Hyperbolic knot theory

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled Hyperbolic knot theory towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Sriram Raghunath at Indian Institute of Science Education and Research under the supervision of Dr. Tejas Kalelkar, Assistant Professor, Department of Mathematics , during the academic year 2019-2020.

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Declaration

I hereby declare that the matter embodied in the report entitled Hyperbolic knot theory are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Tejas Kalelkar and the same has not been submitted elsewhere for any other degree.

Sriram R.

Sriram Raghunath

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Abstract

In this thesis, we study Thurston's approach to finding complete hyperbolic structures on 3-manifolds using ideal triangulations. This approach involves solving a set of equations called the Thurston's gluing equations. These equations are non-linear and difficult to solve, so Casson and Rivin developed the method of angle structures through which they separated Thurston's equations into a linear and a non-linear part and extracted geometric information from each part separately. We also study geometric triangulations of constant curvature manifolds and how they are related by Pachner moves. We specially focus on understanding geometric ideal triangulations of cusped hyperbolic 3-manifolds and prove that any two geometric ideal triangulations have a common geometric subdivision with a finite number of polytopes. As a result, geometric ideal triangulations of a cusped hyperbolic 3-manifold become related by geometric Pachner moves. Along the way, we will discuss some foundational results in the theory of 3-manifolds, triangulations and hyperbolic geometry which we require for studying the central topics in this thesis.

Contents

Chapter 0

Preface

This thesis is of an expository nature and no claim is made to the originality of any of the results in Chapters 1-6. Chapter 7 is a small extension of known results and is original.

Chapter 1 This chapter gives a basic introduction to the theory of 3-manifolds and their triangulations. We state important results in 3-manifold theory such as the prime decomposition, the JSJ decomposition, and Thurston's geometrization theorem, and explain their consequences for knot theory and hyperbolic geometry. We also define basic concepts in the theory of triangulations of 3-manifolds and state classic theorems such as Pachner's theorem.

Chapter 2 In this chapter, we give a basic introduction to hyperbolic geometry in two and three dimensions. We discuss the different models of hyperbolic geometry and the classification of isometries of $\mathbb{H}^2$ and $\mathbb{H}^3$. We also state Mostow-Prasad rigidity and its important consequences to finding invariants of hyperbolic knots.

Chapter 3 This chapter is devoted to the statement and proof of the Margulis lemma and the thick-thin decomposition for hyperbolic 3-manifolds. Along the way, we state and prove properties of Kleinian groups which are required to prove these theorems.

Chapter 4 In this chapter, we explain Thurston's edge gluing consistency and gluing completeness equations. We solve these equations for the standard triangulation of the figure eight knot complement and obtain a complete hyperbolic structure on the knot complement with respect to the standard ideal triangulation. We also discuss a result by Dadd and Duan [DD16] which shows that there are infinitely many geometric triangulations of the figure eight knot complement.

Chapter 5 In this chapter, we discuss Casson and Rivin's approach of separating Thurston's equations into linear and non-linear parts using angle structures. We prove that the existence of an angle structure on an ideal triangulation of the manifold implies that the manifold admits a complete hyperbolic structure. We state the formula for calculating the hyperbolic volume of an ideal tetrahedron and define a volume functional on the space of angle structures using this formula. Finally, we show that the critical point of the volume functional on the space of angle structures corresponds to a complete hyperbolic structure on the 3-manifold.

Chapter 6 We state results by Tejas Kalelkar and Advait Phanse on geometric triangulations of compact constant curvature manifolds and how they are related by Pachner moves in this chapter. We discuss how geometric triangulations are related by geometric Pachner moves, which is a result proved in [KP19b]. We also state an upper bound on Pachner moves required to relate two geometric triangulations of the same manifold proved in [KP19a].

Chapter 7 In this chapter, we prove a minor result about geometric ideal triangulations of complete cusped hyperbolic 3-manifolds. We show that any two geometric ideal triangulations of a cusped hyperbolic 3-manifold admit a common geometric subdivision which has only finitely many polytopes. In the future, this result can help us to extend the theorems in [KP19b] and [KP19a] to complete cusped hyperbolic 3-manifolds.

Chapter 1

Basics of 3-manifolds and their triangulations

In this chapter, we shall define certain basic terms relating to 3-manifolds and state a few classic theorems about the topology and geometry of 3-manifolds. Triangulations are an important tool for studying 3-manifolds and the study of surfaces which are embedded 'nicely' with respect to a given triangulation of the 3-manifold forms the content of normal surface theory. So, we shall also cover the fundamentals of triangulations and normal surface theory. We have borrowed many definitions and the statements of many theorems from [Hat07], [Mar16], [Pur20], [Lic99] and [KP19b].

1.1 Triangulations of 3-manifolds

We first define basic terms related to triangulations of manifolds. We let Δ^n denote the standard n-simplex.

Definition 1.1.1 (Simplicial complex)**.** *An abstract simplicial complex K consists of a finite set K^0 of vertices and a family K of subsets of K^0 such that if $A \in K$ and $B \subset A$, then $B \in K$. We call the sets in this family the simplexes of the simplicial complex. Also, if $A, B \in K$ and $A \subset B$, we say that A is a face of B.*

A realisation of a simplicial complex K is a subspace $|K|$ in some $\mathbb{R}^n$, where the vertices are realised as points of $\mathbb{R}^n$ such that the vertices of each simplex are in general position and *the simplexes are realised as the convex hull of the vertices. The realisation $|K|$ inherits the subspace topology from $\mathbb{R}^n$.*

We shall abuse terminology and refer to both the abstract simplicial complex K and its realisation $|K|$ as the simplicial complex K. The dimension of a simplicial complex is the dimension of the highest dimensional simplex it contains.

Definition 1.1.2 (Join, star and link)**.** *If A and B are two disjoint simplexes of a simplicial complex K, we call the simplex $A \cup B$ as the join of A and B, which we denote as $A \star B$. We define the join of two simplicial complexes K and L to be the simplicial complex*

$$K \star L = \{A \star B \mid A \in K, B \in L\}$$

The link of a simplex A in a simplicial complex K is defined to be the simplicial complex

$$\mathrm{lk}(A, K) = \{B \in K \mid A \star B \in K\}$$

The star of a simplex A in the simplicial complex K is defined to be the simplicial complex

$$\mathrm{st}(A, K) = A \star \mathrm{lk}(A, K)$$

Definition 1.1.3 (Piecewise linear homeomorphism)**.** *A homeomorphism $\phi\colon U \to V$, where U and V are open sets in $\mathbb{R}^n$, is said to be piecewise linear if there exists a neighbourhood N_p around each point $p \in U$ such that the restriction $\phi|_{N_p}$ is affine linear, that is, it maps lines to lines.*

Definition 1.1.4 (Piecewise linear manifold)**.** *A topological n-manifold M is said to be a piecewise linear manifold if each point of M has a neighbourhood which is homeomorphic to an open set in $\mathbb{R}^n$ and all the transition maps are given by piecewise linear homeomorphisms. This neighbourhood is called a co-ordinate chart around that point. Two piecewise linear n-manifolds are said to be piecewise linearly homeomorphic if they are homeomorphic as topological manifolds and the homeomorphism induced on the level of co-ordinate charts is piecewise linear.*

We will henceforth use the standard abbreviation 'PL' to refer to the phrase 'piecewise linear'.

