Perturbative Conformal Field Theory in the Mellin space

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

Sourav Sarkar

under the guidance of

Prof Rajesh Gopakumar

Harish Chandra Research Institute, Allahabad

Indian Institute of Science Education and Research
Pune

Certificate

This is to certify that this thesis entitled ”Perturbative Conformal Field Theory in the Mellin space” submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by Sourav Sarkar at Harish Chandra Research Institute, Allahabad, under the supervision of Prof Rajesh Gopakumar during the academic year 2014-2015.

Student
Sourav Sarkar

Supervisor
Rajesh Gopakumar

Declaration

I hereby declare that the matter embodied in the report entitled "Perturbative Conformal Field Theory in the Mellin space" are the results of the investigations carried out by me at the Department of Physics, Harish Chandra Research Institute, Allahabad, under the supervision of Prof Rajesh Gopakumar and the same has not been submitted elsewhere for any other degree.

Student
SOURAV SARKAR

Supervisor
RAJESH
GOPAKUMAR

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Abstract

The aim of this project is to study the Mellin representation of correlation functions in perturbative scalar conformal field theory and investigate the existence of Feynman rules that can be associated with perturbative diagrams. It is known that the Mellin representation of correlation functions in CFTs makes the covariance with all conformal symmetries manifest. The constraints on the Mellin variables that make the covariance of the amplitude with special conformal symmetry manifest, can be interpreted as the conservation of 'Mellin momentum'. The poles of a propagator in the complex Mellin momentum plane correspond to the exchanged primary operator and its descendants. Thus the Mellin space furnishes a spectral representation of n-point functions in conformal field theories. In this project, we have been able to derive the Mellin amplitude for an arbitrary tree level diagram and have found that we can associate a set of Feynman rules to these diagrams. We have been investigating the existence of similar rules for one-loop diagrams. Preliminary investigations indicate that the Mellin amplitude for such diagrams can be expressed as Mellin Barnes integrals analogous to loop integrals in momentum space. However, we have not been able to establish these results yet. We expect that this formulation of perturbative CFT in the Mellin space can be employed in deriving a natural dual wordsheet description of string theory in Anti de Sitter space from a conformal field theory on its boundary in the large N limit.

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Chapter 1

Introduction

The AdS/CFT correspondence is an explicit realisation of the idea of holographic duality which originates from string theory. The notion of holographic duality is motivated in part by a quest for a quantum theory of gravity and in part by a study of the large N limit of gauge field theories. Despite having an explicit example of a gauge string duality with strong evidence in favour of it, we lack a concrete understanding of how such a duality arises.

Open-closed string duality is considered to be the mechanism behind these dualities. R. Gopakumar had attempted to proceed towards proving the AdS/CFT correspondence by implementing the open-closed string duality in the large N limit of the conformal gauge field theory [1][2][3]. The general scheme in [1][2] was to take planar and higher genus diagrams for the field theory, express the amplitude in the form of a (momentum space) Schwinger parameter integral, and then glue up the holes in the planar diagram via a change of variables on the Schwinger parameter space integral resulting in an amplitude in AdS in one higher dimension.

The long term goal at which this project is aimed at is to implement a similar procedure using a Mellin worldline formalism for the field theory correlation functions. Since the Mellin representation makes the covariance of the CFT correlation functions with all conformal symmetries manifest [4][5], it is expected that the AdS symmetries in the dual worldsheet description will also be manifest. This makes the Mellin space a natural choice of representation to implement the open closed string duality. The Mellin representation is a natural choice also because it provides a spectral representation for the correlation functions in a CFT [4][5].

In this project, we have built upon the work done by Mack on CFT in the Mellin space [4][5]. For quantum field theories, the momentum space representation provides a simple set of Feynman rules and the Källeń Lehmann spectral representation. For CFTs neither the position nor the momentum space are very suitable in this regard. In any interacting CFT, an operator product expansion (OPE) involves only a discrete set of operators (the exchanged primary operator and its descendants). So it would be nice to have a representation in which this discrete spectrum is manifest. It had been proposed by Mack (based on properties of conformal field theories and operator product expansions) [4] that the Mellin space can provide a representation for correlation functions in CFTs with the above desired properties, giving evidence from the factorisation of amplitudes. Furthermore, Mack pointed out that there is a correspondence between CFTs and dual resonance models [5] (which were forerunners to string theory) the Mellin amplitudes in the CFTs being analogous to the scattering amplitudes in the dual resonance models.

Mack's work was followed up by Penedones [6].In [7], Penedones et al showed that the Mellin space is a suitable choice to write correlation functions of conformal field theories (CFT) with weakly coupled dual theories in the bulk. This work leads to the speculation that it might be possible to develop a formalism for perturbative CFT (that is in the weak coupling limit, without any reference to the bulk dual) in the Mellin space.

In this project, we have investigated whether we can formulate Feynman rules for perturbative CFT in the Mellin space. This formulation is a prerequisite for the implementation of our long term goal.

This report is organised into different sections in the following manner. In Chapters 2 and 3, we acquaint ourselves with Mellin transforms and the general mathematical environment in which this problem is set. In Chapters 4 and 5 respectively, we shall derive the Mellin amplitude of a one vertex interaction and a two vertex interaction using tricks and techniques developed by Symanzik [8] (streamlined later by Davydychev [9] and Paulos et al [10] [11]). In Chapter 6 we shall derive a set of Mellin space Feynman rules for a general tree level diagram in an interacting CFT. Finally, in Chapter 7 we shall look at a class of relations between loops and trees and a promising approach to one loop diagrams.

Chapter 2

The Mellin Transform

This chapter is devoted to getting acquainted with the Mellin transform and some of its properties.

2.1 Mellin transformation and its properties

The Mellin transform of a function $f(x)$ of a real variable is defined by,

$$\mathcal{M}\{f(x)\} \equiv F(s) = \int_0^\infty x^{s-1} f(x)\, dx \tag{2.1}$$

s is a complex number. It is easy to note from the definition of the Mellin transform that it is closely related to the Laplace transform and the Fourier transform. A very familiar example of a Mellin transform is the transform of e^{-x}, which is the Gamma function.

$$\int_0^\infty x^{s-1} e^{-x} dx = \Gamma(s) \tag{2.2}$$

The Gamma function is meromorphic on the complex plane with simple poles at 0 and all the negative integers.

The inverse Mellin transform is given by,

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, x^{-s} ds \tag{2.3}$$

c is a real number greater than 0.

Some simple properties of the Mellin transform are,

$$\mathcal{M}\{f(ax)\}(s) = a^{-s}F(s) \tag{2.4}$$
$$\mathcal{M}\{x^a f(x)\}(s) = F(s+a) \tag{2.5}$$
$$\mathcal{M}\{f(x^a)\}(s) = |a|^{-1}F(s/a) \qquad a \neq 0 \tag{2.6}$$
$$\mathcal{M}\{\log x^n f(x)\}(s) = F^{(n)}(s) \tag{2.7}$$
$$\mathcal{M}\{f^{(n)}(x)\}(s) = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} F(s-n) \tag{2.8}$$

Now we look at a convolution property of the Mellin transform. Let

$$F(s) = \int_0^\infty x^{s-1} f(x) dx$$
$$G(s) = \int_0^\infty x^{s-1} g(x) dx$$

Now using the definition of mellin transformation

$$\begin{aligned}
\int_0^\infty x^{s-1} f(x) g(x) dx &= \frac{1}{2\pi i}\int_0^\infty x^{s-1}\int_{c-i\infty}^{c+i\infty} F(t) x^{-t} dt g(x) dx \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(t) \int_0^\infty x^{s-t-1} g(x) dx dt \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(t) G(s-t) dt
\end{aligned} \tag{2.9}$$

For the special case of $s = 1$

$$\int_0^\infty f(x) g(x)\, dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(t) G(1-t) dt \tag{2.10}$$

(2.10) is referred to as the Parseval's formula.

2.2 Mellin-Barnes integrals

We shall look at a class of integrals called Mellin-Barnes integrals which shall be occuring quite often in this formalism. A general Mellin-Barnes integral over one variable is of the form,

$$\int_{c-i\infty}^{c+i\infty} \frac{\prod_i \Gamma(t-a_i) \prod_j \Gamma(b_j - t)}{\prod_k \Gamma(t-c_k) \prod_l \Gamma(d_l - t)} x^{-t} dt \tag{2.11}$$

a_i,b_j,c_k,d_l are complex numbers.

The reciprocal Gamma function does not have any poles and hence all the poles in the integrand of (2.11) come from the Gamma functions in the numerator. The real number c should be chosen such that the contour separates the poles of the $\prod_i \Gamma(t-a_i)$ from the poles of the $\prod_j \Gamma(b_j-t)$. If there exists no such contour that is parallel to the imaginary axis, the contour can be bent without passing any poles so that this separation of poles can be achieved.

The inverse of the Mellin transform is a special case of this Mellin-Barnes integral with only Gamma function.

Two special Mellin-Barnes integrals are particularly useful at times. These go by the name of Barnes lemmas.

The First Barnes Lemma is,

$$\int_{c-i\infty}^{c+i\infty} dv\beta\left(v+a,-v+b\right)\beta\left(-v+c,v+d\right)=\beta\left(a+c,b+d\right) \tag{2.12}$$

The Second Barnes Lemma is,

$$\begin{aligned} & \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz\frac{\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3+z)\Gamma(\lambda_4-z)\Gamma(\lambda_5-z)}{\Gamma(\lambda_6+z)} \\ = & \frac{\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_4)\Gamma(\lambda_3+\lambda_4)\Gamma(\lambda_1+\lambda_5)\Gamma(\lambda_2+\lambda_5)\Gamma(\lambda_3+\lambda_5)}{\Gamma(\lambda_1+\lambda_2+\lambda_4+\lambda_5)\Gamma(\lambda_1+\lambda_3+\lambda_4+\lambda_5)\Gamma(\lambda_3+\lambda_2+\lambda_4+\lambda_5)} \end{aligned} \tag{2.13}$$

with the constraint, $\lambda_6=\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5$

2.3 Mellin space delta function

In this section, we shall introduce a delta function that shall be very crucial in the Mellin space representation of correlation functions.

We know that the following integral is not convergent.

$$\int_{-\infty}^{\infty} e^{-kx}dx \tag{2.14}$$

However

$$\int_{-i\infty}^{i\infty} \frac{f(k)}{2\pi i} dk \int_{-\infty}^{\infty} e^{-kx} dx = f(0) \tag{2.15}$$

The above statement can be obtained simply by changing variables from k to ik. Thus we see that inside the complex integral in (2.15), the integral (2.14) acts like a delta function.

(2.14) can be morphed into $\int_0^\infty t^{-k-1} dt$ via a simple variable change putting $e^x = t$. Therefore we can write,

$$\frac{1}{2\pi i} \int_0^\infty t^{-k-1} dt = \bar{\delta}(k) \tag{2.16}$$

The bar over the delta function in (2.16) indicates that it is only a formal delta function, which acts only inside the complex integral in (2.15). It is important to reiterate that the integral on the left hand side of (2.16) is not convergent by itself.

The delta function in (2.16) shall be used extensively in our calculations.

It is also relevant to ask if this usage of (2.16) is valid when the power of t also has a real part (k being the imaginary part). In other words we wish to know whether the integral $\int_0^\infty t^{-k+a-1} dt$, a being real, can also act as a delta function inside a complex integral like the one in (2.15).

It is easy to see that if the contour in (2.15) can be shifted by $+a$ without crossing any poles of $f(k)$, then $\int_0^\infty t^{-k+a-1} dt$ can also act as a delta function inside the complex integral. This is a necessary and sufficient condition.

Chapter 3

The Mellin Amplitude

In this chapter, we shall have an introductory discussion on the Mellin amplitude corresponding to Feynman diagrams in interacting scalar field theories. This will give an overview of some ideas that will be used and elaborated on in the next few chapters.

For a given Feynman diagram, we can write the position space amplitude from the position space Feynman rules that are well known. For example let's consider the following Feynman diagram. We assume that each interac-

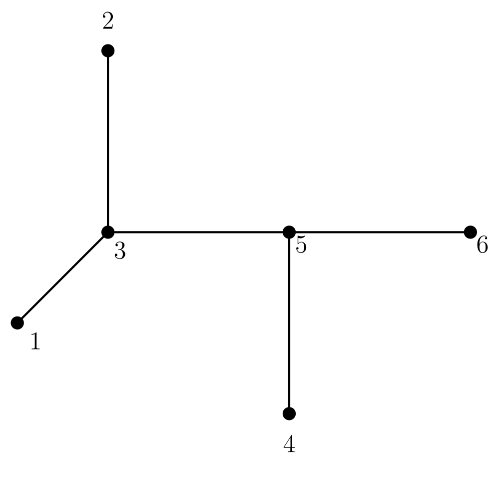


Figure 3.1: A Feynman Diagram

tion vertex comes with a weight equal to the coupling constant. For example in Diagram 3.1, the two interaction vertices have weights g and g', the coupling constants. We shall assume so throughout this work. Let the position space coordinate of the vertex i be x_i. Let the operator exchanged between the vertices ij have a scaling dimension γ_{ij}. Therefore the position space amplitude corresponding to Feynman diagram 3.1 is,

$$gg^{'}\int d^{D}x_{3}\int d^{D}x_{5}\frac{1}{(x_{1}-x_{3})^{2\gamma_{13}}(x_{2}-x_{3})^{2\gamma_{23}}(x_{3}-x_{5})^{2\gamma_{35}}(x_{4}-x_{5})^{2\gamma_{45}}(x_{5}-x_{6})^{2\gamma_{56}}} \tag{3.1}$$

D is the dimension of the spacetime. From now on we shall not write the coupling constants with the amplitudes explicitly.

