Gabber's Presentation Lemma

A thesis

submitted in partial fulfillment of the requirements
of the degree of

Doctor of Philosophy

by

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IISER PUNE

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH PUNE

2020

Dedicated to
My Wife, Parents, Guru & Teachers

Certificate

Certified that the work incorporated in the thesis entitled "*Gabber's Presentation Lemma*", submitted by *Girish M. Kulkarni* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Date: February 17, 2020

Dr. Amit Hogadi
Thesis Supervisor

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Acknowledgements

I begin by thanking my Ph.D. supervisor Prof. Amit Hogadi, for his constant support and encouragement throughout my Ph.D. He provided me an atmosphere in which I could conduct my research in the smoothest way possible. He was always available for discussions, throughout which he was patient enough to explain anything to me. His insights and mathematical guidance have given me a solid base to start my independent research. I consider myself very fortunate to have such a wonderful person as my supervisor.

Besides my supervisor, I would like to thank the rest of my research advisory committee: Prof. S. A. Katre and Dr. Vivek Mohan Mallick for their encouragement. I had the opportunity to discuss mathematics with several people. I thank them for their support and help. In particular, I would like to thank Prof. Chetan Balwe, Dr. Anand Sawant and Prof. Utsav Choudhury.

I thank Dr. Tejas Kalelkar, who was my minor thesis advisor. I also thank my coursework teachers: Dr. Rama Mishra and Dr. Anindya Goswami Dr. Baskar Balasubhramanyam, Dr. Kaneenika Sinha, Dr. Tejas Kalelkar, and Dr. Ronnie Sebastian. I am grateful to all of them.

I am thankful to UGC-CSIR for financial support during my Ph.D. I am also grateful to Dr. Chitrabhanu Choudhary for supporting me financially for the last year of my Ph.D. I acknowledge the support of the institute and its administrative staff members for their cooperation, special thanks to Mrs. Suvarna Bharadwaj, Mr. Tushar Kurulkar, and Mr. Yogesh Kolap.

I am grateful to my teachers starting from my school till date for inspiring me and guiding me in the right direction. I thank all my friends in IISER Pune, with whom I shared a good time. Many of you will recognize yourselves, and I hope that you will forgive me for not naming you individually.

All the students of Mathematics at IISER Pune deserves a note of appreciation for being enthusiastic about discussing mathematics with me. I thank Rohit, Sushil, Manidipa, Pralhad, Prabhat, Makarand, Jatin, Neha, Gunja, Milan, Tathagata, Debangana, Jyotirmoy, Ayesha and others for their help and discussions. A special thanks to Neeraj and Suraj, my 'gurubandhus' with whom I discussed for the last three years of my research. With their help, I could solve a problem which is a part of this thesis.

I must express my deepest gratitude to my wife for providing me with unconditional support and constant encouragement throughout my years of studies and research. I would also like to express my gratitude towards my parents for everything they have done to take me at this level.

Finally, I would like to thank my Guru for his grace and all of those who knowingly and unknowingly have helped me in different ways during my Ph.D.

Contents

Abstract

Gabber's presentation lemma is a foundational result in $\mathbb{A}^1$-homotopy theory. This result can be thought of as an algebro-geometric analog of the tubular neighborhood theorem in differential geometry. Similar to tubular neighbourhood theorem, this lemma gives the local model of the inclusion of a closed subscheme into a smooth scheme. The lemma was proved in 1994 by O. Gabber in the case where the base is a spectrum of an infinite field. We present a proof when the base is a finite field. Further in 2018, S. Schmidt and F. Strunck proved Gabber's presentation lemma over the Henslian discrete valuation rings. We further generalize this result over any noetherian domain with all its residue fields infinite. We also discuss various applications of this lemma in $\mathbb{A}^1$-homotopy theory, which includes a connectivity result.

1

Introduction

In this chapter, we give a brief introduction of Gabber's presentation lemma. In the first section, we present a short history of the lemma. In the next section, we mention the main results of the thesis. We also explain the strategies and the main difficulties in the proofs.

1.1 A short history of Gabber's presentation lemma

The tubular neighbourhood theorem in differential geometry implies that, the smooth immersion of a smooth sub-manifold N, in to a smooth manifold M, given by $i : N \hookrightarrow M$ locally looks like the embedding $\mathbb{A}^n \hookrightarrow \mathbb{A}^m$, where n and m are dimensions of N and M respectively. In algebraic geometry, if the inclusion $i : Y \hookrightarrow X$ is smooth, then (étale) locally i looks like the inclusion $\mathbb{A}^n \hookrightarrow \mathbb{A}^m$, where m and n is are the dimensions of X and Y. Gabber's presentation lemma gives the local picture in the case where Y is not necessarily smooth. The lemma says that there exists V, an open subset of $\mathbb{A}^{m-1}$ such that (Nisnevich) locally Y can be embedded in $\mathbb{A}^1_V$ with Y/V finite.

O. Gabber proved this result in 1994 in [Gab], where he calls it a preparation lemma. He mentioned that the preparation lemma is close to a presentation lemma of Ojanguren [Oja]. This lemma was then used to extend a result of Bloch-Ogus [BO]. In the year 1997 Jean-Louis Colliot-Théléne, Raymond T. Hoobler, and Bruno Kahn proved a little stronger version of the lemma in [CTHK]. They used this lemma to prove Gabber's effacement theorem in general case.

Following is the statement of the Gabber's presentation lemma as proved in [CTHK, Theorem 3.1.1]

Theorem 1.1.1 (Gabber's presentation lemma)**.** *Let X be a smooth, affine, irreducible variety of dimension d over an infinite field k; let $t_1, \dots, t_r \in X$ be a finite set of points and Z a closed subvariety of a positive codimension. Then there exists a map $\phi = (\psi, \nu) : X \to \mathbb{A}^{d-1} \times \mathbb{A}^1$, an open set $V \subset \mathbb{A}^{d-1}$, and an open set $U \subset \psi^{-1}(V)$ containing $t_1, \dots, t_r$ such that*

1. *$Z \cap U = Z \cap \psi^{-1}(V)$;*
2. *$\psi_{|Z}$ is finite;*
3. *$\psi_{|U}$ is étale and defines a closed immersion $Z \cap U \to \mathbb{A}^1_V$;*
4. *$\phi(t_i) \notin \psi(Z)$ if $t_i \notin Z (1 \leq i \leq r)$;*
5. *$\phi^{-1}(\phi(Z \cap U)) \cap U = Z \cap U$.*

1.2 Main Results

Gabber's presentation lemma over finite fields

In the statement of Gabber's presentation lemma, the base scheme is assumed to be a spectrum of an infinite field. In private communications with F. Morel, O. Gabber pointed out that Theorem 1.1.1 is not true in general for $r > 1$, when the base is a finite field, but for $r = 1$ it holds. However, the proof was not published. A proof of this result over finite fields is the main content of this thesis.

Theorem 1.2.1. *[HK, Theorem 1.1] Let X be a smooth variety of dimension $d \geq 1$ over a finite field F and $Z \subset X$ be a closed subvariety. Let $p \in Z$ be a point. Let $\mathbb{A}^d_F \xrightarrow{\pi} \mathbb{A}^{d-1}_F$ denote the projection onto the first $d-1$ coordinates. Then there exists*

(i) *an open neighbourhood* $U \subset X$ *of* p*,*

(ii) *a map* $\Phi : U \to \mathbb{A}^d_F$*,*

(iii) *an open neighbourhood* $V \subset \mathbb{A}^{d-1}_F$ *of* $\Psi(p)$ *where* $\Psi : U \to \mathbb{A}^{d-1}_F$ *denotes the composition*

$$U \xrightarrow{\Phi} \mathbb{A}^d_F \xrightarrow{\pi} \mathbb{A}^{d-1}_F$$

such that

1. Φ *is étale.*

2. $\Psi_{|Z_V} : Z_V \to V$ *is finite, where* $Z_V := Z \cap \Psi^{-1}(V)$*.*

3. $\Phi_{|Z_V} : Z_V \to \mathbb{A}^1_V = \pi^{-1}(V)$ *is a closed immersion.*

In the proof of the lemma over infinite fields (Theorem 1.1.1), it is shown that indeed the generic choices of the maps work. If we try to apply a similar technique in the case of finite fields, the set of maps satisfying necessary condition may be empty, so the same approach fails. The difficulties in proving this result are similar to those encountered in proving Bertini's theorem over finite fields. Hence the initial approach was to use the ideas from Poonen's proof of Bertini's theorem for finite fields. But we could only prove the result for the special case of open subsets of $\mathbb{A}^2$ with those ideas. A substantial part of the proof is to reduce to this special case using dévissage and induction. Firstly, in Lemma 4.1.6 we reduce to the case where X is an open subset of affine space, we use Noether normalization Lemma and Nakayama Lemma to achieve this. Then in Lemma 4.1.14 by using induction, we reduce to the case where X is open subscheme of $\mathbb{A}^2_F$. Now the case of open subsets of $\mathbb{A}^2_F$ is dealt with using Poonen's counting argument. Indeed the argument for points of small degree is quite similar to that of [Poo]. However, for 'high degree points' we could not use these ideas. We fix this with a small trick (see Lemma 4.1.24).

This is joint work with my advisor Amit Hogadi.

Gabber's presentation lemma over noetherian domains

A generalization of Theorem 1.1.1 in a different direction was proved in 2018 by J. Schmidt and F. Strunk [SS]. They proved the lemma when the base is a spectrum of a

Henselian discrete valuation ring with infinite residue field. This result is true Nisnevich locally on the base (see [SS, Theorem 2.4]). As an application, J. Schmidt and F. Strunk prove the shifted stable $\mathbb{A}^1$-connectivity result over Dedekind schemes.

We extend Gabber's presentation lemma to noetherian domains whose all residue fields are infinite (see Theorem 1.2.2). Similar to [SS], this proves the following result, which is Nisnevich local on the base and the source.

Theorem 1.2.2. *[DHKY, Theorem 1.1] Let $S = \mathrm{Spec}\,(R)$ be the spectrum of a noetherian domain with all its residue fields infinite. Let X be a smooth, irreducible, equi-dimensional S-scheme of relative dimension d. Let $Z \subset X$ be a closed subscheme, z be a closed point in Z lying over $s \in S$, such that that $dim(Z_s) < dim(X_s)$. Then after possibly replacing S by a Nisnevich neighbourhood of s and X by a Nisnevich neighbourhood of z, there exists a map $\Phi = (\Psi, \nu) : X \to \mathbb{A}^{d-1}_S \times \mathbb{A}^1_S$, an open subset $V \subset \mathbb{A}^{d-1}_S$ and an open subset $U \subset \Psi^{-1}(V)$ containing z such that*

1. $Z \cap U = Z \cap \Psi^{-1}(V)$
2. $\Psi_{|Z} : Z \to \mathbb{A}^{d-1}_S$ *is finite*
3. $\Phi_{|U} : U \to \mathbb{A}^d_S$ *is étale*
4. $\Phi_{|Z\cap U} : Z \cap U \to \mathbb{A}^1_V$ *is a closed immersion*
5. $\Phi^{-1}(\Phi(Z \cap U)) \cap U = Z \cap U.$

The strategy to prove this result is to follow [SS] roughly. An essential ingredient of the proof of Theorem 1.2.2 and indeed that of [SS] is [Kai, Theorem 3] which is a consequence of Levine's result [Lev, Theorem 10.2.2]. This result states that given an equi-dimensional scheme Y over a Dedekind scheme B with infinite residue fields, Nisnevich locally on B there exists a projective closure $\overline{Y}$ of Y in which Y is fiber-wise dense. Unfortunately, we could not extend the result over a general base in its full strength. However, we observe that a slightly weaker form (see Theorem 4.2.2) can be proved, which is sufficient in our case. After securing the finiteness condition on maps in this way, we follow [CTHK] verbatim, to ensure the remaining requirements.

The condition of residue fields being infinite allows us to make suitable generic choices, as in the original proof of the presentation lemma [CTHK] as well as [SS]. This is joint work with Neeraj Deshmukh, Amit Hogadi and Suraj Yadav.

The following corollary of Gabber's presentation lemma, is crucial for the $\mathbb{A}^1$-homotopy theory as developed by Morel in [Mor3].

Corollary 1.2.3. *With the notation as in Theorem 1.2.1. The map* $\psi_{|Z\cap U} : Z \cap U \to V$ *is finite and the following square commutes*

$$\begin{array}{ccc} Z \cap U & \xhookrightarrow{cl.} & U \\ \Big\downarrow \cong & & \Big\downarrow \phi_{|U} \\ \phi(Z \cap U) & \xhookrightarrow{cl.} & \mathbb{A}^1_V \end{array}$$

where horizontal maps are closed immersions, left vertical map is an isomorphism and the right vertical map is étale. Consequently giving an isomorphism of Nisnevich sheaves

$$U/\Big(U - (Z \cap U)\Big) \to \mathbb{A}^1_V/\Big(\mathbb{A}^1_V \setminus \phi(Z \cap U)\Big).$$

2

A short introduction to $\mathbb{A}^1$-homotopy theory

Gabber's presentation lemma [Gab] is an essential tool to derive several results in $\mathbb{A}^1$-homotopy theory (see [Mor3]). To understand the applications of Gabber's presentation lemma, we now present a brief introduction to the $\mathbb{A}^1$-homotopy theory.

2.1 Model structure

A model structure on a category is a framework to do homotopy theory. The beauty of this framework lies in the freedom to invert certain morphisms without leaving the set-theoretic universe. Let us recall the definition of a model category from [Hov] or [GJ].

Definition 2.1.1. Let M be a category with all small limits and colimits. A model category structure on M consists of three classes W, C, F of morphisms in M, called weak-equivalences, cofibrations, and fibrations respectively, subject to the following set of axioms.

M1 Given two composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in M, if any two are weak equivalences then so is the third.

M2 If f is a retract of g and g is a weak equivalence, fibration or cofibration, then so is f (for the definition of retract see [Hov, Def. 1.1.1]).

M3 Given a diagram of solid arrows, a dotted arrow can be found making the following diagram commutative

$$\begin{array}{ccc} Z & \longrightarrow & E \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ X & \longrightarrow & B \end{array}$$

if either

(a) p is an trivial fibration i.e. $(p \in W \cap F)$ and i is a cofibration, or

(b) i is an trivial cofibration i.e. $(i \in W \cap C)$ and p is a fibration.

M4 Any map $X \to Z$ in M admits two factorizations, $X \xrightarrow{f} E \xrightarrow{p} Z$ and $X \xrightarrow{i} Y \xrightarrow{g} Z$, such that f is an trivial cofibration, p is a fibration, i is a cofibration, and g is an trivial fibration.

Remark 2.1.2. *Any of the two of the three classes of maps weak equivalence, fibrations and cofibrations determine the third (see [Hir, Prop. 7.2.7]).*

Definition 2.1.3. In a model category an object X is called **fibrant** if the map $X \to *$ to the final object is a fibration and X is called **cofibrant** if the map $\emptyset \to X$ from the initial object is a cofibration. Given an object X in M, a trivial fibration $QX \to X$, where QX is cofibrant is called **cofibrant replacement** of X. If $X \to RX$ is a trivial cofibration then it is called **fibrant replacement**.

The **homotopy category** of a model category M is the category obtained by inverting the class of weak equivalences. By a theorem of Quillen [Qui] such a category exists and it is denoted by $\mathcal{H}o(M)$ (see also [Hov, Theorem 1.2.10]).

Example 2.1.4 (Simplicial sets)**.** *Let Δ be the category of finite non-empty ordered sets with order preserving maps of sets. The category of* ***simplicial sets****, $sSet$, is the category of functors $\Delta^{op} \to Sets$. There is a geometric realization functor $|\cdot| : sSet \to Top$, a map $f : X \to Y$ of simplicial sets is a weak equivalence if $|f| : |X| \to |Y|$ is a weak homotopy equivalence and it is cofibration if for each $n \geq 0$, $f(n) : X(n) \to Y(n)$ is an injective map of sets. The fibrant objects in $sSet$ are called Kan complexes (for more* *details see [GJ]). We denote by Δ^n the standard simplicial n-simplex. The 0-simplex S^0 which is also denoted by $*$ is called the point and the quotient of Δ^1 by the boundary $\partial\Delta^1$ is denoted by S^1.*

2.2 Simplicial Model category

If X and Y are simplicial sets, then we can define the **simplicial mapping space** $map_{sSet}(X, Y)$ as a simplicial set with n-simplices given by $Hom_{sSet}(X \times \Delta^n, Y)$. If such a construction is possible in some category it is called a **simplicial category**. In addition to the mapping space functor

$$map_M : M \times M^{op} \to sSet,$$

this category is also equipped with an action of $sSet$,

$$M \times sSet \to M$$

written as $X \otimes S$. For the complete definition of a simplicial category see [GJ, Def II.2.1].

Definition 2.2.1. A model category M is called a **simplicial model category**, if it is a simplicial category and it satisfies the following axiom (**SM7**):
For any cofibration $i : A \to B$ and a fibration $q : X \to Y$

$$map_M(B, X) \to map_M(A, X) \times_{map_M(A,Y)} map_M(B, Y)$$

is a fibration of simplicial sets, which is a weak equivalence if either i or q is.

One of the advantages of having a simplicial model category is, the notion of homotopy can be defined easily. If X is a cofibrant object in a simplicial model category M, we say that the two maps $f, g : X \to Y$ are homotopic if there is a map $H : A \otimes \Delta^1 \to X$ such that

$$\begin{array}{ccc} X \coprod X & \xrightarrow{d_1 \coprod d_0} & X \otimes \Delta^1 \\ {\scriptstyle f \coprod g}\downarrow & \swarrow_{H} & \\ Y & & \end{array}$$

commutes.

2.3 Bousfield localization

Bousfield localization is a handy way to produce a new model category by increasing the class of weak equivalences in a given model category $\mathcal{C}$. Taking the homotopy category of the localization of $\mathcal{C}$ is equivalent to inverting morphisms in the homotopy category $\mathcal{H}o(\mathcal{C})$ of $\mathcal{C}$. For a detail treatment of Bousfield localization see [Hir, I.3.3].

Definition 2.3.1. 1. Let M be a model category, W be the class of weak equivalences in M and I be a set of morphisms in M. An object X is called ***I*-local** if it is fibrant and for all $i : A \to B$ with $i \in I$, the induced map on mapping spaces

$$i^* : map_M(B, X) \to map_M(A, X)$$

is a weak equivalence of simplicial sets.

2. A morphism $f : A \to B$ in M is called ***I*-local weak equivalence** if for every I-local object X

$$f^* : map_M(B, X) \to map_M(A, X)$$

is a weak equivalence of simplicial sets. If we denote the set of I-local weak equivalences by W_I then by **SM7** it follows that $W \subset W_I$.

Denote by F_I the class of morphisms satisfying right lifting property with respect to the class of trivial cofibrations $(W \cap C)$. Now suppose the category (M, W_I, C, F_I) is a model category, we call it as a **left Bousfield localization** of the model category M with respect to I. The following Theorem 2.3.3 guarantees the existence of left Bousfield localization under certain conditions on the model category, the proof of this can be found in [HTT, Prop.A.3.7.3]

Definition 2.3.2. A model category M is called **left proper** if weak equivalences are preserved under pushouts along cofibrations (see [Hir, Def.13.1.1]).

Theorem 2.3.3. *[HTT, Prop. A.3.7.3] If M is a left proper, combinatorial, simplicial model category and I is a set of morphisms in M then the left Bousfield localization $L_I M$*

of M exists and inherits a simplicial model category structure; moreover, it is left proper and combinatorial.

Remark 2.3.4. *The condition of being combinatorial is due to Jeff Smith. Most of the interesting categories are combinatorial. For example, the category of simplicial sets, simplicial sheaves, etc. The combinatorial model categories are in particular cofibrantly generated. See [HTT, Def. A.2.6.1] for details.*

Remark 2.3.5. *The left Bousfield localization also exists if the model category is left proper and cellular, see [Hir, Theoreom 4.1.1] for more details.*

Remark 2.3.6. *The fibrant objects in $L_I M$ are precisely the fibrant objects of M, which are I-local.*

2.4 The category $sPre(\mathcal{S}m_S)$

Let S be a scheme and $\mathcal{S}m_S$ denote the category of smooth schemes of finite type over S. The category of presheaves of simplicial sets on $\mathcal{S}m_S$ is denoted by $sPre(\mathcal{S}m_S)$. We now describe a model structure on this category.

- weak equivalences are objectwise weak equivalences: maps $f : \mathcal{X} \to \mathcal{Y}$, such that for each $X \in \mathcal{S}m_S$, $f(X) : \mathcal{X}(X) \to \mathcal{Y}(X)$ is a weak-equivalence of simplicial sets.

- cofibrations are objectwise cofibrations.

such a model structure is called the injective model structure on $sPre(\mathcal{S}m_S)$ (see [DHI]).

Proposition 2.4.1. *The category of simplicial presheaves on $\mathcal{S}m_S$ with objectwise weak equivalences and objectwise cofibrations is a left proper combinatorial simplicial model category.*

Proof. See [HTT, Prop. A.2.8.2] and [HTT, Remark A.2.2.4]. □

2.5 Nisnevich topology

An étale cover $\mathcal{U} \to X$ of a scheme X is called a Nisnevich cover if it is surjective on k-points for all fields k. These covers give a pretopology on $\mathcal{S}m_S$, the topology generated by this pretopology is called the **Nisnevich topology** on $\mathcal{S}m_S$. For details on Grothendieck topology see [AGV, II.1.1.5], in particular for Nisnevich topology see [MV, Section 3.1.1].

Remark 2.5.1. *There are several reasons to prefer the Nisnevich topology. We list a few of those here.*

- *The Nisnevich cohomological dimension of a noetherian scheme is bounded by the Krull dimension.*
- *K- theory is representable in Nisnevich topology but not in étale topology.*
- *An advantageous property of the Nisnevich topology is that it can be generated from the Cartesian squares of the following type*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{\;j\;} & X \end{array}$$

where p is étale morphism, j is an open embedding and $p^{-1}(X - U) \to X - U$ is an isomorphism. This provides an easy way to check the sheaf condition in the Nisnevich topology.