Definition 1.1.5 (Simplicial triangulation of a manifold)**.** *A simplicial triangulation*

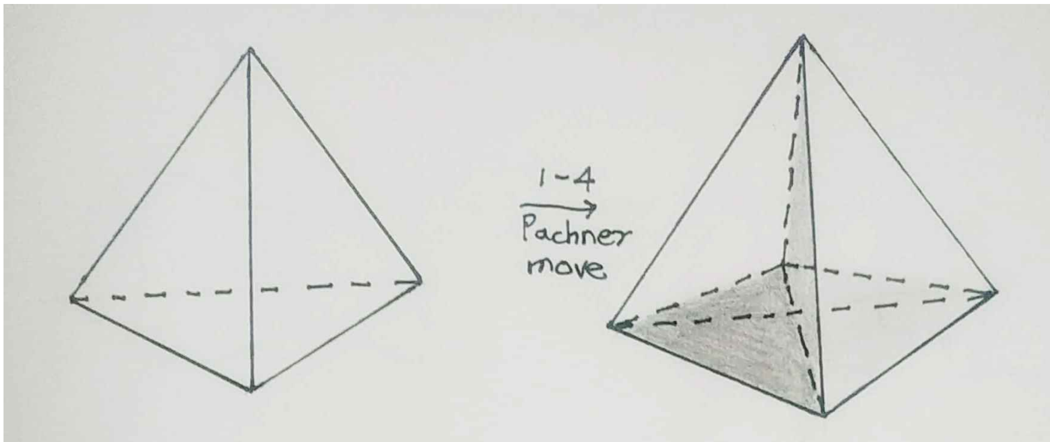


Figure 1.1: 1-4 Pachner move

of a PL-manifold M consists of a simplicial complex K along with a PL-homeomorphism from its linear realisation $|K|$ to M.

We shall now state a few classic theorems in piecewise linear topology which will help us to understand how we can pass between two triangulations of a manifold by using local combinatorial moves on the triangulations.

Definition 1.1.6 (Stellar moves)**.** *Consider a simplicial complex K. Let A be any simplex in K and let a be a point in the interior of A. Then, the stellar subdivision $K \xrightarrow{(A,a)} K'$ consists of removing* $\mathrm{st}(A,K)$ *and replacing it with $a \star \partial A \star \mathrm{lk}(A,K)$. This operation is a local operation which changes K to a new complex K', which is denoted as $K' = (A,a)K$. The opposite of this move is called a stellar weld and we denote this by $K = (A,a)^{-1}K'$.*

Definition 1.1.7 (Pachner moves)**.** *Let K be a simplcial complex of dimension n. Let A be a k-simplex of K and suppose* $\mathrm{lk}(A,K) = \partial B$ *for some $n-k$ simplex B which is not in K. Then, the bistellar move $\kappa(A,B)$ consists of modifying* $\mathrm{st}(A,K)$ *by replacing $A \star \partial B$ with $\partial A \star B$. Bistellar moves are also called as Pachner moves. We can view Pachner moves in another way. Suppose the simplicial complex K has a subcomplex L consisting of r n-simplexes, such that L is simplicially isomorphic to a disk subcomplex of $\partial\Delta^{n+1}$. Then replacing the subcomplex L with its complement $\partial\Delta^{n+1} \setminus L$ in the boundary of the $n+1$-simplex is defined to be the r-$(n+2-r)$ Pachner move.*

1-4 and 2-3 Pachner moves on a 3-dimensional simplicial complex are shown in Figure 1.1 and Figure 1.2. By reversing these moves, we will obtain the 4-1 and 3-2

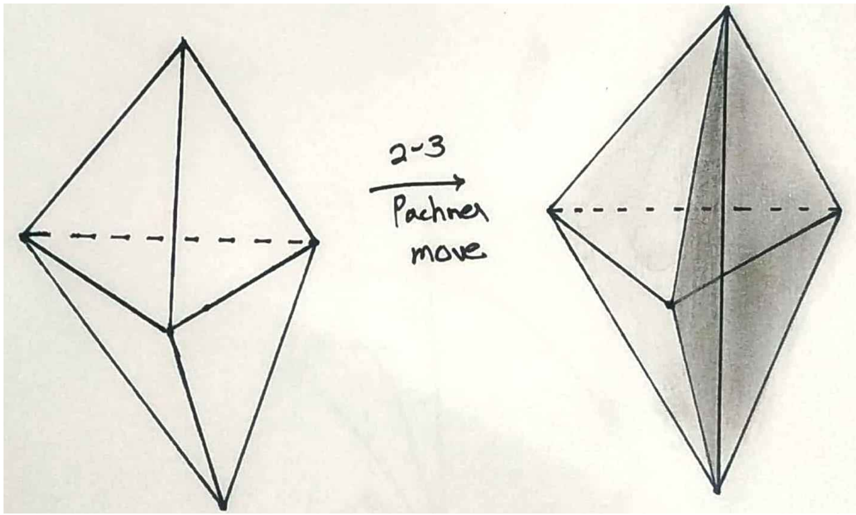


Figure 1.2: 2-3 Pachner move

Pachner moves. We see that both stellar and bistellar moves are local in nature - by definition they take place within the star of one simplex in the simplicial complex K.

Theorem 1.1.1 (Alexander, Newman)**.** *Two n-dimensional simplicial complexes are PL homeomorphic if and only if they are related by a finite sequence of stellar moves.*

Theorem 1.1.2 (Pachner, Newman)**.** *Two closed simplicially triangulated PL n-manifolds are PL homeomorphic if and only if they are related by a sequence of bistellar (Pachner) moves.*

There has been considerable interest in making Pachner's theorem stronger and understanding the connectivity of different sets of triangulations under specific kinds of Pachner moves. The following theorem is a step in this direction. We shall consider more general triangulations in this result, which we define now.