We obtain the corresponding momentum space amplitude by Fourier transforming the position space amplitude. Similarly, we shall obtain the Mellin space amplitude by Mellin transforming the position space amplitude. However, the Mellin space amplitude is not directly the Mellin transform of the position space amplitude.

We shall now define the Mellin amplitude following Mack [4]. Let $A(\{x_i\})$ be the position space amplitude. The x_i here refer to the external vertices only. The Mellin amplitude $M(s_{ij})$ is defined by the following equation, upto some numerical constants (which shall be mentioned explicitly when we come to some concrete examples).

$$A\left(x_{1},\cdot\cdot\cdot,x_{n}\right)=\int_{c_{ij}-i\infty}^{c_{ij}+i\infty}\left[ds_{ij}\right]CM\left(s_{ij}\right)\prod_{1\leq i<j\leq n}\Gamma\left(s_{ij}\right)\left(x_{i}-x_{j}\right)^{-2s_{ij}} \tag{3.2}$$

s_{ij} are the Mellin variables. It is important to remember that the x_i in (3.2) include only the external vertices and not the internal interaction vertices that are integrated over in the position space amplitude. s_{ij} is obvisouly symmetric about its two indices. If we are dealing with a conformal field theory (CFT), the factor C will involve some delta functions between the Mellin variables and hence all the n Mellin variables are not independent. If the theory is not conformal invariant, then $C = 1$.

For example for the Feynman diagram 3.1, (3.2) is,

$$\begin{aligned}A\left(x_{1},x_{2},x_{4},x_{6}\right)= & \left\{\int_{c_{ij}-i\infty}^{c_{ij}+i\infty}ds_{ij}\right\}CM\left(s_{12},s_{14},s_{16},s_{24},s_{26},s_{46}\right)\Gamma\left(s_{12}\right)\Gamma\left(s_{14}\right)\\ & \Gamma\left(s_{16}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{26}\right)\Gamma\left(s_{46}\right)\left(x_{1}-x_{2}\right)^{-2s_{12}}\left(x_{1}-x_{4}\right)^{-2s_{14}}\\ & \left(x_{1}-x_{6}\right)^{-2s_{16}}\left(x_{2}-x_{4}\right)^{-2s_{24}}\left(x_{2}-x_{6}\right)^{-2s_{26}}\left(x_{4}-x_{6}\right)^{-2s_{46}}\end{aligned} \tag{3.3}$$

The delta functions in the factor C of (3.2) and (3.3) (for a CFT) will be n in number, n being the number of external vertices. Thus even though we have $\frac{n(n-1)}{2}$ number of Mellin variables, the n number of constraints imposed by the delta functions on the Mellin variables will reduce the number of Mellin variables to $\frac{n(n-3)}{2}$. These aspects will become transparent once we try to calculate the Mellin amplitude of a CFT Feynman diagram explicitly.

It should be noted that $\frac{n(n-3)}{2}$ is also the number of independent cross-ratios between n points.

The delta functions constraints in C arise when we are dealing with a CFT. But there is nothing in the position space amplitude as stated in (3.1) that is specific to a CFT. We derive a set of constraints on the scaling dimensions of the exchanged operators from the requirement that in a CFT the amplitude will tansform in a given way under re-scaling and inversions. Once we impose these constraints on the scaling dimensions, the amplitude (3.1) indeed bears the signature of a CFT. And that is exactly how the delta function constraints arise. We shall not have the occassion to see how it happens explicitly in this chapter, but we shall derive the constraints on the scaling dimensions that ensure the covariance of the amplitude with re-scaling and inversions.

Let us first consider what happens to $A\left(x_1, x_2, x_4, x_6\right)$ corresponding to the Feynman diagram 3.1 under a re-scaling, given that we are dealing with a CFT.

$$A\left(\lambda x_1, \lambda x_2, \lambda x_4, \lambda x_6\right) = \lambda^{-\gamma_{13}-\gamma_{23}-\gamma_{45}-\gamma_{56}} A\left(x_1, x_2, x_4, x_6\right) \tag{3.4}$$

We know that (3.4) is true from general properties of correlation functions in a CFT. We have not used the form of $A\left(x_1, x_2, x_4, x_6\right)$ explicitly yet. Now we rescale $x \to \lambda x$ in (3.1). It is easy to see that for (3.4) to be true, we must have,

$$\gamma_{13} + \gamma_{23} + \gamma_{45} + \gamma_{56} + 2\gamma_{35} - 2D = 0 \tag{3.5}$$

(3.5) is the constraint on the scaling dimensions that arises because we have demanded the covariance of (3.1) with a rescaling of the position coordinates.

Next we look at what happens to $A\left(x_1, x_2, x_4, x_6\right)$ under an inversion. Under an inversion,

$$x_i^\mu \to \frac{x_i^\mu}{|x_i|^2} \qquad (x_i - x_j)^2 \to \frac{\left(x_i - x_j\right)^2}{(x_i)^2 (x_j)^2} \qquad d^D u \to \frac{d^D u}{(|u|^2)^D} \tag{3.6}$$

$$A\left(\frac{x_1}{|x_1|^2}, \frac{x_1}{|x_1|^2}, \frac{x_2}{|x_2|^2}, \frac{x_4}{|x_4|^2}, \frac{x_6}{|x_6|^2}\right) = |x_1|^{2\gamma_{13}} |x_2|^{2\gamma_{23}} |x_4|^{2\gamma_{45}} |x_6|^{2\gamma_{56}} A\left(x_1, x_2, x_4, x_6\right) \tag{3.7}$$

(3.7) is true owing to the transformation properties of correlation functions in a CFT. However we can now employ (3.6) explicitly on (3.1). For (3.7) to be true, the following two constraints must be satisfied.

$$\gamma_{13} + \gamma_{23} + \gamma_{35} - D = 0 \tag{3.8}$$

$$\gamma_{45} + \gamma_{56} + \gamma_{35} - D = 0 \tag{3.9}$$

We see that the (3.8) can be associated with the internal vertex 3 and (3.9) can be associated with the internal vertex 5. Also the sum of (3.8) and (3.9) is nothing but (3.5). Thus it is sufficient to satisfy the constraints arising from the requirement that (3.1) transforms with inversion in a certain manner defined for correlation functions in a CFT. In general, it is true that corresponding to each internal vertex there is a constraint between the scaling dimensions of the operators interacting at that vertex (that is the sum of these scaling dimensions is equal to D). These constraints together also ensure the covariance of the amplitude with a rescaling.

With this discussion, now we are prepared to explicitly compute the Mellin amplitude corresponding to some interaction diagrams. The next few chapters will be devoted to this.

Chapter 4

One Vertex Interaction

In this chapter, we shall explicitly calculate the Mellin amplitude corresponding to a Feynman diagram for a one vertex interaction in a scalar field theory. We shall start with the position space amplitude and then try to represent it in the form of (3.2). We shall essentially be using tricks used in [8][9][10][11]. The Feynman diagram we are considering is,

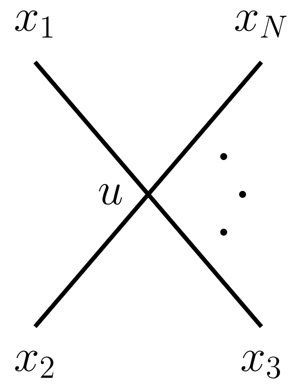


Figure 4.1: One Vertex Interaction

The scaling dimension of the external line correponding to the external vertex x_i is denoted by ν_i. With this convention, the position space amplitude corresponding to this Feynman diagram 4.1 is,

$$\int \frac{d^D u}{\prod_{i=1}^{N} (x_i - u)^{2\nu_i}} \tag{4.1}$$

We shall use the following identity (Schwinger paramterisation) on (4.1)

$$\frac{1}{(x-y)^{2a}} = \frac{1}{\Gamma(a)} \int_0^\infty dt\ t^{a-1} \exp\left[-t(x-y)^2\right] \tag{4.2}$$

This gives us,

$$\frac{1}{\prod_{i=1}^{N}\Gamma(\nu_i)}\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{N}\left[dt_i t_i^{\nu_i-1}\right]\int d^D u \exp\left[-\left(\sum_{i=1}^{N} t_i(x_i-u)^2\right)\right] \tag{4.3}$$

Next we shall perform the Gaussian integral over u. This gives us, upto some possible numerical factors,

$$\frac{1}{\prod_{i=1}^{N}\Gamma(\nu_i)}\int_0^\infty \cdots \int_0^\infty \frac{\prod_i dt_i t_i^{\nu_i-1}}{(\sum_j t_j)^{D/2}} \exp -\left[\sum_k t_k x_k^2 - \frac{(\sum_k t_k x_k)^2}{\sum_k t_k}\right] \tag{4.4}$$

The terms in the exponential factor in (4.4) can be manipulated to obtain,

$$\frac{1}{\prod_{i=1}^{N}\Gamma(\nu_i)}\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{N} dt_i t_i^{\nu_i-1}}{(\sum_i t_i)^{\frac{D}{2}}} \exp\left[-\frac{1}{\sum_i t_i}\left(\sum_j \sum_{i<j} t_i t_j (x_i-x_j)^2\right)\right] \tag{4.5}$$

Now we shall perform a re-scaling trick that will render (4.5) particularly suitable for imposing the contraints discussed in Chapter 3 (that are required for the covariance of the amplitude with inversion).

We introduce a delta function (that can be integrated over to give 1) in (4.5).

$$\frac{1}{\prod_{i=1}^{N}\Gamma(\nu_i)}\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{N} dt_i t_i^{\nu_i-1}}{(\sum_i t_i)^{\frac{D}{2}}} \int_0^\infty dv\delta(v-(\sum t_i))$$
$$\exp\left[-\frac{1}{\sum_i t_i}\left(\sum_j \sum_{i<j} t_i t_j (x_i-x_j)^2\right)\right] \tag{4.6}$$

In the next step, we first take the v integral outside the integrals over the Schwinger parameters t_i. Then we make the change of variables, $t_i \to \frac{t_i}{v}$. This gives us,

$$\frac{1}{\prod_{i=1}^{N}\Gamma(\nu_i)}\int_0^\infty dv\, v^{\sum_i \nu_i - \frac{D}{2}-1}\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{N} dt_i t_i^{\nu_i-1}}{(\sum_i t_i)^{\frac{D}{2}}}\delta(1-(\sum t_i))$$
$$\exp\left[-\frac{v}{\sum_i t_i}\left(\sum_j \sum_{i<j} t_i t_j (x_i-x_j)^2\right)\right] \tag{4.7}$$

By virtue of the delta function in (4.7), we can replace $\sum_i t_i$ by 1 to obtain,

$$\frac{1}{\prod_{i=1}^{N}\Gamma(\nu_i)}\int_0^\infty dv\ v^{\sum \nu_i - \frac{D}{2} - 1}\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{N} dt_i t_i^{\nu_i - 1}\delta(1-(\sum t_i))$$
$$\exp\left[-v\left(\sum_j\sum_{i<j} t_i t_j (x_i - x_j)^2\right)\right] \quad (4.8)$$

In (4.8), we have got rid of the $\sum_i t_i$ from the denominators. Next we perform another rescaling of the Schwinger paramters $t_i \to t_i\sqrt{v}$. This gives us,

$$\frac{1}{\prod_{i=1}^{N}\Gamma(\nu_i)}\int_0^\infty dv\ v^{\sum_i \frac{\nu_i}{2} - \frac{D}{2}}\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{N} dt_i t_i^{\nu_i - 1}$$
$$\delta(v-(\sum t_i)^2)\exp\left[-\left(\sum_j\sum_{i<j} t_i t_j (x_i - x_j)^2\right)\right] \quad (4.9)$$

In (4.9), we have got rid of the v from the exponential term. Now we can take the v integral inside all the Schwinger parameter integrals and perform the integration over the delta function to obtain,

$$\frac{1}{\prod_{i=1}^{N}\Gamma(\nu_i)}\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{N} dt_i t_i^{\nu_i - 1}\left(\sum t_i\right)^{\sum_i \nu_i - D}\exp - \left(\sum_j\sum_{i<j} t_i t_j (x_i - x_j)^2\right) \quad (4.10)$$

The rescaling trick is now complete, and at the end of it we have got rid of clumsy denominators and obtained an extra factor of $(\sum t_i)^{\sum_i \nu_i - D}$.

We wish to bring (4.10) to the form of the right hand side of (3.2). For that we use the fact that the exponential term can be expressed as an inverse Mellin transform of some Gamma functions (refer to Section 2.1).

$$e^{-x} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(s)x^{-s}ds \quad (4.11)$$

The contour of integration is along the imaginary axis, with an infinitesimal curve around the origin to put the origin (a pole of the Gamma function) to the left of the contour.

