Implied Volatility in a Regime-Switching Market: Theory and Computation

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by

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This is to certify that this dissertation entitled Implied Volatility in a Regime-Switching Market: Theory and Computation towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents original research carried out by Shirish Kulhari at Indian Institute of Science Education and Research under the supervision of Dr. Anindya Goswami, Assistant Professor, Department of Mathematics, during the academic year 2014-2015.

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Dedicated to my project guide, Dr. Anindya Goswami, and my parents

Declaration

I hereby declare that the matter embodied in the report entitled Implied Volatility in a Regime-Switching Market: Theory and Computation are the results of the investigations carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Anindya Goswami and the same has not been submitted elsewhere for any other degree.

Shirish Kulhari

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Abstract

Options are starting to gain importance as some of the most widely accepted and used financial instruments for investing purposes. Despite some of their obvious advantages, since the pricing of options depends on the behavior of the underlying stock, it becomes difficult to explicitly determine a fair price of the option if the stock itself is a complicated stochastic process. Since one cannot hope to have an accurate theoretical distribution for the underlying stock, one may attempt to formulate a theory that specifies the market dynamics on the basis of the underlying assets like stocks and bonds. The behavior of these assets is in turn specified by stochastic differential equations that involve parameters like the interest rate and volatility, which can be determined. The particular values of these parameters, which in conjunction with the proposed theory give us the same price of the vanilla option as is directly observed from the market, can be regarded as the implied values of those parameters. These can be used to formulate fair prices of other financial instruments like exotic options.

In this project, we consider vanilla European call options, the simplest of financial instruments, and attempt to show under what conditions we can determine one of these parameters, called the volatility, from available statistical data related to the option price. We start with a few preliminaries and results, like Lévy's Theorem, from stochastic calculus in Chapter 1. In Chapter 2, we discuss basic terminology and results related to discrete-time and continuous-time trading in financial markets, like the No Arbitrage condition in the admissible class for complete markets. We then give a short summary of the Black-Scholes model and the definition of implied volatility. From Chapter 3 onwards, we move on to the more general Markov-modulated market model, where we give a brief description of the model and mention the general PDE satisfied by the price of a European call option in this model. In Chapter 4, we propose some approaches to define the implied volatility. We try out one of these approaches- by treating it as an inverse problem, we prove that in certain conditions, the notion of implied volatility is well-defined in the Markov-modulated model, even though there is no explicit formula for the price of a call option. Finally in Chapter 5, we conduct a numerical experiment that exhibits the methodology of relevant computations to determine the implied volatility.

Contents

Chapter 1

Itō Calculus and Brownian Motion

This chapter contains some prerequisite concepts and results, which we will need in our study of market models of stock prices following stochastic differential equations based on Brownian motion. We shall re-visit Itō calculus and a few properties of Brownian motion. We will restrict ourselves to a particular class of Itō integrals. See [22], [25], [27], [28], [30], [32], [33] and [34].

1.1 Itō Calculus

Definition 1.1.1. *Let $\{\tau_n\}_n$ be a sequence of partitions*

$$\tau_n = \{0 = t_0^n < t_1^n < ... < t_{i_n}^n < \infty\},$$

such that $\lim_{n\to\infty} t_{i_n}^n = \infty$ *and* $\lim_{n\to\infty} \sup_{j\le i_n-1} |t_{j+1}^n - t_j^n| = 0$.

Let X be a càdlàg process on $[0,\infty)$. If $\langle X\rangle_t = \lim_{n\to\infty}\sum_{t_i^n\in\tau_n, t_{i+1}^n<t} |X_{t_{i+1}^n} - X_{t_i^n}|^2$ exists for every t, the function $t \to \langle X\rangle_t$ is called the quadratic variation *of X along $\{\tau_n\}_n$.*

Theorem 1.1.1. *Let $X : [0,\infty) \to \mathbb{R}$ be continuous with continuous quadratic variation $\langle X\rangle_t$, and let F be twice continuously differentiable with all derivatives bounded.*

Then

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)dX_s + \frac{1}{2}\int_0^t F''(X_s)d\langle X\rangle_s,$$

where $\int_0^t F'(X_s)dX_s = \lim_{n\to\infty}\sum_{t_i^n \le t} F'(X_{t_i^n})(X_{t_{i+1}^n} - X_{t_i^n})$. *Note that we choose the leftmost point of the interval* $X_{t_i^n}$ *as the argument of* F'.

Proof. Let $t > 0$. Then by Taylor's theorem,

$$\begin{aligned} F(X_{t_{i+1}^n}) - F(X_{t_i^n}) &= F'(X_{t_i^n})(X_{t_{i+1}^n} - X_{t_i^n}) + \frac{1}{2}F''(X_{\tilde{t}_i^n})\Delta(X_{t_i^n})^2 \\ &= F'(X_{t_i^n})\Delta X_{t_i^n} + F''(X_{t_i^n})\Delta(X_{t_i^n})^2 + \frac{1}{2}(F''(X_{\tilde{t}_i^n}) - F''(X_{t_i^n}))(\Delta X_{t_i^n})^2, \end{aligned}$$

where $\tilde{t}_i^n \in (t_i^n, t_{i+1}^n)$.

Define $R_n = F''(X_{\tilde{t}_i^n}) - F''(X_{t_i^n})$ and $\delta = max_i|\Delta X_{\tilde{t}_i^n}|$. Then

$$|R_n(t_i^n)| \le \frac{1}{2}max_{|x-y|\delta_n}|F''(x) - F''(y)|(\Delta X_{t_i^n})^2 \le \epsilon_n(\Delta X_{t_i^n})^2,$$

since F'' is uniformly continuous on $[0, t]$.

The last term in the Taylor expansion vanishes and we can sum both sides over the t_i^n's in the partition.

1. $\sum(F(X_t^n i + 1) - F(X_{t_i^n})) = F(X_t) - F(x_0)$
2. $\sum \frac{1}{2}F''(X_{t_i^n})(\Delta X_{t_i^n})^2 \to \int_0^t F''(X_s)d\langle X\rangle_s$
3. $\sum F'(X_{t_i^n})\Delta X_{t_i^n} \to \int_0^t F'(X_s)dX_s$.

Plugging these values into the summed-over Taylor expansion gives us Itō's formula.

□

Definition 1.1.2 (Itō Integral)**.** *Let* $\{H_t\}_{t\ge 0}$ *be an adapted càdlàg process and* X_t *a continuous local martingale. If the limit*

$$\lim_{n\to\infty}\sum_{t\ge t_i^n} H_{t_i^n}(\omega)(X_{t_{i+1}^n}(\omega) - X_{t_i^n}(\omega))$$

exists for all $t \ge 0$ *P-almost surely, then the above limit, denoted by* $\int_0^t H_s dX_s$ *is called the stochastic integral of* H *with respect to* X*.*

The following lemma, which we state without proof, will be useful:

Lemma 1.1.1. *Let $\{H_t\}_{t\geq 0}$ and X_t be defined as above. Then $M_t = \int_0^t H_s dX_s$ is a local martingale.*

Lemma 1.1.2. *If $\{X_t\}_{t\geq 0}$ is a local martingale, then $X_t^2 - \langle X\rangle_t$ is also a local martingale.*

Proof. Let $f(x) = x^2$. From Itō's formula,

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle X\rangle_s \\ \implies X_t^2 &= X_0^2 + 2\int_0^t X_s dX_s + \langle X\rangle_t, \\ X_t^2 - \langle X\rangle_t &= X_0^2 + 2\int_0^t X_s dX_s. \end{aligned}$$

In the R.H.S, the integrand is an adapted càdlàg and the integrator is a local martingale. Hence the integral is a local martingale by the previous lemma. □

Lemma 1.1.3. *Let X be a continuous local martingale with $\langle X\rangle = 0$ P-almost surely. Then $X_t = X_0$ almost surely.*

Proof. Set $M_t = X_t^2 - \langle X\rangle_t = X_t^2$. This is a local martingale by the previous lemma.

Let τ_n' and τ_n'' be such that $X_{t\wedge\tau_n'}$ and $X^2_{t\wedge\tau_n''}$ are martingales for all n. Then $X_{t\wedge\tau_n}$ and $X^2_{t\wedge\tau_n}$ are also martingales for all n, where $\tau_n = \tau_n' \wedge \tau_n''$. Now

$$0 \leq E[(X_t - X_0)^2 | X_0] = E(\lim_{n\to\infty}(X_{\tau_n\wedge t} - X_0)^2),$$

by continuity of X_t.

$$\begin{aligned} E(\lim_{n\to\infty}(X_{\tau_n\wedge t} - X_0)^2) &\leq \lim_{n\to\infty} E(X_{\tau_n\wedge t} - X_0)^2 \\ &= \lim_{n\to\infty} E(X^2_{\tau_n\wedge t} - X_0^2 - 2X_0(X_{\tau_n\wedge t} - X_0)), \\ \lim_{n\to\infty} E(X^2_{\tau_n\wedge t} - X_0^2 - 2X_0(X_{\tau_n\wedge t} - X_0)) &= \lim_{n\to\infty} E(X^2_{\tau_n\wedge t} - X_0^2), \end{aligned}$$

since $X_{\tau_n \wedge t}$ is a martingale, and

$$\lim_{n\to\infty} E(X^2_{\tau_n \wedge t} - X^2_0) = \lim_{n\to\infty} 0 = 0,$$

since $X^2_{\tau_n \wedge t}$ is a martingale.

Hence $X_t = X_0$ almost surely for every t. □

Theorem 1.1.2 (Independence of Itō Calculus of the Partition Choice)**.** *The value of the Itō Integral is independent of the choice of partition* $\{\tau_n\}$*.*

Proof. Let X be a local martingale. Fix a partition sequence to be used in the Itō formula:

$$X_t^2 - X_0^2 = 2\int_0^t X_s dX_s + \langle X \rangle_t.$$

The LHS is independent of the partition choice, whereas the RHS is defined using the chosen partition. Consider two different partitions $\{\tau_n^1\}_n$ and $\{\tau_n^2\}_n$ and suppose that the RHS of the above equation corresponding to these choices are $I_t^1 + \langle X \rangle_t^1$ and $I_t^2 + \langle X \rangle_t^2$ respectively. These two are equal to each other.

Now $M_t = \langle X \rangle_t^1 - \langle X \rangle_t^2 = I_t^2 - I_t^1$. Both I_t^1 and I_t^2 are local martingales from lemma 1.2.1. Hence M_t is also a local martingale. But since M_t is a process of finite variation, its quadratic variation $\langle M \rangle_t = 0$ for all t almost surely.

Hence $M_t = M_0 = 0$ from lemma 1.2.3, which implies that $\langle X \rangle_t^1 = \langle X \rangle_t^2$ and $I_t^2 = I_t^1$ for all t almost surely. □

1.2 Brownian Motion

We recall the definition of Brownian motion and its quadratic variation process

Definition 1.2.1 (Standard Brownian Motion)**.** *A real-valued stochastic process* $\{B_t\}_{t\geq 0}$ *on a probability space* $(\Omega, \mathcal{F}, P)$ *is called* Standard Brownian Motion *if*

1. $B_0 = 0$,
2. $t \mapsto B_t(\omega)$ *is continuous almost surely,*

3. *$B_t - B_s$ is independent of $\mathcal{F}_s$ and has normal distribution $N(0, t-s)$, where $0 \leq s < t$.*

Definition 1.2.2 (Geometric Brownian Motion). *A stochastic process S_t is said to follow a* Geometric Brownian Motion (GBM) *if it satisfies the following stochastic differential equation:*

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where W_t is a Wiener process or standard Brownian motion, and μ and σ (respectively called the drift *and* volatility*) are constants.*

If S_t is a geometric brownian motion process, we further assume that S_0 is strictly positive.

Theorem 1.2.1 (Lévy's Theorem). *Fix $\omega \in \Omega$ and consider the path $t \mapsto B_t(\omega)$. For almost every such path, $\langle B \rangle_t(\omega) = t$, for all $t \geq 0$.*

Proof. Let $s_0 > 0$ and $\{\tau_n\}_n$ be a sequence of partitions. Define

$$X_n = \sum_{t_i^n \leq s_0} (B(t_{i+1}^n) - B(t_i^n))^2,$$
$$Y_i^n = B(t_{i+1}^n) - B(t_i^n).$$

Thus, $X_n = \sum_{t_i^n \leq s_0} (Y_i^n)^2$. Now $Y_i^n \sim N(0, t_{i+1}^n - t_i^n)$ by definition, and

$$E((Y_i^n)^2) = V(Y_i^n) = t_{i+1}^n - t_i^n,$$

since $E(Y_i^n = 0)$. Also,

$$\begin{aligned} V((Y_i^n)^2) &= E((Y_i^n)^4) - (E((Y_i^n)^2))^2 \\ &= 3(V(Y_i^n))^2 - (V(Y_i^n))^2 \\ &= 2(t_{i+1}^n - t_i^n)^2. \end{aligned}$$

$$\implies V(X_n) = 2\sum (t_{i+1}^n - t_i^n)^2 \leq ||\tau_n|| s_0.$$

Since the mesh size goes to 0 as $n \to \infty$, $\lim_{n\to\infty} V(X_n) = 0$, or $(X_n - E(X_n)) \to$

0 in L^2. Furthermore, $E(X_n) = \sum_{t_i^n \le s_0} E((Y_i^n)^2) = \sum_{t_i^n \le s_0}(t_{i+1}^n - t_i^n) = max\{t_i^n \le s_0\}$. Thus, we conclude that $\lim_{n\to\infty} E(X_n) = s_0$.

Hence $(X_n - s_0) \to 0$ in L^2, or $X_n \to s_0$ in probability, which implies that there is a subsequence $\{X_{n_k}\}_k$ such that $X_{n_k} \to s_0$ almost surely as $k \to \infty$.

Next, we have to show that $\langle B\rangle_t$ does not depend on the choice of the partition sequence. Hence

$$\langle B\rangle_{s_0} = \lim_{k\to\infty} X_{n_k} = s_0$$

almost surely. Now take the enumeration $\mathbb{Q}_+ = \{s_0, s_1, ...\}$ and let $\mathcal{N}_i \subset \Omega$ such that $\langle B\rangle_{s_i} \neq s_i$ if and only if $\omega \in \mathcal{N}_i$.