Definition 1.1.8 (3 dimensional pseudo-manifold)**.** *Let S be a collection of 3-simplexes and Φ be a collection of affine face-pairing homeomorphisms between the faces of the simplexes in S. Consider the quotient S/Φ; we require that the quotient map when restricted to the interior of each face of each simplex is a homeomorphism. We then say that the quotient S/Φ is a pseudo-manifold.*

Thus, we see that a pseudo-manifold is a manifold if and only if the link of each vertex of the pseudo-manifold is a sphere. Note that as any 3-manifold admits a simplicial triangulation, all 3-manifolds are pseudo-manifolds. A pseudo-manifold is said to be simplicial if it has the structure of a simplicial complex.

Definition 1.1.9 (Singular triangulation of a 3-manifold)**.** *Let M be a manifold and let S/Φ be a pseudo-manifold which is actually a manifold, that is, the link of each vertex of S/Φ is a sphere. Suppose $h\colon S/\Phi \to M$ is a PL homeomorphism. Then, we say that $(S/\Phi, \Phi, h)$ is a singular triangulation of M.*

Definition 1.1.10 (Derived and barycentric subdivisions)**.** *A subdivision of a simplicial complex K is a simplicial complex K' such that $|K| = |K'|$ and each simplex of K' is contained linearly in a simplex of K. The first derived subdivision of a simplicial complex K is obtained by inductively performing stellar subdivisions on the simplexes of K in the order of decreasing dimension. Suppose, for each simplex, we choose the barycentre of the simplex as the interior point at which we perform the stellar subdivision. Then, the derived subdivision is called a barycentric subdivision. The n^{th} derived or barycentric subdivision of a simplicial complex is obtained by iteratively performing derived or barycentric subdivisions on the simplicial complex n times.*

Remark 1.1.1. *We can easily extend Definition 1.1.10 to singular triangulations and perform derived and barycentric subdivisions on them. A singular triangulation can be made simplicial by performing two derived subdivisions.*

Definition 1.1.11 (Ideal and material vertices in a singular triangulation)**.** *A vertex p of a singular triangulation K is said to be material if $\mathrm{lk}(p, K)$ is homeomorphic to a 2-sphere. If $\mathrm{lk}(p, K)$ is not homeomorphic to a 2-sphere, then the vertex p is said to be an ideal vertex.*

Definition 1.1.12 (Ideal triangulation)**.** *A singular triangulation K in which all vertices are ideal, such that K is homeomorphic to a manifold M upon removing all the vertices is called an ideal triangulation of M.*

Theorem 1.1.3 (Amendola, Matveev, Piergallini)**.** *Two triangulations of a 3-dimensional pseudo-manifold having the same number of material vertices are related by 2-3 and 3-2 Pachner moves, except triangulations with a single tetrahedron. In particular, ideal triangulations of a 3-dimensional pseudo-manifold (with more than one tetrahedron) are related by only 2-3 and 3-2 Pachner moves.*

1.1.1 Normal surfaces

We now define and state a few properties about normal surfaces, which are surfaces embedded 'nicely' with respect to a given triangulation of a 3-manifold. Normal surfaces are used crucially in the proof of existence and uniqueness of the prime decomposition and the canonical torus decomposition (JSJ decomposition). Normal surface theory gives us a framework to study a surface embedded in a 3-manifold using the combinatorial data of the intersection of the surface with the triangulation of the 3-manifold.

Definition 1.1.13 (Properly embedded surface)**.** *A surface S embedded in a 3-manifold M is said to be properly embedded in M if $S \cap \partial M = \partial S$, and the surface intersects the boundary of M transversely.*

Definition 1.1.14 (Normal surface)**.** *Let M be a compact 3-manifold with a finite triangulation τ. Let S be a properly embedded compact surface in M which is transverse to the triangulation τ, that is, S does not intersect the vertices of τ and it intersects edges, faces and tetrahedra of T only in finitely many components. Note that any properly embedded compact surface can be perturbed to be transverse to a given triangulation. Then, we say that S is normal with respect to the triangulation τ if it intersects each tetrahedron only in triangles which separate one of the vertices of the tetrahedron from the rest, and squares which separate pairs of disjoint edges in the tetrahedron, as shown in Figure 1.4.*

Definition 1.1.15 (Surgering an embedded surface along a disk)**.** *Let S be a compact surface properly embedded in a 3-manifold M. Let D be a disk in M such that $D \cap S = \partial D$. Then surgering the surface S along the disk D involves removing an annular tubular neighbourhood of ∂D in S and capping the resulting surface by adding two parallel copies of the disk D. This is shown in Figure 1.3. Suppose $D \subset M$ is a disk such that $\partial D = \alpha \cup \beta$, where $D \cap S = \alpha$ and $D \cap \partial M = \beta$. Then, we can also surger S along the disk D in a similar manner.*

Definition 1.1.16 (Elementary transformations)**.** *Let M be a compact 3-manifold with a finite triangulation τ. Let S be a properly embedded compact surface in M which is transverse to the triangulation τ. Then, an elementary transformation on S is one of the following moves:*

1. *We can remove a connected component of S which is contained in a ball.*

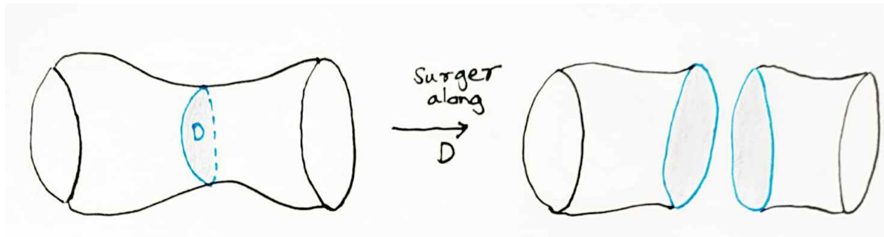


Figure 1.3: Surgering S along a disk D where $D \cap S = \partial D$

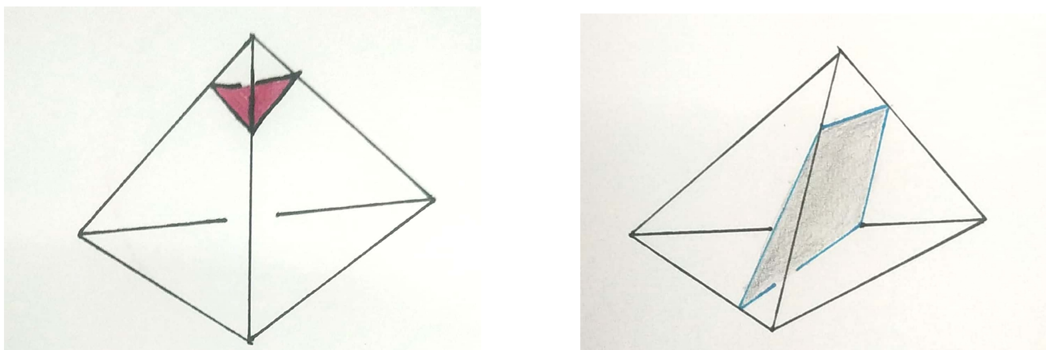


Figure 1.4: Allowed intersections of a normal surface with a tetrahedron in the triangulation

