2b^2+4(\hat{k}_v^2+\hat{\lambda}^2b^2)}}{2}} \end{aligned} \tag{3.38}$$

Here c is a constant from the original hypergeometric equations and $c = 2$. We are looking for normalizable solutions for λ, Ω positive. In this case,

$$\frac{1}{2} < \frac{\sqrt{1+4\hat{\Omega}^2b^2+4(\hat{k}_v^2+\hat{\lambda}^2b^2)}}{2}. \tag{3.39}$$

Hence the near region solution when extended to the overlap region is a growing solution in R with pieces $R^{-1/2+d}$ and $R^{-1/2-d}$ where $d = \frac{\sqrt{1+4\hat{\Omega}^2b^2+4(\hat{k}_v^2+\hat{\lambda}^2b^2)}}{2}$. We shall now argue that this solution will always have the growing piece $R^{-1/2+d}$ since its coefficient does not vanish for any $\hat{\Omega} \geq 0$.

The coefficient of the growing piece is

$$\frac{\Gamma(p+q-c+1)\Gamma(p-q)}{\Gamma(p)\Gamma(p-c+1)},$$

and the Gamma function is non-zero. The coefficient will vanish only at the poles of the Gamma functions in the denominator. Poles of the Gamma functions will occur when either p or $p-c+1$ is a non-positive integer. These two quantities are;

$$p = \frac{1}{2}\left[1 + 2\hat{\Omega}b + \sqrt{1+4\hat{\Omega}^2b^2+4(\hat{k}_v^2+\hat{\lambda}^2b^2)}\right]$$
$$p-c+1 = -\frac{1}{2} + \hat{\Omega}b + \frac{\sqrt{1+4\hat{\Omega}^2b^2+4(\hat{k}_v^2+\hat{\lambda}^2b^2)}}{2}.$$

For the cases we are interested in, i.e. $\hat{\Omega} \geq 0$ and $\hat{k}_v > 0$, these quantities can never be non-positive integers. Thus the solution from the near region will always have the growing piece in the overlap region. From the expressions (3.31) and (3.31), the normalizable solution in the far region is

$$A + B = (Const.)R^{-\frac{1}{2}-\frac{\sqrt{1+4\hat{\Omega}^2b^2+4(\hat{k}_v^2+\hat{\lambda}^2b^2)}}{2}}; \tag{3.40}$$

which is a decaying solution. Therefore we cannot match the solutions from near and far region. There are no unstable solutions for λ, Ω positive that are normalizable in space but growing in time. Similarly, we can show that this statement holds for $(A-B)$. From the near region, its expansion in the overlap region is similar to (3.38) in that it has both pieces $R^{-1/2+d}$ and $R^{-1/2-d}$. The normalizable solution in the far region when extended to the overlap region has only the decaying term $R^{-1/2-d}$. Hence a match is not possible. This shows that in the static limit, with $\Omega = 0$, we have no instability.

So far, we have considered $k_v, \Omega, \lambda \sim O(n)$ at least.To prove stability of the black string for the case when the parameters are of $\mathcal{O}(1)$, we will take this dependence of ℓ, Ω and λ and rewrite the equations (3.9) and (3.10) in the near region. We will have to make similar consideration while solving for and extending the far region solution. We will have to match the two solutions in the overlap region and show that there is no match. We will not explicitly show the calculations in this chapter as they will be explained in detail when we calculate quasinormal modes for the black string. The matching of the solutions in the overlap region will show that there are no vector unstable modes for the black string for Ω non-negative.

Thus we conclude that for $\hat{\Omega}$ real and non-negative, there are no normalizable solutions to the vector perturbation equations, and the black string/brane is stable.

3.3 Non-spherically symmetric scalar perturbations

Let us now look at the scalar perturbations. For the flat black string, the system of three coupled equations for scalar perturbation variables ψ, ϕ and η becomes:

$$\begin{aligned}&-\frac{d^2\psi}{dr^2}+\left[\frac{n^3-2n^2+8n-8}{4nr^2}+\frac{f'^2}{4f^2}-\frac{(n^2+2n-4)}{2n}\frac{f'}{fr}-\frac{f''}{2f}\right.\\&\left.-\frac{2(n-1)}{nr^2f}+\frac{k^2}{fr^2}+\frac{\lambda^2}{f}-\frac{\omega^2}{f^2}\right]\psi=\\&\left[\frac{4}{f}-\frac{2f'r}{f^2}\right](i\omega)\eta+\left[\frac{2(n-1)}{nf}+\frac{2}{n}-\frac{n+2}{n}\frac{rf'}{f}-\frac{r^2f''}{f}+\frac{f'^2r^2}{2f^2}\right]\phi\end{aligned} \tag{3.41}$$

$$\begin{aligned}&-\frac{d^2\phi}{dr^2}+\left[\frac{n^3-2n^2+8n-8}{4nr^2}+\frac{f'^2}{4f^2}-\frac{(n^2+2n-4)}{2n}\frac{f'}{fr}-\frac{f''}{2f}\right.\\&\left.-\frac{2(n-1)}{nr^2f}+\frac{k^2}{fr^2}+\frac{\lambda^2}{f}-\frac{\omega^2}{f^2}\right]\phi=\\&\frac{2f'}{f^2r}\eta(i\omega)+\left[\frac{2(n-1)}{nr^4f}-\frac{2(n-1)}{nr^4}-\frac{2-n}{nr^3}\frac{f'}{f}-\frac{f''}{r^2f}+\frac{f'^2}{2f^2r^2}\right]\psi\end{aligned} \tag{3.42}$$

$$\begin{aligned}-\frac{d^2\eta}{dr^2}&+\left[\frac{n^2-2n}{4r^2}-\frac{(n+2)f'}{2rf}+\frac{3f'^2}{4f^2}-\frac{3f''}{2f}+\frac{k^2}{fr^2}-\frac{\omega^2}{f^2}+\frac{\lambda^2}{f}\right]\eta\\&=\left[\frac{f'}{f}-\frac{2}{r}\right]\frac{r(i\omega)}{f}\phi-\frac{f'}{f^2}\frac{(i\omega)}{r}\psi.\end{aligned} \tag{3.43}$$

As in the vector case, the equations (3.41)—(3.43) cannot be solved analytically. We have to resort to the large n limit of the equations. But unlike in the vector case, the equations do not completely decouple even in this limit. But as we did for the vector case, we can still analyze the large n limit of (3.41)—(3.43) in the near-horizon and far regions. Using matched asymptotic expansions, we will investigate if the two solutions match in the overlap region.

3.3.1 Far region

The far region is defined, as before, by $r \gg b + \frac{b}{n}$. In this limit, we will take $f \to 1$. Using the large n limit, we again neglect terms that have f', f'' in the equations for ψ, ϕ and η. Further, we will keep only the leading order in n pieces in like terms. As in the vector case, we have assumed k^2, ω^2 and λ^2 are at least $\sim \mathcal{O}(n^2)$. For analyzing stability of the black string, we replace $\Omega = i\omega$ in our equations. In the far limit, the equations (3.42)—(3.43) are:

$$-\frac{d^2\psi}{dr^2} + \left[\frac{n^2}{4r^2} + \frac{k^2}{r^2} + \lambda^2 + \Omega^2\right]\psi = 2\phi + 4\Omega\eta \tag{3.44}$$

$$-\frac{d^2\phi}{dr^2} + \left[\frac{n^2}{4r^2} + \frac{k^2}{r^2} + \lambda^2 + \Omega^2\right]\phi = \frac{-f''}{fr^2}\psi + \frac{2f'}{f^2 r}\Omega\eta \tag{3.45}$$

$$-\frac{d^2\eta}{dr^2} + \left[\frac{n^2}{4r^2} + \frac{k^2}{r^2} + \lambda^2 + \Omega^2\right]\eta = -2\Omega\phi - \frac{f'}{f^2 r}\Omega\psi. \tag{3.46}$$

Let us first consider the ϕ equation. We have kept the terms on the left-hand side of the type (ϕ/r^2). The terms on the right-hand side are of the form $(\psi/r^{n+3}), (\eta/r^{n+1})$. For these terms to be significant, for example, (η/r^{n+1}), the magnitude of η must be at least $\eta \sim \phi r^{n-1}$. Similarly, for ψ - its magnitude must be at least $\psi \sim \phi r^{n+1}$ for the term proportional to it on the right-hand side to be considered. We shall later show that there is no solution to the equations (3.41)—(3.43) in the large n limit, corresponding to this situation. Hence, in what follows, we will assume that η and ψ are comparable in in magnitude to ϕ in the large n limit so that the right-hand side of (3.45) can be neglected.

As its equation decouples, we first solve for ϕ and subsequently solve equations for ψ and

η.

$$-\frac{d^2\phi}{dr^2}+\left[\frac{n^2}{4r^2}+\frac{k^2}{r^2}+\lambda^2+\Omega^2\right]\phi=0 \tag{3.47}$$

Let $\nu=\sqrt{\frac{n^2+1}{4}+k^2}$. The general solution of (3.47) is given in terms of modified Bessel functions as:

$$\phi=D_1\sqrt{r}I_\nu(\sqrt{\lambda^2+\Omega^2}r)+D_2\sqrt{r}K_\nu(\sqrt{\lambda^2+\Omega^2}r) \tag{3.48}$$

The order ν of the modified Bessel functions is proportional to n. We have assumed $\lambda,\Omega\sim\mathcal{O}(n)$. Hence the large n limit implies the large order and argument limit of the modified Bessel functions. In this limit, $I_\nu(\nu z)\sim e^{\nu z}$ is a growing solution. Boundary conditions for normalizability dictate $D_1=0$. Hence

$$\phi=D_2\sqrt{r}K_\nu(\sqrt{\lambda^2+\Omega^2}r). \tag{3.49}$$

As in the vector case, we need to find the expansion of this solution in the overlap region to match with the near region solution. We also need overlap region solution for finding the solutions for ψ and η. We use the large argument and order expansion of $K_\nu(\sqrt{\lambda^2+\Omega^2}r)$ and retain terms to leading order in n and then change coordinates from r to R using (3.6) which is valid in the overlap region. This is exactly the calculation done for the far region solution for the vector perturbation. In terms of R, the leading order far solution for ϕ in the overlap region becomes

$$\phi=D_0R^{-\frac{\sqrt{1+4(\hat{k}^2+\hat{\lambda}^2b^2)+4\hat{\Omega}^2b^2}}{2}} \tag{3.50}$$

We henceforth denote scaled k, λ as $k^2/n^2=\hat{k}^2$ and $\lambda^2/n^2=\hat{\lambda}^2$. To solve the equations for ψ and η, we will use (3.49). We will analyze the η equation (3.46) first. In this equation, the term proportional to ψ on the right can be neglected, as it contains f'. $-2\Omega\phi$, with ϕ given by (3.49) appears as a source on the right in this equation. We can find a particular solution to this equation by the method of variation of parameters. As we did in the vector case, we find that the particular solution will decay exponentially as $r\to\infty$, and will have the power law behaviour $\eta=(const.)R^{-\frac{\sqrt{1+4(\hat{k}^2+\hat{\lambda}^2b^2)+4\hat{\Omega}^2b^2}}{2}}$ in the overlap region. A similar statement can be made for ψ.

3.3.2 Near region

For the near region behaviour, it is convenient to analyze equations (3.41)—(3.43) in the R variable given in the near region by (3.6). We also expand ϕ, ψ and η in terms of n as

$$\psi = \sum_{i\geq 0} \frac{\psi_i(R)}{n^i} \qquad \phi = \sum_{i\geq 0} \frac{\phi_i(R)}{n^i} \qquad \eta = \sum_{i\geq 0} \frac{\eta_i(R)}{n^i} \tag{3.51}$$

Taking the large n limit of (3.41)—(3.43), we obtain:

$$\begin{aligned}\frac{d^2\psi}{dR^2} + \frac{1}{R}\frac{d\psi}{dR} - \left[\frac{1}{4R^2} + \frac{1}{4(R-1)^2R^2} - \frac{1}{nR^2(R-1)} + \frac{\hat{k}^2 + \hat{\lambda}^2 b^2}{R(R-1)} + \frac{\hat{\Omega}^2 b^2}{(R-1)^2}\right]\psi = \\ - \left[\frac{2}{n^2(R-1)R} + \frac{2}{n^3R^2} + \frac{1}{R^2(R-1)} + \frac{1}{2R^2(R-1)^2}\right]\phi b^2 \\ - \left[\frac{4}{nR(R-1)} - \frac{2}{R(R-1)^2}\right]\hat{\Omega} b^2 \eta\end{aligned} \tag{3.52}$$

$$\begin{aligned}\frac{d^2\phi}{dR^2} + \frac{1}{R}\frac{d\phi}{dR} - \left[\frac{1}{4R^2} + \frac{1}{4(R-1)^2R^2} - \frac{1}{nR^2(R-1)} + \frac{\hat{k}^2 + \hat{\lambda}^2 b^2}{R(R-1)} + \frac{\hat{\Omega}^2 b^2}{(R-1)^2}\right]\phi = \\ - \left[\frac{2}{n^2(R-1)R} - \frac{2}{n^2R^2} + \frac{1}{R^2(R-1)} + \frac{1}{2R^2(R-1)^2}\right]\frac{\psi}{b^2} - \left[\frac{2}{R(R-1)^2}\right]\hat{\Omega}\eta\end{aligned} \tag{3.53}$$

$$\begin{aligned}\frac{d^2\eta}{dR^2} + \frac{1}{R}\frac{d\eta}{dR} - \left[\frac{1}{4R^2} + \frac{3}{4(R-1)^2R^2} + \frac{1}{R^2(R-1)} + \frac{\hat{k}^2 + \hat{\lambda}^2 b^2}{R(R-1)} + \frac{\hat{\Omega}^2 b^2}{(R-1)^2}\right]\eta = \\ - \left[\frac{1}{(R-1)^2R} - \frac{2}{n(R-1)R}\right]\hat{\Omega}\phi b^2 + \frac{\hat{\Omega}}{R(R-1)^2}\psi.\end{aligned} \tag{3.54}$$

We again use the notation $k^2/n^2 = \hat{k}^2$, $\lambda^2/n^2 = \hat{\lambda}^2$, $i\omega = \Omega$ and $\Omega^2/n^2 = \hat{\Omega}^2$. As in the vector case, the hatted quantities are $\mathcal{O}(1)$.

In the above equations, as we have only kept the leading n-dependence in each term, we have implicitly used the expansion (3.51) in these equations. To analyze the stability of the flat black string, we only need to solve for the leading term in the expansion of ϕ, ψ and η in the large n limit. From the equations (3.52)—(3.54), it is clear that the leading terms ϕ_0, ψ_0 and η_0 in the expansion (3.51) all have to be non-zero. That is, all the variables have same leading n-dependence. For simplicity we drop the subscripts in the following equations.

In the leading order, the equations can be simplified to a great extent by taking linear combinations of ϕ and ψ. Denote $\eta b^2 = \tilde{\eta}$ and $\phi b^2 = \tilde{\phi}$, define

$$H = \psi + \tilde{\phi} \text{ and } G = \psi - \tilde{\phi}. \tag{3.55}$$

The equation for H decouples as η terms cancel out.

$$\frac{d^2H}{dR^2}+\frac{1}{R}\frac{dH}{dR}-\left[\frac{1}{4R^2}-\frac{1}{4R^2(R-1)^2}-\frac{1}{R^2(R-1)}+\frac{(\hat{k}^2+\hat{\lambda}^2b^2)}{R(R-1)}+\frac{\hat{\Omega}^2b^2}{(R-1)^2}\right]H=0 \tag{3.56}$$

This equation can be written as an hypergeometric equation with regular singular points at $0, 1$ and ∞ by making the ansatz $H = R(R-1)^{\frac{1}{2}+\hat{\Omega}b}M$

$$R(1-R)\frac{d^2M}{dR^2}+[3-(4+2\hat{\Omega}b)R]\frac{dM}{dR}+[(\hat{k}^2+\hat{\lambda}^2b^2)-3\hat{\Omega}b-2]M=0 \tag{3.57}$$

If $1 + 2\hat{\Omega}b \neq m$ where m is a positive integer, the solution of this equation near the horizon ($R = 1$) in terms of hypergeometric functions is

$$M = C_1F(p,q,1+2\hat{\Omega}b,1-R)+C_2(1-R)^{-2\hat{\Omega}b}F(3-p,3-q,1-2\hat{\Omega}b,1-R); \tag{3.58}$$

where

$$p = \frac{1}{2}\left[3+2\hat{\Omega}b-\sqrt{1+4(\hat{k}^2+\hat{\lambda}^2b^2)+4\hat{\Omega}^2b^2}\right] \tag{3.59a}$$

$$q = \frac{1}{2}\left[3+2\hat{\Omega}b+\sqrt{1+4(\hat{k}^2+\hat{\lambda}^2b^2)+4\hat{\Omega}^2b^2}\right] \tag{3.59b}$$

In this case, for $\hat{\Omega}b > \frac{1}{2}$, there is an unambiguous way to choose the behaviour of the solution at the horizon. Finiteness of H at the horizon implies that in the general solution for M, we must set $C_2 = 0$. For $\hat{\Omega}b \leq \frac{1}{2}$, both linearly independent solutions for H are finite. Thus there seems to be some ambiguity in the choice of boundary condition at the horizon. As argued in the vector case, it is the original perturbation variables that need to be finite at the horizon. Looking at the definitions of the variables $\phi, \psi, \tilde{\eta}$ in terms of the original variables W, Y, Z from (2.73), we see that finiteness of W, Y, Z at the horizon implies that $C_2 = 0$ for all values of $\hat{\Omega} > 0$.

Imposing this boundary condition in the near region,

$$H = C_1 R(R-1)^{\frac{1}{2}+\hat{\Omega}b} F(p, q, 1+2\hat{\Omega}b, 1-R) \tag{3.60}$$

Even in the case $1+2\hat{\Omega}b = m$, where the hypergeometric equation becomes a degenerate case, (3.60) is the appropriate solution for finiteness of the perturbation at the horizon as the second solution with logarithmic terms and can be ignored by choosing regularity at the horizon. For matching, we need to write the asymptotic expansion of the near region solution in the overlap region. In order to do this, we use the standard transformation formula:

$$H = R(R-1)^{\frac{1}{2}+\hat{\Omega}b} C_1 F(p, q, 1+2\hat{\Omega}b; 1-R) \tag{3.61}$$

$$\begin{aligned} &= R(R-1)^{\frac{1}{2}+\hat{\Omega}b} C_1 \big[\tilde{c_1} R^{-p} F(p, p-2\hat{\Omega}b, p-q+1; 1/R)+ \\ &\qquad \tilde{c_2} R^{-q} F(q, q-2\hat{\Omega}b, q-p+1; 1/R)\big]; \end{aligned} \tag{3.62}$$

where constants $\tilde{c_1}, \tilde{c_2}$ depend on p, q. To obtain asymptotic expansion for H in the overlap region, we approximate $(R-1) \approx R$ far away from the horizon. We evaluate the hypergeometric functions in this large R approximation.

The asymptotic expansion of the near region solution for H in the overlap region is

$$H = C_1 \left[\tilde{c_1} R^{\frac{\sqrt{1+4\hat{\Omega}^2 b^2+4(\hat{k}^2+\hat{\lambda}^2 b^2)}}{2}} + \tilde{c_2} R^{-\frac{\sqrt{1+4\hat{\Omega}^2 b^2+4(\hat{k}^2+\hat{\lambda}^2 b^2)}}{2}}\right] \tag{3.63}$$

The constants $\tilde{c_1}$ and $\tilde{c_2}$ are,

$$\tilde{c_1} = \frac{\Gamma(p+q-c+1)\Gamma(q-p)}{\Gamma(q)\Gamma(q-c+1)} \qquad \tilde{c_2} = \frac{\Gamma(p+q-c+1)\Gamma(p-q)}{\Gamma(p)\Gamma(p-c+1)} \tag{3.64}$$

Here, from the equation (3.57), $c = 3$ and p, q are given by (3.59). The solution that is finite at the horizon again has a growing and a decaying piece when extended to the overlap region. By the same reasoning as employed in vector case (3.38), it can be showed that for

$\hat{\Omega} \geq 0$ and $\hat{k} > 0$, the coefficient of the growing piece $\tilde{c}_1$ is does not vanish for any value of $\hat{\Omega}$.

We would like to solve other perturbation equations for G and $\tilde{\eta}$ in the similar manner. The equations for G and $\tilde{\eta}$ in the near region are:

$$\frac{d^2G}{dR^2} + \frac{1}{R}\frac{dG}{dR} - \left[\frac{1}{4R^2} + \frac{3}{4R^2(R-1)^2} + \frac{1}{R^2(R-1)} + \frac{(\hat{k}^2 + \hat{\lambda}^2 b^2)}{R(R-1)} + \frac{\hat{\Omega}^2 b^2}{(R-1)^2}\right] G = \frac{4\hat{\Omega}}{R(R-1)^2}\tilde{\eta} \tag{3.65}$$

$$\frac{d^2\tilde{\eta}}{dR^2} + \frac{1}{R}\frac{d\tilde{\eta}}{dR} - \left[\frac{1}{4R^2} + \frac{3}{4R^2(R-1)^2} + \frac{1}{R^2(R-1)} + \frac{(\hat{k}^2 + \hat{\lambda}^2 b^2)}{R(R-1)} + \frac{\hat{\Omega}^2 b^2}{(R-1)^2}\right] \tilde{\eta} = \frac{\hat{\Omega} b^2}{R(R-1)^2} G \tag{3.66}$$

Unfortunately the equations for G and $\tilde{\eta}$ do not decouple even in the leading order. We cannot solve these equations analytically in the near region as we did for H. We shall consider various cases for analyzing this set of coupled equations.

Static perturbations:

Let us first consider static perturbations. In this case the equations for G and $\tilde{\eta}$ decouple and resulting equation for both G and $\tilde{\eta}$ is same. This equation, obtained by setting $\hat{\Omega} = 0$ in (3.65) and (3.66), can be written as an hypergeometric equation. The solutions to which are

$$G = \tilde{\eta} = D_1(R-1)^{\frac{3}{2}} F\left(\bar{p}, \bar{q}, 3, 1-R\right) + D_2\left[(R-1)^{\frac{3}{2}} F\left(\bar{p}, \bar{q}, 3, 1-R\right)\log(1-R) + \text{other terms}\right] \tag{3.67}$$

Where

$$\bar{p} = \frac{3}{2} + \frac{\sqrt{1 + 4(\hat{k}^2 + \hat{\lambda}^2 b^2)}}{2} \qquad \bar{q} = \frac{3}{2} - \frac{\sqrt{1 + 4(\hat{k}^2 + \hat{\lambda}^2 b^2)}}{2} \tag{3.68}$$

The normalizable solution, obtained by setting $D_2 = 0$, when extended to the overlap region becomes

$$G = \tilde{\eta} = \frac{\Gamma(\bar{p}+\bar{q})\Gamma(\bar{q}-\bar{p})}{\Gamma(\bar{q})^2} R^{-\frac{\sqrt{1+4(\hat{k}^2+\hat{\lambda}^2 b^2)}}{2}} + \frac{\Gamma(\bar{p}+\bar{q})\Gamma(\bar{p}-\bar{q})}{\Gamma(\bar{p})^2} R^{\frac{\sqrt{1+4(\hat{k}^2+\hat{\lambda}^2 b^2)}}{2}} \tag{3.69}$$

We shall later match this solution with the solution from the far region.

Non-static perturbations:

For non-static perturbations, let us first consider the special case when either one of G or $\tilde{\eta}$ is zero.

Case I: Either G or η is zero

Since in this case the left-hand sides of both (3.65) and (3.66) have the same differential operator, the two possibilities are computationally identical. Let us take the case $\tilde{\eta} = 0$. Then we obtain the following general solution for G with $\tilde{\eta} = 0$ again in terms of hypergeometric functions.

$$\begin{aligned} G =& D_1 (R-1)^{\frac{1}{2}+\sqrt{1+\hat{\Omega}^2 b^2}} F(\tilde{p}, \tilde{q}, 1+2\sqrt{1+\hat{\Omega}^2 b^2}; 1-R) + \\ & D_2 (R-1)^{\frac{1}{2}-\sqrt{1+\hat{\Omega}^2 b^2}} F(1-\tilde{p}, 1-\tilde{q}, 1-2\sqrt{1+\hat{\Omega}^2 b^2}; 1-R). \end{aligned} \tag{3.70}$$

Here,

$$\tilde{p} = \frac{1}{2} + \sqrt{1+\hat{\Omega}^2 b^2} + \frac{\sqrt{1+4\hat{\Omega}^2 b^2 + 4(\hat{k}^2 + \hat{\lambda}^2 b^2)}}{2}$$

and

$$\tilde{q} = \frac{1}{2} + \sqrt{1+\hat{\Omega}^2 b^2} - \frac{\sqrt{1+4\hat{\Omega}^2 b^2 + 4(\hat{k}^2 + \hat{\lambda}^2 b^2)}}{2}.$$

Finiteness at the horizon implies we must set $D_2 = 0$.

This solution can be extended to the overlap region using the asymptotic expansion of the hypergeometric function as with H. In the overlap region, the form of G is nearly identical to (3.63). The solution will again have growing and decaying pieces with different numerical values of the constants $\tilde{c_1}$ and $\tilde{c_2}$ from the H solution. As before coefficient of the growing piece is never zero.

Let us now consider the equations for G and $\tilde{\eta}$ in the general case when they are both coupled.

Case II: Both G and η are non-zero

As we cannot decouple the equations (3.65)and (3.66), we cannot find solutions that are valid in the entire near region. To solve these equations, we use a heuristic argument. We split the near region into the *very near horizon region* and the *far limit of the near region.* In the *very near horizon region* we use the approximation $R - 1 << 1$ so that $R \sim 1$. In this approximation, we keep only the dominant terms with highest powers of $(R - 1)$ in the denominator in (3.65) and (3.66). We define the *far limit of the near region* as the regime where $R >> 1$, so that $(R - 1) \approx R$, which can be applied to the coupled equations (3.65) and (3.66). As we did with our original near and far regions, we will evaluate the solutions to the coupled equations in both regimes and present a matching argument between these two regimes in an 'overlap' region. The definition of the new overlap region here is not as precise as the overlap region between the near and far region in the large n limit. This 'new overlap' region can be thought of as the region where both R and $(R - 1)$ assume equal importance.