Then $P(\cup_i \mathcal{N}_i) = 0$. Let $A \subset \Omega$ such that $B_t(\omega)$ is not continuous for $\omega \in A$. Hence, for ω not belonging to $\mathcal{N} = A \cup_{i\ge1} \mathcal{N}_i$, $\langle B\rangle_{s_i}(\omega) = s_i$ for all i, and $P(\mathcal{N}) = 0$.

If $t > 0$, there exists a subsequence $s_{i_k} \to t$ and from the continuity of $t \mapsto B_t(\omega)$ for all $\omega \in \mathcal{N}^c$,

$$\langle B\rangle_t(\omega) = \lim_{k\to\infty} \langle B\rangle_{s_{i_k}}(\omega) = \lim_{k\to\infty} s_{i_k} = t.$$

□

We can use Itō's formula to obtain an explicit form for a process S_t which follows Geometric Brownian Motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Since the coefficients μS_t and σS_t are Lipschitz in s, the above SDE has a strong solution that is continuous almost surely. Let $f(S_t) := \log(S_t)$ for $t < \tau(\omega)$, where $\tau := \min\{t > 0; S_t \le 0\}$. Then

$$\begin{aligned} d\log S_t = df(S_t) &= f'(S_t)dS_t + \frac{1}{2}f''(S_t)S_t^2\sigma^2 dt \\ &= \frac{1}{S_t}(\sigma S_t dW + \mu S_t dt) - \frac{1}{2}\sigma^2 dt \\ &= \sigma dB_t + (\mu - \frac{\sigma^2}{2})dt. \end{aligned}$$

It follows that for $t < \tau$,

$$\begin{aligned}
\log S_t &= \log S_0 + \sigma B_t + (\mu - \frac{\sigma^2}{2})t \\
\implies S_t &= S_0 \exp(\sigma B_t + (\mu - \frac{\sigma^2}{2})t).
\end{aligned}$$

Now it remains to show that the solution does not exist for $\tau(\omega) < \infty$. Let ω be such that $\tau(\omega) < \infty$ and the solution exists for $\tau(\omega)$, and let $t \uparrow \tau(\omega)$. Then using the continuity of $\{S_t\}_{t\geq 0}$ at $\tau(\omega)$,

$$0 = \lim_{t\uparrow\tau(\omega)} S_t = S_{\tau(\omega)^-} = S_0 \exp(\sigma B_{\tau(\omega)^-} + (\mu - \frac{\sigma^2}{2})\tau(\omega)^-) = S_0 \exp(\sigma B_{\tau(\omega)} + (\mu - \frac{\sigma^2}{2})\tau(\omega)) \neq 0$$

almost surely, which is a contradiction proving that no solution exists for $\tau(\omega) < \infty$.

Chapter 2

Introduction to Trading in Financial Markets

We shall begin by defining several notions in a discrete-time market model, and then go on to generalize these notions in the context of continuous-time markets. For references used herein, see [18], [19], [20], [21], [24] and [22].

2.1 Some Market-Related Definitions

Definition 2.1.1 (Bond)**.** *A riskless securrity earning a fixed interest at rate r in each unit of time. An investor can invest in or borrow a bond a rate of interest r, and it follows deterministic pricing dynamics.*

Definition 2.1.2 (Stock)**.** *Shares of a stock of some specified company are traded in the market. Prices of stocks are modelled as random processes.*

Definition 2.1.3 (Option)**.** *The right, but not the obligation, to buy or sell stock from or to a particular party, at some fixed price K (called strike price). If the holder of the option can buy or sell the stock only at the end of some fixed time period, then that option is called* European option*.*

If the holder of the option can do so before the terminal time as well, then that option is called an American option*.*

Suppose times $t_1, t_2, ..., t_k$ are the only times when stock can be traded (trading times), ξ_k^1 are the number of shares and ξ_k^0 the number of bonds held on the k^{th} day.

Let the price of the stock on the k^{th} day (after the trading time) be S_k. Then ξ_k^i can be set strategically depeding on the history of the stock process in the following way:

$$\xi_k^i = \pi_k^i(S_0, S_1, ..., S_{k-1}),$$

where π_k^i are real functions on $\mathbb{R}^k$, $k \geq 1$.

Definition 2.1.4 (Self-financing Strategy)**.** *A trading strategy in which no money is put in and there is no surplus on any day except the initial investment x, that is,*

$$\xi_{k-1}^1 + \xi_{k-1}^0(1+r)^{k-1} = \xi_k^1 S_{k-1} + \xi_k^0(1+r)^{k-1}.$$

Suppose S_k can take only finitely many values and we consider a time horizon of N days. Let

$$\Omega = \{(s_0, s_1, ..., s_N) \in \mathbb{R}^{N+1} : P(S_0 = s_0, ..., S_N = s_N) > 0\},$$

$$P((S_0, ..., S_N) \in \Omega) = 1.$$

Without loss of generality, let the underlying probability space be $(\Omega, \mathcal{F}, P)$ and $S_i(s_0, ..., s_N) = s_i$.

A self-financing strategy is represented as $\theta = \{x, \pi_1^1, ..., \pi_k^1\}$. The value of the portfolio (on the k^{th} day after the trading time) is

$$V_k(\theta)(s_0, ..., s_N) = [x + \sum_{j=1}^{k} \pi_j^1(s_0, ..., s_{j-1})(s_j\beta^j - s_{j-1}\beta^{j-1})](1+r)^k,$$

where $\beta = (1+r)^{-1}$.

Definition 2.1.5 (Arbitrage Opportunity)**.** *A self-financing strategy $\theta = (0, \pi^1)$ such that*

1. $V_n(\theta)(s_0, ..., s_N) \geq 0 \ \forall \ (s_0, ..., s_N) \in \Omega,$
2. $V_n(\theta)(s_0^1, ..., s_N^1) > 0$ *for some* $(s_0^1, ..., s_N^1) \in \Omega.$

2.2 Continuous Time Trading

Let there be k stocks whose prices at $t \in [0, T]$ are $(S_t^1, ..., S_t^k)$ and a bond priced S_t^0.

We assume that $S_u^0 \leq S_t^0$ for $u < t$, and S_t^i are r.c.l.l processes. The discounted prices are $\tilde{S}_t^i = S_t^i (S_t^0)^{-1}$. Let $\mathcal{G}_t$ be the smallest σ-algebra with respect to which S_u^i are measurable ($0 \leq i \leq k$, $0 \leq u \leq t$).

A self-financing *simple* strategy has an initial investment x and $a_{t_j}^i$, $1 \leq i \leq k$, $0 \leq j \leq m$, are bounded $\mathcal{G}_{t_j}$-measurable random variables.

We assume that the portfolio changes at times $0 \leq t_0 < t_1 < ... < t_m \leq T$, and $\mathcal{F}_t$ denotes the σ-algebra obtained on completing $\mathcal{G}_t$ and forcing it to satisfy the usual conditions.

$a_{t_j}^i$ denotes the number of shares of the i^{th} stock held during $(t_j, t_{j+1}]$.

$\pi_t^i = \sum_j \pi_{t_j}^i \mathbb{1}_{(t_j, t_{j+1}]}(t)$, π_t^i is $\mathcal{F}_t$-measurable and left-continuous.

Let the strategy be denoted by $\theta = (x, \pi^1, ..., \pi^k)$. The discounted value of the portfolio is

$$\tilde{V}_t(\theta) = x + \sum_{j=1}^{k} \sum_{i=0}^{m-1} a_{t_i}^j (\tilde{S}_{t_{i+1} \wedge t}^j - \tilde{S}_{t_i \wedge t}^j).$$

A contingent claim is *attainable* via a simple strategy θ at time T if $X = V_T(\theta) = S_T^0 \tilde{V}_T(\theta)$.

Definition 2.2.1 (Arbitrage Opportunity (Continuous trading))**.** *A simple strategy* $\theta = (0, \pi)$ *is an* arbitrage opportunity *if*

$$P(\tilde{V}_T(\theta) \geq 0) = 1; P(\tilde{V}_T(\theta) > 0) > 0.$$

We assume that S_t^i are continuous semi-martingale processes.

Definition 2.2.2 (Trading Strategy)**.** *Let* $\mathcal{F}_t$ *be the filtration generated by* $S = (S^0, ..., S^k)$*.* $\theta = (\pi^0, ..., \pi^k)$ *is a* trading strategy *if*

1. *each* π_t^i *is* $\mathcal{F}_t$*-predictable,*
2. *exists for* $i = 0, ..., k$*.*

Definition 2.2.3 (Value of the Portfolio)**.** *The* value of a portfolio *is the stochastic process*

$$V_t(\theta) = \sum_{i=0}^{k} \pi_t^i S_t^i,$$

where $t > 0$*.*

Definition 2.2.4 (Gains Process)**.** *The stochastic process:*

$$G_t(\theta) = \sum_{i=1}^{k} \int_0^t \pi_u^i dS_u^i.$$

The discounted versions of the above processes can be obtained by replacing the stock price with the discounted stock price in the respective formulae.

Definition 2.2.5 (Self-Financing Strategy)**.** $\theta = (\pi^0, ..., \pi^k)$ *is self-financing if* $\tilde{V}_t(\theta) = \tilde{V}_0(\theta) + \tilde{G}_t(\theta)$ *almost surely, for* $0 \leq t \leq T$*.*

Definition 2.2.6 (Admissible Strategy)**.** *A self-financing strategy* θ *is* admissible *if* $\exists m < \infty$ *such that*

$$P(\tilde{V}_t(\theta) \geq -m \; \forall \; t) = 1.$$

An arbitrage opportunity is an admissible strategy such that $\tilde{V}_T(\theta) \geq 0$ almost surely, and $P(\tilde{V}_T(\theta) > 0) > 0$.

$\tilde{S} = (\tilde{S}^1, ..., \tilde{S}^k)$ has the *no arbitrage property* if no arbitrage opportunity exists.

Definition 2.2.7 (Equivalent Measure)**.** *Let* (X, σ) *be a measurable space and let* μ *and* ν *be measures defined on* σ*. Then* μ *and* ν *are said to be* equivalent *iff for every measurable set* A*,* $\mu(A) = 0 \iff \nu(A) = 0$*.*

Definition 2.2.8. *Let* P *be the underlying probability measure with respect to which the market model is presented. Let the discounted stock price process be denoted by* $\tilde{S}_t^i$*. Then the* class of equivalent martingale measures *with respect to* P *is defined as* $\mathcal{M}(P) := \{Q : Q \equiv P$ *and* $\tilde{S}_t^i$ *is a* Q*-local martingale,* $1 \leq i \leq k\}$*.*

Theorem 2.2.1. *Let* Q *be an equivalent martingale measure to* P*. For an admissible strategy* $\theta = (x, \pi)$*,*

$$U_t = \sum_{i=1}^{k} \int_0^t \pi_u^i d\tilde{S}_u^i$$

is a Q-local martingale and Q-supermartingale, that is, $\mathcal{M}(P) \neq \phi \implies$ no arbitrage in the admissible class.

Proof. Since the stochastic integral with respect to a local martingale is itself a local martingale, U_t is a local martingale.

If $\{\tau_n\}$ is an increasing sequence of stopping times such that $P(\tau_n = T) \to 1$ and $U_t^n = U_{t\wedge\tau_n}$ is a Q-martingale, then for $s \leq t$, $E^Q(U_t^n|\mathcal{F}_s) = U_s^n$.

$U_t^n \geq -m$ for some m (admissibility), so we can use Fatou's lemma on it:

$$E^Q(U_t|\mathcal{F}_s) = E^Q(\liminf U_t^n|\mathcal{F}_s) \leq \liminf E^Q(U_t^n|\mathcal{F}_s) = \liminf U_s^n = U_s,$$

so U_t is a Q-supermartingale. Therefore

$$E^Q(U_t) \leq E^Q(U_0) = 0,$$

$$P(U_T \geq 0) = 1 \implies Q(U_T \geq 0) = 1.$$

$$E^Q(U_T) \leq 0 \implies Q(U_T = 0) = 1 \implies P(U_T = 0) = 1.$$

Thus, the no arbitrage condition is satisfied. □

2.3 Implied Volatility

Definition 2.3.1 (Black-Scholes Equation for a Call Option)**.** *Let the stock price process be denoted by S_t. If $S_t = S$ for a particular time t, let us denote the price of a European call option at that time t by $C(t, S)$. We assume that the stock price follows a geometric Brownian motion:*

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ and σ are constant parameters. From this, we obtain the following PDE for the call option price, the Black-Scholes PDE *for the call option:*

$$rS_t \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} - rC = 0, \tag{2.3.1}$$

where r is the rate of interest on the bond.

Definition 2.3.2 (Implied Volatility)**.** *Let r be the interest rate of the bond and C the observed initial market price of the call option with strike price K and terminal time T. Using boundary conditions appropriate to the European call option, we obtain a solution $\mathbb{B}(S, T, r, K, \sigma)$ to the Black-Scholes equation. If we can solve the equation $\mathbb{B}(S, T, r, K, \sigma) = C$ for σ, then the solution $\sigma(K, T)$ is known as the* ***implied volatility*** *and its graphical representation with respect to K and T is called the* ***volatility surface****.*

The volatility is constant in the default Black-Scholes model, but actually varies with time and strike price. For a general option-pricing model, the implied volatility is defined similarly - we will have a different function ϕ instead of $\mathbb{B}$.

Definition 2.3.3 (Greeks)**.** Greeks *measure the sensitivity of option prices to various market parameters. Following are the greeks and their respective formulae for call options under the default Black-Scholes model:*

1. *Delta* $= \frac{\partial C}{\partial S} = e^{-qT}\phi(d_1)$,
   where q is the dividend paid by the stock and
   $d_1 = \frac{log(S_t/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$.
2. *gamma* $= \frac{\partial^2 C}{\partial S^2} = e^{-qT}\frac{\phi(d_1)}{\sigma S\sqrt{T}}$
3. *vega* $=\frac{\partial C}{\partial \sigma} = e^{-qT}\phi(d_1)S\sqrt{T}$
4. *theta* $=-\frac{\partial C}{\partial T} = -e^{-qT}S\phi(d_1)\frac{\sigma}{2\sqrt{T}} + qe^{-qT}SN(d_1) - rKe^{-rT}N(d_2)$,
   where $d_2 = d_1 - \sigma\sqrt{T-t}$.