2. *Suppose $D \subset M$ is a disk such that $D \cap S = \partial D$. Then we can surger S along the disk D.*

3. *Suppose $D \subset M$ is a disk such that $\partial D = \alpha \cup \beta$, where $D \cap S = \alpha$ and $D \cap \partial M = \beta$, then we can surger S along D.*

All the elementary transformations are local moves - they occur inside a ball in M. An elementary transformation changes the surface S into a new surface S'.

The following theorem is a classic theorem on transforming properly embedded surfaces into normal surfaces and is originally due to Kneser and Haken. [Hat07] and [Mar16] are good references for the basic theory of normal surfaces.

Theorem 1.1.4. *Let M be a 3-manifold with triangulation τ. Let S be a properly embedded surface in M. Then S can be transformed into a surface which is in normal form with respect to the triangulation τ by a finite sequence of elementary transformations and isotopies.*

1.2 Cutting 3-manifolds along essential surfaces

To understand the topology and geometry of 3-manifolds, it is often useful to cut them into simpler pieces which are more easily tractable. For this purpose, 3-manifolds are cut along certain special surfaces called essential surfaces. We define a few terms to make this idea more rigorous.

Definition 1.2.1 (∂-parallel surface)**.** *A properly embedded surface S in a 3-manifold M is said to be boundary parallel if it is isotopic to a surface contained in ∂M, where the isotopy fixes ∂S.*

Definition 1.2.2 (Incompressible surface)**.** *Let S be a properly embedded surface in a 3-manifold M. A disk $D \subset M$ such that $D \cap S = \partial D$ and ∂D does not bound a disk in S is said to be a compressing disk for S. Surgering the surface S along the disk is called a compression. The surface S is said to be incompressible if it does not have any compressing disk in M.*

Definition 1.2.3 (∂-incompressible surface)**.** *Let S be a properly embedded surface in a 3-manifold M. We say that a disk $D \subset M$ is a ∂-compressing disk for S if $\partial D = \alpha \cup \beta$, where $\alpha \subset S$ and $\beta \subset \partial M$, such that there is no disk D' in S with $\partial D' = \alpha \cup \beta'$, where $\beta' \subset \partial S$. Surgering S along a ∂-compressing disk is called a ∂-compression. The surface S is called ∂-incompressible if it does not have any ∂-compressing disk in M.*

Definition 1.2.4 (Essential surfaces)**.** *We assume that all the surfaces under consideration are properly embedded in the 3-manifold M.*

- *A 2-sphere S embedded in M is said to be an essential sphere if it does not bound a 3-ball in M.*
- *A disk D in M is called an essential disk if it is not ∂-parallel.*
- *A torus T embedded in M is said to be essential if it is incompressible and not ∂-parallel.*
- *An annulus A embedded in M is said to be essential if it is incompressible, ∂-incompressible and not ∂-parallel.*

In general, any surface S, with $\chi(S) \leq 0$, which is properly embedded in M is said to be essential if it is is incompressible, ∂-incompressible and not boundary parallel. A manifold

M is said to be irreducible, ∂-irreducible, atoroidal, or anannular if it has no essential sphere, disk, torus, or annulus respectively.

We can thus cut 3-manifolds along various essential surfaces and decompose them into simpler parts. We state a few classic theorems to this end.

Definition 1.2.5 (Connected sum)**.** *Let M_1 and M_2 be two oriented 3-manifolds. Remove 3-balls from the interior of M_1 and M_2 and denote the resulting manifolds with boundary as $\widehat{M_1}$ and $\widehat{M_2}$. Glue $\widehat{M_1}$ and $\widehat{M_2}$ along the spherical boundary component corresponding to the removed ball by an orientation reversing diffeomeorphism. The resulting manifold is called the connected sum of M_1 and M_2 and is denoted as $M_1 \# M_2$.*

Definition 1.2.6 (Prime 3-manifold)**.** *A 3-manifold M is said to be prime if it cannot be expressed as a non-trivial connected sum of two other 3-manifolds. Here, we refer to the connected sum of any 3-manifold M with S^3 as a trivial connected sum, as this operation does not affect M, that is, $M \# S^3 = M$.*

Clearly any irreducible 3-manifold is prime, while $S^2 \times S^1$ is the only orientable prime 3-manifold which is not irreducible. The following classic theorem is due to Kneser and can be found in [Mil62].

Theorem 1.2.1 (Prime decomposition)**.** *Any compact orientable 3-manifold M decomposes as the connected sum of prime 3-manifolds*

$$M = M_1 \# M_2 \# .... \# M_k$$

The M_i are unique up to ordering and insertion and deletion of copies of S^3.

We can also further decompose irreducible manifolds by cutting along essential disks. This theorem can be found in [Mil62] and a stronger result can be found in [Gro69].

Theorem 1.2.2 (Decomposition along essential disks)**.** *Any compact, orientable, irreducible 3-manifold M can be cut along a system of non-parallel essential disks to obtain irreducible and ∂-irreducible manifolds M_1,..., M_k. This decomposition is unique up to permutation of the M_i and adding or removing balls.*

Now, given a compact, irreducible, ∂-irreducible 3-manifold M, Jaco-Shalen and Johannson showed that it can be further decomposed along essential tori into atoroidal pieces [JS79]. The uniqueness part of this theorem is tricky and involves understanding the theory of Seifert-fibered manifolds, so we state only the existence part.

Theorem 1.2.3 (JSJ decomposition - existence)**.** *Let M be a compact, irreducible, ∂-irreducible 3-manifold. Then, there exists a system of non-parallel essential torii in M such that cutting along these tori decomposes M into atoroidal pieces M_1,..., M_k.*

All the decomposition theorems stated above are proved by transforming the essential surfaces under consideration into normal surfaces and by using the combinatorial structure of these normal surfaces.