Let $\mathcal{S}_{Nis}$ denote the set of covering sieves for Nisnevich topology, here we view X as a representable presheaf. The left Bousfield localization of $sPre(\mathcal{S}m_S)$ with respect to $\mathcal{S}_{Nis}$ gives the Nisnevich local model structure, we denote this new model category by $L_{Nis}sPre(\mathcal{S}m_S)$. The fibrant objects in $L_{Nis}sPre(\mathcal{S}m_S)$ are those presheaves of Kan complexes which take the Cartesian squares in Remark 2.5.1 for all schemes $X \in \mathcal{S}m_S$ and all U, V as per in the diagram to homotopy Cartesian squares.

Remark 2.5.2. *When S is noetherian of finite Krull dimension, the fibrant objects in $L_{Nis}sPre(\mathcal{S}m_S)$ can be identified via hypercover descent as well (see [DHI] for more details).*

2.6 Unstable $\mathbb{A}^1$-homotopy category

Let I be the morphism $pt \to \mathbb{A}^1$. The left Bousfield localization of $L_{Nis}sPre(\mathcal{S}m_S)$ with respect to I gives the category $L_{\mathbb{A}^1}(L_{Nis}sPre(\mathcal{S}m_S))$, the homotopy category of this category is called the unstable $\mathbb{A}^1$-homotopy category $\mathcal{H}(S)$.

The fibrant objects in this category are the Nisnevich sheaves of Kan complexes that are $\mathbb{A}^1$-local. Recall that, $\mathcal{Y} \in sPre(\mathcal{S}m_S)$ is $\mathbb{A}^1$-local if for any $\mathcal{Y} \times \mathbb{A}^1 \to \mathcal{Y}$, with $\mathcal{Y} \in sPre(\mathcal{S}m_S)$, the induced map

$$Hom_{L_{Nis}sPre(\mathcal{S}m_S)}(\mathcal{Y}, \mathcal{X}) \to Hom_{L_{Nis}sPre(\mathcal{S}m_S)}(\mathcal{Y} \times \mathbb{A}^1, \mathcal{X})$$

is a bijection.

There exists a functor $L^{\mathbb{A}^1} : sPre(\mathcal{S}m_S) \to sPre(\mathcal{S}m_S)$ such that given any simplicial presheaf $\mathcal{X}$, $L^{\mathbb{A}^1}(\mathcal{X})$ is a fibrant object in $\mathcal{H}(S)$ (for construction see [MV, page 107]), this functor is called $\mathbb{A}^1$**-fibrant replacement functor**.

Let $\pi_i^s(X, x)$ denote the i^{th} homotopy group of a pointed simplicial set (X, x). Then the i^{th} $\mathbb{A}^1$**-homotopy sheaf** of a pointed simplicial sheaf $(\mathcal{X}, x)$ is defined to be

$$\pi_i^{\mathbb{A}^1}(\mathcal{X}, x)(U) = \pi_i^s(L^{\mathbb{A}^1}(\mathcal{X})(U), x)$$

.

The following definition plays a central role in $\mathbb{A}^1$-homotopy as developed by F. Morel in [Mor3].

Definition 2.6.1. Let k be a field and $S = Spec(k)$

1. A sheaf of sets $\mathcal{F}$ on $\mathcal{S}m_S$ in the Nisnevich topology is said to be $\mathbb{A}^1$**-invariant** if for any $X \in \mathcal{S}m_S$, the map $\mathcal{F}(X) \to \mathcal{F}(\mathbb{A}^1 \times X)$ induced by the projection $\mathbb{A}^1 \times X \to X$, is a bijection.

2. A sheaf of groups G on $\mathcal{S}m_S$ in the Nisnevich topology is said to be **strongly** $\mathbb{A}^1$ **-invariant** if for any $X \in \mathcal{S}m_S$ the map

$$H^i_{Nis}(X; G) \to H^i_{Nis}(X \times \mathbb{A}^1; G)$$

induced by the projection $\mathbb{A}^1 \times X \to X$, is a bijection for $i \in \{0, 1\}$.

3. A sheaf M of abelian groups on $\mathcal{S}m_S$ in the Nisnevich topology is said to be **strictly $\mathbb{A}^1$-invariant** if for any $X \in \mathcal{S}m_S$ the map

$$H^i_{Nis}(X; M) \to H^i_{Nis}(X \times \mathbb{A}^1; M)$$

induced by the projection $\mathbb{A}^1 \times X \to X$, is a bijection for any $i \in \mathbb{N}$.

2.7 Stable $\mathbb{A}^1$-homotopy category

For detailed treatment of the stable $\mathbb{A}^1$-homotopy category we refer to [Mor1, Section 2].

Let $sShv(\mathcal{S}m_S)$ be the category simplicial sheaves on $\mathcal{S}m_S$ in the Nisnevich topology and $sShv_\bullet(\mathcal{S}m_S)$ be the pointed objects in this category, these are called as pointed simplicial sheaves.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two pointed simplicial sheaves, the **wedge** of $\mathcal{X}$ and $\mathcal{Y}$ is given by $\mathcal{X} \vee \mathcal{Y} = \mathcal{X} \times * \cup * \times \mathcal{Y}$. The quotient of pointed simplicial sheaves of sets $(\mathcal{X} \times \mathcal{Y})/(\mathcal{X} \vee \mathcal{Y})$ is called the **smash product** of $\mathcal{X}$ and $\mathcal{Y}$ and is denoted by $\mathcal{X} \wedge \mathcal{Y}$.

Definition 2.7.1. An S^1**- spectrum** E in $\mathcal{S}m_S$ with Nisnevich topology is a collection $\{E_n, \sigma_n\}_{n \in \mathbb{N}}$ such that for each $n \geq 0$, E_n is a pointed simplicial sheaf and $\sigma_n : \Sigma(E_n) = E_n \wedge S^1 \to E_{n+1}$ is a map of pointed simplicial sheaves. The category of S^1-spectra in $\mathcal{S}m_S$ is denoted by $Sp^{S^1}(\mathcal{S}m_S)$.

Example 2.7.2. *Let $\mathcal{X}$ be any pointed simplicial sheaf, its suspension spectrum $\Sigma^\infty(\mathcal{X})$ is the spectrum with n^{th} term $\mathcal{X} \wedge S^n$, where $S^n = S^1 \wedge \cdots \wedge S^1$ (see Example 2.1.4) and structure morphisms are identity maps. This construction defines a suspension functor $\Sigma^\infty : sShv_\bullet(\mathcal{S}m_S) \to Sp^{S^1}(\mathcal{S}m_S)$. When $\mathcal{X} := *$ is the point, we set $S_0 := \Sigma^\infty(*_+)$.*

We will denote the suspension spectrum $\Sigma^\infty(\mathcal{X})$ simply by $(\mathcal{X})$, wherever there is no room for confusion.

Definition 2.7.3. Let E be an S^1-spectrum and $n \in \mathbb{Z}$, then the sheaf associated to the presheaf

$$X \mapsto \pi_n(E(X)) = coilm_{r>>0}\pi_{n+r}(E_r)$$

is called the **n-th homotopy sheaf** of E.

The notion of a stable weak equivalence between S^1-spectra is defined using these homotopy sheaves, as follows.

Definition 2.7.4. A morphism $f : E \to F$ of S^1-spectra is called a **stable weak equivalence** if it induces an isomorphism of sheaves $\pi_n(E) \cong \pi_n(F)$ for all $n \in \mathbb{Z}$.

The **stable homotopy category of S^1-spectra**, $\mathcal{SH}_s^{S^1}(\mathcal{S}m_S)$, is obtained by inverting stable weak equivalences in $Sp^{S^1}(\mathcal{S}m_S)$ (i.e. taking the homotopy category). Indeed there exists a combinatorial model structure on $Sp^{S^1}(\mathcal{S}m_S)$, where the stable weak equivalences are the weak equivalences. The set of morphisms in $\mathcal{SH}_s^{S^1}(\mathcal{S}m_S)$ between E and F is denoted by $[E, F]$.

Definition 2.7.5.

1. An S^1-spectrum $E \in Sp^{S^1}(\mathcal{S}m_S)$ is called **$\mathbb{A}^1$-local** if for any $F \in Sp^{S^1}(\mathcal{S}m_S)$, the projection $F \wedge (\mathbb{A}^1_+) \to F$ induces an isomorphism of abelian groups
$$[F, E] \to [F \wedge (\mathbb{A}^1_+), E].$$

2. A morphism $f : X \to Y$ in $Sp^{S^1}(\mathcal{S}m_S)$ is called a **stable $\mathbb{A}^1$-weak equivalence** if for any $\mathbb{A}^1$-local spectra E, the map
$$[Y, E] \to [X, E]$$
is an isomorphism.

3. The homotopy category of the left Bousfield localization of the category $Sp^{S^1}(\mathcal{S}m_S)$ with respect to the map $pt \to (\mathbb{A}^1_+)$ yields the **stable $\mathbb{A}^1$-homotopy category**, $\mathcal{SH}_{\mathbb{A}^1}^{S^1}(\mathcal{S}m_S)$.

Remark 2.7.6. *Every stable weak equivalence is a stable $\mathbb{A}^1$-weak equivalence.*

Similar to unstable case the $\mathbb{A}^1$-fibrant replacement functor for $Sp^{S^1}(\mathcal{S}m_S)$ also exists as a consequence of the left Bousfield localization. For explicit construction of $L_{\mathbb{A}^1}$ in the stable case, see [Mor2, Theorem 4.2.1].

3

Applications of Gabber's presentation lemma

In this chapter, we will see various applications of Gabber's presentation lemma. Most of the applications are extensions of the already established results to the new base where Gabber's presentation lemma now holds. In Section 3.1, we see many applications in the unstable $\mathbb{A}^1$-homotopy theory, mainly these results are from [Mor3]. In Section 3.2, a result about representability of G-torsors for certain groups G is mentioned. In the next Section 3.3, the stable $\mathbb{A}^1$-connectivity result of [SS] is extended to a more general base. In the last Section, we mention the Effacement theorem of Gabber for which he proves the presentation lemma in the original paper. For the statements of Gabber's presentation lemma refer to Theorems 1.2.1 and 1.2.2.

3.1 Results in unstable $\mathbb{A}^1$-homotopy theory

An important $\mathbb{A}^1$-homotopic consequence of Gabber's presentation lemma is due to F. Morel. It says that, for any smooth scheme X and a divisor Z of X, Nisnevich locally in Z the map $X \to X/(X - Z)$ is null homotopic.

More precisely, the above statement can be stated as follows (see [Mor3, Lemma 6.6]):

Theorem 3.1.1 (Morel)**.** *Let k be a field. Let X be a smooth scheme, $p \in X$ be a point and $Z \subset X$ be a closed subscheme of codimension $d > 0$. Then there exists an open* *subscheme $\Omega \subset X$ containing p and a closed subscheme $Z' \subset \Omega$ of codimension $d-1$, containing $Z_\Omega = Z \cap \Omega$ and such that the map of pointed sheaves*

$$\Omega/(\Omega - Z') \to \Omega/(\Omega - Z_\Omega)$$

is the trivial map in $\mathcal{H}_\bullet(k)$.

Proof. By Gabber's presentation lemma there exist

1. an open neighbourhood Ω of p
2. an étale morphism $\phi : \Omega \to \mathbb{A}^1_V$, where V is an open subset in some affine space over k

such that

1. $Z_\Omega := Z \cap \Omega \to \mathbb{A}^1_V$ is closed immersion.
2. $\phi^{-1}(Z_\Omega) = Z_\Omega$
3. $Z_\Omega \to V$ is a finite map.

Let F be the image of Z_Ω in V and $Z' = \phi^{-1}(\mathbb{A}^1_F)$. Since $\dim(F) = \dim(Z)$, *codim*$(Z') = d - 1$. We get the following isomorphism of Nisnevich sheaves

$$\Omega/(\Omega - Z_\Omega) \to \mathbb{A}^1_V/(\mathbb{A}^1_V - Z_\Omega).$$

Furthermore, we have the following commutative square

$$\begin{array}{ccc} \Omega/(\Omega - Z') & \longrightarrow & \Omega/(\Omega - Z_\Omega) \\ \downarrow & & \downarrow \simeq \\ \mathbb{A}^1_V/(\mathbb{A}^1_V - \mathbb{A}^1_F) & \longrightarrow & \mathbb{A}^1_V/(\mathbb{A}^1_V - Z_\Omega) \end{array}$$

hence is is sufficient to show that

$$\mathbb{A}^1_V/(\mathbb{A}^1_V - \mathbb{A}^1_F) \to \mathbb{A}^1_V/(\mathbb{A}^1_V - Z_\Omega)$$

is a trivial map in $\mathcal{H}_\bullet(k)$. Since $Z \to F$ is finite, the composition $Z \to \mathbb{A}^1_F \hookrightarrow \mathbb{P}^1_F$ is a closed immersion. Hence it does not intersects the section at infinity $s_\infty : V \to \mathbb{P}^1_V$. It follows from the Mayer-Vietoris property of the morphism $\mathbb{A}^1_V/(\mathbb{A}^1_V - Z_\Omega) \to \mathbb{P}^1_V/(\mathbb{P}^1_V - Z_\Omega)$ is an isomorphism of pointed sheaves. Indeed this isomorphism follows by verifying the bijection on Henselian local ring valued points. Thus it is sufficient to verify that

$$\mathbb{A}^1_V/(\mathbb{A}^1_V - \mathbb{A}^1_F) \to \mathbb{P}^1_V/(\mathbb{P}^1_V - Z_\Omega)$$

is a trivial map in $\mathcal{H}_\bullet(k)$.

We note that the morphism induced by the zero section

$$s_0 : V/(V - F) \to \mathbb{A}^1_V/(\mathbb{A}^1_V - \mathbb{A}^1_F)$$

is $\mathbb{A}^1$-weak equivalence. Further, the composition

$$s_0 : V/(V - F) \to \mathbb{A}^1_V/(\mathbb{A}^1_V - \mathbb{A}^1_F) \to \mathbb{P}^1_V/(\mathbb{P}^1_V - Z_\Omega)$$

is $\mathbb{A}^1$-homotopic to the section at infinity $s_\infty : V/(V - F) \to \mathbb{P}^1_V/(\mathbb{P}^1_V - Z_\Omega)$, the result follows from the previous observation that s_∞ is disjoint from Z_Ω hence

$$s_\infty : V/(V - F) \to \mathbb{P}^1_V/(\mathbb{P}^1_V - Z_\Omega)$$

is equal to the point. □

Following is a demonstration of the usefulness of this result.

Corollary 3.1.2. *Let X be a scheme over a field and $i : U \hookrightarrow X$ be an open set of X. Then for any $\mathbb{A}^1$-invariant sheaf M, the induced map $i^* : M(X) \to M(U)$ is injective.*

Proof. We have the exact sequence

$$0 \to M(X/U) \to M(X) \xrightarrow{i^*} M(U).$$

Let $Z = X/U$ from Theorem 3.1.1 we know that the map $X \to X/X - Z$ is null homotopic. Hence the kernel is trivial. □

More generally Theorem 3.1.1 is used to establish the strong and strict $\mathbb{A}^1$-invariance of $\mathbb{A}^1$-homotopy sheaves (see [Mor3, Cor.6.2]).

Theorem 3.1.3 (Morel)**.** *For any pointed space* $(\mathcal{X}, x)$*,*

1. $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ *is strongly* $\mathbb{A}^1$*-invariant and*

2. *for* $n \geq 2$ $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ *is strictly* $\mathbb{A}^1$*-invariant.*

3.2 Representability of Nisnevich locally trivial G-torsors

A. Asok, M. Hoyois and M. Wendt use Gabber's presentation lemma over a finite field in [AHW2] to extend the representability result over finite fields. Let us first recall the definition of an isotropic reductive algebraic group scheme.

Definition 3.2.1. A reductive algebraic group scheme G over a field k is said to be **isotropic** if each of the almost k-simple components of the derived group of G contains a k-subgroup scheme isomorphic to $\mathbb{G}_m$.

Theorem 3.2.2. *[AHW2, Theorem 2] Suppose k is a field, and G is an isotropic reductive k-group. For every smooth affine k-scheme X, there is a bijection*

$$H^1_{Nis}(X, G) \cong Hom_{\mathcal{H}(k)}(X, BG)$$

that is functorial in X.

This result was proved in [AHW1, Theorem 4.1.3] under the assumption that k is an infinite field. The proof of this theorem follows once one establish that for any smooth affine k-scheme X, the projection $X \times \mathbb{A}^1 \to X$ induces a bijection $H^1_{Nis}(X, G) \to H^1_{Nis}(X \times \mathbb{A}^1, G)$. We give an outline of its proof here.

Proposition 3.2.3. *[AHW2, Theorem 2.4] Let k be a field, and G be an isotropic reductive k-group scheme, then, for any smooth k-algebra A and any integer $n \geq 0$, the map*

$$H^1_{Nis}(Spec(A), G) \longleftrightarrow H^1_{Nis}(Spec(A[t_1, \ldots, t_n]), G)$$

is a pointed bijection.

Outline of the proof. Let $\mathcal{P}$ be a Nisnevich locally trivial G-torsor over $A[t_1,\ldots,t_n]$. By a local to global principle for torsors of reductive group schemes, [AHW1, Cor.3.2.6] it is sufficient to show that for every maximal ideal m of A, the G-torsor $\mathcal{P}_m$ over $A_m[t_1,...,t_n]$ is extended from A_m. In fact, we will show that $\mathcal{P}_m$ is a trivial torsor.

Let $\mathcal{F}$ be a presheaf of pointed sets on the category $\mathcal{C}$ of essentially smooth schemes over a field k. Under certain conditions (see [AHW1, Prop. 2.2]), which are satisfied by the functor from k-algebras to pointed sets given by $A \to H^1_{Nis}(Spec(A), G)$, as a consequence of Gabber's presentation lemma it follows that, for $Spec(B) \in \mathcal{C}$, with B local and for any integer $n \geq 0$ the restriction map

$$\mathcal{F}(B[t_1,\ldots,t_n]) \to \mathcal{F}\Big(Frac(B)(t_1,\ldots,t_n)\Big)$$

has trivial kernel (this follows from Corollary 3.1.2).

Now since a field has no nontrivial Nisnevich covering sieves, $\mathcal{P}_m$ becomes trivial over the field $Frac(A_m)(t_1,\ldots,t_n)$. □

3.3 Stable $\mathbb{A}^1$-connectivity

After proving the Gabber's presentation lemma over Dedekind domains with infinite residue field, in [SS] J. Schmidt and F. Strunk prove the shifted $\mathbb{A}^1$-connectivity theorem over Dedekind schemes with the infinite residue field. We observed that once we have Gabber's presentation lemma over noetherian domains, the shifted stable $\mathbb{A}^1$-connectivity holds over noetherian domains with infinite residue fields. The statement for the connectivity result is as follows.

Theorem 3.3.1. *Let R be a noetherian domain of dimension d such that all its residue fields are infinite and $Spec(R) = S$. Then S has the shifted stable $\mathbb{A}^1$-connectivity property that is if $E \in \mathcal{SH}^s_{S^1 \geq i}(S)$ then $L_{\mathbb{A}^1}E \in \mathcal{SH}^s_{S^1 \geq i-d}(S)$.*

The proof of Theorem 3.3.1 is precisely the same as the one given [SS, Theorem 4.16], except for the input from Gabber's presentation lemma (Theorem 1.2.2). We produce here the outline of the proof, for complete arguments refer to [SS, Theorem 4.16].

Outline of the proof. From [SS, Prop.4.5] it is sufficient to prove that for all $X \in \mathcal{S}m_S$, all integers $k < i$ and for all $f \in [\Sigma^{\infty}_{S^1} X_+, L^{\mathbb{A}^1} E[-k]]$, Nisnevich locally in X there exists a Zariski open set $\Omega \hookrightarrow X$ such that

1. $f_{|\Sigma^{\infty}_{S^1}\Omega_+} = 0$

2. $\pi_0^{\mathbb{A}^1}(X/\Omega) = 0$.

The existence of Ω follows essentially from the homotopy exactness of L^{∞}, which is a model for the functor $L^{\mathbb{A}^1}$ and the fact that the Nisnevich cohomological dimension is bounded by the Krull dimension, for details see [SS, Lemma 4.9].

To show that $\pi_0^{\mathbb{A}^1}(X/\Omega) = 0$, we see that $X \to \pi_0(L_{\mathbb{A}^1}(X/\Omega))$ is an epimorphism, hence it is sufficient to show that any point $x \in X$ has an open neighbourhood U such that $U \to \pi_0(L_{\mathbb{A}^1}(U/(\Omega \cap U)))$ is trivial.

Let $i : Z \hookrightarrow X$ be the reduced closed complement of Ω. From the proof of Theorem 3.1.1, for any point $x \in X$, there exist subsets U and V such that,

$$U/(U - Z_U) \to \mathbb{A}^1_V/(\mathbb{A}^1_V - Z_U)$$

is an isomorphism of Nisnevich sheaves. Hence it is sufficient to check that

$$\pi_0(L_{\mathbb{A}^1}(\mathbb{A}^1_V/(\mathbb{A}^1_V - Z_U)))$$

is trivial. By Mayer-Vietoris excision,

$$\mathbb{A}^1_V/(\mathbb{A}^1_V - Z_U) \to \mathbb{P}^1_V/(\mathbb{P}^1_V - Z_U)$$

is an isomorphism. Hence $\mathbb{A}^1_V \to \mathbb{P}^1_V/(\mathbb{P}^1_V - Z_U)$ is onto, further, $L_{\mathbb{A}^1}(\mathbb{A}^1_V) = L_{\mathbb{A}^1}(V)$. Consequently, the following composition is surjective for any section $V \to \mathbb{A}^1_V$

$$V \to \mathbb{A}^1_V \to \pi_0\Big(L_{\mathbb{A}^1}(\mathbb{P}^1_V/(\mathbb{P}^1_V - Z_U))\Big).$$

But in $\mathbb{P}^1_V$ the zero section is $\mathbb{A}^1$-homotopic to the section of infinity and $s_{\infty}(V) \subset \mathbb{P}^1_V - Z_U$, hence

$$V \to \pi_0\Big(L_{\mathbb{A}^1}(\mathbb{P}^1_V/(\mathbb{P}^1_V - Z_U))\Big)$$

is trivial morphism. □

3.4 Gabber's Effacement theorem

This is the first application of Gabber's presentation lemma over infinite fields (see [CTHK, Thereom 2.2.7] and [Gab]).