In the *very near horizon region* approximation, keeping only the terms with dominant behaviour in $(R - 1)$, the equations (3.65) and (3.66) reduce to

$$\frac{d^2G}{dR^2} + \frac{dG}{dR} + \left[-\frac{3}{4} - \hat{\Omega}^2 b^2\right] \frac{1}{(R-1)^2} G = \frac{4\hat{\Omega}}{(R-1)^2}\tilde{\eta} \tag{3.71a}$$

$$\frac{d^2\tilde{\eta}}{dR^2} + \frac{d\tilde{\eta}}{dR} + \left[-\frac{3}{4} - \hat{\Omega}^2 b^2\right] \frac{1}{(R-1)^2}\tilde{\eta} = \frac{\hat{\Omega} b^2}{(R-1)^2} G. \tag{3.71b}$$

These equations are still coupled. By rewriting (3.71) in terms of the new coordinate $y = \ln(R - 1)$, we can simplify the equations further. Also we write G and $\tilde{\eta}$ as

$$\tilde{\eta} = e^{y/2} P \qquad G = e^{y/2} Q$$

In the very near horizon limit $y \to -\infty$, we now obtain coupled differential equations with constant coefficients which can be solved analytically.

$$\frac{d^2Q}{dy^2} = \left(1+\hat{\Omega}^2 b^2\right) Q + 4\hat{\Omega} P$$
$$\frac{d^2P}{dy^2} = \left(1+\hat{\Omega}^2 b^2\right) P + \hat{\Omega} b^2 Q$$

In the case $\hat{\Omega} b \neq 1$, the general solutions for $\tilde{\eta}$ and G are:

$$\begin{aligned} G =& C_1(4\hat{\Omega})(R-1)^{\frac{3}{2}+\hat{\Omega}b} + C_2(4\hat{\Omega})(R-1)^{-\frac{1}{2}-\hat{\Omega}b} \\ &+ C_3(4\hat{\Omega})(R-1)^{-\frac{1}{2}+\hat{\Omega}b} + C_4(4\hat{\Omega})(R-1)^{\frac{3}{2}-\hat{\Omega}b} \end{aligned} \tag{3.72}$$

$$\begin{aligned} \tilde{\eta} =& C_1(2\hat{\Omega}b)(R-1)^{\frac{3}{2}+\hat{\Omega}b} + C_2(2\hat{\Omega}b)(R-1)^{-\frac{1}{2}-\hat{\Omega}b} \\ &+ C_3(-2\hat{\Omega}b)(R-1)^{-\frac{1}{2}+\hat{\Omega}b} + C_4(-2\hat{\Omega}b)(R-1)^{\frac{3}{2}-\hat{\Omega}b} \end{aligned} \tag{3.73}$$

The general solutions for these two coupled ordinary differential equations is characterized by four arbitrary constants. Let us now discuss the boundary conditions. A natural choice is finiteness of G and $\tilde{\eta}$ at the horizon. However, we recall that it is the original perturbations variables W, Y, Z that need to be finite at the horizon for consistency of linearized perturbation theory. If we again refer back to (2.73) for the definitions of the variables $\phi, \psi, \tilde{\eta}$ in terms of W, Y, Z, we see that finiteness of W, Y, Z at the horizon makes the choice of boundary conditions very simple. W, Y are related to G by a factor $(R-1)^{-\frac{1}{2}}$. This implies the following: Regardless of the value of $\hat{\Omega}b$, we must set $C_2 = 0$. For $0 < \hat{\Omega}b < 1$, $C_3 = 0$ and $C_1, C_4 \neq 0$. For $\hat{\Omega}b > 1$, $C_4 = 0$ and $C_1, C_3 \neq 0$.

A special case is $\hat{\Omega}b = 1$. The solutions for G and $\tilde{\eta}$ for which the original perturbation variables are finite at the horizon are:

$$\begin{aligned} G &= C_1(R-1)^{\frac{5}{2}} + C_4(4\hat{\Omega})\sqrt{R-1}; \\ \tilde{\eta} &= C_1\frac{b}{2}(R-1)^{\frac{5}{2}} - 2C_4\sqrt{R-1}. \end{aligned} \tag{3.74}$$

We now try to solve the equations (3.65),(3.66) in the *far limit of the near region* where we consider R to be large. In this limit $(R-1) \approx R$.

$$\frac{d^2G}{dR^2} + \frac{1}{R}\frac{dG}{dR} - \left[\frac{1}{4} + \hat{k}^2 + \hat{\lambda}^2 b^2 + \hat{\Omega}^2 b^2\right]\frac{1}{R^2}G = \frac{4\hat{\Omega}}{R^3}\tilde{\eta} \tag{3.75}$$

$$\frac{d^2\tilde{\eta}}{dR^2} + \frac{1}{R}\frac{d\tilde{\eta}}{dR} - \left[\frac{1}{4} + \hat{k}^2 + \hat{\lambda}^2 b^2 + \hat{\Omega}^2 b^2\right]\frac{1}{R^2}\tilde{\eta} = \frac{\hat{\Omega}b^2}{R^3}G \tag{3.76}$$

To analyze this equation, we first consider G and $\tilde{\eta}$ to have a similar R dependence in the large R limit. In this case we can neglect the right hand side of both the equations as it will be subleading for large R as it decays faster than the terms on the left hand side. The resulting equations are decoupled. We notice that the final equations are Euler differential equations whose solutions are

$$G = \tilde{\eta} = a_1 R^{-\frac{\sqrt{1+4\hat{\Omega}^2 b^2 + 4(\hat{k}^2+\hat{\lambda}^2 b^2)}}{2}} + a_2 R^{\frac{\sqrt{1+4\hat{\Omega}^2 b^2 + 4(\hat{k}^2+\hat{\lambda}^2 b^2)}}{2}}; \tag{3.77}$$

and similarly for $\tilde{\eta}$.

Consider the cases where the R dependence of G and $\tilde{\eta}$ in the far region is not equal. First alternative scenario in the far limit of the near region is one where G has a higher power of R than $\tilde{\eta}$ in the far region. We can then neglect the right hand side in (3.75). The solution for G in this case will be same as (3.77). Using this solution as a source term in (3.76), we can then solve for $\tilde{\eta}$. By doing this using standard Green's function methods, we obtain a solution for $\tilde{\eta}$ which does not tally with the form of the solution for $\tilde{\eta}$ from the far region. Specifically, the exponents of R for $\tilde{\eta}$ do not match those from the far region. This indicates we cannot have this alternative scenario. The possibility of $\tilde{\eta}$ having a higher power of R than G can be ruled out in a similar way.

3.3.3 Matching of Solutions

We are investigating if the black string is unstable under scalar perturbations $(\lambda, \Omega > 0)$. In the overlap region, the solution from the near region for H is a growing solution with both exponents of R, $R^{\frac{\sqrt{1+4\hat{\Omega}^2 b^2 + 4(\hat{k}^2+\hat{\lambda}^2 b^2)}}{2}}$ and $R^{-\frac{\sqrt{1+4\hat{\Omega}^2 b^2 + 4(\hat{k}^2+\hat{\lambda}^2 b^2)}}{2}}$ present and the coefficient of the growing piece $\tilde{c}_1$ (3.64) is always non-zero.

However, the normalizable solution from the far region is of the following form in the overlap region:

$$\phi = D_0 R^{-\frac{\sqrt{1+4\hat{\Omega}^2 b^2+4(\hat{k}^2+\hat{\lambda}^2 b^2)}}{2}}; \tag{3.78}$$

with the same exponent for ψ and η as well. Since G, H and $\tilde{\eta}$ are sums, differences or scalar multiples of ϕ, ψ, η, the form of H from the far region has only the decaying piece $R^{-\frac{\sqrt{1+4\hat{\Omega}^2 b^2+4(\hat{k}^2+\hat{\lambda}^2 b^2)}}{2}}$. Thus there are no unstable modes of the type H for the black string/brane for scalar perturbations with angular momentum $l \neq 0$.

Further, in the static limit $\hat{\Omega} = 0$, the same matching argument can be used to conclude from (3.69) that there are no unstable modes of any type (H, G or $\tilde{\eta}$). The same statement can also be made in the non-static case when either one of G or $\tilde{\eta}$ is zero.

Let us now consider the general case when both G and $\tilde{\eta}$ are non-zero. We have obtained solutions for G and η in the *very near horizon* region and in the *far limit of the near region.* Our heuristic argument is to match the behaviour of these solutions in the 'new overlap' region.

In the far limit of the near region, G is of the form (3.77), and a similar expression holds for $\tilde{\eta}$ since the equations decouple. In the 'new overlap' region, where both R and $(R-1)$ assume equal importance, the far region (decoupled) solution is of the form (3.70) with arbitrary constants D_1 and D_2. The solution for G from the far limit of the near region, in the overlap region will have the form

$$\begin{aligned} G =& D_1(R-1)^{\frac{1}{2}+\sqrt{1+\hat{\Omega}^2 b^2}} F(\tilde{p}, \tilde{q}, 1+2\sqrt{1+\hat{\Omega}^2 b^2}; 1-R) + \\ & D_2(R-1)^{\frac{1}{2}-\sqrt{1+\hat{\Omega}^2 b^2}} F(1-\tilde{p}, 1-\tilde{q}, 1-2\sqrt{1+\hat{\Omega}^2 b^2}; 1-R). \end{aligned} \tag{3.79}$$

In this expression for G in the new overlap region, the hypergeometric functions are a series in $R-1$. Now, from the very near horizon region, the solutions for G and $\tilde{\eta}$ are given by (3.73), with finiteness of the perturbation required at the horizon. For example, let us consider $\hat{\Omega} b > 1$. In the very near horizon region,

$$G = C_1(4\hat{\Omega})(R-1)^{\frac{3}{2}+\hat{\Omega} b} + C_3(4\hat{\Omega})(R-1)^{-\frac{1}{2}+\hat{\Omega} b}. \tag{3.80}$$

$$\tilde{\eta} = C_1(2\hat{\Omega} b)(R-1)^{\frac{3}{2}+\hat{\Omega} b} + C_3(-2\hat{\Omega} b)(R-1)^{-\frac{1}{2}+\hat{\Omega} b}. \tag{3.81}$$

If we consider $C_1, C_3 > 0$, then in the very near horizon region, both G and G' are positive. Similarly, if both $C_1, C_3 < 0$ then both G and G' are negative. If $C_1 > 0, C_3 < 0$, then both $\tilde{\eta}$ and $\tilde{\eta}'$ are positive. If $C_1 < 0, C_3 > 0$, then both $\tilde{\eta}$ and $\tilde{\eta}'$ are negative. Thus, irrespective of the sign of the constants, either one of the two functions, G or $\tilde{\eta}$ will be such that the function and its derivative are of the same sign in the very near horizon region. For example, let this function be G. It needs to match with the expression for G from the far limit of the near region (3.79) in the overlap region for some D_1 and D_2. We cannot match exact powers of $(R-1)$ from both sides, as this new overlap region is not very precisely defined and we do not know the exact solution in this region. The exponents coming from the very near horizon region depend crucially on the coupling terms, whereas in the far limit of the near region, the solutions are decoupled.

However, if $\hat{\Omega}$ is large and $b > 1$, the coupling terms will not be significant in the overlap region. This can be seen from the equation for G, (3.65) for example, where the coupling term is $\frac{4\hat{\Omega}}{R(R-1)^2}\tilde{\eta}$. A comparable term on the left is $\frac{\hat{\Omega}^2 b^2}{(R-1)^2}G$ which in the overlap region could be larger than the coupling term for sufficiently large $\hat{\Omega}$. We can match features of the solutions from both sides. Let G and G' have the same sign from the very near horizon region. Then at leading order in $(R-1)$, in order to match this feature with (3.79), we need $D_1 \neq 0$, since a solution with $D_1 = 0$ will have a sign opposite to its derivative. If $D_1 \neq 0$, then in the far limit of the near region, we will have an expansion containing a piece $R^{\frac{\sqrt{1+4\hat{\Omega}^2 b^2+4(\hat{k}^2+\hat{\lambda}^2 b^2)}}{2}}$ which will not match with the solution from the asymptotic region. Thus G must be the trivial solution. Plugging this in the equation for $\tilde{\eta}$, we can conclude the same for it.

This heuristic argument is not as water-tight as the case of the perturbation H. This is due to splitting of the near region into the very near horizon region, the far limit of the near region and the overlap region between them is not very precisely defined. However, since the problematic coupling terms are not significant for $\hat{\Omega}$ large, we believe that there are no unstable modes of the form G or $\tilde{\eta}$ at least in that case.

As discussed in the vector case, we can naively analyze the stability in the case of λ and Ω being lower order in n by setting them to zero in our final solutions. This will show that there is still no match between the near and far region solutions. To prove stability of the black string in these cases, one would have to rewrite the perturbation equations for $\Omega, \lambda, \ell \sim \mathcal{O}(1)$ and solve the system of three coupled equations order by order.

3.4 Summary

In this chapter, we have investigated stability of the black string/brane for non-spherically symmetric perturbations. To analyze the stability, we have assumed a time behaviour $e^{\Omega t}$ for the perturbations, and investigated if there are normalizable solutions to the perturbation equations with Ω real and positive. We have employed the large n limit of general relativity [29], [30] to analyze vector and scalar perturbations. The vector equations decouple in the near-horizon region and the asymptotic region. In the large n limit, these regions are well-defined and contain an overlap region which allows us to employ the technique of matched asymptotic expansions to rule out instabilities. We require the perturbations to be finite at the horizon for consistency of perturbation theory and normalizable asymptotically.Static perturbations with $\hat{\Omega} = 0$ do not lead to instabilities.

In the case of the three scalar perturbation equations for H, G and $\tilde{\eta}$, the equation for H decouples in the near-horizon and asymptotic regions for the case $\Omega, \lambda, \ell \sim \mathcal{O}(n)$. As in the vector case, we show this does not lead to instabilities. The other two perturbations remain coupled in the near-horizon region, although they can be solved asymptotically. We have analyzed these perturbations in various cases. If any one of them is zero, the other does not lead to instabilities. In the case when both G and $\tilde{\eta}$ are non-zero, we employ a two step matching procedure. We split the near-horizon region into two regions with an overlap, solve the two coupled equations in the two regions, and match their features in the new overlap region. We then argue that this solution does not match with the asymptotic solution, and that these perturbations cannot also lead to instability. The split of the near-horizon region into two, and the overlap region of the two is not as neat as the large n split of the spacetime into a near-horizon and far region. However as discussed in the previous section, we believe these perturbations do not lead to instability. In the static limit $\Omega = 0$, we can show that none of the three scalar perturbations leads to an instability.

Taken together, these results in the large n limit provide direct evidence from the analysis of the equations themselves that the Gregory-Laflamme instability is the only instability of the flat black brane. We have also shown that the corresponding Gross-Perry-Yaffe mode for semiclassical black hole perturbations is the *unique* unstable mode in the large n limit.

In this chapter, as we were able to prove the stability of black string using only the leading order solutions, we have not considered next order corrections. We have also not explicitly shown that there are no unstable modes for the case where the parameters in the equation are of $\mathcal{O}(1)$.

The results of this chapter have demonstrated the power of the large D limit of general relativity in tackling difficult problems in black brane perturbation theory.

Chapter 4

Quasinormal modes of black string and black hole

In this chapter we will find the quasinormal modes of the black string and Schwarzschild-Tangherlini black hole. In the large D limit, there are two distinct types of quasinormal modes. The modes with frequencies of $\mathcal{O}(1)$ are termed decoupled modes, and those with $\mathcal{O}(D)$ non-decoupled modes. We will find the decoupled vector quasinormal modes of black holes using a $1/D$ expansion of both the mode functions and mode frequencies. The decoupled vector quasinormal modes of the Schwarzschild-Tangherlini black holes are also obtained in the large D limit, without assuming a $1/D$ expansion. Finally, we also calculate non-decoupled quasinormal modes for both the black string and black hole case.

4.1 Quasinormal modes

Quasinormal modes are the characteristic response of a black hole to a perturbation. In the case of black holes in general relativity, these frequencies are completely characterized by the black hole parameters mass, charge and angular momentum. They are independent of the initial configuration of the perturbation. We shall now give a mathematical definition of quasinormal modes. Equations governing linearized perturbations of the form $\chi(t,r) = e^{i\omega t}\bar{\chi}(r)$ on the black hole background can be written as

$$-\frac{d^2\bar{\chi}}{dx^2}+V(r)\bar{\chi}=\omega^2\bar{\chi} \tag{4.1}$$

where x is the tortoise coordinate given by $dx = dr/f(r)$. In our case x goes to $-\infty$ at the horizon and ∞ at the spatial infinity. In asymptotically flat spacetimes, the potential V is usually positive, $V \to 0$ at both the boundaries, and does not allow bound states. For such a potential, we can have plane wave solution at both the boundaries. At both $x \to \infty$ and $x \to -\infty$,

$$\bar{\chi} \to e^{\pm i\omega x}.$$

The quasinormal modes(QNM) are the perturbations that are purely ingoing at the horizon and purely outgoing at the infinity. For a perturbation $\chi(t, r)$ these boundary conditions are

$$\chi \sim e^{i\omega(t+x)} \quad \text{at horizon} \qquad \chi \sim e^{i\omega(t-x)} \quad \text{at infinity} \tag{4.2}$$

These boundary conditions are physically motivated. The purely ingoing condition at the horizon means entering the black hole. These boundary conditions merely specify the desired class of perturbations that have aforementioned functional behaviour at both the ends. The quasinormal mode frequencies ω are usually complex such that the quasinormal modes decay exponentially in time. This implies that the spacetime is stable.

Let us look at boundary condition at infinity. Adding a small amount of normalizable ingoing part $e^{i\omega(t+x)}$ to the dominant blowing up solution $e^{i\omega(t-x)}$ does not alter its functional behaviour at infinity. Hence we cannot uniquely distinguish a purely outgoing solution from an outgoing solution with small ingoing part at this boundary. A similar problem exists for boundary condition at the horizon. Such a contamination may lead to inaccuracies in determination of quasinormal mode frequencies.

To address this issue, Nollert and Schmidt [80] use the method of Laplace transforms for calculating these frequencies. The perturbation $\chi(r,t)$ obeys the equation

$$-\frac{d^2\chi}{dx^2}+\frac{d^2\chi}{dt^2}+V(r)\chi=0 \tag{4.3}$$

where r can be considered a function of the tortoise coordinate x. Computationally, the method involves evaluating the Green's function obtained from taking the Laplace transform of the equation (4.3) . We define a Laplace transform of the perturbation χ as,

$$y(\Omega,x)=\int_0^\infty e^{-\Omega t}\chi(t,x)dt \tag{4.4}$$

Taking the Laplace transform on both sides of (4.3), we find it satisfies the differential equation,

$$y''(\Omega,x)-\Omega^2 y(\Omega,x)-V(x)y(\Omega,x)=J(\Omega,x) \tag{4.5}$$

where $J(\Omega,x)$ is determined by the data at $t=0$. For initial data χ having compact support, the Laplace transform exists and is holomorphic for $\Omega=\Omega_R+i\Omega_I$, where Ω_R is positive. As the Laplace transform exists, the solutions to the Laplace transformed perturbation equations are bounded at both ends. Let y_+ be the solution of homogeneous part of the transformed equations which is square integrable at infinity. Similarly let y_- be the solution which is square integrable at the horizon. The general solution of the full inhomogeneous equation obeyed by $y(\Omega,x)$, is given by

$$y(\Omega,x)=\int_{-\infty}^{\infty} G(\Omega,x,x')J(\Omega,x')dx' \tag{4.6}$$

Where the Green's function is defined as,

$$G(\Omega,x,x')=\frac{1}{W(\Omega,x')}\begin{cases} y_-(\Omega,x')y_+(\Omega,x) & (x'<x) \\ y_-(\Omega,x)y_+(\Omega,x') & (x'>x) \end{cases} \tag{4.7}$$

The boundary conditions for the bounded solutions are,

$$y_-(\Omega,x)\approx e^{\Omega x} \quad x\to-\infty; \qquad\qquad y_+(\Omega,x)\approx e^{-\Omega x} \quad x\to\infty. \tag{4.8}$$

$W(\Omega, x)$ is the Wronskian of y_- and y_+. For the equation (4.5), the Wronskian $W(\Omega)$ is independent of x. Notice that for $\Omega = i\omega$, these boundary conditions are exactly the quasinormal boundary conditions described in (4.2).

Quasinormal modes simultaneously satisfy both the boundary conditions. Hence quasinormal mode frequencies are defined as the values of Ω for which the solutions y_- and y_+ become linearly dependent. That is

$$W(\Omega, x) = 0.$$

Vector perturbations on the black string are described by a system of two coupled differential equations (2.53),(2.54). In a limit where number of dimensions in metric (2.13), n, is large, we are able to decouple these equations in leading order in n and solve for quasinormal modes. We use the Laplace Transform procedure in our problem.

We define Laplace transforms of F_t and F_r.

$$\tilde{F}_t(\Omega, r) = \int_0^\infty e^{-\Omega t} F_t(t, r) dt \tag{4.9}$$

$$\tilde{F}_r(\Omega, r) = \int_0^\infty e^{-\Omega t} F_r(t, r) dt \tag{4.10}$$

Taking Laplace transforms of equation (2.53) and (2.54),

$$f\tilde{F}_t'' - \frac{1}{f}\Omega^2 \tilde{F}_t + \frac{nf}{r}\tilde{F}_t' - \left[\frac{nf}{r^2} + \frac{\alpha}{r^2}\right]\tilde{F}_t + \left[f' - \frac{2f}{r}\right]\Omega\tilde{F}_r = \lambda^2 \tilde{F}_t + \text{source terms} \tag{4.11}$$

$$\begin{aligned} f\tilde{F}_r'' - \frac{1}{f}\Omega^2 \tilde{F}_r + \left[2f' + \frac{(n-2)f}{r}\right]\tilde{F}_r' \\ + \left[f'' + \frac{(n-2)f'}{r} - \frac{2(n-1)f}{r^2} - \frac{\alpha}{r^2}\right]\tilde{F}_r + \frac{f'}{f^2}\Omega\tilde{F}_t = \lambda^2 \tilde{F}_r + \text{source terms} \end{aligned} \tag{4.12}$$

where the source terms depend on initial data at $t = 0$. Let us define

$$\tilde{F}_t = A, \quad \tilde{F}_r = \frac{B}{f}. \tag{4.13}$$

The factor $f(r)$ in $B(r)$ is chosen for convenience and $f(r) = \left(1 - \frac{b^{n-1}}{r^{n-1}}\right)$. Substituting in the the equations for $\tilde{F}_t$ we get a set of coupled ordinary differential equations for A and B.

$$\frac{d^2A}{dr^2} + \frac{n}{r}\frac{dA}{dr} + \left(-\frac{n}{r^2} - \frac{\alpha}{fr^2} - \frac{\lambda^2}{f} - \frac{\Omega^2}{f^2}\right) A = \left(\frac{2}{rf} - \frac{(n-1)b^{n-1}}{f^2 r^n}\right) \Omega B + \text{source terms} \tag{4.14}$$

$$\frac{d^2B}{dr^2} + \frac{(n-2)}{r}\frac{dB}{dr} + \left(-\frac{2(n-1)}{r^2} - \frac{\alpha}{fr^2} - \frac{\lambda^2}{f} - \frac{\Omega^2}{f^2}\right) B = -\frac{(n-1)b^{n-1}}{f^2 r^n} \Omega A + \text{source terms} \tag{4.15}$$

We have denoted all the source terms in the non-homogeneous differential equations by the general tag 'source terms' as their exact form is irrelevant in the computation that follows. Here n is the number of dimensions of the sphere in metric (2.13). These are exactly the equations we used to analyze stability of the black string. Again, we resort to large n limit in order to analyze these equations.

In the large D limit, there exist two distinct sets of black hole quasinormal modes called decoupled and non-decoupled modes due to the shape of potential V. In the large D limit, the potential shows a steep rise creating a potential barrier very near the horizon and then falls off far from it. The height of the barrier is $\mathcal{O}(D^2)$ [32]. The quasinormal modes that have frequencies $\omega \sim \mathcal{O}(1)$ are localised very near the horizon. They are ingoing at the horizon, but decay in the far region. They have been calculated in a $1/D$ expansion using matched asymptotic expansions in [31] and by effective theory methods in [48]. These modes have been calculated for black holes in AdS and dS in [52] and for Gauss-Bonnet black holes in [58, 63]. The non-decoupled modes have $\omega \sim \mathcal{O}(D)$ and obey outgoing boundary conditions at infinity. These modes have been calculated by various methods by Emparan et. al. in [31], [32]. The imaginary correction to the real leading order frequencies occurs is found to be $\mathcal{O}(D^{1/3})$. Both these modes have been numerically obtained by Dias et. al. in [34].

We will employ two different strategies to calculate the non-decoupled and the decoupled modes. For non-decoupled modes we employ the technique of Nollert and Schmidt. In this calculation we will decouple the equations (4.14) and (4.15) in a $1/n$ expansion. The

Green's functions of the so decoupled equations are obtained in the usual way from the solutions of the corresponding homogeneous equations. Then we compute the poles of the Green's function to obtain quasinormal mode frequencies in the large D limit. For decoupled modes, as they decay far from the horizon , we employ a matching procedure between the solutions coming from the near region and the far region in the overlap region.

4.2 Non-Decoupled Modes

We have now established the procedure to find quasinormal modes of black strings/holes in large n approximation. Let us now look at a class of quasinormal modes having all Ω, λ and ℓ scaling as n. These modes are called non-decoupled modes. In contrast to decoupled modes these modes do not decay in far region and are outgoing at infinity. For Schwarzschild-Tangherlini black holes in large n limit these modes have been studied in [31], [32]. They also have been numerically obtained for n dimensional black holes by Dias et al in [34].

We use the Laplace transform method to compute these modes. The computation involves decoupling the equations (4.14) and (4.15) in the large n limit. Then one has to find the Green's function from solutions to the corresponding homogeneous equation. The poles of the Green's function are non-decoupled quasinormal modes. Computationally it turns out that one can simply neglect the source terms in the equations (4.14) and (4.15) since one is anyway only interested in the corresponding homogeneous equation.