Chapter 3

Regime-Switching Market Model

3.1 Drawbacks of the Black-Scholes Model

Even though the Black-Scholes model is widely used, it suffers from a few drawbacks. In the Black-Scholes model, the parameters μ and σ (that describe the stock price process) are constant and the stock price of a European call option follows geometric Brownian motion as mentioned in the definition of the Black-Scholes equation. In other words, the implied volatility is supposed to be independent of the strike price and time of maturity. One way to depict the relation of implied volatility to the strike price and time of maturity is by using the volatility surface, which is flat in the Black-Scholes model (figure 3.1). However, empirical observations show that the volatility surface is actually skewed, a property that is referred to as the *volatility smile* (figure 3.2).

Among the various generalizations of the Black-Scholes model (see [2], [3], [4], [5], [9], [10], [11], [12], [13], [26], [31] and [35]), we consider a particular model, as given in [1], [7], [8], [14], [15] and [17].

3.2 A Generalization of the Black-Scholes Model

We assume that the state of the market corresponds to that of a finite state, continuous time Markov chain; in other words, we say that the market is composed of several "regimes", and each regime characterizes such a state. We denote such a Markov chain by $\{X_t, t \geq 0\}$ that takes the values $\{1, 2, \ldots, k\}$ (used in above equation), and for which

$$P(X_{t+\delta t} = j | X_t = i) = \lambda_{ij}\delta t + o(\delta t),$$

where $\Lambda = [\lambda_{ij}]$ is the generating Q-matrix of the Markov chain, satisfying the properties $\lambda_{ij} \geq 0$ for $i \neq j$ and $\lambda_{ii} = -\sum_{i \neq j}^{k} \lambda_{ij}$. The transition probabilities between states are given by $p_{ij} := \frac{|\lambda_{ij}|}{\lambda_{ii}}$.

We assume that the market consists of a single stock S_t, whose price is a stochastic process that follows a Markov-modulated geometric Brownian motion:

$$dS_t = r(X_t-)S_t dt + \sigma(X_t-)S_t dW_t,$$

and of a single risk-free asset B_t that satisfies:

$$dB_t = B_t r(X_t)dt,$$

where $S_0, B_0 > 0$, $t \geq 0$, W_t is a standard Weiner process independent of the Markov chain $\{X_t, t \geq 0\}$, and $r, \sigma \in \mathbb{R}^k$.

The derivation of the price equation for a European call option is more involved in this case, since the market described by the regime-switching model is *incomplete*. A market is called incomplete when not all contingent claims can be attained using only self-financing strategies. In an incomplete market, there are several price functions that satisfy No Arbitrage.

Therefore we employ an *optimal strategy* to hedge the call option, which we try to approximate as close as possible to a self-financing by minimizing its *quadratic residual risk* (which is a measure of the cash flow), subject to a constraint. Using this procedure, we obtain a unique, locally risk-minimizing price function of a call option that satisfies the following parabolic PDE:

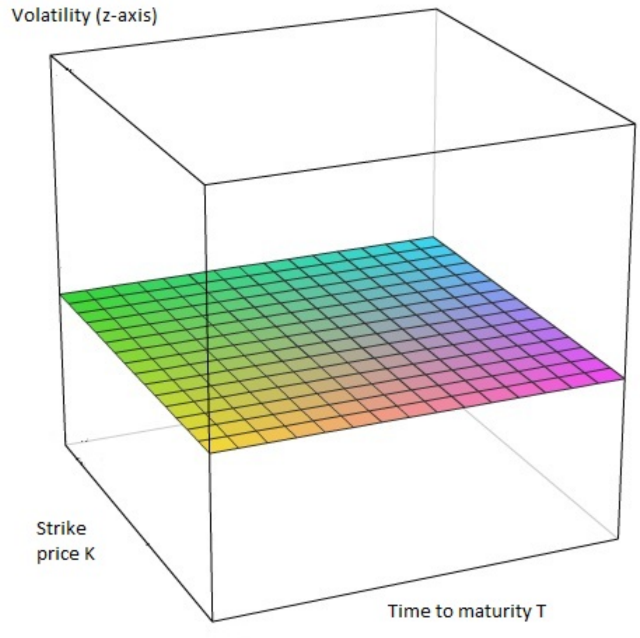

Figure 3.1: Flat volatility surface

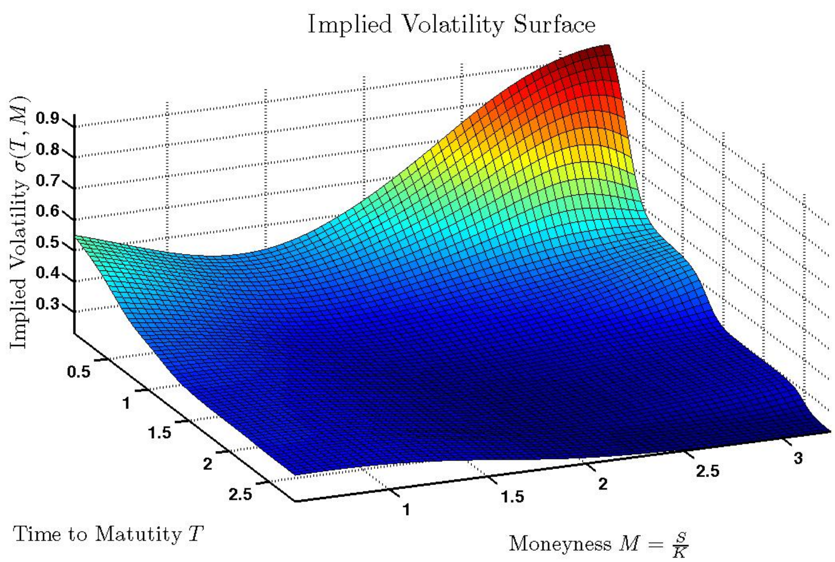

Figure 3.2: Volatility "smile" (source- `http://relavalue.blogspot.in/2013_12_01_archive.html`)

$$\frac{\partial \phi(t,s)}{\partial t} + sR\frac{\partial \phi(t,s)}{\partial s} + \frac{1}{2}s^2 \mathrm{diag}(\sigma^2)\frac{\partial^2 \phi(t,s)}{\partial s^2} + \Lambda\phi(t,s) = R\phi(t,s); \qquad (3.2.1)$$

1. $\phi_i(T, s) = (s - K)^+$,
2. $\phi_i(t, 0) = 0$,

for all s and $i = 1, 2, \ldots, k$, and where K is the strike price of the option and Λ is a $k \times k$ rate matrix, i.e., its off-diagonal entries are non-negative and the sum of each of its row is zero. The solution $\phi(t, s)$, a vector-valued function $\phi : [0, T] \times \mathbb{R}_+ \to \mathbb{R}^k$, is the price function in the regime-switching model. It denotes the price of a European call option at time t when the stock price $S_t = s$.

In the above equation, we define $R \in \mathbb{R}^{k \times k}$ such that $R(i, i) = r(i)$ for $i = 1, 2, \ldots, k$ and $R(i, j) = 0$ for $i \neq j$. Furthermore, if $A, B \in \mathbb{R}^k$, then we denote the k-vector with i^{th} component $A(i)B(i)$ by AB, and we use the convention that $\operatorname{diag}(AB) \in \mathbb{R}^{k \times k}$ such that $\operatorname{diag}(AB)(i, i) = A(i)B(i)$ for $i = 1, 2, \ldots, k$ and $\operatorname{diag}(AB)(i, j) = 0$ for $i \neq j$.

Remark. *No closed-form explicit solution is known, but it can be proved that a smooth unique solution exists for the above PDE. However, there exists an integral representation of the price function, which we have mentioned in section 5.1.*

Chapter 4

Notion of Implied Volatility in the Regime-Switching Model

4.1 Introduction

The implied volatility is an important parameter that is widely used in considerations related to transactions involving and pricing of options. In this chapter, we seek to establish that in a Markov-modulated market, and under certain conditions, it is possible to determine the implied volatility if we are given the price function of a European call option.

Let us first discuss an intuitive approach to define implied volatility in the regime-switching market. We assume that we can observe X_t, r and Λ from the market, in addition to S_t and $\phi(t, S_t, X_t)$. Suppose we observe the values of these quantities at specific points of time $t_1, t_2, \ldots, t_k$ close enough so that $X_{t_i} = x$ for all $i = 1, 2, \ldots, k$. Let the observed value of ϕ at time t_i be denoted by y_i. Thus, we can potentially obtain k equations:

$$\begin{aligned}
\phi(t_1, S_{t_1}, x, \sigma) &= y_1 \\
\phi(t_2, S_{t_2}, x, \sigma) &= y_2 \\
&\vdots \\
\phi(t_k, S_{t_k}, x, \sigma) &= y_k,
\end{aligned}$$

which can be collectively written using a vector function Φ as

$$\Phi(\mathbf{t}, \mathbf{s}, x, \sigma) = \mathbf{y},$$

where we define $\mathbf{t} := (t_1, t_2, \ldots, t_k)$, $\mathbf{s} := (S_{t_1}, S_{t_2}, \ldots, S_{t_k})$ and $\mathbf{y} := (y_1, y_2, \ldots, y_k)$.

So now we have a vector-valued function Φ and a map $\sigma \mapsto \mathbf{y}$. If the latter map is invertible, we can obtain the implied volatility using $\mathbf{y}$ and Φ. However, proving that the inverse function theorem can indeed be used for this map is mathematically intractable and we consider a different approach where we just have a single time t instead of an n-tuple $\mathbf{t}$. Therefore we must only deal with a single $\phi(t, S_t)$ and attempt to prove that the map $\sigma \mapsto \phi$ is invertible.

4.2 A Few Results

For this section, we will require the following result:

Lemma 4.2.1. *Consider the following PDE:*

$$\left(\frac{\partial}{\partial t} + \mathcal{A}_\sigma\right)\phi = f(t, s) + Rg(t, s),$$

on $(0, T) \times (0, \infty)$, *with the following boundary conditions:*

1. $\phi(T, s)(i) = 0$ *for all* $s \geq 0$ *and for all* i,
2. $\phi(t, 0)(i) = 0$ *for all* $t \in [0, T]$ *and for all* i,

where $\mathcal{A}_\sigma \phi(t, s, i) := R_i s \frac{\partial \phi(t,s,i)}{\partial s} + \frac{1}{2} s^2 \sigma_i^2 \frac{\partial^2 \phi(t,s,i)}{\partial s^2} + \sum_j \Lambda_{ij} \phi(t, s, j)$, *and* $\phi(t, s), f(t, s) \in \mathbb{R}^k$. *Let the function* $f(t, s)$ *have at most quadratic growth with respect to* s. *Then the above initial boundary value problem has a unique classical solution* ϕ *that also has at most quadratic growth with respect to* s.

Proof. Let $\tau_n := \inf\{\tau > t; |S_\tau - s| \geq n\}$ and define

$$g_1^{(n)}(t, s, i) := \mathbf{E}\left[\int_t^{\tau_n \wedge T} e^{-\int_t^{t'} R(X_u)\, du} |f(t', S_{t'}, X_{t'})|\, dt' \middle| S_t = s, X_t = i\right].$$

Using the definition of τ_n and the quadratic growth condition on f, we can say that the above quantity

$$g_1^{(n)}(t,s,i) \leq c\mathbf{E}\left[\int_t^T |1+S_{t'}|^2\, dt' \middle| S_t = s, X_t = i\right].$$

Now if we can show that

$$\mathbf{E}\left[\int_0^T S_{t'}^2\, dt'\right] < \infty,$$

then $g_1^{(n)}(t,s,i)$ will also be bounded and we will be done. We know that

$$S_{t'} = S_0 \exp\left[\int_0^{t'} \{\mu(X_u) - \frac{1}{2}\sigma^2(X_u)\}\, du + \int_0^{t'} \sigma(X_u)\, dW_u\right].$$

Let $c := \max_{i\in S}\{\mu(i) - \frac{1}{2}\sigma^2(i)\}$ and $d := \max_{i \in S}\{\sigma^2(i)\}$; then clearly,

$$S_{t'}^2 \leq S_0^2 \exp\left(2\int_0^{t'} c\, du\right) \exp\left(2\int_0^{t'} \sigma(X_u) dW_u\right). \tag{4.2.1}$$

It is enough to show the RHS has a finite expectation. Consider:

$$\begin{aligned}\int_0^{t'} \sigma(X_u) dW_u &= \sum_{n=1}^{\infty} \int_{T_{n-1}\wedge t'}^{T_n \wedge t'} \sigma(X_{T_{n-1}})\, dW_u \\ &= \sum_{n=1}^{\infty} \int_{T_{n-1}\wedge t'}^{T_n \wedge t'} \sigma(X_{T_{n-1}})(W_{T_n\wedge t'} - W_{T_{n-1}\wedge t'}).\end{aligned}$$

If $\mathcal{F}_{t'}^X$ is the filtration generated by $\{X_t\}_{t\geq 0}$, then the conditional distribution of $\int_0^{t'} \sigma(X_u) dW_u$ given $\mathcal{F}_{t'}^X$ is a normal distribution with zero mean and variance

$$V = \sum_{n=1}^{\infty} \sigma^2(X_{T_n-1})[(T_n \wedge t') - (T_{n-1} \wedge t')].$$

From the expression of the variance of a lognormal random variable, we obtain

$$\begin{aligned}
\mathbf{E}\Big[\exp\Big(2\int_0^{t'}\sigma(X_u)dW_u\Big)\Big] &= \mathbf{E}\Big[\exp\Big(\sum_{n=1}^{\infty}\sigma^2(X_{T_n-1})[(T_n\wedge t')-(T_{n-1}\wedge t')]\Big)\Big] \\
&\le \mathbf{E}\Big[\exp\Big(d\sum_{n=1}^{\infty}[(T_n\wedge t')-(T_{n-1}\wedge t')]\Big)\Big] \\
&= \exp(d\tau).
\end{aligned}$$

Now using the above and equation (4.2.1),

$$\mathbf{E}(S_{t'}^2) \le S_0^2 e^{2ct'}e^{dt'} = S_0 e^{(2c+d)t'},$$

and therefore,

$$\begin{aligned}
\int_0^T \mathbf{E}(S_{t'})\,dt' &= S_0\int_0^T e^{(2c+d)t'}\,dt' \\
&= \frac{S_0}{2c+d}(e^{(2c+d)T}-1) < \infty.
\end{aligned}$$

□

We also state a particular version of the Feynman-Kac theorem here without proof (refer to Theorem 7.6, [23] for a proof of the generalized version). The theorem is stated under the assumptions given as (7.2) to (7.4) in [23].

