Construction of (1+1)D Analogue gravity model of BEC with specific velocity field

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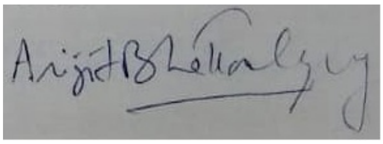
Certificate

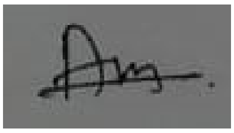
This is to certify that this dissertation entitled Global solution for acoustic fluctuation in Analogue gravity model should appear heretowards the partial fulfilment of the MS degree programme at the Indian Institute of Science Education and Research, Pune represents work carried out by Akash Mukherjee, under the supervision of Dr.Arijit Bhattacharyay, Associate Professor, Department of Physics, during the academic year 2017-2019.



Dr.Arijit Bhattacharyay (Supervisor)

Declaration

I hereby declare that the matter embodied in the report entitled Global solution for acoustic fluctuation in Analogue gravity model are the results of the work carried out by me at the Department of Physics,Indian Institute of Science Education and Research, Pune, under the supervision of Dr.Arijit Bhattacharyay and the same has not been submitted elsewhere for any other degree.



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Abstract

The aim of the thesis is to construct a (1+1)D analogue model of gravity for a BEC. Analogue gravity is observed in such models based on a background profile and that is where it is needed to know what all it takes to experimentally get a particular velocity profile. In this work we have tried to develop a model with hyperbolic tangent velocity profile and indicate the particular confining potential which will result in such profile. Here we derive what PDE the fluctuation of velocity potential follows at independent order of length scale in a stationary, non-homogeneous Bose-Einstein condensation. A global solution of this model will help continuous velocity tracking backward from flat infinity to the analogue event horizon in such system . Solutions of these PDEs i.e fluctuation fields, are significant to calculate Hawking radiation in analogue model.

Contents

Chapter 1

Introduction

1.1 Brief history of Analogue gravity

Analogue model of General Relativity in condensed matter systems have considerable indication where curved space-time emerges from background quantum many-body systems. Key point to understand that some collective properties of these condensed matter systems satisfy equation of motion equivalent to relativistic fields in curved spacetime (please refer to [1]).

In this thesis we have discussed about how acoustic fluctuation behaves around a sonic horizon (analogous to black-hole horizon). It had been already known that this kind of analogy can be drawn for propagation of sonic waves in inviscid fluid however the possibility of blackhole configuration in this kind of fluid systems was first explained by Unruh in a seminal paper, 1981(refer to [2]). This publication was taken up by T. Jacobson, 10 years later ([3]). After 7 years in 1998 a significant work has been done by Visser (please refer to [4]) to establish a rigorous theorem on gravitational analogy with fluid model and discussed analogy in quantities like horizon, ergosphere and surface-gravity.

1.1.1 Emergent curved spacetime

In static homogeneous inviscid fluid sound waves follows

$$\partial_t^2 \psi = c^2 \nabla^2 \psi$$

For sound wave in fluid medium this ψ represents acoustic pressure (local deviation from ambient pressure) and c is the velocity of sound. But how acoustic perturbation propagates through "non-homogeneous flowing fluid" (check [4]) is more subtle than expected, this is addressed by Visser's 1998 paper which is discussed as follows:

" **Theorem**[1] **:** *If a fluid is barotropic and inviscid, and the flow is irrotational (though possibly time dependent) then the equation of motion for the velocity potential describing an acoustic disturbance is identical to the d'Alembertian equation of motion for a minimally coupled massless scalar field propagating in a (3 + 1)–dimensional Lorentzian geometry*

$$\Delta\psi \equiv \frac{1}{\sqrt{-\mathbf{g}}}\partial_\mu(\sqrt{-\mathbf{g}}\mathbf{g}^{\mu\nu}\partial_\nu)\psi = 0."$$

So basically propagation of acoustic fluctuation depends on underlying metric-$\mathbf{g}_{\mu\nu}(t, \mathbf{r})$ (where $\mathbf{g} = \det[\mathbf{g}_{\mu\nu}]$), as acoustic perturbation travel along the null geodesics of the metric (also known as acoustic metric). Speed of this fluctuation in the field (speed of sound) plays the fundamental role as of light speed in spacetime (see [5]). Here the stucture of the metric is

$$\mathbf{g}_{\mu\nu}(t, \mathbf{r}) = \Lambda \begin{bmatrix} -(c^2 - v^2) & -\mathbf{v} \\ -\mathbf{v} & I_{3\times 3} \end{bmatrix} \tag{1.1}$$

[1]refer to [4]

which explicitly depends on velocity of fluid $\mathbf{v}$ and velocity of sound in the fluid c. Λ is a conformal factor (will be discussed) and $I_{3\times 3}$ is the 3 by 3 identical matrix. If the fluid is moving and non-homogeneous (velocity and density profile changes from point to point) then Riemannian tensor associated with metric will be non-zero which imply that background is effectively curved ([4]).

1.1.2 Emergence of curvature from fluctuation in fluid dynamics

We will show proof of the following theorem and equation of motion for the acoustic fluctuation by Visser's ([4]) formulation. In a classical non-relativistic fluid of local density $\rho(t, \mathbf{r})$, velocity $\mathbf{v}(t, \mathbf{r})$ and pressure $p(t, \mathbf{r})$ the fundamental equations are (as follows in [6])

$$\partial_t \rho + \nabla.(\rho \mathbf{v}) = 0 \tag{1.2}$$

$$-\partial_t \psi + \frac{1}{2}(\nabla \psi)^2 + h(p) + V = 0 \tag{1.3}$$

respectively known as Continuity equation and Euler equation. Here

$$\mathbf{v} = -\nabla \psi \tag{1.4}$$

ψ is velocity potential, $h(p)$ is specific enthalpy and V is external potential. Now we can linearize (refer to [4]) above two equations (1.2 and 1.3) around a classical background solution (ρ_0, p_0, ψ_0) i.e by

$$\rho = \rho_0 + \epsilon\rho_1 + O(\epsilon^2) \tag{1.5}$$

$$p = p_0 + \epsilon p_1 + O(\epsilon^2) \tag{1.6}$$

$$\psi = \psi_o + \epsilon\psi_1 + O(\epsilon^2) \tag{1.7}$$

By linearizing (1.2) we get at $O(1)$ and $O(\epsilon)$

$$\partial_t\rho_0 + \nabla.(\rho_0\mathbf{v}_0) = 0 \tag{1.8}$$

$$\partial_t\rho_1 + \nabla.(\rho_0\mathbf{v}_1 + \rho_1\mathbf{v}_0) = 0 \tag{1.9}$$

Now as $\nabla h = \frac{1}{\rho}\nabla p$ and using it in Euler equation we get at $O(1)$ and $O(\epsilon)$

$$-\partial_t\psi_0 + \frac{1}{2}(\nabla\psi_0)^2 + h(p_0) + V = 0 \tag{1.10}$$

$$-\partial_t\psi_1 + \frac{p_1}{\rho_0} - \mathbf{v}_0.\nabla\psi_1 = 0 \tag{1.11}$$

As the fluid is barotropic we can find linearized density fluctuation as

$$\rho_1 = \frac{\partial\rho}{\partial p}p_1 = \frac{\partial\rho}{\partial p}\rho_0\big(\partial_t\psi_1 + \mathbf{v}_0.\nabla\psi_1\big) \tag{1.12}$$