Theorem 3.4.1. *Let X be a smooth, affine variety over k, $t_1, \ldots, t_r \in X$ be a finite number of points, $p \geq 0$ an integer and Z a closed subvariety of codimension $\geq p+1$. Let A be a sheaf of torsion abelian groups over the (small) étale site of X. Assume that $A = p^* A_0$, where $p : X \to Spec(k)$ is the structural morphism and A_0 is a $Gal(k_s/k)$-module. If k is infinite, then there exists an open subset U of X containing all t_i and a closed subvariety $Z' \subset X$ containing Z such that*

1. *$codim_X(Z') \geq p$;*

2. *the map $H^n_{Z\cap U}(U, A) \to H^n_{Z'\cap U}(U, A)$ is 0 for all $n \geq 0$. If k is finite, then there exists (U, Z') as above such that (at least) the composite*

$$H^n_Z(X, A) \to H^n_{Z\cap U}(U, A) \to H^n_{Z'\cap U}(U, A)$$

is 0 for all $n \geq 0$.

4

Gabber's Presentation Lemma

In this chapter we present the proof of Gabber's presentation lemma over finite fields in 4.1.1 and over noetherian domains with all residue fields infinite in 4.2.1. Though the statements are similar the strategies for the proofs are totally different (see Section 1 for the strategies). The first result is a joint work with my advisor Amit Hogadi which appears in [HK] and the second is a joint work with Neeraj Deshmukh, Amit Hogadi and Suraj Yadav which appears in [DHKY].

4.1 Gabber's Presentation Lemma over finite fields

Theorem 4.1.1. *Let X be a smooth variety of dimension $d \geq 1$ over a finite field F and $Z \subset X$ be a closed subscheme. Let $p \in Z$ be a point. Let $\mathbb{A}^d_F \xrightarrow{\pi} \mathbb{A}^{d-1}_F$ denote the projection onto the first $d-1$ coordinates. Then there exists*

(i) an open neighbourhood $U \subset X$ of p,

(ii) a map $\Phi : U \to \mathbb{A}^d_F$,

(iii) an open neighbourhood $V \subset \mathbb{A}^{d-1}_F$ of $\Psi(p)$ where $\Psi : U \to \mathbb{A}^{d-1}_F$ denotes the composition

$$U \xrightarrow{\Phi} \mathbb{A}^d_F \xrightarrow{\pi} \mathbb{A}^{d-1}_F$$

such that

1. Φ *is étale.*

2. $\Psi_{|Z_V} : Z_V \to V$ *is finite where* $Z_V := Z \cap \Psi^{-1}(V)$.

3. $\Phi_{|Z_V} : Z_V \to \mathbb{A}^1_V = \pi^{-1}(V)$ *is a closed immersion.*

Remark 4.1.2. *Without loss of generality, we may (and will) assume henceforth that* X *is affine. Moreover, by [CTHK, 3.2], we may also assume that* Z *is a principal divisor defined by* $f \in \mathcal{O}(X)$ *and* p *is a closed point.*

4.1.1 Reduction to open subsets of $\mathbb{A}^d_F$

In this section we prove Lemma 4.1.6. This lemma reduces Theorem 4.1.1 to the case where X is an open subset of $\mathbb{A}^d_F$ and $p \in Z \subset X$ is a closed point with first $d-1$ coordinates equal to 0.

Notation 4.1.3. Throughout this section we work over a fixed finite field F. We further fix the following notation.

1. Let Y be a subset of a scheme X/F. We let $Y_{\leq r} := \{x \in Y \mid \mathsf{deg}(x) \leq r\}$ and similarly $Y_{<r} := \{x \in Y \mid \mathsf{deg}(x) < r\}$ and $Y_{=r} := \{x \in Y \mid \mathsf{deg}(x) = r\}$.

2. For $f_1, ..., f_i \in F[X_1, ..., X_n]$ we let $\mathsf{Z}(f_1, ..., f_i)$ denote the closed subscheme of $\mathbb{A}^n_F$ defined by the ideal $(f_1, ..., f_i)$.

We first recall the following standard trick (see [Mum]) used in the proof of Noether's normalization lemma.

Lemma 4.1.4. *[Mum, pg. 2] Let* k *be any field and* $n \geq 1$ *be any integer. Let* Z/k *be a finitely generated affine scheme of dimension at most* $n-1$*. Let*

$$Z \xrightarrow{(\phi_1,\ldots,\phi_n)} \mathbb{A}^n_k$$

be a finite map. Let $Q(T) \in k[T]$ *be a non constant monic polynomial and* $Q = Q(\phi_n)$*. Then for* $\ell >> 0$*, the map*

$$Z \xrightarrow{(\phi_1 - Q^{\ell^{n-1}},\ldots,\phi_{n-1}-Q^{\ell})} \mathbb{A}^{n-1}_k$$

is finite.

Remark 4.1.5. *We claim that finiteness of* $Z \xrightarrow{(\phi_1,\ldots,\phi_n)} \mathbb{A}^n_k$ *implies that of* $Z \xrightarrow{(\phi_1,\ldots,Q(\phi_n))} \mathbb{A}^n_k$*. This is because the later map is a composition of the following two finite maps*

$$Z \xrightarrow{(\phi_1,\ldots,\phi_n)} \mathbb{A}^n_k \xrightarrow{(Y_1,\ldots,Q(Y_n))} \mathbb{A}^n_k.$$

One can thus easily reduce the proof of the above general case to the case where $Q(T) = T$*. Unless explicitly mentioned, we will usually assume* $Q(T) = T$ *while applying the lemma. As in the proof of Noether normalization, the above lemma is usually applied repeatedly until one gets a map from* Z *to* $\mathbb{A}^{\dim(Z)}_k$*.*

Lemma 4.1.6. *Let* $p \in Z \subset X$ *be as in Theorem 4.1.1. Further, assume that* X *is affine,* Z *is a principal divisor and* p *is a closed point (see Remark 4.1.2). Then there exists a map* $\varphi : X \to \mathbb{A}^d_F$ *and an open neighbourhood* W *of* $\varphi(p)$ *such that*

1. $\varphi^{-1}(W) \to W$ *is étale.*
2. $Z_W := Z \cap \varphi^{-1}(W) \to W$ *is a closed immersion.*
3. *The first* $d-1$ *coordinates of* $\varphi(p)$ *are* 0*.*

In particular, it suffices to prove Theorem 4.1.1 where X *is an open subset of* $\mathbb{A}^d_F$ *and the first* $d-1$ *coordinates of* p *are zero.*

Proof. Let

- $X = \mathrm{Spec}\,(A)$.
- $Z = \mathrm{Spec}\,(A/(f))$ and let $\overline{A} := A/(f)$.
- $\mathfrak{m} \subset A$ be the maximal ideal of the closed point p.
- $F(p)$ denote the residue field of p.

Step 1: Since X/F is smooth, $\dim_{F(p)}(\mathfrak{m}/\mathfrak{m}^2) = d$. Choose $\{x_1, \ldots, x_{d-1}\} \subset \mathfrak{m}$ such that they span a $d-1$ dimensional $F(p)$-subspace of $\mathfrak{m}/\mathfrak{m}^2$. In this step we claim that there exists $y \in A$ such that

1. y mod $\mathfrak{m}$ is a primitive element of $F(p)/F$.

2. The set $\{x_1, ..., x_{d-1}, h(y)\}$ (modulo $\mathfrak{m}^2$) gives a $F(p)$-basis of $\mathfrak{m}/\mathfrak{m}^2$, where h is the minimal polynomial of y mod $\mathfrak{m}$.

3. The map $(x_1, \ldots, x_{d-1}, y) : X \xrightarrow{\eta} \mathbb{A}^d$ is étale at p.

4. The map η induces an isomorphism on residue fields $F(\eta(p)) \to F(p)$.

Now let $w \in \mathfrak{m}$ be an element such that $\{x_1, ..., x_{d-1}, w\}$ span $\mathfrak{m}/\mathfrak{m}^2$ as a $F(p)$-vector space. Let c be a primitive element of $F(p)/F$ and h be its minimal polynomial. Choose $\hat{y} \in A$ such that

$$\hat{y} \equiv c \mod \mathfrak{m}.$$

Since c is separable over F, $h'(c) \neq 0$. Thus $h'(\hat{y}) \notin \mathfrak{m}$ or equivalently $h'(\hat{y})$ is a unit in the ring $A/\mathfrak{m}^2$. Choose $\epsilon \in \mathfrak{m}$ such that

$$\epsilon \equiv \frac{w - h(\hat{y})}{h'(\hat{y})} \operatorname{\mathsf{mod}} \mathfrak{m}^2.$$

Thus the $F(p)$-span of $\{x_1, ..., x_{d-1}, h(\hat{y}) + \epsilon h'(\hat{y})\}$ is $\mathfrak{m}/\mathfrak{m}^2$. Let

$$y = \hat{y} + \epsilon.$$

We note that

$$h(y) = h(\hat{y} + \epsilon) \equiv h(\hat{y}) + \epsilon h'(\hat{y}) \mod \mathfrak{m}^2.$$

Hence $\{x_1, \ldots, x_{d-1}, h(y)\}$ gives a $F(p)$-basis for $\mathfrak{m}/\mathfrak{m}^2$.

Now let η be the map $(x_1, \ldots, x_{d-1}, y) : X \to \mathbb{A}^d_F$. Since $y \operatorname{\mathsf{mod}} \mathfrak{m}$ is a primitive element of $F(p)$, one observes that $F(\eta(p)) \to F(p)$ is an isomorphism. It remains to show that η is étale at p. The maximal ideal of $\eta(p)$ in $F[X_1, ..., X_d]$ is $\mathfrak{n} = (X_1, \ldots, X_{d-1}, h(X_d))$. As $\{x_1, \ldots, x_{d-1}, h(y)\}$ is a $F(p)$-basis for $\mathfrak{m}/\mathfrak{m}^2$, that η is étale at p follows from the surjectivity of

$$\mathfrak{n}/\mathfrak{n}^2 \xrightarrow{\eta^*} \mathfrak{m}/\mathfrak{m}^2.$$

Step 2: Let U be an open neighbourhood of p in X such that $\eta_{|U}$ is étale. Let

$$B = (X \setminus U) \sqcup Z.$$

In this step we modify $x_1, \ldots, x_{d-1}$ to $z_1, \ldots, z_{d-1}$ so that

1. The map $\widetilde{\eta} = (z_1, \dots, z_{d-1}, y) : X \to \mathbb{A}^d_F$ is étale on U.

2. The set $\{z_1, \dots, z_{d-1}, h(y)\}$ is a $F(p)$ basis for $\mathfrak{m}/\mathfrak{m}^2$.

3. The map $B \xrightarrow{(z_1,\dots,z_{d-1})} \mathbb{A}^{d-1}_F$ is finite.

Let $\widetilde{A} := A/I(B)$ and $\widetilde{\mathfrak{m}}$ denote the image of $\mathfrak{m}$ in $\widetilde{A}$. For any element $\alpha \in A$, let $\widetilde{\alpha}$ denote its image in $\widetilde{A}$. Choose $y_1, ..., y_m \in A$ which generate A as an F algebra. We expand this generating set to include the x_i's. In particular

$$A = F[x_1, ..., x_{d-1}, y_1, ..., y_m],$$
$$\widetilde{A} = F[\widetilde{x}_1, ..., \widetilde{x}_{d-1}, \widetilde{y}_1, ..., \widetilde{y}_m].$$

The image of y_i in $A/\mathfrak{m}$ satisfies a non-constant monic polynomial, say f_i, over F. Let

$$\begin{aligned} y_{i,0} &:= f_i(y_i) \in \mathfrak{m}. \\ x_{i,0} &:= x_i \\ A_0 &:= F[x_{1,0}, .., x_{d-1,0}, y_{1,0}, ..., y_{m,0}] \\ \widetilde{A}_0 &:= F[\widetilde{x}_{1,0}, .., \widetilde{x}_{d-1,0}, \widetilde{y}_{1,0}, ..., \widetilde{y}_{m,0}] \end{aligned}$$

Clearly, $\widetilde{A}$ is finite over $\widetilde{A}_0$.

For $0 \leq r \leq m-1$, we inductively define A_{r+1} and elements $x_{i,r+1}, y_{i,r+1}$ as follows. By 4.1.4, we choose an integer $\ell_r > 1$ such that the following definitions make $\widetilde{A}_r$ a finite $\widetilde{A}_{r+1}$-algebra. Since any sufficiently large choice of ℓ_r works, we assume that ℓ_r is a multiple of the char(F). Let

$$\begin{aligned} x_{i,r+1} &:= x_{i,r} - (y_{m-r,r})^{\ell_r^i} && \forall\ 1 \leq i \leq d-1 \\ y_{i,r+1} &:= y_{i,r} - (y_{m-r,r})^{\ell_r^{d-1+i}} && \forall\ 1 \leq i \leq m-r-1 \\ A_{r+1} &:= F[x_{1,r+1}, .., x_{d-1,r+1}, y_{1,r+1}, ..., y_{m-r-1,r+1}] \\ \widetilde{A}_{r+1} &:= F[\widetilde{x}_{1,r+1}, .., \widetilde{x}_{d-1,r+1}, \widetilde{y}_{1,r+1}, ..., \widetilde{y}_{m-r-1,r+1}] \end{aligned}$$

Since, $x_{i,0}$ and $y_{i,0}$ belong to $\mathfrak{m}$, inductively one can observe

$$y_{i,r} \in \mathfrak{m}$$
$$x_{i,r} \in \mathfrak{m}$$
$$x_{i,r+1} \equiv x_{i,r} \text{ mod } \mathfrak{m}^2$$

For ease of notation, let us rename

$$z_i := x_{i,m}.$$

Note that for all $i \leq d-1$, $z_i - x_i$ is of the form $\beta_i^{k_i}$ for $\beta_i \in \mathfrak{m}$ and an integer k_i divisible by char(F). This ensures requirements (1) and (2) of Step 2. Recall that m is an integer such that $\{y_1, ..., y_m\}$ are the chosen generators of A as an F algebra. It is now straightforward to see that $\{z_1, ..., z_{d-1}\} \subset \mathfrak{m}$ such that $\widetilde{A}$ is a finite algebra over $\widetilde{A}_m = F[\widetilde{z}_1, ..., \widetilde{z}_{d-1}]$.

Step 3: In this step we will further modify y while ensuring that (1) and (2) of the above step continue to hold. Since the map $\widetilde{\eta}_{|B} : B \to \mathbb{A}_F^{d-1}$ is finite, there exists finitely many points $\{p, p_1, ..., p_t\} \subset B$ which are contained in the zero locus $\mathsf{Z}(z_1, ..., z_{d-1})$. Let $\mathfrak{m}_i$ be the maximal ideal corresponding to p_i for $1 \leq i \leq t$. By Chinese remainder theorem, choose $\delta \in A$ such that

$$\delta \equiv 0 \text{ mod } \mathfrak{m}$$
$$\delta^{\text{char}(F)} \equiv -y \text{ mod } \mathfrak{m}_i \quad \forall\, 1 \leq i \leq t \qquad \text{...(note that } A/\mathfrak{m}_i \text{ is perfect)}$$

Let

$$z = y + \delta^{\text{char}(F)}.$$

For later use, we note that

$$z \equiv 0 \text{ mod } \mathfrak{m}_i \quad \forall\, 1 \leq i \leq t.$$

Using the fact that $z - y$ is char(F)-th power of an element of $\mathfrak{m}$, it is straightforward to deduce the following from (1) and (2) of the above step.

1. The map $\varphi : X \to \mathbb{A}_F^d$ defined by $(z_1, ..., z_{d-1}, z)$ is étale at p.

2. $\{z_1, ..., z_{d-1}, h(z)\}$ is an $F(p)$-basis of $\mathfrak{m}/\mathfrak{m}^2$.

3. $z \bmod \mathfrak{m}$ is the primitive element c of $F(p)/F$.

We further claim that we have the following equality of ideals of $\widetilde{A} = A/(I(B))$:

$$\sqrt{(\tilde{z}_1, ..., \tilde{z}_{d-1}, h(\tilde{z}))} = \tilde{\mathfrak{m}}.$$

To see the claim, we first observe

$$h(z) \in \tilde{\mathfrak{m}}$$
$$h(z) \notin \tilde{\mathfrak{m}}_i \quad \forall\ 1 \leq i \leq t.$$

The first containment follows as h is the irreducible polynomial of $z \bmod \mathfrak{m}$. Moreover, since $h(0) \neq 0$, the second statement follows from the fact that $z \equiv 0 \bmod \mathfrak{m}_i$.

As $\{\tilde{\mathfrak{m}}, \tilde{\mathfrak{m}}_1, ..., \tilde{\mathfrak{m}}_t\}$ are the only prime ideals of $\widetilde{A}$ containing the ideal $(\tilde{z}_1, ..., \tilde{z}_{d-1})$, and $h(z) \notin \mathfrak{m}_i\ \forall\ i$, we conclude that $\tilde{\mathfrak{m}}$ is the unique prime ideal of $\widetilde{A}$ containing the ideal $(\tilde{z}_1, ..., \tilde{z}_{d-1}, h(\tilde{z}))$. Therefore

$$\sqrt{(\tilde{z}_1, ..., \tilde{z}_{d-1}, h(\tilde{z}))} = \tilde{\mathfrak{m}}.$$

Step 4: We claim that in fact

$$\Big(\tilde{z}_1, ..., \tilde{z}_{d-1}, h(\tilde{z})\Big) = \tilde{\mathfrak{m}}.$$

Note that both are $\tilde{\mathfrak{m}}$-primary ideals and hence it is enough to show the equality in the localization $\widetilde{A}_{\tilde{\mathfrak{m}}}$. But the equality holds in this local ring by Nakayama's Lemma since it holds modulo $\tilde{\mathfrak{m}}^2$ as $\{z_1, ..., z_{d-1}, h(z)\} \bmod \mathfrak{m}^2$ gives a basis of $\mathfrak{m}/\mathfrak{m}^2$ (see condition (2) of the the above Step).

Step 5: Recall that $\varphi : X \to \mathbb{A}^d_F$ is the map defined by $(z_1, ..., z_{d-1}, z)$. We claim that p is the unique point in $\varphi^{-1}\varphi(p) \cap Z$. In fact we have that p is the unique point of $\varphi^{-1}\varphi(p) \cap B$. This is a direct consequence of Step 3, since the ideal defining $\varphi^{-1}\varphi(p) \cap B$ in $B = \operatorname{Spec}(\widetilde{A})$ is equal to $(\tilde{z}_1, ..., \tilde{z}_{d-1}, h(\tilde{z})) = \tilde{\mathfrak{m}}$. Indeed, what we have observed is that the scheme $\varphi^{-1}\varphi(p) \cap B$ is reduced and has p as the only underlying point. Thus the same holds for $\varphi^{-1}\varphi(p) \cap Z$. If $\mathfrak{n}$ denotes the maximal ideal in the coordinate ring of $\mathbb{A}^d_F$ of the point $\varphi(p)$, then $\mathfrak{n}\overline{A} = \overline{\mathfrak{m}}$. Recall that $\overline{A} := A/(f)$ and $Z = \mathrm{Spec}\,(\overline{A})$.

Step 6: In this step we prove the rest of the theorem using a trick used in the proof of [CTHK, 3.5.1]. In fact, the argument in this step has been directly taken from *loc. cit.* The map $\varphi : Z \to \mathbb{A}^d_F$ is finite. Let $\mathfrak{n}$ be the maximal ideal of $\varphi(p)$ in $F[X_1, ..., X_d]$. By Step 5, $\mathfrak{n}\overline{A} = \overline{\mathfrak{m}}$ and the map

$$\frac{F[X_1, ..., X_d]}{\mathfrak{n}} \xrightarrow{\varphi^*} \frac{\overline{A}}{\mathfrak{n}\overline{A}}$$

is an isomorphism, in particular surjective. By Nakayama's lemma, there exists a $g \in F[X_1, ..., X_d] \backslash \mathfrak{n}$ such that the map

$$F[X_1, ..., X_d]_g \to \overline{A}_g$$

is surjective. In particular, if $V = \mathbb{A}^d_F \backslash \mathsf{Z}(g)$, then

$$Z \cap \varphi^{-1}(V) \to V$$

is a closed immersion. Note that V is an open neighbourhood of $\varphi(p)$.

Let $D \subset X$ be the maximal closed subset on which the map φ is not étale. Clearly $p \notin D$. Also, since D is a subset of B (see Step 2) and p is the only point in $\varphi^{-1}\varphi(p) \cap B$, we must have $\varphi(p) \notin \varphi(D)$. However, the map $\varphi_{|B}$ is finite, we have that $\varphi(D)$ is a closed subset of $\mathbb{A}^d_F$. Let

$$W := \left(\mathbb{A}^d_F \backslash \varphi(D)\right) \cap \left(\mathbb{A}^d_F \backslash \mathsf{Z}(g)\right).$$

Thus $\varphi^{-1}(W) \to W$ is étale. Moreover, $\varphi^{-1}(W)$ is an open neighbourhood of p. It is now clear that φ and W satisfy conditions (1) and (2) of the Lemma. Condition (3) is also immediate since the map φ is defined by $(z_1, ..., z_{d-1}, z)$ and z_i vanish on p for $1 \le i \le d-1$. □

4.1.2 Reduction to open subsets of $\mathbb{A}^2_F$

In the previous section we reduced Theorem 4.1.1 to the case, in which X is an open subset of $\mathbb{A}^d_F$. In this section we reduce further to the case $d = 2$ (see Lemma 4.1.14). This reduction is achieved by using induction argument. The following version of the Noether normalization trick (see (4.1.4)) is used crucially in the reduction.