Theorem 4.2.1 (Feynman-Kac Representation)**.** *Suppose that $v(t,s):[0,T]\times\mathbb{R}^k\to\mathbb{R}^k$ is continuous, is of class $C^{1,2}([0,T)\times\mathbb{R}^k)$ and satisfies the Cauchy problem*

$$-\frac{\partial v}{\partial t} + rv = \mathcal{A}v + g;$$

$$v(T,s) = 0,$$

as well as the polynomial growth condition

$$\max_{0\le t\le T}|v(t,s)| \le M(1+||s||^{2\mu});$$

for some $M>0$ and $\mu\ge 1$. Then $v(t,x)$ admits the stochastic representation

$$v(t,x) = \mathbf{E}^{t,x}\left[\int_t^T g(\tau, X_s)e^{-\int_t^\tau r(\theta,X_\theta)\,d\theta}\,d\tau\right].$$

Remark. *From here on, we refer to the solution of the PDE (3.2.1) as ϕ. In the following theorem, we write ϕ as ϕ_σ to emphasize its dependence on the parameter σ. Similarly, $\phi_{\sigma+h}$ denotes the solution of a PDE obtained by replacing σ in (3.2.1) by $\sigma+h$.*

Theorem 4.2.2. *The price function, ϕ, is continuous with respect to the volatility coefficient σ.*

Proof. We can re-write equation (3.2.1) as:

$$\frac{\partial \phi(t,s)}{\partial t} + \mathcal{A}\phi(t,s) = R\phi(t,s),$$

where $\mathcal{A} := Rs\frac{\partial}{\partial s} + \frac{1}{2}s^2\mathrm{diag}(\sigma^2)\frac{\partial^2}{\partial s^2} + \Lambda$ and $h \in \mathbb{R}^k$. Now define $g_1(h;t,s) := \phi_{\sigma+h}(t,s) - \phi_\sigma(t,s)$. We want to show that $g_1(h) \to 0$ as $h \to 0$. Therefore ($g_1(h)$, ϕ_σ, $\phi_{\sigma+h}$, $\mathcal{A}_\sigma$ and $\mathcal{A}_{\sigma+h}$ depend on t and s, and we will not be explicitly specifying that in the following equations),

$$\begin{aligned}\frac{\partial g_1(h)}{\partial t} &= \frac{\partial \phi_{\sigma+h}}{\partial t} - \frac{\partial \phi_\sigma}{\partial t}\\ &= \mathcal{A}_\sigma\phi_\sigma - \mathcal{A}_{\sigma+h}\phi_{\sigma+h} + R(\phi_{\sigma+h} - \phi_\sigma)\\ &= (\mathcal{A}_\sigma - \mathcal{A}_{\sigma+h})\phi_{\sigma+h} - \mathcal{A}_\sigma(\phi_{\sigma+h} - \phi_\sigma) + R(\phi_{\sigma+h} - \phi_\sigma)\\ &= (\mathcal{A}_\sigma - \mathcal{A}_{\sigma+h})\phi_{\sigma+h} - \mathcal{A}_\sigma g_1(h) + Rg_1(h).\end{aligned}$$

The terminal conditions on $\phi(t,s)$ in equation (3.2.1) do not depend on σ. In other words, $\phi_\sigma(t,s)$ and $\phi_{\sigma+h}(t,s)$ take the same values and therefore $g_1(h;t,s)$ vanishes at the terminal points (T,s) (for all $s \geq 0$) and $(t,0)$ (for all $t \in [0,T]$). Therefore we get the following PDE in $g_1(h;t,s)$:

$$\frac{\partial g_1(h;t,s)}{\partial t} + \mathcal{A}_\sigma g_1(h;t,s) = (\mathcal{A}_\sigma - \mathcal{A}_{\sigma+h})\phi_{\sigma+h}(t,s) + Rg_1(h;t,s),$$

with the following boundary conditions:

1. $g_1(h;T,s)(i) = 0$ for all $s \geq 0$ and for all i,
2. $g_1(h;t,0)(i) = 0$ for all $t \in [0,T]$ and for all i.

We also have

$$\begin{aligned}\mathcal{A}_{\sigma+h} - \mathcal{A}_\sigma &= \frac{1}{2}s^2\mathrm{diag}[(\sigma+h)^2]\frac{\partial^2}{\partial s^2} - \frac{1}{2}s^2\mathrm{diag}(\sigma^2)\frac{\partial^2}{\partial s^2} \\ &= \frac{1}{2}s^2\mathrm{diag}[h^2+2\sigma h]\frac{\partial^2}{\partial s^2}.\end{aligned} \tag{4.2.2}$$

Therefore the equation in $g_1(h;t,s)(i)$ can finally be written as

$$\frac{\partial g_1(h;t,s)(i)}{\partial t} + \mathcal{A}_\sigma g_1(h;t,s)(i) = \frac{-s^2}{2}\mathrm{diag}(h^2+2\sigma h)\frac{\partial^2}{\partial s^2}\phi_{\sigma+h}(t,s)(i) + Rg_1(h;t,s)(i). \tag{4.2.3}$$

$Rg_1(h;t,s)(i)$ can be brought over from the RHS of (4.2.3) to the LHS and incorporated into the operator $\mathcal{A}_\sigma$. In the remaining term on the RHS, $\frac{\partial^2\phi}{\partial s^2}$ is bounded in s and multiplied by s^2. Thus, the source term has at most quadratic growth in s, and by Lemma 4.2.1, there exists a unique classical solution $g_1(h;t,s)$ to the PDE above, that also has at most quadratic growth in s. Therefore we can use Remark 3.5.5 in [29] to specify $g_1(h;t,s)$ explicitly as follows:

$$\begin{aligned}g_1(h;t,s)(i) = \mathbf{E}\Bigg[\int_t^T & e^{-\int_t^\tau r(X_u)du}\left(\frac{-S_\tau^2}{2}\right)(h(X_\tau)^2 + 2\sigma(X_\tau)h(X_\tau))\frac{\partial^2}{\partial s^2}\phi_{\sigma+h}(\tau,S_\tau)(X_\tau)\,d\tau \Bigg| \\ & S_t = s, X_t = i\Bigg],\end{aligned}$$

where $S_0 > 0$, X_t is a Markov chain with rate matrix Λ and S_t satisfies the SDE

$$dS_t = S_t[r(X_t)dt + \sigma(X_t)dW_t],$$

and $h(X_t)$ denotes the X_t^{th} component of the k-vector h.

By the Feynman-Kac theorem, the expectation above exists. The norm of g_1

(for convenience, we omit the function parameters for the time being) is:

$$\|g_1\| = \left\| \mathbf{E}\left[\int_t^T e^{-\int_t^\tau r(X_u)du} \left(\frac{-S_\tau^2}{2}\right)(h^2+2\sigma h)\frac{\partial^2}{\partial s^2}\phi_{\sigma+h}\, d\tau \middle| S_t = s, X_t = i\right] \right\|$$
$$\leq \mathbf{E}\left[\int_t^T \left| e^{-\int_t^\tau r(X_u)du} \left(\frac{-S_\tau^2}{2}\right)(h^2+2\sigma h)\frac{\partial^2}{\partial s^2}\phi_{\sigma+h}\right| d\tau \middle| S_t = s, X_t = i\right].$$

Since the expectation exists, the integrand in the expectation above, say η, is in L^1. Suppose we have a decreasing sequence $h_1 > h_2 > \ldots$ in $\mathbb{R}^k$ that converges to 0. Let us denote by η_j and $(g_1)_j$ the functions obtained by substituting h with h_j in η and g_1 respectively. Then $|\eta_n| < \eta_1$ for all j, where η_1 is in L^1 and $\eta_n \to 0$ for each (τ, ω). Therefore by the dominated convergence theorem, $(g_1)_n \to 0$, or $g_1(h) \to 0$ as $h \to 0$. □

Lemma 4.2.2. *The double derivative of the price function with respect to s, say ψ, is continuous with respect to σ.*

Proof. After differentiating equation (3.2.1) twice with respect to s, we get

$$\frac{\partial \psi}{\partial t} + s(R + 2\mathrm{diag}(\sigma^2))\frac{\partial \psi}{\partial s} + \frac{s^2}{2}\mathrm{diag}(\sigma^2)\frac{\partial^2 \psi}{\partial s^2} + \Lambda\psi = -(R + \mathrm{diag}(\sigma^2))\psi; \quad (4.2.4)$$

1. $\psi(T, s)(i) = 0$ for all $s \geq 0$ and for all i,
2. $\psi(t, 0)(i) = 0$ for all $t \in [0, T]$ and for all i.

The above equation can be re-written as

$$\frac{\partial \psi}{\partial t} + \mathcal{A}'\psi = -(R + \mathrm{diag}(\sigma^2))\psi,$$

where $\mathcal{A}' := s(R + 2\mathrm{diag}(\sigma^2))\frac{\partial}{\partial s} + \frac{1}{2}s^2\mathrm{diag}(\sigma^2)\frac{\partial^2}{\partial s^2} + \Lambda$. Therefore,

$$\mathcal{A}'_{\sigma+h} - \mathcal{A}'_\sigma = \mathrm{diag}(h^2 + 2\sigma h)\left(2s\frac{\partial}{\partial s} + \frac{s^2}{2}\frac{\partial^2}{\partial s^2}\right).$$

Let us define $g_2(h; t, s)(i) := \psi_{\sigma+h}(t, s)(i) - \psi_\sigma(t, s)(i)$ (as earlier, we write ψ as

ψ_σ to emphasize its dependence on σ). Then

$$\begin{aligned}\frac{\partial g_2(h)}{\partial t} &= \frac{\partial \psi_{\sigma+h}}{\partial t} - \frac{\partial \psi_\sigma}{\partial t} \\ &= \mathcal{A}'_\sigma \psi_\sigma + R\psi_\sigma + \mathrm{diag}(\sigma^2)\psi_\sigma - \mathcal{A}'_{\sigma+h}\psi_{\sigma+h} - R\psi_{\sigma+h} - \mathrm{diag}(\sigma^2 + h^2 + 2\sigma h)\psi_{\sigma+h} \\ &= (\mathcal{A}'_\sigma - \mathcal{A}'_{\sigma+h})\psi_{\sigma+h} - (\mathcal{A}'_\sigma + R + \mathrm{diag}(\sigma^2))(\psi_{\sigma+h} - \psi_\sigma) - \mathrm{diag}(h^2 + 2\sigma h)\psi_{\sigma+h}.\end{aligned}$$

Thus, we obtain the following PDE in $g_2(h;t,s)(i)$:

$$\begin{aligned}\frac{\partial g_2(h;t,s)(i)}{\partial t} + \mathcal{A}'_\sigma g_2(h;t,s)(i) = -(R + \mathrm{diag}(\sigma^2))g_2(h;t,s)(i) - \Bigg[\mathrm{diag}(h^2 + 2\sigma h) \\ \times \left(\psi_{\sigma+h}(t,s)(i) + 2s\frac{\partial \psi_{\sigma+h}(t,s)(i)}{\partial s} + \frac{s^2}{2}\frac{\partial^2 \psi_{\sigma+h}(t,s)(i)}{\partial s^2}\right)\Bigg];\end{aligned}$$

1. $g_2(h;T,s)(i) = 0$ for all $s \geq 0$ and for all i,
2. $g_2(h;t,0)(i) = 0$ for all $t \in [0,T]$ and for all i.

Once again, we can segregate the source term, which only involves the second or higher derivatives of ϕ with respect to s. These derivatives are bounded in s and are multiplied by 1, s or s^2. Therefore the source term as a whole has at most quadratic growth in s, and we can use Lemma 4.2.1 and Remark 3.5.5 in [29] to explicitly write $g_2(h;t,s)(i)$ as:

$$\begin{aligned}g_2(h;t,s)(i) = \mathbf{E}\Bigg[\int_t^T e^{-\int_t^\tau r(X_u)+\sigma^2(X_u)\,du} - (h(X_\tau)^2 + 2\sigma(X_\tau)h(X_\tau)) \\ \times \left\{\phi_{\sigma+h}(\tau,S_\tau)(X_\tau) + 2S_\tau\frac{\partial \phi_{\sigma+h}(\tau,S_\tau)(X_\tau)}{\partial s} + \frac{S_\tau^2}{2}\frac{\partial^2 \phi_{\sigma+h}(\tau,S_\tau)(X_\tau)}{\partial s^2}\right\} d\tau \Bigg| \\ S_t = s, X_t = i\Bigg],\end{aligned}$$

where $S_0 > 0$, X_t is a Markov chain with rate matrix Λ and S_t satisfies the SDE

$$dS_t = S_t[(r(X_t) + 2\sigma^2(X_t))dt + \sigma(X_t)dW_t].$$

As before,

$$
\begin{aligned}
g_2(h;t,s)(i) \leq \mathbf{E}\Bigg[\int_t^T \Bigg| & e^{-\int_t^\tau r(X_u)+\sigma^2(X_u)\,du} - (h(X_\tau)^2 + 2\sigma(X_\tau)h(X_\tau)) \\
& \times \left\{ \phi_{\sigma+h}(\tau,S_\tau)(X_\tau) + 2S_\tau \frac{\partial \phi_{\sigma+h}(\tau,S_\tau)(X_\tau)}{\partial s} + \frac{S_\tau^2}{2}\frac{\partial^2 \phi_{\sigma+h}(\tau,S_\tau)(X_\tau)}{\partial s^2} \right\} \Bigg|\, d\tau \Bigg| \\
& S_t = s, X_t = i \Bigg],
\end{aligned}
$$

We can now use the same reasoning as in Theorem 4.2.2 to establish that $g_2 \to 0$ as $h \to 0$ for each (t,s,i). □

Lemma 4.2.3. *ψ is continuous with respect to Λ.*

Proof. The equation in $\psi(t,s)(i)$ and the operator definition are the same as those in Lemma 4.2.2. This time, we write ψ as ψ_Λ since we are studying how it changes with a small perturbation in the parameter Λ. Let $h \in \mathbb{R}^{k\times k}$ such that all its off-diagonal terms are non-negative and the sum of each of its rows is zero. Define $g_3(h;t,s)(i) := \psi_{\Lambda+h}(t,s)(i) - \psi_\Lambda(t,s)(i)$; thus we get

$$
\begin{aligned}
\frac{\partial g_3(h)}{\partial t} &= \frac{\partial \psi_{\Lambda+h}}{\partial t} - \frac{\partial \psi_\Lambda}{\partial t} \\
&= \mathcal{A}'_\Lambda \psi_\Lambda + (R + \mathrm{diag}(\sigma^2))\psi_\Lambda - \mathcal{A}'_{\Lambda+h}\psi_{\Lambda+h} - (R+\mathrm{diag}(\sigma^2))\psi_{\Lambda+h} \\
&= (\mathcal{A}'_\Lambda - \mathcal{A}'_{\Lambda+h})\psi_{\Lambda+h} - (\mathcal{A}'_\Lambda + R + \mathrm{diag}(\sigma^2))(\psi_{\Lambda+h} - \psi_\Lambda).
\end{aligned}
$$

Therefore, the PDE in $g_3(h;t,s)(i)$ is

$$
\frac{\partial g_3(h;t,s)(i)}{\partial t} + \mathcal{A}'_\Lambda g_3(h;t,s)(i) = -h\psi_{\Lambda+h} - (R + \mathrm{diag}(\sigma^2))g_3(h;t,s)(i);
$$

1. $g_3(h;T,s)(i) = 0$ for all $s \geq 0$ and for all i,
2. $g_3(h;t,0)(i) = 0$ for all $t \in [0,T]$ and for all i.