4.2.1 Near Region Solutions

As $\lambda \sim (n-1)$, $\ell \sim (n-1)$ and $\Omega \sim (n-1)$, we introduce notation $\Omega/(n-1) = \hat{\Omega}$. Similarly we denote $\hat{\lambda}, \hat{\ell}$ for $\lambda/(n-1)$ and $\ell/(n-1)$ respectively. We postulate an expansion of the variables A, B and Ω in terms of $(n-1)$ as

$$A = \sum_{i\geq 0} \frac{A_i}{(n-1)^i} \qquad B = \sum_{i\geq 0} \frac{B_i}{(n-1)^i} \qquad \Omega = \sum_{i\geq 0} \frac{\Omega_i}{(n-1)^i} \tag{4.16}$$

$$\frac{d^2A_0}{dR^2}+\frac{2}{R}\frac{dA_0}{dR}-\left[\frac{\hat{\lambda}^2b^2+\hat{\ell}^2+\hat{\ell}}{R(R-1)}+\frac{\hat{\Omega}_0^2b^2}{(R-1)^2}\right]A_0=-\frac{\hat{\Omega}_0B_0b}{R(R-1)^2} \tag{4.17}$$

$$\frac{d^2B_0}{dR^2}+\frac{2}{R}\frac{dB_0}{dR}-\left[\frac{\hat{\lambda}^2b^2+\hat{\ell}^2+\hat{\ell}}{R(R-1)}+\frac{\hat{\Omega}_0^2b^2}{(R-1)^2}\right]B_0=-\frac{\hat{\Omega}_0A_0b}{R(R-1)^2} \tag{4.18}$$

Let us first look at the leading order equations to determine Ω_0. We observe that the equations for the combination (A_0+B_0) and (A_0-B_0) decouple.

$$\begin{aligned}&\frac{d^2(A_0+B_0)}{dR^2}+\frac{2}{R}\frac{d(A_0+B_0)}{dR}+\\&\left[-\frac{\hat{\lambda}^2b^2+\hat{\ell}^2+\hat{\ell}}{R(R-1)}-\frac{\hat{\Omega}_0^2b^2}{(R-1)^2}+\frac{\hat{\Omega}_0b}{R(R-1)^2}\right](A_0+B_0)=0 \qquad (4.19)\\&\frac{d^2(A_0-B_0)}{dR^2}+\frac{2}{R}\frac{d(A_0-B_0)}{dR}+\\&\left[-\frac{\hat{\lambda}^2b^2+\hat{\ell}^2+\hat{\ell}}{R(R-1)}-\frac{\hat{\Omega}_0^2b^2}{(R-1)^2}-\frac{\hat{\Omega}_0b}{R(R-1)^2}\right](A_0-B_0)=0 \qquad (4.20)\end{aligned}$$

Solutions to these equations are given in terms of hypergeometric functions.

$$\begin{aligned}A_0+B_0=&C_2(R-1)^{\hat{\Omega}_0b}(1-R)^{1-2\hat{\Omega}_0b}F(2-p,2-q,2-2\hat{\Omega}_0b,1-R)+\\&C_1(R-1)^{\hat{\Omega}_0b}F(p,q,2\hat{\Omega}_0b,1-R) \qquad (4.21a)\\A_0-B_0=&C_4(R-1)^{-\hat{\Omega}_0b}(1-R)^{1+2\hat{\Omega}_0b}F(2-p_1,2-q_1,2+2\hat{\Omega}_0b,1-R)+\\&C_3(R-1)^{-\hat{\Omega}_0b}F(p_1,q_1,-2\hat{\Omega}_0b,1-R) \qquad (4.21b)\end{aligned}$$

where,

$$p = \frac{1}{2} + \hat{\Omega}_0 b + \frac{\Delta}{2} \qquad\qquad q = \frac{1}{2} + \hat{\Omega}_0 b - \frac{\Delta}{2}$$
$$p_1 = \frac{1}{2} - \hat{\Omega}_0 b + \frac{\Delta}{2} \qquad\qquad q_1 = \frac{1}{2} - \hat{\Omega}_0 b - \frac{\Delta}{2}$$

and

$$\Delta = \sqrt{1 + 4(\hat{\lambda}^2 b^2 + \hat{\ell}^2 + \hat{\ell}) + 4\hat{\Omega}_0^2 b^2} \tag{4.22}$$

Very near the horizon, where $R \approx 1$, behaviour of A, B is dictated by coordinate $(R - 1)$. For $(R - 1)$ small, all the hypergeometric functions in (4.21) can be approximated as $F(a, b, c, 1 - R) \approx 1$. We wish to impose boundary condition $B = e^{\Omega b x}$ at the horizon where the coordinate x is defined by (4.56). For Ω, λ and ℓ of the order n, this condition takes following form.

$$e^{\Omega b x} = (R - 1)^{\hat{\Omega}_0 b} \tag{4.23}$$

Hence we impose the condition $C_2 = C_3 = 0$ in (4.21). To simplify calculations further, without loss of generality, we additionally fix $C_1 = (-1)^{1+2\hat{\Omega}_0 b} C_4$. The solutions B_0 and A_0 very near horizon become,

$$A_0 = \frac{C_1}{2}(R - 1)^{\hat{\Omega}_0 b}[1 + (R - 1)] \quad B_0 = \frac{C_1}{2}(R - 1)^{\hat{\Omega}_0 b}[1 - (R - 1)] \tag{4.24}$$

Let us now concentrate on B_0. The solutions (4.21) of the hypergeometric equations are valid near the point $R = 1$. In order to calculate the Wronskian, we need to extend the solution B_0 in overlap region. We can do so by using the standard formulae for analytic continuation of hypergeometric functions. These formulae relate hypergeometric functions of the form $F(\alpha, \beta, \gamma, 1 - R)$ to linear combinations of the functions of form $F(\alpha_1, \beta_1, \gamma_1, 1/R)$. We then take the large R limit of the continued solution. All the hypergeometric functions of the form $F(\alpha_1, \beta_1, \gamma_1, 1/R) \to 1$ in this limit. In the overlap region, we also approximate $(R - 1)$ by R and ignore the subleading R terms.

$$B_0 = \frac{C_1}{2}(R-1)^{\hat{\Omega}_0 b} F(p, q, 2\hat{\Omega}_0 b, 1-R) - \frac{C_4}{2}(R-1)^{-\hat{\Omega}_0 b}(1-R)^{1+2\hat{\Omega}_0 b} F(2-p_1, 2-q_1, 2+2\hat{\Omega}_0 b, 1-R) \quad (4.25)$$

$$B_0 = \frac{C_1}{2}\left[(\tilde{c}_1^1 - \tilde{c}_4^1)R^{-\frac{1}{2}-\frac{\Delta}{2}} + (\tilde{c}_1^2 - \tilde{c}_4^2)R^{-\frac{1}{2}+\frac{\Delta}{2}}\right] \quad (4.26)$$

The constants $\tilde{c}_j^i$s are Gamma functions which depend on parameters p, q and $\hat{\Omega}_0$ and Δ is given by (4.22). From the following formulae we can see that the constants $(\tilde{c}_1^1 - \tilde{c}_4^1)$ and $(\tilde{c}_1^2 - \tilde{c}_4^2)$ are always nonzero.

$$\tilde{c}_1^1 = \frac{\Gamma(2\hat{\Omega}_0 b)\Gamma(q-p)}{\Gamma(q)\Gamma(q-1)} \qquad \tilde{c}_1^2 = \frac{\Gamma(2\hat{\Omega}_0 b)\Gamma(p-q)}{\Gamma(p)\Gamma(p-1)} \quad (4.27)$$

$$\tilde{c}_4^1 = \frac{\Gamma(2+2\hat{\Omega}_0 b)\Gamma(q_1-p_1)}{\Gamma(1-p_1)\Gamma(2-p_1)} \qquad \tilde{c}_4^2 = \frac{\Gamma(2+2\hat{\Omega}_0 b)\Gamma(p_1-q_1)}{\Gamma(1-q_1)\Gamma(2-q_1)} \quad (4.28)$$

4.2.2 Far Region Solution

Far region is described by limit $r \gg b + \frac{b}{n-1}$. In this limit, the ratio $b^{(n-1)}/r^{(n-1)} \sim e^{-n \ln r}$ is a small quantity when both n and r are large. Hence we can approximate $f(r) \approx 1$ and neglect the terms including $f'(r)$ and $f''(r)$ in the equations (4.14) and (4.15). We can neglect the coupling term in the B equations as it falls off much faster compared to the terms in the left hand side. The far region equations are,

$$\frac{d^2A}{dr^2} + \frac{n}{r}\frac{dA}{dr} + \left(-\frac{\ell^2 + \ell(n-1)}{r^2} - \Omega^2 - \lambda^2\right)A = \frac{2\Omega}{r}B \quad (4.29)$$

$$\frac{d^2B}{dr^2} + \frac{n-2}{r}\frac{dB}{dr} + \left(-\frac{\ell^2 + \ell(n-1) + n - 2}{r^2} - \Omega^2 - \lambda^2\right)B = 0. \quad (4.30)$$

General solution of (4.30) is given in terms of modified Bessel functions of order $\nu = \frac{n-1}{2}\left(1 + \frac{2\ell}{n-1}\right)$

$$B = r^{\frac{3-n}{2}}\left[D_1 I_\nu(\sqrt{\lambda^2+\Omega^2}\,r) + D_2 K_\nu(\sqrt{\lambda^2+\Omega^2}\,r)\right] \quad (4.31)$$

For non-decoupled modes, both λ and Ω are $\mathcal{O}(n)$. Let us introduce a new coordinate z

such that $\sqrt{\lambda^2+\Omega^2}r = \nu z$. In terms of z, the modified Bessel functions $I_\nu(\nu z)$ and $K_\nu(\nu z)$ have large order and large argument. The asymptotic behaviour of $I_\nu(\nu z)$ and $K_\nu(\nu z)$ can hence be approximated as

$$I_\nu(\nu z) \sim e^{\nu z}; \quad K_\nu(\nu z) \sim e^{-\nu z}. \tag{4.32}$$

For quasinormal modes, we wish to choose a purely outgoing solution at infinity. This solution corresponds to the decaying solution in terms of Ω. We impose the boundary condition by choosing $D_1 = 0$. Standard asymptotic expansion for $K_\nu(\nu z)$ for large ν is,

$$K_\nu(\nu z) = \sqrt{\frac{\pi}{2\nu}}\frac{e^{-\nu\eta}}{(1+z^2)^{1/4}}\left[1+\sum_{m=1}^{\infty}(-1)^m\frac{u_m(\tilde{t})}{\nu^m}\right] \tag{4.33}$$

where

$$\eta = \sqrt{1+z^2}+\ln\left[\frac{z}{1+\sqrt{1+z^2}}\right] \qquad \tilde{t} = \frac{1}{\sqrt{1+z^2}}$$

and $u_m(\tilde{t})$ are polynomials in $\tilde{t}$ of degree $3m$.The first three polynomials are

$$u_0(\tilde{t}) = 1, \quad u_1(\tilde{t}) = \frac{1}{24}(3\tilde{t}-5\tilde{t}^3), \quad u_2(\tilde{t}) = \frac{1}{1152}(81\tilde{t}^2-462\tilde{t}^4+385\tilde{t}^6) \tag{4.34}$$

In order to compute the Wronskian with the near region solution, we need to extend this solution to overlap region. The form of (4.33) in the overlap region is obtained by changing variables from r to R using (3.6). Let us simplify the notation by introducing β such that $z = \beta r$. Hence $\beta = \frac{\sqrt{\lambda^2+\Omega^2}}{\nu}$.

$$\begin{aligned} r^{\frac{3-n}{2}}K_\nu(\nu z) = R^{-\frac{1}{2}-\frac{\nu\sqrt{1+\beta^2b^2}}{(n-1)}}\Bigg[1 - \frac{1}{24\nu\sqrt{1+\beta^2b^2}}\left(3-\frac{5}{1+\beta^2b^2}\right) \\ +\left(1-\frac{\beta^2b^2}{1+\beta^2b^2}\right)\frac{\ln R}{(n-1)} - \frac{\nu\beta^2b^2}{2\sqrt{1+\beta^2b^2}}\frac{(\ln R)^2}{(n-1)^2}+\mathcal{O}\left(\frac{1}{(n-1)^2}\right)\Bigg] \end{aligned} \tag{4.35}$$

This calculation follows exactly the same procedure as that discussed in the section (3.2.3) of the previous chapter. We will not repeat the details of calculation here. We did not need to go beyond the leading order for the stability analysis, but here we keep track of the next to leading order terms in both order ν and argument of the modified Bessel function. The final expression for B in overlap region for non decoupling case is obtained by substituting $(1+\beta^2 b^2)$ for Ω, λ and ℓ of the order n. For non-decoupled modes,

$$\sqrt{1+\beta^2 b^2} = \frac{(n-1)\Delta}{2\nu} \tag{4.36}$$

Here Δ is given by (4.22). Substituting in (4.35), we get the overlap region solution as following. The constant D is arbitrary.

$$B_{f0} = DR^{-\frac{1}{2}-\frac{\Delta}{2}} \tag{4.37}$$

4.2.3 The Wronskian Calculation

We shall now calculate the Wronskian between the leading order solutions B_0 from the near region given by (4.26) and B_{f0} from far region given by (4.37) in overlap region.

$$W = B_0 B_{f0}^{'} - B_{f0} B_0^{'} = -D(\tilde{c}_1^2 - \tilde{c}_4^2)\frac{\sqrt{1+4(\hat{\lambda}^2 b^2 + \hat{\ell}^2 + \hat{\ell}) + 4\hat{\Omega}_0^2 b^2}}{R^2} \tag{4.38}$$

The frequencies $\hat{\Omega}_0$ for which $W=0$ are the quasinormal mode frequencies. The constant D and $(\tilde{c}_1^2 - \tilde{c}_4^2)$ are always non-zero. Hence the zeros of the Wronskian can only occur at $1+4(\hat{\lambda}^2 b^2 + \hat{\ell}^2 + \hat{\ell}) + 4\hat{\Omega}_0^2 b^2 = 0$.

$$\hat{\Omega}_0 b = \frac{i\sqrt{(1+2\hat{\ell})^2 + 4\hat{\lambda}^2 b^2}}{2} \tag{4.39}$$

$$\omega_0 b = \frac{(n-1)\sqrt{(1+2\hat{\ell})^2 + 4\hat{\lambda}^2 b^2}}{2} \tag{4.40}$$

In the last step we have replaced $\Omega = i\omega$. For $\lambda = 0$ we recover the real part of black hole quasinormal frequencies obtained in [32], and [34]. For Schwarzschild black holes in large n,

$$\omega_0 b = \frac{(n-1)+2\ell}{2} \tag{4.41}$$

4.2.4 Next to leading order corrections

The main disadvantage of using the method of Laplace transforms is that we cannot compute higher order corrections to the quasinormal mode frequency. This is because in the near region, the solutions of the leading hypergeometric equations (4.19) and (4.20) become degenerate after substituting the leading order value of Ω_0. These degenerate solutions become the homogeneous solutions in the source integrals in the next to leading order calculations making the integrals very complicated. These degenerate solution are defined in the neighbourhood of a singular point and cannot be extended beyond that. Due to this, the solutions near horizon cannot reliably be extended to the overlap region.

Although we cannot completely determine it, we can put a bound on the n-dependence of the correction to Ω just by looking at the behaviour of the asymptotic solution extended to the overlap region. The decaying far region solution in our case is $\sim K_\nu(\nu z)$ (4.33). When extended to overlap region, this solution behaves as $\sim R^{-\frac{1}{2}-\frac{\Delta}{2}}$ where $\Delta = \sqrt{1+4(\hat{\lambda}^2 b^2 + \hat{\ell}^2 + \hat{\ell}) + 4\hat{\Omega}_0^2 b^2}$. In obtaining this leading order solution, we have neglected the series terms in the expansion (4.33). Without loss of generality, in the following discussion we will consider $\lambda = 0$ for convenience.

The leading order quasinormal modes obtained by calculating the Wronskian are $\Delta = 0$. Which, for $\nu = \frac{n-1}{2}\left(1+\frac{2\ell}{n-1}\right)$ can be written as

$$\left(1+\frac{\Omega_0^2 b^2}{\nu^2}\right) = 0. \tag{4.42}$$

Let us assume a next order correction to Ω as $\Omega = \Omega_0 + \frac{\Omega_1}{(D-3)^\alpha}$. asymptotic expansion for the far region solution diverges in the overlap region in the large D limit for $\alpha > \frac{2}{3}$. To the next to leading order, the expansion for $K_\nu(\nu z)$ is,

$$K_\nu(\nu z) \approx \sqrt{\frac{\pi}{2\nu}}\frac{\left(1+\sqrt{1+z^2}\right)^\nu e^{-\nu\sqrt{1+z^2}}}{z^\nu(1+z^2)^{1/4}}\left[1-\frac{1}{24\nu}\left(\frac{3}{(1+z^2)^{\frac{1}{2}}}-\frac{5}{(1+z^2)^{\frac{3}{2}}}\right)\right] \tag{4.43}$$

To extend this solution to overlap region, we write z in terms of R using (3.6).

$$
\begin{aligned}
(1+z^2) &= \left[1+\frac{\Omega_0^2 b^2}{\nu^2}+\frac{2\Omega_1\Omega_0 b^2}{\nu^2(n-1)^\alpha}+\frac{\Omega_0^2 b^2}{\nu^2}\frac{2\ln R}{(n-1)}\right] \\
&= \left(1+\frac{\Omega_0^2 b^2}{\nu^2}\right)\left[1+\frac{2\Omega_1\Omega_0 b^2}{\nu^2\left(1+\frac{\Omega_0^2 b^2}{\nu^2}\right)(n-1)^\alpha}+\frac{2\frac{\Omega_0^2 b^2}{\nu^2}}{1+\frac{\Omega_0^2 b^2}{\nu^2}}\frac{\ln R}{(n-1)}\right]. \qquad (4.44)
\end{aligned}
$$

Using this expansion, we get the leading order answer, which leads to the leading order quasinormal mode (4.42). To see the effect of leading order matching on the subleading terms of (4.33), let us express the polynomials $u_m(\tilde{t})$ in the coordinate R. The polynomials $u_m(\tilde{t})$ are in the variable $\tilde{t}=(\sqrt{1+z^2})^{-1}$. Using (4.44) we get

$$
\frac{1}{\sqrt{1+z^2}} = \left(1+\frac{\Omega_0^2 b^2}{\nu^2}\right)^{-\frac{1}{2}}\left[1-\frac{\Omega_1\Omega_0 b^2}{\nu^2\left(1+\frac{\Omega_0^2 b^2}{\nu^2}\right)(n-1)^\alpha}-\frac{\frac{\Omega_0^2 b^2}{\nu^2}}{1+\frac{\Omega_0^2 b^2}{\nu^2}}\frac{\ln R}{(n-1)}\right]. \qquad (4.45)
$$

For the Ω_0 obtained in (4.42), denominator of all the terms becomes zero. We therefore have to evaluate (4.45) carefully beyond leading order. Substituting for Ω_0,

$$
\begin{aligned}
\frac{1}{\sqrt{1+z^2}} &= \left[\frac{2\Omega_1\Omega_0 b^2}{\nu^2(n-1)^\alpha}+\frac{\Omega_0^2 b^2}{\nu^2}\frac{2\ln R}{(n-1)}\right]^{-\frac{1}{2}} \\
&= (n-1)^{\frac{\alpha}{2}}\left(\frac{2\hat{\Omega}_1\hat{\Omega}_0 b^2}{(1+2\hat{\ell})^2}\right)^{-\frac{1}{2}}\left[1+\frac{\hat{\Omega}_0}{\hat{\Omega}_1}\frac{\ln R}{(n-1)^{1-\alpha}}\right]^{-\frac{1}{2}} \qquad (4.46)
\end{aligned}
$$

The hatted quantities denote $\hat{\Omega}=\frac{\Omega}{(n-1)}$ and $\hat{\ell}=\frac{\ell}{(n-1)}$. As we are in the overlap region, $1 \ll \ln R \ll \frac{(n-1)}{2}$. Let us now look at the subleading term in (4.43) . The subleading series term, denoted by u_1, after substituting 4.46 is,

$$\frac{1}{\nu}\left\{U_1\left(\frac{1}{\sqrt{1+z^2}}\right)\right\} =$$
$$\frac{(n-1)^{\frac{\alpha}{2}}(1+2\hat{\ell})}{(n-1)}\left(\frac{2\hat{\Omega}_1\hat{\Omega}_0 b^2}{(1+2\hat{\ell})^2}\right)^{-\frac{1}{2}}\frac{1}{8}\left[1+\frac{\hat{\Omega}_0}{\hat{\Omega}_1}\frac{\ln R}{(n-1)^{1-\alpha}}\right]^{-\frac{1}{2}} -$$
$$\frac{(n-1)^{\frac{3\alpha}{2}}(1+2\hat{\ell})}{(n-1)}\frac{5}{24}\left(\frac{2\hat{\Omega}_1\hat{\Omega}_0 b^2}{(1+2\hat{\ell})^2}\right)^{-\frac{3}{2}}\left[1+\frac{\hat{\Omega}_0}{\hat{\Omega}_1}\frac{\ln R}{(n-1)^{1-\alpha}}\right]^{-\frac{3}{2}} \tag{4.47}$$

Consider the n dependence of the second term. The coefficient of the second term on (4.47) is proportional to $(n-1)^{\frac{3\alpha-2}{2}}$. For $\alpha > \frac{2}{3}$, the exponent $\frac{3\alpha-2}{2}$ becomes positive. We can further see that from the expressions for (4.34), for each m the series terms in (4.33) , the leading term will be $\approx (n-1)^{\frac{(3\alpha-2)m}{2}}$. For $\alpha > \frac{2}{3}$, $(3\alpha-2)m$ is positive and both the series diverge for large $(n-1)$. For $\alpha = \frac{2}{3}$, the leading order term for each m is $\mathcal{O}(1)$. Hence we cannot truncate the asymptotic expansion for large D. The asymptotic expansion of the far region solution thus breaks down for $\alpha > \frac{2}{3}$. Interestingly, $\alpha = \frac{2}{3}$ corresponds to the correction $\omega_1 \sim \mathcal{O}(n^{1/3})$ obtained in [30] for the Schwarzschild-Tangherlini black hole. The value $\alpha = \frac{1}{2}$ found numerically in [34] is also consistent with convergence of the asymptotic expansion. This argument is valid for $\lambda \neq 0$.

4.3 Decoupled Modes

We shall now analyse the decoupled modes of the black strings. This is work in progress. In this case all the parameters Ω, ℓ and λ in (4.14) and (4.15) are $\mathcal{O}(n^0)$ quantities. Decoupled modes are localized in the near region where $f(r)$ is steeply increasing and decay rapidly in the far region. Decoupled modes for Schwarzschild-Tangherlini black holes have previously been studied analytically in [31], [48] and by numerical methods in [34].

Here we do not use the Laplace transform method. Instead we employ a simple matching procedure to obtain modes that are ingoing at the horizon and decaying in the far region. To this end we define new variables A and B by modal decomposition of our original perturbation

F_r and F_t.

$$F_t = e^{i\omega t}A(r), \quad F_r = e^{i\omega t}\frac{B(r)}{f} \tag{4.48}$$

These are distinct from A and B defined in the previous section. For convenience and consistency, we maintain the notation and denote variable related to F_r by B and that related to F_t by A. We replace $i\omega = \Omega$ for convenience so that our equations for A and B look exactly like (4.14) and (4.15) without the source terms.

4.3.1 Near Region Solution

To get the near region equations, we first write the equations (4.14) and (4.15) (without the source terms) in terms of the near region coordinate $R = (\frac{r}{b})^{n-1}$ using (3.6).

$$\begin{aligned}
&\frac{d^2A}{dR^2} + \frac{2}{R}\frac{dA}{dR} + \Bigg[-\frac{n}{(n-1)^2R^2} - \frac{\ell^2 + \ell(n-1) - (1+(n-1))}{(n-1)^2}\frac{1}{R(R-1)} \\
&- \frac{\lambda^2 b^2}{(n-1)^2R(R-1)}\left(1 + \frac{2\ln R}{(n-1)}\right) - \frac{\Omega^2 b^2}{(n-1)^2(R-1)^2}\left(1+\frac{2\ln R}{(n-1)}\right)\Bigg] A \\
&= \left[\frac{2}{(n-1)^2R(R-1)} - \frac{1}{(n-1)R(R-1)^2}\right]\Omega b\left(1+\frac{\ln R}{(n-1)}\right) B
\end{aligned} \tag{4.49}$$

$$\begin{aligned}
&\frac{d^2B}{dR^2} + \frac{2}{R}\left(1 - \frac{1}{(n-1)}\right)\frac{dB}{dR} + \Bigg[-\frac{2}{(n-1)R^2} - \frac{\ell^2 + \ell(n-1) - 1}{(n-1)^2}\frac{1}{R(R-1)} \\
&+ \frac{1}{(n-1)R(R-1)} - \frac{\lambda^2 b^2}{(n-1)^2R(R-1)}\left(1+\frac{2\ln R}{(n-1)}\right) - \\
&\frac{\Omega^2 b^2}{(n-1)^2(R-1)^2}\left(1+\frac{2\ln R}{(n-1)}\right)\Bigg] B = -\frac{\Omega b}{(n-1)R(R-1)^2}\left(1+\frac{\ln R}{(n-1)}\right) A
\end{aligned} \tag{4.50}$$

We wish to write these equations as a series in parameter n. We postulate an expansion of the variables A, B and Ω in terms of $(n-1)$ as

$$A = \sum_{i\geq 0}\frac{A_i}{(n-1)^i} \qquad B = \sum_{i\geq 0}\frac{B_i}{(n-1)^i} \qquad \Omega = \sum_{i\geq 0}\frac{\Omega_i}{(n-1)^i} \tag{4.51}$$

All the A_is, B_is and Ω_is in the expansion are independent of n. Our choice of expansion parameter, $(n-1)$, is partly dictated by the structure of above equations. In principle, we could have expanded these quantities as $A = \sum A_i((n-1)^\alpha)^i$ for any $\alpha \leq 1$. But looking at the equations order by order, we can see that the expansion in $(n-1)^\alpha$ will only change the overall n-dependent scaling constant multiplying the full solutions. We can also expand the perturbation variables A, B and Ω in two different powers of $(n-1)^\alpha$. But such an expansion would lead to inconsistencies.