Using (1.11) in (1.9) we will get the wave equation for linearised velocity potential as

$$\partial_t\left(\frac{\partial\rho}{\partial p}\rho_0\big(\partial_t\psi_1 + \mathbf{v}_0.\nabla\psi_1\big)\right) + \nabla.\left(\rho_0\nabla\psi_1 - \left(\frac{\partial\rho}{\partial p}\rho_0\big(\partial_t\psi_1 + \mathbf{v}_0.\nabla\psi_1\big)\right)\mathbf{v}_0\right) = 0 \tag{1.13}$$

We can algebraically simplify equation (1.13) by identifying local speed of sound

$$c^{-2} \equiv \frac{\partial\rho}{\partial p} \tag{1.14}$$

Now we can construct a symmetric 4×4 matrix as

$$f^{\mu\nu}(t, \mathbf{r}) = \frac{\rho_0}{c^2}\begin{bmatrix} -1 & -v_0^j \\ -v_0^j & (c^2\delta^{ij} - v_0^i v_0^j) \end{bmatrix} \tag{1.15}$$

Where δ is kronecker delta. Using (3+1) dimensional space time coordinate we can write (1.12) as

$$\partial_\mu(f^{\mu\nu}\partial_\nu\psi_1) = 0 \tag{1.16}$$

where greek indices run for 0-3 and Roman indices run for 1-3. Now it is easy to show that

$$f = \det[f^{\mu\nu}] = -\frac{\rho_0^4}{c^2} \tag{1.17}$$

If $\mathbf{g}_{\mu\nu}$ be the effective metric then we identify the analogy with gravity by noticing that $f^{\mu\nu} = \sqrt{-\mathbf{g}}\mathbf{g}^{\mu\nu}$ where $\mathbf{g} = \det[\mathbf{g}_{\mu\nu}]$ This imples that

$$\det[f^{\mu\nu}] = \det[\sqrt{-\mathbf{g}}\mathbf{g}^{\mu\nu}] = \left(\sqrt{-\mathbf{g}}\right)^4 \det[\mathbf{g}^{\mu\nu}] = \left(\sqrt{-\mathbf{g}}\right)^4 \mathbf{g}^{-1} = \mathbf{g} \tag{1.18}$$

And $\mathbf{g} = -\frac{\rho_0^4}{c^2}$. Explicit form of metric can be written as

$$\mathbf{g}_{\mu\nu}(t, \mathbf{r}) = \frac{\rho_0}{c}\begin{bmatrix} -(c^2 - v_0^2) & -v_0^j \\ -v_0^i & \delta_{ij} \end{bmatrix} \tag{1.19}$$

If we generalise this determinant to (d+1) dimensional model we will get

$$\mathbf{g} = f^{\frac{2}{d-1}} = \left[(-1)^d \frac{\rho_0^{d+1}}{c^2}\right]^{\frac{2}{d-1}} \tag{1.20}$$

It is clear from (1.19) that this formalism breaks down for (1+1)D model ([7]). However if we consider some symmetry in (3+1) dimensional model for example a three dimensional system with planer symmetry that transforms this (3+1)D

system into a effective (1+1)D model (we will discuss this in context of velocity profile).
So (3+1) dimensional d'Alembertian equation of motion is nothing but the wave equation for ψ_1 in (1.15) which can be written as

$$\Delta\psi_1 \equiv \frac{1}{\sqrt{-\mathbf{g}}}\partial_\mu(\sqrt{-\mathbf{g}}\mathbf{g}^{\mu\nu}\partial_\nu)\psi_1 = 0 \tag{1.21}$$

And Acoustic interval can be written as

$$ds^2 = \mathbf{g}_{\mu\nu}dx^\mu dx^\nu = \frac{\rho_0}{c}\Big[-c^2dt^2 + (dx^i - v_0^i dt)\delta_{ij}(dx^j - v_0^j dt)\Big] \tag{1.22}$$

So the theorem is proved. It is to be noted that signature of the metric is indeed $(-,+,+,+)$ i.e it is a Lorentzian. It should also be noted that here we are dealing with a bi-metric system where background fluid couples to flat-Minkowski metric ($\eta_{\mu\nu} = diag[-c_{light}^2, 1, 1, 1]$) and acoustic fluctuation couples to effective acoustic metric $\mathbf{g}_{\mu\nu}$. This acoustic metric $\mathbf{g}_{\mu\nu}$ suggest that there will be 3 independent scaler $\psi_0(t, \mathbf{x})$, $\rho_0(t, \mathbf{x})$ and $c(t, \mathbf{x})$, however continuity equation reduce it to two, $\psi_0(t, \mathbf{x})$ and $c(t, \mathbf{x})$ ([4]).

1.2 Ideal fluid for analogue gravity models: BEC

After 2000 when Garay *et.al* proposed to simulate black-hole in Bose-Einstein Condensate (please refer to [8]) it was first Barcelo, Liberati and Visser (see [9]) who described how black-holes created in BEC can be used to simulate Hawking radiation[1]. Among other condensed matter system, BEC is preferred as background fluid because of the following reasons

- Temperature for forming BEC [11] is in nK range so it is easier to detect

[1] for Hawkinng radiation see [10]

Hawking temperature and Hawking Radiation.

- Local speed of sound is much smaller than velocity of light and experimentally achievable.
- High degree of quantum coherence.

Chapter 2

Theoretical overview of Bose-Einstein condensates (BEC)

For the past two or three decades the rapid experimental advancements in the study of Bose-Einstein condensates (BEC) compelled the scientific community to rethink about it's applicability in various aspects of statistical physics as well as other branches. However the theoretical development started back in the year of 1924-25 with the help of Albert Einstein and Prof S.N.Bose. Einstein predicted with the help of a scientific article written by Prof S.N.Bose that a phase-transition occurs in a gas of non-interacting particles (now called Bosons) with integral spin[12]. First experimental evidence for the long predicted theory came in 1995, where BEC had been actually seen in dilute atomic gases[13].

2.1 The elementary statistics of Bosons

Bosons follow an elegently unique probability distribution which results in numerous natural phenomena in quantum regime so quantum statistics of Bosons is an intriguing as well as useful subject to learn and nurture.

2.1.1 Ideal Bose gas

In this section we will discuss about the statistics of ideal Bose-gas using grand canonical partition function. For most of this part we'll use the formalism followed by Pitaevskii and Stringari in their text-book on BEC [14].

Suppose in a system, attached to a reservoir of temperature T and chemical potential μ, there are N' number of particles in a state k with energy E_k. In grand canonical ensemble, probability of occurrence of this configuration is

$$P_{N'}(E_k) = \frac{e^{(\mu N' - E_k)}}{Z(\beta, \mu)}, \tag{2.1}$$

Where $Z(\beta, \mu)$ is the the well known grand canonical partition function and β is the inverse temperature defined as $\beta = \frac{1}{k_B T}$. The grand partition function can be calculated as

$$Z(\beta, \mu) = \sum_{N'=0}^{\infty} e^{\beta \mu N'} \sum_k e^{-\beta E_k}. \tag{2.2}$$

Summation over k includes complete set of eigenstate of the Hamiltonian with energy E_k. Now let us consider that the above mention system is characterised by independent particle Hamiltonian

$$\hat{H} = \sum_i \hat{H}_i \tag{2.3}$$

and now the eigenstates k are specified by the microscopic occupation number $\{n_i\}$ of the single particle state. These single particle state are described by solving single particle Schrodinger equations

$$\hat{H}_i \psi_i = \epsilon_i \psi_i \tag{2.4}$$

where ϵ_i is the energy eigenvalue of ith single particle state. Now from our above arrangement we can wright

$$N' = \sum_i n_i \tag{2.5}$$

$$E_k = \sum_i \epsilon_i n_i \tag{2.6}$$

Calculating the grand partition function itself is very tricky and tough job, however from the previous setup we can calculate it after some thermodynamical consideration as elaborately discussed in [14]. Partition function comes out to be

$$Z(\beta, \mu) = \prod_i \frac{1}{1 - e^{\beta(\mu - \epsilon_i)}} \tag{2.7}$$

Now it is quite straightforward to calculate the total number of particles N in the system

$$N(\beta, \mu) = -\frac{\partial}{\partial \mu}\Bigg(- k_B T ln Z(\beta, \mu)\Bigg)_T \tag{2.8}$$