Lemma 4.1.7. *Let $n \geq 2$ be any integer, k be any field and Z/k be an affine variety of dimension $n-1$. Let*

$$Z \xrightarrow{(\phi_1,\ldots,\phi_n)} \mathbb{A}^n_k$$

be a finite map. Let $Q \in k[\phi_n]$ be a non constant monic polynomial. Then for an integer $\ell >> 0$, the map

$$Z \xrightarrow{(\phi_1-Q_1^\ell,\ldots,\phi_{n-1}-Q_{n-1}^\ell)} \mathbb{A}^{n-1}_k$$

is finite, where Q_i's are inductively defined by

$$Q_{n-1} := Q.$$

$$Q_i := \phi_{i+1} - Q_{i+1}^\ell \quad \forall\ 1 \leq i \leq n-2.$$

Proof. The proof is similar to that of 4.1.4 (see [Mum, page 2]) and hence we only give a sketch. Since $\dim(Z) = n-1$, $\phi_1, \phi_2, \ldots, \phi_n$ cannot be algebraically independent. Thus there exists a non-zero polynomial $f \in k[Y_1, ..., Y_n]$ such that $f(\phi_1, \phi_2, \ldots, \phi_n) = 0$. Let ℓ be any integer greater than $n\mathsf{deg}(f)$ where $\mathsf{deg}(f)$ is the total degree of f. Let $\widetilde{Q} \in k[Y_n]$ be a polynomial such that $Q = \widetilde{Q}(\phi_n)$. Inductively define $\widetilde{Q}_i$ for $1 \leq i \leq n-1$ as follows:

$$\widetilde{Q}_{n-1} := \widetilde{Q}$$
$$\widetilde{Q}_i := Y_{i+1} - \widetilde{Q}_{i+1}^\ell \qquad \forall\ 1 \leq i\ \leq n-2.$$

Notice that the polynomials $\widetilde{Q}_i$'s are defined such that

$$\widetilde{Q}_i(\phi_1, ..., \phi_n) = Q_i.$$

Moreover, we note that if $d = Y_n$-degree of $\widetilde{Q}$ then each $\widetilde{Q}_i$ is monic in Y_n of degree

$\ell^{n-i-1}d$. Consider the elements $Z_1, ..., Z_{n-1} \in k[Y_1, ..., Y_n]$ defined as follows:

$$Z_i := Y_i - \widetilde{Q}_i^{\ell} \quad \forall 1 \leq i \leq n-1.$$

We leave it to the reader to check that

$$k[Z_1, ..., Z_{n-1}, Y_n] = k[Y_1, ..., Y_n].$$

For future reference, we note that the map

$$\eta : \mathbb{A}_F^n \xrightarrow{(Y_1 - \widetilde{Q}_1^{\ell}, ..., Y_{n-1} - \widetilde{Q}_{n-1}^{\ell}, Y_n)} \mathbb{A}_F^n$$

is an automorphism. It is enough to show that the polynomial f, expressed in the variables $Z_1, ..., Z_{n-1}, Y_n$ is monic in Y_n. Let us write f as

$$f = \Sigma_{I=(i_1, ... i_n)} \alpha_I \cdot m_I$$

where m_I's are monomials in $Y_1, ..., Y_n$ and $\alpha_I \in k$. We leave it to the reader to verify that when expressed in new coordinates $Z_1, \ldots Z_{n-1}, Y_n$, each monomial m_I becomes a polynomial which is monic in Y_n of Y_n-degree equal to $i_n + \Sigma_{k=1}^{n-1} i_k \cdot \ell^{n-k} \cdot d$. Since $\ell > n\mathsf{deg}(f)$, one can show that these Y_n-degrees are distinct. Thus in the coordinates $Z_1, ..., Z_{n-1}, Y_n$, f remains monic in Y_n. □

Notation 4.1.8. Let $d \geq 2$ be an integer and $f, g \in F[X_1, ..., X_d]$ be nonzero polynomials with no common irreducible factors (see Remark 4.1.11). Let $X := \mathbb{A}_F^d \backslash \mathsf{Z}(g)$ and $Z := \mathsf{Z}(f) \cap X$. Let $p \in Z$ be a closed point (see Remark 4.1.2) whose first $d-1$ coordinates are 0.

Recall from Lemma 4.1.6, it is sufficient to prove Theorem 4.1.1 for (X, Z, p) as above. To prove this, as a first step we need a map $\Phi : X \to \mathbb{A}_F^d$. Actually, we will try to find maps Φ whose domain of definition is whole of $\mathbb{A}_F^d$. That is we will find polynomials $\{\phi_1, ..., \phi_d\} \subset F[X_1, ..., X_d]$. The following definition lists the conditions on the polynomials, which will provide (see Lemma 4.1.10) the resulting map Φ is as per (4.1.1).

Definition 4.1.9. Let f, g, X, Z, p be as in Notation 4.1.8. For $\{\phi_1, ..., \phi_d\} \subset F[X_1, ..., X_d]$, let

(i) $\Phi : \mathbb{A}^d_F \xrightarrow{(\phi_1,...,\phi_d)} \mathbb{A}^d_F$.

(ii) $\Psi : \mathbb{A}^d_F \xrightarrow{(\phi_1,...,\phi_{d-1})} \mathbb{A}^{d-1}_F$.

We say that $(\phi_1, ..., \phi_d)$ presents $(X, \mathsf{Z}(f), p)$ if

1. $\Psi_{|\mathsf{Z}(f)}$ is finite and $\Psi(p) = (0, ..., 0)$.

2. $\Psi^{-1}\Psi(p) \cap \mathsf{Z}(f) \subset Z$

3. Φ is étale at $S := \Psi^{-1}\Psi(p) \cap Z$.

4. Φ is radicial at S.

Recall that Φ is said to be *radicial* [Sta, Tag 01S2] if $\Phi_{|S}$ is injective and for all $x \in S$ the residue field extension $F(x)/F(\Phi(x))$ is trivial.

Due to the following lemma, to prove Theorem 4.1.1 for X, Z, p as per Notation 4.1.8 it is sufficient to find $\phi_1, ..., \phi_d$ which presents $(X, \mathsf{Z}(f), p)$.

Lemma 4.1.10. *Let X, Z, p be as above. Assume there exists $\{\phi_1, ..., \phi_d\}$ which presents $(X, \mathsf{Z}(f), p)$ and Φ, Ψ be as in Definition 4.1.9. Then there exist open neighborhoods $V \subset \mathbb{A}^{d-1}_F$ of $\Psi(p)$ and $U \subset X$ of p, such that $\Phi_{|U}, \Psi_{|U}, U, V$ satisfy conditions (1),(2),(3) of Theorem 4.1.1. Moreover, $\Psi^{-1}(V) \cap \mathsf{Z}(f) \subset U$.*

Proof. The argument here is directly taken from [CTHK, 3.5.1]. We construct an open neighbourhood V of $\Psi(p)$ in $\mathbb{A}^{d-1}_F$, such that if $Z_V := \Psi^{-1}(V) \cap \mathsf{Z}(f)$ then

(i) $Z_V \subset Z$

(ii) Φ is étale at all points in Z_V

(iii) $\Phi|_{Z_V} : Z_V \to \mathbb{A}^1_V$ is closed immersion

Let B be the smallest closed subset of $\mathsf{Z}(f)$ containing all points of $\mathsf{Z}(f)$ at which Φ is not étale and also containing $\mathsf{Z}(f) \backslash Z$. Since $\Psi_{|\mathsf{Z}(f)}$ is a finite map, $\Psi(B)$ is closed in $\mathbb{A}^{d-1}_F$.

Moreover, because of conditions (2) and (3) of Definition 4.1.9, we have $\Psi(p) \notin \Psi(B)$. Thus, we can choose affine open subset $W \subset \mathbb{A}_F^{d-1}$ such that $\Psi(p) \in W \subset \mathbb{A}_F^{d-1} \backslash \Psi(B)$. Let $Z_W = Z \cap \Psi^{-1}(W)$. We have following commutative diagram of affine schemes and consequently their coordinate rings.

Let $\Psi(p) = q$ and $\mathfrak{m}_q$ be the maximal ideal in $F[W]$ corresponding to q. Thus the ideal corresponding to $S = \Psi^{-1}(q) \cap Z$ in $F[Z_W]$ is $\mathfrak{m}_q \cdot F[Z_W]$. Since Φ is radicial as well as étale at S,

$$\Phi_{|S} : S \hookrightarrow \mathbb{A}^1_W$$

is a closed immersion. Thus the map on the coordinate rings

$$F[\mathbb{A}^1_W] \twoheadrightarrow \frac{F[Z_W]}{\mathfrak{m}_q F[Z_W]}$$

is surjective. The surjectivity of the above map is equivalent to

$$C \otimes_{F[W]} \frac{F[W]}{\mathfrak{m}_q} = 0$$

where

$$C := \mathrm{Coker}\Big(F[\mathbb{A}^1_W] \to F[Z_W]\Big).$$

But C is a finite $F[W]$ module. Hence by Nakayama's lemma $C_{\mathfrak{m}_q} = 0$. Thus there exists $h \in F[W] \backslash \mathfrak{m}_q$ such that $C_h = 0$ or equivalently

$$F[\mathbb{A}^1_W]_h \twoheadrightarrow F[Z_W]_h$$

is surjective. Let $V := W \backslash \mathsf{Z}(h)$. The coordinate ring of $Z_V := \Psi^{-1}(V) \cap \mathsf{Z}(f)$ is $F[Z_W]_h$ and that of $\pi^{-1}V$ is $F[\mathbb{A}^1_W]_h$. Thus the surjectivity of the above map implies that

$$Z_V \hookrightarrow \mathbb{A}^1_V$$

is a closed immersion as required.

Let $U \subset X$ be the maximal open subset containing points at which Φ is étale. To finish the proof, we need to show that $U, V, \Phi_{|U}, \Psi_{|U}$ satisfy conditions $(1), (2), (3)$ of Theorem 4.1.1. (1) is clearly satisfied by the definition of U. To see (2), note that $\Psi_{|Z(f)}$ is finite, and hence, as $Z_V = \Psi^{-1}(V) \cap Z(f)$, $\Psi_{|Z_V} : Z_V \to V$ is finite. (3) is precisely the condition (iii) mentioned at the beginning of the proof. By the construction of W, subsequently V, it follows that $\Psi^{-1}(V) \cap \mathsf{Z}(f) \subset U$. □

Remark 4.1.11. *The justification for the assumption in Remark 4.1.8 that f, g have no common irreducible factors is the following: If f and g have common irreducible factors, dividing f by the g.c.d. of f and g does not change $\mathsf{Z}(f)\backslash\mathsf{Z}(g)$. Eventually we want to prove Theorem 4.1.1 for (X, Z, p), we are allowed to change $\mathsf{Z}(f)$ as long as it does not change Z.*

We use an easy coordinate change to prove the following lemma. This will be a part of the proof of the main result (4.1.14) in this section.

Lemma 4.1.12. *Let $(\phi_1, ..., \phi_d)$ present $(X, \mathsf{Z}(f), p)$. as in Lemma 4.1.10. Then there exist $(\widetilde{\phi}_1, ..., \widetilde{\phi}_d)$ which present $(X, \mathsf{Z}(f), p)$ such that there exists an open subset $V \subset \mathbb{A}_F^{d-1}$ satisfying the conclusion of Lemma 4.1.10 for $(\widetilde{\phi}_1, ..., \widetilde{\phi}_d)$ and which satisfies the following additional condition:*

$$\dim \left(\mathsf{Z}(\widetilde{\phi}_1, ..., \widetilde{\phi}_{d-2}) \cap \Psi^{-1}_{|Z(f)} \left(\mathbb{A}_F^{d-1} \backslash V \right) \right) = 0.$$

Proof. We note that if $d = 2$, by convention,

$$\mathsf{Z}(\widetilde{\phi}_1, ..., \widetilde{\phi}_{d-2}) \cap \Psi^{-1}_{|Z(f)}(\mathbb{A}_F^1 \backslash V) = \Psi^{-1}_{|Z(f)}(\mathbb{A}_F^1 \backslash V)$$

which is of zero dimension since V is non-empty. Thus we may assume $d \geq 3$. For an integer ℓ, consider the automorphism $\rho : \mathbb{A}_F^{d-1} \to \mathbb{A}_F^{d-1}$ induced by

$$(X_1, ..., X_{d-1}) \mapsto (X_1 - X_{d-1}^{\ell^{(d-1)-1}}, X_2 - X_{d-1}^{\ell^{(d-1)-2}}, ..., X_{d-2} - X_{d-1}^{\ell^1}, X_{d-1}).$$

We choose $\ell >> 0$, such that by (4.1.4), $(X_1, ..., X_{d-2})_{|\rho(\mathbb{A}^{d-1} \setminus V)}$ is a finite map. Let

$$\widetilde{\phi}_i := \phi_i - \widetilde{\phi}_{d-1}^{\ell^{d-1-i}} \quad \text{for} \quad i \leq d-2$$
$$\widetilde{\phi}_i := \phi_i \quad \text{for } i = d-1, d$$

It is then straightforward to check that $(\widetilde{\phi}_1, ..., \widetilde{\phi}_d)$ presents $(X, \mathsf{Z}(f), p)$ (since it is obtained by a coordinate change from the original ϕ_i's) and moreover

$$\dim \left(\mathsf{Z}(\widetilde{\phi}_1, ..., \widetilde{\phi}_{d-2}) \cap \Psi^{-1}_{|Z(f)}(\mathbb{A}^{d-1}_F \setminus \rho(V)) \right) = 0.$$

□

Lemma 4.1.13. *Let $d \geq 3$, and $f, g \in F[X_1, ..., X_d]$ be two non-zero polynomials with no common factors. Let p be a closed point of $\mathbb{A}^d_F$ such that $X_i(p) = 0$ for all $i \leq d-1$. Then there exists a coordinate change of $F[X_1, ..., X_d]$, i.e., elements $Y_i \in F[X_1, ..., X_d]$ with*

$$F[X_1, ..., X_d] = F[Y_1, ..., Y_d]$$

such that $f(0, Y_2, ..., Y_d)$ and $g(0, Y_2, ..., Y_d)$ are nonzero polynomials with no common irreducible factors and $Y_i(p) = 0$ for all $i \leq d-1$.

Proof. The condition that $f(0, Y_2, ..., Y_d)$ and $g(0, Y_2, ..., Y_d)$ are nonzero polynomials with no common irreducible factors is equivalent to the condition that no irreducible component of $\mathsf{Z}(f) \cap \mathsf{Z}(g)$ is contained in $\mathsf{Z}(Y_1)$.

By Noether normalization trick 4.1.4, we may assume, by a suitable coordinate change, that the projection

$$(X_2, ..., X_d) : \mathsf{Z}(f) \cap \mathsf{Z}(g) \xrightarrow{\eta} \mathbb{A}^{d-1}_F$$

is finite. Note that since $d \geq 3$, the image of every irreducible component of $\mathsf{Z}(f) \cap \mathsf{Z}(g)$ under η is of dimension at least one. Thus we may choose closed points $z_1, ..., z_\tau$, one in each irreducible component of $\mathsf{Z}(f) \cap \mathsf{Z}(g)$ such that $\eta(z_i)$ are pairwise distinct and also different from $\eta(p)$. For every closed point x of $\mathbb{A}^d_F$, either X_1 or $X_1 + 1$ is non-vanishing on x. Thus for each z_i, we choose $\epsilon_i = 0$ or 1, such that $X_1 + \epsilon_i$ does not vanish on z_i.

By Chinese remainder theorem, there exists a polynomial $\gamma \in F[X_2, ..., X_d]$ such that

$$\gamma(\eta(z_i)) = \epsilon_i \quad \text{and} \quad \gamma(p) = 0.$$

It is now straightforward to check that

$$Y_1 := X_1 - \gamma \quad \text{and} \quad Y_i := X_i \;\; \forall\, 2 \leq i \leq d$$

satisfies our requirement. □

Lemma 4.1.14. *[Reduction to $d = 2$] Assume that for $d = 2$ and every f, g, X, Z, p as in Notation 4.1.8, there exist $\phi_1, \phi_2 \in F[X_1, X_2]$ which presents $(X, \mathsf{Z}(f), p)$. Then the same holds for every $d \geq 2$.*

Proof. We prove this lemma by induction on d. Assume $d \geq 3$.

Step 0: As before, we let $F[X_1, ..., X_d]$ be the coordinate ring of $\mathbb{A}^d_F$. Let $\overline{f}(X_2, ..., X_d) := f(0, X_2, ..., X_d)$ and $\overline{g}(X_2, ..., X_d) := g(0, X_2, ..., X_d)$. By Lemma 4.1.13, we may assume that then $\overline{f}$ and $\overline{g}$ are non-zero and have no common factors. We let

- $\overline{X} := X \cap \mathsf{Z}(X_1)$.
- $\overline{Z} := Z \cap \overline{X}$.

Note that $p \in \overline{Z}$ and $\overline{X} = \mathsf{Z}(X_1) \backslash \mathsf{Z}(g)$ where $\mathsf{Z}(X_1) \cong \mathbb{A}^{d-1}_F$ with coordinate ring $F[X_2, ..., X_d]$. By induction, there exists $\{\overline{\phi}_2, ..., \overline{\phi}_d\} \subset F[X_2, ..., X_d]$ which presents $(\overline{X}, \mathsf{Z}(\overline{f}), p)$. Let

$$\overline{\Phi} := (\overline{\phi}_2, ..., \overline{\phi}_d) \quad \text{and} \quad \overline{\Psi} := (\overline{\phi}_2, ..., \overline{\phi}_{d-1}).$$

By Lemma 4.1.10, there exist neighbourhoods $\overline{V} \subset \mathbb{A}^{d-1}_F$ and $\overline{U} \subset \overline{X}$ of $\overline{\Psi}(p)$ and p respectively such that if

$$\overline{Z}_{\overline{V}} := \overline{Z} \cap \overline{\Psi}^{-1}(\overline{V})$$

then the following conditions of Theorem 4.1.1

1. $\overline{\Phi}_{|\overline{U}}$ is étale
2. $\overline{\Psi}_{|\overline{Z}_{\overline{V}}} : \overline{Z}_{\overline{V}} \to \overline{V}$ is finite

3. $\overline{\Phi}_{|\overline{Z}_{\overline{V}}} : \overline{Z}_{\overline{V}} \to \mathbb{A}^1_{\overline{V}}$ is a closed immersion

are satisfied.
Further, by Lemma 4.1.12, we also assume (without loss of generality) that if E is the closed subset of $Z(\bar{f})$ defined by

$$E := Z(\bar{f}) \backslash \overline{\Psi}^{-1}(\overline{V})$$

then

(4) $\dim(E \cap \mathsf{Z}(\overline{\phi}_2, ..., \overline{\phi}_{d-2})) = 0$.

Note that (4) is vacuously satisfied unless $d \geq 4$. Indeed for $d = 3$, $\mathbb{A}^{d-2}_F \setminus V$ is a finite set, and since $\Psi_{|Z(f)} : Z(\bar{f}) \to \mathbb{A}^{d-2}_F$ is finite, E is thus a finite set.
<u>Step 1</u>: Since $\mathsf{Z}(\overline{f}) \xrightarrow{(\overline{\phi}_2,...,\overline{\phi}_{d-1})} \mathbb{A}^{d-2}_F$ is finite (see 4.1.9(1)), for $2 \leq i \leq d$, the image of X_i in $F[X_2, ..., X_d]/(\overline{f})$ satisfies a monic polynomial

$$P_i(T) := T^{m_i} + a_{m_i-1,i}T^{m_i-1} + \cdots + a_{0,i}$$

where each $a_{i,j} \in F[\overline{\phi}_2, ..., \overline{\phi}_{d-1}]$. So $P_i(X_i)$ is zero in $F[X_2, ..., X_d]/(\bar{f})$. Note that each $\overline{\phi}_i$ is an element of $F[X_2, ..., X_d]$. Thus we have a map of algebras

$$F[\overline{\phi}_2, ..., \overline{\phi}_{d-1}][T] \to F[X_1, ..., X_d][T]/(f).$$

We let $\widetilde{P}_i(T)$ be the image of the polynomial $P_i(T)$ under this map. Since $P_i(X_i)$ is zero in $F[X_2, ..., X_d]/(\bar{f})$, $\widetilde{P}_i(X_i)$ maps to zero via the map

$$F[X_1, ..., X_d]/(f) \xrightarrow{X_1 \mapsto 0} F[X_2, ..., X_d]/(\overline{f}).$$

Therefore

$$\widetilde{P}_i(X_i) = X_1 g_i$$

for some $g_i \in F[X_1, ..., X_d]/(f)$. We claim that the map

$$\mathsf{Z}(f) \xrightarrow{(\overline{\phi}_2,...,\overline{\phi}_d,X_1,X_1g_2,...,X_1g_d)} \mathbb{A}^{2d-1}_F$$

is finite. This is clear because for $i \geq 2$, each X_i satisfies the monic polynomial $\widetilde{P}_i(T) -$

$X_1 g_i$ with coefficients which are polynomial expressions in the functions defining the above map. Applying 4.1.4 repeatedly to this map (see Remark 4.1.5), we get $\phi_2, ..., \phi_d \in F[X_1, ..., X_d]$ such that

$$\phi_i \equiv \overline{\phi}_i \mod X_1$$

and the map $(\phi_2, ..., \phi_d)_{|\mathsf{Z}(f)}$ is finite.