The leading order equation for A, B and their solutions are,

$$\frac{d^2B_0}{dR^2} + \frac{2}{R}\frac{dB_0}{dR} = 0 \qquad \frac{d^2A_0}{dR^2} + \frac{2}{R}\frac{dA_0}{dR} = 0 \tag{4.52}$$

$$B_0 = \frac{d_1}{R} + d_2 \qquad A_0 = \frac{c_1}{R} + c_2 \tag{4.53}$$

The next order equations are obtained by collecting all the terms in (4.49) and (4.50)that are proportional to $1/(n-1)$. These equations have B_0 and A_0 as source terms. The equation and solution for B_1 are

$$\frac{d^2B_1}{dR^2} + \frac{2}{R}\frac{dB_1}{dR} = \frac{dB_0}{dR} + \left(\frac{2}{R^2} + \frac{\ell-1}{R(R-1)}\right)B_0 - \frac{\Omega_0 b}{R(R-1)^2}A_0 \tag{4.54}$$

$$\begin{aligned} B_1 =& (\ell-1)d_1\left[\ln(R-1) - \ln R - \frac{\ln(R-1)}{R}\right] + \Omega_0 b c_1\left[\ln(R-1) - \ln R + \frac{1}{R}\right] \\ &+ d_2\left[-2 + 2\ln R + (\ell-1)\left(-1 - \frac{\ln(R-1)}{R} + \ln(R-1)\right)\right] \\ &+ \Omega_0 b c_2\left[\frac{\ln(R-1)}{R} + \frac{1}{R}\right] + \frac{d_1^1}{R} + d_2^1 \end{aligned} \tag{4.55}$$

We now look at the boundary condition at the event horizon. The near horizon dynamics of A and B are described by the tortoise coordinate x. In terms of R, x is written using relation between r and R,(3.6), as

$$x = \int \frac{dr}{f(r)} \approx \frac{b}{(n-1)}\ln(R-1) \tag{4.56}$$

This coordinate has range $-\infty \leq x \leq \infty$. We are interested in purely ingoing solutions at the horizon. In terms of Ω, such a solution corresponds to an exponentially decaying solution as $x \to -\infty$. This boundary condition implies $B = Ce^{\Omega x}$. The constant C is arbitrary. Using the expansion of Ω and x, this condition in the very near horizon region becomes

$$B = C\left[1 + \frac{\Omega_0 b}{(n-1)}\ln(R-1) + \frac{\Omega_1 b}{(n-1)^2}\ln(R-1) + \frac{\Omega_0^2 b^2}{2}\frac{(\ln(R-1))^2}{(n-1)^2}\right]. \tag{4.57}$$

Let us look at B_1 very near the horizon by taking the limit $R = 1$ and $(R-1) << 1$. At the horizon the only significant contribution comes from the terms proportional to $\ln(R-1)$ and the pure constant terms. Very near the horizon, B_1 becomes,

$$B_1 = \Omega_0 b(c_1 + c_2)\ln(R-1) + \Omega_0 b(c_1 + c_2) - (l+1)d_2 + d_1^1 + d_2^1 \tag{4.58}$$

To the first correction in $(n-1)$, B at the horizon is

$$B = (d_1 + d_2)\left[1 + \frac{1}{n-1}\left(\Omega_0 b\frac{(c_1+c_2)}{(d_1+d_2)}\ln(R-1) - \frac{(l+1)d_2 - (d_1^1 + d_2^1)}{(d_1+d_2)} + \Omega_0 b\frac{(c_1+c_2)}{(d_1+d_2)}\right)\right] \tag{4.59}$$

We wish to write (4.59) as a pure ingoing solution of the form (4.57). Comparing the two we get following conditions on the constants.

$$c_1 + c_2 = d_1 + d_2 \tag{4.60}$$

$$(l+1)d_2 - (d_1^1 + d_2^1) = \Omega_0 b(c_1 + c_2) \tag{4.61}$$

We shall treat the second condition as the necessary condition for obtaining an ingoing solution. In order to match with the far region solution, we need to look at large R behaviour of the near region solution(4.55). The leading order solutions A_0 and B_0 are unchanged in this limit. The form of B_1 in overlap region for large R is,

$$B_1 = \left[-(\ell-1)(d_1+d_2)+\Omega_0 b c_2 + d_1^1\right]\frac{1}{R} + \left[-(\ell-1)(d_1+d_2)+ +\Omega_0 b c_2\right]\frac{\ln R}{R} + \left[2d_2+(\ell-1)d_2\right]\ln R - \left[2d_2+(\ell-1)d_2-d_2^1\right] \tag{4.62}$$

By similar process we can find the solution of A. Expression for A till $\mathcal{O}(1/(n-1))$ is,

$$\begin{aligned} A =& \frac{c_1}{R}+c_2+\frac{c_1}{n-1}\left[-\frac{\ln R}{R}-\frac{1}{R}+(\ell-1)\left(\ln(R-1)-\ln R-\frac{\ln(R-1)}{R}\right)\right] \\ &+\frac{\Omega_0 b d_1}{n-1}\left[\ln(R-1)-\ln R+\frac{1}{R}\right]+\frac{\Omega_0 b d_2}{n-1}\left[\frac{\ln(R-1)}{R}+\frac{1}{R}\right] \\ &+\frac{c_2}{n-1}\left[-1+\ln R+(\ell-1)\left(-1-\frac{\ln(R-1)}{R}+\ln(R-1)\right)\right]+\frac{c_1^1}{R}+c_2^1 \end{aligned} \tag{4.63}$$

Then we extend the near region solution to overlap region and match with the far region solution also extended to overlap region.

4.3.2 Far Region Solution

The far region equations for A and B, (4.29) and (4.30), have been obtained by considering only the far region approximation $r >> b$. Hence these equations and their solutions are valid for both decoupled modes and non-decoupled modes. The solution for B is,

$$B = r^{\frac{3-n}{2}}\left[D_1 I_\nu(\sqrt{\lambda^2+\Omega^2}r)+D_2 K_\nu(\sqrt{\lambda^2+\Omega^2}r)\right] \tag{4.64}$$

For decoupled modes with ω, ℓ, λ of $\mathcal{O}(1)$, the order of modified Bessel functions ν is large. For large order, asymptotic expansion of the Bessel function are $I_\nu(z) \sim z^\nu$ and $K_\nu(z) \sim z^{-\nu}$. Normalizable solution at infinity is chosen by setting $D_1 = 0$. We wish to extend this solution to the overlap region. To this end, we use the asymptotic expansion for modified Bessel function for large order,

$$K_\nu(\sqrt{\lambda^2+\Omega^2}r) = \sqrt{\frac{\pi}{2\nu}}\left(\frac{e\sqrt{\lambda^2+\Omega^2}r}{2\nu}\right)^{-\nu}. \tag{4.65}$$

We extend this solution to the overlap region by writing r in terms of R using (3.6). In the overlap region the next to leading order expansion for B, denoted by B_f is

$$B_f = D\frac{1}{R} + \frac{(1-\ell)}{n-1}\frac{\ln R}{R} \tag{4.66}$$

Here, D is the overall constant.

This expansion now has similar form as the solution from near region. Writing (4.66) as a series in $(n-1)$ similar to (4.16), we get the first two terms,

$$B_{f_0} = \frac{D}{R} \qquad B_{f_1} = \frac{D(1-\ell)\ln R}{R} \tag{4.67}$$

We can similarly solve for A using B as source term in (4.29). As the source integrals for A are very complicated, we cannot get the full solution for A in the far region. Instead we solve for A in the overlap region in terms of R. The solution for A till order $1/n$ turns out to be,

$$A_f = D\frac{1}{R}\left[1 - \frac{\ell}{(n-1)}\ln R\right]. \tag{4.68}$$

4.3.3 The Matching Calculation

We have obtained solutions for B in both near and far region. The quasinormal modes are the values of Ω for which both the solutions match. Therefore the solutions B from near region and B_f from far region must match at all orders in n. We shall perform matching at each order in n and impose this condition. At the leading order,

$$B_0 = \frac{d_1}{R} + d_2 \qquad B_{f_0} = \frac{D}{R}$$

The two solutions match when $d_2 = 0$ and $d_1 = D$. By a similar calculation for A, we can show $c_2 = 0$ and $c_1 = D$. Plugging these values of constants in (4.62), the overlap region solution becomes,

$$B_1 = \left[-(\ell-1)D + d_1^1\right]\frac{1}{R} - (\ell-1)D\frac{\ln R}{R} + d_2^1 \tag{4.69}$$

Let us now match the solutions at $\mathcal{O}((n-1)^{-1})$. The solutions B_1 and B_{f_1} at this order are given by (4.69) and (4.67) respectively.

Imposing matching gives us $d_2^1 = 0$ and $-(\ell - 1)D + d_1^1 = 0$. But value of d_1^1 is fixed by the boundary condition at horizon (4.61). Putting $d_2 = d_2^1 = 0$ in the boundary condition at horizon gives us $d_1^1 = -\Omega_0 bD$. Combining the two conditions on d_1^1 we get,

$$\Omega_0 b = 1 - \ell \tag{4.70}$$

We get the same result by using near region and far region solutions of variable A. We may match the near region solution A,(4.63), with the far region solution A_f, (4.68). In this case, imposing the boundary condition (4.57) on (4.63) fixes $c_1^1 = D - \Omega_0 bD$ and demanding matching of the solutions implies $c_2^1 = 0$ and $c_1^1 = D\ell$. All these conditions combine to give us same value for Ω_0 as (4.70).

We see that at the leading order the parameter λ does not contribute to Ω_0. Hence the leading order quasinormal frequency for decoupling mode is same for black string and black hole. We expect λ to contribute at the next order. This is work in progress and we hope to report on the results soon.

Replacing $\Omega = i\omega$ we get the quasinormal mode frequencies for decoupled mode.

$$\omega b = i\,(\ell - 1) \tag{4.71}$$

We see that the ωs are pure imaginary.

4.4 Other cases

Apart from decoupled modes and non-decoupled modes, we can have several other cases of quasinormal modes. For $\Omega \sim \mathcal{O}(1)$, we can have either one or both $\ell, \lambda \sim \mathcal{O}(n)$. Similarly for $\Omega \sim \mathcal{O}(n)$, we can have modes with either one or both $\ell, \lambda \sim \mathcal{O}(1)$.

Let us first look at the cases for $\Omega \sim \mathcal{O}(1)$. We can have three cases. First where both $\lambda, \ell \sim \mathcal{O}(n)$, second having $\lambda \sim \mathcal{O}(1)$ and $\ell \sim \mathcal{O}(n)$ and a third with $\lambda \sim \mathcal{O}(n)$ and $\ell \sim \mathcal{O}(1)$. We here sketch details only for the first case, the second and third cases can be dealt the same way and lead to the same conclusion. We shall re-use the notation $\hat{\ell} = \ell/(n-1)$ and

$\hat{\lambda} = \lambda/(n-1)$. We obtain the equations for A and B using the series expansion in (4.16) for A, B and Ω . Plugging these expansions in (4.49) and (4.50), we find the leading order equations containing A_0 and B_0.

$$\frac{d^2 A_0}{dR^2} + \frac{2}{R}\frac{dA_0}{dR} + \left[-\frac{\hat{\lambda}^2 b^2 + \hat{\ell}^2 + \hat{\ell}}{R(R-1)}\right] A_0 = 0 \tag{4.72}$$

$$\frac{d^2 B_0}{dR^2} + \frac{2}{R}\frac{dB_0}{dR} + \left[-\frac{\hat{\lambda}^2 b^2 + \hat{\ell}^2 + \hat{\ell}}{R(R-1)}\right] B_0 = 0 \tag{4.73}$$

Let us denote $(\hat{\lambda}^2 b^2 + \hat{\ell}^2 + \hat{\ell}) = \delta$. These are hypergeometric equations with solution

$$\begin{aligned} B_0 =& e_1(1-R) F\left(\frac{3+\sqrt{1+4\delta}}{2}, \frac{3-\sqrt{1+4\delta}}{2}, 2; (1-R)\right) \\ &+ e_2(1-R)\left[F\left(\frac{3+\sqrt{1+4\delta}}{2}, \frac{3-\sqrt{1+4\delta}}{2}, 2; (1-R)\right)\ln(1-R)\right. \\ &\left. - \sum_{k=1}^{2} \alpha_k (R-1)^{-k} \sum_{k=1}^{2} \gamma_k (1-R)^k\right] \end{aligned} \tag{4.74}$$

Here α_k and γ_k are constants. As the second solution blows up at the horizon for $R \to 1$, we choose $e_2 = 0$. To match with the far region solution, we extend B_0 to overlap region using standard analytical continuation formulas for hypergeometric functions. The leading order solution in the overlap region is,

$$B_0 = c_1 R^{-\frac{1}{2} - \frac{\sqrt{1+4(\hat{\lambda}^2 b^2 + \hat{\ell}^2 + \hat{\ell})}}{2}} + c_2 R^{-\frac{1}{2} + \frac{\sqrt{1+4(\hat{\lambda}^2 b^2 + \hat{\ell}^2 + \hat{\ell})}}{2}} \tag{4.75}$$

Again c_1 and c_2 are constants containing Gamma functions that depend on δ which have form similar to (4.27)–(4.28) . It can be shown that both the c_1 and c_2 are non-zero. The far region solution can be obtained by expanding $1 + \beta^2 b^2$ in terms of $\Omega, \hat{\lambda}$ and $\hat{\ell}$ and plugging it in the far region solution (4.35). The overlap region form of far region solution in this case upto the leading order is

$$B_{f_0} = D R^{-\frac{1}{2} - \frac{\sqrt{1+4(\hat{\lambda}^2 b^2 + \hat{\ell}^2 + \hat{\ell})}}{2}} \tag{4.76}$$

As D and c_1 are non-zero, the the two solutions will only match for for $1+4(\hat{\lambda}^2 b^2 + \hat{\ell}^2 + \hat{\ell}) =$

0. But both the parameters λ^2 and ℓ are non-negative. Hence there are no such modes possible. The calculation for the two other cases obtains the exact same conclusion.

For modes having $\Omega \sim \mathcal{O}(n)$, we again have three cases. First with both $\lambda, \ell \sim \mathcal{O}(1)$, second having $\lambda \sim \mathcal{O}(1)$ and $\ell \sim \mathcal{O}(n)$ and third with $\lambda \sim \mathcal{O}(n)$ and $\ell \sim \mathcal{O}(1)$. Here we shall first look at the quasinormal mode frequencies for the second case. The leading order equations and their solution in this case are exactly the equations (4.19) and (4.20) and their solutions (4.21) with $\hat{\lambda} = 0$. Hence the value of Ω_0 is (4.39) with λ set to zero.

4.5 Scalar Non-Decoupled Quasinormal Modes

For the scalar perturbations, we need to analyze the coupled system of equations for ϕ, ψ and η (3.41)–(3.43). Looking at the vector quasinormal mode calculations, for non-decoupled modes with $\Omega, \lambda, \ell \sim \mathcal{O}(n)$, we see that the leading order equations for the quasinormal mode calculations are exactly the equations we obtained for analyzing stability of the black string. For the scalar perturbations, to analyze stability, we defined new variables $H, G, \tilde{\eta}$ by taking linear combinations of (ψ, ϕ, η).

$$H = \psi + b^2\phi, \quad G = \psi - b^2\phi \quad \tilde{\eta} = \eta b^2 \tag{4.77}$$

The equation of H decouples but the equations for $G, \tilde{\eta}$ remain coupled. We shall skip the details of the calculation here. The H equation (3.56), can be solved completely in the near region. The normalizable solution at the horizon according to the boundary condition (4.23) is,

$$H = C_1 R(R-1)^{\frac{1}{2}+\hat{\Omega}_0 b} F(p, q, 1 + 2\hat{\Omega}_0 b, 1 - R) \tag{4.78}$$

where

$$\begin{aligned} p &= \frac{1}{2}\left[3 + 2\hat{\Omega}_0 b - \sqrt{1 + 4(\hat{k}^2 + \hat{\lambda}_0^2 b^2) + 4\hat{\Omega}_0^2 b^2}\right] \\ q &= \frac{1}{2}\left[3 + 2\hat{\Omega}_0 b + \sqrt{1 + 4(\hat{k}^2 + \hat{\lambda}_0^2 b^2) + 4\hat{\Omega}_0^2 b^2}\right] \end{aligned} \tag{4.79}$$

and $\hat{k}^2 = \hat{\ell}^2 + \hat{\ell}$. $\hat{\Omega}_0$ and $\hat{\lambda}_0$ are leading order values of Ω and λ. To calculate the Wronskian, we extend this solution to the overlap region.

$$H = C_1 \left[\frac{\Gamma(p+q-2)\Gamma(q-p)}{\Gamma(q)\Gamma(q-2)} R^{\frac{\sqrt{1+4\hat{\Omega}_0^2 b^2+4\hat{k}^2+4\hat{\lambda}_0^2}}{2}} + \frac{\Gamma(p+q-2)\Gamma(p-q)}{\Gamma(p)\Gamma(p-2)} R^{-\frac{\sqrt{1+4\hat{\Omega}^2 b^2+4\hat{k}^2+4\hat{\lambda}_0^2}}{2}} \right] \tag{4.80}$$

Equations for $(G, \tilde{\eta})$ (3.65) and (3.66) do not decouple. Using the arguments in the section (ref) it can be shown that the normalizable solution to G and η in the overlap region will be of the form

$$G = \eta = a_1 R^{\frac{\sqrt{1+4\hat{\Omega}_0^2 b^2+4\hat{k}^2+4\hat{\lambda}_0^2}}{2}} + a_2 R^{-\frac{\sqrt{1+4\hat{\Omega}_0^2 b^2+4\hat{k}^2+4\hat{\lambda}_0^2}}{2}} \tag{4.81}$$

The constants a_1 and a_2 are both non-zero.

In the far region, taking $f(r) \approx 1$ and neglecting terms having with $f'(r)$ and $f''(r)$ in equations (3.41)—(3.43), we get the set of equations (3.44)–(3.45). The equation for ϕ decouples and its solutions are given in terms of modified Bessel functions. Outgoing solution at infinity is of the form

$$\phi = D_2 \sqrt{r} K_\nu(\sqrt{\lambda^2 + \Omega^2} r). \tag{4.82}$$

This solution can be used as a source term in the equations for ψ and η. The solution for ϕ when extended to the overlap region takes the form

$$\phi = D_0 R^{-\frac{\sqrt{1+4(\hat{k}^2+\hat{\lambda}_0^2 b^2)+4\hat{\Omega}_0^2 b^2}}{2}} \tag{4.83}$$

Now we calculate the Wronskian of this solution with the near region solution for ϕ obtained by linear combination of (4.80), (4.81). The zeros of the Wronskian give

$$\hat{\Omega}_0 b = \frac{i\sqrt{(1+2\hat{\ell})^2 + 4\hat{\lambda}^2 b^2}}{2} \tag{4.84}$$

$$\omega_0 b = \frac{(n-1)\sqrt{(1+2\hat{\ell})^2 + 4\hat{\lambda}^2 b^2}}{2} \tag{4.85}$$

Note that the leading order quasinormal mode frequency in both the vector case (4.40) and scalar case (4.85) is equal. We will obtain the same condition by solving the equations for G and η. For $\lambda = 0$, we obtain the leading order quasinormal frequency of the Schwarzschild black hole.

For decoupled modes, these equations are difficult to work with since the quasinormal mode is sub-leading in D and at that order, all the equations (3.41)—(3.43) are coupled.

4.6 Quasinormal Modes of Schwarzschild Black Hole

The decoupled quasinormal modes of Scwarzschild black holes have previously been calculated by assuming the standard $1/D$ expansion [31, 48]. In this section, we evaluate the vector quasinormal modes of D-dimensional Schwarzschild-Tangherlini black holes in the large D limit without a $1/D$ expansion for the quasinormal mode functions or the mode frequency $\omega \sim \mathcal{O}(1)$. Our aim is to understand the nature of the large D limit of the perturbation equation and its solutions.

For this calculation, we shall use the perturbation equations for black hole by Ishibashi and Kodama [24]. Denoting the master variable corresponding to vector perturbations as Ψ_V, the equation governing vector perturbations is

$$\frac{d^2}{dr_*^2}\Psi_V + (\omega^2 - V_V)\Psi_V = 0. \tag{4.86}$$

Here $dr_* = \frac{dr}{f(r)}$ and

$$V_V = \frac{(D-3)^2 f(r)}{4r^2}\left[\left(1+\frac{2\ell}{D-3}\right)^2 - \frac{1}{(D-3)^2} - 3\left(1+\frac{1}{D-3}\right)^2\left(\frac{b}{r}\right)^{D-3}\right]. \tag{4.87}$$

For a D dimensional black hole, $f(r) = \left(1 - \frac{b^{D-3}}{r^{D-3}}\right)$. We change to coordinate $R =$

$(\frac{r}{b})^{D-3}$. From (4.86), the equation for $\Psi_V(R)$ is:

$$R(1-R)\Psi_V''(R) + \left[-R + \frac{R-1}{D-3}\right]\Psi_V'(R) + \left[\frac{1}{4}\left(1+\frac{2\ell}{D-3}\right)^2 - \frac{1}{4}\frac{1}{(D-3)^2} - \frac{3}{4R}\left(1+\frac{1}{D-3}\right)^2 - \frac{\omega^2 r^2}{(D-3)^2} - \frac{\omega^2 r^2}{(D-3)^2(R-1)}\right]\Psi_V = 0. \tag{4.88}$$

4.6.1 Near Region analysis

This equation cannot be solved exactly for the entire range of $1 \leq R < \infty$ due to the fact that we need to write r as a function of R in (4.88) using the near region expansion 3.6, which contains $\log R$ terms. Recall that the near region expansion of r in terms of R is obtained simply by assuming $r - b \ll b$.

In this region,

$$\begin{aligned} \ln R &= (D-3)\ln\left(\frac{r}{b}\right) = (D-3)\ln\left(1+\frac{r-b}{b}\right) \\ &\sim (D-3)\left(\frac{r-b}{b}\right). \end{aligned} \tag{4.89}$$

Inverting this relation leads to near region expansion of r in terms of R. Hence we can approximate r by b in (4.88) as the error $(r-b) \ll b$.

In this analysis, unlike the previous black string case, we will not explicitly expand the function Ψ_V as a series in $1/D$ in the perturbation equation. Instead, in this section, we will derive the approximate equation valid in the entire near region. Let us now write $\Psi_V = R^\alpha (R-1)^\beta \chi$ where

$$\alpha = \frac{3}{2}\left(1+\frac{1}{D-3}\right); \quad \beta = \frac{i\omega b}{D-3}. \tag{4.90}$$

The equation for χ reduces in the near region to a hypergeometric differential equation, using which we can write in this region;

$$\begin{aligned}
\Psi_V &= R^{\alpha}(R-1)^{\beta}\Big[c_1 F(a,\tilde{b},a+\tilde{b}-c+1;1-R) \\
&+ c_2(1-R)^{c-a-\tilde{b}}F(c-a,c-\tilde{b},c-a-\tilde{b}+1;1-R)\Big]; \\
a &= \alpha+\beta-\frac{1}{2(D-3)}+\sqrt{\bar{\omega}_\ell^2-\frac{\omega^2 b^2}{(D-3)^2}}; \\
\tilde{b} &= \alpha+\beta-\frac{1}{2(D-3)}-\sqrt{\bar{\omega}_\ell^2-\frac{\omega^2 b^2}{(D-3)^2}}; \\
c &= 2\alpha-\frac{1}{D-3}.
\end{aligned} \tag{4.91}$$

Here, $\bar{\omega}_\ell = (\frac{1}{2}+\frac{\ell}{D-3})$.

In the case of non-decoupled modes we have to neglect the $1/D$ corrections in the constants. This is due to the fact that in this case, we are neglecting the $\log R$ terms which multiply ω^2 in the equation and occur at $\mathcal{O}(1/D)$. To employ the Nollert-Schmidt procedure, we replace $\omega = -i\Omega$. The bounded solution at the horizon has $c_2 = 0$. Let us call this solution y_-. In order to calculate the Wronskian, this solution is extended to the overlap region. Using standard formulae and taking R large, the leading behaviour of the solution is

$$\begin{aligned}
y_- \sim &\Bigg[\frac{\Gamma(a+\tilde{b}+1-c)\Gamma(\tilde{b}-a)}{\Gamma(\tilde{b})\Gamma(\tilde{b}-c+1)}R^{\frac{1}{2(D-3)}-\sqrt{\bar{\omega}_\ell^2+\frac{\Omega^2 b^2}{(D-3)^2}}} \\
&+\frac{\Gamma(a+\tilde{b}+1-c)\Gamma(a-\tilde{b})}{\Gamma(a)\Gamma(a-c+1)}R^{\frac{1}{2(D-3)}+\sqrt{\bar{\omega}_\ell^2+\frac{\Omega^2 b^2}{(D-3)^2}}}\Bigg].
\end{aligned} \tag{4.92}$$

4.6.2 Far Region Analysis

In the far region we approximate $f(r) \approx 1$ and $dr_* = dr$. The far region equation becomes,

$$-\frac{d^2\Psi_V}{dr^2}+\frac{(D-3)^2}{4r^2}\left[\left(1+\frac{2\ell}{D-3}\right)^2-\frac{1}{(D-3)^2}\right]\Psi_V = -\Omega^2\Psi_V \tag{4.93}$$

Solutions of this equation are the modified Bessel functions of order $\nu = \frac{D-3}{2}\left(1 + \frac{2\ell}{(D-3)}\right)$.

$$\Psi_V = (\Omega r)^{1/2}\left[d_1 I_\nu(\Omega r) + d_2 K_\nu(\Omega r)\right] \tag{4.94}$$

The normalizable solution is with $d_1 = 0$. We again use the uniform asymptotic expansion of the modified Bessel functions to extend the solution to the overlap region. As in the black string case, the leading order far region solution, denoted by y_+ is

$$y_+ = \tilde{C} R^{\frac{1}{2(D-3)} - \sqrt{\bar{\omega}_\ell^2 + \frac{\Omega^2 b^2}{(D-3)^2}}} \tag{4.95}$$

We note that the solution y_+ decays as $r \to \infty$ and it is impossible for its asymptotic expansion to be contaminated by the other growing solution. We also observe that the exponent contains a term of order $1/D$, namely $\frac{1}{2(D-3)}$. This also matches exactly with one of the linearly independent near region solutions in (4.92).