If $\bar{n_i}$ is the average occupation number in i^{th} single particle state then we can wright from (2.7) and (2.8) as

$$\sum_i \bar{n_i} = N = \sum_i \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}$$

which implies

$$\bar{n_i} = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}. \tag{2.9}$$

This is the renowned **Bose-Einstein distribution** function. Inspecting (2.9) it is quite evident that chemical potential μ can never be less than that of single-particle ground state energy. If ϵ_0 be the ground state energy then $\mu < \epsilon_0$ implies a negative occupation number, which is not physical. However if $\mu \to \epsilon_0$ then occupation number

$$N_0 = \bar{n_0} = \frac{1}{e^{\beta(\epsilon_0 - \mu)} - 1} \tag{2.10}$$

of the lowest energy state increases exponentially. This is the starting point of Bose- Einstein Condensation. Note that if μ become identical with ϵ_0 then occupation number diverges.

2.1.2 Theory of condensation

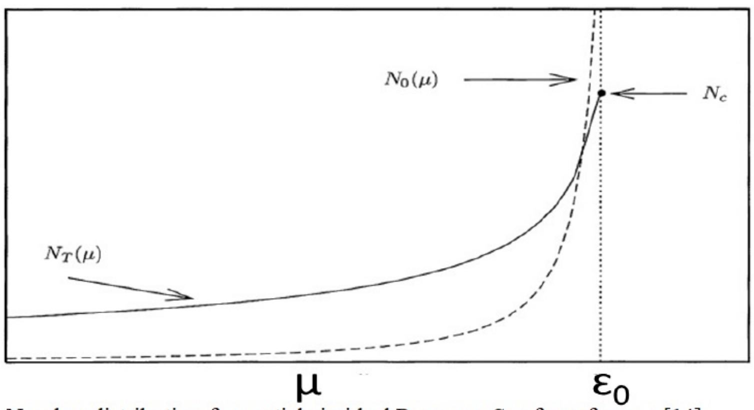
Above treatment suggest that N_0 and $N - N_0$(let's call it N_T) behave very differently(see eqn 3.23 of [14]) and we can always separate them out and wright

$$N = N_0 + N_T \tag{2.11}$$

where

$$N_T(\beta, \mu) = \sum_{i \neq 0} \bar{n}_i(\beta, \mu)$$

is the part of bosonic gase, out of the condensate. This N_T is a smooth function of μ, for a fixed temperature T and reaches maximum N_c at $\mu = \epsilon_0$ (see fig below).



Number distribution for particle in ideal Bose gas. See for reference [14].

Now if at temperature T_c (maximum temperature the system permit to satisfy (2.11)) $N_c(T = T_c, \mu = \epsilon_0) = N$ and then effect of N_0 is negligible. However at $T < T_c$ effect of N_0 is significant in order to satisfy the normalization condition (2.11). So in thermodynamic limit ($\frac{N_0}{N} \neq 0$) as $N_c < N$ then μ is almost equal to ϵ_0 and the system exhibits BEC. This temperature T_c is the critical temperature below which Bose-Einstein condensation takes place. One can find detailed theoretical discussions on how the occupancy fraction for an ideal Bose gases in BEC changes with temperature in [14]. For a elaborate discussion on how a BEC is experimentally formed in an-isotropic harmonic trap please refer to [15].

2.2 Many-body formalism for weakly-interacting Bose gases

Bosonic gases trapped in an external potential below a critical temperature $T < T_c$ macroscopic number of atoms occupy the ground state and condensation takes place($N_0 \sim N$) as we discussed in the previous section.

2.2.1 Dilute Bose gas

This macroscopic occupency in BEC can be represented as quantum field operator $\Psi(\mathbf{r})$ and $\Psi^\dagger(\mathbf{r})$. They respectively annihilate and create particle at a point $\mathbf{r}$. In Bosonic system field operator satisfy well-known equal-time commutation relation

$$[\Psi(\mathbf{r}), \Psi^\dagger(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}')$$

$$[\Psi(\mathbf{r}), \Psi(\mathbf{r}')] = 0 = [\Psi^\dagger(\mathbf{r}), \Psi^\dagger(\mathbf{r}')] \tag{2.12}$$

Hamiltonian of the system can be written as

$$\hat{H} = \int \left(\frac{\hbar^2}{2m} \nabla\Psi^\dagger \nabla\Psi \right) d\mathbf{r} + \int \left(\nabla\Psi^\dagger V_{ext} \nabla\Psi \right) d\mathbf{r} + \frac{1}{2} \int \Psi^\dagger \Psi^{\dagger\prime} V(\mathbf{r}' - \mathbf{r}) \Psi \Psi' d\mathbf{r} d\mathbf{r}' \tag{2.13}$$

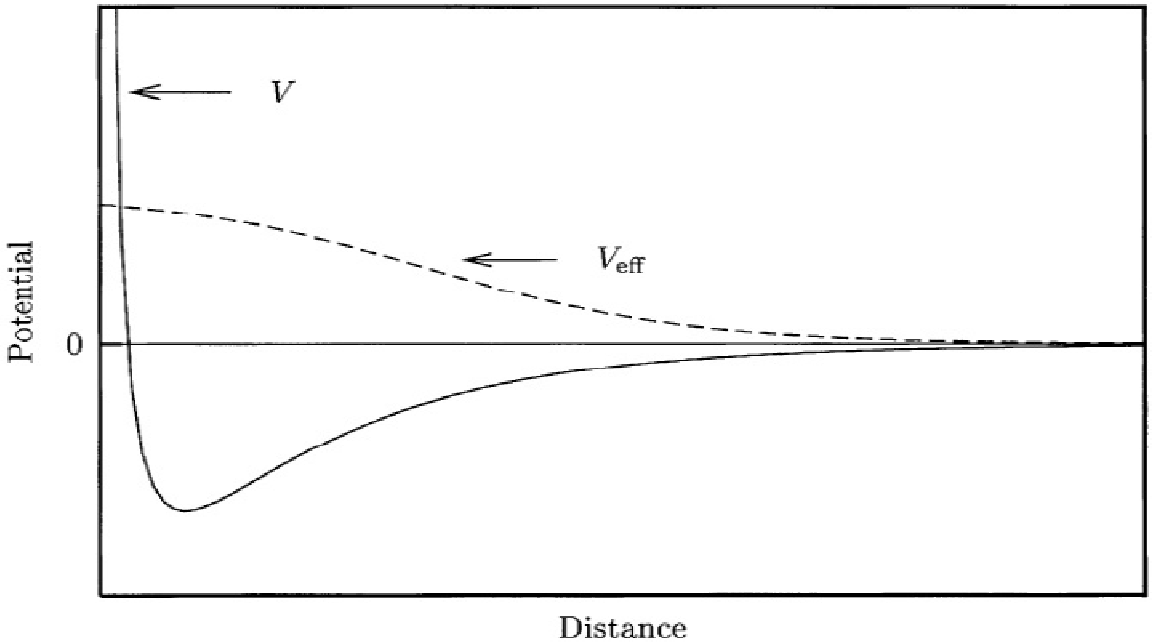
Where V_{ext} is the external trapping potential, m atomic mass and $V(\mathbf{r}' - \mathbf{r})$ is the two body interaction potential. These field operator can be written as (in momentum space)