1.3 Geometrization of 3-manifolds

We have described how 3-manifolds can be cut along essential surfaces into simpler pieces. Thurston conjectured that every 3-manifold M possesses a geometric decomposition, which is obtained in a similar way by cutting along essential spheres, disks and tori, and that each of the resulting pieces of the geometric decomposition admits one of eight geometries. We shall describe this idea in more detail in this section, and explain a few of its consequences. We shall however skip some technical details in the exposition.

Definition 1.3.1. *A connected Riemannian manifold M is said to be homogeneous if for any two points p and q in M, there exists an isometry f of M such that $f(p) = q$.*

Definition 1.3.2. *We say that a Riemannian manifold M has a geometric structure modelled on a model geometry N if M is locally isometric to N.*

Thurston's eight model geometries are all complete, simply connected and homogeneous Riemannian manifolds. The eight geometries are S^3, $\mathbb{R}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, Sol and $\widetilde{SL_2}$. Out of these, the first three geometries are the ones with constant sectional curvature. Of the eight geometries, all but Sol and $\mathbb{H}^3$ are Seifert fibered. We will study hyperbolic geometry in more detail in the following chapters. We now state the geometrization theorem, which was proved by Thurston for Haken manifolds, and proved in general by Perelman using Hamilton's Ricci flow.

Theorem 1.3.1 (Thurston's geometrization theorem)**.** *Every 3-manifold admits a geo-*

metric decomposition by cutting along essential spheres, disks and tori. Each of the pieces of the geometric decomposition admits a complete finite volume geometric structure modelled on one of Thurston's eight geometries.

Each of the model geometries has a specific characterization in terms of the essential surfaces it allows and its fundamental group which enables it to be distinguished from the others easily. We shall now state a few consequences of Thurston's geometrization theorem which will prove useful for us in later chapters.

Corollary 1.3.1.1 (Hyperbolization)**.** *A compact 3-manifold M with torus boundary components has an interior admitting a complete hyperbolic metric (a complete geometric structure modelled on $\mathbb{H}^3$) of finite volume if and only if M is irreducible, ∂-irreducible, atoroidal and anannular.*

Thurston also used the geometrization theorem to prove a classification of knots based on the structure of their complements. We first define the different kinds of knots which appear in this classification.

Definition 1.3.3. *Embed a torus T in S^3 as the boundary of a tubular neighbourhood of an unknot. There are two compressing disks for T which are on opposite sides of the torus in S^3. Both these disks intersect T only in their boundary circles, which are non-trivial simple closed curves on T by construction. We shall choose one of these to be the meridian m of T and the other to be the longitude l of T. We let m and l be the generators for the homology $H_1(T;\mathbb{Z})$. Let p and q be coprime integers. Then a simple closed curve on T from the class $(p,q) = pm + ql \in H_1(T;\mathbb{Z})$ is an oriented knot in S^3, with the orientation determined by the signs of p and q. This is called the (p,q)-torus knot and is denoted by $T(p,q)$.*

The complement of a torus knot contains an essential annulus and does not contain any essential tori.

Definition 1.3.4 (Satellite knot)**.** *Consider a knot K' in a solid torus V such that K' is not contained in a 3-ball in V and K' is not isotopic to the core of V. Let K'' be a non-trivial knot in S^3. Drill out a regular neighbourhood of K'' from S^3 and fill the hole with the solid torus V with K'' embedded in it without twisting, that is, glue V such that the meridian curve of K'' still bounds a disk in V. The embedding of K' we get from this procedure is called a satellite knot. The knot K'' is called the companion of the satellite knot. Thus, the satellite knot obtained through this construction lies in a regular neighbourhood of the* *companion knot.*

The complement of a satellite knot contains an essential torus, the boundary of V.

Definition 1.3.5 (Hyperbolic knot)**.** *A knot is said to be hyperbolic if its complement admits a complete hyperbolic metric, that is, the complement admits a complete geometric structure modelled on* $\mathbb{H}^3$.

From the hyperbolization theorem, we see that the complement of a hyperbolic knot does not contain any essential tori or annuli.

Corollary 1.3.1.2 (Classification of knots)**.** *Any knot* $K\colon S^1 \to S^3$ *is either a torus knot, a satellite knot, or a hyperbolic knot. These three classes are mutually exclusive.*

Torus knots are well understood and many of their invariants have been explicitly calculated. The complement of torus knots are Seifert fiber spaces, which are well understood. Hyperbolic knots are abundant among the three classes and the hyperbolic structure on a hyperbolic knot complement is a complete invariant of the knot due to Mostow-Prasad rigidity (Theorem 2.3.1) and Gordon-Luecke theorem (Theorem 2.3.2). Thus, geometric invariants of hyperbolic knot complements are knot invariants. So, finding and studying invariants of the hyperbolic structure on hyperbolic knot complements can help us to completely distinguish between hyperbolic knots. In the rest of this thesis, we will be restricting our attention to hyperbolic 3-manifolds.

Chapter 2

Basic hyperbolic geometry

In this chapter, we describe a few basic results in hyperbolic geometry that will aid us in our study of geometric triangulations of hyperbolic manifolds. We will describe different models of hyperbolic space and understand the classification of isometries of hyperbolic space in dimensions 2 and 3. We shall also state the Mostow-Prasad rigidity theorem and understand its consequences.

2.1 Models of hyperbolic space

In this section, we shall describe the various models of hyperbolic space that we will need in later chapters. We will try to understand the metric on these spaces, the geodesics in these models and the boundary of the hyperbolic space in these models.

2.1.1 The upper half space model

In this model of hyperbolic space, which we denote by $\mathbb{H}^n$, the underlying set is given by

$$\mathbb{H}^n = \{x \in \mathbb{R}^n \mid x_n > 0\}$$

The metric on this space is given by

$$ds^2 = \frac{\|dx\|^2}{x_n^2}$$

We find that the geodesics in $\mathbb{H}^n$ induced by this metric are semicircular arcs and vertical lines perpendicular to the $x_n = 0$ hyperplane. The boundary of hyperbolic space in this model consists of the $x_n = 0$ hyperplane and the point at ∞.

$$\partial\mathbb{H}^n = \{x \in \mathbb{R}^n \mid x_n = 0\} \cup \{\infty\}$$

2.1.2 The Poincare disk model

The underlying set for this model of hyperbolic space, which we shall denote as $\mathbb{D}^n$, is the unit disk in n-dimensions.

$$\mathbb{D}^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$$

Here we have used the symbol $\|.\|$ to denote the usual Eulcidean norm in $\mathbb{R}^n$. The metric on the Poincare disk is

$$ds^2 = \frac{4\|dx\|^2}{(1 - \|x\|^2)^2}$$

The geodesics in this model are the diameters of the disk and circular arcs which are perpendicular to the boundary of the disk. The boundary of hyperbolic space in this model is clear - it is simply the boundary of the disk.