Again, the source term has at most quadratic growth in s, and Lemma 4.2.1 and Remark 3.5.5 in [29] allow us to obtain an expression for $g_3(h;t,s)(i)$:

$$
g_3(h;t,s)(i) = \mathbf{E}\left[\int_t^T e^{-\int_t^\tau r(X_u)+\sigma^2(X_u)\,du}(-h\psi_{\Lambda+h}(\tau,S_\tau)(X_\tau))\,d\tau \,\middle|\, S_t = s, X_t = i \right],
$$

where $S_0 > 0$, X_t is a Markov chain with rate matrix Λ and S_t satisfies the SDE

$$dS_t = S_t[(r(X_t) + 2\sigma^2(X_t))dt + \sigma(X_t)dW_t].$$

Similarly as in previous results, we get the following inequality involving g_3:

$$g_3(h;t,s)(i) \leq \mathbf{E}\left[\int_t^T |e^{-\int_t^\tau r(X_u)+\sigma^2(X_u)\,du}(-h\psi_{\Lambda+h}(\tau,S_\tau)(X_\tau))|\,d\tau \middle| S_t = s, X_t = i\right],$$

and similarly as in those results, we can prove that $g_3 \to 0$ as $h \to 0$ for each (t,s,i). One important difference is in the definition of the sequence h_n: this time we have $h_{n+1}(i,j) < h_n(i,j)$ for all $i,j = 1,2,\ldots,k$ and for all $n = 1,2,\ldots$. □

Theorem 4.2.3 (Existence of Vega). *The price function ϕ is differentiable with respect to the volatility coefficient σ.*

Proof. Consider the following Cauchy problem:

$$\frac{\partial V(t,s)(i)}{\partial t} + \mathcal{A}_{\sigma_0}V(t,s)(i) + \left(\frac{\partial \mathcal{A}}{\partial \sigma}\right)_{\sigma_0}\phi_{\sigma_0}(t,s)(i) = 0, \tag{4.2.5}$$

with the boundary condition $V(T,s)(i) = 0$ for all i and $s \geq 0$, and for some $\sigma = \sigma_0$. Now, define $g_4(h;t,s)(i)$ as follows:

$$g_4(h;t,s)(i) := \frac{1}{\|h\|}\left(\phi_{\sigma_0+h}(t,s)(i) - \phi_{\sigma_0}(t,s)(i) - V(t,s)(i)h\right).$$

Note that by setting $t = T$ in the above, we get $g_4(h;T,s)(i) = 0$. To prove differentiability of $\phi(t,s)(i)$ with respect to σ, we must show that $g_4(h;t,s)(i)$ vanishes as $h \to 0$. Differentiating the above equation with respect to t,

$$\begin{aligned}
\frac{\partial g_4(h)}{\partial t} &= \frac{1}{\|h\|}\left(\frac{\partial \phi_{\sigma_0+h}}{\partial t} - \frac{\partial \phi_{\sigma_0}}{\partial t} - \frac{\partial V}{\partial t}h\right) \\
&= -\frac{1}{\|h\|}\left(\mathcal{A}_{\sigma_0+h}\phi_{\sigma_0+h} - \mathcal{A}_{\sigma_0}\phi_{\sigma_0} - \mathcal{A}_{\sigma_0}Vh - \left(\frac{\partial \mathcal{A}}{\partial \sigma}\right)_{\sigma_0}\phi_{\sigma_0}h\right) \\
&= -\frac{1}{\|h\|}\left((\mathcal{A}_{\sigma_0+h} - \mathcal{A}_{\sigma_0})\phi_{\sigma_0+h} + (\mathcal{A}_{\sigma_0}\phi_{\sigma_0+h} - \mathcal{A}_{\sigma_0}\phi_{\sigma_0} - \mathcal{A}_{\sigma_0}Vh) - \left(\frac{\partial \mathcal{A}}{\partial \sigma}\right)_{\sigma_0}\phi_{\sigma_0}h\right) \\
&= -\mathcal{A}_{\sigma_0}g_4(h) + \frac{1}{\|h\|}\left(\left(\frac{\partial \mathcal{A}}{\partial \sigma}\right)_{\sigma_0}\phi_{\sigma_0}h - (\mathcal{A}_{\sigma_0+h} - \mathcal{A}_{\sigma_0})\phi_{\sigma_0+h}\right).
\end{aligned}$$

Thus, we get the following PDE for $g_4(h)$:

$$\left(\frac{\partial}{\partial t} + \mathcal{A}_{\sigma_0}\right)g_4(h) = \frac{1}{\|h\|}\left[\left(\frac{\partial \mathcal{A}}{\partial \sigma}\right)_{\sigma_0}\phi_{\sigma_0}.h - (\mathcal{A}_{\sigma_0+h} - \mathcal{A}_{\sigma_0})\phi_{\sigma_0+h}\right] \tag{4.2.6}$$

Next, we shall expand some of the terms in the above equation. Differentiating $\mathcal{A}$ with respect to σ,

$$\begin{aligned}
\left(\frac{\partial \mathcal{A}}{\partial \sigma}\right)_{\sigma_0}\phi_{\sigma_0} &= \frac{1}{2}s^2\left(\frac{\partial}{\partial \sigma}[\mathrm{diag}(\sigma^2)]\right)_{\sigma_0}\frac{\partial^2 \phi_{\sigma_0}}{\partial s^2} \\
&= s^2\mathcal{M}(t,s),
\end{aligned}$$

where we define $\mathcal{M}(t,s)$ as a diagonal matrix $\mathcal{M}_{ii} = (\sigma_0)_i\frac{\partial^2(\phi_{\sigma_0})_i}{\partial s^2}$. Also, replacing σ in equation (4.2.2) with σ_0 gives us

$$\mathcal{A}_{\sigma_0+h} - \mathcal{A}_{\sigma_0} = \frac{1}{2}s^2\mathrm{diag}[h^2 + 2\sigma_0 h]\frac{\partial^2}{\partial s^2}.$$

Plugging in the above expressions for $\mathcal{A}_{\sigma_0+h} - \mathcal{A}_{\sigma_0}$ and $\left(\frac{\partial \mathcal{A}}{\partial \sigma}\right)_{\sigma_0}\phi_{\sigma_0}$ in equation

(4.2.6), we get

$$\begin{aligned}
\left(\frac{\partial}{\partial t}+\mathcal{A}_{\sigma_0}\right) g_4(h) &= \frac{1}{2\|h\|}s^2\left[2\mathcal{M}.h-\operatorname{diag}(h^2+2\sigma_0 h)\frac{\partial^2\phi_{\sigma_0+h}}{\partial s^2}\right]\\
&= \frac{1}{2\|h\|}s^2\left[-\operatorname{diag}(h^2)\frac{\partial^2\phi_{\sigma_0+h}}{\partial s^2}+2\left(\mathcal{M}h-\operatorname{diag}(\sigma_0 h)\frac{\partial^2\phi_{\sigma_0+h}}{\partial s^2}\right)\right]\\
&= \frac{1}{2\|h\|}s^2\left[-\operatorname{diag}(h^2)\frac{\partial^2\phi_{\sigma_0+h}}{\partial s^2}+2\left(\mathcal{M}h-\operatorname{diag}\left(\sigma_0\frac{\partial^2\phi_{\sigma_0+h}}{\partial s^2}\right)h\right)\right]\\
&= \frac{1}{2\|h\|}s^2\left[-\operatorname{diag}(h^2)\frac{\partial^2\phi_{\sigma_0+h}}{\partial s^2}+2\left(\mathcal{M}-\operatorname{diag}\left(\sigma_0\frac{\partial^2\phi_{\sigma_0+h}}{\partial s^2}\right)\right)h\right]
\end{aligned}$$

If we denote the right-hand side of the above equation as f_h, then we get an alternative representation of $g_4(h)$ in the form of the following PDE, where f_h acts as the source term:

$$\left(\frac{\partial}{\partial t}+\mathcal{A}_{\sigma_0}\right) g_4(h) = f_h, \tag{4.2.7}$$

with $g_4(h;T,s)(i)=0$ for all s and i, and $g_4(h;t,0)(i)=0$ for all i and $t\leq T$.

Now, let us look at

$$\begin{aligned}
f_h(t,s) &= \frac{1}{2\|h\|}s^2\left[-\operatorname{diag}(h^2)\frac{\partial^2\phi_{\sigma_0+h}}{\partial s^2}+2\left(\operatorname{diag}\left(\sigma_0\frac{\partial^2\phi_{\sigma_0}}{\partial s^2}\right)-\operatorname{diag}\left(\sigma_0\frac{\partial^2\phi_{\sigma_0+h}}{\partial s^2}\right)\right)h\right]\\
&= s^2\left[-\frac{1}{2\|h\|}\operatorname{diag}(h^2)\frac{\partial^2\phi_{\sigma_0+h}}{\partial s^2}+\operatorname{diag}\left(\sigma_0\frac{\partial^2\phi_{\sigma_0}}{\partial s^2}-\sigma_0\frac{\partial^2\phi_{\sigma_0+h}}{\partial s^2}\right)\frac{h}{\|h\|}\right]
\end{aligned}$$

as $h\to 0$. The first term, $\frac{-\operatorname{diag}(h^2)}{2\|h\|}$ vanishes as $h\to 0$. The second term also vanishes as $h\to 0$ since $\frac{\partial^2\phi}{\partial s^2}$ is continuous with respect to σ by Lemma 4.2.2. Therefore, $f_h\to 0$ as $h\to 0$. Since f_h has at most quadratic growth in s, it satisfies the hypothesis of Lemma 4.2.1 and therefore, we can conclude that the above PDE has a unique classical solution $g_4(h;t,s)(i)$, which has at most quadratic growth with respect to s. Therefore by Remark 3.5.5 of [29], $g_1(h;t,s)(i)$ can be explicitly specified using the Feynman-Kac formula:

$$g_4(h;t,s)(i) = \mathbf{E}\left[\int_t^T f_h(\tau,S_\tau)(X_\tau)\,d\tau\,\middle|\,S_t=s,X_t=i\right].$$

Again, $g_4(h;t,s)(i)$ satisfies

$$\|g_4(h;t,s)(i)\| \leq \mathbf{E}\left[\int_t^T |f_h(\tau,S_\tau)(X_\tau)|\, d\tau \middle| S_t = s, X_t = i\right],$$

where we know that the integrand converges to zero as $h \to 0$ and we can employ the same route, utilizing the dominated convergence theorem, as seen before to establish that $g_4 \to 0$ as $h \to 0$ for each (t,s,i). □

As seen in definition 2.3.3, the derivative of the price function with respect to the implied volatility is known as **vega**.

4.3 Obtaining Implied Volatility from Price Function

Theorem 4.3.1. *The Vega, V, is continuous with respect to Λ.*

Proof. As we saw in Theorem 4.2.3 the PDE in vega is as follows:

$$\frac{\partial V(t,s)(i)}{\partial t} + \mathcal{A}V(t,s)(i) + \frac{\partial \mathcal{A}}{\partial \sigma} = RV(t,s)(i);$$

1. $V(T,s)(i) = 0$ for all $s \geq 0$ and for all i,
2. $V(t,0)(i) = 0$ for all $t \in [0,T]$ and for all i.

Let $g_5(h;t,s)(i) := V_{\Lambda+h}(t,s)(i) - V_\Lambda(t,s)(i)$. We also note that $\frac{\partial \mathcal{A}_{\Lambda+h}}{\partial \sigma} = \frac{\partial \mathcal{A}_\Lambda}{\partial \sigma} = \frac{\partial \mathcal{A}}{\partial \sigma}$. Then we have

$$\begin{aligned}
\frac{\partial g_5(h)}{\partial t} &= \frac{\partial V_{\Lambda+h}}{\partial t} - \frac{\partial V_\Lambda}{\partial t} \\
&= RV_{\Lambda+h} - \mathcal{A}_{\Lambda+h}V_{\Lambda+h} - \frac{\partial \mathcal{A}}{\partial \sigma}\sigma_{\Lambda+h} - RV_\Lambda - \mathcal{A}_\Lambda V_\Lambda + \frac{\partial \mathcal{A}}{\partial \sigma}\sigma_\Lambda \\
&= (\mathcal{A}_\Lambda - \mathcal{A}_{\Lambda+h})V_{\Lambda+h} - \mathcal{A}_\Lambda(V_{\Lambda+h} - V_\Lambda) + R(V_{\Lambda+h} - V_\Lambda) - \frac{\partial \mathcal{A}}{\partial \sigma}(\phi_{\Lambda+h} - \phi_\Lambda).
\end{aligned}$$

The resultant PDE in $g_5(h;t,s)(i)$ is

$$\begin{aligned}\frac{\partial g_5(h;t,s)(i)}{\partial t}+\mathcal{A}_\Lambda g_5(h;t,s)(i) =& Rg_5(h;t,s)(i)-hV_{\Lambda+h}(t,s)(i)\\ &-\frac{\partial\mathcal{A}}{\partial\sigma}(\phi_{\Lambda+h}(t,s)(i)-\phi_\Lambda(t,s)(i))\end{aligned}$$

As seen in Theorem 4.2.3, $\frac{\partial\mathcal{A}}{\partial\sigma}\phi(t,s)(i)$ is a diagonal matrix with $\left(\frac{\partial\mathcal{A}}{\partial\sigma}\phi(t,s)(i)\right)_{ii}=s^2\sigma_i\frac{\partial^2\phi(t,s)(i)}{\partial s^2}$. Therefore the above PDE becomes

$$\begin{aligned}\frac{\partial g_5(h;t,s)(i)}{\partial t}+\mathcal{A}_\Lambda g_5(h;t,s)(i) =& Rg_5(h;t,s)(i)-hV_{\Lambda+h}(t,s)(i)\\ &-s^2\sigma_i\left(\frac{\partial^2\phi_{\Lambda+h}(t,s)(i)}{\partial s^2}-\frac{\partial\phi_\Lambda(t,s)(i)}{\partial s^2}\right);\end{aligned}$$

1. $g_5(h;T,s)(i)=0$ for all $s\geq 0$ and for all i,
2. $g_5(h;t,0)(i)=0$ for all $t\in[0,T]$ and for all i.