Using (4.11) in the exponential term in (4.10), we can obtain,

$$\frac{1}{\prod_{i=1}^{N}\Gamma(\nu_i)}\left(\frac{1}{2\pi i}\right)^{\frac{N(N-1)}{2}}\int_{-i\infty}^{i\infty}\cdots\int_{-i\infty}^{i\infty}\prod_{j}\prod_{i<j}\left[ds_{ij}((x_i-x_j)^2)^{-s_{ij}}\Gamma(s_{ij})\right]$$
$$\int_0^\infty\cdots\int_0^\infty\prod_k dt_k t_k^{\rho_k-1}\left(\sum t_k\right)^{\sum_k \nu_k - D} \tag{4.12}$$

where

$$\rho_i = \nu_i - \sum_{j<i} s_{ij} - \sum_{l>i} s_{li} \tag{4.13}$$

s_{ij} are our Mellin variables.

We are interesd in CFTs. Therefore, we can demand that our amplitude transform in a prescribed manner, ie be covariant with conformal transformations.

We recall our discussion of constraints from covariance with inversion and re-scaling in Chapter 3. Following that discussion, it is easy to figure out that there is only one constraint here (corresponding to one interaction vertex). This constraint is,

$$\sum_i \nu_i - D = 0 \tag{4.14}$$

Implementing (4.14) on (4.12), we are left with,

$$\frac{1}{\prod_{i=1}^{N}\Gamma(\nu_i)}\left(\frac{1}{2\pi i}\right)^{\frac{N(N-1)}{2}}\int_{-i\infty}^{i\infty}\cdots\int_{-i\infty}^{i\infty}\prod_{j}\prod_{i<j}\left[ds_{ij}((x_i-x_j)^2)^{-s_{ij}}\Gamma(s_{ij})\right]$$
$$\int_0^\infty\cdots\int_0^\infty\prod_k dt_k t_k^{\rho_k-1} \tag{4.15}$$

We look at the factor

$$\int_0^\infty\cdots\int_0^\infty\prod_i dt_i t_i^{\rho_i-1} \tag{4.16}$$

We should remember that this expression (4.16) is sitting inside the Mellin inverse integral (4.15). This is where our discussion on the special Mellin space delta function in Section 2.3 comes in handy.

First we consider the contours of the s_{ij} integrals in (4.15). They are along the imaginary axis with a little curve around the origin to put the origin to the left of the contours. Now we consider the poles in the integrand of (4.15). They are all contributed by the Gamma functions whose poles are at the origin and the negative integers. Hence the contours of the s_{ij} integrals can be shifted to the right freely.

Now we look at the terms ρ_i. The Mellin variables s_{ij} have a negative sign in it, while the scaling dimensions ν_i have a positive sign (a negative scaling dimension does not make any physical sense). From our dicussion in 2.3, we know that (4.16) will act as N delta functions if we can shift each contour to the right by an appropriate amount. Since we can shift our contours to the right freely, we do not need to worry about the details of the contour shifts. Thus we can write (formally, as indicated by the bar on the delta function),

$$\int_0^\infty \cdots \int_0^\infty \prod_i dt_i t_i^{\rho_i - 1} = (2\pi i)^N \prod_i \bar{\delta}(\rho_i) \tag{4.17}$$

Using (4.17) in (4.15), we have,

$$\frac{1}{\prod_{i=1}^N \Gamma(\nu_i)} \left(\frac{1}{2\pi i}\right)^{\frac{N(N-3)}{2}} \int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \prod_j \prod_{i<j} \left[ds_{ij}((x_i - x_j)^2)^{-s_{ij}} \Gamma(s_{ij})\right] \prod_i \delta(\rho_i) \tag{4.18}$$

The delta functions constitute the factor C in (3.2). These delta functions effectively reduce the number of independent Mellin variables to $\frac{N(N-3)}{2}$.

The delta functions in C have originated from our demand that the amplitude be covariant with scaling and inversion. Now we can have a careful look at (4.18). The N delta functions in (4.18) force the $(x_i - x_j)^{-2s_{ij}}$ terms to combine and form $\frac{N(N-3)}{2}$ cross ratios between the external vertices x_i. Thus the position space amplitude is an inverse Mellin transform, the arguments of the position space amplitude being a set of independent cross ratios between the external vertices, which is indeed how it should be for a CFT. In fact this is the significance of having as many independent Mellin variables as the number of independent cross ratios that can be constructed from the external vertices. Thus the delta functions have expressed the position space amplitude as a function of independent cross rations, thereby making it manifestly covariant with special conformal transformation (and other conformal transformations), a feature that is absent in the position space representation. This is an important virtue of the Mellin space representation of CFT correlation functions (true in general for all Feynman diagrams), that the covariance with special conformal transformations is manifest.

The numerical factor that is absorbed into the Mellin measure is $\frac{1}{\prod_{i=1}^{N}\Gamma(\nu_i)}\left(\frac{1}{2\pi i}\right)^{\frac{N(N-3)}{2}}$.

Let us compare (4.18) with (3.2) now. We can read off the Mellin ampltitude corresponding to the Feynman diagram 4.1.

We see that the Mellin amplitude is just 1.

All the jugglery in this chapter has thus given us an exceedingly simple answer, that the Mellin amplitude of a one vertex interaction in a CFT is 1. However this exercise will prove to be very useful as we shall repeat these steps to calculate the Mellin amplitude of diagrams with higher number of interaction vertices in the next few chapters.

Chapter 5

Two Vertex Interaction

In this chapter, we shall calculate the Mellin amplitude corresponding to two vertex tree level interactions. This will give us the propagator in Mellin space. Once again, we shall follow the method by Paulos et al in [10] The Feynman diagram we are considering is

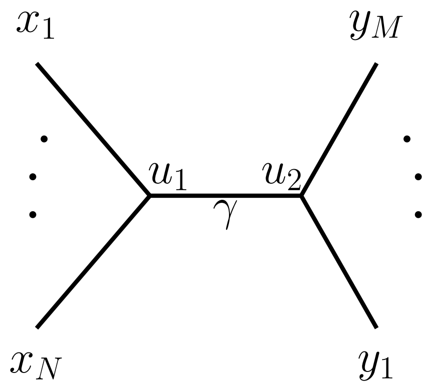


Figure 5.1: Two vertex tree

The scaling dimension of the internal line is γ as shown in Diagram 5.1. The scaling dimension of the external line correponding to the external vertex x_i is denoted by ν_i, and that corresponding to the external vertex y_i is denoted by δ_i. The position space amplitude is then given by,

$$\int\int \frac{d^D u_1 d^D u_2}{\prod_i (x_i - u_1)^{2\nu_i} \prod_j (y_j - u_2)^{2\delta_j} (u_2 - u_1)^{2\gamma}} \tag{5.1}$$

We shall repeat the same set of steps as in the one vertex case in Chapter 4, but for one internal vertex at a time. First we introduce the Schwinger parameters for all lines connected to vertex u_1. We have,

$$\int d^D u_2 \frac{1}{\prod (y_i - u_2)^{2\delta_i}} \frac{1}{\prod \Gamma(\nu_i)\Gamma(\gamma)} \int d^D u_1 \int_0^\infty \cdots \int_0^\infty \prod da_i dc \prod a_i^{\nu_i - 1} c^{\gamma - 1}$$
$$\exp - \left[\sum a_i (x_i - u_1)^2 + c(u_2 - u_1)^2 \right] \tag{5.2}$$

a_i are the Schwinger parameters corresponding to the external lines while c is the Schwinger parameter corresponding to the internal line. Now we shall carry out the Gaussian integral over u_1 and introduce a delta function to obtain,

$$\int d^D u_2 \frac{1}{\prod (y_i - u_2)^{2\delta_i}} \frac{1}{\prod \Gamma(\nu_i)\Gamma(\gamma)} \int_0^\infty \cdots \int_0^\infty \frac{\prod da_i dc \prod a_i^{\nu_i - 1} c^{\gamma - 1}}{(\sum a_i + c)^{D/2}}$$
$$\exp \left[- \frac{1}{\sum a_i + c} \left(\sum \sum a_i a_j (x_i - x_j)^2 + \sum a_i c (x_i - u_2)^2 \right) \right]$$
$$\int_0^\infty dv \delta(v - (\sum a_i + c)) \tag{5.3}$$

Next we shall perform the same re-scaling trick on the Schwinger parameters a_i and c as in Chapter 4 (refer to steps (4.6) to (4.10)). We shall directly write the result we obtain from this.

$$\int d^D u_2 \frac{1}{\prod (y_i - u_2)^{2\delta_i}} \frac{1}{\prod \Gamma(\nu_i)\Gamma(\gamma)} \int_0^\infty \cdots \int_0^\infty \prod da_i dc \prod a_i^{\nu_i - 1} c^{\gamma - 1}$$
$$\left(\sum a_i + c \right)^{\nu_i + \gamma - D} \exp \left[- \left(\sum \sum a_i a_j (x_i - x_j)^2 + \sum a_i c (x_i - u_2)^2 \right) \right] \tag{5.4}$$

We repeat the same steps for the vertex u_2 now. We introduce the Schwinger parameters (say b_i) for the remaining external lines and integrate over u_2. Then we perform the re-scaling trick on the Schwinger parameters b_i and c (it is important to note that we do not involve the a_i in this round of the re-scaling trick). All these steps finally give us,

$$\frac{1}{\prod \Gamma(\nu_i) \prod \Gamma(\delta_j) \Gamma(\gamma)} \int_0^\infty \cdots \int_0^\infty \prod da_i \prod db_j dc \prod a_i^{\nu_i - 1} \prod b^{\delta_j - 1} c^{\gamma - 1}$$
$$\left(\sum b_i + \sum a_i c \right)^{\sum \delta_i + \gamma - D} \left[\sum a_i + (\sum b_i + \sum a_i c) c \right]^{\sum \nu_i + \gamma - D}$$
$$\exp \left[- \left(c \sum \sum b_i a_j (y_i - x_j)^2 + \sum \sum b_i b_j (y_i - y_j)^2 \right) \right]$$
$$\exp \left[- \left((1 + c^2) \sum \sum a_i a_j (x_i - x_j)^2 \right) \right] \tag{5.5}$$

We can now follow step (4.12) of Chapter 4 to express (5.5) as an inverse Mellin transform. We obtain, upto a factor of $\left(\frac{1}{2\pi i}\right)^{\frac{(N+M)(N+M-1)}{2}}$,

$$\frac{1}{\prod\Gamma(\nu_i)\prod\Gamma(\delta_j)\Gamma(\gamma)}\int_{-i\infty}^{i\infty}\cdots\int_{-i\infty}^{i\infty}\left[\prod_j\prod_{i<j}dp_{ij}(x_i-x_j)^{-2p_{ij}}\Gamma(p_{ij})\right]$$
$$\left[\prod_j\prod_{i<j}dq_{ij}(y_i-y_j)^{-2q_{ij}}\Gamma(q_{ij})\right]\left[\prod_i\prod_j dr_{ij}(x_i-y_j)^{-2r_{ij}}\Gamma(r_{ij})\right]$$
$$\int_0^\infty\cdots\int_0^\infty\prod da_i\prod a_i^{\rho_i-1}\prod db_i\prod b_i^{\sigma_i-1}dcc^{\gamma-1}(1+c^2)^{-\sum_j\sum_{i<j}p_{ij}}c^{-\sum_i\sum_j r_{ij}}$$
$$\left(\sum b_i+\sum a_ic\right)^{\sum\delta_i+\gamma-D}\left[\sum a_i+(\sum b_i+\sum a_ic)c\right]^{\sum\nu_i+\gamma-D} \tag{5.6}$$

where,

$$\rho_i=\nu_i-\sum_{j\neq i}p_{ij}-\sum_j r_{ij}$$
$$\sigma_j=\delta_j-\sum_{i\neq j}q_{ij}-\sum_i r_{ij} \tag{5.7}$$

The conditions from the requirement of covariance with inversion for this case are,

$$\sum\delta_i+\gamma-D=0$$
$$\sum\nu_i+\gamma-D=0 \tag{5.8}$$

Using (5.8) in (5.6) we get rid of the factors $(\sum b_i+\sum a_ic)^{\sum\delta_i+\gamma-D}$ and $[\sum a_i+(\sum b_i+\sum a_ic)c]^{\sum\nu_i+\gamma-D}$.

Now the a_i and b_i integrals give the $N+M$ delta functions in the factor C of (3.2). These reduce the number of Mellin variables from $\frac{(N+M)(N+M-1)}{2}$ to $\frac{(N+M)(N+M-3)}{2}$.

After these steps, the Mellin amplitude corresponding to the two vertex tree can be read off. It is,

$$\frac{1}{\Gamma(\gamma)}\int_0^\infty dcc^{\gamma-1}(1+c^2)^{-\sum_j\sum_{i<j}p_{ij}}c^{-\sum_i\sum_j r_{ij}} \tag{5.9}$$

Evaluating the Schwinger parameter integral in (5.9), we get,

$$\frac{1}{2\Gamma(\gamma)}\beta\left(\frac{\gamma-\sum_j\sum_i r_{ij}}{2},\sum_j\sum_{i<j}p_{ij}-\frac{\gamma-\sum_i\sum_j r_{ij}}{2}\right) \tag{5.10}$$

Now we shall introduce some notation to simplify the look of (5.10). We introduce the convention,

$$K_{IJ}=\sum_{i\in I}\sum_{j\in J}s_{ij} \qquad \text{with no overcounting} \tag{5.11}$$

Here s_{ij} represents a general Mellin variable between external vertices i and j. I,J represent internal vertices or equivalently, the set of external vertices connected to them.