$$\partial\mathbb{D}^n = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

Remark 2.1.1. *Both the upper half space model and the Poincare disk model are conformal models of hyperbolic space, that is, the hyperbolic angle between two curves is the same as the Euclidean angle between the representations of the two curves in these models.*

2.1.3 The hyperboloid model

The underlying set for this model of hyperbolic space is the upper sheet of the n-dimensional hyperboloid, which we denote as $\mathbb{L}^n$.

$$\mathbb{L}^n = \{x \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + ... + x_n^2 - x_{n+1}^2 = -1, x_{n+1} > 0\}$$

The metric in the hyperboloid model is given by

$$ds^2 = dx_1^2 + ... + dx_n^2 - dx_{n+1}^2$$

This metric is actually the metric induced by the Minkowksi inner product on $\mathbb{R}^{n+1}$, which is given by

$$\langle x, y \rangle = x_1 y_1 + ... + x_n y_n - x_{n+1} y_{n+1}$$

The Minkowski inner product gives a Riemannian metric on the hyperboloid since the inner product is positive definite when restricted to the tangent space at any point on the upper sheet of the hyperboloid.

The geodesics in the hyperboloid model are given by the intersection of two dimensional subspaces of $\mathbb{R}^{n+1}$ with the hyperboloid $\mathbb{L}^n$. The boundary of hyperbolic space is represented by rays on the light cone.

$$\partial \mathbb{L}^n = \{x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle = 0\} / x \sim \lambda x, \lambda \in \mathbb{R}, \lambda > 0$$

2.1.4 The Klein model

The underlying set of the Klein model of hyperbolic space, which we denote as $\mathbb{K}^n$, is also the unit disc in $\mathbb{R}^n$.

$$\mathbb{K}^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$$

However, the metric is different and is given by

$$ds^2 = \frac{\|dx\|^2}{1 - \|x\|^2} + \frac{(x_1 dx_1 + ... + x_n dx_n)^2}{(1 - \|x\|^2)^2}$$

The Klein model is obtained from the hyperboloid model by radial projection to the unit disk in the $x_{n+1} = 0$ hyperplane and the metric is induced by pullback from the metric on $\mathbb{L}^n$.

The geodesics in the Klein model are the Euclidean straight lines in the disk. The boundary of the Klein model is the usual topological boundary of the unit disk.

Remark 2.1.2. *Both the hyperboloid and the Klein model are not conformal models of hy-*

perbolic space.

Remark 2.1.3. *All the models of hyperbolic space are isometric to each other. The isometries take the geodesics in one model to corresponding geodesics in the other model.*

2.2 Isometries of hyperbolic space

2.2.1 Classification of isometries of $\mathbb{H}^2$

If we view the hyperbolic plane in the upper half plane model, we see that transformations of the form $z \mapsto \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$ are isometries of the hyperbolic plane. These are known as real Mobius transformations, and the group of such transformations is isomorphic to the group $PSL(2, \mathbb{R})$. In fact, these are all the orientation preserving isometries of $\mathbb{H}^2$. We shall classify the orientation preserving isometries of $\mathbb{H}^2$ on the basis of the number of their fixed points and their trace. Note that the trace of the isometry is defined only up to sign.

Proposition 2.2.1. *The group of orientation preserving isometries of $\mathbb{H}^2$ is isomorphic to $PSL(2, \mathbb{R})$. Any isometry A can be classified on the basis of its fixed points and its trace.*

1. *If $tr(A) = \pm 2$, then A fixes only one point on $\partial\mathbb{H}^2$ and no points in $\mathbb{H}^2$. Such an isometry is called a parabolic isometry and is conjugate to $z \mapsto z + 1$ in the upper half plane model.*

2. *If $tr(A) \in (-2, 2)$, then A will not fix any point on $\partial\mathbb{H}^2$ and fixes one point in the interior of $\mathbb{H}^2$. Such an isometry is called an elliptic isometry and in the Poincare disk model, it is conjugate to $z \mapsto e^{2i\theta} z$, where $2\theta \neq 2n\pi, \forall n \in \mathbb{N}$.*

3. *If $tr(A) \notin [-2, 2]$, then A fixes two points on $\partial\mathbb{H}^2$, leaving the geodesic joining these two points invariant. It does not fix any point in $\mathbb{H}^2$. Such an isometry is called a loxodromic isometry and is conjugate to $z \mapsto \rho^2 z$, where $\rho \in \mathbb{R}$ and $\rho > 1$.*

2.2.2 Classification of isometries of $\mathbb{H}^3$

Each orientation preserving isometry of $\mathbb{H}^3$ is an extension of a Mobius transformation from $\partial\mathbb{H}^3 = \mathbb{C} \cup \infty$ to the interior of hyperbolic space. We describe how to extend Mobius transformations on $\partial\mathbb{H}^3$ to isometries of $\mathbb{H}^3$. Given a Mobius transformation T, let $\widehat{T}$ be the corresponding isometry of $\mathbb{H}^3$. Consider a point $x \in \mathbb{H}^3$ - it can be obtained as the intersection of two geodesics l and m in $\mathbb{H}^3$. Let p and q be the endpoints of the geodesic l and r and s be the endpoints of the geodesic m. Then, suppose the Mobius transformation T takes the endpoints p, q, r and s to the points p', q', r' and s' on $\partial\mathbb{H}^3$ respectively. We define $\widehat{T}$ such that it takes the geodesics l and m to the geodesics l' and m' which have their endpoints at p', q', r' and s' respectively. Then, we define $\widehat{T}(x) = y$, where $y \in \mathbb{H}^3$ is the point at which l' and m' intersect. This definition of $\widehat{T}$ is independent of the choice of the geodesics l and m, that is, $\widehat{T}$ is well defined.

Isometries of $\mathbb{H}^3$ can be classified by the number of points they fix on the boundary $\partial\mathbb{H}^3$.