The RHS has at most quadratic growth in s, therefore we can use Lemma 4.2.1 and Remark 3.5.5 in [29] to write $g_5(h;t,s)(i)$ explicitly as

$$\begin{aligned}g_3(h;t,s)(i)=\mathbf{E}\Bigg[\int_t^T e^{-\int_t^\tau r(X_u)\,du}\Big\{&-h(X_\tau)V_{\Lambda+h}(\tau,S_\tau)(X_\tau)-s^2\sigma(X_\tau)\\ \times\left(\frac{\partial^2\phi_{\Lambda+h}(\tau,S_\tau)(X_\tau)}{\partial s^2}-\frac{\partial\phi_\Lambda(\tau,S_\tau)(X_\tau)}{\partial s^2}\right)\Big\}\,d\tau\Bigg|S_t=s,X_t=i\Bigg],\end{aligned}$$

where $S_0>0$, X_t is a Markov chain with rate matrix Λ and S_t satisfies the SDE

$$dS_t=S_t[r(X_t)dt+\sigma(X_t)dW_t].$$

The proof follows from the same argument as at the end of Theorem 4.2.3. □

When $\Lambda=0$, there are no transitions between the regimes, i.e., $X_t=X_0$ for all t. The stock therefore follows the SDE:

$$dS_t=S_t(r(X_0)dt+\sigma(X_0)dW_t),$$

which is just the geometric Brownian motion followed by the stock price in the Black-Scholes model, since $r(X_0)$ and σ_{X_0} are just constants. In fact, if we plug in $\Lambda=0$

in equation (3.2.1), we just get k different decoupled equations for the Black-Scholes model (compare equations (3.2.1) and (2.3.1)). It is known that in the Black-Scholes model, the partial derivative of the solution with respect to the scalar volatitlity parameter is positive. Furthermore due to the decoupling, for $\Lambda = 0$, $\frac{\partial \phi_i}{\partial \sigma_j} = 0 \; \forall i \neq j$, and thus the vega matrix is just a diagonal matrix with positive entries and therefore non-singular.

Let $\mathbb{N}_\delta$ be a small open neighborhood in "vega-space" around $\left(\frac{\partial \phi}{\partial \sigma}\right)_{\Lambda=0}$, so that all $V \in \mathbb{N}_\delta$ are also non-singular. Due to the continuity of the $\Lambda \mapsto V$ map (Theorem 4.3.1), the pre-image of $\mathbb{N}_\delta$ under this map, say $\mathbb{U}$, is also an open set in Λ-space that contains $\Lambda = 0$, which is the inverse image of $\left(\frac{\partial \phi}{\partial \sigma}\right)_{\Lambda=0}$.

Hence, there exists an open neighborhood $\mathbb{U}$ around $\Lambda = 0$ such that for all $\Lambda \in \mathbb{U}$, the corresponding vega $\left(\frac{\partial \phi}{\partial \sigma}\right)_{\Lambda}$ is non-singular and therefore invertible. We already established in Theorem 4.2.2 that the map $\sigma \mapsto \phi$ is continuous; therefore a sufficient condition for its invertibility at some point $\sigma = \sigma_0$ is for the Jacobian matrix $\left(\frac{\partial \phi}{\partial \sigma}\right)_{\sigma_0}$ to be invertible, which is exactly the case for all $\Lambda \in \mathbb{U}$.

We can thus say that for sufficiently small Λ, we can invert the map $\sigma \mapsto \phi$ and obtain the implied volatility if the price function is given. We summarize this in the following theorem:

Theorem 4.3.2. *There exists a $\delta > 0$ such that if $\Lambda \in B_\delta(0) \subset \mathbb{R}^{k\times k}$, where Λ is a rate matrix, then the notion of implied volatility is well-defined for a market model (as described in section 3.2) characterized by such values of Λ.*

Chapter 5

Numerical Experiment

5.1 Overview

We wish to conduct a numerical experiment to see if the implied volatility, as proposed in Chapter 4, can be computed for typical values of market parameters with some standard numerical techniques with reasonable efficiency. We consider a typical initial value of σ (say σ_0) and numerically compute the option price ϕ using some results given in [16]. Then we assume that option price is an observed quantity and work backwards to get the implied volatility, as described in the definition of implied volatility in Chapter 2.

5.2 Theory and Numerical Scheme

We consider a typical set of values for σ $(= \sigma_0)$, rate matrix Λ, time to maturity T, strike price K and the transition probablility matrix P. The first part of the program involves direct computation of the Black-Scholes price η from the given quantities, which is then plugged into the following integral equation (refer to Theorem 2.1 and equations (4), (5) in [16]) to numerically compute the option price:

$$
\begin{aligned}
\phi(t,s,i) = e^{-\lambda_i(T-t)}\eta_i(t,s) + \int_0^{T-t} & \lambda_i e^{-(\lambda_i + r(i))v} \\
\times \sum_j p_{ij} \int_0^\infty \phi(t+v,x,j) & \frac{e^{-\frac{1}{2}\left(\left(\ln(\frac{x}{s}) - \left(r(i) - \frac{\sigma^2(i)}{2}\right)v\right)\frac{1}{\sigma(i)\sqrt{v}}\right)^2}}{\sqrt{2\pi}\sigma(i)\sqrt{v}x}\, dx\, dv;
\end{aligned}
\tag{5.2.1}
$$

1. $\phi(T,s,i) = (s-K)^+$ for all s and i,
2. $\phi(t,0,i) = 0$ for all $t \in [0,T]$ and for all i,

where $\eta_i(t,s)$ is the Black-Scholes price for a call option with interest rate $r(i)$ and volatility $\sigma(i)$, and $\lambda_i := -[\Lambda]_{ii}$.

To numerically perform the integrations in the above formula, we must first compute the multiplier of ϕ in the integrand for integration with respect to x above. Then we employ the step-by-step quadrature method over the variables x and v (refer to Section 4 in [16]). Suppose we obtain the price function $C = \phi(0,s,i)$ as a result of these computations; we can now pretend that this is the **observed** option price at time zero. We define the following vector function in σ:

$$f(\sigma) := \phi_\sigma(0,s,i) - C = 0,$$

and we want to solve $f(\sigma) = 0$ in σ, i.e., to numerically determine the zero of $f(\sigma)$.

Since this is a vector-valued equation, we use a generalization of the Newton-Raphson method (see [6]), where we start with a guess σ_1 for a zero of $f(\sigma)$, and a better approximation for the actual zero is given iteratively by:

$$\sigma_{n+1} = \sigma_n - V^{-1}f,$$

where V is the Jacobian matrix $\frac{\partial f}{\partial \sigma} = \frac{\partial(\phi - C)}{\partial \sigma} = \frac{\partial \phi}{\partial \sigma}$ evaluated at $\sigma = \sigma_n$.

After several iterations, we expect σ_n to converge to the "actual value" σ_0 that we started with. If that happens, we would have obtained the implied volatility for a typical market using standard numerical techniques.

5.3 Numerical Results

We consider a two-regime market, which means that it is modulated by a two-state Markov chain. The variables t and s are discretized, with the number of discrete steps being 16 and 100 respectively. The following values for the various given quantities are considered:

1. $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, the transition probability matrix for the two-state Markov chain,
2. Time to maturity $T = 1$,
3. Strike price $K = 1$,
4. Rate of interest for bond $r = [0.3, 0.3]$,
5. Starting guess for σ, $\sigma_1 = [0.01, 0.01]$.

Also, we run different instances of the program with different sets of values of σ_0 and Λ in each instance. The different combinations of values for σ_0 and Λ are as follows:

1. $\Lambda = [0.1, 0.2]$, $\sigma_0 = [0.3, 0.4]$
2. $\Lambda = [0.1, 0.2]$, $\sigma_0 = [0.4, 0.3]$
3. $\Lambda = [0.1, 0.2]$, $\sigma_0 = [0.4, 0.5]$
4. $\Lambda = [0.1, 0.2]$, $\sigma_0 = [0.5, 0.3]$
5. $\Lambda = [0.2, 0.1]$, $\sigma_0 = [0.3, 0.4]$
6. $\Lambda = [0.2, 0.1]$, $\sigma_0 = [0.4, 0.3]$
7. $\Lambda = [0.2, 0.1]$, $\sigma_0 = [0.5, 0.4]$
8. $\Lambda = [0.1, 0.2]$, $\sigma_0 = [0.2, 0.4]$

Table 5.1 lists the values of σ_0 and Λ for which our guess eventually converges to the actual value of volatility, the numerically-obtained option price (the "observed" price C) and the number of steps taken for it to converge (error less than 10^{-3}). The convergence of σ vector is also shown graphically for some of these cases.

Table 5.1: Numerical Experiment Results

Λ	σ_0	Observed Price C	Number of Steps for Convergence
$[0.1, 0.2]$	$[0.3, 0.4]$	$[0.134, 0.167]$	12
$[0.1, 0.2]$	$[0.4, 0.3]$	$[0.169, 0.135]$	7
$[0.1, 0.2]$	$[0.4, 0.5]$	$[0.172, 0.205]$	4
$[0.1, 0.2]$	$[0.5, 0.3]$	$[0.206, 0.139]$	15
$[0.2, 0.1]$	$[0.3, 0.4]$	$[0.135, 0.169]$	7
$[0.2, 0.1]$	$[0.4, 0.3]$	$[0.167, 0.134]$	12
$[0.2, 0.1]$	$[0.5, 0.4]$	$[0.205, 0.172]$	4
$[0.1, 0.2]$	$[0.2, 0.4]$	$[0.098, 0.165]$	6