Step 2: Consider the maps

$$\widetilde{\Phi} : \mathbb{A}^d_F \xrightarrow{(X_1, \phi_2, ..., \phi_d)} \mathbb{A}^d_F$$

$$\widetilde{\Psi} : \mathbb{A}^d_F \xrightarrow{(X_1, \phi_2, ..., \phi_{d-1})} \mathbb{A}^{d-1}_F.$$

Note that for all points $x \in \mathsf{Z}(X_1)$, $\widetilde{\Phi}$ is étale at x iff $\mathsf{Z}(X_1) \xrightarrow{\overline{\phi}_2, ..., \overline{\phi}_d} \mathbb{A}^{d-1}_F$ is étale at x. Let E be the closed subset of $\mathsf{Z}(\overline{f}) \subset \mathsf{Z}(f)$ defined in Step 0. We have the following:

1. $\widetilde{\Phi}_{|\mathsf{Z}(f)}$ is finite. In fact, the map $(\phi_2, ..., \phi_d)_{|\mathsf{Z}(f)}$ is finite.
2. $\widetilde{\Psi}(p) \notin \widetilde{\Psi}(E)$ (this follows from the definition of E)
3. $\widetilde{\Phi}$ restricted to $\mathsf{Z}(\overline{f}) \backslash E$ is a locally closed immersion.
4. $\widetilde{\Phi}$ is étale at all points in $\mathsf{Z}(\overline{f}) \backslash E$.

By condition (4) of Step 0,

$$E \cap \mathsf{Z}(\overline{\phi}_2, ..., \overline{\phi}_{d-2}) = E \cap \mathsf{Z}(\phi_2, ..., \phi_{d-2})$$[1]

is finite. Let Q be any non-constant polynomial expression in ϕ_d which vanishes on the finite set

$$\Big(E \cap \mathsf{Z}(\phi_2, ..., \phi_{d-2})\Big) \cup \big\{p\big\}.$$

Let ℓ be a large enough integer which is divisible by $\mathrm{char}(F)$. Let $\phi_1 = X_1$ and as in Lemma 4.1.7, let $Q_{d-1} := Q$ and

$$Q_i := \phi_{i+1} - Q^{\ell}_{i+1} \quad \forall\, i \leq d-2.$$

Let

$$\Phi := (\phi_1 - Q^{\ell}_1, \ldots, \phi_{d-1} - Q^{\ell}_{d-1}, \phi_d) : \mathbb{A}^d_F \longrightarrow \mathbb{A}^d_F$$

[1] where by convention $\mathsf{Z}(\phi_2, ..., \phi_{d-3})$ is the whole of $\mathbb{A}^d_F$ if $d \leq 3$.

$$\Psi := (\phi_1 - Q_1^\ell, \ldots, \phi_{d-1} - Q_{d-1}^\ell) : \mathbb{A}_F^d \to \mathbb{A}_F^{d-1}.$$

By Lemma 4.1.7 $\Psi_{|\mathsf{Z}(f)}$ is finite. We let S be the finite set of points in $\Psi^{-1}\Psi(p) \cap \mathsf{Z}(f)$. To finish the proof, it suffices to verify the conditions (2)-(4) of Definition (4.1.9). We first note that $S \subset \mathsf{Z}(\phi_1, ..., \phi_{d-2})$. This is because if $x \in S$, then by definition of S,

$$\phi_{i+1} - Q_{i+1}^\ell(x) = Q_i(x) = 0 \quad \forall \ \ i \leq d-2.$$

And thus

$$\phi_i - Q_i^\ell(x) = \phi_i(x) = 0 \quad \forall \ \ i \leq d-2.$$

We now show that S is disjoint from E. First note that $S \subset \mathsf{Z}(\phi_1) = \mathsf{Z}(X_1)$. Also $\Psi(p) = 0$ since $Q(p) = 0$ and $\phi_i(p) = 0$ for $1 \leq i \leq d-1$. Let $x \in S \cap E$ if possible. Hence x is necessarily in $E \cap \mathsf{Z}(\phi_2, ..., \phi_{d-2})$ by the above argument. In particular we note that $\phi_{d-2}(x) = 0$. Now we claim that

$$\phi_{d-1}(x) = 0.$$

Since $\Psi(x) = 0$ we have $(\phi_{d-2} - Q_{d-2}^\ell)(x) = 0$. But as $\phi_{d-2}(x) = 0$, we conclude that

$$Q_{d-2}(x) = 0.$$

Thus

$$\phi_{d-1}(x) = (Q_{d-2} - Q^\ell)(x) = 0.$$

This proves the claim. Consequently, $x \in Z(\phi_2, \ldots \phi_{d-1})$. By definition of E, $x \in E$ implies $\bar{\Psi}(x) \notin \bar{V}$ where $\bar{V}$ is as defined in Step 0. As $\bar{V}$ is a neighborhood of $0 = \Psi(p)$, we have $\bar{\Psi}(x) \neq 0$. But as $x \in E \subset Z(X_1)$, we have

$$\bar{\Psi}(x) = (\phi_2, ..., \phi_{d-1})(x) = (\bar{\phi}_2, ..., \bar{\phi}_{d-1})(x).$$

Hence $\phi_i(x) \neq 0$ for some i with $2 \leq i \leq d-1$. This is a contradiction to the fact that $x \in Z(\phi_2, \ldots \phi_{d-1})$. Hence S must be disjoint from E. Hence $\widetilde{\Phi}$ is a locally closed immersion on S by property (3) of Step 2.

As in the proof of Lemma 4.1.7, we let

$$\mathbb{A}_F^{d-2} \xrightarrow{\eta} \mathbb{A}_F^{d-2}$$

be the automorphism defined by

$$\eta = (Y_1 - \widetilde{Q}_1^\ell, \ldots, Y_{d-1} - \widetilde{Q}_{d-1}^\ell, Y_d)$$

where $\widetilde{Q}_i \in F[Y_1, ..., Y_d]$ are polynomials satisfying $Q_i = \widetilde{Q}_i(\phi_1, ..., \phi_d)$. It is straightforward to check that

$$\Phi = \eta \circ \widetilde{\Phi}.$$

Hence Φ is a locally closed immersion on S, this proves condition (4) of Definition 4.1.9.

From Lemma 4.1.10 we have $Z(f) \cap \Psi^{-1}(V) \subset X$. This with the fact that $Z = Z(f) \cap X$ implies conditions (2) of Definition 4.1.9. For checking condition (3), i.e. to check Φ is étale at all points in S, we note that since ℓ is divisible by $\mathrm{char}(F)$, Φ is étale precisely at those points where $\widetilde{\Phi}$ is étale. In particular Φ is étale at all points of $\mathsf{Z}(\overline{f})\backslash E$. □

4.1.3 Open subsets of $\mathbb{A}_F^2$

By reduction using lemmas 4.1.6, 4.1.14 we only need to handle the case of open subsets of $\mathbb{A}_F^2$. The low degree points are dealt with in the similar way as in [Poo], whereas for the high degree point we use a different technique (see Lemma 4.1.24).

Lemma 4.1.15. *Let F be a finite field as before, and $C \subset \mathbb{A}_F^2$ be a closed curve such that the projection onto the Y-coordinate $Y_{|C} : C \to \mathbb{A}_F^1$ is finite. Let $C^{(1)}$ denote the set of closed points of C. Then the following set of points is dense in C*

$$\left\{x \in C^{(1)} \mid \mathsf{deg}_F(Y(x)) = \mathsf{deg}_F(x)\right\}.$$

Proof. Without loss of generality, we may assume C is irreducible and hence we simply have to show that the set

$$\left\{x \in C^{(1)} \mid \mathsf{deg}_F(Y(x)) = \mathsf{deg}_F(x)\right\}$$

is infinite. Let $x_1, ..., x_q$ be the F-rational points of $\mathbb{A}^1_F$. Let $C' := C \backslash Y^{-1}(\{x_1, ..., x_q\})$. C' is a dense open subset of C as $Y_{|C}$ is finite. Now, any point $x \in C'$ of prime degree satisfies $\mathsf{deg}_F(Y(x)) = \mathsf{deg}_F(x)$. By Lang-Weil estimates [LW], for all large enough prime number ℓ, there is a point $x \in C'^{(1)}$ of degree ℓ. Hence, since ℓ is a prime, we must have $\mathsf{deg}_F(Y(x)) = \mathsf{deg}_F(x)$. This proves the lemma. □

Notation 4.1.16. Let

1. $A = F[X, Y]$ and for $d \geq 0$ let $A_{\leq d} = \{h \in A \mid \mathsf{deg}(h) \leq d\}$. Here $\mathsf{deg}(h)$ denotes the total degree.

2. $f, g \in A$ be two non-constant polynomials, with no common irreducible factors. By performing a change of coordinates if necessary, we will assume that f is monic in X of degree m.

3. $W := \mathbb{A}^2_F \backslash \mathsf{Z}(g)$. In this section, we call our variety W instead of X, since the later will denote a coordinate function on $\mathbb{A}^2_F$.

4. $Z := \mathsf{Z}(f) \cap W$. Note that $\mathsf{Z}(f) \backslash Z$ is finite as f, g have no common irreducible components.

5. $p \in Z$ be a closed point such that its X-coordinate is 0. We also choose a set of closed points $\{p_1, ..., p_t\}$ in Z such that the set $T := \{p, p_1, ..., p_t\}$ satisfies

   (a) T contains at least one point from each irreducible component of Z.

   (b) No two points in T have same degrees and for all $p_i \in T$, $\mathsf{deg}(Y(p_i)) = \mathsf{deg}(p_i)$. This can be ensured by Lemma 4.1.15. Note that since X-coordinate of p is 0, we also have $\mathsf{deg}(Y(p)) = \mathsf{deg}(p)$.

6. Let $D = \{q_1, \ldots, q_s\}$ be a finite set of closed points in $\mathsf{Z}(f)$ satisfying:

   (a) D contains all points in $\mathsf{Z}(f) \backslash Z$.

   (b) D contains at least one point from each irreducible component of $\mathsf{Z}(f)$.

   (c) D does not contain any point of $\{p, p_1, ..., p_t\}$.

Moreover, for a point x in $\mathsf{Z}(f)$, the notation $\mathcal{O}_x$ (resp. $\mathfrak{m}_x$) will denote $\mathcal{O}_{\mathbb{A}^2_F, x}$ (resp. $\mathfrak{m}_{\mathbb{A}^2_F, x}$) i.e. the local ring (resp. maximal ideal) of x as a point of $\mathbb{A}^2_F$.

The main result of this section is the following.

Theorem 4.1.17. *There exists* $(\phi_1, \phi_2) \in F[X, Y]$ *which presents* $(W, \mathsf{Z}(f), p)$.

This is enough to prove Theorem 4.1.1.

Proof of 4.1.1. This follows from Lemmas 4.1.6, 4.1.10 and 4.1.14 and Theorem 4.1.17. □

In order to prove Theorem 4.1.17, ϕ_1 is arranged in Lemma 4.1.18 and ϕ_2 is arranged in Lemma 4.1.24. The counting techniques by Poonen [Poo] are the backbone of the proofs of these lemmas.

Recall from 4.1.3, for Y a subset of a scheme X/F, $Y_{\leq r} := \{x \in Y \mid \mathsf{deg}(x) \leq r\}$.

Lemma 4.1.18. *Let the notation be as in 4.1.16. There exists* $c \in \mathbb{N}$*, such that for every* $d >> 0$*, there exists a* $\phi \in A_{\leq d}$ *satisfying*

1. $\phi(p) = \phi(p_i) = 0$ *for all* $i = 1, ..., t$ *and* $\phi(q_i) \neq 0$ *for all* $i = 1, \ldots, s$.

2. (ϕ, Y) *is étale at all* $x \in S$ *where* $S := \mathsf{Z}(\phi) \cap Z$.

3. *The projection* $Y : \mathbb{A}^2_F \to \mathbb{A}^1_F$ *is radicial at* $S_{\leq (d-c)/3}$.

Remark 4.1.19. *We could prove only some of the conditions on* ϕ *such that* (ϕ, Y) *presents* $(W, \mathsf{Z}(f), p)$*. The above lemma is basically formulated using those conditions. Suppose* $\phi_{|Z(f)}$ *is a finite map and* Y *is radicial at whole of* S *(as opposed to* $S_{(d-c)/3}$ *above), then* (ϕ, Y) *would present* (W, Z, p) *thereby proving* (4.1.17).

Remark 4.1.20. *The set* $S = \mathsf{Z}(\phi) \cap Z$ *in the statement of the above Lemma is necessarily finite. Since, in each irreducible component of* Z*, on at least one* q_i *(see Notation* (4.1.16)*(6)(b))* ϕ *does not vanish. Since* T *intersects each irreducible component of* $\mathsf{Z}(f)$ *(see Notation* (4.1.16)*(5)(a)), we know that any open neighbourhood of* S *is dense in* $\mathsf{Z}(f)$.

Following [Poo] define the density of a subset $\mathcal{C} \subset A$ by

$$\mu(\mathcal{C}) := \lim_{d \to \infty} \frac{\#\, (\mathcal{C} \cap A_{\leq d})}{\#\, A_{\leq d}}$$

provided the limit exists. Similarly, the upper and lower densities of $\mathcal{C}$, denoted by $\overline{\mu}(\mathcal{C})$ and $\underline{\mu}(\mathcal{C})$, are defined by replacing limit in the above expression by lim sup and lim inf respectively.

We show the existence of ϕ as in Lemma 4.1.18 by proving that the density of such ϕ is positive. The proof of the Lemma 4.1.18 is in two steps, in the first step we prove (Lemma 4.1.22) that ϕ satisfying conditions $(1),(3)$ and condition (2) for points upto certain degree, exists. Then in Lemma 4.1.23 we show that the set of ϕ which does not satisfy condition (2) for points of higher degrees is of zero density.

Let $\phi \in A$ and $r \geq 1$ be an integer. Consider the following conditions on ϕ, which are closely related to the conditions (1), (2) and (3) of Lemma 4.1.18.

(a) $\phi(p) = \phi(p_i) = 0$ for all $1 \leq i \leq t$ and $\phi(q_i) = 1$ for all $1 \leq i \leq s$.

(b_r) For all $x \in \mathsf{Z}(f)_{\leq r}$ such that $\phi(x) = 0$, $\frac{\partial \phi}{\partial X}(x) \neq 0$.

(c_r) For all points $x_1, x_2 \in \mathsf{Z}(f)_{\leq r}$, such that $\mathsf{deg}(x_1) = \mathsf{deg}(x_2) = \mathsf{deg}(Y(x_1)) = \mathsf{deg}(Y(x_2))$ and $\phi(x_1) = \phi(x_2) = 0$, we have $Y(x_1) \neq Y(x_2)$.

(d_r) For all $x \in \mathsf{Z}(f)_{\leq r}$ such that $\phi(x) = 0$, $\mathsf{deg}(Y(x)) = \mathsf{deg}(x)$.

Remark 4.1.21. *The main motivation for introducing the above conditions, are the following straightforward implications between them and the conditions of 4.1.18*

- ϕ *satisfies* (4.1.18)*(1) if* ϕ *satisfies (a).*
- ϕ *satisfies* (4.1.18)*(2) iff* ϕ *satisfies* (b_r) *for all* $r \geq 1$.
- ϕ *satisfies* (4.1.18)*(3) iff* ϕ *satisfies* (c_r) *and* (d_r) *for all* $r \leq (d-c)/3$.

Lemma 4.1.22. *There exist integers* $r_0, c \in \mathbb{N}$, *with*

$$r_0 > \max\Big\{\mathsf{deg}(p), \mathsf{deg}(p_1), ..., \mathsf{deg}(p_t), \mathsf{deg}(q_1), ..., \mathsf{deg}(q_s)\Big\}$$

such that the lower density of the set

$$\mathcal{P} := \bigcup_{d > c + 2r_0} \Big\{\phi \in A_{\leq d} \mid \phi \text{ satisfies } (a), (b_{(d-c)/3}), (c_{(d-c)/3}), (d_{(d-c)/3}) \text{ and } \phi(x) = 1 \ \forall\, x \in \mathsf{Z}(f)_{\leq r_0} \backslash T\Big\}$$

is positive.

Proof. By Lang-Weil estimates [LW] there exists $c' \in \mathbb{N}$ such that for all $n \geq 1$,

$$\#\left(\mathsf{Z}(f)_{=n}\right) \leq c' \cdot q^n.$$

For reasons which will be clear during the course of the calculations below, we choose r_0 and c as follows. Recall that m is the X-degree of f. Let r_0 be any integer satisfying

(i) $r_0 > \max\left\{\mathsf{deg}(p), \mathsf{deg}(p_1), ..., \mathsf{deg}(p_t), \mathsf{deg}(q_1), ..., \mathsf{deg}(q_s)\right\}$.

(ii) $\left(\sum_{i > r_0/m} \frac{1}{q^i}\right) \cdot \left(c' + \binom{m}{2} + \frac{m}{2}\right) < 1 - \sum_{x \in T} q^{-\mathsf{deg}(x)}$.

Note that it is always possible to ensure (ii) as

$$\left(\sum_{i > r_0/m} \frac{1}{q^i}\right) \to 0 \quad \text{as} \quad r_0 \to \infty$$

and as degrees of points in T are distinct we have

$$\sum_{x \in T} q^{-\mathsf{deg}(x)} < \sum_{i=1}^{\infty} q^{-i} \leq 1.$$

Let

$$c = \sum_{x \in \mathsf{Z}(f)_{\leq r_0}} \mathsf{deg}(x).$$

Let $d \geq c + 2r_0$ be any integer and $r := (d - c)/3$. Let

$$\begin{aligned}
\mathcal{T} &:= \left\{\phi \in A_{\leq d} \mid \phi \text{ satisfies (a) and } \phi(x) = 1 \ \forall\ x \in \mathsf{Z}(f)_{\leq r_0} \backslash T\right\}. \\
\mathcal{T}_b &:= \left\{\phi \in \mathcal{T} \mid \phi \text{ does not satisfy} (b_r)\right\} \\
\mathcal{T}_c &:= \left\{\phi \in \mathcal{T} \mid \phi \text{ does not satisfy} (c_r)\right\} \\
\mathcal{T}_d &:= \left\{\phi \in \mathcal{T} \mid \phi \text{ does not satisfy} (d_r)\right\}
\end{aligned}$$

Let

$$\delta := \frac{\#\,\mathcal{T}}{\#\,A_{\leq d}}, \quad \delta_b := \frac{\#\,\mathcal{T}_b}{\#\,A_{\leq d}}, \quad \delta_c := \frac{\#\,\mathcal{T}_c}{\#\,A_{\leq d}}, \quad \delta_d := \frac{\#\,\mathcal{T}_d}{\#\,A_{\leq d}}.$$

In the following steps we will estimate $\delta, \delta_b, \delta_c, \delta_d$.

Step 1 : (Estimation for δ) : Note that the condition that ϕ belongs to $\mathcal{T}$ depends solely on the image of ϕ in the zero dimensional ring

$$\prod_{x \in \mathsf{Z}(f)_{\leq r_0}} (\mathcal{O}_x/m_x).$$

Since the dimension over F of the above ring is c and since $d \geq c$, by [Poo, Lemma 2.1] the map

$$A_{\leq d} \xrightarrow{\rho} \prod_{x \in \mathsf{Z}(f) \leq r_0} (\mathcal{O}_x/m_x)$$

is surjective. One can easily see that $\mathcal{T}$ is a coset of $\mathrm{Ker}(\rho)$. Therefore

$$\delta = \prod_{x \in \mathsf{Z}(f)_{\leq r_0}} q^{-\mathsf{deg}(x)}.$$

Step 2 : (Estimation for δ_b) : Let $x \in \mathsf{Z}(f)_{\leq r}$ where recall that $r = (d-c)/3$. The following are equivalent :

(i) $\phi \in \mathcal{T}$ and $\phi(x) = 0$ and $\frac{\partial \phi}{\partial X}(x) = 0$.

(ii) $\phi \in \mathcal{T}$ and ϕ mod $\mathfrak{m}_x^2$ lies in the kernel of the linear map $\frac{\partial}{\partial X} : \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2} \to F(x)$.

Let us first consider the case when $\mathsf{deg}(x) > r_0$. In this case, each of the above condition for ϕ depends only on its image in the zero dimensional ring

$$\left(\prod_{q \in \mathsf{Z}(f)_{\leq r_0}} (\mathcal{O}_q/m_q) \right) \times (\mathcal{O}_x/m_x^2).$$

The cardinality of the above ring is

$$\left(\prod_{y \in \mathsf{Z}(f)_{\leq r_0}} q^{\mathsf{deg}(y)} \right) \cdot q^{3\mathsf{deg}(x)}.$$

Let us call an element ξ in the above ring as a favorable value iff all ϕ mapping to ξ

satisfy the above conditions. It is an easy exercise to check that the set of all favorable values has cardinality $q^{\mathsf{deg}(x)}$. Thus the ratio of the number of favorable values to the cardinality of the ring is nothing but $\delta q^{-2\mathsf{deg}(x)}$. The dimension over F of this ring is $c + 3{\cdot}\mathsf{deg}(x)$. Since $d \geq c + 3{\cdot}\mathsf{deg}(x)$, by [Poo, Lemma 2.1], $A_{\leq d}$ surjects onto this ring. Due to this, the ratio of $\phi \in A_{\leq d}$ satisfying the above two conditions to the $\#A_{\leq d}$ is nothing but $\delta q^{-2\mathsf{deg}(x)}$. In other words,

$$\frac{\#\left\{\phi \in \mathcal{T} \mid \phi(x) = 0,\ \frac{\partial\phi}{\partial X}(x) = 0\right\}}{\#\ A_{\leq d}} = \delta \cdot q^{-2\mathsf{deg}(x)}.$$

Now let us consider the case where $\mathsf{deg}(x) \leq r_0$. We claim that unless $x \in T$, there is no $\phi \in \mathcal{T}$ which vanishes on x. This follows from the definition of $\mathcal{T}$. So let us assume $x \in T$. In this case, the above two conditions for ϕ depend solely on the image of ϕ in the ring

$$\left(\prod_{\substack{q \in \mathsf{Z}(f)_{\leq r_0} \\ q \neq x}} (\mathcal{O}_q/m_q)\right) \times (\mathcal{O}_x/m_x^2).$$

Proceeding in a manner similar to the case where $\mathsf{deg}(x) > r_0$, we find that for $x \in T$,

$$\frac{\#\left\{\phi \in \mathcal{T} \mid \phi(x) = 0,\ \frac{\partial\phi}{\partial X}(x) = 0\right\}}{\#\ A_{\leq d}} = \delta \cdot q^{-\mathsf{deg}(x)}.$$

Since

$$\mathcal{T}_b = \bigcup_{\substack{x \in \mathsf{Z}(f)_{\leq r} \text{ such that} \\ x \in T \text{ or } \mathsf{deg}(x) > r_0}} \left\{\phi \in \mathcal{T} \mid \phi(x) = 0,\ \frac{\partial\phi}{\partial X}(x) = 0\right\}$$

we get an estimate

$$\begin{aligned} \delta_b &\leq \sum_{x \in T} \delta q^{-\mathsf{deg}(x)} + \sum_{\substack{x \in \mathsf{Z}(f)_{\leq r} \text{ such that} \\ \mathsf{deg}(x) > r_0}} \delta \cdot q^{-2\mathsf{deg}(x)} \\ &\leq \delta\left(\sum_{x \in T} q^{-\mathsf{deg}(x)} + \sum_{r_0 < i \leq r} c' q^{-i}\right) \end{aligned}$$

where recall that c' was the constant in Lang-Weil estimates such that $\#\mathsf{Z}(f)_{=n} \leq c'q^n$.