$$\Psi(t, \mathbf{r}) = \sum_{\mathbf{p}} \hat{a}_{\mathbf{p}} \frac{1}{\sqrt{V}} e^{i\mathbf{p}.\mathbf{r}/\hbar} \tag{2.14}$$

where $\hat{a}_{\mathbf{p}}$ annihilate a particle in single particle state with momentum $\mathbf{p}$ and the gas takes a volume V uniformly. From (2.14) and (2.13) we get

$$\hat{H} = \sum_{\mathbf{p}} \frac{p^2}{2m} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \frac{1}{2V} \sum_{\mathbf{p_1}, \mathbf{p_2}, \mathbf{q}} V_{\mathbf{q}} \hat{a}_{\mathbf{p_1}+q} \hat{a}_{\mathbf{p_2}-q} \hat{a}_{\mathbf{p_1}} \hat{a}_{\mathbf{p_2}} \tag{2.15}$$

where $\mathbf{q} = \mathbf{r}' - \mathbf{r}$ and $V_{\mathbf{q}} = \int V(\mathbf{r}) \exp(-i\mathbf{q}.\mathbf{r}/\hbar) d\mathbf{r}$. For BEC we consider low momentum (low energy) so major interaction is due to s wave scattering. Hence instead of $V(\mathbf{r}' - \mathbf{r})$ which shows a spike when interaction distance is short (where quantum correlation are strong) (see fig in the next page), we can use an effective potential $V_{eff}(\mathbf{r})$ which retains the s wave scattering length a. This $V_{eff}(\mathbf{r})$ is a soft potential (slowly varying) in which perturbation theory can be applied. So using this potential well balance the many-body problem.



Graphical reprentation of real velocity profile (solid line) and effective potential (dashed line). For reference see [14]

Hamiltonian can be written as

$$\hat{H} = \sum_{\mathbf{p}} \frac{p^2}{2m} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \frac{V_0}{2V} \sum_{\mathbf{p_1},\mathbf{p_2},\mathbf{q}} V_{\mathbf{q}} \hat{a}_{\mathbf{p_1}+q} \hat{a}_{\mathbf{p_2}-q} \hat{a}_{\mathbf{p_1}} \hat{a}_{\mathbf{p_2}} \tag{2.16}$$

Where $V_0 = \int V_{eff}(r) d\mathbf{r}$. Now according to Bogoliubov formulation we substitute $\hat{a}_0 = \sqrt{N_0}$ $(N_0 \sim N)$ which supports the effective potential consideration.

In a weakly interacting gas, at $T = 0$, occupancy in states with $\mathbf{p} \neq 0$ is negligible. Neglecting all $\mathbf{p} \neq 0$ terms and putting $\hat{a} = \sqrt{N}$ (as $N_0 \sim N$) in equation (2.16) we get ground state energy as

$$E_0 = \frac{N^2 V_0}{2V} \tag{2.17}$$

Now V_0 can be written as a $V_0 = \frac{4\pi\hbar^2 a}{m}$ (see [14]). So now

$$E_0 = \frac{Nng}{2} \tag{2.18}$$

where $g = \frac{4\pi\hbar^2 a}{m}$ is the interaction strength (as a function of a) and $n = \frac{N}{V}$. (2.18) shows that for dilute gases pressure for ground state doesn't reach zero at zero temperature.

$$P = -\frac{\partial E_0}{\partial V} = \frac{gn^2}{2}.$$

If c is the velocity of sound then from hydrodynamic relation we get

$$\frac{1}{mc^2} = \frac{\partial n}{\partial P} = \frac{1}{gn}$$

So $c = \sqrt{\frac{gn}{m}}$. This c infact coincides with the form one can get from the dispersion relation of the elementary fluctuation in low energy limit[14] .

2.2.2 Quantum fluctuation and local speed of sound

In the previous section we ignore $\mathbf{p} \neq 0$ in (2.16). As a one step ahead if we consider higher order fluctuation then we have to retain upto quadratic terms for particle operator[1]. Taking this approximation we can find the corrected ground state energy as

$$E_0 = \frac{Nng}{2} + \frac{1}{2}\sum_{\mathbf{p}\neq 0}\left[\epsilon(p) - gn - \frac{p^2}{2m} + \frac{mg^2n^2}{p^2}\right] \tag{2.19}$$

where

$$\epsilon(p) = \left(\frac{gn}{m}p^2 + (\frac{p^2}{2m})^2\right)^{1/2} \tag{2.20}$$

[1]for a detailed analysis refer to section 4.2 in [14]

is the well known **Bogoliubov dispersion** relation. This is the dispersion relation of non-interacting quasi particle (phonons) which describe the fluctuation in the interacting Bose gas. For small momentum regime dispersion relation for these quasi particles takes the form

$$\epsilon(p) = cp \tag{2.21}$$

where $c = \sqrt{\frac{gn}{m}}$ is velocity of sound and it is same as what we get in the previous section.

2.2.3 Healing length

For high momentum regime dispersion relation takes the form

$$\epsilon(p) \approx \frac{p^2}{2m} + gn \tag{2.22}$$

The transition occurs between this two regime is when $p \sim mc$ i.e when $\frac{p^2}{2m} = gn$. For $p = \frac{\hbar}{\xi_0}$, we can find

$$\xi_0 = \sqrt{\frac{\hbar^2}{2mgn}} \tag{2.23}$$

where this ξ_0 is known as **Healing length**. This is also known as characteristics interaction length. We will discuss it's application in analogue gravity perspective in next chapter.

2.3 Local Gross-Pitaevskii equation

If the temperature of systems bring down below the critcal temperature (T_c), then it condenses and according to Bogoliubov prescription [16] Ψ can be seperated as a macroscopic wavefunction ψ plus a fluctuation.

$$\Psi(t, \mathbf{r}) = \psi(t, \mathbf{r}) + \Delta\psi(t, \mathbf{r}) \tag{2.24}$$

In dilute atomic cases at very low temperature the non-condesate part $\Delta\psi(t, \mathbf{r}) \to 0$. This mean-field approximation is an excellent approach to investigate the evolution of ψ in weakly interacting gases (s wave scattering length is much smaller than average inter-atomic distance) in thermodynamic limit. For those gases dynamics of ψ can be found by following prescription.
Effective potential consideration in the 2.2 section will be again helpful here, as we will substitute $V(\mathbf{r}' - \mathbf{r})$ by $g\delta(\mathbf{r}' - \mathbf{r})$ where $g = \frac{4\pi\hbar^2 a}{m}$

$$V(\mathbf{r}' - \mathbf{r}) = g\delta(\mathbf{r}' - \mathbf{r}) \tag{2.25}$$

Using (2.13) and (2.25) we get

$$\hat{H} = \int \left(\frac{\hbar^2}{2m}\nabla\Psi^\dagger\nabla\Psi\right) d\mathbf{r} + \int \left(\nabla\Psi^\dagger V_{ext}\nabla\Psi\right) d\mathbf{r} + \frac{g}{2}\int \left(\Psi^\dagger\Psi^\dagger\Psi\Psi\right) d\mathbf{r} \tag{2.26}$$

Using (2.24) and by Heisenberg's evolution of $\hat{H}$ we get

$$i\hbar\partial_t\psi = \left(\frac{\hbar^2}{2m}\nabla^2 + V_{ext} + g|\psi|^2\right)\psi \tag{2.27}$$

This is a non-linear Schrodinger equation known as **Gross-Pitaevskii equation**. This equation shows the complex dynamics of order paremeter ψ for a in-homogeneous condensate [17],[18].

Chapter 3

Analogue gravity model in BEC system

As we discussed in previous chapters that a BEC consists of high degree of quantum coherence, very cold temperature, low velocity of sound and as is convenient to work with in experimental setup, it provide the best test field for semi-classical gravity phenomenon such as hawking radiation. As a matter of fact BEC is extensively used in analogue models of gravity [7].