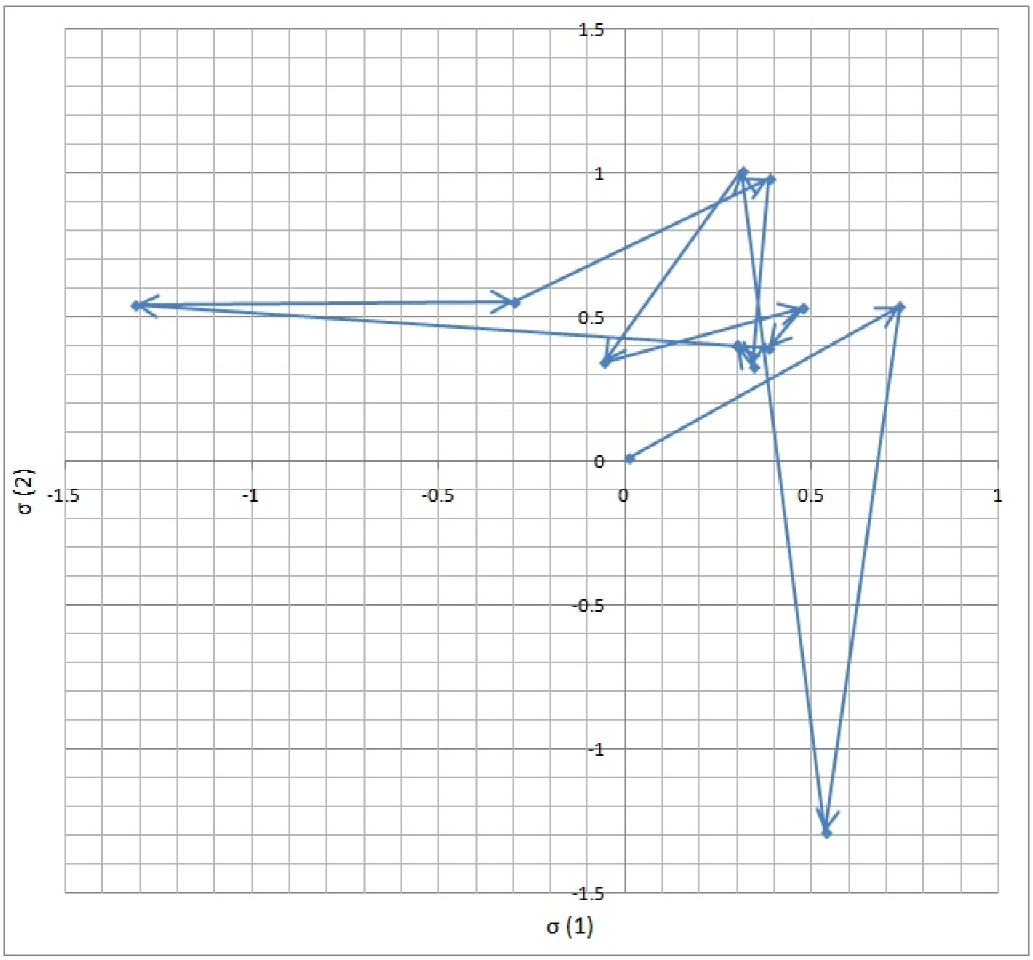

Figure 5.1: $\Lambda = [0.1, 0.2]$; $\sigma_0 = [0.3, 0.4]$

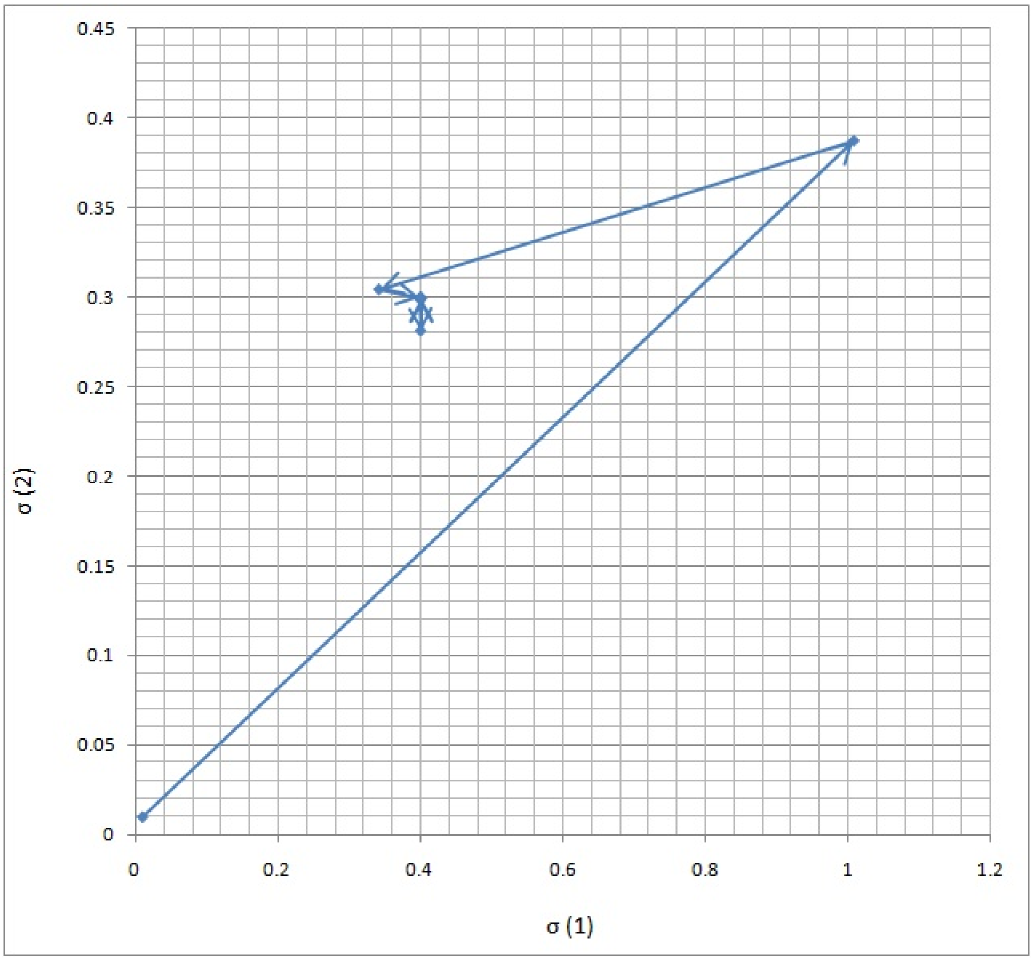

Figure 5.2: $\Lambda = [0.1, 0.2]$; $\sigma_0 = [0.4, 0.3]$

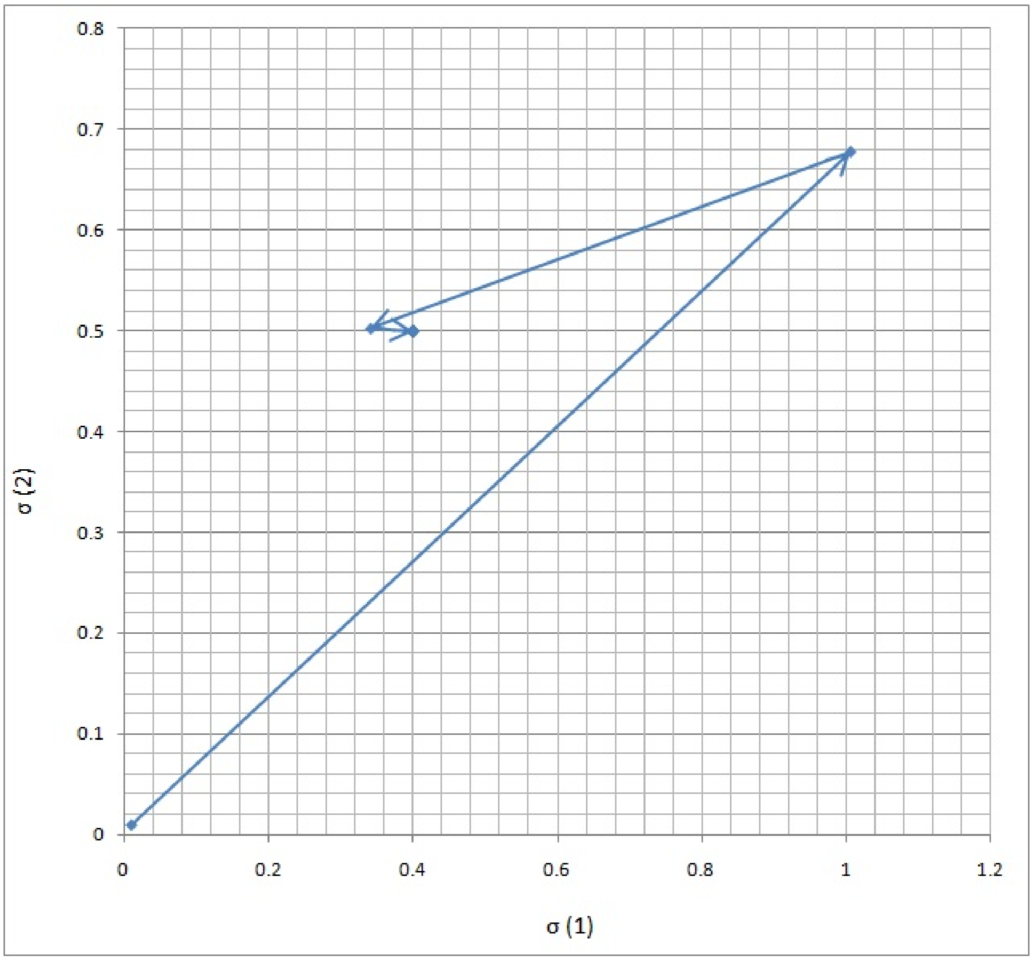

Figure 5.3: $\Lambda = [0.1, 0.2]$; $\sigma_0 = [0.4, 0.5]$

Figure 5.4: $\Lambda = [0.1, 0.2]$; $\sigma_0 = [0.5, 0.3]$

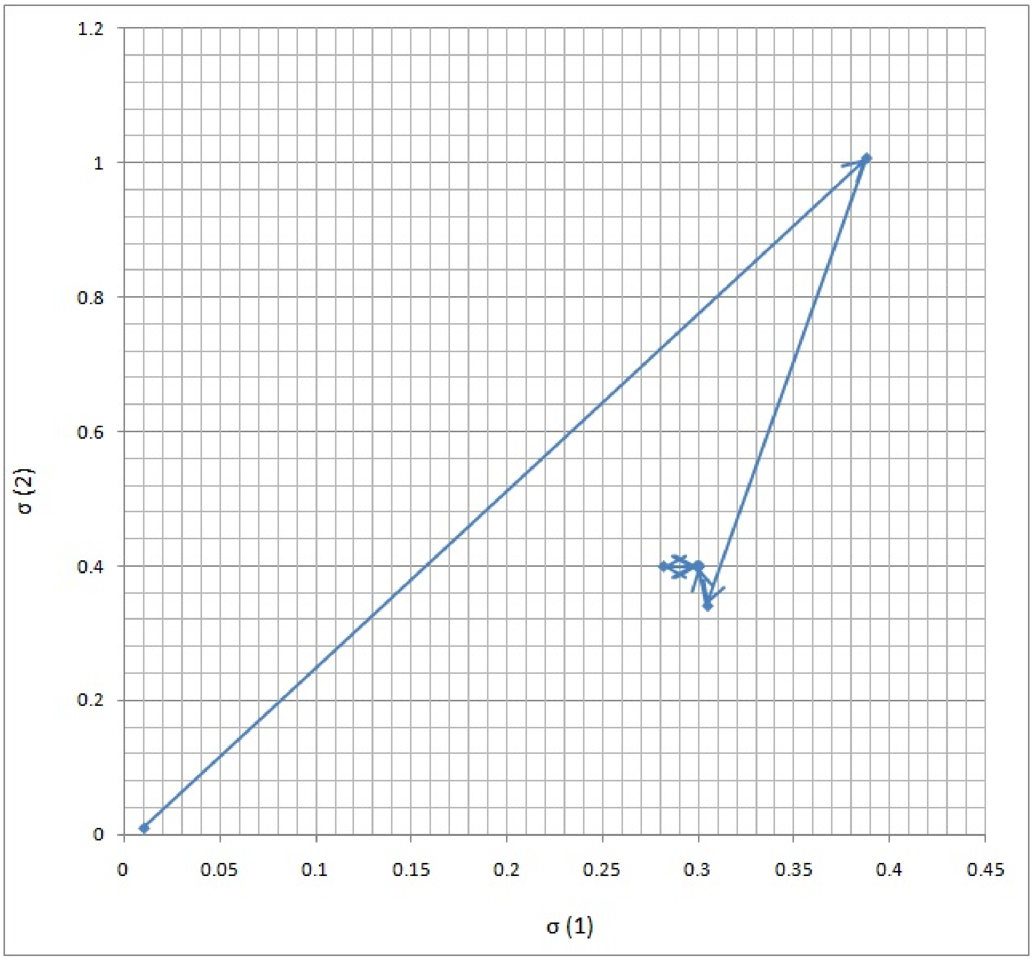

Figure 5.5: $\Lambda = [0.2, 0.1]$; $\sigma_0 = [0.3, 0.4]$

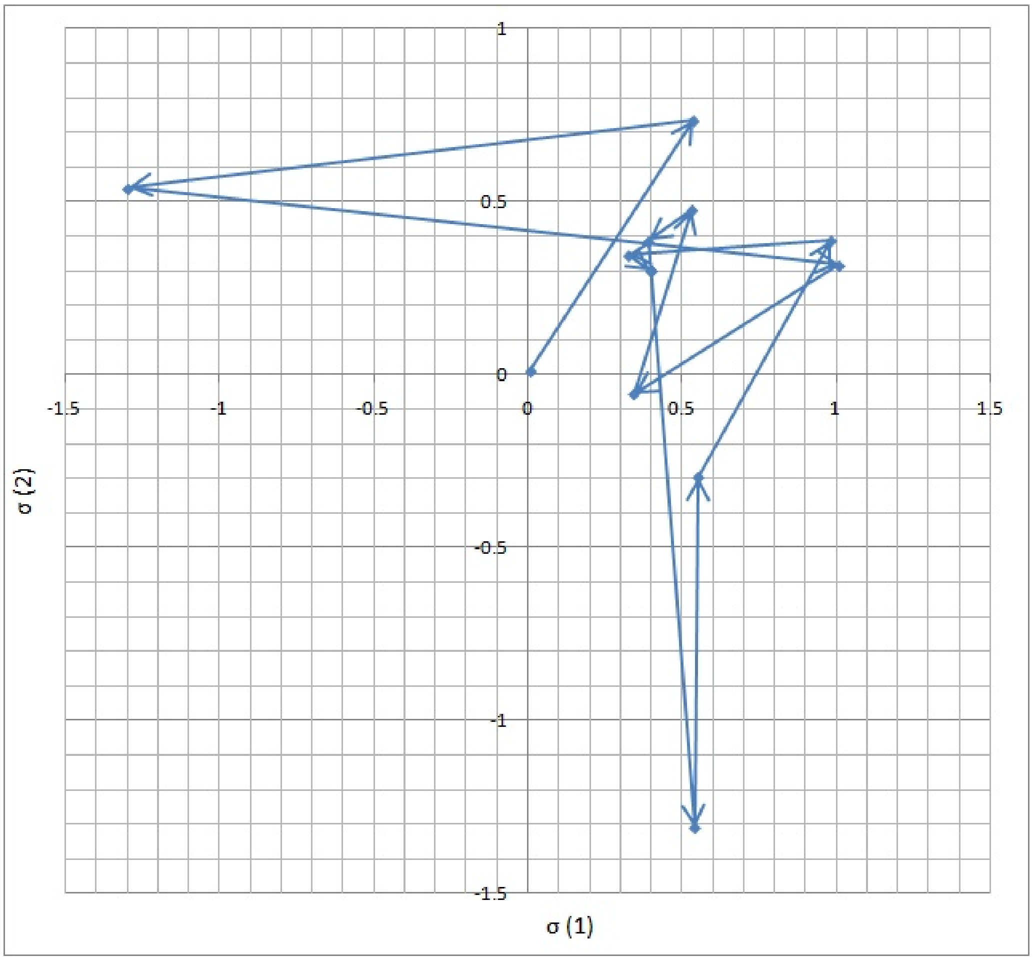

Figure 5.6: $\Lambda = [0.2, 0.1]$; $\sigma_0 = [0.4, 0.3]$

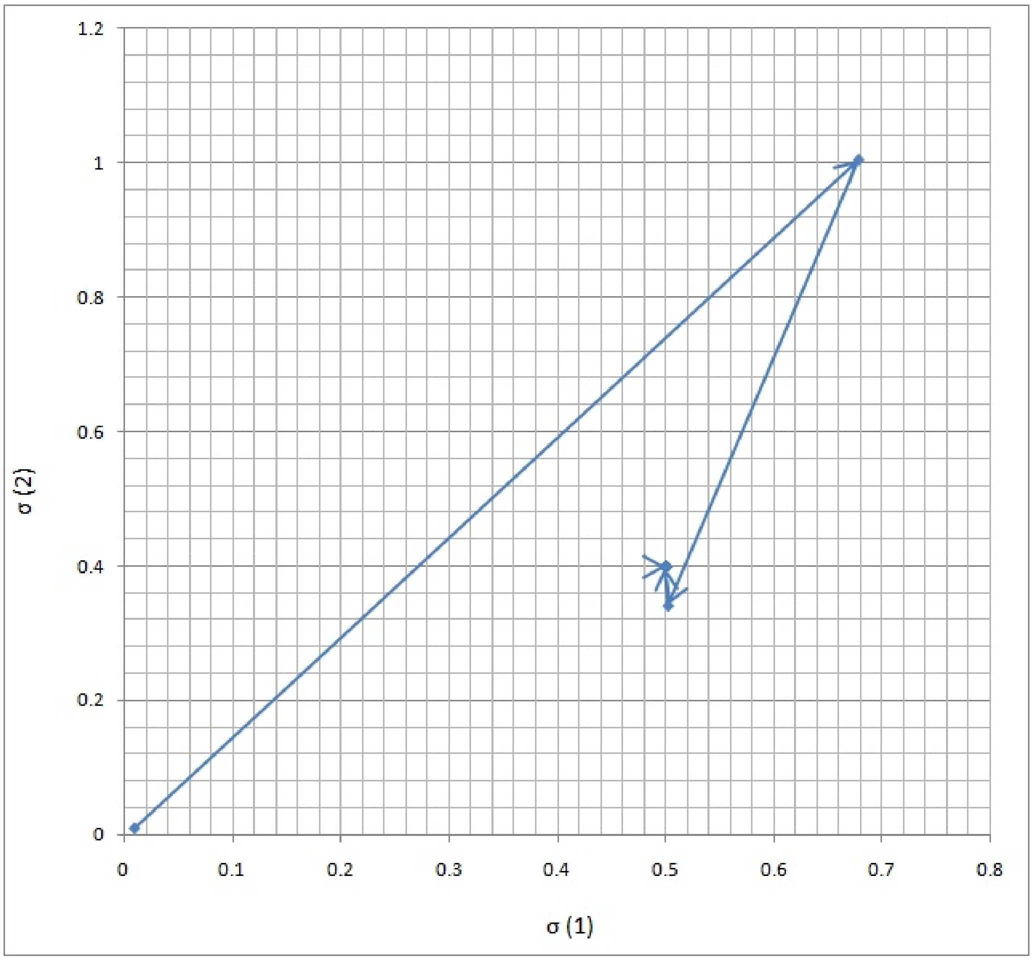

Figure 5.7: $\Lambda = [0.2, 0.1]$; $\sigma_0 = [0.5, 0.4]$

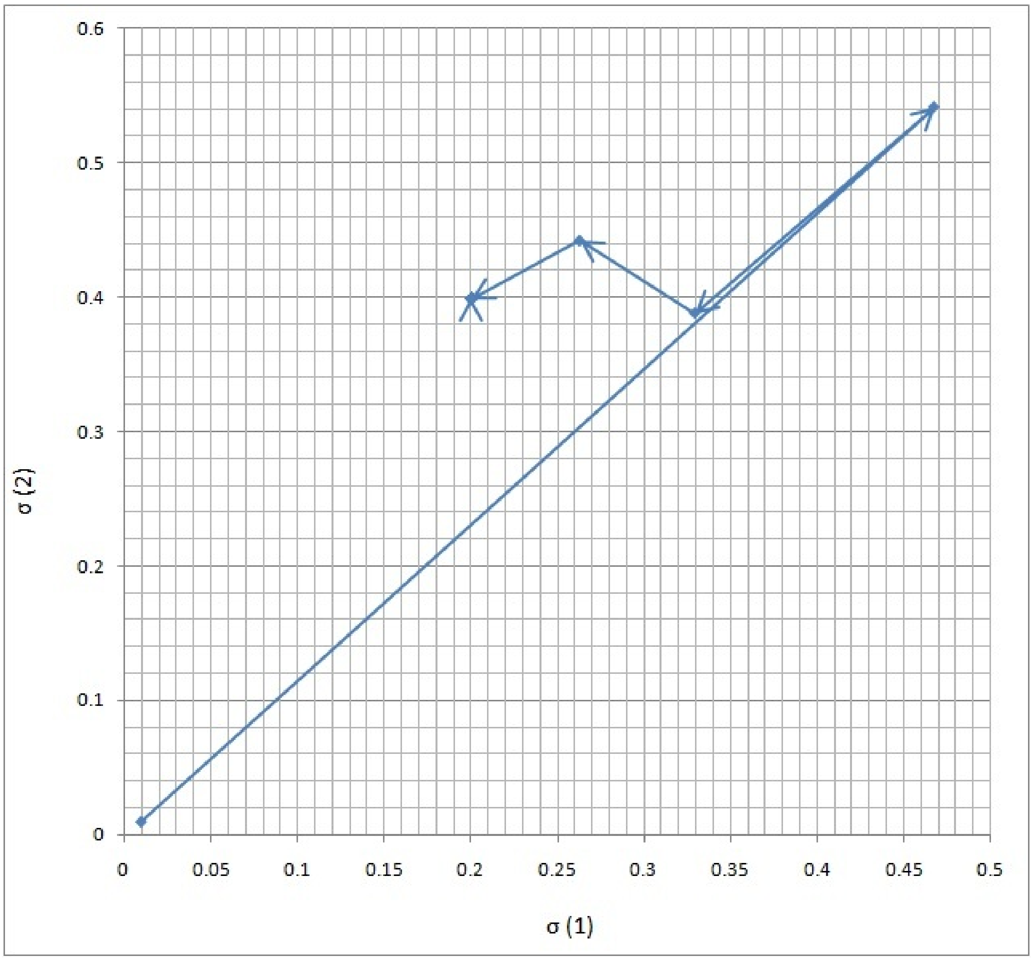

Figure 5.8: $\Lambda = [0.1, 0.2]$; $\sigma_0 = [0.2, 0.4]$

Appendix A

MATLAB Source Code

The MATLAB source code used to implement the program in Chapter 5 is as follows:

```
clear
tic;
t_steps=16; s_steps=100;
BSprice=zeros(t_steps,s_steps,2);
price=zeros(t_steps,s_steps,2,3);
C1=zeros(t_steps,2);
C2=zeros(1,2);
C3=zeros(t_steps,2);
G=zeros(t_steps,s_steps,s_steps,2);
x_int=zeros(t_steps,s_steps,2); %Integrand w.r.t. x
V=zeros(20,20);

T=1; %Time to maturity
dt=T/(t_steps-1); %Time step-size
K=1.0; %Strike price
X0=0;
k1=20;
dx=(5*K)/s_steps;
P= [0,1;1,0]; %Transition probability matrix
lambda=[.1,.2]; %Lambda
```

```
R=[0.03,0.03]; %Interest rate
sigma0= [0.2,0.4]; %"Correct" value of volatility
for j=1:s_steps
    for i=1:2
        for p=1:3
            price(1,j,i,p)=max(0.0,dx*(j-1)-K);
        end
    end
end

SND= 0.5+ 0.5*erf((1/sqrt(2))*(-4+0.001*(1:4000)));

p=1; sigma=sigma0;
%Calculation of Black-Scholes price; numerical solution of Volterra
%integral equation
for k=1:2
    rp=R(k)+(1.0/2)*sigma(k)^2;
    rm=R(k)-(1.0/2)*sigma(k)^2;
    for i=2:t_steps
        tm=(i-1)*dt; %tm: Time to expiry := T-t
        dn= sigma(k)*sqrt(tm);
        for j=1:s_steps
            s=j*dx;
            x= (log(s/K)+rm*tm)/dn;
            if x > 4.0
                ph= 1.0;
            elseif x<-4.0
                ph= 0.0;
            elseif x<0.0
                x2=floor(1000*(4.0+x))+1;
                ph= SND(x2);
            elseif x>0.0
                x2=floor(1000.0*(4.0 - x))+1;
                ph= 1 - SND(x2);
```

```
            end
            x3= K*exp(-R(k)*tm)*ph;
            y= (log(s/K)+rp*tm)/dn;
            if y > 4.0
                ph= 1.0;
            elseif y<-4.0
                ph= 0.0;
            elseif y<0.0
                x2=floor(1000*(4.0+y))+1;
                ph= SND(x2);
            elseif y>0.0
                x2=floor(1000.0*(4.0 - y))+1;
                ph= 1 - SND(x2);
            end
            BSprice(i,j,k)= s*ph - x3;
        end
    end
end

for i=1:2
    C2(i)= (1.0/(sqrt(2*pi)))/sigma(i);
    for kk=2:t_steps
        C1(kk,i) = lambda(i)*(exp(-(R(i)+lambda(i))*((kk-1)*dt)) /
            sqrt((kk-1)*dt));
        C3(kk,i) = exp(-lambda(i)*(kk-1)*dt);
        for j=1:s_steps
            for jj=1:s_steps
                G(kk,j,jj,i)=exp(-0.5*((
                    log(jj/j)-(R(i)-0.5*sigma(i)^2)*((kk-1)*dt))/
                                   (sigma(i)*sqrt((kk-1)*dt) ))^2);
            end
        end
    end
end
```

```
%Calculation of price function for all t, s, i
for k=2:t_steps
    for j=1:s_steps
        for i=1:2
            v_int=0;
            for kk=2:k
                jj=1;
                %------------- Integration w.r.t. x ---------------
                x3 =0;
                for ii=1:2
                    x3=x3 + price(k-kk+1,jj,ii,p)* P( i, ii);
                end
                x_int(kk,j,i) = 0.5 * x3 * G(kk,j,jj,i)/jj;
                for jj=2:s_steps-1
                    x3 =0;
                    for ii=1:2
                        x3=x3 + price(k-kk+1,jj,ii,p)* P( i, ii);
                    end
                    x_int(kk,j,i) = x_int(kk,j,i)+x3*G(kk,j,jj,i)/jj;
                end
                jj=s_steps;
                x3 =0;
                for ii=1:2
                    x3=x3 + price(k-kk+1,jj,ii,p)* P( i, ii);
                end