If we had worked with the perturbation equation (4.93) with the quasinormal mode frequency ω and picked the outgoing solution at ∞ instead of the Laplace transform approach, our solutions would be given in terms of Bessel functions instead of modified Bessel functions. The outgoing solution at ∞ in this case would be a Hankel function. For complex frequencies ω, the outgoing mode ($e^{i\omega(t-r_*}$) is non-normalizable, whereas the ingoing mode ($e^{i\omega(t+r_*}$) is exponentially decaying. A small contamination of the outgoing mode by an ingoing piece is does not affect the leading terms in the asymptotic expansion. The asymptotic expansion of the Hankel function differs in different domains of the complex plane, and in the overlap region of two domains, we have two different expansions for the Hankel function which differ by exponentially decaying pieces (c.f p.238-240, [95]). However, when continuing the asymptotic expansion to the overlap region of the near and the far region, we have to be cautious. In the overlap region, the ingoing piece becomes significant. For this reason, we prefer the Laplace transform approach to obtain quasinormal mode frequencies.

4.6.3 Non-decoupled quasinormal modes

We have now obtained the form of both the solutions f_-, bounded at the horizon, and f_+, bounded at ∞ in the overlap region, given by (4.92) and (4.95) respectively. We look for complex values of Ω when their Wronskian is zero. One way this can happen is that the coefficient of the term increasing in R in (4.92) must go to zero. The coefficient is $\frac{\Gamma(a+\tilde{b}+1-c)\Gamma(a-\tilde{b})}{\Gamma(a)\Gamma(a-c+1)}$, which can only go to zero at the poles of the Gamma functions in the denominator. This possibility is ruled out in this case. The only other way for f_- and f_+ to be linearly dependent is the limiting case $\sqrt{\bar{\omega}_\ell^2 + \frac{\Omega^2 b^2}{(D-3)^2}} \to 0$, which happens for $\Omega b \to i(D-3)\bar{\omega}_\ell$. This corresponds precisely to the leading order quasinormal mode computed in [30], $\omega b = (D-3)\bar{\omega}_\ell$. It is important to note that this is true only in a limiting sense as the we only obtain leading order frequency. However, in this approach we cannot go beyond leading order in D in the computation of the non-decoupled quasinormal modes.

As shown for the black string case, we can put a bound on the D dependence of the correction to Ω just by looking at the behaviour of the asymptotic solution extended to the overlap region. Let $\omega = D\hat{\omega}$. We can conjecture that next order correction to the quasinormal modes can be written as $(D-3)[\hat{\omega}_0 + \hat{\omega}_1/(D-3)^k]$ and look at the behaviour of corrections to the leading order f_+ for $\omega_0 b = (D-3)\bar{\omega}_\ell$. As argued in section (4.2.4), we see that from the far region, for a range of k, the terms in the asymptotic expansion for the modified Bessel function become ill-defined in the overlap region upon plugging in the leading order value for ω_0 that we have obtained in the large D limit. Looking at the correction terms, the necessary condition for the asymptotic expansion of the modified Bessel to converge is $k \leq 2/3$.

The correction beyond leading order has been computed analytically in [31], [32] to be $\mathcal{O}(D^{\frac{1}{3}})$, whereas numerical results predict a correction $\mathcal{O}(D^{\frac{1}{2}})$ [34].

4.6.4 Decoupled modes

The decoupled modes $\omega \sim \mathcal{O}(1)$ are ingoing at the horizon and decaying far from the horizon for large R. For these modes $\ell \sim \mathcal{O}(1)$. From (4.92), we observe that the solution that is ingoing at the horizon has a piece growing as $R^{\frac{1}{2}}$ in the overlap region at leading order in

D. The coefficient of this piece must vanish for the decoupled quasinormal mode i.e. the Gamma functions in the denominator must have a pole. This leads to two possibilities: $a = -m$ or $a - c + 1 = -m$ where m is a non-negative integer. There is no decoupled mode corresponding to the first possibility. But there is a mode for $a - c + 1 = 0$.

We have

$$-\frac{1}{2} - \frac{1}{(D-3)} + \frac{\Omega b}{(D-3)} + \frac{1}{2}\sqrt{\left(1 + \frac{2\ell}{(D-3)}\right)^2 + \frac{4\Omega^2 b^2}{(D-3)^2}} = -m \tag{4.96}$$

For $\Omega, l \sim \mathcal{O}(1)$, the square root can be approximated by a series.

$$-\frac{1}{2} - \frac{1}{(D-3)} + \frac{\Omega b}{(D-3)} + \frac{1}{2}\left[1 + \frac{2\ell}{(D-3)} + \frac{2\Omega^2 b^2}{(D-3)^2}\right] = -m \tag{4.97}$$

This equation then has a solution only for $m = 0$. This is the decoupled mode with frequency upto $\mathcal{O}(\frac{1}{(D-3)})$

$$\omega = i\left[(\ell - 1) + \frac{1}{(D-3)}(\ell - 1)^2\right]. \tag{4.98}$$

This answer agrees with past numerical work by Dias, Hartnett and Santos [34]. It also agrees with the computation of Emparan,Suzuki and Tanabe in the $1/D$ expansion [31] to $\mathcal{O}(\frac{1}{(D-3)})$.Our computation only uses the approximation $(r - b) << b$ without an explicit $1/D$ expansion for mode functions or frequencies. Here we neglect terms of $\mathcal{O}(\frac{1}{(D-3)^3})$ as they are multiplied by $\ln R$ factors which make the equation hard to solve. These neglected terms possibly have an impact on the quasinormal mode frequencies at $\mathcal{O}(\frac{1}{(D-3)^2})$ and we are unable to calculate these with our methods. Thus our answer yields the quasinormal mode to next-to-leading order.

Taking a large D limit of the hypergeometric equation for the mode function, we find that the parameters in the leading order lead to a degenerate case of the hypergeometric equation in this limit. The general solution (4.91) to the complete equation beyond leading order cannot be analytically expanded about the solution to the leading order degenerate equation, as it has a logarithmic singularity, in powers of $1/D$. Out of the two linearly independent solutions, the problem lies with the solution which is *outgoing* at the horizon.

We would like our solution to be ingoing at the horizon. However, upon choosing the ingoing solution, the problem recurs in the far limit (4.92). One of the Gamma functions in (4.92) cannot be naively expanded about the leading order expression in powers of $1/D$. The reason is that the leading order expression is a pole of the Gamma function (at leading order, $\tilde{b} - a = -1$). This is a manifestation of the fact that the hypergeometric equation reduces to a degenerate case at leading order. In [31], in a $1/D$ expansion, since the ingoing solution is chosen and the quasinormal mode identified at the horizon, this does not affect the computation.

4.7 Scalar quasinormal modes

We investigate the equations governing scalar quasinormal modes next. As shown in [24], these can be reduced to one Schrodinger-type equation

$$\frac{d^2}{dr_*^2}\Psi_S + (\omega^2 - V_S)\Psi_S = 0. \tag{4.99}$$

Here, the effective potential V_S is given by

$$V_S(r) = \frac{f(r)Q(r)}{16r^2H^2}, \tag{4.100}$$

with

$$\begin{aligned} Q(r) =& n^4(n+1)^2x^3 + n(n+1)\left[4(2n^2-3n+4)m + n(n-2)(n-4)(n+1)\right]x^2 \\ &- 12n\left[(n-4)m + n(n+1)(n-2)\right]mx + 16m^3 + 4n(n+2)m^2, \end{aligned} \tag{4.101}$$

$$H(r) = m + \frac{1}{2}n(n+1)x; \quad m = k^2 - n, \quad x = \frac{b^{n-1}}{r^{n-1}}. \tag{4.102}$$

It is not possible to solve (4.99) exactly for all r. We will therefore do a near and far region analysis of this equation. We work with the near region R coordinate. Then the equation obeyed by Ψ_S in the near region is of the form

$$\begin{aligned} &\frac{d^2}{dR^2}\Psi_S + \left[\frac{1}{R-1} - \frac{1}{(D-3)R}\right]\frac{d}{dR}\Psi_S \\ + &\left[\frac{\omega^2 b^2}{(D-3)^2(R-1)^2} - \frac{(c_3R^3 + c_4R^2 + c_5R + c_6)}{4R^2(R-1)(pR+q)^2}\right]\Psi_S = 0. \end{aligned} \tag{4.103}$$

Here, c_3, c_4, c_5, c_6, p, q are constants that depend on the angular momentum mode ℓ and dimension D.

$$p = \frac{2(-1+\ell)}{(D-3)} + \frac{2\left(-1+\ell^2\right)}{(D-3)^2} \tag{4.104}$$

$$q = 1 + \frac{3}{(D-3)} + \frac{2}{(D-3)^2} \tag{4.105}$$

Constants c_{3---6} are:

$$\begin{aligned} c_3 =& \frac{4(-1+\ell)^2}{d^2} + \frac{8\left(1+\ell-5\ell^2+3\ell^3\right)}{d^3} + \frac{4\left(10\ell-7\ell^2-16\ell^3+13\ell^4\right)}{d^4} + \\ & \frac{8\left(-1+3\ell+5\ell^2-9\ell^3-4\ell^4+6\ell^5\right)}{d^5} + \frac{4\left(-1+6\ell^2-9\ell^4+4\ell^6\right)}{d^6} \end{aligned} \tag{4.106}$$

$$\begin{aligned} c_4 =& -\frac{12(-1+\ell)}{d} - \frac{12\left(-3+\ell+2\ell^2\right)}{d^2} - \frac{12\left(-4+3\ell-\ell^2+2\ell^3\right)}{d^3} - \\ & \frac{12\left(-4+7\ell-4\ell^3+\ell^4\right)}{d^4} + \frac{12\left(3-4\ell-7\ell^2+6\ell^3+2\ell^4\right)}{d^5} + \\ & \frac{12\left(1-4\ell^2+3\ell^4\right)}{d^6} \end{aligned} \tag{4.107}$$

$$\begin{aligned} c_5 =& 1 + \frac{2(-3+4\ell)}{d} + \frac{4\left(-11+7\ell+2\ell^2\right)}{d^2} + \frac{2\left(-45+20\ell+14\ell^2\right)}{d^3} \\ & + \frac{-89+44\ell+40\ell^2}{d^4} + \frac{4\left(-12+6\ell+11\ell^2\right)}{d^5} + \frac{12\left(-1+2\ell^2\right)}{d^6} \end{aligned} \tag{4.108}$$

$$c_6 = 1 + \frac{8}{d} + \frac{26}{d^2} + \frac{44}{d^3} + \frac{41}{d^4} + \frac{20}{d^5} + \frac{4}{d^6} \tag{4.109}$$

The equation (4.103) has four regular singular points at $R = -q/p,\ 0,\ 1,\ \infty$ and can therefore be rewritten as a Heun differential equation. However, the solutions to this equation are harder to analyze. Specifically, it is difficult to find solutions satisfying specified boundary conditions at both the horizon and infinity. This is because unlike the hypergeometric equation, whose solutions around the singular points are connected by linear transformations, a similar result, termed the 'connection problem' for the Heun equation is as yet, unsolved except in specific cases.

To get the R equation in the form of a Heun equation we define

$$\Psi_S = R^{\alpha}(R-1)^{\beta}(pR+q)^{\gamma}\chi \tag{4.110}$$

The equation for χ is

$$\frac{d^2\chi}{dR^2} + \left[\frac{2\alpha+d_1}{R} + \frac{1+2\beta}{R-1} + \frac{2\gamma}{R+\frac{q}{p}}\right]\frac{d\chi}{dR} + \frac{ABR-C}{R(R-1)(R+\frac{q}{p})}\chi = 0 \tag{4.111}$$

Here,

$$\alpha = \frac{1}{2} + \frac{1}{2(D-3)} \qquad \beta = \frac{i\omega b}{(D-3)} \qquad \gamma = 2 \qquad d_1 = -\frac{1}{(D-3)} \tag{4.112}$$

The constants A, B, C are

$$A + B = 2(\alpha+\beta+\gamma) + d_1 \tag{4.113}$$

$$AB = (\alpha+\gamma)(1+2\beta) + 2\alpha\gamma + (\gamma+\beta)d_1 + \frac{(c_5+c_6)}{4pq} - \frac{c_6}{2q^2} - \frac{(c_3+c_4+c_5+c_6)}{4p(p+q)} \tag{4.114}$$

$$C = -\frac{\alpha(1+2\beta)q}{p} + 2\alpha\gamma - \frac{\beta d_1 q}{p} + \gamma d_1 + \frac{c_5+c_6}{4pq} - \frac{c_6}{2q^2} \tag{4.115}$$

At leading order in D, the various constants in the equation can be evaluated, and $p \to 0$ in this limit ($q \to 1$). This results from setting $\ell \sim \mathcal{O}(1)$ for decoupled modes. The singular points in the Heun equation are at $R = -q/p,\ 0,\ 1,\ \infty$, and in this limit, the singular point at $-q/p$ approaches the point at infinity.[1] We now examine this limit carefully. We will employ a rescaling that is often used to obtain the confluent Heun equation from the Heun equation. We define a new coordinate $\bar{R}$ by $R = \frac{-q}{p}\bar{R}$. We also define the constant $\bar{C}$ by $C = \frac{-q}{p}\bar{C}$. We first perform the rescaling and then take the limit $p \to 0$ and $q \to 1$ (the

[1] For non-decoupled modes for which $\ell \sim \mathcal{O}(D)$, we do not have $p \to 0$.

values at leading order in D). This yields the equation

$$\frac{d^2\chi}{d\bar{R}^2} + \left[\frac{2\alpha + d_1 + 1 + 2\beta}{\bar{R}} + \frac{2\gamma}{\bar{R}-1}\right]\frac{d\chi}{d\bar{R}} + \frac{AB\bar{R} - \bar{C}}{\bar{R}^2(\bar{R}-1)}\chi = 0. \tag{4.116}$$

However, we also need to evaluate all the constants in this equation in the large D limit. We find that in this limit, $\bar{C} = 0$ in (4.116). The equation (4.111),upon taking the limit $D \to \infty$ reduces to a hypergeometric equation

$$\frac{d^2\chi}{d\bar{R}^2} + \left[\frac{2\alpha + d_1 + 1 + 2\beta}{\bar{R}} + \frac{2\gamma}{\bar{R}-1}\right]\frac{d\chi}{d\bar{R}} + \frac{AB}{\bar{R}(\bar{R}-1)}\chi = 0. \tag{4.117}$$

It must be noted that all the constants in (4.117) take their leading order in D values, and do not contain the quasinormal mode frequency ω which is not of $\mathcal{O}(D)$. Thus this equation cannot be used to obtain the decoupled mode frequency ω as this equation is valid only at leading order.

For any finite D, in the Heun equation (4.111), the singular point at $R = -q/p$ has not actually merged with the singular point at infinity. Rather, they are 'nearby' and merge in the large D limit. To compute decoupled modes with $\omega \sim \mathcal{O}(1)$, if they are present , we need to take into account sub-leading corrections in D in the various constants in (4.111). We re-write the equation in terms of $z = \frac{1}{R}$. In the new coordinates, the black hole horizon lies at $z = 1$ and infinity is mapped to $z = 0$.

The singularity at $z = -\frac{p}{q}$ tends to zero in the decoupled mode case in the large D limit. To get the equation (4.111) in canonical Heun form in terms of z, we define $\chi = z^\delta \Phi$. Then, for an appropriate value of δ, Φ obeys a Heun equation

$$\frac{d^2\Phi}{dz^2} + \left[\frac{2\delta + 1 - 2(\alpha+\beta+\gamma) - d_1}{z} + \frac{1+2\beta}{z-1} + \frac{2\gamma}{z+\frac{p}{q}}\right]\frac{d\Phi}{dz} + \frac{(MNz - L)\Phi}{z(z-1)\left(z+\frac{p}{q}\right)} = 0 \tag{4.118}$$

In terms of the constants in the original equation (4.111),

$$\delta = \frac{1}{2}\left[2(\alpha+\beta+\gamma) + d_1 + \sqrt{(2(\alpha+\beta+\gamma)+d_1)^2 - 4AB}\right] \tag{4.119}$$

$$L = -\frac{p}{q}C + \delta\left(2\gamma - \frac{p(1+2\beta)}{q}\right) - AB\left(1 - \frac{p}{q}\right) \tag{4.120}$$

$$M = \delta; \quad N = \delta + 1 - (2\alpha + d_1) \tag{4.121}$$

We are interested in finding those quasinormal mode frequencies for which the mode function is ingoing at the horizon and bounded at ∞ (since we are working with the near region equation, we are interested in solutions that decay for large R). We let $\Omega = i\omega$ and map an ingoing solution at the horizon to a bounded solution. Then, we are interested in finding Ω for which the solution to (4.118) is bounded both at $z = 0$ and $z = 1$. The complication is that Ω is sub-leading in D, but in the large D limit, the singular point at $z = -\frac{p}{q} \to 0$ and merges with the singular point at $z = 0$. Unfortunately, results on merging singular points of the Heun equation use an expansion of the solution as a series of hypergeometric functions. At leading order in D, some parameters of these hypergeometric functions become degenerate and it is not possible to use these results. It is hence not possible to compute the quasinormal modes without assuming a $1/D$ expansion as was done for the vector case. The decoupled scalar quasinormal modes have been evaluated in a $1/D$ expansion in [31]. This reproduces past numerical results for these modes in [34] at finite (large) D at leading order and computes higher order corrections.

For non-decoupled quasinormal modes, we recover the leading order frequency by taking $\lambda = 0$ in the black string solution.

4.8 Summary and Discussion

In this chapter, we have obtained quasinormal mode frequencies for black string and revisited the analysis of quasinormal of D-dimensional Schwarzschild-Tangherlini black holes in the large D limit. In this limit, there are two distinct types of quasinormal modes (i) decoupled modes with $\omega \sim \mathcal{O}(1)$ and (ii) non-decoupled modes with $\omega \sim \mathcal{O}(D)$.

For the black string case, we have obtained the vector decoupled quasinormal mode frequency to leading order. We see that the effect of the mode depending on the extra dimension, signified by λ will be seen only from the next to leading order.

We have calculated both vector and scalar non-decoupled modes for the black string. We have used the method of Laplace Transforms by Nollert and Schmidt for this purpose. We have obtained the leading order frequencies in both the cases. We discuss the limitations of the Laplace transform method to obtaining the next to leading order corrections to the non-decoupled modes. For $\lambda = 0$ our results match the leading order frequency obtained in [31].

We have addressed the problem of black hole quasinormal modes without assuming an expansion of the mode function as a series in $1/D$.

For the vector decoupled quasinormal modes, the perturbation equation reduces to a degenerate case of the hypergeometric equation at leading order in D and its general solution cannot be obtained from that of the leading order equation as a series in $1/D$. We compute these modes without an assumption of a series expansion. We obtain an equation for the decoupled vector quasinormal modes from which we evaluate the modes to $\mathcal{O}(\frac{1}{(D-3)^2})$. It agrees with previous computations in the $1/D$ expansion [31] as well as numerical results [34] up to the next to leading order.

We study the equation governing scalar quasinormal modes. This is a Heun equation and hard to analyze analytically as relatively little is known about solutions to Heun equations. For decoupled modes, in the large D limit, two of the singular points of the associated Heun equation are 'nearby' and merge in the $D \to \infty$ limit. Due to this reason, it is not possible to compute the decoupled scalar quasinormal modes without a $1/D$ expansion as done in [31]. For nondecoupled scalar modes, an analysis of the Heun equation is difficult.

Chapter 5

Semiclassical stability of SAdS black holes

In this chapter we study the semiclassical stability of D-dimensional Schwarzschild-AdS (SAdS) black holes. For a certain class of non-spherically symmetric scalar perturbations, we prove that they do not lead to instability for all values of D for both small and large black holes. In some cases of non-spherically symmetric scalar perturbations, we show the large black holes are semiclassically stable in the large D limit. We then analyze the spherically symmetric perturbations and show that in the large D limit, the large black holes are semiclassically stable. For small black holes, which do have an unstable mode, we find the eigenvalue corresponding to this mode in a $1/D$ expansion.

5.1 Semiclassical stability of SAdS black holes

We have discussed the notion of semiclassical stability in detail in the chapter (2). To reiterate, a spacetime is semi-classically unstable if there exist perturbations that reduce the action from its background value. Computationally, such perturbations are normalizable solutions to the eigenvalue equation

$$2\delta G_{\mu\nu} + 2\Lambda h_{\mu\nu} = \lambda h_{\mu\nu} \tag{5.1}$$

with $\lambda < 0$. Here $\delta G_{\mu\nu}$ is variation of Einstein tensor evaluated for transverse-traceless perturbations $h_{\mu\nu}$.

Existence of such perturbations for Schwarzschild black holes in four dimension was shown by Gross, Perry and Yaffe in [83]. This negative mode is shown to be spherically symmetric. The eigenvalue of this mode for higher dimensional Schwarzschild-Tangherlini black holes was calculated in [29]. Absence of such unstable modes for non-spherically symmetric perturbations has been shown in Chapter 3 in the large D limit. In this chapter we investigate the problem of semi-classical instability of the Schwarzschild-AdS black holes in the large D limit. This analysis has been performed, for spherically symmetric perturbations, analytically for four dimensional SAdS black holes and numerically for n-dimensional SAdS black holes by Prestidge in [91]. The large black holes ($r\sigma \gg 1$, σ is length scale associated with the cosmological constant (2.30)) are found to be stable whereas small black holes with ($r\sigma \ll 1$) are unstable. For n-dimensional black holes, Prestidge found a bound on the eigenvalue corresponding to the unstable mode for small black hole to be $\lambda < (n^2/4)$.

Thermodynamic properties of black holes in finite isothermal cavities and their relation to the semiclassical stability of the black holes was studied by Whiting and York in [86, 87, 88]. In this setup sign of the second variation of the reduced Euclidean action with respect to the horizon radius r_+ is same as the sign of the heat capacity of the black hole [88]. A black hole is thermodynamically unstable if it has negative specific heat. In the case of SAdS black holes, the dynamical (semiclassical) (in)stability mimics the features of thermodynamical (in)stability of the (small) large SAdS black holes found by Hawking and Page in [90]. In [91] Prestidge shows that the eigenvalue λ in equation (5.1) undergoes a transition from being positive for large black holes to negative for small black holes. These results are obtained numerically and there is no analytical formula obtained for the eigenvalue. We shall use the large D limit as an analytical tool to study this problem.

The SAdS black hole metric in $D = n + 2$ dimensions is given by,

$$g_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_n^2; \tag{5.2}$$

where

$$f(r) = \left(1 - \frac{2\Lambda}{n(n+1)}r^2 - \frac{b^{n-1}}{r^{n-1}}\right). \tag{5.3}$$

We can write $\frac{2\Lambda}{n(n+1)} = -\sigma^2$ such that for AdS spacetime with $\Lambda < 0$, σ is positive. For the case $\Lambda = 0$, $\sigma = 0$. In this notation,

$$f(r) = \left(1 + \sigma^2 r^2 - \frac{b^{n-1}}{r^{n-1}}\right) \tag{5.4}$$

The horizon of the SAdS black hole is at $r = r_+$, where r_+ is defined by $f(r_+) = 0$.

We will concentrate on the static perturbations for our analysis. This is because the Gross-Perry-Yaffe unstable mode for Schwarzschild black hole [83] ($\Lambda = 0$) is static ($\omega = 0$). For static perturbations on SAdS black hole, equations (2.74)–(2.76) for the scalar perturbation variables ψ, ϕ end η reduce to

$$\begin{aligned}
&-\frac{d^2\psi}{dr^2} + \left[\frac{n^3 - 2n^2 + 8n - 8}{4nr^2} + \frac{f'^2}{4f^2} - \frac{(n^2+2n-4)}{2n}\frac{f'}{fr} - \frac{f''}{2f}\right. \\
&\left. + \frac{2(n^2-1)\sigma^2}{nf} - \frac{2(n-1)}{nr^2 f} + \frac{k^2}{fr^2} + \frac{\lambda}{f}\right]\psi = \\
&\left[\frac{2(n-1)}{nf} + \frac{2}{n} - \frac{n+2}{n}\frac{rf'}{f} - \frac{r^2 f''}{f} + \frac{f'^2 r^2}{2f^2} + \frac{2(n+1)}{n}\frac{\sigma^2 r^2}{f}\right]\phi
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
&-\frac{d^2\phi}{dr^2} + \left[\frac{n^3 - 2n^2 + 8n - 8}{4nr^2} + \frac{f'^2}{4f^2} - \frac{(n^2+2n-4)}{2n}\frac{f'}{fr} - \frac{f''}{2f}\right. \\
&\left. + \frac{2(n^2-1)\sigma^2}{nf} - \frac{2(n-1)}{nr^2 f} + \frac{k^2}{fr^2} + \frac{\lambda}{f}\right]\phi = \\
&\left[\frac{2(n-1)}{nr^4 f} - \frac{2(n-1)}{nr^4} - \frac{2-n}{nr^3}\frac{f'}{f} - \frac{f''}{r^2 f} + \frac{f'^2}{2f^2 r^2} + \frac{2(n+1)}{n}\frac{\sigma^2}{r^2 f}\right]\psi
\end{aligned} \tag{5.6}$$

$$-\frac{d^2\eta}{dr^2} + \left[\frac{n^2-2n}{4r^2} - \frac{(n+2)f'}{2rf} + \frac{3f'^2}{4f^2} - \frac{3f''}{2f} + \frac{2(n+1)\sigma^2}{f} + \frac{k^2}{fr^2} + \frac{\lambda}{f}\right]\eta = 0 \tag{5.7}$$

In this chapter we shall replace the eigenvalue λ^2 in the original equations (2.74)–(2.76) by λ for consistency of notation. For stability analysis, we wish to find solutions for these equations with $\lambda > 0$ that are normalizable at both the horizon and infinity. Observe that the η equation decouples but the equations for ψ and ϕ remain coupled. We will analyze the equations for η and the coupled equations for (ϕ, ψ) using two different strategies.

5.2 The η equation

To show the stability of the η equation, we will use the S-deformation argument described in [25],[96]. Let us define $\xi = f^{-\frac{1}{2}}\eta$. The equation for ξ can be written in Schrödinger form as

$$-\frac{d^2\xi}{dr_*^2} + V\xi = 0 \tag{5.8}$$

where

$$V = \frac{(n^2 - 2n)}{4r^2}f^2 - \frac{(n+2)}{2}\frac{f'f}{r} + f'^2 - 2f''f + \frac{k^2}{r^2}f + \lambda f + 2(n+1)\sigma^2 f. \tag{5.9}$$

The tortoise coordinate r_* is defined by $dr_* = \frac{dr}{f(r)}$ and for f given by (5.4) has range $-\infty \leq r_* \leq 0$. We assume λ to be positive. Next we multiply the equation (5.8) by the complex conjugate of ξ and integrate the equation over the range of r_*. The equation (5.8) then becomes

$$-\xi^* \frac{d\xi}{dr_*}\bigg|_{-\infty}^{0} + \int_{-\infty}^{0}\left(\left|\frac{d\xi}{dr_*}\right|^2 + V(r)|\xi|^2\right) dr_* = 0. \tag{5.10}$$

For stability, we want solutions ξ to equation (5.8) which vanishes at both the boundaries. Let us first obtain these solutions. As we cannot fully integrate dr_* to find the tortoise coordinate r_* for the entire range of r, we analyze the equation (5.8) at the horizon $(r = r_+)$ and infinity.