Step 3 : (Estimation for δ_c): Let $y \in \mathbb{A}^1_F$ with $i := \mathsf{deg}(y) \leq r$. Let

$$\mathcal{T}_c^y := \Big\{\phi \in \mathcal{T} \mid \exists \text{ distinct } x_1, x_2 \in \mathsf{Z}(f)_{=i} \text{ with } Y(x_1) = Y(x_2) = y \text{ and } \phi(x_1) = \phi(x_2) = 0\Big\}.$$

First, note that $\mathcal{T}_c^y$ is empty unless $i > r_0$. This is because the only points of degree $\leq r_0$ on which a $\phi \in \mathcal{T}$ vanishes are the points in T. However, by choice, all points in $x \in T$ have different degrees and satisfy $\deg(x) = \deg(Y(x))$. Thus, let us assume $i > r_0$. In this case, we claim that

$$\frac{\#\ \mathcal{T}_c^y}{\#\ A_{\leq d}} \leq \delta \cdot \binom{m}{2} \cdot q^{-2i}.$$

For fixed x_1, x_2 with $Y(x_1) = Y(x_2) = y$,

$$\Big\{\phi \in \mathcal{T} \mid \phi(x_1) = \phi(x_2) = 0\Big\}$$

is a coset of the kernel of the following map

$$A_{\leq d} \longrightarrow \Bigg(\prod_{q \in \mathsf{Z}(f)_{\leq r_0}} (\mathcal{O}_q/m_q)\Bigg) \times (\mathcal{O}_{x_1}/m_{x_1}) \times (\mathcal{O}_{x_2}/m_{x_2})$$

which is surjective by [Poo, 2.1]. Thus

$$\frac{\#\ \Big\{\phi \in \mathcal{T} \mid \phi(x_1) = \phi(x_2) = 0.\Big\}}{\#\ A_{\leq d}} \leq \delta \cdot q^{-2i}.$$

To prove the claim we now simply observe that since f is monic in X of degree m there are atmost $\binom{m}{2}$ possible choices for a pair $\{x_1, x_2\}$ as above.

As discussed above, since $\mathcal{T}_c^y$ is empty unless $i > r_0$, we have

$$\mathcal{T}_c = \bigcup_{\substack{y \in \mathbb{A}^1_F \\ r_0 < \mathsf{deg}(y) \leq r}} \mathcal{T}_c^y.$$

For a fixed i,

$$\#\ \Big\{y \in \mathbb{A}^1_F \mid \mathsf{deg}(y) = i\Big\} \leq q^i.$$

From this, it is elementary to deduce

$$\delta_c = \frac{\#\ \mathcal{T}_c}{\#\ A_{\leq d}} \leq \delta\Big(\sum_{r_0 < i \leq r} \binom{m}{2} q^{-i} \Big).$$

Step 4 : (Estimation for δ_d): As in the above step, let $y \in \mathbb{A}^1_F$ with $i := \mathsf{deg}(y) \leq r$. Let

$$\mathcal{T}_d^y := \Big\{ \phi \in \mathcal{T} \mid \exists\ x \in \mathsf{Z}(f)_{\leq r} \text{ with } \phi(x) = 0\ ,\ Y(x) = y \text{ and } \mathsf{deg}(x) > i. \Big\}.$$

We first claim that $\mathcal{T}_d^y$ is empty unless $\mathsf{deg}(y) > r_0/m$. Otherwise, there would exist a $\phi \in \mathcal{T}$ and an $x \in \mathsf{Z}(f)_{\leq r}$ with $Y(x) = y$, $\phi(x) = 0$ and $\mathsf{deg}(x) > \mathsf{deg}(y)$. But as f is monic in X of degree m, the maximum degree of a point x lying over y is $m \cdot \mathsf{deg}(y) \leq r_0$. Which means $x \in \mathsf{Z}(f)_{\leq r_0}$. However as $\phi \in \mathcal{T}$, the only points in $\mathsf{Z}(f)_{\leq r_0}$ on which ϕ vanishes are those in T. Thus $x \in T$. But by (4.1.16)(5)(c), for such x, $\mathsf{deg}(Y(x)) = \mathsf{deg}(y) = \mathsf{deg}(x)$ which is a contradiction.

We will now estimate

$$\frac{\#\ \mathcal{T}_d^y}{\#\ A_{\leq d}}.$$

Fix a point $x \in \mathsf{Z}(f)_{\leq r}$ with $\mathsf{deg}(x) > i$ and $Y(x) = y$. For this x, we first note that because of (4.1.16)(5)(c), $x \notin T$.

For $\mathsf{deg}(x) > r_0$ we note that

$$\frac{\#\ \Big\{ \phi \in \mathcal{T} \mid \phi(x) = 0. \Big\}}{\#\ A_{\leq d}} \leq \delta \cdot q^{-\mathsf{deg}(x)} \leq \delta \cdot q^{-2i}.$$

This is deduced, as before, from the surjectivity of

$$A_{\leq d} \longrightarrow \Big(\prod_{q \in \mathsf{Z}(f)_{\leq r_0}} (\mathcal{O}_q/m_q) \Big) \times (\mathcal{O}_x/m_x)$$

If $\mathsf{deg}(x) \leq r_0$, $\Big\{ \phi \in \mathcal{T} \mid \phi(x) = 0 \Big\}$ is empty as there is no points in $\mathsf{Z}(f) \backslash T$ on which a $\phi \in \mathcal{T}$ vanishes. And hence, the above estimate trivially holds in this case also.

As f is monic in X of degree m, and $\mathsf{deg}(x) \geq 2i$, there are at most $\frac{m}{2}$ possible choices for $x \in \mathsf{Z}(f)$ such that $Y(x) = y$. This shows that

$$\frac{\#\ \mathcal{T}_d^y}{\#\ A_{\leq d}} \leq \delta \cdot \frac{m}{2} \cdot q^{-2i}.$$

Since, as discussed above, $\mathcal{T}_d^y$ is empty unless $\mathsf{deg}(y) > r_0/m$, we have

$$\mathcal{T}_d = \bigcup_{y \in (\mathbb{A}_F^1)_{\geq r_0/m}} \mathcal{T}_d^y.$$

For a fixed i,

$$\#\ \left\{ y \in \mathbb{A}_F^1 \mid \mathsf{deg}(y) = i \right\} \leq q^i.$$

Thus

$$\delta_d = \frac{\#\ \mathcal{T}_d}{\#\ A_{\leq d}} \leq \delta \Big(\sum_{r_0/m < i \leq r} \frac{m}{2}\ q^{-i} \Big).$$

<u>Step 5</u> : (Estimation for $\mathcal{P}$): If we let

$$\mathcal{P}_d := \Big\{ \phi \in A_{\leq d} \mid \phi \text{ satisfies } (a), (b_{(d-c)/3}), (c_{(d-c)/3}), (d_{(d-c)/3}) \text{ and } \phi(x) = 1\ \forall\ x \in \mathsf{Z}(f)_{\leq r_0} \backslash T \Big\},$$

then

$$\mathcal{P}_d = \mathcal{T} \backslash (\mathcal{T}_b \cup \mathcal{T}_c \cup \mathcal{T}_d).$$

Therefore

$$\begin{aligned} \frac{\#\ \mathcal{P}_d}{\#\ A_{\leq d}} &\geq \delta - \delta_b - \delta_c - \delta_d \\ &\geq \delta \left[1 - \sum_{x \in T} q^{-\mathsf{deg}(x)} - \sum_{r_0 < i \leq r} c' q^{-i} - \sum_{r_0 < i \leq r} \binom{m}{2} q^{-i} - \sum_{r_0/m < i \leq r} \frac{m}{2}\ q^{-i} \right] \\ &\geq \delta \left[1 - \sum_{x \in T} q^{-\mathsf{deg}(x)} - \left(\sum_{r_0/m < i \leq r} \frac{1}{q^i} \right) \cdot \left(c' + \binom{m}{2} + \frac{m}{2} \right) \right] \end{aligned}$$

Note that in the above expression $r = (d-c)/3$. As $d \to \infty$, so does r. Hence we observe

that
$$\mathsf{inf}\left(\frac{\#\,\mathcal{P}_d}{\#\,A_{\leq d}}\right) \geq \delta\left[1-\sum_{x\in T} q^{-\mathsf{deg}(x)}-\left(\sum_{i>r_0/m}\frac{1}{q^i}\right)\cdot\left(c'+\binom{m}{2}+\frac{m}{2}\right)\right]$$
which is positive, thanks to the definition of r_0. Thus the lower density of
$$\mathcal{P}=\bigcup_d \mathcal{P}_d$$
is positive as required. □

Lemma 4.1.23. *Let c be as in Lemma 4.1.22 and let*
$$\mathcal{Q} := \bigcup_{d\geq 0}\left\{\phi\in A_{\leq d} \mid \exists\ x\in \mathsf{Z}(f)_{>(d-c)/3} \text{ such that } \phi(x)=\frac{\partial\phi}{\partial X}(x)=0\right\}.$$
Then $\mu(\mathcal{Q})=0$.

Proof. The proof is identical to that of [Poo, 2.6]. We reproduce the argument verbatim here for the convenience of the reader. We will bound the probability of ϕ constructed as
$$\phi=\phi_0+g^pX+h^p$$
and for which there is a point $x\in\mathsf{Z}(f)_{>(d-c)/3}$ with $\phi(x)=\frac{\partial\phi}{\partial X}(x)=0$. Note that if ϕ is of the above form, then
$$\frac{\partial\phi}{\partial X}=\frac{\partial\phi_0}{\partial X}+g^p.$$
Further, if $\phi_0\in A_{\leq d}$, $g\in A_{\leq d-1/p}$ and $h\in A_{\leq d/p}$, then $\phi\in A_{\leq d}$. Define
$$W_0:=\mathsf{Z}(f) \quad\text{and}\quad W_1:=\mathsf{Z}\left(f,\frac{\partial\phi}{\partial X}\right).$$
Note that $\dim(W_0)=1$.
Let
$$\gamma:=\lfloor\frac{d-1}{p}\rfloor \quad\text{and}\quad \eta=\lfloor\frac{d}{p}\rfloor.$$
Claim 1: The probability (as a function of d) of choosing $\phi_0\in A_{\leq d}$ and $g\in A_{\leq(d-1)/p}$ such that $\dim(W_1)=0$ is $1-o(1)$ as $d\to\infty$.

Let $V_1, ..., V_\ell$ be F irreducible components of W_0. Clearly $\ell \leq \mathsf{deg}(f)$ (where $\mathsf{deg}(f)$ is the total degree). Since the projection onto the Y coordinate is finite on $\mathsf{Z}(f)$ (by (4.1.16)(2)), we know that $Y(V_k)$ is of dimension one for all k. We will now bound the set

$$G_k^{\mathrm{bad}} := \left\{ g \in A_{\leq \gamma} \mid \frac{\partial \phi}{\partial X} = \frac{\partial \phi_0}{\partial X} + g^p \text{ vanishes identically on } V_k \right\}.$$

If $g, g' \in G_k^{\mathrm{bad}}$, then $g - g'$ vanishes on V_k. Thus if G_k^{bad} is non-empty, it is a coset of the subspace of functions in A_γ which vanish identically on V_k. The codimension of that subspace, or equivalently the dimension of the image of A_γ in the regular functions on V_k is at least $\gamma + 1$, since no polynomial in Y vanishes on V_k. Thus the probability that $\frac{\partial \phi}{\partial X}$ vanishes on V_k is at most $q^{-\gamma-1}$. Thus, the probability that $\frac{\partial \phi}{\partial X}$ vanishes on some V_k is at most $\ell q^{-\gamma-1} = o(1)$. Since $\dim(W_1) = 0$ iff $\frac{\partial \phi}{\partial X}$ does not identically vanish on any component V_k, the claim follows.

We will now estimate the probability of choosing h such that there is no bad point in $Z(f)$, i.e., a point in $\mathsf{Z}(f)_{>(d-c)/3}$ where both ϕ and $\frac{\partial \phi}{\partial X}$ vanish. Note that the set of such bad points is precisely

$$\mathsf{Z}(\phi) \cap W_1 \cap \mathsf{Z}(f)_{>(d-c)/3}.$$

Claim 2: Conditioned on the choice of ϕ_0 and g such that W_1 is finite, the probability of choosing h such that

$$\mathsf{Z}(\phi) \cap W_1 \cap \mathsf{Z}(f)_{>(d-c)/3} = \emptyset$$

is $1 - o(1)$ as $d \to \infty$.

It is clear by the Bezout theorem that $\# W_1 = O(d)$. For a given $x \in W_1$, the set

$$H^{\mathrm{bad}} = \left\{ h \in A_\eta \mid \phi = \phi_0 + g^p X + h^p \text{ vanishes on } x \right\}$$

is either $\emptyset$ or a coset of $\mathrm{Ker}\left(A_\eta \xrightarrow{ev_x} F(x)\right)$ where $F(x)$ is the residue field of x. For the purpose of this claim, we only need to consider x such that $\mathsf{deg}(x) > (d-c)/3$. In this case, [Poo, Lemma 2.5] implies that

$$\frac{\# H^{\mathrm{bad}}}{\# A_\eta} \leq q^{-\nu} \quad \text{where} \quad \nu = \mathsf{min}(\eta + 1, (d-c)/3).$$

Thus, the probability that both ϕ and $\frac{\partial\phi}{\partial X}$ vanish at such x is at most $q^{-\nu}$. There are at most $\#\, W_1$ many possibilities for x. Thus the probability that there exists a 'bad point', i.e. point in $x \in W_1$ with $\mathsf{deg}(x) > (d-c)/3$ such that both ϕ and $\frac{\partial\phi}{\partial X}$ vanish at such x is at most $\left(\#\, W_1\right)q^{-\nu} = O(dq^{-\nu})$. Since as $d \to \infty$, ν grows linearly in d, $O(dq^{-\nu}) = o(1)$. In other words, the probability of choosing h such that there is no bad point is $1 - o(1)$.

Combining the above two claims, it follows that the probability of choosing $\phi = \phi_0 + g^p X + h^p$ such that

$$\mathsf{Z}(\phi) \cap W_1 \cap \mathsf{Z}(f)_{>(d-c)/3} = \emptyset$$

is equal to $(1 - o(1))(1 - o(1)) = 1 - o(1)$. This shows that $\mu(Q) = 0$. □

Proof of Lemma 4.1.18. Let $\overline{Q}$ denote the complement of Q in A. Let $\mathcal{P}$ be as in Lemma 4.1.22. To prove Lemma 4.1.18 we need to show that $\mathcal{P} \cap \overline{Q}$ is non-empty. However, combining the above two lemmas, we in fact get that $\mu(\mathcal{P} \cap \overline{Q}) > 0$. This finishes the proof. □

Condition (3) of Lemma 4.1.18 ensures that $Y : \mathbb{A}^2_F \to \mathbb{A}^1_F$ is radicial at $S_{\leq (d-c)/3}$. We would have ideally liked to have S instead of $S_{\leq (d-c)/3}$ here. If this was the case, and if $\phi_{|Z(f)}$ was finite, as mentioned in Remark 4.1.19, we would be able to deduce that (ϕ, Y) presents $(W, \mathsf{Z}(f), p)$. However we are unable to handle points in S of degree greater than $(d-c)/3$. In order to rectify that, we replace the map (ϕ, Y) with a map (ϕ, h) for a suitable h as found by the following lemma. Finiteness of ϕ will be handled later using a Noether normalization argument.

Lemma 4.1.24. *Let $c \in \mathbb{N}$ be as in Lemma 4.1.18. Let $d >> 0$ be an integer such that for every $i > (d-c)/3$,*

$$\#\, (\mathbb{A}^1_F)_{=i} > dm.$$

Let $\phi \in A_{\leq d}$ be as given by (4.1.18) and $S := \mathsf{Z}(\phi) \cap Z$. Then, there exists $h \in F[X, Y]$ such that

1. *$h_{|S} : S \to \mathbb{A}^1$ is radicial, i.e. injective and preserves the degree.*
2. *The map $\mathbb{A}^2_F \xrightarrow{(\phi,h)} \mathbb{A}^2_F$ is étale at all $x \in S$.*
3. *$h_{|\mathsf{Z}(f)} : \mathsf{Z}(f) \to \mathbb{A}^1_F$ is a finite map.*

Proof. Step(1): In this step we will show that with the given choice of d, it is possible to choose h_1 which satisfies condition (1) of the Lemma.
We claim that

$$\#\, S_{=i} \leq \#\, (\mathbb{A}^1_F)_{=i} \quad \forall\, i \geq 1.$$

As explained in Remark 4.1.20, $\mathsf{Z}(\phi) \cap \mathsf{Z}(f)$ is finite. By Bezout theorem, $\#\, S \leq \mathsf{deg}(\phi)\mathsf{deg}(f) = dm$. Thus the above claim holds for all $i > (d-c)/3$ by the choice of d. On the other hand, the claim also holds for $i \leq (d-c)/3$, since by Lemma 4.1.18, Y is radicial at $S_{\leq (d-c)/3}$. Thus we can choose a set theoretic map $S \xrightarrow{\widetilde{h}} \mathbb{A}^1_F$ which is injective and preserves degree of points. By Chinese remainder theorem, there exists an $h_1 \in F[X,Y]$ such that for all $x \in S$

$$h_1(x) = \widetilde{h}(x).$$

Step(2): Now, using the h_1 from above step, we will find a $h_2 \in F[X,Y]$ which satisfies conditions (1) and (2) of the Lemma. It is sufficient to find an $h_2 \in F[X,Y]$ such that

$$(i) \quad h_2 \equiv h_1 \mod \mathfrak{m}_x \quad \forall\, x \in S$$
$$(ii) \quad \frac{\partial h_2}{\partial X}(x) = 0 \quad \forall\, x \in S$$
$$(iii) \quad \frac{\partial h_2}{\partial Y}(x) = 1 \quad \forall\, x \in S$$

First, we claim that for any closed point $x \in \mathbb{A}^2_F$, there exists an $h_x \in F[X,Y]$ such that

$$h_x \equiv h_1 \mod \mathfrak{m}_x$$
$$\frac{\partial h_x}{\partial X}(x) = 0$$
$$\frac{\partial h_x}{\partial Y}(x) = 1$$

We choose a polynomial $f_1 \in F[X]$ such that $f_1(x) = 0$ and $\partial f_1/\partial X(x) \neq 0$. To see that such a choice is possible, let $\pi_1 : \mathbb{A}^2_F \to \mathbb{A}^1_F$ be the projection on to the X-coordinate. The minimal polynomial of any primitive element of the residue field of $\pi_1(x)$ satisfies our requirement. Similarly, we choose $f_2 \in F[Y]$ such that $f_2(x) = 0$ and $\partial f_2/\partial Y(x) \neq 0$. Using Chinese remainder theorem and the fact that the residue field $F(x)$ is perfect, we choose $g_1, g_2 \in F[X,Y]$ such that

$$g_1^p(x) = -\frac{\partial h_1/\partial X(x)}{\partial f_1/\partial X(x)},$$

$$g_2^p(x) = \frac{\big(1 - \partial h_1/\partial Y(x)\big)}{\partial f_2/\partial Y(x)}.$$

We leave it to the reader that

$$h_x = h_1 + g_1^p f_1 + g_2^p f_2$$

satisfies the requirement of our claim. Now, by Chinese remainder theorem, there exists $h_2 \in F[X,Y]$ such that

$$h_2 \equiv h_x \mod \mathfrak{m}_x^2 \;\; \forall\, x \in S.$$

It is straightforward to see that h_2 satisfies conditions (1) and (2) of the Lemma.

Step (3): Choose a non-constant polynomial $\beta \in F[Y]$ such that $\beta(x) = 0$ for all $x \in S$. Since f is monic in X, $\mathsf{Z}(f) \xrightarrow{Y} \mathbb{A}^1_F$ is a finite map. Thus $\beta : \mathsf{Z}(f) \to \mathbb{A}^1_F$ is also a finite map. As $\dim(\mathsf{Z}(f)) = 1$, for a sufficiently large integer ℓ,

$$h := h_2 - \beta^{p\ell}$$

defines a finite map $\mathsf{Z}(f) \xrightarrow{h} \mathbb{A}^1_F$ by Noether normalization trick (see (4.1.4)). Clearly h continues to satisfy conditions (1) and (2) of the Lemma since $\beta^{p\ell} \in \mathfrak{m}_x^2$ for all $x \in S$.