Proposition 2.2.2. *Any orientation preserving isometry of $\mathbb{H}^3$ is an extension of a Mobius transformation of the form $z \mapsto \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. The group of orientation preserving isometries of $\mathbb{H}^3$ is thus isomorphic to the group of Mobius transformations, which is isomorphic to $PSL(2, \mathbb{C})$. Any orientation preserving isometry A thus has a representation as a matrix in $PSL(2, \mathbb{C})$ and can be classified by the number of points it fixes and its trace.*

1. *If $tr(A) = \pm 2$, then A fixes only one point on $\partial\mathbb{H}^3$ and no points in $\mathbb{H}^3$. Such an isometry is called a parabolic isometry and is conjugate to $z \mapsto z + 1$.*

2. *If $tr(A) \in (-2, 2)$, then A fixes two points on $\partial\mathbb{H}^3$ and the geodesic joining these two points. Such an isometry is called an elliptic isometry and is conjugate to $z \mapsto e^{2i\theta}z$, where $2\theta \neq 2n\pi, \forall n \in \mathbb{N}$.*

3. *If $tr(A) \notin [-2, 2]$, then A fixes two points on $\partial\mathbb{H}^3$, leaving the geodesic joining these two points invariant. It does not fix any point in $\mathbb{H}^3$. Such an isometry is called a loxodromic isometry and is conjugate to $z \mapsto \rho^2 z$, where $\rho \in \mathbb{C}$ and $|\rho| > 1$.*

2.3 Mostow-Prasad rigidity

Mostow-Prasad rigidity theorem restricts the possible complete hyperbolic structures on a finite volume hyperbolic manifold of dimensions greater than two.

Mostow had proved the theorem for finite volume closed hyperbolic manifolds and Prasad extended it for the case of finite volume cusped hyperbolic manifolds. As a consequence of the thick-thin decomposition (Theorem 3.4.1), we will see that all finite volume hyperbolic 3-manifolds are either closed or have finitely many cusps .

Theorem 2.3.1 (Mostow-Prasad rigidity)**.** *Let M_1 and M_2 be complete finite volume hyperbolic n-manifolds, where $n \geq 3$. Then any isomorphism between the fundamental groups of M_1 and M_2 is realised by a unique isometry between M_1 and M_2. Thus, any complete finite volume hyperbolic manifold of dimension $n \geq 3$ has a unique complete hyperbolic structure up to isometry.*

Thus, any invariant of the complete hyperbolic structure on a complete finite volume hyperbolic 3-manifold is also a topological invariant of the 3-manifold. This gives rise to a rich variety of geometric invariants such as hyperbolic volume, cusp shape, maximal cusp volume, length spectrum of geodesics, etc.

Hyperbolic geometry is intimately related with knot theory. This is because the Gordon-Luecke theorem states that knots are determined by knot complements.

Theorem 2.3.2 (Gordon-Luecke theorem)**.** *Two knots K_1 and K_2 are isotopic if and only if their complements in the 3-sphere $S^3 \setminus K_1$ and $S^3 \setminus K_2$ are homeomorphic. Furthermore, they are ambient isotopic if and only if $S^3 \setminus K_1$ and $S^3 \setminus K_2$ are homeomorphic via an orientation preserving homeomorphism.*

Combining the Mostow-Prasad rigidity theorem and the Gordon-Luecke theorem, we see that the geometry of a hyperbolic knot complement is actually a complete invariant of the knot. Thus, the geometric invariants mentioned above are knot invariants for the class of hyperbolic knots. Studying the rich interplay between the combinatorics of hyperbolic knot diagrams and the topology and the geometry of the knot complements has been an active area of research in recent years. The main focus has been on relating diagrammatic invariants of hyperbolic knots with the geometric invariants of the knot complements. The volume conjecture by Kashaev and Murakami [MY18] is one such attempt which tries to relate the Colored Jones polynomial of hyperbolic knots with the hyperbolic volume of the knot complements.

Chapter 3

Margulis lemma and the thick-thin decomposition

3.1 Introduction

A complete hyperbolic manifold M is the quotient of $\mathbb{H}^3$ by a subgroup of isometries $\Gamma < PSL(2, \mathbb{C})$. This is because M has a hyperbolic structure and the holonomy group acts on $\mathbb{H}^3$ (which is the universal cover of M) by deck transformations as M is complete. We shall see that the subgroup Γ must be discrete for M to be a complete hyperbolic manifold. So, we would like to study discrete subgroups of $PSL(2, \mathbb{C})$, which are called Kleinian groups, to understand the structure of complete hyperbolic 3-manifolds. In most of this chapter, we will follow the exposition by Purcell [Pur20]. Marden's book [Mar07] also contains an excellent treatment of Kleinian groups. We have also referred to the excellent book by Benedetti and Petronio [BP12] for the proof of the thick-thin decomposition.

3.2 Kleinian groups

Definition 3.2.1 (Kleinian group)**.** *A subgroup G of $PSL(2, \mathbb{C})$ is said to be discrete if it does not contain any sequence of distinct elements converging to the identity in $PSL(2, \mathbb{C})$. Discrete subgroups of $PSL(2, \mathbb{C})$ are called Kleinian groups.*

Lemma 3.2.1. *The following are equivalent:*

- *A subgroup G of $PSL(2, \mathbb{C})$ is discrete*

- *G does not contain any sequence of distinct elements A_n converging to an element A in $PSL(2,\mathbb{C})$.*

We shall now state a technical lemma about sequences of elements of $PSL(2,\mathbb{C})$ which will be useful later in the chapter.

Lemma 3.2.2. *Let T_n be a sequence in $PSL(2,\mathbb{C})$. Then either one of the following must be true:*

1. *T_n contains a subsequence which converges to some element $T \in PSL(2,\mathbb{C})$*

2. *There exists a point $p \in \partial\mathbb{H}^3$, such that given any $x \in \mathbb{H}^3$, $T_n(x)$ has a subsequence which converges to p.*

We will now see that the holonomy group Γ of a complete hyperbolic manifold is discrete by considering its action on $\mathbb{H}^3$. We first define a few terms to describe the actions of subgroups of $PSL(2,\mathbb{C})$ on $\mathbb{H}^3$.

Definition 3.2.2 (Properly discontinuous action)**.** *The actions of a subgroup $\Gamma \leq PSL(2,\mathbb{C})$ on $\mathbb{H}^3$ is said to be properly discontinuous if for any closed ball $B \subset \mathbb{H}^3$, the set $\{\gamma \in \Gamma \mid \gamma B \cap B \neq \varnothing\}$ is a finite set.*

Definition 3.2.3 (Free action)**.** *The actions of a subgroup $\Gamma \leq PSL(2,\mathbb{C})$ on $\mathbb{H}^3$ is said to be free if no element of Γ other than the identity has a fixed point in $\mathbb{H}^3$.*

Among all isometries of $\mathbb{H}^3$, only the elliptics have fixed points in $\mathbb{H}^3$; the parabolics and the loxodromics only fix points on the boundary $\partial\mathbb{H}^3$. So, the action of a subgroup Γ on $\mathbb{H}^3$ is free if and only if it contains no elliptics. We now describe a condition which is equivalent to the discreteness of the holonomy subgroup $\Gamma \leq PSL(2,\mathbb{C})$.