In this notation, the Mellin amplitude becomes,

$$\frac{1}{2\Gamma(\gamma)}\beta\left(\frac{\gamma-K_{12}}{2},K_{11}-\frac{\gamma-K_{12}}{2}\right) \tag{5.12}$$

We can already guess that (5.12) should be our propagator in the Mellin space. However, this expression is not symmetric about the two internal vertices u_1 and u_2. It seems to depend on which vertex among the two we integrate over first. But obviously the order of integration should not create a difference since we started out with the same expression. And it better not create any difference if we intend to carry forward with our programme of deriving Feynman rules for CFTs in the Mellin space because an ambiguity of this sort in the propagator is not acceptable.

The expression (5.12) can be simplified using the equations of constraint that we have used for ensuring covariance with inversion and re-scaling, and also the constraints between the Mellin variables imposed by the $N+M$ delta functions.

Summing over all the constraints imposed by the delta functions obtained from the a_i integrals in (5.6) (in the conformal case), we get,

$$\sum_{i\in 1}\nu_i-2K_{11}-K_{12}=0 \tag{5.13}$$

The constraint from the conformal symmetry of our theory, corresponding to vertex u_1, is

$$\sum_{i\in 1}\nu_i+\gamma-D=0 \tag{5.14}$$

Using (5.13) and (5.14), we can simplify (5.12) to obtain,

$$\frac{1}{2\Gamma(\gamma)}\beta(\frac{\gamma - K_{12}}{2}, \frac{D}{2} - \gamma) \tag{5.15}$$

(5.15) is clearly symmetric about the two internal vertices. Thus we have our Mellin space propagator in expression (5.15).

But this is of little use unless we can show that for any Feynman diagram (at least at the tree level) the Mellin space amplitude factorises into a product of propagators corresponding to each edge of the graph. We shall seek to do so in the next chapter.

Chapter 6

A General Tree

In this chapter we shall prove that for any tree level interaction graph in an interacting CFT, the Mellin amplitude is a product of propagators each of which can be associated with one edge of the graph. Before coming to the proof itself, we have to first develop a general algorithmic approach to find the Mellin amplitude of any Feynman diagram as an integral over the Schwinger parameters for the internal lines in the diagram.

We shall be using lower case indices when referring to external lines or vertices and upper case indices when referring to internal lines and vertices. The scaling dimensions of internal lines will be referred to as γ_I.

6.1 A Diagrammatic Algorithm

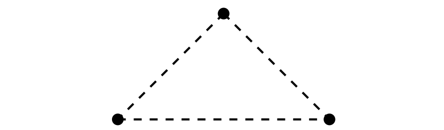
In this section, we discuss a diagrammatic algrithm to write down the Mellin amplitude of any Feynman diagram (for tree and loop diagrams both) as an integral over the internal Schwinger parameters. This algorithm mimicks the steps that we have had some practice with in Chapters 4 and 5.

We basically construct a diagram (which we shall call a **connection diagram**) via the algorithm and then write the amplitude as an integral over the internal Schwinger paramters from this connection diagram.

To begin with, we have to follow the following two steps to obtain the **skeleton**:

- Suppres all external lines to a point (represented by a dot).
- Represent the internal lines with dashed lines.

The following diagrams explain these steps.



(1)

(2)

Figure 6.1: Skeleton: (1) Three vertex tree (2) Three vertex loop

Henceforth we shall never draw the external lines explicitly (for any Feynman diagram, we shall only draw the skeleton).

We integrate over each of the internal vertices. When integrating over any internal vertex, we have to do the following:

- Draw a loop at that vertex representing all the contractions between external lines at that vertex and assign it a value of 1.
- Draw a solid line over each of the dashed line connections (if one is not present already), representing a connection with the connected vertices.
- Assign a value equal to the Schwinger parameter to each of the solid internal lines.
- Connect all vertices connected to the vertex in question with each other, assigning a value which is the product of the two (corresponding) Schwinger parameters (will be clear in the example below).
- If there are more than one solid line connections between two points, replace it with one with a value which is the sum of all these.
- If there is already a loop at any of the vertices connected to the vertex in question with a solid line, draw another loop at it, and assign it a value which is the square of the Schwinger parameter (or the value) of the line connecting the two vertices.

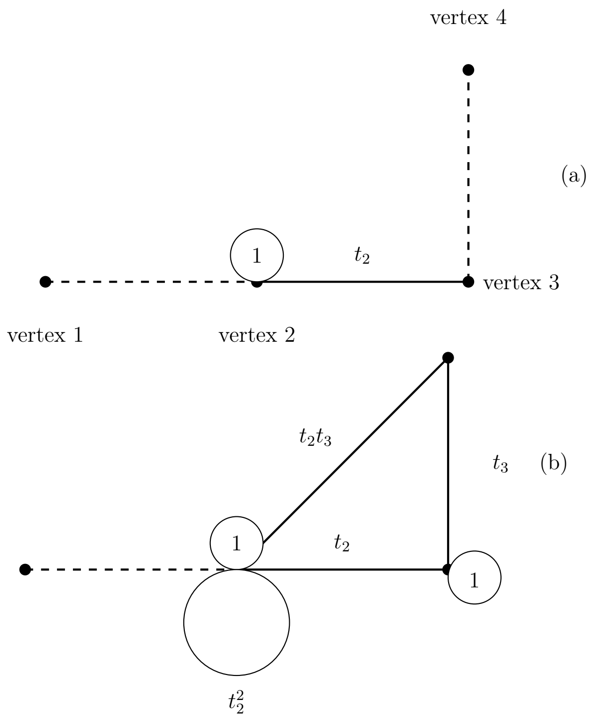


Figure 6.2: (a)Before and (b) after integrating over vertex 3

The figures 6.2 explain the steps outlined above that need to be followed while integrating over an interaction vertex.

We have to complete the steps corresponding to integration over a vertex for all the vertices. Then we can write the amplitude by the following rules:

- Include a factor of $t_I^{\gamma_I - 1}$ for each internal line.
- Sum over the values of loops at each vertex, and raise the sum to the power $-K_{II}$.
- For each solid line, raise the net value of the line to the power $-K_{IJ}$.
- Integrate over all internal Schwinger parameters, with the integrand being a product of the above three contributions.
- Multiply by the reciprocal of $\Gamma(\gamma_I)$ for each of the internal lines.

It should be noted that this algorithm has nothing to do with the conventions used for indexing the Schwinger parameters or the scaling dimenions. We can safely follow these steps for any self consistent convention for the indices.

6.2 Mellin amplitude of a general tree

We are finally ready to calculate the Mellin amplitude of a general tree level graph and show that we indeed arrive at a set of Feynman rules for the tree level Feynman diagrams in an interacting CFT. We shall start from the Mellin amplitude as an integral over the internal Schwinger parameters and show that this is equal to a product of beta functions, each of which can be interpreted as a propagator in the tree.

We already understand the rules to draw a connection diagram. We shall now look closely at the integral over the Schwinger parameters (the amplitude) which we call M.

$$M = \int \left(\prod_P dt_P \frac{t_P^{\gamma_P - 1}}{\Gamma(\gamma_P)} \right) \mathcal{I} \tag{6.1}$$

$$= \int \prod_P Dt_P \mathcal{I} \tag{6.2}$$

The index P runs over all the internal lines. In the first step, we have called the contribution to the integrand from the connection diagram as $\mathcal{I}$, and in the second step, we have redefined the measure of the integral.

One can easily find out from a few simple examples that $\mathcal{I}$, for any given graph, depends of the order of integration over the vertices while making the connection diagram. For a straight chain of propagators the natural choice is to go from left to right. For an arbitrary tree, we shall specify an order that we shall consider in this derivation.

We have been referring to the Feynman diagram without the external lines as the **skeleton**. All vertices in the skeleton shall be interchangably referred to as **node** or **vertex**. Thus nodes are those vertices in the Feynman diagram that are integrated over in the position space amplitude. Any node at which more than two lines are connected shall be called a **branch node**. The term **line** and the corresponding Schwinger parameter shall be used interchangably.

The order of integration is shown on a skeleton below with arrows. It is fixed by the rule that there should be only one line at any branch node which goes out, and all other lines should go in. Each branch node is integrated over when all but one of the neighbouring nodes have been integrated over. Thus there will be only one end line in the skeleton on which the arroe goes out.

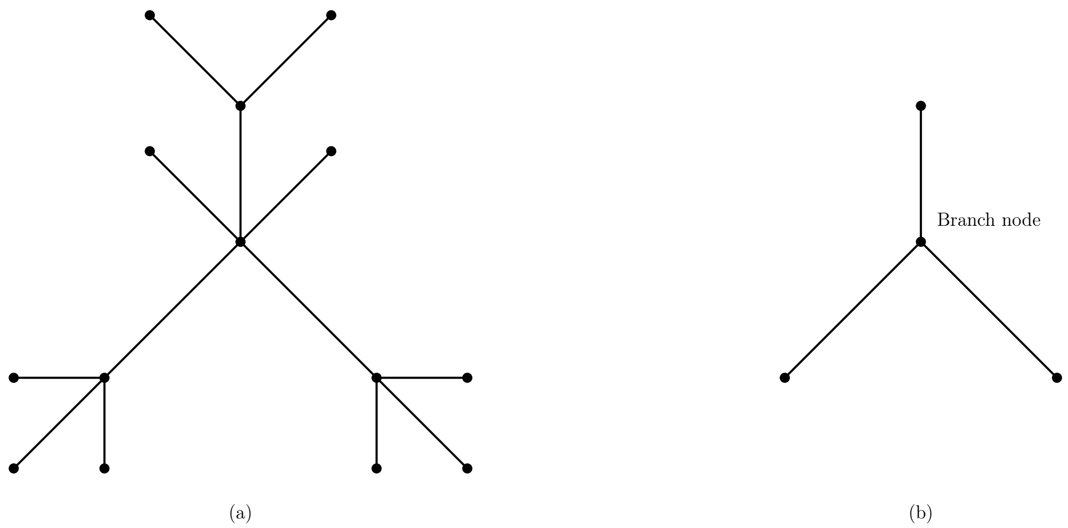


Figure 6.3: (a)A Feynman diagram and (b) the corresponding skeleton

This is the end point of the skeleton in a sense, and shall be referred to as the **exit**. The Diagram 6.4 illustrates a compatible order of integration.

Each pair of nodes on the skeleton represents a connection. Suppose we draw a continuous line (without raising the pen) between any two nodes on the skeleton via the nodes that come in between. We shall call this the **connect route** between the two points. Since we are considering a tree, there will exist a node on the connect route that is nearest to the exit. The continuous route from this node to the exit shall be referred to as the **exit route** for the given connection ie the given pair of nodes (refer to Diagram 6.5).

A connection is denoted by (I, J). Since we shall have to deal with both nodes and propagators on the skeleton and external lines and vertices, we shall use upper case indices for the former and lower case indices for the latter. However, in other discussions, when we shall have no need for such disambiguation, we shall use lower case letters for both types of indices.

The integrand $\mathcal{I}$, by construction of the connection diagram is given by the product of all possible connections (self connections at each node and cross connections between two different nodes) each raised to some appropriate power.

We use a convention that the line coming out of a node P (as per the ar-

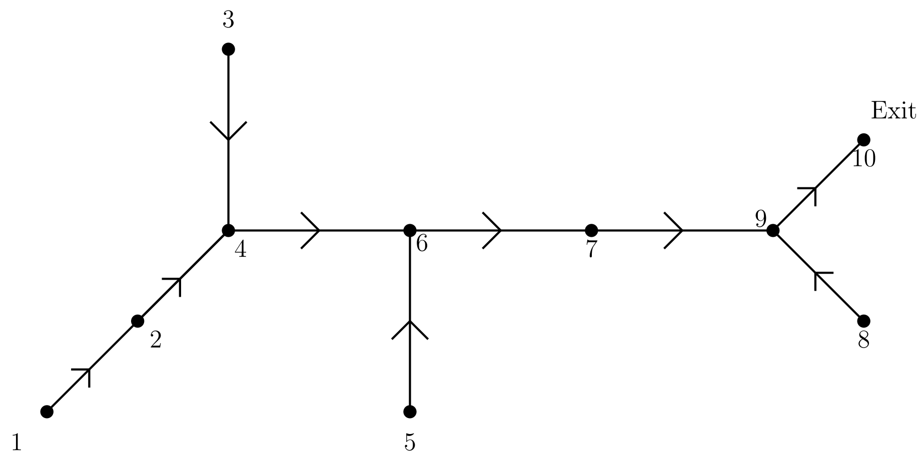


Figure 6.4: Order of integration over the nodes depicted by the numbers (in increasing order) and arrows.

rows) will be denoted by t_P. Let I also denote the set of all external vertices connected to the node I with the external lines.