3.1 Dynamics of condensate

We can express Gross-Pitaevskii equation in terms of independent quantities (as we discussed in first chapter), velocity of sound c and velocity of fluid flow v for a hydrodynamic system. Using Madelung ansatz for the order parameters

$$\psi(t, \mathbf{r}) = \sqrt{n(t, \mathbf{r})}e^{i\theta(t,\mathbf{r})/\hbar} \tag{3.1}$$

where $|\psi(t,\mathbf{r})|^2 = n(t,\mathbf{r})$ and $v \equiv \frac{\nabla\theta}{m}$, GP equation can be written as two coupled equations

$$\partial_t n = -\frac{1}{m}\nabla.(\nabla\theta) \tag{3.2}$$

$$\partial_t \theta = -\frac{1}{2m}(\nabla\theta)^2 - gn - V_{ext} - V_{quantum} \tag{3.3}$$

where

$$V_{quantum} = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}. \tag{3.4}$$

This $V_{quantum}$ is known as **'quantum potential'**. Generally this term is neglected in standard literature, instead we will use it to investigate dynamics at different order of length scale which we will discuss in next chapter elaborately. (3.2) and (3.3) are Continuity equation and Euler equation respectively, which describe the condensate in a hydrodynamic context.

3.2 Dumb-hole configuration in BEC

To construct an analogue gravity model first we will carefully construct a background velocity field and will use the formalism described in previous section to analyze the dynamics of the condensate.

3.2.1 $(1+1)$ dimensional model

In this section we will construct a effective (1+1) dimensional[1] BEC configuration which consists of varying velocity and density profile which will be stationary i.e time independent. We will choose the velocity background field in such a way that

[1]Please refer to section (1.1.2)

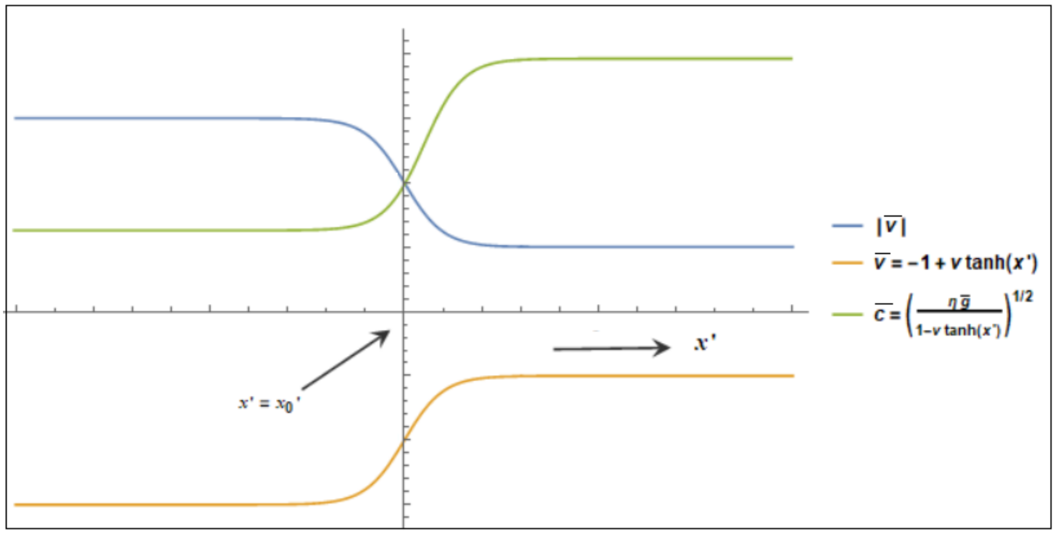
- Parameters are so chosen that there is a single accoustic black-hole at $x = x_0$ where x_0 is adjustable. This is analogues to general relativistic black-hole horizon.
- Left side of the horizon behaves as general relativistic singularity, from which nothing can escape.
- Right side of the horizon mimics the asymptotic behaviour (as $x \to +\infty$) of the general relativistic black-hole i.e velocity and density will be uniform.
- We will also include smooth transition between this two region to avoid any discontinuity at horizon x_0 .
- We will also like to include parameters which help us to control the analytic structure of the background fluid.
- As per the convention we will choose the fluid motion from right to left. So velocity will always be negative.

Major work has been done on this by Corley and Jacobson (refer to [19]) in 1996 and by Parentani (see [20]) in 2009. Unlike these, in this work we will discuss about possibility of a global solution (where $x \in (-\infty, +\infty)$) at different order of length scale.

So let's write a guessed velocity profile and check whether they follow the rules previously mentioned. This velocity can be written as

$$v(x) = -\alpha + \beta \tanh \gamma x \tag{3.5}$$

where α ensure that overall velocity remain negative ($\alpha > \beta$). $|v|$ is the magnitude of the velocity. β defines asymptotic value of velocity and γ defines how fast the velocity changes in transition region or in other word what will be the width of the transition region. It is evident that γ has a dimension of inverse of length and α, β have dimension of velocity i.e LT^{-1} .



Graphical represntation of velocity of fluid and velocity of sound in primed coordinate

Here if $\gamma \to \infty$ i.e a step-like discontinuity which is discussed in many papers (for example see [19]). Considering this velocity profile, the velocity of sound (refer to section 2.2.1) comes out to be of the form $c \propto \frac{1}{\sqrt{\alpha - \beta \tanh \gamma x}}$ (we will discuss this in next chapter). If we consider $|v| = c$ then we will get a sonic horizon at $x = x_0$ (explicit form will be calculated in next chapter). Left hand asymptotic region of this x_0 point is the singularity (subsonic region) as nothing can escape towards horizon. However there is no intrinsic singularity in this region unlike general relativity. tanh is a smooth function for $x \in (-\infty, +\infty)$ so there is no discontinuity in velocity field. And most of all fluid velocity and density is uniform for $x \to \infty$.

3.2.2 Nondimensionalization

It is always useful to non-dimensionalize an equation so that number of parameters remain minimum. We will use γ and α to make Euler and continuity equation dimensionless in our model.

$$\partial_{t'}\bar{n} = -\nabla'.(\nabla'\bar{\theta}) \tag{3.6}$$

$$\partial_{t'}\bar{\theta} = -\frac{1}{2}(\nabla'\bar{\theta})^2 - \bar{g}\bar{n} - \bar{V_{ext}} + \frac{\hbar^2}{2l^2}\frac{\nabla'^2\sqrt{\bar{n}}}{\sqrt{\bar{n}}} \tag{3.7}$$

where $x'_i = \gamma x_i$ and $t' = \alpha\gamma t$ are scaled time and space coordinate where $x_i \equiv (x, y, z)$. And

$$\bar{n} = \frac{n}{\gamma^3} \tag{3.8}$$

$$\bar{\theta} = \frac{\theta\gamma}{m\alpha} \tag{3.9}$$

$$\bar{g} = \frac{\gamma^3 g}{m\alpha^2} \tag{3.10}$$

$$\bar{V_{ext}} = \frac{V_{ext}}{m\alpha^2} \tag{3.11}$$

$$l = \frac{m\alpha}{\gamma} \tag{3.12}$$

are new dependent parameters. In new coordinate velocity can be written as (from (3.9) and (3.5))

$$\bar{v}(x') = \frac{v}{\alpha} = -1 + \nu \tanh x' \tag{3.13}$$

where $\nu = \frac{\beta}{\alpha} < 1$. These bared quantities are new dimensionless parameters of our model. We can easily revert back to traditional parameters by using (3.8) to (3.12). $\bar{n}$ is dimensionless density, $\bar{\theta}$ is dimensionless phase parameter which gives rise to velocity in scaled coordinate (3.13), $\bar{g}$ and $\bar{V_{ext}}$ are dimensionless potential strength and external potential respectively. l has a dimension of angular momentum which makes $\frac{\hbar^2}{2l^2}$ in (3.7) dimensionless.