Very near the horizon, we can approximate

$$f(r) \approx f'(r_+)(r - r_+) = \frac{(n+1)\sigma^2 r_+^2 - 2}{r_+}(r - r_+) = \alpha(r - r_+) \tag{5.11}$$

Consequently, very near the horizon r_* becomes

$$r_* = \int \frac{dr}{f(r)} \approx \frac{\ln(r - r_+)}{\alpha} \tag{5.12}$$

In this limit as $r \to r_+$, the coordinate $r_* \to -\infty$. Therefore, very near the horizon the potential (5.9) can be approximated by constant $v(r) \approx f'(r_+)^2 = \alpha^2$. The Schrödinger equation for ξ in this limit is,

$$-\frac{d^2\xi}{dr_*^2} + \alpha^2\xi = 0 \tag{5.13}$$

Solutions to the above equation are $\xi = c_1 e^{\alpha r_*} + c_2 e^{-\alpha r_*}$. As $r_* \to -\infty$, the second solution $e^{-\alpha r_*} \to \infty$ whereas $e^{\alpha r_*} \to 0$. The normalizable solution ξ at the horizon is

$$\xi = c_1 e^{\alpha r_*} = c_1(r - r_+) \tag{5.14}$$

Let us now find r_* near the infinity. In this limit $r_* = \frac{1}{\sigma r}$. As $r \to \infty$, $r_* \to 0$. Substituting $f(r)$ in the potential (5.9) equation for ξ near $r \to \infty$ becomes

$$-\frac{d^2\xi}{dr_*^2} + \left(\frac{n^2}{4} + \frac{n}{2} + \frac{\lambda}{\sigma^2}\right)\frac{1}{r_*^2}\xi = 0 \tag{5.15}$$

Solutions to this equation are

$$\xi = d_1 r_*^{\frac{1}{2}+\frac{1}{2}\sqrt{(n+1)^2+\frac{4\lambda}{\sigma^2}}} + d_2 r_*^{\frac{1}{2}-\frac{1}{2}\sqrt{(n+1)^2+\frac{4\lambda}{\sigma^2}}} \tag{5.16}$$

For $\lambda > 0$ and $n \geq 2$ the quantity $(n+1)^2 + \frac{4\lambda}{\sigma^2} > 1$. The normalizable solution at $r_* = 0$ is thus obtained by $d_2 = 0$. The normalizable solution $\xi \to 0$ as $r \to \infty$.

$$\xi = d_1 r_*^{\frac{1}{2}+\frac{1}{2}\sqrt{(n+1)^2+\frac{4\lambda}{\sigma^2}}} \tag{5.17}$$

We have now shown that at both the horizon and infinity, there are solutions to equation for ξ that vanish at the boundaries. For normalizable solutions, the boundary term in the equation (5.10) is zero. However from the equation (5.10), it is obvious that there cannot exist normalizable solutions for any $V(r) > 0$. Thus to establish stability of the spacetime, one needs to show that $V(r)$ is positive .

Even if $V(r)$ has a small region where it is negative, in some cases, one can use the S-deformation method to show stability. Here we give a brief outline of the method. As shown in [25], [96] we define a new operator with an arbitrary continuous function $S(r_*)$

$$D = \frac{d}{dr_*} + S(r_*). \tag{5.18}$$

Given these conditions, for normalizable solutions to the equation (5.8), we can write

$$-\left[\xi^* \frac{d\xi}{dr_*} + S|\xi|^2\right]_{-\infty}^{0} + \int_{-\infty}^{0} \left(\left| \frac{d\xi}{dr_*} + S\xi \right|^2 + \left(V(r) + \frac{dS}{dr_*} - S^2 \right) |\xi|^2 \right) dr_* = 0 \tag{5.19}$$

The function S is such that the boundary terms vanish for the normalizable solutions ξ. To show stability it suffices to show that the deformed potential

$$\tilde{V} = V(r) + \frac{dS}{dr_*} - S^2 \tag{5.20}$$

is positive for the aforementioned boundary conditions. This can be done by choosing S appropriately. In our case we find

$$S = f'(r) = 2\sigma^2 r + (n-1)\frac{b^{n-1}}{r^n}. \tag{5.21}$$

The chosen S obeys the all the requirements of this method. As $f(r) \geq 0$ and $f'(r) > 0$ the deformed potential $\tilde{V} > 0$.

$$\tilde{V} = \frac{(n^2-2n)}{4r^2} f^2 + \frac{(n-2)}{2}\frac{f' f}{r} + \frac{k^2}{r^2} f + \lambda f \tag{5.22}$$

This argument is valid for all values of n and is true for all the values of black hole parameters. The argument relies on the assumption $\lambda > 0$. This implies there are no normalizable solutions to η for both small and large black holes with $\lambda > 0$.

This analysis is difficult to extend to the coupled equations for ϕ, ψ [1]. Here we have to resort to the large n limit in which these equations decouple.

[1] See [97] for extending the S deformation method to coupled equations.

5.3 The Large n Limit

As in the case of Schwarzschild-Tangherlini black holes, in the large n limit, the function $f(r) = 1 + \sigma^2 r^2 - \frac{b^{n-1}}{r^{n-1}}$ is steeply increasing in the region close to the black hole horizon r_+. This creates two distinct regions in the black hole spacetime, near the black hole where the $f(r)$ rapidly increases and far from the black hole where $f(r)$ can be approximated to that of AdS. In the large n limit, these two regions share an overlap region.

In the case of SAdS black holes, we have an additional length scale appears due to the non-zero cosmological constant, namely $1/\sigma$.

$$\frac{1}{\sigma} = \sqrt{\frac{n(n+1)}{2|\Lambda|}} \sim \frac{n}{\sqrt{2|\Lambda|}} \tag{5.23}$$

In this case, if we keep Λ fixed, in the limit $n \to \infty$, the effect of the cosmological constant vanishes from the geometry. To avoid this situation, we consider Λ such that the σ is finite. Specifically, in all our discussion, we shall treat the quantity $\sigma r_+ \sim \mathcal{O}(1)$ rather than $\sim \mathcal{O}(D)$. Following Hawking [90], even in the large n limit we can define small black holes with $\sigma r_+ < 1$ and large black holes with $\sigma r_+ > 1$.

Following (emparan), we define a *near region* where $r - r_+ \ll r_+$. Far away from the horizon $f(r) \approx 1 + \sigma^2 r^2$. Let us define far region as $r - r_+ \gg \frac{r_+}{n}$. These definitions create an overlap region between the two regions. We can motivate the definitions of far and overlap region by looking at the derivatives of $f(r)$. The first derivative of $f(r)$ has a minimum at $r = r_f$ where r_f is given by $\frac{r_f}{b} = \left(\frac{n(n-1)}{2\sigma^2 b^2}\right)^{\frac{1}{n+1}}$. For this value of r_f the function $f(r)$ can be approximated to $f(r_f) \approx 1 + \sigma^2 r_f^2$, which is the far region approximation. Writing b in terms of r_+ we get,

$$\begin{aligned}\frac{r_f}{r_+} &= \exp\left\{\frac{1}{n+1}\ln\left[\frac{n(n-1)}{2}\left(1 + \frac{1}{\sigma^2 r_+^2}\right)\right]\right\} \\ &\approx 1 + 2\frac{\ln n}{n+1} + \frac{1}{n+1}\ln\left(1 + \frac{1}{\sigma^2 r_+^2}\right) - \frac{\ln 2}{n+1} + \mathcal{O}\left(\frac{1}{n^2}\right)\end{aligned} \tag{5.24}$$

Taking n to be large it can be seen that both in the case of small black holes ($\sigma r_+ < 1$) and large black holes ($\sigma r_+ > 1$) , the quantity $r_f - r_+ \ll r_+$. That is, r_f always lies in the near region. This creates an overlap region. In both the cases r_f can be approximately written as $r_f \approx r_+(1 + 2\frac{\ln n}{n})$.

To re-iterate, the two distinct regions in the space time are defined as

$$\begin{array}{ll} \text{Near region} & r - r_+ \ll r_+ \\ \text{Far region} & r - r_+ \gg \dfrac{r_+}{n} \end{array}$$

With an overlap region given by $\frac{r_+}{n} \ll r - r_+ \ll r_+$.

In terms of the coordinate $R = \frac{r^{n-1}}{b^{n-1}}$, the horizon is situated at R_h. Where R_h is given by

$$R_h = \left(\frac{r_+}{b}\right)^{n-1} = \frac{1}{1 + \sigma^2 r_+^2} \tag{5.25}$$

We again write r in the near region $r - r_+ \ll r_+$ in terms of R by expanding $r = bR^{1/(n-1)}$ around R_h as (5.26).

$$r = r_+ \left[1 + \ln\left(\frac{R}{R_h}\right)\frac{1}{n-1} + \ln\left(\frac{R}{R_h}\right)^2 \frac{1}{2(n-1)^2} + \ldots\right] \tag{5.26}$$

The expression (5.26) is not a $1/n$ expansion as r_+ is the root of relation $r_+ = b(1 + \sigma^2 r_+^2)^{-\frac{1}{n-1}}$ and is n-dependent. In the large n limit we can write,

$$r_+ = b \exp\left[\frac{-\ln(1 + \sigma^2 r_+^2)}{n-1}\right] \sim b\left[1 - \frac{\ln(1 + \sigma^2 r_+^2)}{n-1}\right] \tag{5.27}$$

For $\sigma r_+ \sim \mathcal{O}(1)$, the difference $r_+ - b \sim \mathcal{O}\left(\frac{1}{n}\right)$. Thus in the leading order we can approximate $r_+ \approx b$. Incorporating the expansion of r_+, (5.26) becomes,

$$r = b\left[1 + \frac{\ln R}{n-1} + \frac{(\ln R)^2}{2(n-1)^2} + \ldots\right] \tag{5.28}$$

We shall use this expansion of r in the following sections.

5.4 Solving the ψ and ϕ equations

We will now solve the equations for ψ and ϕ in this limit in both the near and far region approximation. We then extend their solution to the overlap region and using matched asymptotic expansions, compare the two solutions.

5.4.1 Near region solutions

In the near region, we write the equations (5.5) and (5.6) in terms of the near region coordinate R. As we are in the region where $r - r_+ \ll r_+$, we replace r by b in the leading order using (5.28).We will restrict to the perturbations with $k^2, \lambda \sim \mathcal{O}(n^2)$. We find that the leading order equations decouple for the variables H and G defined as,

$$H = \psi + \phi b^2 \quad G = \psi - \phi b^2$$

The leading order equations for H and G are,

$$\begin{aligned}
&-\left(1+\sigma^2 b^2 - \frac{1}{R}\right)^2 \left[R^2 \frac{d^2 H}{dR^2} + R\frac{dH}{dR}\right] + \\
&\left[-\frac{1}{4R^2} + \frac{1}{4}\left(1+\sigma^2 b^2 - \frac{1}{R}\right)^2 + \left(\frac{k^2+\lambda b^2}{(n-1)^2} - \frac{1}{R}\right)\left(1+\sigma^2 b^2 - \frac{1}{R}\right)\right] H = 0 \qquad (5.29)\\
&-\left(1+\sigma^2 b^2 - \frac{1}{R}\right)^2 \left[R^2 \frac{d^2 G}{dR^2} + R\frac{dG}{dR}\right] + \\
&\left[\frac{3}{4R^2} + \frac{1}{4}\left(1+\sigma^2 b^2 - \frac{1}{R}\right)^2 + \left(\frac{k^2+\lambda b^2}{(n-1)^2} + \frac{1}{R}\right)\left(1+\sigma^2 b^2 - \frac{1}{R}\right)\right] G = 0 \qquad (5.30)
\end{aligned}$$

Let us redefine $(1+\sigma^2 b^2)R = \tilde{R}$. The above equations can be written as hypergeometric equations with singularities at $\tilde{R} = 0, 1$ and ∞ by redefinition of variable. The horizon is at $\tilde{R} = 1$. The solution to H in terms of hypergeometric functions is

$$H = \tilde{R}(\tilde{R}-1)^{\frac{1}{2}} \left[C_1 F(p,q,1,1-\tilde{R}) + C_2 F(p,q,1,1-\tilde{R}) \ln(1-\tilde{R})\right] \qquad (5.31)$$

C_1 and C_2 are arbitrary constants. Let us denote $\frac{k}{(n-1)} = \hat{k}$ and $\frac{\lambda}{(n-1)^2} = \hat{\lambda}$ The constants

p, q are

$$p = \frac{3}{2} + \frac{1}{2}\sqrt{1 + \frac{4(\hat{k}^2 + \hat{\lambda} b^2)}{(1+\sigma^2 b^2)}} \quad q = \frac{3}{2} - \frac{1}{2}\sqrt{1 + \frac{4(\hat{k}^2 + \hat{\lambda} b^2)}{(1+\sigma^2 b^2)}}$$

We want perturbations to be normalizable at the horizon, $\tilde{R} = 1$. Hence we set $C_2 = 0$. The normalizable solution to (5.29) is,

$$H = C_1 \tilde{R} (\tilde{R} - 1)^{\frac{1}{2}} F(p, q, 1, 1 - \tilde{R}) \tag{5.32}$$

We extend this solution to the overlap region to match with the far region solution. Using standard formulae for hypergeometric functions, the solution near $\tilde{R} = 1$ is written as a linear combination of hypergeometric functions of the form $F(\alpha, \beta, \gamma, 1/\tilde{R})$.

$$\begin{aligned} H = C_1 \tilde{R} (\tilde{R} - 1)^{\frac{1}{2}} \Bigg[& \frac{\Gamma(q-p)}{\Gamma(q)\Gamma(q-2)} \tilde{R}^{-p} F\left(p, p-2, p-q+1, \frac{1}{\tilde{R}}\right) + \\ & \frac{\Gamma(p-q)}{\Gamma(p)\Gamma(p-2)} \tilde{R}^{-q} F\left(q, q-2, q-p+1, \frac{1}{\tilde{R}}\right) \Bigg]. \end{aligned} \tag{5.33}$$

Taking $\tilde{R}$ large, the solution for H in the overlap region is,

$$H = \frac{C_1 \Gamma(q-p)}{\Gamma(q)\Gamma(q-2)} \tilde{R}^{-\frac{1}{2}\sqrt{1 + \frac{4(\hat{k}^2 + \hat{\lambda} b^2)}{(1+\sigma^2 b^2)}}} + \frac{C_1 \Gamma(p-q)}{\Gamma(p)\Gamma(p-2)} \tilde{R}^{\frac{1}{2}\sqrt{1 + \frac{4(\hat{k}^2 + \hat{\lambda} b^2)}{(1+\sigma^2 b^2)}}} \tag{5.34}$$

Similarly we can solve the G equation (5.30). The normalizable solution at the horizon is,

$$G = D_1 (\tilde{R} - 1)^{\frac{3}{2}} F(p, q, 3, 1 - \tilde{R}) \tag{5.35}$$

which extended to the overlap region becomes

$$G = \frac{D_1 2\Gamma(q-p)}{\Gamma(q)^2} \tilde{R}^{-\frac{1}{2}\sqrt{1 + \frac{4(\hat{k}^2 + \hat{\lambda} b^2)}{(1+\sigma^2 b^2)}}} + \frac{D_1 2\Gamma(p-q)}{\Gamma(p)^2} \tilde{R}^{\frac{1}{2}\sqrt{1 + \frac{4(\hat{k}^2 + \hat{\lambda} b^2)}{(1+\sigma^2 b^2)}}} \tag{5.36}$$

The expressions for ψ and ϕ in the overlap region can be found by adding and subtracting (5.34) and (5.36).

5.4.2 Far region solution

In the far region, as $r >> r_+$, the quantity $b^{n-1}/r^{n-1} \sim e^{-n\ln r}$ is a small when r and n are both large. For asymptotically AdS spacetimes, we can then approximate the function $f(r) \approx 1+\sigma^2 r^2$. In the large n limit, the equations (3.41) and (3.42) become

$$-\frac{d^2\psi}{dr^2}+\left(\frac{n^2+4k^2}{4r^2}+\frac{\lambda-k^2\sigma^2}{1+\sigma^2r^2}-\frac{\sigma^2}{(1+\sigma^2r^2)^2}\right)\psi=\left(\frac{2}{n}+\frac{2}{(1+\sigma^2r^2)^2}\right)\phi \tag{5.37}$$
$$-\frac{d^2\phi}{dr^2}+\left(\frac{n^2+4k^2}{4r^2}+\frac{\lambda-k^2\sigma^2}{1+\sigma^2r^2}-\frac{\sigma^2}{(1+\sigma^2r^2)^2}\right)\phi=\frac{2\sigma^4}{(1+\sigma^2r^2)^2}\psi \tag{5.38}$$

Here we have kept only the leading order coefficient for each different r-dependent term. As in the near region, we assume $k^2, \lambda \sim \mathcal{O}(n^2)$. The equations do not decouple in an obvious manner at this stage. Looking at the near region solution $H=\psi+b^2\phi$ in (5.34) and $G=\psi-b^2\phi$ in (5.36), we know that the leading order r dependence of the two solutions is the same in terms of $R\sim r^n$. Writing the near region solutions for ψ and ϕ, we also notice that there is an extra factor of b^2 multiplying the solution ϕ in the near horizon region where we approximate $r\sim b$. In view of this observation, in the far region, let us consider the special case when $\psi\sim r^\gamma\phi$. From our previous observations of ϕ and ψ, we know that $\gamma \ll n$. It is easy to see that for $\gamma>2$, the right hand side of (5.37) can be neglected for both large r and large n. Similarly the coupling terms in (5.38) can be neglected for $\gamma\leq 2$ in the large n approximation. First consider the latter case. The normalizable solution of the decoupled ϕ equation is given in terms of a hypergeometric function. For large n,

$$\phi = d_1 r^{-\frac{n}{2}\sqrt{1+\frac{4\lambda}{\sigma^2}}}\times$$
$$F\left(\frac{1}{4}\left(\sqrt{n^2+4k^2}+\sqrt{n^2+\frac{4\lambda}{\sigma^2}}\right),\frac{1}{4}\left(\sqrt{n^2+\frac{4\lambda}{\sigma^2}}-\sqrt{n^2+4k^2}\right),\frac{1}{2}\sqrt{n^2+\frac{4\lambda}{\sigma^2}},-\frac{1}{\sigma^2r^2}\right) \tag{5.39}$$

For matching with the near region, we would like to extend this solution to the overlap region by writing r in terms of R using (5.28). For this purpose, we would also have to use the transformation formulae for hypergeometric functions. In our case, in the large n limit, the transformation of the hypergeometric function becomes unwieldy as all the parameters of the function become large.

To address this issue, we approximate $f(r) \approx \sigma^2 r^2$ for $r \to \infty$ in (3.41) and (3.42). Using this approximation, we get the solution for ψ, ϕ as Bessel functions, which can be extended to the overlap region. This approximation is invalid for small black holes ($\sigma r_+ << 1$).[2] This is because in the overlap region we cannot approximate $1 + \sigma^2 r^2 \sim \sigma^2 r^2$. Substituting $f(r)$ in (3.41) and (3.42).

$$-\frac{d^2\psi}{dr^2} + \left[\left(\frac{\lambda}{\sigma^2} + \frac{n^2}{4}\right)\frac{1}{r^2} + \frac{k^2}{\sigma^2 r^4}\right]\psi = \left(2 - \frac{2}{n}\right)\frac{1}{\sigma^2 r^2}\phi \tag{5.40}$$

$$-\frac{d^2\phi}{dr^2} + \left[\left(\frac{\lambda}{\sigma^2} + \frac{n^2}{4}\right)\frac{1}{r^2} + \frac{k^2}{\sigma^2 r^4}\right]\phi = \left[\frac{2}{r^4} + \left(2 - \frac{2}{n}\right)\frac{1}{\sigma^2 r^6}\right]\psi \tag{5.41}$$

We again consider perturbations that can be written as $\psi \sim r^\gamma \phi$. The equations decouple for various values of γ. For $\gamma > 2$, we can neglect the coupling terms in the ψ equation (5.40). For $\gamma = 2$ or $\gamma < 1$ we can ignore the coupling terms in the ϕ equation (5.41) in the large n limit. The case $\gamma = 1$ can be decoupled only if we keep the leading terms proportional to r^{-2} in the right hand sides of both the equations. We will examine the case $\gamma = 2$ or $\gamma < 1$ here. As the right hand sides of both the equations have the same form, leading order solutions for the case $\gamma > 2$ can be obtained by replacing ϕ by ψ in the following calculation. The leading order equation for ϕ is,

$$-\frac{d^2\phi}{dr^2} + \left[\left(\frac{\lambda}{\sigma^2} + \frac{n^2}{4}\right)\frac{1}{r^2} + \frac{k^2}{\sigma^2 r^4}\right]\phi = 0 \tag{5.42}$$

The solution to this equation is

$$\phi = \sqrt{r}\left[c_1 I_\nu\left(\frac{k}{\sigma r}\right) + c_2 K_\nu\left(\frac{k}{\sigma r}\right)\right] \tag{5.43}$$

where $\nu = \sqrt{\frac{n^2+1}{4} + \frac{\lambda}{\sigma^2}}$. As $r \to \infty$, the two solutions behave as $I_\nu(\frac{1}{r}) \sim r^{-\nu}$ and $K_\nu(\frac{1}{r}) \sim r^\nu$. As we want the solution to be normalizable at infinity, we choose $c_2 = 0$. The final solution

[2] Large black holes are those with $\sigma r_+ > 1$. We follow the terminology of Hawking and Page here [90].

is

$$\phi = c_1\sqrt{r}I_\nu\left(\frac{k}{\sigma r}\right). \tag{5.44}$$

We now want to extend this solution to the overlap region. As we are dealing with $k^2, \lambda \sim n^2$, both the argument and order of the modified Bessel function $I_\nu\left(\frac{k}{\sigma r}\right)$ are large in the overlap region. We can hence use the uniform asymptotic expansion for modified Bessel functions to extend this solution to overlap region. Let us define z such that the argument of the Bessel function $\frac{k}{r\sigma} = \nu z$. In terms of z, ϕ is written as,

$$\phi = \frac{c_1 I_\nu(\nu z)}{\sqrt{z}} = \frac{1}{\sqrt{z}}\frac{e^{\nu\eta}}{(1+z^2)^{1/4}\sqrt{2\pi\nu}}\left[1+\sum_{m=1}^{\infty}\frac{U_m(\tilde{t})}{\nu^m}\right] \tag{5.45}$$

where

$$\begin{aligned}
\eta &= \sqrt{1+z^2} + \ln\left[\frac{z}{1+\sqrt{1+z^2}}\right]; \\
\tilde{t} &= \frac{1}{\sqrt{1+z^2}}.
\end{aligned} \tag{5.46}$$

and $U_m(\tilde{t})$ are polynomials in $\tilde{t}$. To find the solution in the overlap region, we write r in terms of R by using (3.6) which is valid in the entire near region, and therefore, in particular, in the overlap region. This calculation follows exact same procedure as described in (3.2.3). The leading order solution in terms of R is,

$$\phi = c_1 R^{-\frac{1}{2}\sqrt{1+\frac{4(\hat{k}^2+\hat{\lambda}b^2)}{\sigma^2 b^2}}} \tag{5.47}$$

We can similarly solve for ψ. The leading order solution for ψ in the overlap region is again,

$$\psi = d_1 R^{-\frac{1}{2}\sqrt{1+\frac{4(\hat{k}^2+\hat{\lambda}b^2)}{\sigma^2 b^2}}} \tag{5.48}$$

Let us consider the cases where λ, k^2 are $\mathcal{O}(n)$ or less. As the coupling terms in the equations (5.40) and (5.41) are $\mathcal{O}(1)$, we can decouple the equations using the same logic as the $\lambda, k^2 \sim \mathcal{O}(n^2)$ case. The normalizable solutions for ϕ and ψ are $\sim I_\nu\left(\frac{k}{\sigma}r\right)$. The order of the modified Bessel function $I_\nu\left(\frac{k}{\sigma}r\right)$ is $\nu \sim n$ and the argument is $k \sim \sqrt{n}$. As the order and argument are large but of different orders, we cannot use the uniform asymptotic expansions to extend the solutions to the overlap region. For this reason we cannot comment on cases where $\lambda, k^2 \sim \mathcal{O}(n)$ or less.

5.4.3 Matching the solutions

From the solutions (5.34) and (5.36), we can write ϕ and ψ in the overlap region as,

$$\phi = e_1 \tilde{R}^{-\frac{1}{2}\sqrt{1+\frac{4(\hat{k}^2+\hat{\lambda}b^2)}{(1+\sigma^2 b^2)}}} + e_2 \tilde{R}^{\frac{1}{2}\sqrt{1+\frac{4(\hat{k}^2+\hat{\lambda}b^2)}{(1+\sigma^2 b^2)}}} \tag{5.49}$$

$$\psi = e_3 \tilde{R}^{-\frac{1}{2}\sqrt{1+\frac{4(\hat{k}^2+\hat{\lambda}b^2)}{(1+\sigma^2 b^2)}}} + e_4 \tilde{R}^{\frac{1}{2}\sqrt{1+\frac{4(\hat{k}^2+\hat{\lambda}b^2)}{(1+\sigma^2 b^2)}}} \tag{5.50}$$

Where

$$e_1 = \frac{C_1\Gamma(q-p)}{2b^2\Gamma(q)\Gamma(q-2)} - \frac{D_1 2\Gamma(q-p)}{2b^2\Gamma(q)^2} \quad e_2 = \frac{C_1\Gamma(p-q)}{2b^2\Gamma(p)\Gamma(p-2)} - \frac{D_1 2\Gamma(p-q)}{2b^2\Gamma(p)^2} \tag{5.51}$$

$$e_3 = \frac{C_1\Gamma(q-p)}{2\Gamma(q)\Gamma(q-2)} + \frac{D_1 2\Gamma(q-p)}{2\Gamma(q)^2} \quad e_4 = \frac{C_1\Gamma(p-q)}{2\Gamma(p)\Gamma(p-2)} + \frac{D_1 2\Gamma(p-q)}{2\Gamma(p)^2} \tag{5.52}$$

We now want to match these solutions to the solutions obtained from the far region (5.47) and (5.48). We see that the exponents in the two solutions become the same only for the large black holes i.e. $\sigma b \gg 1$. This is expected as our far region approximation is only valid for these black holes. Solutions coming from the near region have both a growing and a decaying piece, whereas the solution from far is only decaying. For the matching of solutions, we need the coefficient of the growing piece from the near region to vanish both for ψ and ϕ simultaneously. This translates to making both $e_2, e_4 = 0$. Looking at the coefficients in (5.51), we see that by choosing C_1 and D_1 appropriately, we may only be able to match either ψ or ϕ solution to the far solution, but not both. For both the constants e_2 and e_4 to vanish simultaneously, we will need the gamma function $\Gamma(p)$ to have a pole, i.e. p to be a nonpositive integer. For the parameter range that we are interested in, namely $k > 0, \lambda > 0$ and $k^2, \lambda \sim \mathcal{O}(n^2)$, we can eliminate this possibility. Hence there are no normalizable modes for SAdS black holes for $\lambda \sim \mathcal{O}(n^2)$.