From the discussion on how to write the amplitude as an integral over Schwinger parameters from the connection diagram, we know that for any two nodes I,J the value contributed by the corresponding connection is,

$$(I,J)^{-K_{IJ}} \tag{6.3}$$

Next, we wish to know the functional dependence of a given (I,J) on the Schwinger parameters.

Following the rules of the constructing the connection diagram, we can find that,

$$\begin{aligned}(I,J) = \quad & [\text{product of the Schwinger parameters on the } (I,J) \text{ connect route}] \\ & \left[1 + t_P^2(1 + t_{P+1}^2(\dots \text{ and so on along the } (I,J) \text{ exit route}) \cdots)\right]\end{aligned} \tag{6.4}$$

In (6.4), P is the node on (I,J) connect route that is nearest to the exit. The chain in the second factor in (6.4) terminates at the second last node, ie at the node nearest to the exit on the exit route.

Note that in the second factor on the right hand side of the formula above,

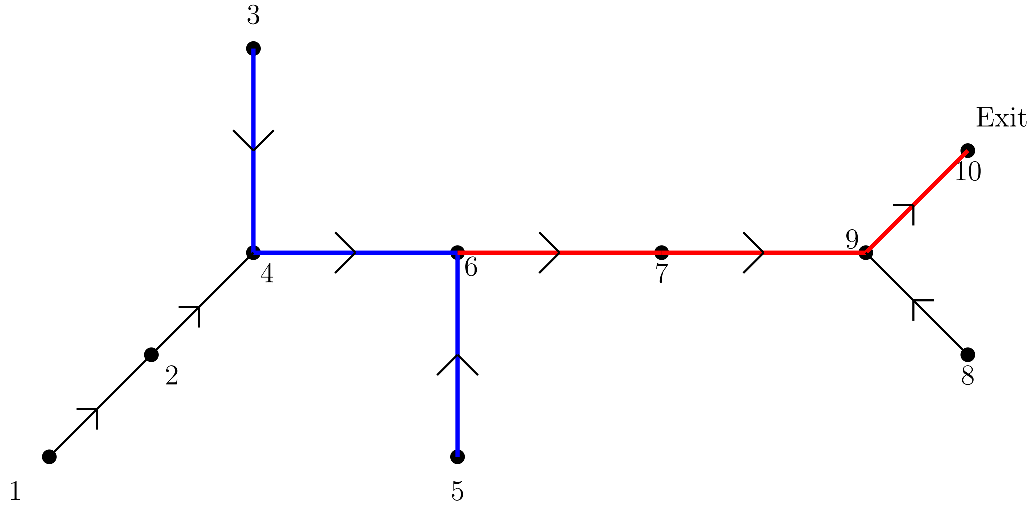


Figure 6.5: Connect route (blue) and Exit route (red) for (3,5)

only those Schwinger parameters are involved that fall in the exit route of the connection, and +1 in the index means one step next in the exit route, and not necessarily in the general order of integration. For self contractions at a node, the first factor is just one.

It should be emphasized that this functional dependence of (I, J) is true only for the chosen order of integration and the formula does not hold in general.

As an example, for the skeleton in the diagram (3), the connection $(3, 5)$ is equal to,

$$(3, 5) = t_3 t_4 t_5 \left(1 + t_6^2 \left(1 + t_7^2 \left(1 + t_9^2\right)\right)\right) \tag{6.5}$$

At this point, we introduce one more notation. If we cut any line in the skeleton, it is divided into two parts. We call the part of the diagram for which the arrow at the cut is going into the cut line say t_I, to be the Left, and the set of nodes in this part as L_I. Similarly we define the Right and the corresponding set R_I.

Keeping in mind the previous discussion, we now can easily write down the factors in $\mathcal{I}$ and their respective powers to which they are raised.

Firstly we have each Schwinger parameter t_P in the skeleton raised to the

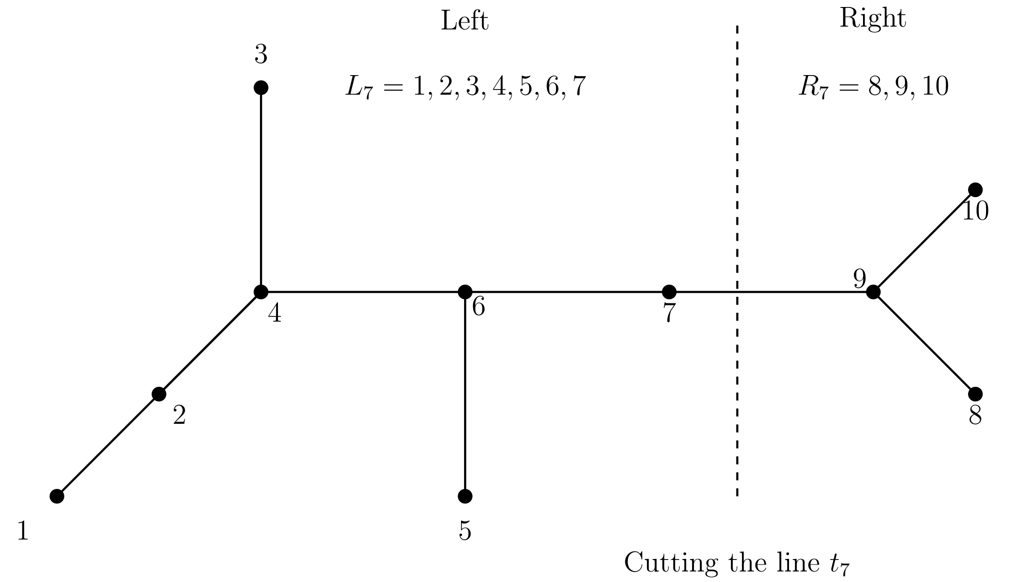


Figure 6.6: Left and Right of a cut line

power,

$$- O_P = -\sum_{I \in L_P} \sum_{J \in R_P} K_{IJ} \tag{6.6}$$

Now we shall make a notational simplification to the second factor in (6.4). We define A_I corresponding to a line t_I recursively as,

$$A_I = 1 + t_I^2 A_{I+1} \tag{6.7}$$

Here too, the +1 in the index of A indicates a step next along an exit route.

Therefore the second factor in (6.4) for an exit route starting at node P is,

$$1 + t_P^2(1 + t_{P+1}^2(\ldots \text{ and so on along the } (I, J) \text{ exit route }) \cdot \cdot\cdot) = A_P \tag{6.8}$$

So the power to which each A_P in $\mathcal{I}$ will be raised is,

$$- Q_P = -\bar{\sum_{P \neq I \in L_P}} \bar{\sum_{P \neq J \in L_P}} K_{IJ} - \sum_{J \in L_p} K_{PJ} \tag{6.9}$$

The bar over the summation is to denote that I,J are not on the same branch. Hence if P is not a branch node, then this entire term is absent.

So now, we have,

$$\mathcal{I} = \prod_{\text{all internal lines}} t_I^{-O_I} A_I^{-Q_I} \tag{6.10}$$

We shall now carry out the integral over the internal Schwinger parameters. The order of integration over the lines is the order followed for integration over the vertices. Say we are integrating out t_I. We have,

$$\frac{1}{\Gamma(\gamma_I)} \int dt_I t_I^{\gamma_I - O_I - 1} A_I^{-Q_I'} \tag{6.11}$$

We have taken an arbitrary Q_I' here for reasons that will be clear soon. We have to compute,

$$\begin{aligned}
& \int dt_I \frac{t_I^{\gamma_I - O_I - 1}}{(1 + t_I^2 A_{I+1})^{Q_I'}} && (6.12)\\
= \quad & \frac{1}{2} \int dr_I \frac{r_I^{\frac{\gamma_I - O_I}{2} - 1}}{(1 + r_I A_{I+1})^{Q_I'}} \qquad\qquad r_I = t_I^2 && (6.13)\\
= \quad & \left(\frac{1}{A_{I+1}}\right)^{\frac{\gamma_I - O_I}{2}} \frac{1}{2} \int dr_I \frac{r_I^{\frac{\gamma_I - O_I}{2} - 1}}{(1 + r_I)^{Q_I'}} && (6.14)\\
= \quad & \left(\frac{1}{A_{I+1}}\right)^{\frac{\gamma_I - O_I}{2}} \frac{1}{2} \beta\left(\frac{\gamma_I - O_I}{2}, Q_I' - \frac{\gamma_I - O_I}{2}\right) && (6.15)
\end{aligned}$$

The factor of $A_{I+1}^{-\frac{\gamma_I - O_I}{2}}$ goes into the integrand of the integral of the next line along an exit route. That is why we had taken an arbitrary Q_i' in (6.11). In fact, that tells us,

$$Q_I' = Q_I + \sum_a \frac{\gamma_{I-1}^{(a)} - O_{I-1}^{(a)}}{2} \tag{6.16}$$

Here a labels the lines that go into the node I (necessary when I is a branch node).

Thus we see that the integral over a line t_I gives us the following factor for that line,

$$\frac{1}{2\Gamma(\gamma_P)} \beta\left(\frac{\gamma_I - O_I}{2}, Q_I' - \frac{\gamma_I - O_I}{2}\right) \tag{6.17}$$

This is the propagator for the line t_I (in the special case of a two vertex tree we have our result (5.12) of Chapter 5).

Carrying out this process over the entire skeleton clearly factories the M into propagators for each line. Next we look at the beta function in the propagator carefully.

We consider the beta function,

$$\frac{1}{2}\beta\left(\frac{\gamma_P - O_P}{2}, Q_P^{'} - \frac{\gamma_P - O_P}{2}\right) \tag{6.18}$$

But,

$$\begin{aligned} & Q_P^{'} - \frac{\gamma_P - O_P}{2} \\ = \quad & Q_P - \frac{\gamma_P - O_P}{2} + \sum_a \frac{\gamma_{P-1}^{(a)} - O_{P-1}^{(a)}}{2} \qquad (6.19) \\ = \quad & \frac{1}{2}(\sum_a \gamma_{P-1}^{(a)} - \gamma_P) + \frac{1}{2}\sum_{J\in L_P, J\neq P} K_{JP} + \frac{1}{2}\sum_{J\in R_P, J\neq P} K_{PJ} + K_{PP} \qquad (6.20) \end{aligned}$$

From the delta function constraints, we have at each node P,

$$\sum_{i\in P}\nu_i - \sum_{J\neq P} K_{PJ} - 2\sum K_{PP} = 0 \tag{6.21}$$

Also from the conformal covariance of the amplitude, we have,

$$\sum_{i\in P}\nu_i + \gamma_P + \sum_a \gamma_{P-1}^{(a)} = D \tag{6.22}$$

Using (6.21) and (6.22), we finally get the propagator for t_P as,

$$\frac{1}{2\Gamma(\gamma_P)}\beta\left(\frac{1}{2}\left(\gamma_P - \sum_{I\in L_P}\sum_{J\in R_P} K_{IJ}\right), \frac{D}{2} - \gamma_P\right) \tag{6.23}$$

Thus the full Mellin amplitude of an arbitrary tree is given by,

$$\prod_P \frac{1}{2\Gamma(\gamma_P)}\beta\left(\frac{1}{2}\left(\gamma_P - \sum_{I\in L_P}\sum_{J\in R_P} K_{IJ}\right), \frac{D}{2} - \gamma_P\right) \tag{6.24}$$

Thus we have shown that the Mellin amplitude corresponding to a general tree level Feynman diagram is a product of some factors (propagators) each of which can be associated with an internal line in the diagram. We have arrived at a set of Feynman rules for tree level perturbative diagrams for CFTs in the Mellin space.

These Feynman rules are:

- The propagator for any internal line is

$$\frac{1}{2\Gamma(\gamma_p)}\beta\left(\frac{1}{2}\left(\gamma_p - \sum_{I\in L_p}\sum_{J\in R_p} K_{IJ}\right), \frac{D}{2} - \gamma_p\right)$$

- Multiply the propagators of all the internal lines.

Since we are now convinced about the Mellin space propagator (6.23), we wish to know about the pole structure of the propagator. This is what we shall look into in the next section.

6.3 Spectral representation

For a general quantum field theory, the isolated poles of the momentum space propagator correspond to single particle states and the branch cuts give the multi-particle states. In a CFT we don't really have single particle states characterised by the masses since mass is a dimensionful parameter. Therefore the momentum space does not provide any spectral representation for CFT correlation functions. An operator product expansion (OPE) in an interacting CFT involves a discrete set of operators, which are the exchanged primary field and its descendants. So it is desirable to have a representation for correlation functions in CFTs that makes this discrete spectrum manifest. Mack had proposed that [4] [5] that Mellin space provides such a representation. This is exactly what we seek to show in this section using our Mellin space Feynman rules.

We interpret a Mellin variable s_{ij} as,

$$s_{ij} = k_i \cdot k_j \tag{6.25}$$

with

$$k_i^2 = -\nu_i \tag{6.26}$$

ν_i being the scaling dimension of the external line denoted by i. We shall refer to k_i as Mellin momentum flowing through the external vertex i. It is a vector in any dimension.