□

Proof of Theorem 4.1.17. Let ϕ, h be as in Lemmas 4.1.18 and 4.1.24 respectively. Let $\widetilde{\Phi}$ be the map $\mathbb{A}^2_F \xrightarrow{(\phi,h)} \mathbb{A}^2_F$, and $\widetilde{\Psi} := \phi$. Recall that $\widetilde{S} := \phi^{-1}(0) \cap \mathsf{Z}(f)$ (with reduced scheme structure). By Remark 4.1.20 it is finite.

Step 1: We claim that there exists a $g \in F[X,Y]$ such that if $W_g := \mathbb{A}^2_F \backslash \mathsf{Z}(g)$, then

$\widetilde{\Phi}(\widetilde{S}) \subset W_g$ and

$$\widetilde{\Phi}_{|\widetilde{\Phi}^{-1}(W_g)\cap \mathsf{Z}(f)} : \widetilde{\Phi}^{-1}(W_g) \cap \mathsf{Z}(f) \longrightarrow W_g$$

is a closed immersion. The proof of this claim is a repetition of the argument in [CTHK, 3.5.1] (see also (4.1.10)). Let $\{p, x_1, ..., x_n\}$ be the set of points in $\widetilde{S}$. Since $\widetilde{\Phi}$ is étale and radicial at all points of $\widetilde{S}$ (see (4.1.18)(2) and (4.1.24)(1)) we have $\widetilde{\Phi}^{-1}(\widetilde{\Phi}(\widetilde{S})) \to \mathbb{A}^2_F$ is a closed immersion. Let $y_0, ..., y_n$ be the points in $\widetilde{\Phi}(\widetilde{S})$. Let η_i be the maximal ideal in $F[X, Y]$ corresponding to the closed point y_i. Thus the above closed immersion gives us a surjective map

$$F[X, Y] \twoheadrightarrow \frac{F[\mathsf{Z}(f)]}{\eta_0 \cdots \eta_n}$$

where $F[\mathsf{Z}(f)]$ is the coordinate ring of $\mathsf{Z}(f)$. If C denotes the cokernel of $F[X, Y] \to F[\mathsf{Z}(f)]$ (as $F[X, Y]$ modules), then the above surjective map implies that

$$C \otimes \frac{F[X, Y]}{\eta_0 \cdots \eta_n} = 0.$$

Note that $\widetilde{\Phi}_{|\mathsf{Z}(f)}$ is a finite map, since h is a finite map ((4.1.24)(3)). Thus $F[\mathsf{Z}(f)]$ is a finite $F[X, Y]$ module. Thus, by Nakayama's lemma, there exists an element $g \in F[X, Y]$ such that $g \notin \eta_0 \cdots \eta_n$ and $C_g = 0$. In other words, the map

$$F[X, Y]_g \twoheadrightarrow F[\mathsf{Z}(f)]_g$$

is surjective. This proves the claim since if $W_g := \mathbb{A}^2_F \backslash \mathsf{Z}(g)$, the above surjectivity is equivalent to the following being a closed immersion

$$\widetilde{\Phi} : \mathsf{Z}(f) \cap \widetilde{\Phi}^{-1}(W_g) \to W_g.$$

Step 2: Let E be the smallest closed subset of $\mathsf{Z}(f)$ satisfying the following three conditions

(i) $x \in E$ if $x \in \mathsf{Z}(f)$ and $\widetilde{\Phi}$ is not étale at x.

(ii) $\mathsf{Z}(f)\backslash Z \subset E$.

(iii) $\mathsf{Z}(f)\backslash\left(\widetilde{\Phi}^{-1}(W_g)\cap \mathsf{Z}(f)\right)\subset E.$

Since $\widetilde{S}$ contains at least one point in each irreducible component of $\mathsf{Z}(f)$, (iii) implies that E is finite (see also Remark 4.1.20). Moreover, by the above step and condition (iii) we have

$$\mathsf{Z}(f)\backslash E \longrightarrow \mathbb{A}^2_F$$

is a locally closed immersion. Moreover, $\widetilde{S}$ and E are disjoint, and hence $\phi(p)\notin\phi(E)$. Since E is finite, we choose a non-constant polynomial expression Q in h which vanishes on p as well as E. For an integer $\ell >> 0$ and divisible by char(F), we claim that $(\phi - Q^\ell, h)$ presents $(W, \mathsf{Z}(f), p)$. Let

$$\Phi := (\phi - Q^\ell, h) \quad \text{and} \quad \Psi := \phi - Q^\ell.$$

To prove the claim we need to verify the conditions of the Definition (1)-(4) 4.1.9. Condition (1), i.e. finiteness of $\Psi_{|\mathsf{Z}(f)}$, follows by (4.1.4) since ℓ is large, and $h_{|\mathsf{Z}(f)}$ is finite. As Q vanishes on p and E, $\Psi(p)\notin\Psi(E)$ follows from $\phi(p)\notin\phi(E)$. Thus if $S := \Psi^{-1}\Psi(p)\cap Z$, then $S\subset \mathsf{Z}(f)\backslash E$. Conditions (2) to (4) of (4.1.9) follow from the conditions (i) to (iii) of E in the beginning of this step. □

4.2 Gabber's Presentation Lemma over Noetherian domains

The second result in the thesis is Theorem 4.2.1, which is a Gabber's presentation lemma over noetherian domains with infinite residue fields. The statement of the theorem is as follows:

Theorem 4.2.1. *Let $S = \operatorname{Spec}(R)$ be the spectrum of a noetherian domain with all its residue fields infinite. Let X be a smooth, irreducible, equi-dimensional S-scheme of relative dimension d. Let $Z\subset X$ be a closed subscheme, z be a closed point in Z lying over $s\in S$, such that that $\dim(Z_s) < \dim(X_s)$. Then after possibly replacing S by a Nisnevich neighbourhood of s and X by a Nisnevich neighbourhood of z, there exists a map $\Phi = (\Psi,\nu) : X\to \mathbb{A}^{d-1}_S\times\mathbb{A}^1_S$, an open subset $V\subset\mathbb{A}^{d-1}_S$ and an open subset $U\subset\Psi^{-1}(V)$ containing z such that*

1. $Z \cap U = Z \cap \Psi^{-1}(V)$

2. $\Psi_{|Z} : Z \to \mathbb{A}_S^{d-1}$ *is finite*

3. $\Phi_{|U} : U \to \mathbb{A}_S^d$ *is étale*

4. $\Phi_{|Z\cap U} : Z \cap U \to \mathbb{A}_V^1$ *is a closed immersion*

5. $\Phi^{-1}(\Phi(Z \cap U)) \cap U = Z \cap U$.

4.2.1 Fiber-wise denseness

In this section we establish a technical result, Theorem 4.2.2, which is a an essential part of the proof of Theorem 4.2.1. This result is a slight modification of [Kai, Theorem 3] (see also [Lev, Theorem 10.2.2]). Throughout this section, $\dim_B(Y)$ denotes the supremum of dimensions of all the fibers of $Y \to B$.

Theorem 4.2.2. *Let B be the spectrum of a noetherian domain. Let Y/B be either a smooth scheme or a divisor in a smooth scheme X. Let $y \in Y$ be a point lying over a point $b \in B$ with $\dim_B(Y_b) = n$. Assume $k(b)$ is an infinite field. Then there exist Nisnevich neighborhoods $(Y', y) \to (Y, y)$ and $(B', b) \to (B, b)$, fitting into the following commutative diagram,*

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

and a closed immersion $Y' \to \mathbb{A}_{B'}^N$ for some $N \geq 0$ such that if $\overline{Y'}$ is its closure in $\mathbb{P}_{B'}^N$ then Y'_y is dense in the union of n-dimensional irreducible components of $(\overline{Y'})_y$.

The following lemma is an ingredient which will be used repeatedly (see also [Lev, 10.1.4]).

Lemma 4.2.3. *Let X be an affine scheme. Choose a closed embedding $X \to \mathbb{A}_B^N$ and a point $x \in X$. Let $\overline{X}$ be the projective closure of X in $\mathbb{P}_B^N$ with fiber dimension n. Then, there exists*

1. *a projective scheme $\widetilde{X}$,*

2. an open neighbourhood X_0 of x (in X),

3. an open immersion $X_0 \hookrightarrow \widetilde{X}$ and

4. a projective morphism $\psi : \widetilde{X} \to \mathbb{P}^{n-1}_B$

such that ψ has fiber dimension one.

Proof. We follow the arguments given in [Kai, Theorem 3] verbatim (see also [Lev, Theorem 10.1.4]). After possibly shrinking B, we can find n hyperplanes $\Psi = \{\psi_1, \ldots, \psi_n\}$ which are part of a basis of $\Gamma(\mathbb{P}^N_B, \mathcal{O}(1))$ as a B-module. The choice is such that $V(\Psi)$ does not contain x and it meets X fiber-wise properly over B, so that $\overline{X} \cap V(\Psi)$ is finite over B. Let $p : \widetilde{\mathbb{P}^N} \to \mathbb{P}^N$ be the blowup of $\mathbb{P}^N$ along $V(\Psi)$, and $\widetilde{X}$ the strict transform of $\overline{X}$ in the blowup. This gives us a map $\psi : \widetilde{\mathbb{P}^N_B} \to \mathbb{P}^{n-1}_B$. Let $X_0 := \overline{X} \setminus V(\Psi)$. We have the following commutative diagram:

$$\begin{array}{ccccccc} & \nearrow & \widetilde{X} & \overset{cl.}{\hookrightarrow} & \widetilde{\mathbb{P}^N_B} & \xrightarrow{\psi} & \mathbb{P}^{n-1}_B \\ X_0 & & \downarrow & & \downarrow & & \\ & \searrow & \overline{X} & \overset{cl.}{\hookrightarrow} & \mathbb{P}^N_B & & \end{array}$$

We claim that $\psi : \widetilde{X} \to \mathbb{P}^{n-1}_B$ has fiber dimension one. To see this, choose any point $y \in \mathbb{P}^{n-1}_B$, and consider the composite $a : \operatorname{Spec}(\Omega) \xrightarrow{y} \mathbb{P}^{n-1}_B \to B$. Then, the fiber of ψ over y may be identified with a linear subscheme $V(y)$ of $\mathbb{P}^N_a$, of dimension $N - n + 1$. Furthermore, $V(y)$ contains the base change $V(\Psi)_a$, which has dimension $N - n$, by construction. Again by construction, the intersection $V(y) \cap \overline{X} \cap V(\Psi)_a$ is finite in $\mathbb{P}^N_a$. This means that $V(y) \cap \overline{X}$ has dimension 1 in the projective space $V(y)$.

Further note that for $x \in V(\Psi)$, $p^{-1}(y) \simeq \mathbb{P}^{n-1}$. Also, the exceptional divisor of $\widetilde{X}$ is an irreducible subscheme. Therefore, for any point $x \in V(\psi) \cap X$, the fiber $\widetilde{X}_x$ is an irreducible subscheme of $\mathbb{P}^{n-1}$ of dimension $n-1$. Therefore, $p^{-1}(\overline{X}) = \widetilde{X}$, so that $p : \psi^{-1}(y) \cap \widetilde{X} \to V(y) \cap \overline{X}$ is a bijection. Thus, $\psi : \widetilde{X} \to \mathbb{P}^{n-1}_B$ has 1-dimensional fibers. □

We now give a proof of Theorem 4.2.2.

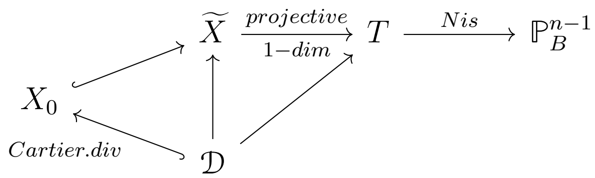
Proof of 4.2.2. We first prove the case when $Y = X$ is a smooth scheme. The proof is by induction on n. The case $n = 0$ follows from a version of Hensel's lemma.

Step 1: As X is smooth, Zariski locally on B, we write X as a hypersurface in some $\mathbb{A}^N_B$. Let $\overline{X}$ denote its reduced closure in $\mathbb{P}^N_B$. Note that $\overline{X}$ also has fiber-dimension n over B. By applying, Lemma 4.2.3, we get a projective morphism $\psi : \widetilde{X} \to \mathbb{P}^{n-1}_B$ with 1-dimensional fibers.

Step 2: Set $T = \mathbb{P}^{n-1}_B$ and $t = \psi(x)$. Choose any projective embedding $\widetilde{X} \hookrightarrow \mathbb{P}^{N_2}_T$. Let $(\widetilde{X})_t$ and $(X_0)_t$ denote the fibers over t of $\widetilde{X}$ and X_0 respectively. Then choose a hyperplane $H_t \subset \mathbb{P}^{N_2}_T$ satisfying the next three conditions.

1. (if x is closed point in $(X_0)_t$) $x \in H_t$
2. $(\widetilde{X})_t$ and H_t meet properly in $\mathbb{P}^{N_2}_t$.
3. H_t does not meet $\overline{(X_0)_t} \setminus (X_0)_t$.

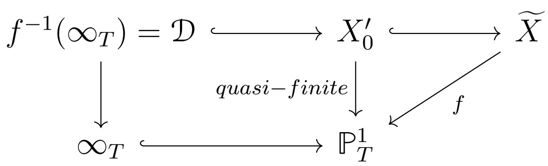
Now after restricting to a suitable Nisnevich neighbourhood of T, which we denote again by T (and after base changing everything to T) using the hyperplane H_t, we can choose a Cartier divisor $\mathcal{D}$ which fits into the following diagram



For sufficiently large m we can find a section s_0 of $\Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(m\mathcal{D}))$ which maps to nowhere vanishing section of $\Gamma(\mathcal{D}, \mathcal{O}_{\mathcal{D}})$. Let $s_1 : \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{X}}(m\mathcal{D})$ be the canonical inclusion. Since the zero-loci of s_0 and s_1 are disjoint we get a map

$$f = (s_0, s_1) : \widetilde{X} \to \mathbb{P}^1_T.$$

Since the quasi-finite locus of a morphism is open on the source [Gro, IV$_3$, 13.1.4], shrink T around t such that $\mathcal{D}$ is contained in the quasi finite locus of f after the base change. Let X_0' be the quasi-finite locus of the base change.



Then the subset $W = f(\widetilde{X} \setminus X_0') \subset \mathbb{P}^1_T$ is proper over T and is contained in $\mathbb{P}^1_T \setminus H_t = \mathbb{A}^1_T$ hence it is finite over T. The map $\widetilde{X} \setminus f^{-1}(W) \to \mathbb{P}^1_T \setminus W$, being proper and quasi-finite, is finite. By condition (i) we see that $\widetilde{X} \setminus f^{-1}(W)$ contains x.

Step 3: Now by induction there exist Nisnevich neighborhoods $B_1 \to B$ and $T_1 \to T$ such that the projective compactification $T_1 \to \overline{T_1}$ is fiber-wise dense in union of n-dimensional irreducible components over B_1. Take a factorization of f of the form $\widetilde{X} \hookrightarrow \mathbb{P}^{N_3}_{T_1} \times_{T_1} \mathbb{P}^1_{T_1} \to \mathbb{P}^1_{T_1}$. Let $\overline{X_1}$ denote the reduced closure of $\widetilde{X}$ in $\mathbb{P}^{N_3}_{\overline{T_1}} \times_{\overline{T_1}} \mathbb{P}^1_{\overline{T_1}}$. We get the following diagram where every square is Cartesian

$$\begin{array}{ccccc}
X_2 := \widetilde{X} \setminus f^{-1}(W) & \hookrightarrow & \widetilde{X} & \hookrightarrow & \overline{X_1} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{P}^{N_3}_{T_1} \times_{T_1} (\mathbb{P}^1_{T_1} \setminus W) & \longrightarrow & \mathbb{P}^{N_3}_{T_1} \times_{T_1} \mathbb{P}^1_{T_1} & \hookrightarrow & \mathbb{P}^{N_3}_{\overline{T_1}} \times_{\overline{T_1}} \mathbb{P}^1_{\overline{T_1}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{P}^1_{T_1} \setminus W & \longrightarrow & \mathbb{P}^1_{T_1} & \longrightarrow & \mathbb{P}^1_{\overline{T_1}}
\end{array}$$

By Stein factorization we decompose the map $\overline{f_1} : \overline{X_1} \to \mathbb{P}^1_{\overline{T_1}}$ as

$$\overline{f_1} : \overline{X_1} \to \overline{X_2} \xrightarrow{finite} \mathbb{P}^1_{\overline{T_1}},$$

where the first map has geometrically connected fibers. Since $\overline{f_1}$ is finite over the open set $\mathbb{P}^1_{T_1} \setminus W$, $\overline{X_2} \times_{\mathbb{P}^1_{\overline{T_1}}} (\mathbb{P}^1_{T_1} \setminus W)$ is isomorphic to $X_2 := \widetilde{X} \setminus f^{-1}(W)$. Since X_2 is open in $\overline{X_2}$, the fiber dimension of $\overline{X_2}$ is at least n. Combining this with the fact that $\overline{X_2}$ is finite over $\mathbb{P}^1_{\overline{T_1}}$, we conclude that the fiber dimension of $\overline{X_2}$ over B_1 is exactly n.

We observe that since T_1 is fiberwise dense in the union of n-dimensional irreducible components of $\overline{T_1}$, so is $\mathbb{P}^1_{T_1}$ (in $\mathbb{P}^1_{\overline{T_1}}$). Also as W is finite over T_1, $\mathbb{P}^1_{T_1} \setminus W$ is fiberwise dense in $\mathbb{P}^1_{T_1}$. Hence it is dense in the union of n-dimensional irreducible components of $\mathbb{P}^1_{\overline{T_1}}$. Now we claim X_2 intersects the fiber of $\overline{X_2}$ over any point b_1 of B_1. Let X_2' be the irreducible component of dimension n of the fiber $(\overline{X_2})_{b_1}$. Then the induced map $X_2' \to (\mathbb{P}^1_{\overline{T_1}})_{b_1}$ is a finite morphism of schemes of the same dimension. Hence it is a surjection to an irreducible component say, U of $(\mathbb{P}^1_{\overline{T_1}})_{b_1}$. Further $\mathbb{P}^1_{T_1} \setminus W$ intersects U by denseness. Taking inverse image of its intersection with irreducible component proves that X_2 intersects Y.

As $\overline{X_2}$ is projective over B_1, we choose any embedding of it in projective space $\mathbb{P}^N_{B_1}$. Then for the closed subscheme $\overline{X_2} \backslash X_2$ (with reduced structure) there exists a hypersurface H of $\mathbb{P}^N_{B_1}$ of degree, say d, containing $\overline{X_2} \setminus X_2$, not containing the point x and such that H_{b_1} intersects $(X_2)_{b_1}$ properly in $\mathbb{P}^N_{b_1}$. Hence by discussion in previous paragraph, H_{b_1} also intersects $(\overline{X_2})_{b_1}$ properly. Replacing X_2 by $\overline{X_2} \setminus H$ and taking d fold Vernose embedding we may assume H to be $\mathbb{P}^{N-1}_\infty$. Now we have the embedding $\overline{X_2} \setminus H \hookrightarrow \mathbb{A}^N_{B_1} = \mathbb{P}^N_{B_1} \setminus \mathbb{P}^{N-1}_\infty$ thereby proving the smooth case.

We shall now consider the case when Y is a divisor in a smooth scheme.

Step 4: Let Y be a divisor in a smooth scheme X. We will produce a map, $\psi : \widetilde{Y} \to \mathbb{P}^{d-1}$ whose fibers are 1-dimensional.

Since X is smooth, by Steps 1-3, Nisnevich locally, we have a closed embedding $Y \to X \to \mathbb{A}^N_B$ such that all fibers of $\overline{Y} \to B$ are n-dimensional. Then by Lemma 4.2.3, we have a commutative diagram,

$$\begin{array}{ccccccc} & \nearrow & \widetilde{Y} & \overset{cl.}{\hookrightarrow} & \widetilde{\mathbb{P}^N_B} & \xrightarrow{\psi} & \mathbb{P}^{n-1}_B \\ Y_0 & & \downarrow & & \downarrow & & \\ & \searrow & \overline{Y} & \overset{cl.}{\hookrightarrow} & \mathbb{P}^N_B & & \end{array}$$

Then as in Step 2 of the theorem, we obtain a morphism Nisnevich locally on Y, $\phi : Y \to \mathbb{P}^1_T$, where T is a Nisnevich neighbourhood of $\mathbb{P}^{n-1}$. Since T is a smooth B-scheme, our theorem holds for T. The rest of the proof is the same as in Step 3. □

4.2.2 Gabber's presentation lemma over noetherian domains

We now present a proof of the Gabber's presentation lemma over noetherian domains with all its residue fields infinite, recall the statement from Theorem 4.2.1

We first reduce to the case that z is a closed point and Z is a principal divisor.

Lemma 4.2.4. *(See [CTHK, Lemma 3.2.1]) With the notation as in Theorem 4.2.1, there exists a closed point $z' \in X$ such that z' is a specialization of z and there exists a non-zero $f \in \Gamma(X, \mathcal{O}_X)$ such that $Z \subset V(f)$.*

Remark 4.2.5. *Since Theorem 4.2.1 is Nisnevich locally true so, henceforth we assume that the ring R is Henselian local with the closed point σ and an infinite residue field k.*

Let $S = Spec(R)$ with $\mathbb{A}^n_S = R[x_1, \ldots, x_n]$. Let E be R span of $\{x_1, \ldots, x_n\}$ and consider $\mathcal{E} := \underline{\mathrm{Spec}}\,(Sym^\bullet E^\vee)$ (note that $\mathcal{E}(R) = E$). For any integer $d > 0$ and R algebra A, $\mathcal{E}^d(A)$ parametrizes all linear morphisms $v = (v_1, \ldots, v_d) : \mathbb{A}^n_T \to \mathbb{A}^d_T$, where $T = Spec(A)$. Considering $\mathbb{A}^n_S \hookrightarrow \mathbb{P}^n_S = \mathrm{Proj}\ S[X_0, \ldots, X_n]$, as a distinguished open subscheme $D(X_0)$, we extend such a linear morphism to a rational map $\overline{v} : \mathbb{P}^n_S \dashrightarrow \mathbb{P}^d_S$ whose locus of indeterminacy L_v is given by $V_+(X_0, v_1, \ldots, v_d) \subseteq \mathbb{P}^n_S$(We will use this notation throughout what follows). Given any closed subscheme Y in $\mathbb{A}^n_S$ we denote by $\overline{Y}$ its projective closure in $\mathbb{P}^n_S$. For the following lemma we refer to Lemma 2.3 of [SS].