We cannot apply this analysis for other parameter ranges as the equations remain coupled in the far region.

5.5 Spherically symmetric ($\ell = 0$) perturbation

The equations used in the previous section for non-spherically symmetric perturbations use IK variables which are only valid for $\ell \geq 2$. IK provide a prescription to extend these variables to the spherically symmetric $\ell = 0$ perturbations via gauge fixing. But the resulting equations obtained by this procedure are unwieldy. We use a much simpler equation for these perturbations has been obtained in [91] for our analysis. This equation has been analysed numerically for various n values for SAdS black holes in [91] and analytically for large n for the $\Lambda = 0$ black hole in [29]. As discussed in Chapter (2), only the terms operating on the transverse traceless modes can have negative eigenvalues. Following [91] we consider transverse and traceless perturbations $h_{\mu\nu}$ of the form

$$h_{\mu}^{\nu} = diag\left(\zeta(r), \chi(r), H_L(r), \ldots, H_L(r)\right) \tag{5.53}$$

where $(H_L(r), \ldots H_L(r))$ are n terms. For the transverse traceless mode

$$H_L = -\frac{\zeta + \chi}{n}. \tag{5.54}$$

Subsequently, using $\nabla^{\mu} h_{\mu\nu} = 0$, the relation between ζ and χ can be written as

$$\zeta = \frac{2rf}{rf' - 2f}\chi'(r) + \frac{rf' + 2(n+1)f}{rf' - 2f}\chi(r). \tag{5.55}$$

With this ansatz the relevant perturbation equation reduces to a linear second order ordinary differential equation for χ [91].

$$\begin{aligned} -f\chi'' + &\frac{2r^2(ff'' - f'^2) - rnff' + 2(n+2)f^2}{r(rf' - 2f)}\chi' \\ &+ \frac{r^2 f'f'' + r[2(n+1)ff'' - (n+4)f'^2] + 4ff'}{r(rf' - 2f)}\chi = -\lambda\chi \end{aligned} \tag{5.56}$$

To find unstable modes, we want solutions to this equation that are normalizable at both the horizon and infinity for $\lambda > 0$. This equation has four singular points. They are $0, r_+, \infty$ and r_s where r_s is the solution to $(rf' - 2f) = 0$. Hence, along with the regularity at both the boundaries, it is desirable for the perturbation to be well-behaved at r_s. This implies, from the equation (5.55), at r_s

$$\frac{\chi'(r_s)}{\chi(r_s)} = -\frac{rf' + 2(n+1)f}{2rf}\bigg|_{r_s} = -\frac{n+2}{r_s}. \tag{5.57}$$

We will later see that both the linearly independent solutions to (5.56) around the point r_s satisfy this equation, so this is *not* in fact an extra condition on the solution χ.

The equation (5.56) cannot be solved for the entire range of r analytically. We use the large n limit as an analytical approximation tool to study (in)stability.

5.5.1 Far region solution

We first solve equation (5.56) in the far region. We proceed by substituting for $f(r)$ in (5.56) and neglecting the terms of order r^{-n} in the far region. Note that this is just the far region approximation, we have not assumed a large n limit to obtain the following equation:

$$(1 + \sigma^2 r^2)\chi'' + \frac{(n+2) + (n+6)\sigma^2 r^2}{r}\chi' + [(2n+6)\sigma^2 - \lambda]\chi = 0. \tag{5.58}$$

The solutions of this equation are,

$$\begin{aligned}\chi =& D_1(\sigma^2 r^2)^{-p} F\left(p, p-s+1, p-q+1, -\frac{1}{\sigma^2 r^2}\right) + \\ & D_2(\sigma^2 r^2)^{-q} F\left(q, q-s+1, q-p+1, -\frac{1}{\sigma^2 r^2}\right)\end{aligned} \tag{5.59}$$

Here $s = \frac{n+3}{2}$ and

$$p = \frac{(n+5)}{4} + \frac{1}{2}\sqrt{\frac{(n+1)^2}{4} + \frac{\lambda}{\sigma^2}}; \quad q = \frac{(n+5)}{4} - \frac{1}{2}\sqrt{\frac{(n+1)^2}{4} + \frac{\lambda}{\sigma^2}}. \tag{5.60}$$

At $r \to \infty$, the hypergeometric function can be can be truncated to its leading term.

$$\chi \approx D_1 r^{-\frac{n}{2}-\frac{n}{2}\sqrt{1+4\frac{\hat{\lambda}}{\sigma^2}}} + D_2 r^{-\frac{n}{2}+\frac{n}{2}\sqrt{1+4\frac{\hat{\lambda}}{\sigma^2}}} \tag{5.61}$$

Here $\hat{\lambda} = \frac{\lambda}{(n-1)^2}$. We want to choose the solution that is normalizable at infinity. For $\hat{\lambda} > 0$, this condition dictates $D_2 = 0$. In terms of R, the leading order solution in the far region for the case $\lambda \sim \mathcal{O}(n^2)$ is

$$\chi \approx D_1 R^{-\frac{1}{2}-\frac{1}{2}\sqrt{1+4\frac{\hat{\lambda}}{\sigma^2}}} \tag{5.62}$$

For the cases λ of order $\mathcal{O}(n)$ or lower, the far region solution at the leading order in n is

$$\chi \approx D_1 \frac{1}{R}. \tag{5.63}$$

5.5.2 Near region solution

We now turn our attention to the near region. As before, to get the equation in the near region, we write (5.56) in terms of the R coordinate.

$$\begin{aligned} \frac{d^2\chi}{dR^2} + \left[\frac{2n}{(n-1)R} - \frac{4}{2R-(n+1)} + \frac{2((n-1)(1+\sigma^2r^2)+2\sigma^2r^2)}{(n-1)(R(1+\sigma^2r^2)-1)}\right]\frac{d\chi}{dR} \\ + \frac{2(ABR-C)}{R(2R-(n+1))(R(1+\sigma^2r^2)-1)}\chi = 0 \end{aligned} \tag{5.64}$$

Where

$$C = (n-1)(1+\sigma^2r^2) - \frac{\lambda r^2}{2(n-1)} + 3 + 6\sigma^2r^2 + \frac{12\sigma^2r^2}{(n-1)} + \frac{8\sigma^2r^2 - \lambda r^2}{(n-1)^2} \tag{5.65}$$

$$AB = \frac{-\lambda r^2 + (8+2(n-1))\sigma^2r^2}{(n-1)^2} \tag{5.66}$$

Even for the leading value of $r \approx b$ in the near horizon region, equation (5.64) is a Heun equation with singularities at $0, \frac{1}{1+\sigma^2b^2}, \frac{n+1}{2}$ and ∞. Solutions of these type of equations can generically be written as a power series around each singular point. There are no connection formulae between the solutions at two singular points and thus not much can be inferred from them. In the equation (5.64), different terms assume importance very near the horizon at $\frac{1}{1+\sigma^2 b^2}$ where $R \sim \mathcal{O}(1)$ and near the singular point $R = \frac{n+1}{2}$ corresponding to r_s where $R \sim \mathcal{O}(n)$. To get the near region solution, we follow a two step matching procedure. We will solve the equation in the two regimes, $R \sim \mathcal{O}(1)$ and $R \sim \mathcal{O}(n)$. In both the regimes, we employ a $(1/n)$ expansion for χ and λ. As both the regimes are still in the original near region $r - r_+ \ll r_+$, the full near region solution must be obtained by matching the two solutions. To this end, we propose an overlap region between the two singular points such that $1 \ll R \ll n$. We then match the solutions in this new overlap region. The whole near region solution obtained by this matching procedure is then matched with the far region solution for $R \to \infty$. This two step matching procedure gives us the value of λ. This matching scheme has been used by Emparan, Suzuki, Tanabe in [31] to obtain decoupled quasinormal modes for Schwarzschild-Tangherlini black holes.

Henceforth we will denote

$$n - 1 = m. \tag{5.67}$$

The perturbation variable χ and eigenvalue λ are expanded as,

$$\chi = \sum_{i=0}^{\infty} \frac{\chi_i}{m^i}; \qquad \lambda = \sum_{i=0}^{\infty} \frac{\lambda_i}{m^i} \tag{5.68}$$

Let us first consider the case $\lambda \sim \mathcal{O}(n^2)$.

The case $\lambda \sim \mathcal{O}(n^2)$

Very near the horizon, $R \sim \mathcal{O}(1)$. We rewrite the equation (5.64) in terms of the coordinate $x = 1 - (1 + \sigma^2 b^2)R$ and denote χ as $\tilde{\chi}$ in this region. In terms of x, the horizon lies at $x = 0$. In the leading order, the equation becomes

$$x(1-x)\frac{d^2\tilde{\chi}_0}{dx^2} + (2-4x)\frac{d\tilde{\chi}_0}{dx} - \left(2 - \frac{\hat{\lambda}b^2}{(1+\sigma^2 b^2)}\right)\tilde{\chi}_0 = 0 \tag{5.69}$$

Here we have written $\hat{\lambda} = \frac{\lambda}{(n-1)^2}$ such that $\hat{\lambda} \sim \mathcal{O}(1)$. Let us denote

$$\delta = \frac{\hat{\lambda}b^2}{(1+\sigma^2 b^2)}.$$

Solutions of this equation are

$$\begin{aligned}\tilde{\chi}_0 =& C_1 F\left(\frac{3}{2}+\frac{\sqrt{1+4\delta}}{2}, \frac{3}{2}-\frac{\sqrt{1+4\delta}}{2}, 2, x\right) + \\ & C_2\left(F\left(\frac{3}{2}+\frac{\sqrt{1+4\delta}}{2}, \frac{3}{2}-\frac{\sqrt{1+4\delta}}{2}, 2, x\right)\ln x + \frac{a_1}{x} + \sum_{i=0}^{k} c_k x^k\right);\end{aligned} \tag{5.70}$$

where a_1 and c_k are constants.[3] We choose $C_2 = 0$ as we want χ_0 to be normalizable at $x = 0$. In order to compare with the solution at $R \approx m/2$, let us extend this solution to the overlap region within the near region $1 \ll R \ll n$ by taking x large. The far limit of the near horizon solution is

$$\begin{aligned}\tilde{\chi}_0 =& C_1 \frac{\Gamma(-\sqrt{1+4\delta})x^{-\frac{3}{2}-\frac{\sqrt{1+4\delta}}{2}}}{\Gamma\left(\frac{3}{2}-\frac{\sqrt{1+4\delta}}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\sqrt{1+4\delta}}{2}\right)} F\left(\frac{3}{2}+\frac{\sqrt{1+4\delta}}{2}, \frac{1}{2}+\frac{\sqrt{1+4\delta}}{2}, 1+\sqrt{1+4\delta}, \frac{1}{x}\right) + \\ & C_1 \frac{\Gamma(\sqrt{1+4\delta})x^{-\frac{3}{2}+\frac{\sqrt{1+4\delta}}{2}}}{\Gamma\left(\frac{3}{2}+\frac{\sqrt{1+4\delta}}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\sqrt{1+4\delta}}{2}\right)} F\left(\frac{3}{2}-\frac{\sqrt{1+4\delta}}{2}, \frac{1}{2}-\frac{\sqrt{1+4\delta}}{2}, 1-\sqrt{1+4\delta}, \frac{1}{x}\right).\end{aligned} \tag{5.71}$$

For large x, we can replace $x = -(1+\sigma^2 b^2)R$. Taking the limit x large in the hypergeometric functions, χ_0 simplifies to

$$\tilde{\chi}_0 = C_1 \frac{\Gamma(-\sqrt{1+4\delta})R^{-\frac{3}{2}-\frac{\sqrt{1+4\delta}}{2}}}{\Gamma\left(\frac{3}{2}-\frac{\sqrt{1+4\delta}}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\sqrt{1+4\delta}}{2}\right)} + C_1 \frac{\Gamma(\sqrt{1+4\delta})R^{-\frac{3}{2}+\frac{\sqrt{1+4\delta}}{2}}}{\Gamma\left(\frac{3}{2}+\frac{\sqrt{1+4\delta}}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\sqrt{1+4\delta}}{2}\right)} \tag{5.72}$$

This is the far limit of the near horizon solution which is normalizable at the horizon. Near the singularity r_s we write the equation (5.64) in terms of y, such that $R = my$ and $y \sim \mathcal{O}(1)$. We denote χ as $\hat{\chi}$ in this region. The singular point r_s translates to $y = \frac{1}{2}$. The leading order equation is

$$\frac{d^2\hat{\chi}_0}{dy^2} + \left(\frac{4}{y} - \frac{4}{2y-1}\right)\frac{d\hat{\chi}_0}{dy} + \frac{(-2\delta y - (2-\delta))}{y^2(2y-1)}\hat{\chi}_0 = 0. \tag{5.73}$$

[3]The values of these constants can be found in chapter 15 of Handbook of Mathematical Functions by M Abrahmowitz and I Stegun.

The general solution of this equation is

$$\hat{\chi}_0 = d_1 y^{\frac{-3+\sqrt{1+4\delta}}{2}}((2y-1)(\sqrt{1+4\delta}-1)-2)+d_2 y^{\frac{-3-\sqrt{1+4\delta}}{2}}((2y-1)(\sqrt{1+4\delta}+1)+2) \quad (5.74)$$

Both the linearly independent solutions in (5.74) are finite at $y = \frac{1}{2}$. The solutions at $y = \frac{1}{2}$ must satisfy the condition (5.57). This condition rewritten in terms of y simply becomes $\chi'(y) = -2\chi$ at the leading order .The condition is satisfied by any combination of d_1 and d_2. An analogous observation was made in [28] for $\sigma = 0$. To extend this solution to the near horizon region, we rewrite y in terms of R and let $R \ll n$. The leading order solution then becomes,

$$\hat{\chi}_0 = d_1(-\sqrt{1+4\delta}-1)\left(\frac{R}{m}\right)^{\frac{-3+\sqrt{1+4\delta}}{2}} + d_2(-\sqrt{1+4\delta}+1)\left(\frac{R}{m}\right)^{\frac{-3-\sqrt{1+4\delta}}{2}}. \quad (5.75)$$

Let us now compare the two solutions $\tilde{\chi}_0$ (5.72) and $\hat{\chi}_0$ (5.75). The solutions coming from both sides can be matched for all values of λ by identifying the constants d_1 and d_2 with those in (5.72). The solution at r_s can then be written as

$$\begin{aligned}\hat{\chi}_0 = & \frac{\Gamma(\sqrt{1+4\delta})m^{\frac{-3+\sqrt{1+4\delta}}{2}}}{\Gamma\left(\frac{3}{2}+\frac{\sqrt{1+4\delta}}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\sqrt{1+4\delta}}{2}\right)}\frac{y^{\frac{-3+\sqrt{1+4\delta}}{2}}((2y-1)(\sqrt{1+4\delta}-1)-2)}{(-\sqrt{1+4\delta}-1)} \\ & + \frac{\Gamma(-\sqrt{1+4\delta})m^{\frac{-3-\sqrt{1+4\delta}}{2}}}{\Gamma\left(\frac{3}{2}-\frac{\sqrt{1+4\delta}}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\sqrt{1+4\delta}}{2}\right)}\frac{y^{\frac{-3-\sqrt{1+4\delta}}{2}}((2y-1)(\sqrt{1+4\delta}+1)+2)}{(-\sqrt{1+4\delta}+1)}\end{aligned} \quad (5.76)$$

Thus we have extended the normalizable solution near the horizon to r_s. In order to match with the solution in the far region (5.62), we take the limit $r \to \infty$ of the solution (5.76).

$$\begin{aligned}\hat{\chi}_0 \approx & \frac{\Gamma(\sqrt{1+4\delta})m^{\frac{-3+\sqrt{1+4\delta}}{2}}}{\Gamma\left(\frac{3}{2}+\frac{\sqrt{1+4\delta}}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\sqrt{1+4\delta}}{2}\right)}\frac{R^{\frac{-1+\sqrt{1+4\delta}}{2}}}{(-\sqrt{1+4\delta}-1)} \\ & + \frac{\Gamma(-\sqrt{1+4\delta})m^{\frac{-3-\sqrt{1+4\delta}}{2}}}{\Gamma\left(\frac{3}{2}-\frac{\sqrt{1+4\delta}}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\sqrt{1+4\delta}}{2}\right)}\frac{R^{\frac{-1-\sqrt{1+4\delta}}{2}}}{(-\sqrt{1+4\delta}+1)}.\end{aligned} \quad (5.77)$$

We can see that the solution has a growing and decaying part whereas the far region solution (5.62) is only decaying. For the solutions to match, the coefficient of the growing piece must be zero. To this end, the gamma functions in the denominator of the growing piece in (5.77) must have a pole, with the numerator remaining finite. This will never be the case for coefficients of the growing part. We further notice that the exponents of R in the two solutions take the same form only for large black holes where $1+\sigma^2 b^2$ can be approximated by $\sigma^2 b^2$. For the case of small black holes we cannot make a concrete statement. To do so, we would have to extend the solution in the far region to the overlap region. As in the case $\ell > 2$ (5.39), the transformation formulae for hypergeometric functions become unwieldy. We conclude that there are no normalizable modes for $\lambda \sim \mathcal{O}(n^2)$ for the cases considered of the large black hole. This conclusion matches with the bound on the value of λ obtained in [91]. Considering this result and in view of the value of λ found in [29] for the $\sigma = 0$ case i.e. the Schwarzschild-Tangherlini black holes, we shall now investigate the case $\lambda \sim \mathcal{O}(n)$.

The case $\lambda \sim \mathcal{O}(n)$

As in the previous case, we solve the equation (5.64) in the limit $R \sim \mathcal{O}(1)$ for $\tilde{\chi}$, and in the limit $R \sim \mathcal{O}(n)$ for $\hat{\chi}$. We denote $\lambda = mL$ such that $L \sim \mathcal{O}(1)$. The expansion of λ then becomes

$$\lambda = \sum_{i=0}^{\infty} \frac{\lambda_i}{m^i} = mL_0 + L_1 + \frac{L_2}{m} + \dots . \tag{5.78}$$

(i) For $R \sim \mathcal{O}(1)$:

Substituting r in terms of R using (5.28), and expanding $\tilde{\chi}$ as a series in m, we write the equation (5.64) in orders of m. For $R \sim \mathcal{O}(1)$, we can approximate $\left(R - \frac{m+2}{2}\right) \approx -\frac{m}{2}$. For convenience, we divide the whole equation (5.64) by m. Equation (5.64) then is written in orders of m. The leading order equation can be written as $\mathcal{L}\tilde{\chi}_0 = 0$. The operator $\mathcal{L}$ is

$$\mathcal{L} = \frac{R}{2}(1-(1+\sigma^2 b^2)R)\frac{d^2}{dR^2} + (1-2(1+\sigma^2 b^2)R)\frac{d}{dR} - (1+\sigma^2 b^2) \tag{5.79}$$

The next order equations then are written as

$$\frac{\mathcal{L}\tilde{\chi}_i}{m^i} = \frac{\mathcal{S}(R)_i}{m^i} \tag{5.80}$$

Source terms $\mathcal{S}(R)_i$ depend on the solutions of the previous order equations $(\tilde{\chi}_0, \ldots, \tilde{\chi}_{i-1})$. The leading order equation is

$$\frac{R}{2}(1-(1+\sigma^2 b^2)R)\frac{d^2\tilde{\chi}_0}{dR^2} + (1-2(1+\sigma^2 b^2)R)\frac{d\tilde{\chi}_0}{dR} - (1+\sigma^2 b^2)\tilde{\chi}_0 = 0. \tag{5.81}$$

Solutions to this equation are,

$$\tilde{\chi}_0 = \frac{d_1}{R} + \frac{d_2}{(1-(1+\sigma^2 b^2)R)} \tag{5.82}$$

Normalizability at the horizon tells us $d_2 = 0$ so that

$$\tilde{\chi}_0 = \frac{d_1}{R} \tag{5.83}$$

The source term $\mathcal{S}(R)_1$ at $\mathcal{O}(1/m)$ depends on the solution to the leading order equation $\tilde{\chi}_0$.

$$\begin{aligned}\mathcal{L}\tilde{\chi}_1 = &- \left(-3 + \frac{L_0 b^2}{2} - 6\sigma^2 b^2 - 2\sigma^2 b^2 \ln R\right)\tilde{\chi}_0 - \\ &\left(3 - 5R + 2R^2 - 7\sigma^2 b^2 R + 2\sigma^2 b^2 R^2 - 4\sigma^2 b^2 R \ln R\right)\tilde{\chi}_0' - \\ &\left(R - 2R^2 + R^3 - \sigma^2 b^2 R^2 + \sigma^2 b^2 R^3 - \sigma^2 b^2 R^2 \ln R\right)\tilde{\chi}_0''\end{aligned} \tag{5.84}$$

Solving for $\tilde{\chi}_1$, the solution is

$$\begin{aligned}\tilde{\chi}_1 = &\frac{d_1^1}{R} + \frac{d_2^1}{(R(1+\sigma^2 b^2)-1)} - \frac{2d_1 + 2d_1 \ln R}{R} \\ &- \frac{(-6 + b^2(L_0 - 4\sigma^2))d_1 + (1+\sigma^2 b^2)d_1^1 + (2 - L_0 b^2)d_1 \ln R}{(R(1+\sigma^2 b^2)-1)}\end{aligned} \tag{5.85}$$

Here d_1^1, d_2^1 are arbitrary constants corresponding to the homogeneous solution. Regularity at the horizon fixes d_2^1 to be

$$d_2^1 = \left((-6 + b^2(L_0 - 4\sigma^2))d_1 + (1+\sigma^2 b^2)d_1^1 - (2 - L_0 b^2)d_1 \ln(1+\sigma^2 b^2)\right). \tag{5.86}$$

The next to leading order solution $\tilde{\chi}_1$ after substituting for d_2^1 becomes

$$\tilde{\chi}_1 = \frac{-2d_1 + d_1^1 - 2d_1 \ln R}{R} - \frac{(2 - L_0 b^2) d_1 (\ln(1 + \sigma^2 b^2) + \ln R)}{(R(1 + \sigma^2 b^2) - 1)} \tag{5.87}$$

We take the full solution $\tilde{\chi}_0 + \frac{\tilde{\chi}_1}{m}$ to the overlap region by taking R large such that $1 \ll R \ll m$.

$$\begin{aligned}\tilde{\chi} = \frac{d_1}{R} + \frac{1}{m}\Bigg[&\frac{-2d_1 + d_1^1 - 2d_1 \ln R}{R} - \\ &\frac{(2 - L_0 b^2) d_1 (\ln(1 + \sigma^2 b^2) + \ln R)}{R(1 + \sigma^2 b^2)} \left(1 + \frac{1}{R(1 + \sigma^2 b^2)} + \frac{1}{R^2(1 + \sigma^2 b^2)^2} + \ldots\right)\Bigg]\end{aligned} \tag{5.88}$$

Here d_1, d_1^1 are arbitrary constants. We will match this solution to the one obtained from the region where $R \sim \mathcal{O}(n)$.