Now we consider the constraint imposed by the delta function arising out of the integration of this particular line (integration over the corresponding Schwinger paramter),

$$\nu_i - \sum_{j\in\{\text{all other external vertices}\}} s_{ij} = 0 \tag{6.27}$$

$$\Rightarrow \quad k_i \cdot \left(\sum_{j\in\{\text{all external vertices}\}} k_j \right) = 0 \tag{6.28}$$

This can be satisfied if,

$$\sum_{j\in\{\text{all external vertices}\}} k_j = 0 \tag{6.29}$$

This is the condition that the sum of all the Mellin momenta flowing into the diagram is zero. In other words, Mellin momentum is conserved in any interaction.

It is important to remember that the conservation of Mellin momentum is not a necessary condition for (6.28) to be satisfied. We are just interpreting (6.28) in this manner.

We consider the propagator corresponding to the internal line P

$$\frac{1}{2\Gamma(\gamma_P)}\beta\left(\frac{1}{2}\left(\gamma_P - \sum_{I\in L_P}\sum_{J\in R_P} K_{IJ}\right), \frac{D}{2} - \gamma_P\right) \tag{6.30}$$

$$\begin{aligned}\gamma_P - \sum_{I\in L_P}\sum_{J\in R_P} K_{IJ} = \gamma_P - \sum_{i\in L_P}\sum_{j\in R_P} s_{ij} &= \gamma_P - \sum_{i\in L_P}\sum_{j\in R_P} k_i \cdot k_j \\ &= \gamma_P - \left(\sum_{i\in L_P} k_i\right)\cdot\left(\sum_{j\in R_P} k_j\right)\end{aligned} \tag{6.31}$$

Due to the overall conservation of Mellin momenta, this is equal to,

$$\gamma_P + \left(\sum_{i\in L_P} k_i\right)^2 = \gamma_P + k_P^2 \tag{6.32}$$

Thus the K_{IJ} are like Mandelstam variables here. k_P is the total Mellin momentum flowing through the internal line P. Putting this back into the propagator, we have,

$$\frac{1}{2\Gamma(\gamma_P)}\beta\left(\frac{1}{2}\left(\gamma_P + k_P^2\right), \frac{D}{2} - \gamma_P\right) = \frac{1}{\Gamma(\gamma_P)}\sum_{n=0}^{\infty} {}^{n-\frac{D}{2}+\gamma_P}C_n \frac{1}{k_P^2 + \gamma_P + 2n} \tag{6.33}$$

Thus the total Mellin momentum (squared) flowing through the line P has poles at $-\gamma_P - 2n$. These correspond to the primary field corresponding to the propagator and its descendants.

The Mellin space indeed furnishes a spectral representation for correlation functions in conformal field theory.

We would like to wrap up this discussion with a brief recapitulation of the most important points we have learnt so far. Mellin space provides a manifestly covariant (with all conformal transformations) representation for correlation functions in a CFT. At tree level (at least), there exists a set of Mellin space Feynman rules that can be associated with Feynman diagrams in an interacting CFT. The Mellin variables can be interpreted as Mandelstam variables constructed out of the (hypothetical) external Mellin momenta flowing into the diagram. The covariance of the amplitude with special conformal transformations allows for a statement of conservation of Mellin momentum. All the Mellin variables (or equivalently all the Mandelstam variables) are not independent (because of the conservation of Mellin momentum) and the number of independent Mellin variables is equal to the number of independent cross rations between the external vertices in the diagram. Mellin space also allows a spectral representation for the correlation functions as any propagator in the diagram has a discrete infinite set of poles corresponding to the exchanged primary field and its descendants.

We shall conclude our discussion of the tree level here and move on to loop diagrams.

Chapter 7

One Loop Diagrams

This chapter is devoted to calculating the Mellin amplitude of one loop Feynman diagrams. Since we have already understood the method to calculate the Mellin amplitude of any diagram and also the Feynman rules corresponding to tree level diagrams, the primary challenge lies in expressing the Mellin amplitude of an n-gon (skeleton) in the best possible way. It is particularly difficult because we have no prior clue of what would be the best way to express the Mellin amplitude of an n-gon. In fact we cannot be sure that there exists a set of Feynman rules for the Mellin amplitude in this case. We have not been able to prove or disprove the existence of these rules explicitly as of yet. However we shall present a calculation here which gives us a major hint about the desired form of the Mellin amplitude. Before that we look at a class of relations between loops and trees.

7.1 Star-delta relation

There exists a very obvious relation between a loop and a tree each with two interaction vertices.

$$\begin{aligned} & \int\int d^D u_1 d^D u_2 \frac{1}{\prod_i (x_i - u_1)^{-2\nu_i} (u_1 - u_2)^{-2\gamma} \prod_j (y_j - u_2)^{-2\delta_j}} \\ = \quad & \int\int d^D u_1 d^D u_2 \frac{1}{\prod_i (x_i - u_1)^{-2\nu_i} (u_1 - u_2)^{-2\gamma_1} (u_2 - u_1)^{-2\gamma_2} \prod_j (y_j - u_2)^{-2\delta_j}} \end{aligned} \tag{7.1}$$

where,

$$\gamma_1 + \gamma_2 = \gamma$$

The left hand side of (7.1) is the position space amplitude for the tree, while the right hand side is the position space amplitude for the two vertex loop. (By vertex, we shall mean an interaction vertex unless stated otherwise.) (7.1) tells us that the amplitude for the two vertex tree and loop are exactly equal with the scaling dimension of the internal line in the tree being equal to the sum of the scaling dimensions of the two internal lines in the loop.

In this section, we shall loop at a (not so obvious) relation between a four vertex tree (a three pronged star) and a three vertex loop. Though this is a well known relation, we shall derive it in our own way using the tricks used in the calculation of Mellin amplitudes (the relation itself has no intrinsic connection to the Mellin space though and is valid in any space). Furthermore we shall consider a possible generalisation of this relation to a maximally connected n-gon (K_n). (It is worth reiterating once that we are always talking about the skeleton and ignoring the external lines for convenience. For example the n-gon just mentioned is only the skeleton and there are external lines feeding into each of the n vertices.)

We shall also loosen up our conventions for indexing the internal lines from here onwards. Since we shall not have much use of the external lines in this chapter, we shall use lower case indices for the internal lines as well. We shall still use γ_i for the scaling dimensions of the internal lines unless otherwise stated.

Let us consider the position space amplitude of a star. We are consider-

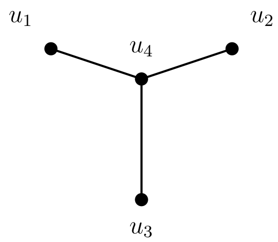


Figure 7.1: A three pronged star

ing the case where there are no external vertices to u_4. The position space amplitude is given by,

$$\int \cdots \int d^D u_1 \cdots d^D u_4 \frac{\prod (x_i - u_1)^{-2a_i} \prod (y_j - u_2)^{-2b_j} \prod (z_k - u_3)^{-2c_k}}{(u_1 - u_4)^{2\gamma_1} (u_2 - u_4)^{2\gamma_2} (u_3 - u_4)^{2\gamma_3}} \quad (7.2)$$

We shall carry the integral over u_4 following Schwinger parametrisation of the three internal lines, leaving the rest of the expression intact. After this Schwinger parametrisation we have,

$$A\int d^D u_4 \int_0^{\infty}\cdots\int_0^{\infty} dt_1 dt_2 dt_3 \frac{t_1^{\gamma_1-1}t_2^{\gamma_2-1}t_3^{\gamma_3-1}}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)}$$
$$\exp-\left[t_1(u_1-u_4)^2+t_2(u_2-u_4)^2+t_3(u_3-u_4)^2\right] \tag{7.3}$$

where,

$$A=\int\int\int d^D u_1 d^D u_2 d^D u_3 \frac{1}{\prod(x_i-u_1)^{2a_i}\prod(y_j-u_2)^{2b_j}\prod(z_k-u_3)^{2c_k}} \tag{7.4}$$

We know from our experience with the tree level that (7.3) is equal to (ignoring factors of $\frac{1}{2\pi i}$),

$$A\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} ds_{12}ds_{13}ds_{23}\frac{\Gamma(s_{12})\Gamma(s_{23})\Gamma(s_{13})}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)}$$
$$\frac{\delta(\gamma_1-s_{12}-s_{13})\delta(\gamma_2-s_{12}-s_{23})\delta(\gamma_3-s_{13}-s_{13})}{(u_1-u_2)^{2s_{12}}(u_2-u_3)^{2s_{23}}(u_3-u_1)^{2s_{13}}} \tag{7.5}$$

Let's consider the system of equations enforced by the delta functions,

$$\begin{aligned}\gamma_1-s_{12}-s_{13}&=0\\ \gamma_2-s_{12}-s_{23}&=0\\ \gamma_3-s_{23}-s_{13}&=0\end{aligned} \tag{7.6}$$

There is a unique solution. Let it be $s_{12}=p_1$,$s_{23}=p_2$ and $s_{13}=p_3$. We can find p_i from the γ_i and vice versa.

Carrying out the s_{ij} integrals in (7.5), we get,

$$A\frac{\Gamma(p_1)\Gamma(p_2)\Gamma(p_3)}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)}\frac{1}{(u_1-u_2)^{2p_1}(u_2-u_3)^{2p_2}(u_3-u_1)^{2p_3}}$$
$$= \frac{\Gamma(p_1)\Gamma(p_2)\Gamma(p_3)}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)}\int\int\int d^D u_1 d^D u_2 d^D u_3 \frac{\prod(x_i-u_1)^{-2a_i}\prod(y_j-u_2)^{-2b_j}\prod(z_k-u_3)^{-2c_k}}{(u_1-u_2)^{2p_1}(u_2-u_3)^{2p_2}(u_3-u_1)^{2p_3}} \tag{7.7}$$

The position space amplitude of the three vertex loop (triangle) is given by (7.7) (with a multiplicative factor that does not depend on any kinematical variables). Thus from the above discussion we come to know that the amplitude for the triangle loop in Diagram 7.2 is proportional to the amplitude for the three pronged star (7.2).

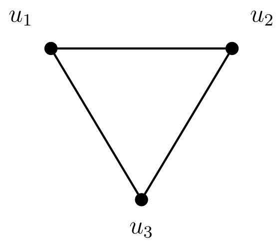


Figure 7.2: Triangle loop

We should note that the γ_i are the scaling dimensions of the internal lines of the star, while p_i are the scaling dimensions of the internal lines of the triangle. The convention followed for the triangle (evident from (7.7)) is that the scaling dimension of the line opposite to the vertex i is p_i.

Thus the duality can be formally stated in the position space as,

$$\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)A_{\text{3-pt star}}(\gamma_i) = \Gamma(p_1)\Gamma(p_2)\Gamma(p_3)A_{K_3}(p_i) \tag{7.8}$$

K_3 is the graph theoretic notation for the maximally connected 3 vertex graph, which is the triangle. A refers to the position space amplitude. The scaling dimensions on the external lines are the same for both the diagrams (there are no external lines with the central vertex of the star), and the relation between γ_i and p_i is given by (7.6). It is now trivial to state the duality in any other space as well.

This relation (7.7) tells us the Mellin amplitude for the triangle loop directly (although this is not a preferable way to express Mellin amplitude of loops since it does not generalize to other loops as we shall see later). We we can write the Mellin amplitude of the triangle loop from the Mellin amplitude for Diagram 7.1 using the discussion above,

$$\begin{aligned} &\frac{1}{\Gamma(p_1)\Gamma(p_2)\Gamma(p_3)}\frac{1}{2}\beta\left(\frac{p_2+p_3-K_{12}-K_{13}}{2}, \frac{D}{2}-\gamma_1\right) \\ &\frac{1}{2}\beta\left(\frac{p_1+p_3-K_{23}-K_{12}}{2}, \frac{D}{2}-\gamma_2\right)\frac{1}{2}\beta\left(\frac{p_1+p_2-K_{23}-K_{13}}{2}, \frac{D}{2}-\gamma_3\right) \end{aligned} \tag{7.9}$$

(7.8) has a generalized version that gives a relation between the amplitude of an n pronged star and K_n. The derivation is essentially the same as in Section 7.1. However in this case there shall not be enough delta functions at the step (7.5) to fix the scaling dimensions of all internal lines and do away with all the s_{ij} integrals. Therefore we shall have to solve for certain of the scaling dimensions of K_n in terms of the others that will be integrated over and the scaling dimensions of the internal lines in the n pronged star. This generalisation of (7.8) is,

$$\prod_{i=1}^{n}\Gamma(\gamma_i)A_{\text{n-pt star}}(\{\gamma_i\}) = \int_{-i\infty}^{i\infty}\cdots\int_{-i\infty}^{i\infty} dc_1\cdots dc_m \prod_{i=1}^{n}\Gamma(b_i)\prod_{j=1}^{\frac{n(n-3)}{2}}\Gamma(c_i)A_{K_n}(\{b_i\},\{c_i\}) \tag{7.10}$$

b_i are the scaling dimensions of some of the internal lines of K_n that are fixed in terms of γ_i and c_i, c_i being the scaling dimensions (that are not fixed and are integrated over) of the remaining internal lines of K_n.

A natural question to ask at this juncture is which are the internal lines in K_n for which we can fix the corresponding scaling dimensions, or in other words how to choose the two sets of lines corresponding to b_i and c_i.