Generally the last term in (3.7) is neglected as $\frac{\hbar^2}{2l^2}$ is of the order of ξ_0^2 (Thomas-

Fermi limit) where ξ_0 is the Healing length of condensate. However we will keep it and use it as an expansion parameter to perturbatively expand fluctuation field $\bar{\theta}$ in order to follow it's dynamics at different order of length scale.

Chapter 4

Analysis of (1+1) dimensional model on multiple independent length scale

We discussed in previous chapter that as ξ_0 is a small quantity so we will use $\epsilon = \gamma\xi_0$ as a expansion parameter to expand $\bar{\theta}$ and $\bar{n}$. Dynamics at different order of ϵ are independent of each other. Here $\xi_0 = \frac{\hbar}{\sqrt{2mgN}}$, where N is the far away ($x \to \infty$) density of the condensate. Explicit form of N will be calculated in next section. Now we can perturbatibely expand $\bar{n}$ and $\bar{\theta}$ around a classical background solution $\bar{n_0}$ and $\bar{\theta_0}$ as

$$\bar{n} = \bar{n}_0 + \epsilon\bar{n}_1 + \epsilon^2\bar{n}_2 \tag{4.1}$$

$$\bar{\theta} = \bar{\theta}_0 + \epsilon\bar{\theta}_1 + \epsilon^2\bar{\theta}_2 \tag{4.2}$$

Many significant work has been done on *linearized* Gross-Pitavskii equation[7] i.e keeping term upto 1st order of ϵ. We will keep expansion upto order ϵ^2 as the quantum potential term in (3.7) is itself upto ϵ^2 as $\frac{\hbar^2}{2l^2} = \bar{g}\bar{N}\epsilon^2$, and we want to see dynamics upto ϵ^2 order. $\bar{N}$ in previous line is dimensionless far away density which will be dicussed in next section. Now we will substitute (4.1) and (4.2) in (3.7) and (3.6) and will get 3 set of coupled equation in different order of ϵ. We will discuss them one by one.

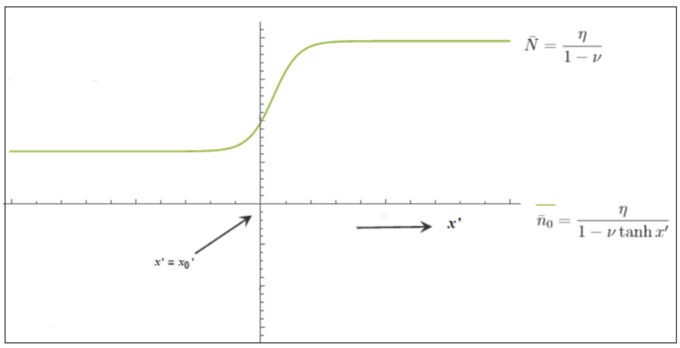
4.1 Background velocity and density: ϵ^0 order

At 0^{th} order, coupled equation comes of as

$$\frac{\partial \bar{n}_0}{\partial t'} + \nabla'.(\bar{n}_0 \nabla' \bar{\theta}_0) = 0 \tag{4.3}$$

$$\frac{\partial \bar{\theta}_0}{\partial t'} + \frac{1}{2}(\nabla' \bar{\theta}_0)^2 + \bar{V_{ext}} + \bar{g}\bar{n}_0 = 0 \tag{4.4}$$

We assume that background velocity and density are time-independent. As we have taken an effective (1+1) dimensional model we will consider velocity to be varying in only one direction i.e along x-axis (as we have already discussed this in previous chapters). We can take $v(y) = 0$ and $v(z) = 0$ and hence $\frac{\partial \bar{\theta}_0}{\partial y'} = 0$ and $\frac{\partial \bar{\theta}_0}{\partial z'} = 0$ and $v(x) = \frac{\partial \bar{\theta}_0}{\partial x'} = -1 + \nu \tanh x'$. So from equation (4.3) we can calculate the background density.



Graphical representation of density profile

$$\bar{n}_0 = \frac{\eta}{1 - \nu \tanh x'} \tag{4.5}$$

Here η is a positive real number. Magnitude of η can be found from normalisation of $\bar{n_0}$ (please refer to [14]). Quite evident from (4.5) that far away (where $x \to \infty$ i.e $x' = \gamma x \to \infty$) dimensionless density is (see figure in previous page)

$$\bar{N} = \frac{\eta}{1 - \nu} \tag{4.6}$$

and $N = \gamma^3 \bar{N}$. If we have to maintain this background formation we have to keep pushing it by a external potential, which can be formulated from (4.4) as

$$\bar{V_{ext}} = -\frac{\bar{g}\eta}{1 - \nu \tanh x'} - \frac{1}{2}(1 - \nu \tanh x')^2 \tag{4.7}$$

Above description of background condensate from a hydrodynamic context shows what it takes to fix a stationary velocity profile in it.

4.2 Acoustic metric: ϵ^1 order

4.2.1 Effective metric from underlying BEC

At 1st order of ϵ

$$\frac{\partial \bar{n}_1}{\partial t'} + \bigtriangledown'.(\bar{n}_0 \bigtriangledown' \bar{\theta_1} + \bar{n}_1 \bigtriangledown' \bar{\theta_0}) = 0 \tag{4.8}$$

$$\frac{\partial \bar{\theta}_1}{\partial t'} + (\bigtriangledown' \bar{\theta_0} . \bigtriangledown' \bar{\theta_1}) + \bar{g}\bar{n}_1 = 0 \tag{4.9}$$

If we substitute $\bar{n}_1$ from (4.9) to (4.8) we will get a 2nd order PDE of $\bar{\theta}_1$. Explicitly we can write the PDE as,

$$\frac{\partial}{\partial t'}(-\frac{1}{\bar{g}}\frac{\partial\bar{\theta}_1}{\partial t'})+\frac{\partial}{\partial t'}(-\frac{1}{\bar{g}}\frac{\partial\bar{\theta}_0}{\partial x'}\frac{\partial\bar{\theta}_1}{\partial x'})+\frac{\partial}{\partial x'}(-\frac{1}{\bar{g}}\frac{\partial\bar{\theta}_0}{\partial x'}\frac{\partial\bar{\theta}_1}{\partial t'})+$$
$$\frac{\partial}{\partial x'}[(\bar{n}_0-\frac{1}{\bar{g}}(\frac{\partial\bar{\theta}_0}{\partial x'})^2)\frac{\partial\bar{\theta}_1}{\partial x'}]+\bar{n}_0(\frac{\partial^2\bar{\theta}_1}{\partial y'^2}+\frac{\partial^2\bar{\theta}_1}{\partial z'^2})=0$$

Introducing (3+1) dimensional space-time coordinate (Greek indices run from 0-3 and Roman indices run from 1-3),

$$x'^{\mu}\equiv(t',x'^{i}) \tag{4.10}$$

the wave equation for $\bar{\theta}_1$ can be easily written as,

$$\partial'_{\mu}(f^{\mu\nu}\partial'_{\nu}\bar{\theta}_1)=0 \tag{4.11}$$

where $f^{\mu\nu}$ is a symmetric 4×4 matrix and can be written as

$$f^{\mu\nu}=\begin{bmatrix}-\frac{1}{\bar{g}} & -\frac{1}{\bar{g}}\frac{\partial\bar{\theta}_0}{\partial x'} & 0 & 0\\ -\frac{1}{\bar{g}}\frac{\partial\bar{\theta}_0}{\partial x'} & \bar{n}_0-\frac{1}{\bar{g}}(\frac{\partial\bar{\theta}_0}{\partial x'})^2 & 0 & 0\\ 0 & 0 & \bar{n}_0 & 0\\ 0 & 0 & 0 & \bar{n}_0\end{bmatrix} \tag{4.12}$$