                x_int(kk,j,i) = x_int(kk,j,i)+0.5* x3*G(kk,j,jj,i)/jj;
                %------------------------------------------------
                v_int = v_int + x_int(kk,j,i) * C1(kk,i) * dt;
            end
            price(k,j,i,p)=C3(k,i)*BSprice(k,j,i) + v_int * C2(i);
        end
    end
end
```

```
c=[price(t_steps,k1,1,1), price(t_steps,k1,2,1)]; %Observed price
   established

sigma1= [0.01,0.01]; %Starting guess for volatility
h=.001; %Small error in either component of volatility
for l=1:15 %15 iterations in the Newton-Raphson method
   for p=1:3
      if p==1
         sigma=sigma1;
      elseif p==2
         sigma=sigma1+h*[1,0];
      else
         sigma=sigma1+h*[0,1];
      end

      for k=1:2
         rp=R(k)+(1.0/2)*sigma(k)^2;
         rm=R(k)-(1.0/2)*sigma(k)^2;
         for i=2:t_steps
            tm=(i-1)*dt; %tm is time to expiry:=T-t
            dn= sigma(k)*sqrt(tm);
            for j=1:s_steps
               s=j*dx;
               x= (log(s/K)+rm*tm)/dn;
               if x > 4.0
                  ph= 1.0;
               elseif x<-4.0
                  ph= 0.0;
               elseif x<0.0
                  x2=floor(1000*(4.0+x))+1;
                  ph= SND(x2);
               elseif x>0.0
                  x2=floor(1000.0*(4.0 - x))+1;
```

```
                ph= 1 - SND(x2);
            end
            x3= K*exp(-R(k)*tm)*ph;
            y= (log(s/K)+rp*tm)/dn;
            if y > 4.0
                ph= 1.0;
            elseif y<-4.0
                ph= 0.0;
            elseif y<0.0
                x2=floor(1000*(4.0+y))+1;
                ph= SND(x2);
            elseif y>0.0
                x2=floor(1000.0*(4.0 - y))+1;
                ph= 1 - SND(x2);
            end
            BSprice(i,j,k)= s*ph - x3;
        end
    end
end

for i=1:2
    C2(i)= (1.0/(sqrt(2*pi)))/sigma(i);
    for kk=2:t_steps
        C1(kk,i) =
           lambda(i)*(exp(-(R(i)+lambda(i))*((kk-1)*dt)) /
           sqrt((kk-1)*dt));
        C3(kk,i) = exp(-lambda(i)*(kk-1)*dt);
        for j=1:s_steps
            for jj=1:s_steps
                G(kk,j,jj,i)=exp(-0.5*((
                    log(jj/j)-(R(i)-0.5*sigma(i)^2)*((kk-1)*dt))/
                                  (sigma(i)*sqrt((kk-1)*dt) ))^2);
            end
        end
```

```
        end
    end

    for k=2:t_steps
        for j=1:s_steps
            for i=1:2
                v_int=0;
                for kk=2:k
                    jj=1;
                    x3 =0;
                    for ii=1:2
                        x3=x3 + price(k-kk+1,jj,ii,p)* P( i, ii);
                    end
                    x_int(kk,j,i) = 0.5 * x3 * G(kk,j,jj,i)/jj;
                    for jj=2:s_steps-1
                        x3 =0;
                        for ii=1:2
                            x3=x3 + price(k-kk+1,jj,ii,p)* P( i, ii);
                        end
                        x_int(kk,j,i) =
                            x_int(kk,j,i)+x3*G(kk,j,jj,i)/jj;
                    end
                    jj=s_steps;
                    x3 =0;
                    for ii=1:2
                        x3=x3 + price(k-kk+1,jj,ii,p)* P( i, ii);
                    end
                    x_int(kk,j,i) = x_int(kk,j,i)+0.5*
                        x3*G(kk,j,jj,i)/jj;
                    v_int = v_int + x_int(kk,j,i) * C1(kk,i) * dt;
                end
                price(k,j,i,p)=C3(k,i)*BSprice(k,j,i) + v_int *
                    C2(i);
            end
```

```
            end
        end
    end
    V11=(price(t_steps,k1,1,2)-price(t_steps,k1,1,1))/h;
    V12=(price(t_steps,k1,1,3)-price(t_steps,k1,1,1))/h;
    V21=(price(t_steps,k1,2,2)-price(t_steps,k1,2,1))/h;
    V22=(price(t_steps,k1,2,3)-price(t_steps,k1,2,1))/h;
    V=[V11,V12;V21,V22]; %Jacobian matrix, effectively vega
    f=[price(t_steps,k1,1,1), price(t_steps,k1,2,1)] - c;
    hh=(inv(V)*f')'; % One iteration step of
    error(l,:)=hh; % the Newton-Raphson
    sigma1 = sigma1 - hh; % method
    sigma_estimate(l,:)= sigma1;
end
xlswrite('result.xls',[P; lambda; R; sigma0; c; sigma_estimate;
   error])
```

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