The answer to our question is that we can choose the n b_i and the $\frac{n(n-3)}{2}$ c_j freely as long as the corresponding set of equations (from the delta functions) that determine b_i in terms of γ_i and c_i is solvable.

Classifying the allowed choices graph theoretically is an interesting exercise. I shall directly state the answer here. The set of n lines whose scaling dimensions would be b_i should be chosen such that these lines form a connected graph (with n vertices and n edges) which is an n-gon with odd n or has a part that is a l-gon with odd l, or a disconnected graph each of whose connected components is a one-loop graph of the same type.

7.2 n-gon

Now we come to our task of manipulating the Mellin amplitude of an n-gon to extract something useful from it. As mentioned in the Section 7.1 we shall use lower case indices for Schwinger parameters and scaling dimensions of internal lines (unlike in Chapter 6). We shall denote the Schwinger parameter for the line connecting the vertices i and $i+1$ in the n-gon by t_{i+1} ($n+1 \equiv 1$). The corresponding scaling dimension will be denoted by γ_{i+1}. We shall ignore factors of $\frac{1}{2\pi i}$ everywhere apart from stating independent identities or results.

The n-gon amplitude (for a cyclical order of integration over the interaction vertices) can be written as an integral over the Schwinger parameters (can be deduced from the diagrammatic algorithm in Section 6.1 or otherwise),

$$\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n} dt_i t_i^{\gamma_i - 1} \prod_{j=1}^{n-1} \left[\prod_{k=j}^{n} \left(C_j^k + B_j B_k \right)^{-K_{jk}} \right] \tag{7.11}$$

with

$$\begin{aligned}
&C_j^k = t_{j+1} \cdots t_k A_k && 1 \leq j < k < n \\
&C_j^j = A_j && 1 \leq j < n \\
&C_j^n = 0 && \\
&B_i = t_1 \cdots t_i A_i + t_{i+1} \cdots t_n && i \leq n-1 \\
&B_n = 1 && \\
&A_i = 1 + t_{i+1}^2 A_{i+1} && 1 \leq i < n-1 \\
&A_{n-1} = 1 &&
\end{aligned} \tag{7.12}$$

We shall ingore the reciprocal Gamma functions (of the scaling dimensions of the lines in the skeleton) in the Mellin amplitude in this entire chapter. It is just a multiplicative constant, and since we shall not have the occasion to calculate a final answer to our problem in this chapter (derive Feynman rules for one-loop diagrams), this choice made for convenience will not come in the way of what we wish to achieve here.

Now we shall introduce an identity that will be very important to us in this chapter. This is nothing but the result for the Mellin inverse of the beta function.

$$\frac{1}{(1+x)^a} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-w} \beta(w, a-w) dw \tag{7.13}$$

Using (7.13), we can write (7.11) as,

$$\begin{aligned}
\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n} dt_i t_i^{\gamma_i - 1} & \left[\prod_{j=1}^{n-1} \prod_{k=j}^{n-1} \int_{a_{jk}-i\infty}^{a_{jk}+i\infty} du_{jk} \beta(u_{jk}, K_{jk} - u_{jk}) \right] \\
& \left[\prod_{j=1}^{n-1} \prod_{k=j}^{n-1} \left(C_j^k \right)^{-K_{jk} + u_{jk}} (B_j B_k)^{-u_{jk}} \right] \prod_{j=1}^{n-1} (B_j B_n)^{-K_{jn}}
\end{aligned} \tag{7.14}$$

Using (7.12), we can simplify (7.14) to the form,

$$\left(\prod_{j=1}^{n-1}\prod_{k=j}^{n-1}\int_{a_{jk}-i\infty}^{a_{jk}+i\infty} du_{jk}\beta(u_{jk},K_{jk}-u_{jk})\right)\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{n} dt_i t_i^{\gamma_i-1}$$
$$\prod_{i=1}^{n-1}\left(B_i^{-2u_{ii}-\sum_{n>j\neq i}u_{ij}-K_{in}}\right)\prod_{i=1}^{n-2}\left(A_i^{-\sum_{j\in\bar{L}_{i+1}}(K_{ij}-u_{ij})}t_{i+1}^{-\sum_{j\in\bar{L}_{i+1}}\sum_{k\in\bar{R}_{i+1}}(K_{jk}-u_{jk})}\right) \tag{7.15}$$

$\bar{L}_i$ is the set of all vertices from 1 to $i-1$ and $\bar{R}_i$ is the set of the rest of the vertices except n, the order of integration over the vertices being 1 to n cyclically. Please consider the usage of these symbols here to be independent of the usage of similar symbols in the Chapter 6.

Now we introduce some notation for the powers. Let,

$$\begin{aligned} P_i^1 &= 2u_{ii}+\sum_{n>j\neq i}u_{ij}+K_{in}\\ P_i^2 &= \sum_{j\in L_{i+1}}(K_{ij}-u_{ij})\\ P_i^3 &= \sum_{j\in L_{i+1}}\sum_{k\in R'_{i+1}}(K_{jk}-u_{jk}) \end{aligned} \tag{7.16}$$

So we can write (7.15) as,

$$\prod_{j=1}^{n-1}\prod_{k=j}^{n-1}\left(\int_{a_{jk}-i\infty}^{a_{jk}+i\infty} du_{jk}\beta(u_{jk},K_{jk}-u_{jk})\right)\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{n} dt_i t_i^{\gamma_i-1}\prod_{i=1}^{n-1}$$
$$\left(B_i^{-P_i^1}\right)\prod_{i=1}^{n-2}\left(A_i^{-P_i^2}t_{i+1}^{-P_i^3}\right) \tag{7.17}$$

After using (7.13) on (7.17), and doing necessary simplifications, we obtain,

$$\prod_{j=1}^{n-1}\prod_{k=j}^{n-1}\left(\int_{a_{jk}-i\infty}^{a_{jk}+i\infty} du_{jk}\beta(u_{jk},K_{jk}-u_{jk})\right)\prod_{i=1}^{n-1}\left(\int_{b_i-i\infty}^{b_i+i\infty} dv_i\beta(v_i,P_i^1-v_i)\right)$$
$$\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{n} dt_i t_i^{\gamma_i-1}\prod_{i=1}^{n-2}\left(A_i^{-P_i^2-v_i}t_{i+1}^{-P_i^3-\sum_{i+1\leq j\leq n-1}v_j-\sum_{j<i+1}(P_j^1-v_j)}\right)$$
$$\prod_{i=1}^{n-2}\left(A_i^{-P_i^2-v_i}t_{i+1}^{-P_i^3-\sum_{i+1\leq j\leq n-1}v_j-\sum_{j<i+1}(P_j^1-v_j)}\right)t_1^{-\sum_{j<n}v_j}t_n^{-\sum_{j<n}(P_j^1-v_j)} \tag{7.18}$$

Evaluating the integrals over the Schwinger parameters in (7.18) and simplifying the results, we can get,

$$\prod_{j=1}^{n-1}\prod_{k=j}^{n-1}\left(\int_{a_{jk}-i\infty}^{a_{jk}+i\infty} du_{jk}\beta(u_{jk},K_{jk}-u_{jk})\right)\prod_{i=1}^{n-1}\left(\int_{b_i-i\infty}^{b_i+i\infty} dv_i\beta(v_i,P_i^1-v_i)\right)$$
$$\prod_{i=3}^{n-1}\left[\frac{1}{2}\beta\left(\frac{\gamma_i-P_i^4}{2},P_{i-1}^2+v_{i-1}+\frac{\gamma_{i-1}-P_{i-1}^4}{2}-\frac{\gamma_i-P_i^4}{2}\right)\right]\delta\left(\gamma_1-\sum_{j<n}v_j\right)$$
$$\frac{1}{2}\beta\left(\frac{\gamma_2-P_2^4}{2},P_1^2+v_1-\frac{\gamma_2-P_2^4}{2}\right)\delta\left(\gamma_n-\sum_{j<n}(P_j^1-v_j)\right) \tag{7.19}$$

where,

$$P_{i+1}^4=P_i^3+\sum_{i+1\le j\le n-1}v_j+\sum_{j<i+1}(P_j^1-v_j) \tag{7.20}$$

We can see that the number of independent loop variables in (7.19) is $\frac{(n-2)(n+3)}{2}$.

We wish to simplify the arguments of the beta functions in (7.19) for the edges $i=\{2,n-1\}$. Using the definitions of P_i^1,P_i^2 and P_i^3 (these cannot be further simplified), we can find that,

$$P_i^4=\sum_{j\in L_i^{'}}\sum_{k\in R_i^{'}}K_{jk}+\sum_{j\in L_i^{'}}\sum_{k\in L_i^{'}}u_{jk}+\sum_{i\le j\le n-1}v_j-\sum_{j<i}v_j+\sum_{j\in L_i^{'}}u_{jj} \tag{7.21}$$

$R_i^{'}$ is the set containing the vertex n and all the vertices contained in $\bar{R}_i$, $\bar{L}_i=L_i^{'}$. Now we should put down clearly what $L_i^{'}$ and $R_i^{'}$ mean in this context (refer to Diagram 7.3).

In diagram (7.3) the A_1A_i line cuts the diagram into two parts. The $L_i^{'}$ and $R_i^{'}$ contain the vertices on their respective sides as indicated in the diagram above with $i=4$.

We can simplify the arguments of the beta functions in (7.19) for the edges

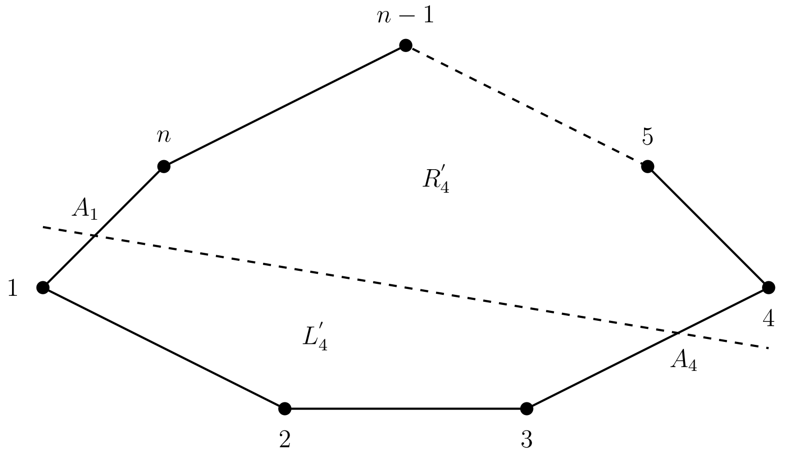


Figure 7.3: $L_i^{'}$ and $R_i^{'}$

$i = \{2, n-1\}$, and obtain,

$$\prod_{j=1}^{n-1}\prod_{k=j}^{n-1}\left(\int_{a_{jk}-i\infty}^{a_{jk}+i\infty} du_{jk}\beta(u_{jk}, K_{jk}-u_{jk})\right)\prod_{i=1}^{n-1}\left(\int_{b_i-i\infty}^{b_i+i\infty} dv_i\beta(v_i, P_i^1-v_i)\right)$$
$$\prod_{i=3}^{n-1}\left[\frac{1}{2}\beta\left(\frac{\gamma_i-\sum_{j\in L_i^{'}}\sum_{k\in R_i^{'}}K_{jk}-E_i}{2}, \frac{D}{2}-\gamma_i\right)\right]\delta\left(\gamma_n-\sum_{j<n}(P_j^1-v_j)\right)$$
$$\frac{1}{2}\beta\left(\frac{\gamma_2-\sum_{k\geq 2}K_{1k}-E_2}{2}, \frac{D}{2}-\gamma_2-\frac{\gamma_1}{2}+\sum_{j\leq n-1}\frac{v_j}{2}\right)\delta\left(\gamma_1-\sum_{j<n}v_j\right) \tag{7.22}$$

where,

$$E_i = \sum_{j\in L_i^{'}}\sum_{k\in L_i^{'}}u_{jk} + \sum_{i\leq j\leq n-1}v_j - \sum_{j<i}v_j + \sum_{j\in L_i^{'}}u_{jj} \tag{7.23}$$

Since we have a $\delta\left(\gamma_1-\sum_{j<n}v_j\right)$ inside the integral, we can replace $\sum_{j\leq n-1}\frac{v_j}{2}$ with $\frac{\gamma_1}{2}$ in the propagator for t_2. Therefore we shall have from (7.22), with

some manipulation,

$$\prod_{j=1}^{n-1}\prod_{k=j}^{n-1}\left(\int_{a_{jk}-i\infty}^{a_{jk}+i\infty} du_{jk}\beta(u_{jk},K_{jk}-u_{jk})\right)\prod_{i=1}^{n-1}\left(\int_{b_i-i\infty}^{b_i+i\infty} dv_i\beta(v_i,P_i^1-v_i)\right)$$
$$\prod_{i=2}^{n-1}\left[\frac{1}{2}\beta\left(\frac{\gamma_i-\sum_{j\in L_i'}\sum_{k\in R_i'}K_{jk}-E_i}{2},\frac{D}{2}-\gamma_i\right)\right]\delta\left(\gamma_1-\sum_{j<n}v_j\right)$$
$$\delta\left(\gamma_n-\sum_{j\in L_n'}\sum_{k\in R_n'}K_{jk}-E_n\right) \tag{7.24}$$

If we define $L_1^{'}$ and $R_1^{'}$ to be empty sets, then we have a meaning for E_1, and we can write the amplitude as,

$$\prod_{j=1}^{n-1}\prod_{k=j}^{n-1}\left(\int_{a_{jk}-i\infty}^{a_{jk}+i\infty} du_{jk}\beta(u_{jk},K_{jk}-u_{jk})\right)\prod_{i=1}^{n-1}\left(\int_{b_i-i\infty}^{b_i+i\infty} dv_i\beta(v_i,P_i^1-v_i)\right)$$
$$\prod_{i=2}^{n-1}\left[\frac{1}{2}\beta\left(\frac{\gamma_i-\sum_{j\in L_i'}\sum_{k\in R_i'}K_{jk}-E_i}{2},\frac{D}{2}-\gamma_i\right)\right]\delta\left(\gamma_n-\sum_{j\in L_n'}\sum_{k\in R_n'}K_{jk}-E_n\right)$$
$$\delta\left(\gamma_1-\sum_{j\in L_1'}\sum_{k\in R_1'}K_{jk}-E_1\right) \tag{7.25}$$

We shall not proceed further with the Mellin amplitude for the n-gon, but rather look at the Mellin amplitude of the n vertex straight chain. Eventually, we would compare the two.