(ii) For $R \sim \mathcal{O}(n)$:

We use the coordinate $y = \frac{R}{m}$ and solve for $\hat{\chi}(y)$ in this region. As before, we expand $\hat{\chi}$ as a series in terms of m. The equation (5.64) in a $1/m$ expansion in terms of y can be written as

$$\frac{\bar{\mathcal{L}}\hat{\chi}_i}{m^i} = \frac{\bar{\mathcal{S}}(y)_i}{m^i} \tag{5.89}$$

As in equation (5.80) for $R \sim \mathcal{O}(1)$, the $\bar{\mathcal{S}}(y)_i$ are the source terms with $\bar{\mathcal{S}}(y)_0 = 0$.The operator $\bar{\mathcal{L}}$ is

$$\bar{\mathcal{L}} = (1 + \sigma^2 b^2)\left[\left(y^3 - \frac{y^2}{2}\right)\frac{d^2}{dy^2} + (2y^2 - 2y)\frac{d}{dy} - 1\right] \tag{5.90}$$

The leading order equation is,

$$(1 + \sigma^2 b^2)\left[\left(y^3 - \frac{y^2}{2}\right)\hat{\chi}_0'' + (2y^2 - 2y)\hat{\chi}_0' - \hat{\chi}_0\right] = 0 \tag{5.91}$$

Solutions to this equation are

$$\hat{\chi}_0 = \frac{e_1}{y} + e_2 \frac{(-1 + 4y^2 - 4y \ln y)}{y^2} \tag{5.92}$$

where e_1 and e_2 are constants. The equation (5.57) is again satisfied by both the solutions at the leading order. We can now match the solution with the behaviour of solution coming from far region (5.63). This implies $e_2 = 0$. The leading order solution is

$$\hat{\chi}_0 = \frac{e_1}{y} \tag{5.93}$$

At the next order,

$$\begin{aligned}\bar{\mathcal{L}}\hat{\chi}_1 = &-\left(-3 + \frac{b^2 L_0}{2} - 6\sigma^2 b^2 + y(-L_0 b^2 + 2\sigma^2 b^2) - 2\sigma^2 b^2(\ln m + \ln y)\right)\hat{\chi}_0 - \\ &\left(1 + y(-5 - 7\sigma^2 b^2 - 4\sigma^2 b^2(\ln m + \ln y)) + y^2(2 + 6\sigma^2 b^2 + 4\sigma^2 b^2(\ln m + \ln y))\right)\hat{\chi}_0' \\ &- \left(\frac{y}{2} + 2\sigma^2 b^2 y^3(\ln m + \ln y) + y^2(-2 - \sigma^2 b^2 - \sigma^2 b^2(\ln m + \ln y))\right)\hat{\chi}_0''\end{aligned} \tag{5.94}$$

The solution to this equation is

$$\begin{aligned}\hat{\chi}_1 = &\left(\frac{e_1(1 - L_0 b^2 - \sigma^2 b^2)}{2(1+\sigma^2 b^2)} - e_2^1\right)\frac{1}{y^2} + \frac{-e_1(2 + L_0 b^2 + 4\sigma^2 b^2) + e_1^1(1+\sigma^2 b^2)}{y(1+\sigma^2 b^2)} \\ &+ 4e_2^1 + \left(-e_1(2 + L_0 b^2 + 4\sigma^2 b^2) - 4e_2^1(1+\sigma^2 b^2)\right)\frac{\ln y}{y(1+\sigma^2 b^2)}\end{aligned} \tag{5.95}$$

The constants e_1^1, e_2^1 are arbitrary constants corresponding to the solution to the homogeneous equation (added to the particular solution of the inhomogeneous equation). We want to match this solution to the near region solution. We write the solution in terms of R.

$$\begin{aligned}\hat{\chi} &= \hat{\chi}_0(y) + \frac{\hat{\chi}_1(y)}{m} \\ &= \frac{me_1}{R} + \frac{m}{R^2}\left(\frac{e_1(1 - L_0 b^2 - \sigma^2 b^2)}{2(1+\sigma^2 b^2)} - e_2^1\right) + \frac{-e_1(2 + L_0 b^2 + 4\sigma^2 b^2) + e_1^1(1+\sigma^2 b^2)}{R(1+\sigma^2 b^2)} \\ &+ 4\frac{e_2^1}{m} + \left(-e_1(2 + L_0 b^2 + 4\sigma^2 b^2) - 4e_2^1(1+\sigma^2 b^2)\right)\frac{\ln R - \ln m}{R(1+\sigma^2 b^2)}\end{aligned} \tag{5.96}$$

Let us look at this solution carefully.

(i) We have a constant term proportional to e_2^1 which has no counterpart in the next to leading order solution (5.88). Also, when extended to $y \approx R \to \infty$, the constant term may dominate the solution making it non-normalizable at the boundary. We hence put $e_2^1 = 0$.

(ii) Notice that in (5.96), the term proportional to $(1/y^2)$ from $\hat{\chi}_1$ (5.95) gets written as

$$\frac{\hat{\chi}_1}{m} \sim \frac{1}{my^2} \sim \frac{m}{R^2}. \tag{5.97}$$

Thus it contributes to the leading order solution after replacing y as $\frac{R}{m}$ in (5.96). This implies that terms from the higher orders in $\hat{\chi}(y)$ can contribute at the leading order in $\hat{\chi}(R)$. For example upon replacing $y = \frac{R}{m}$, terms of the form $(1/m^i y^{i+1})$ at each order contribute to the leading order solution and similarly $(1/m^i y^i)$ contribute in the next to leading order and so on.

$$\frac{1}{m^i y^{i+1}} \sim \frac{m}{R^{i+1}}; \quad \frac{1}{m^i y^i} \sim \frac{1}{R^i}$$

To obtain the contributions from the higher order solutions to $\hat{\chi}$ in (5.96), we first solve for $\hat{\chi}$ up to $\mathcal{O}(1/m^4)$ in terms of y. Then we replace $y = \frac{R}{m}$ in this solution. The full solution in y is too lengthy hence we will not give it here. We will write the solution $\hat{\chi}$ up to $\mathcal{O}(1/m)$ after replacing y in terms of R. This is because we want to match the solution $\hat{\chi}$ to the solution from the near region $\tilde{\chi}$ obtained in (5.88) which is only up to $\mathcal{O}(1/m)$.

$$
\begin{aligned}
\hat{\chi}(R) = {} & \frac{e_1}{R} + \frac{\left(-b^2 e_1 L_0 - b^2 e_1 \sigma^2 + e_1\right)}{2\left(b^2\sigma^2+1\right)} \left(\frac{1}{R^2} + \frac{1}{R^3\left(b^2\sigma^2+1\right)} + \frac{1}{R^4\left(b^2\sigma^2+1\right)^2} \right) + \\
& \frac{1}{m} \left\{ \frac{b^2 e_1^1 \sigma^2 + b^2 e_1 L_0 \ln m - b^2 e_1 L_0 + 4b^2 e_1 \sigma^2 \ln m - 4b^2 e_1 \sigma^2 + e_1^1 + 2e_1 \ln m - 2e_1}{R\left(b^2\sigma^2+1\right)} \right. \\
& + \frac{\ln R\left(-b^2 e_1 L_0 - 4b^2 e_1 \sigma^2 - 2e_1\right)}{R\left(b^2\sigma^2+1\right)} \\
& - \frac{1}{2R^2\left(b^2\sigma^2+1\right)^2} \left[b^4 e_1^1 L_0 \sigma^2 + b^4 e_1^1 \sigma^4 - 3b^4 e_1 L_0^2 - 7b^4 e_1 L_0 \sigma^2 + b^4 e_1 L_1 \sigma^2 \right. \\
& - 3b^4 e_1 \sigma^4 - e_1^1 + 3e_1 + b^2 e_1^1 L_0 + b^2 e_1 L_0 + b^2 e_1 L_1 + 4b^2 e_1 \sigma^2 \\
& \left. + e_1 \ln m \left(b^4 \left(L_0^2 + 5L_0\sigma^2 + 4\sigma^4 \right) + b^2 \left(L_0 - 2\sigma^2 \right) - 2 \right) \right] \\
& - \frac{e_1 \ln R\left(b^4\left(-\left(L_0^2 + 5L_0\sigma^2 + 4\sigma^4\right)\right) + b^2\left(L_0 + 6\sigma^2\right) + 2\right)}{2R^2\left(b^2\sigma^2+1\right)^2} \\
& - \frac{1}{4R^3\left(b^2\sigma^2+1\right)^3} \left[2b^4 e_1^1 L_0 \sigma^2 + 2b^4 e_1^1 \sigma^4 - 5b^4 e_1 L_0^2 - 13b^4 e_1 L_0 \sigma^2 + 2b^4 e_1 L_1 \sigma^2 \right. \\
& - 6b^4 e_1 \sigma^4 + 2b^2 e_1^1 L_0 + b^2 e_1 L_0 + 2b^2 e_1 L_1 + 8b^2 e_1 \sigma^2 - 2e_1^1 + 6e_1 \\
& \left. + 2e_1 \ln m \left(b^4 \left(L_0^2 + 5L_0\sigma^2 + 4\sigma^4 \right) + b^2 \left(L_0 - 2\sigma^2 \right) - 2 \right) \right] \\
& + \frac{e_1 \ln R\left(b^4\left(L_0^2 + 7L_0\sigma^2 + 6\sigma^4\right) - b^2\left(L_0 + 8\sigma^2\right) - 2\right)}{2R^3\left(b^2\sigma^2+1\right)^3} \\
& - \frac{1}{6R^4\left(b^2\sigma^2+1\right)^4} \left[3b^4 e_1^1 L_0 \sigma^2 + 3b^4 e_1^1 \sigma^4 - 7b^4 e_1 L_0^2 - 19b^4 e_1 L_0 \sigma^2 + 3b^4 e_1 L_1 \sigma^2 \right. \\
& - 9b^4 e_1 \sigma^4 + 3b^2 e_1^1 L_0 + b^2 e_1 L_0 + 3b^2 e_1 L_1 + 12b^2 e_1 \sigma^2 - 3e_1^1 + 9e_1 \\
& \left. + 3e_1 \ln m \left(b^4 \left(L_0^2 + 5L_0\sigma^2 + 4\sigma^4 \right) + b^2 \left(L_0 - 2\sigma^2 \right) - 2 \right) \right] \\
& \left. + \frac{e_1 \ln R\left(b^4\left(L_0^2 + 9L_0\sigma^2 + 8\sigma^4\right) - b^2\left(L_0 + 10\sigma^2\right) - 2\right)}{2R^4\left(b^2\sigma^2+1\right)^4} \right\}
\end{aligned}
\tag{5.98}
$$

In obtaining this solution, we have used the boundary condition that $\hat{\chi}_i$ is normalizable at infinity at each order.[4] This lets us put various constants to zero similar to how we set $e_2 = 0$ in the leading order solution (5.92).

We see that after replacing y as $\frac{R}{m}$ the leading order solution acquires an m factor while the leading order solution from the $R \sim \mathcal{O}(1)$ (5.88) is of $\mathcal{O}(1)$. As we want to match the two solutions, we have scaled the entire solution $\hat{\chi}$ by an overall factor of m (recall that we

[4] We have used Mathematica for detailed computations.

had scaled the near region equations by a factor of m). $\hat{\chi}$ solves the homogeneous equation (5.64) near r_s.

(iii) Matching the solutions

Compare the leading order terms from both $\tilde{\chi}$ (5.88) and $\hat{\chi}$ (5.98). The leading order term in (5.98) is

$$\hat{\chi}_0(R) = \frac{e_1}{R} + \frac{(-b^2 e_1 L_0 - b^2 e_1 \sigma^2 + e_1)}{2\,(b^2\sigma^2 + 1)} \left(\frac{1}{R^2} + \frac{1}{R^3\,(b^2\sigma^2+1)} + \frac{1}{R^4\,(b^2\sigma^2+1)^2} \right) \tag{5.99}$$

The leading order term in (5.88) is $\tilde{\chi}_0 = \frac{d_1}{R}$. The two solutions match after setting $-b^2 e_1 L_0 - b^2 e_1 \sigma^2 + e_1 = 0$. This condition fixes L_0.

$$L_0 b^2 = 1 - \sigma^2 b^2 \tag{5.100}$$

We are looking for the modes that have $\lambda = mL > 0$. Recall (5.78). We can see that for large black holes with $\sigma b > 1$, λ becomes negative. This signifies that there are no normalizable modes for large black holes having $\lambda > 0$. For the small black hole with $\sigma b < 1$, there is a negative mode.For the Schwarzschild black hole case with $\sigma = 0$, this answer matches with the leading order value of λ obtained in [29].

We make the coefficients of $\frac{1}{R}$ terms in both the leading order solutions equal by setting $d_1 = e_1 = 1$. Substituting for L_0 in the solutions (5.88) and (5.98) we get,

$$\begin{aligned}
\tilde{\chi} =& \frac{1}{R} + \frac{1}{m}\left[\frac{-2 + d_1^1 - \ln(1+\sigma^2 b^2)}{R} - \frac{3\ln R}{R} \right. \\
& - \ln(1+\sigma^2 b^2) \left(\frac{1}{(1+\sigma^2 b^2)R^2} + \frac{1}{(1+\sigma^2 b^2)^2 R^3} + \frac{1}{(1+\sigma^2 b^2)^3 R^4} \right) \\
& \left. - \ln R \left(\frac{1}{(1+\sigma^2 b^2)R^2} + \frac{1}{(1+\sigma^2 b^2)^2 R^3} + \frac{1}{(1+\sigma^2 b^2)^3 R^4} \right) \right].
\end{aligned} \tag{5.101}$$

We have truncated the infinite series in (5.88) at $\sim R^{-4}$. The solution $\hat{\chi}$ becomes,

$$\begin{aligned}\hat{\chi} =& \frac{1}{R} + \frac{1}{m}\left[\frac{-3+e_1^1+3\ln m}{R} - \frac{3\ln R}{R}\right.\\ &- \frac{(1+L_1b^2+\sigma^2b^2)}{2}\left(\frac{1}{(1+\sigma^2b^2)R^2} + \frac{1}{(1+\sigma^2b^2)^2R^3} + \frac{1}{(1+\sigma^2b^2)^3R^4}\right)\\ &\left.- \ln R\left(\frac{1}{(1+\sigma^2b^2)R^2} + \frac{1}{(1+\sigma^2b^2)^2R^3} + \frac{1}{(1+\sigma^2b^2)^3R^4}\right)\right]\end{aligned} \tag{5.102}$$

Let us match the coefficients of the next to leading order terms. Matching $\frac{1}{R}$ terms, we get

$$-2+d_1^1-\ln(1+\sigma^2b^2) = -3+e_1^1+3\ln m \tag{5.103}$$

As both d_1^1 and e_1^1 are arbitrary constants, this equation can be satisfied by choosing them to make both sides equal to zero. We see that all the terms proportional to $\ln R$ match. The next condition comes from matching the coefficients of higher powers of $1/R$. This fixes L_1.

$$L_1b^2 = -1-\sigma^2b^2+2\ln(1+\sigma^2b^2) \tag{5.104}$$

In the solution $\tilde{\chi}$ (5.101), we have truncated the series in (5.88) and matched the solutions. Looking at both the solutions (5.101) and (5.102), we expect the rest of the terms in the series in (5.88) will be reproduced by calculating higher order corrections for $\hat{\chi}(R)$ in (5.98). We would expect the coefficients of the terms in higher powers of $\frac{1}{(1+\sigma^2b^2)R}$ will be $-\frac{(1+L_1b^2+\sigma^2b^2)}{2}$ and that the $(\log R)$ terms will match upon substituting L_0 at all orders.

The value of λ up to next to leading order is,

$$\lambda b^2 = m(1-\sigma^2b^2) - \left[1+\sigma^2b^2-2\ln(1+\sigma^2b^2)\right]. \tag{5.105}$$

For $\sigma = 0$, i.e. the Schwarzschild black hole case, this value of λ matches that obtained in [29] till next to leading order. In fact for small black holes taking $\sigma b \ll 1$, we can write our eigenvalue

$$\lambda b^2 = (m-1)(1-\sigma^2b^2). \tag{5.106}$$

We have calculated λ only up to two orders here. We would need to calculate higher order solutions of both $\tilde{\chi}$ and $\hat{\chi}$ in order to calculate higher order corrections to λ. The solution for $\tilde{\chi}$ at the next to leading order contains polylogarithmic functions in R. The solutions then become very hard to analyse. Hence we have not proceeded further.

5.6 Results and Discussion

In this chapter we studied the semiclassical stability problem for the SAdS black hole under scalar perturbations. for non-spherically symmetric perturbations, we have used the equations obtained in (chapter 2). In the case of static perturbation, one of the three resulting equations decouples from the others. We analyze the decoupled equation using the S-deformation approach [24], [96]. We have showed semiclassical stability for this class of perturbations. For this class of perturbations, we have not used the large D limit to analyze our equation. Further, this result is valid for both large and small black holes.

We next analyze the two coupled equations in the non-spherically symmetric perturbation sector using the large D limit. For a class of perturbations we prove the absence of unstable modes with eigenvalue of $\mathcal{O}(D^2)$ in the case of the large black hole. We cannot make concrete statements about the small black hole. This is because while the near region analysis is possible for the small black hole, the far region solution cannot be continued to a simple expression in the overlap region.

We finally analyze the spherically symmetric perturbation using the large D approach. We have analyzed the equation in the near and far regions as before. The equation in the near region is a Heun equation making it difficult to find solutions. There are two singular points in the near region, one being the horizon where $R \sim \mathcal{O}(1)$ and the other at $R \sim \mathcal{O}(D)$. We employ a procedure first used in [31] to find decoupled quasinormal modes of Schwarzschild-Tangherlini black holes in the large D limit. We find the normalizable solution for the subregion with $R \sim \mathcal{O}(1)$ and the general solution in the subregion with $R \sim \mathcal{O}(D)$, then extend them to an overlap region $1 \ll R \ll D$ and match the two solutions. After this matching procedure, the solution around the second singular point is compared with the normalizable solution in the far region. We found that there is no unstable mode with eigenvalue of $\mathcal{O}(D^2)$. This is consistent with the bound on this eigenvalue obtained by Prestidge [91]. On the other hand, there is indeed an unstable mode with eigenvalue of $\mathcal{O}(D)$ for the small black hole which disappears on increasing the mass. Thus the large black hole is stable. This is the analogue of the Hawking-Page phase transition in semiclassical stability analysis. We obtain this eigenvalue corresponding to this unstable mode for the first time in a large D limit in a $1/D$ expansion. We compute the eigenvalue to next to leading order and for $\Lambda = 0$, this agrees with the value of the unstable mode in [29] for the Schwarzschild black hole in the large D limit. The agreement is to next to leading order. As shown in [29], the $\Lambda = 0$ value corresponds to the Gross-Perry-Yaffe unstable mode [83] for the D-dimensional Schwarzschild instanton which is also mathematically related to the Gregory-Laflamme unstable mode [6] for the $(D+1)$-dimensional flat black string.

Chapter 6

Summary

In this thesis we have studied the linearized non-spherically symmetric perturbations of black strings and black holes in D dimensions. We have analyzed classical stability and quasinormal modes of the black string/brane. We have also studied semiclassical stability of Schwarzschild and Schwarzschild-Anti-de Sitter (SAdS) black holes in the Euclidean path integral approach of quantum gravity. Most of the previous research on the classical stability of black string and semiclassical stability of black holes has been focused on the spherically symmetric perturbations. The non-spherically symmetric perturbations were relatively unexplored. Gregory and Laflamme showed that the black string is unstable under spherically symmetric perturbations [6]. Existence of a single spherically symmetric, static semiclassically unstable mode for Schwarzschild black holes in four dimensions was proven by Gross, Perry and Yaffe in [83]. Such an unstable mode was numerically found for the D-dimensional SAdS black holes by Prestidge in [91]. The unstable modes of $(D+1)$ dimensional black string are related to semiclassical unstable modes of D-dimensional Schwarzschild black holes [12]. In this thesis our aim was to study these instabilities in the sector of non-spherically symmetric perturbations. We have also studied quasinormal modes of black strings and Schwarzschild black holes in D dimensions.

To this end we first formulated equations governing these perturbations by extending the formalism of gauge invariant variables by Ishibashi and Kodama. These gauge invariant variables were formulated to study classical perturbations of D-dimensional black holes. To use these variable for studying the $(D+1)$ dimensional black string, where these variables are no longer gauge invariant, we made an appropriate gauge choice which made the perturbation equations on the string to reduce to an eigenvalue equation on the corresponding D-dimensional black hole background. The perturbations were then decomposed in terms of the scalar, vector and tensor spherical harmonics on the $(D-2)$-sphere part of the string metric. We then wrote the eigenvalue equations for scalar and vector perturbations in terms of the Ishibashi-Kodama variables. In our formulation, the vector perturbations reduce to a system of two coupled ODEs. The scalar perturbations reduce to three coupled ODEs. In his set of notes [26], Kodama formulated a set of gauge invariant variables for the black strings. On imposing our gauge the equations governing vector perturbations in the two approaches match. The tensor perturbations which reduce to an ODE for a single function have already been discussed in [26]. The scalar and vector perturbation equations form a set of coupled ODEs which cannot be solved analytically. Our simplified perturbation equations are amenable to a numerical study to investigate several aspects of stable black brane perturbations. We hope this work also serves as a pointer for tackling more difficult problems such as a direct study of non-spherically symmetric perturbations of curved black branes.

We have used the large D limit as an analytical tool to study our equations. The large D limit was first used by Asnin et al in [29] and Kol et al in [29] to analyze the spherically symmetric black string perturbations. Effect of this limit on black holes and other black objects in general relativity was studied in detail by Emparan et al [30]. In the large D limit, the gravitational field of the black hole becomes localized only very near the horizon. This introduces a length scale (r_0/D) in the theory (r_0 is the horizon). This creates two distinct regions in the black hole spacetime called *near region* and *far region* with a small overlap region between them. This feature of spacetime simplifies the equations governing perturbations and enables use of the matched asymptotic expansions. Further simplification is achieved by expanding the perturbations as a series in the small parameter $(1/D)$ in the near region. The equations then can be solved order by order in D. Emparan et al pioneered the use of matched asymptotic expansions to study perturbations of black holes in this limit [30].

To analyze the stability of the black string, following Gregory and Laflamme [6], we have assumed a time behaviour $e^{\Omega t}$ for the perturbations. We investigated if there are normalizable solutions to the perturbation equations with Ω positive. We require the perturbations to be finite at the horizon for consistency of perturbation theory and normalizable asymptotically. Defining $\Omega = D\hat{\Omega}$, we have concentrated on the perturbations where all the parameters in the equation $\Omega, \lambda, \ell \sim \mathcal{O}(D)$. The vector equations decouple in the near-horizon region and the asymptotic region. Using the technique of matched asymptotic expansions, we have ruled out instabilities in this sector. In the case of the three scalar perturbation equations for perturbation variables H, G and $\tilde{\eta}$, the equation for H decouples in the near-horizon and asymptotic regions.As in the vector case, we show this does not lead to instabilities. The other two perturbations remain coupled in the near-horizon region, although they can be solved asymptotically. We have analyzed these perturbations in various cases. For the special case when one of them is zero, the other does not lead to instabilities. In the case when both G and $\tilde{\eta}$ are non-zero, we argue that there are no instabilities by a heuristic two step matching procedure. These results provide direct evidence from the analysis of the equations themselves that the Gregory-Laflamme unstable mode is the only instability of the flat black string in the large D limit. In the static limit $\hat{\Omega} = 0$, we have showed that none of the vector or scalar perturbations lead to an instability. In the static limit, this result also proves that the corresponding Gross-Perry-Yaffe mode for semiclassical black hole perturbations is the *unique* unstable mode in the large D limit.

A striking feature of the large D limit is the existence of two distinct sets of quasinormal modes decoupled modes with $\omega \sim \mathcal{O}(1)$ and non-decoupled modes with $\omega \sim \mathcal{O}(D)$. First found by Emparan et al in [31], the decoupled quasinormal modes are ingoing at the horizon and decay far from it. These modes are localized near the horizon. We have obtained quasinormal mode frequencies for black string and revisited the analysis of quasinormal of D-dimensional Schwarzschild-Tangherlini black holes in this limit.

We have used the method of Laplace transforms by Nollert and Schmidt [80] to study the non-decoupled modes. We have obtained the leading order frequencies for both vector and scalar non-decoupled modes for the black string. We have discussed the limitations of the Laplace transform method to obtaining the next to leading order corrections to the non-decoupled modes. For $\lambda = 0$ our results match the leading order frequency obtained in [31]. For the black string case, we have obtained the vector decoupled quasinormal mode frequency to leading order. We see that the effect of the mode depending on the extra dimension, signified by λ will be seen only from the next to leading order. This work is in progress.

To understand the effect of large D limit as it pertains to the perturbation equation and its solutions, we have addressed the problem of black hole quasinormal modes without assuming an expansion of the mode function as a series in $1/D$. We found that the vector decoupled quasinormal modes, the perturbation equation reduces to a degenerate case of the hypergeometric equation at leading order in D and its general solution cannot be obtained from that of the leading order equation as a series in $1/D$. We have evaluated the decoupled vector quasinormal modes to $\mathcal{O}(\frac{1}{(D-3)^2})$. It agrees with previous computations in the $1/D$ expansion [31] as well as numerical results [34] up to the next to leading order. The equation governing scalar quasinormal modes is a Heun equation and is hard to analyze analytically.We found that it is not possible to compute the decoupled scalar quasinormal modes without a $1/D$ expansion.

We have finally studied the semiclassical stability problem for the SAdS black hole under static scalar perturbations. In the case of non-spherically symmetric perturbations, one of the three resulting equations decouples from the others. We analyze the decoupled equation using the S-deformation approach [24], [96]. We have showed semiclassical stability for this class of perturbations for all D. Further, this result is valid for both large and small black holes. We analyzed the two coupled equations in the non-spherically symmetric perturbation sector using the large D limit. For a class of perturbations we proved the absence of unstable modes with eigenvalue of $\mathcal{O}(D^2)$ in the case of the large black hole. As in the case of small black holes the near region analysis is possible but the far region solution cannot be continued to a simple expression in the overlap region, we cannot make concrete statements.

We have also analyzed the spherically symmetric semiclassical perturbations of the SAdS black holes. In this case the equation in the near region is a Heun equation that has two singular points in the near region, one being the horizon where $R \sim \mathcal{O}(1)$ and the other at $R \sim \mathcal{O}(D)$. We employed a two step matching procedure which was first used in [31] to find decoupled quasinormal modes of Schwarzschild-Tangherlini black holes in the large D limit. Using this procedure, we found that there is no unstable mode with eigenvalue of $\mathcal{O}(D^2)$.

This is consistent with the bound on this eigenvalue obtained by Prestidge [91].

We showed that there is indeed an unstable mode with eigenvalue of $\mathcal{O}(D)$ for the small black hole which disappears on increasing the mass. Thus the large black hole is stable. This is the analogue of the Hawking-Page phase transition in semiclassical stability analysis. A delicate calculation in a $1/D$ expansion is needed to obtain the eigenvalue corresponding to this unstable mode. We computed the eigenvalue to next to leading order and for $\Lambda = 0$, this agrees with the value of the unstable mode in [29] for the Schwarzschild black hole in the large D limit. The agreement is to next to leading order. The significance of the unstable mode is that there exists a metric perturbation of the small black hole that decreases the Euclidean action. Thus, in the Euclidean path integral approach to quantum gravity, the small black hole is a saddle point of the action, not a true minimum.

In this thesis we have demonstrated the power of the large D limit of general relativity in tackling difficult problems in black string perturbation theory using simple technique of matched asymptotic expansion. Although we have concentrated on the flat black strings, our results are extendable to the flat black branes. A possible extension of this work is to use our techniques to formulate and study the perturbations on curved black branes.

The large D limit has opened a new avenue in the analytical study of various higher dimensional black objects like black branes, black rings etc. The simplified structure of spacetime and the use of $(1/D)$ as a small perturbation parameter has been used to formulate fully nonlinear effective field approaches that capture dynamics of black objects at a scale $1/D$. These approaches have been successfully applied to study various instabilities in the higher dimensional black objects. A natural way forward is to apply this limit to study black hole solutions in various modified theories of gravity. Efforts in this direction are already underway [58]–[64]. In the case of general relativity, it would be interesting to see if one can use the formalism to study the non-linear processes like black hole mergers of which we currently do not have much analytical understanding. We hope that the looking at such processes in the large D limit helps us get insight into their dynamics in four dimensions.

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