If we identify $f^{\mu\nu}=\sqrt{-\mathbf{g}}\mathbf{g}^{\mu\nu}$, where $\mathbf{g}_{\mu\nu}$ is the effective metric generated from the background dynamics of BEC and $\mathbf{g}=\det[\mathbf{g}_{\mu\nu}]$, equation (4.11) can be written in covarient form which is

$$\Delta\bar{\theta}_1 \equiv \frac{1}{\sqrt{-\mathbf{g}}}\partial'_\mu(\sqrt{-\mathbf{g}}\mathbf{g}^{\mu\nu}\partial'_\nu)\bar{\theta}_1 = 0 \tag{4.13}$$

where

$$\mathbf{g}_{\mu\nu} = \sqrt{\frac{\bar{n}_0}{\bar{g}}}\begin{bmatrix} \bar{v}^2 - \bar{n}_0\bar{g} & -\bar{v} & 0 & 0 \\ -\bar{v} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.14}$$

From equation (4.13) we can see that acoustic perturbation propagate according to a d'Alembertian equation of motion which is similar to the wave equation of a mass-less scalar field propagating in a curved space-time (as we discussed in first chapter). Determinant of metric can be found from equation (4.14) as

$$\mathbf{g} = \det[\mathbf{g}_{\mu\nu}] = -\frac{\bar{c}^6}{\bar{g}^4} \tag{4.15}$$

As we have already identified that $f^{\mu\nu} = \sqrt{-\mathbf{g}}\mathbf{g}^{\mu\nu}$ hence,

$$\det[f^{\mu\nu}] = \det[\sqrt{-\mathbf{g}}\mathbf{g}^{\mu\nu}] = \left(\sqrt{-\mathbf{g}}\right)^4 \det[\mathbf{g}^{\mu\nu}] = \left(\sqrt{-\mathbf{g}}\right)^4 \mathbf{g}^{-1} = \mathbf{g} \tag{4.16}$$

4.2.2 Velocity of sound and sonic horizon

If we compare equation (4.14) and equation (1.18), it is quite simple to identify that

$$\bar{c}^2 = \bar{n}_0\bar{g} = \frac{\bar{g}\eta}{1 - \nu\tanh x'} \tag{4.17}$$

Here $\bar{c} = \frac{c}{\alpha}$ is the dimensionless velocity of sound in BEC(please refer to the image in previous chapter). We can revert back to our traditional parameter by using

(3.8) and (3.10) and then structure of c comes out to be same as we discussed in section (2.2.1). It is quite straightforward that at the horizon

$$\bar{v}^2 = \bar{c}^2 \tag{4.18}$$

$$\Rightarrow |\bar{v}| = +\bar{c} \tag{4.19}$$

So if horizon is at $x' = x'_0$ then at x'_0

$$\frac{\bar{g}\eta}{1 - \nu \tanh x'_0} = (1 - \nu \tanh x'_0)^2$$

Then we have

$$x'_0 = \tanh^{-1}\left(\frac{1 - (\bar{g}\eta)^{1/3}}{\nu}\right) \tag{4.20}$$

So now the exact location of horizon is identified in terms of already fixed quantities. At x'_0 there is a transition between subsonic region to supersonic region. We can change the position of sonic horizon by changing parameters in system. If we choose $\bar{g}\eta = 1$ then horizon will be at $x = 0$. Analytically this choosing may seem easy but to experimentally achieving this is a different scenario. there will be a constraint on η which will be imposed on by the normalization condition of atomic density (see [14]).

4.2.3 Dynamics of phase fluctuation

In equation (4.11) the $\bar{\theta}_1$ term is the "Quantum acoustic representation" of perturbation (see [7]) which means they are complex quantum field. Here we will treat $\bar{\theta}_1$ as a classical wave which can be quantized later. These algebric classical solutions indeed help us to understand the nature of in going and outgoing modes.

If we write (4.11) in more explicit form

$$\frac{\partial}{\partial t'}\left(\frac{\partial\bar{\theta}_1}{\partial t'}\right)+\frac{\partial}{\partial t'}\left((-1+\nu\tanh x')\frac{\partial\bar{\theta}_1}{\partial x'}\right)+\frac{\partial}{\partial x'}\left((-1+\nu\tanh x')\frac{\partial\bar{\theta}_1}{\partial t'}\right) \tag{4.21}$$

$$+\frac{\partial}{\partial x'}\left[\left((1-\nu\tanh x')^2-\frac{\eta\bar{g}}{(1-\nu\tanh x')}\right)\frac{\partial\bar{\theta}_1}{\partial x'}\right]=0$$

We can see that algebraically the equation is a polynomial of infinite order. Mathematically this is a problem however we can get rid of the tanh function by a substitution $\tanh x' = z$, then the equation becomes finite order polynomial. So $\bar{\theta}_1$ become a function of z and t'. The first term in (4.21) remain as it is

$$\frac{\partial}{\partial t'}\left(\frac{\partial\bar{\theta}_1(x',t')}{\partial t'}\right)=\frac{\partial^2\bar{\theta}_1(z,t')}{\partial t'^2}.$$

2nd term

$$\frac{\partial}{\partial t'}\left((-1+\nu\tanh x')\frac{\partial\bar{\theta}_1}{\partial x'}\right)=(-1+\nu z)(1-z^2)\frac{\partial^2\bar{\theta}_1}{\partial z\partial t'}$$

3rd term

$$\frac{\partial}{\partial x'}\left((-1+\nu\tanh x')\frac{\partial\bar{\theta}_1}{\partial t'}\right)=\nu(1-z^2)\frac{\partial\bar{\theta}_1}{\partial t'}+(-1+\nu z)(1-z^2)\frac{\partial^2\bar{\theta}_1}{\partial z\partial t'}$$

4th term

$$\frac{\partial}{\partial x'}\left[\left((1-\nu\tanh x')^2-\frac{\eta\bar{g}}{(1-\nu\tanh x')}\right)\frac{\partial\bar{\theta}_1}{\partial x'}\right]=\left(1-z^2\right)^2\left(-\frac{\eta\nu\bar{g}}{(1-\nu z)^2}-2\nu(1-\nu z)\right)\frac{\partial\bar{\theta}_1}{\partial z}$$

$$+(1-z^2)\left((1-\nu z)^2-\frac{\eta\bar{g}}{1-\nu z}\right)\left((1-z^2)\frac{\partial^2\bar{\theta}_1}{\partial z^2}-2z\frac{\partial\bar{\theta}_1}{\partial z}\right)$$