7.3 n vertex chain

The integrand for the n-gon amplitude (the part coming from the Gaussian integral over the vertices) is given by (7.11) and (7.12).

We know that setting any t_i to zero in (7.11) makes the integrand reduce to the corresponding result for the tree level. The form of the resulting integrand depends on which t_i we set equal to zero. The different possible integrands that one could obtain by putting different t_i s to zero correspond to different order of (Gaussian) integration over the internal vertices of the chain. However all of the integrals are equal as they are the same amplitude. We set t_n to zero in (7.11). We can see from (7.12) that this changes only the B_i for $i \leq n-1$ keeping the form of (7.11) still valid. We get,

$$B_i = t_1 \cdots t_i A_i \qquad i \leq n-1 \tag{7.26}$$

We should note that the diagram we are considering is Diagram 7.4

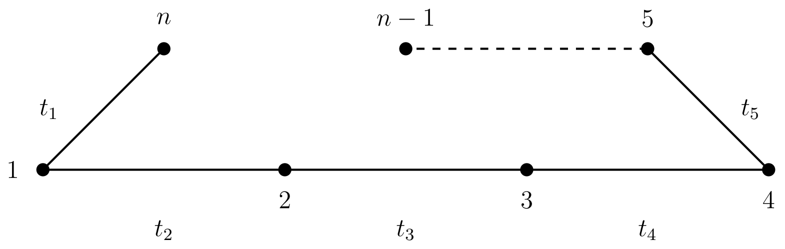


Figure 7.4: n vertex chain

The order of integration over the vertices that gives the integrand we are dealing with here is 1 to n. It is important to note that in this order of integration over the vertices, we are not integrating from one end to the other, in which case we get a rather simple integral that directly gives a product of $n-1$ beta functions. Due to the changed order of integration, the integrand in the Schwinger paramter integral is more complicatd here. But it can indeed be shown that both these Schwinger parameter integrals are equal (though we shall not show it explicitly here).

The amplitude for the skeleton in Diagram 7.4 is,

$$\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n-1} dt_i t_i^{\gamma_i - 1} \prod_{j=1}^{n-1} \left[\prod_{k=j}^{n} \left(C_j^k + B_j B_k\right)^{-K_{jk}} \right] \tag{7.27}$$

(7.27) can be written as,

$$\prod_{j=1}^{n-1} \prod_{k=j}^{n-1} \int_{a_{jk}-i\infty}^{a_{jk}+i\infty} du_{jk} \beta(u_{jk}, K_{jk} - u_{jk}) \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n-1} \left(dt_i t_i^{\gamma_i - 1} A_i^{-Q_i^1} t_i^{-Q_i^2} \right) \tag{7.28}$$

where,

$$\begin{aligned} Q_i^2 = &\quad \sum_{j \in L_i} \sum_{k \in R_i} K_{jk} + \sum_{j \in L_i'} \sum_{k \in L_i'} u_{jk} + \sum_{j \in L_i'} u_{jj} \\ Q_i^1 = &\quad \sum_{j \in L_i} K_{ij} + K_{ii} + \sum_{j \in R_i} u_{ij} \end{aligned} \tag{7.29}$$

L_i' and R_i' have the same meaning as explained in Section 7.2, and L_i and R_i have the same meaning as defined in the Section 6.2 for the tree level diagrams. One should however be careful with the meaning of the index i here. Cutting the ith line means cutting the line between the vertices i and $i-1$ as we have already stated at the beginning of this chapter. In the discussion on the tree level diagrams, by the ith line we had meant the line connecting the vertices i and $i+1$.

We can perform manipulations similar to those in Section 7.2 on (7.28) and then simplify it to obtain the Mellin amplitude,

$$\prod_{j=1}^{n-1}\prod_{k=j}^{n-1}\int_{a_{jk}-i\infty}^{a_{jk}+i\infty} du_{jk}\beta(u_{jk}, K_{jk}-u_{jk})\delta(\gamma_1 - \sum_{j\in L_1}\sum_{k\in R_1} K_{jk} - H_1)$$
$$\prod_{i=2}^{n-1}\frac{1}{2}\beta\left(\frac{\gamma_i - \sum_{j\in L_i}\sum_{k\in R_i} K_{jk} - H_i}{2}, \frac{D}{2} - \gamma_i\right) \quad (7.30)$$

where,

$$H_i = \sum_{j\in L_i'}\sum_{k\in L_i'} u_{jk} + \sum_{j\in L_i'} u_{jj} \quad (7.31)$$

We do not know yet how to integrate over the u_{jk} in (7.30) to obtain the tree level Mellin amplitude that we know from Section 6.2 to be a product of beta functions each of which can be associated with one of the edges in the skeleton. However that is not what we seek to do in this chapter. We are rather interested in the result (7.30) itself as it is similar to result (7.25) for the n-gon in some ways. We wish to look at the similarities and differences between (7.25) and (7.30) and learn something about the Mellin amplitude of the n-gon. This is what the next section is about.

7.4 A clue to the n-gon

First we rerwrite the amplitudes of the n vertex chain and the n gon that we wish to compare. The n vertex chain amplitude is,

$$\prod_{j=1}^{n-1}\prod_{k=j}^{n-1}\int_{a_{jk}-i\infty}^{a_{jk}+i\infty} du_{jk}\beta(u_{jk}, K_{jk}-u_{jk})\delta(\gamma_1 - \sum_{j\in L_1}\sum_{k\in R_1} K_{jk} - H_1)$$
$$\prod_{i=2}^{n-1}\frac{1}{2}\beta\left(\frac{\gamma_i - \sum_{j\in L_i}\sum_{k\in R_i} K_{jk} - H_i}{2}, \frac{D}{2} - \gamma_i\right) \quad (7.32)$$

where,

$$H_i = \sum_{j \in L'_i} \sum_{k \in L'_i} u_{jk} + \sum_{j \in L'_i} u_{jj}$$

The n gon amplitude is,

$$\prod_{j=1}^{n-1} \prod_{k=j}^{n-1} \left(\int_{a_{jk}-i\infty}^{a_{jk}+i\infty} du_{jk} \beta(u_{jk}, K_{jk} - u_{jk}) \right) \prod_{i=1}^{n-1} \left(\int_{b_i - i\infty}^{b_i + i\infty} dv_i \beta(v_i, P_i^1 - v_i) \right)$$
$$\prod_{i=2}^{n-1} \left[\frac{1}{2} \beta \left(\frac{\gamma_i - \sum_{j \in L'_i} \sum_{k \in R'_i} K_{jk} - E_i}{2}, \frac{D}{2} - \gamma_i \right) \right] \delta \left(\gamma_n - \sum_{j \in L'_n} \sum_{k \in R'_n} K_{jk} - E_n \right)$$
$$\delta \left(\gamma_1 - \sum_{j \in L'_1} \sum_{k \in R'_1} K_{jk} - E_1 \right) \tag{7.33}$$

where,

$$E_i = \sum_{j \in L'_i} \sum_{k \in L'_i} u_{jk} + \sum_{i \le j \le n-1} v_j - \sum_{j < i} v_j + \sum_{j \in L'_i} u_{jj}$$

Let us consider the form of the Mellin amplitude of the n vertex chain in (7.32). This is a Mellin Barnes type integral in effectively $\frac{(n+1)(n-2)}{2}$ independent variables u_{jk}. Though we do not know how to compute this integral and obtain a product of $n-1$ beta functions, from our knowledge of the Mellin amplitude of a tree, we know that (7.32) should be equal to,

$$\prod_{i=1}^{n-1} \frac{1}{2} \beta \left(\frac{1}{2} (\gamma_i - \sum_{j \in L_i} \sum_{k \in R_k} K_{jk}), \frac{D}{2} - \gamma_i \right) \tag{7.34}$$

We get the form (7.34) quite easily if we integrate over the internal vertices from one end to the other instead of the order we chose in Section 7.3. Therefore u_{ij} integrals have resulted due to this difference in the order of integration over the interaction vertices.

Now we consider the Mellin amplitude for the n-gon in the form (7.33) and the Mellin amplitude for the n vertex chain in the form (7.32). We see that the ways in which u_{ij} appear in the two Mellin Barnes integrals are very similar. In fact, the $n-2$ beta functions $\beta \left(\frac{\gamma_i - \sum_{j \in L_i} \sum_{k \in R_i} K_{jk} - H_i}{2}, \frac{D}{2} - \gamma_i \right)$ in (7.32) are very similar to the $n-2$ beta functions $\beta \left(\frac{\gamma_i - \sum_{j \in L'_i} \sum_{k \in R'_i} K_{jk} - E_i}{2}, \frac{D}{2} - \gamma_i \right)$

in (7.33). There is an extra delta function in (7.33) (which is probably associated with the extra edge in the n-gon) but the most striking difference between (7.33) and (7.32) is the presence of $n-1$ number of v_i integrals in the the Mellin amplitude of the n-gon (7.33). Also we realize that there is no natural order of integration for the n-gon analogous to the natural order (from one end to the other) for the chain. These factors lead us to guess that the u_{ij} integrals in the n-gon amplitude must also be due to the order of integration over the vertices (which always has to start at an intermediate vertex). It should not be essentially due to the fact that the diagram we are considering is a one-loop diagram. The $n-1$ v_i integrals on the other hand should be intrinsically connected with one-loop diagrams.

What gives us some amount of confidence on this guess is that there is a way these extra $n-1$ v_i integrals could be interpreted for one loop diagrams. From our experience with general quantum field theories, we know that there is an undetermined loop momentum associated with a loop. The Mellin momentum flowing throught the edges in the loop can be put to the form p and $q_i + p$ ($n-1$ in number), where p is the loop Mellin momentum and q_i are from the external lines. Therefore we can have $n-1$ variables of the form $p \cdot q_i$. These are what we think the v_i may be. These can also be related to Mandelstam like variables $(q_i + p)^2$.

Thus for one loop diagrams, we expect that our Feynman rules involve an integration over $n-1$ undetermined variables of the form $p \cdot q_i$ just like usual momentum space Feynman rules for loop diagrams involve integration over the unconstrained loop momentum. In principle, we should be able to integrate over the u_{ij}s in (7.32) and obtain the answer (7.34). Based on that, it is our guess that we can integrate over the u_{ij}s in (7.33) and obtain an answer for the Mellin momentum of the n-gon which has $n-1$ v_i integrals which can be interpreted as a loop integral.

Thus our problem of deriving Feynman rules for the n-gon is now reduced to (in principle) integrating over u_{ij}s in (7.33) and getting an answer which can be expressed as a Mellin Barnes integral over $n-1$ variables, the integrand having at least the n beta functions that can be associated with the n internal lines of the n-gon. We have not been able to do this as of yet.

Chapter 8

Conclusion

In this report we have studied the Mellin space representation of correlation functions in perturbative CFT in some detail. The most important result that we have obtained here is a set of Mellin space Feynman rules that can be associated with tree level perturbative diagrams in any CFT. This work also reviews two very important features of the Mellin representation. Firstly, the Mellin amplitude makes all the conformal symmetries in the theory manifest and the covariance with special conformal symmetry can be interpreted as the conservation of 'Mellin momentum'. Secondly, the Mellin amplitude is meromorphic with the poles of a propagator in the Mellin momentum space corresponding to the scaling dimensions of the exchanged primary field and its descendants. We have also cited some reason to believe that the Mellin amplitude of a one loop diagram is a Mellin Barnes integral analogous to a momentum space loop integral.

We wish to continue our work on loop diagrams and derive a set of Feynman rules (at least in the one loop case). Thereafter we should be able to associate a worldline interpretation to the Mellin amplitudes. This shall provide us with the tools necessary to implement open-closed string duality in the large N limit using this Mellin worldline formalism. Another interesting direction to pursue would be to study the Mellin representation itself in further detail and try to use this formalism in CFTs describing a physical system. This can help us gain further understanding of Mellin momentum and its physical significance (if any).

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