Lemma 4.2.6. *In the setting of previous paragraph if $L_v \cap \overline{Y} = \emptyset$, then $\overline{v} : \overline{Y} \to \mathbb{P}^d_S$ and $v : Y \to \mathbb{A}^d_S$ are finite maps.*

Following is a standard result about existence of a certain hyperplane.

Lemma 4.2.7. *Let W be a closed subscheme of $\mathbb{P}^N_k$ then there exists a hyperplane $H \subset \mathbb{P}^N_k$ such that $\dim_k(H \cap W) = \dim_k(W) - 1$.*

Proof. Let $\zeta_1, \ldots \zeta_r$ be the generic points of W corresponding to homogeneous prime ideals $\wp_1, \ldots \wp_r$. Viewing $\wp_i$'s and $\Gamma(\mathcal{O}(1), \mathbb{P}^N_k)$ as vector spaces over the infinite field k, we find a hyperplane H not containing ζ_i's; as no non trivial vector space over an infinite field can be written as a finite union of proper subspaces. Hence by Krull's principal ideal theorem $\dim_k(H \cap W) = \dim_k(W) - 1$. □

We now prove that the condition in Lemma 4.2.6 are met in our case.

Proposition 4.2.8. *Let Y be as in Theorem 4.2.2 and $\overline{Y}$ be its projective closure, then there exist $v_1, \ldots, v_n$ in k-span of $\{X_1, \ldots, X_N\}$ such that $(\overline{Y})_\sigma \cap L_v = \emptyset$, where $L_v = V_+(X_0, v_1, \ldots, v_n)$.*

Proof. Without loss of generality, we assume $\mathbb{A}^n_k = D(X_0)$. Let $H_\infty = V_+(X_0)$ denote the hyperplane at infinity of $\mathbb{P}^N_k$. Generic points of irreducible components of $\overline{Y_\sigma}$ lie in $\mathbb{A}^n_k = D(X_0)$. Therefore $\dim(\overline{Y_\sigma} \cap H_\infty) = n - 1$. By Theorem 4.2.2, we have $\dim((\overline{Y})_\sigma \cap H_\infty) = n - 1$. Now applying Lemma 4.2.7 repeatedly, proves the claim. □

In the following theorem we will produce maps with properties listed in Theorem 4.2.1 and in Lemmas 4.2.13 and 4.2.14, we get the open sets U and V. These constructions give Φ, U and V as required in Theorem 4.2.1.

Theorem 4.2.9. *Let $X = Spec(A)/S$ be a smooth, affine, irreducible scheme of relative dimension d, let $Z = Spec(B)$ be a principal divisor of X and z be a closed point in Z. Then there exists an open subset $\Omega \subset \mathcal{E}^d$ with $\Omega(k) \neq \emptyset$ such that for all $\Phi = (\Psi, \nu) \in \Omega(k)$ the following hold*

1. *$\Psi_{|Z} : Z \to \mathbb{A}_S^{d-1}$ is finite.*
2. *Ψ is étale at all points of $F := \psi^{-1}(\psi(z)) \cap Z$.*
3. *$\Phi_{|F} : F \to \Phi(F)$ is radicial.*

Recall that $\Phi : F \to \Phi(F)$ is said to be radicial [Sta, Tag 01S2] if Φ is injective and for all $x \in F$ the residue field extension $k(x)/k(\Phi(x))$ is trivial.
To prove this theorem, we first get an open set of finite maps in proposition 4.2.11. Then we get a non-empty open set of étale and radicial maps in Lemma 4.2.12.

Remark 4.2.10. *By proposition 2.6 and lemma 2.7 of [SS] we have a closed embedding $i_0 : X \hookrightarrow \mathbb{A}_S^n$ such that Z (Nisnevich locally around z) satisfies Theorem 4.2.2.*

In the following proposition, we prove that a generic choice works to get a finite map.

Proposition 4.2.11. *Let X and Z be as in Theorem 4.2.9 with S a spectrum of a Henselian ring R. Then there is an open subset $\Omega \subset \mathcal{E}^d$ with $\Omega(R) \neq \emptyset$ such that for all $\Psi \in \Omega(R)$, $\Psi_{|Z} : Z \to \mathbb{A}_S^{d-1}$ is finite.*

Proof. We proceed as in Lemma 2.11 of [SS]. By remark 4.2.10 we have closed embedding $X \hookrightarrow \mathbb{A}_S^N$. Viewing $\mathcal{E}^{d-1}$ as a closed subscheme of $\mathcal{E}^d$ by taking first $d-1$ factors we consider the closed subscheme

$$V = \mathcal{E}^{d-1} \times_S H_\infty \hookrightarrow \mathcal{E}^d \times_S H_\infty$$

where H_∞ is the hyperplane at infinity in $\mathbb{P}_S^N$. Note that $V \to \mathcal{E}^d$ has fiber $V_v = L_{(v_1,\ldots,v_{d-1})}$ for any $v = (v_1, \ldots, v_d) \in \mathcal{E}^d(R)$. Consider the open subscheme Ω of $\mathcal{E}^d$ defined as

$$\mathcal{E}^d \setminus p_1(V \cap (\mathcal{E}^d \times_S (\overline{Z} \cap H_\infty))),$$

where p_1 is projection of $\mathcal{E}^{d-1} \times_S H_\infty$ onto the first factor. By construction every point in $\Omega(R)$ consists of a linear map $v = (v_1, \ldots, v_d) : \mathbb{A}_S^N \to \mathbb{A}_S^d$ such that $L_{v'} \cap \overline{Z} = \emptyset$, where $v' = (v_1, \ldots, v_{d-1})$. By Lemma 4.2.6, this will be our required finite map, thus proving $\Omega(R) \neq \emptyset$ will finish the proposition. As R is Henselian, the induced map from $\Omega(R)$ to $\Omega(k)$ is surjective, hence it suffices to prove $\Omega(k) = \Omega_\sigma(k) \neq \emptyset$. By construction we have, $\Omega_\sigma(k) = \mathcal{E}^d_\sigma \setminus p_1(V_\sigma \cap (\mathcal{E}^d_\sigma \times_S ((\overline{Z})_\sigma \cap H_\infty)))$ and any point in $\Omega(k)$ gives a linear map $u = (u_1, \ldots, u_d) : \mathbb{A}^N_k \to \mathbb{A}^d_k$ such that $L_{u'} \cap (\overline{Z})_\sigma = \emptyset$, where $u' = (u_1, \ldots, u_{d-1})$. By Lemma 4.2.8 such a map exists. □

The following proposition is lemma 2.12 of [SS]. We reproduce the proof here for the sake of completeness.

Proposition 4.2.12. *(cf. [SS, Lemma 2.12]) Let $\phi = (\psi, \nu) = (u_1, \ldots, u_d) : X \to \mathbb{A}^{d-1}_S \times \mathbb{A}^1_S$ and $F := \psi^{-1}(\psi(z)) \cap Z$. There exists an open set $\Omega_2 \subset \mathcal{E}^d$ such that $\Omega_2(R) \neq \emptyset$ and for any $\phi \in \Omega_2(R)$*

1. *ϕ is étale at all points of F.*

2. *$\phi_{|F} : F \to \phi(F)$ is radicial.*

Proof. We will again use the fact that the induced map from $\Omega(R)$ to $\Omega(k)$ is surjective. The map $i_0 : X \to \mathbb{A}^n_S$ (see Remark 4.2.10) is a closed immersion between smooth R-schemes hence it is regular. Denote by $I := (f_1, \ldots, f_{n-d}) \subset R[X_1, \ldots, X_n]$ the ideal of i_0 given by the regular sequence $f_1, \ldots, f_{n-d}$. Using the map ϕ, we may write the coordinate ring of X, $A := \mathcal{O}(X)$ as an $R[t_1, \ldots, t_d]$-algebra

$$A = R[\underline{t}][X_1, \ldots, X_n]/\Big(f_i, u_j - t_j \mid 1 \leq i \leq n-d \ 1 \leq j \leq d\Big).$$

To check that ϕ is étale at a point $x \in X$, it suffices to show that the Jacobian-determinant

$$\det\left(\left\{\frac{\partial f_i}{\partial x_s}\right\}_{i,s} | \left\{\frac{\partial u_j}{\partial x_s}\right\}_{j,s}\right)$$

s invertible in $\mathcal{O}_{X,x}$.
We can write this determinant as $df_1 \wedge \ldots \wedge df_{n-d} \wedge du_1 \wedge \ldots \wedge du_d$ in $\Omega^n_{R[\underline{X}]/R} \otimes_{R[\underline{X}]} \mathcal{O}_{X,x}$. Further, the conormal sequence corresponding to i_0 is split exact.

$$0 \to I/I^2 \otimes_A \mathcal{O}_{X,x} \to \Omega^1_{R[\underline{X}]/R} \otimes_{R[\underline{X}]} \mathcal{O}_{X,x} \to \Omega^1_{A/R} \otimes_A \mathcal{O}_{X,x} \to 0$$

hence we have the isomorphism

$$\Omega^n_{R[\underline{X}]/R} \otimes_{R[\underline{X}]} \mathcal{O}_{X,x} \cong \bigwedge^{n-d}(I/I^2 \otimes_A \mathcal{O}_{X,x}) \otimes_{\mathcal{O}_{X,x}} \Omega^d_{A/R} \otimes_A \mathcal{O}_{X,x}.$$

Since I/I^2 is free over A with a basis given by the regular sequence $f_1, \ldots, f_{n-d}$, the element $f_1 \wedge \ldots \wedge f_{n-d}$ is invertible in $\bigwedge^{n-d}(I/I^2 \otimes_A \mathcal{O}_{X,x}) = \mathcal{O}_{X,x}$. Write $\bar{u}_j$ as the image of t_j under $R[\underline{t}] \to A$. So,

$$df_1 \wedge \ldots \wedge df_{n-d} \wedge du_1 \wedge \ldots \wedge du_d = (f_1 \wedge \ldots \wedge f_{n-d}) \otimes (d\bar{u}_1 \wedge \ldots \wedge d\bar{u}_d)$$

is invertible if and only if $d\bar{u}_1 \wedge \ldots \wedge d\bar{u}_d$ is invertible in $\Omega^d_{A/R} \otimes_A \mathcal{O}_{X,x} = \mathcal{O}_{X,x}$. By Nakayama, this is equivalent to $d\bar{u}_1 \wedge \ldots \wedge d\bar{u}_n \neq 0$ in $\Omega^d_{A/R} \otimes_A k(x)$.
Hence to check that ϕ is étale at x it sufficient to check that $d\bar{u}_1 \wedge \ldots \wedge d\bar{u}_n \neq 0$ in $\Omega^d_{A/R} \otimes_A k(x)$. From [CTHK, Lemma 3.4.2] there exists $W'_1 \subset \mathcal{E}^d \otimes_k \overline{k}$ such that $W'_1(\overline{k}) \neq \emptyset$ and for all $\phi \in W'_1(\overline{k})$ ϕ is étale at all points of F.
Now, by descent (cf. [CTHK, Lemma 3.4.3]) for W'_1, we get $W_1 \subset \mathcal{E}^d$ such that $W_1(k) \neq \varphi$ and each ϕ in $\mathcal{E}^d(k)$ is étale at all points of F. This proves (1).

For universal injectivity, by [CTHK, Lemma 3.4.2] we get a non-empty open subset $W'_2 \subseteq \mathcal{E}^d \otimes_k \overline{k}$ with $\phi|_{F \otimes_k \overline{k}}$ is universally injective. Using similar descent argument as above we get $W_2 \subset \mathcal{E}^d$ satisfying (2). Then setting $\Omega_2 = W_1 \cap W_2$ is the desired open set. □

Proof of Theorem 4.2.9. Let Ω_1 and Ω_2 be as in the Propositions 4.2.11 and 4.2.12. Then the set $\Omega = (\Omega_1 \times \mathcal{E}) \cap \Omega_2$ satisfies all the required conditions. □

In the following two lemmas, the sets U and V satisfying conditions in Theorem 4.2.1 are constructed. These lemmas are Lemma 3.5.1 and Lemma 3.6.1 of [CTHK]. We reproduce them here, verbatim.

Lemma 4.2.13. *Let $\Phi = (\Psi, \nu)$ satisfy the condition of Theorem 4.2.9. Then there exists an open neighborhood $V \subset \mathbb{A}^{d-1}_S$ of $\Psi(z)$ such that*

1. *Φ is étale at all points of $Z \cap \Psi^{-1}(V)$.*
2. *$\Phi|_{Z \cap \Psi^{-1}(V)} \to \mathbb{A}^1_V$ is a closed immersion.*

Proof. (cf. [SS, Lemma 2.13]) We find a neighborhood V_1 of $\Psi(z)$ satisfying condition (1) and another neighborhood V_2 satisfying condition (2). We then obtain V be intersecting V_1 and V_2.

Construction of V_1: Let Z_Φ be the intersection of Z with the non-étale locus of Φ. Then, $Z_\Phi \cap \Psi^{-1}\Psi(z) = \emptyset$. Further, $\Psi|_Z$ is finite and $\Psi(Z_\Phi)$ is closed in $\mathbb{A}_S^{d-1}$. The complement $V_1 \subset \mathbb{A}_S^{d-1}$ of $\Psi(Z_\Phi)$ is such that Φ is étale at all points of $Z \cap \Psi^{-1}(V_1)$, and that $\Psi(z) \in V_1$.

Let $B = R[Z]$ and consider the maps induced by Φ and Ψ on the coordinate rings.

Construction of V_2: Let $\mathfrak{p}$ be the maximal ideal of $R[U_1, \ldots, U_{d-1}]$ corresponding to $\Psi(z)$. Since the map Φ is étale at all points of F and $\Phi_{|F} : F \to \Phi(F)$ is radicial

$$R[Z_1, \ldots Z_d]/\mathfrak{p}R[Z_1, \ldots, Z_d] \to D/\mathfrak{p}D$$

is an isomorphism. Since D is a finite $R[U_1, \ldots, U_{d-1}]$-module, by Nakayama lemma there is an $f \in R[U_1, \ldots, U_{d-1}] \setminus (\mathfrak{p})$ such that $R[U_1, \ldots, U_{d-1}][1/f] \to B[1/f]$ is surjective. Let $V_2 = \{f \neq 0\} \subseteq \mathbb{A}_S^{d-1}$. Then V_2 contains $\Psi(z)$ and has the property that $\Phi|_{Z\cap\Psi^{-1}(V_2)} \to \mathbb{A}^1_{V_2}$ is a closed immersion.

Let $V = V_1 \cap V_2$. Then Φ is étale at all points of $Z \cap \Psi^{-1}(V)$. Furthermore $\Phi|_{Z\cap\Psi^{-1}(V)} \to \mathbb{A}^1_V$ is a closed immersion and $\Psi(z) \in V$. □

Lemma 4.2.14. *There exists a closed subset $\mathfrak{U} \subset \Psi^{-1}(V)$ such that*

1. *$\mathfrak{U}$ is closed in $\Psi^{-1}(V)$*

2. *$U_1 = \Psi^{-1}(V) \setminus \mathfrak{U}$ contains z*

3. *U_1 satisfies $Z \cap \Psi^{-1}(V) = Z \cap U_1$ and $\Phi^{-1}(\Phi(Z \cap U_1)) \cap U_1 = Z \cap U_1$.*

Proof. (cf. [SS, Lemma 2.14]) Let $T = Z \cap \Psi^{-1}(V)$ and $\mathfrak{U} = \Phi^{-1}(\Phi(T)) - T$. By Lemma 4.2.13 $\Phi_{|\Phi^{-1}(\Phi(T))} : \Phi^{-1}(\Phi(T)) \to \Phi(T)$ is étale and $T \to \Phi(T)$ is an isomorphism. Hence T is open in $\Phi^{-1}(\Phi(T))$ and $\mathfrak{U} = \Phi^{-1}(\Phi(T)) - T$ is closed in $\Phi^{-1}(\Phi(T))$. Since $\Psi_{|Z}$ is finite $\Psi_{|T}$ is finite over V and Φ_T is finite over $\mathbb{A}^1_V$. So $\Phi(T)$ is closed in $\mathbb{A}^1_V$ hence in $\mathbb{A}^1_{\Psi(T)}$. Hence $\Phi^{-1}(\Phi(T))$ is closed in $\Phi^{-1}(\mathbb{A}^1_{\Psi(T)}) = \Psi^{-1}(\Psi(T))$. Thus $\mathfrak{U}$ is closed in $\Psi^{-1}(\Psi(T))$.

By finiteness of $\Psi_{|Z}$, $\Psi^{-1}(\Psi(T))$ is closed in $\Psi^{-1}(V)$ This proves (1). (2) and (3) are clear from the construction. □

Now Theorem 4.2.1 follows:

Proof of Theorem 4.2.1. Let U_2 be the open locus where Φ is étale. From Lemma 4.2.13 $z \in U_2$ and $Z \cap \Psi^{-1}(V) \subset U_2$. Now let $U = U_1 \cap U_2$, with U_1 as in Lemma 4.2.14. Then U also satisfies conditions (2) and (3) of Lemma 4.2.14. Furthermore Ψ_U is étale. Hence we get Φ, Ψ, U, V satisfying all the conditions of Theorem 4.2.1. □

Bibliography

[AGV] M Artin, A Grothendieck, and JL Verdier. Théorie des topos et cohomologie étale des schémas (sga 4), tome 3. *Lecture Notes in Math*, 269, 1973.

[AHW1] Aravind Asok, Marc Hoyois, and Matthias Wendt. Affine representability results in $\mathbb{A}^1$-homotopy theory, ii: Principal bundles and homogeneous spaces. *Geometry & Topology*, 22(2):1181–1225, 2018.

[AHW2] Aravind Asok, Marc Hoyois, and Matthias Wendt. Affine representability results in $\mathbb{A}^1$-homotopy theory iii: finite fields and complements. *arXiv preprint arXiv:1807.03365*, 2018.

[BO] Spencer Bloch and Arthur Ogus. Gersten's conjecture and the homology of schemes. In *Annales scientifiques de l'École Normale Supérieure*, volume 7, pages 181–201, 1974.

[CTHK] Jean-Louis Colliot-Thélene, Raymond T Hoobler, and Bruno Kahn. The bloch-ogus-gabber theorem. *Algebraic K-Theory (Toronto, ON, 1996)*, 16:31–94, 1997.

[DHI] Daniel Dugger, Sharon Hollander, and Daniel C Isaksen. Hypercovers and simplicial presheaves. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 136, pages 9–51. Cambridge University Press, 2004.

[DHKY] Neeraj Deshmukh, Amit Hogadi, Girish Kulkarni, and Suraj Yadav. Gabber presentation lemma over noetherian domains. *arXiv preprint arXiv:1906.09931*, 2019.

[Gab] Ofer Gabber. Gersten's conjecture for some complexes of vanishing cycles. *manuscripta mathematica*, 85(1):323–343, 1994.

[GJ] Paul G Goerss and John F Jardine. *Simplicial homotopy theory*. Springer Science & Business Media, 2009.

[Gro] Alexander Grothendieck. Éléments de géométrie algébrique (rédigés avec la collaboration de jean dieudonné): Iv. étude locale des schémas et des morphismes de schémas, quatrieme partie. *Publications mathématiques de l'IHES*, 32:5–361, 1967.

[Hir] Philip S Hirschhorn. *Model categories and their localizations*. Number 99. American Mathematical Soc., 2003.

[HK] Amit Hogadi and Girish Kulkarni. Gabber's presentation lemma for finite fields. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2018.

[Hov] Mark Hovey. *Model categories*. Number 63. American Mathematical Soc., 2007.

[HTT] J Lurie HTT. Higher topos theory. *Annals of Mathematics Studies. Princeton, NJ: Princeton University Press*, 170, 2009.

[Kai] Wataru Kai. A Moving Lemma for Algebraic Cycles With Modulus and Contravariance. *International Mathematics Research Notices*, 05 2019. rnz018.

[Lev] Marc Levine. Chow's moving lemma and the homotopy coniveau tower. *K-theory*, 37(1):129–209, 2006.

[LW] Serge Lang and André Weil. Number of points of varieties in finite fields. *American Journal of Mathematics*, 76(4):819–827, 1954.

[Mor1] Fabien Morel. An introduction to $\mathbb{A}^1$-homotopy theory. contemporary developments in algebraic k-theory 357–441, ictp lect. notes, xv, abdus salam int. cent. *Theoret. Phys., Trieste*, 2004.

[Mor2] Fabien Morel. The stable-connectivity theorems. *K-theory*, 35(1):1–68, 2005.

[Mor3] Fabien Morel. $\mathbb{A}^1$-*algebraic topology over a field*, volume 2052. Springer, 2012.

[Mum] David Mumford. *The red book of varieties and schemes: includes the Michigan lectures (1974) on curves and their Jacobians*, volume 1358. Springer Science & Business Media, 1999.

[MV] Fabien Morel and Vladimir Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 90(1):45–143, 1999.

[Oja] Manuel Ojanguren. Quadratic forms over regular rings. *Journal of the Indian Mathematical Society. New Series*, 44:109–116, 1980.

[Poo] Bjorn Poonen. Bertini theorems over finite fields. *Annals of mathematics*, pages 1099–1127, 2004.

[Qui] D Quillen. Springer lecture notes in math. *Homotopical algebra*, 43, 1967.

[SS] Johannes Schmidt and Florian Strunk. Stable $\mathbb{A}^1$-connectivity over dedekind schemes. *Annals of K-Theory*, 3(2):331–367, 2018.

[Sta] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2019.