Now dynamics of $\bar{\theta}_1$ becomes

$$(1-2z\nu+z^2\nu^2)\frac{\partial^2\bar{\theta}_1}{\partial t'^2}+\Big(-2\nu^3z^5+6\nu^2z^4+\left(2\nu^3-6\nu\right)z^3+\left(2-6\nu^2\right)z^2+6\nu z-2\Big)\frac{\partial^2\bar{\theta}_1}{\partial z\partial t'}+$$

$$\Big(\nu^4z^8-4\nu^3z^7+\left(6\nu^2-2\nu^4\right)z^6+z^5\left(\eta\nu\bar{g}+8\nu^3-4\nu\right)+z^4\left(-\eta\bar{g}+\nu^4-12\nu^2+1\right)$$

$$+z^3\left(-2\eta\nu\bar{g}-4\nu^3+8\nu\right)+z^2\left(2\eta\bar{g}+6\nu^2-2\right)+z\left(\eta\nu\bar{g}-4\nu\right)+1-\eta\bar{g}\Big)\frac{\partial^2\bar{\theta}_1}{\partial z^2}+$$

$$\Big(4\nu^4z^7-14\nu^3z^6+\left(18\nu^2-6\nu^4\right)z^5+z^4\left(\eta\nu\bar{g}+20\nu^3-10\nu\right)+z^3\left(-2\eta\bar{g}+2\nu^4-24\nu^2+2\right)+$$

$$\left(12\nu-6\nu^3\right)z^2+z\left(2\eta\bar{g}+6\nu^2-2\right)-2\nu-\eta\nu\bar{g}\Big)\frac{\partial\bar{\theta}_1}{\partial z}.$$

$$+\Big(2z\nu^2+\nu(\nu^2-1)z^2+2z^3\nu^2-z^4\nu^3+\nu\Big)\frac{\partial\bar{\theta}_1}{\partial t'}=0 \qquad (4.22)$$

Here we have used

$$\frac{\partial z}{\partial x'}=sech^2(x')=1-z^2$$

If we want to set horizon at $x'_0=0$ above equation becomes

$$(1-2z\nu+z^2\nu^2)\frac{\partial^2\bar{\theta}_1}{\partial t'^2}+\Big(-2\nu^3z^5+6\nu^2z^4+\left(2\nu^3-6\nu\right)z^3+\left(2-6\nu^2\right)z^2+6\nu z-2\Big)\frac{\partial^2\bar{\theta}_1}{\partial z\partial t'}+$$

$$\Big(\nu^4z^8-4\nu^3z^7+\left(6\nu^2-2\nu^4\right)z^6+z^5\left(8\nu^3-3\nu\right)+z^4\left(+\nu^4-12\nu^2\right)$$

$$+z^3\left(+6\nu-4\nu^3\right)+6\nu^2z^2-3\nu z\Big)\frac{\partial^2\bar{\theta}_1}{\partial z^2}+\Big(-3\nu+4\nu^4z^7-14\nu^3z^6+\left(18\nu^2-6\nu^4\right)z^5+z^4\left(20\nu^3-9\nu\right)$$

$$+z^3\left(2\nu^4-24\nu^2\right)+\left(12\nu-6\nu^3\right)z^2+6\nu^2z\Big)\frac{\partial\bar{\theta}_1}{\partial z}+\Big(\nu-2z\nu^2+\nu(\nu^2-1)z^2+2z^3\nu^2-z^4\nu^3\Big)\frac{\partial\bar{\theta}_1}{\partial t'}=0. \qquad (4.23)$$

This above choosing can help investigate the dynamics very near to the horizon as one can possibly pertubatively expand z very near to $x' = 0$ and retain lower orders of z. Solutions of this $\bar{\theta}_1$ are actually relevent to hawking effect (as discussed first chapter) as after imposing some intelligent boundary condition we can get hawking mode solution and at the horizon, outgoing mode solutions gives rise to Hawking effect (see [10]).

4.3 Correction to the fluctuation: ϵ^2 order

At this order

$$\frac{\partial \bar{n}_2}{\partial t'} + \nabla'.(\bar{n}_0 \nabla' \bar{\theta}_2 + \bar{n}_2 \nabla' \bar{\theta}_0 + \bar{n}_1 \nabla' \bar{\theta}_1) = 0 \tag{4.24}$$

$$\frac{\partial \bar{\theta}_2}{\partial t'} + (\nabla' \bar{\theta}_0 . \nabla' \bar{\theta}_2) + \frac{1}{2}(\nabla' \bar{\theta}_1)^2 + \bar{g}\bar{n}_2 - \underbrace{\bar{g}\bar{N}\frac{\nabla'^2 \sqrt{\bar{n}_0}}{\sqrt{\bar{n}_0}}} = 0 \tag{4.25}$$

So we will actually get to know about the effect of quantum potential on fluctuation in local Gross-Pitaevskii model. However this will be a small correction (of the order of ξ_0^2) to the condensate wavefunction. In equation (4.25) the term underlined by curly bracket is the effect of quantum potential in the dynamics of Hawking fluctuation. As this term is consists of zeroth order density function, can be analytically calculated.

4.4 Discussion and outlook

Hawking radiation is essentially the thermal spectrum of outgoing particles (in a given outgoing packet) created from initial vaccum state, calculated in free-fall frame of black-hole[10]. So at horizon there is a "mode-conversion" between in-going wave-packet and outgoing wave-packet. To accurately calculate the number of particles in wave-packet (solution of $\bar{\theta}_1$) one have to propagate the packet back in time and calculate the norm of negative energy part of the solution ([19]). These formulation are discussed in [3] and [20] rigorously however, there is a discontinuity in space (to be precise in x direction) considered in these two papers (and this is a general consideration in standard literature of analogue gravity[1]) which makes the dynamics obviously incomplete and doubtful (at horizon). In our model, however transition around horizon is smooth which makes the model more complete. From equation (4.24) and (4.25) it is evident that at ϵ^2 order we can get a space dependent dispersion relation which may indicate interesting property about the dumb-hole configuration. So the Analogue model described in this thesis not only eases the challanges in calculation of Hawking radiation but also looks deeper (to the ξ_0^2 order) in the dynamics around a smooth horizon. It is important to mention that whole formulation is from a hydrodynamic context and background is generated from the dynamics of the fluid unlike general relativity where Einstein's equation represents the dynamics. However the Analogue model which establishes Einstein's equation is yet to find and still an open question (see [7]).

[1] see for reference [7]

References

[1] N. D. Birrell and P. C. W. Davies, *Quantum Fields In Curved Space*. 1

[2] W. G. Unruh, "Experimental black-hole evaporation?," (1981). 1

[3] T. Jacobson, "Black hole evaporation and ultrashort distances," 1991. 1, 33

[4] M. Visser, "Acoustic black holes: Horizons, ergospheres, and hawking radiation," 1998. 1, 2, 3, 6

[5] S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*. 2

[6] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*. 3

[7] Barcelo, Libareti, and Visser, "Analogue gravity, living reviews in relativity," 2002. 5, 18, 24, 29, 33

[8] J. L.J.Garay, J.R.Anglin and P.Zoller, "Black holes in bose–einstein condensates," 2000. 6

[9] S. L. C. Barceló and M. Visser, "Towards the observation of hawking radiation in bose–einstein condensates," 2003. 6

[10] S. Hawking, "Black hole explosions?," 1974. 6, 32

[11] J. Anglin and W. Ketterle, "Bose–einstein condensation of atomic gases," 2002. 6

[12] A. Einstein, "Quantum theory of ideal monoatomic gases:atranslation of quantentheorie des einatomigen idealen gases," 2015. 8

[13] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, "Observation of bose-einstein condensation in a dilute atomic vapor," vol. 269, no. 5221, pp. 198–201, 1995. 8

[14] L. Pitaevskii and S.Stringari, *Bose-Einstein Condensation.* 9, 10, 11, 12, 15, 26, 29

[15] C. Pethic and H. Smith, *BOSE-EINSTEIN CONDENSATION IN DILUTE GASES.* 12

[16] N. Bogoliubov, "On the superfluidity," 1947. 16

[17] E. P. Gross, "Structure of a quantized vortex in boson systems," 1961. 17

[18] L. Pitaevskii, "Vortex lines in an imperfect bose gas," 1961. 17

[19] S. Corley and T. Jacobson, "Hawking spectrum and high frequency dispersion," 1996. 20, 21, 33

[20] J. Macher and R. Parentani, "Hawking spectrum and high frequency dispersion," 2009. 20, 33