Proposition 3.2.1. *A subgroup Γ of $PSL(2,\mathbb{C})$ is discrete if and only if its action on $\mathbb{H}^3$ is properly discontinuous.*

Theorem 3.2.1. *The action of a group $\Gamma \leq PSL(2,\mathbb{C})$ on $\mathbb{H}^3$ is free and properly discontinuous if and only if $\mathbb{H}^3/\Gamma$ is a complete hyperbolic manifold and the quotient map from $\mathbb{H}^3 \mapsto \mathbb{H}^3/\Gamma$ is a covering projection.*

3.3 Elementary groups

In what follows, we shall assume that Γ is a discrete subgroup of $PSL(2, \mathbb{C})$ without elliptics, as only then is the quotient $\mathbb{H}^3/\Gamma$ a complete hyperbolic manifold, by Theorem 3.2.1. To understand the structure of complete hyperbolic 3-manifolds by studying their holonomy groups, we shall consider subgroups of $PSL(2, \mathbb{C})$ generated by a small number of generators and understand their properties.

Definition 3.3.1 (Elementary groups)**.** *Let Γ be a Kleinian group without elliptics and let S be the set consisting of the union of the fixed points on $\partial\mathbb{H}^3$ of all its non-identity elements. Then we say that Γ is elementary if $|S| \leq 2$.*

We now prove a few properties of elementary subgroups.

Proposition 3.3.1. *Let Γ be a non-trivial elementary Kleinian group without elliptics. Then, either of the following must be true:*

1. *The set S consists of a single point on $\partial\mathbb{H}^3$. Then Γ is generated by one or two parabolics which fix the same point on $\partial\mathbb{H}^3$ and is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$.*

2. *The set S consists of two points on $\partial\mathbb{H}^3$. Then Γ is generated by a single loxodromic which leaves the geodesic joining the two points invariant. In this case, Γ is isomorphic to $\mathbb{Z}$.*

Proof. Suppose the set S consists of exactly one point on $\partial\mathbb{H}^3$. We can conjugate Γ so that the fixed point is ∞ on $\partial\mathbb{H}^3$. Then, Γ can contain only parabolics which fix $\infty \in \partial\mathbb{H}^3$. The parabolics in Γ act as Euclidean translations on any horosphere about ∞. As Γ is discrete, the group of these Euclidean translations must be generated by either one or two linearly independent translations, that is, Γ must be generated by one or two independent parabolics fixing ∞. If Γ is generated by one parabolic fixing ∞, then $\Gamma \cong \mathbb{Z}$, and if Γ is generated by two independent parabolics fixing ∞, then $\Gamma \cong \mathbb{Z} \times \mathbb{Z}$.

Now suppose the set S consists of two points on $\partial\mathbb{H}^3$. We can conjugate Γ so that the fixed points are 0 and ∞. Now, we claim that Γ cannot consist only of parabolics. Suppose A is a parabolic fixing 0 and B is a parabolic fixing ∞, then the product AB belongs to Γ and fixes neither 0 nor ∞. Also AB cannot be the identity element, as $A \neq B^{-1}$. This contradicts the fact that the union of the fixed points on $\partial\mathbb{H}^3$ of all the non-identity elements of Γ consists of only 0 and ∞. Thus, Γ must contain a loxodromic element L which fixes the points 0 and ∞ and leaves the axis joining 0 and ∞ invariant. Now, we claim that Γ cannot contain any parabolic elements. Assume without loss of generality that Γ contains a parabolic element P which fixes 0. Then, from the proof of the second claim of Proposition 3.3.2, we see that the group generated by P and L is not discrete, contradicting the fact that Γ is discrete.

Thus, Γ consists only of loxodromics which fix 0 and ∞ and leave the axis joining 0 and ∞ invariant. Each loxodromic γ in Γ has a translation distance along the axis, which is the distance $d(x, \gamma(x))$, where x is a point on the axis. As Γ is discrete and non-trivial, the minimum translation distance of all the non-identity elements of Γ exists and is non-zero. Let d be the minimum translation distance and A be an element in Γ corresponding to it. Then, we claim that Γ is generated by A, that is $\Gamma = \langle A \rangle$. We first show that the translation distance of any other non-identity element C in Γ is a multiple of d. Suppose this is not true, then we see that the translation distance of C must lie strictly between nd and $(n+1)d$ for some $n \in \mathbb{N}$. Then, $CA^{-n} \in \Gamma$ will have a translation distance strictly less than d, which is a contradiction. Now, we show that any element C in Γ is generated by A. Suppose the translation distance of C is nd, for some $n \in \mathbb{N}$, then CA^{-n} has translation distance 0, which means that CA^{-n} fixes each point on the axis, and hence must be the identity element of Γ, as Γ contains no elliptics. Hence, $C = A^n$, and $\Gamma = \langle A \rangle \cong \mathbb{Z}$. ■

We also need to understand the structure of non-elementary Kleinian groups without elliptics. So, we will state the following key proposition:

Proposition 3.3.2. *Let Γ be a non-elementary Kleinian group without elliptics. Then, Γ satisfies the following properties:*

1. *Γ is infinite*
2. *Given a loxodromic element $A \in \Gamma$, there does not exist another non-trivial element $B \in \Gamma$ such that B shares exactly one fixed point on $\partial\mathbb{H}^3$ with A.*
3. *Given a non-trivial element $C \in \Gamma$, there exists a loxodromic element $D \in \Gamma$ which does not share any fixed point on $\partial\mathbb{H}^3$ with C.*
4. *Γ contains two loxodromics with no shared fixed points.*

Proof.

1. As Γ is a non-elementary Kleinian group without elliptics, it must contain a parabolic or a loxodromic element, both of which have infinite order. Hence, Γ is infinite.

2. Suppose there exists a non-trivial element $B \in \Gamma$ such that it shares exactly one fixed point in common with the loxodromic A. We shall show that the group generated by A and B is not discrete. We can conjugate the group Γ such that A is of the form $\left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{smallmatrix}\right)$, where $|\rho| < 1$. As B shares exactly one fixed point with A, B is of the form $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$. Then,

$$A^n B A^{-n} B^{-1} = \begin{pmatrix} 1 & ab(\rho^{2n} - 1) \\ 0 & 1 \end{pmatrix}$$

   Letting $n \to \infty$, we see that $A^n B A^{-n} B^{-1}$ converges to the parabolic $\left(\begin{smallmatrix} 1 & -ab \\ 0 & 1 \end{smallmatrix}\right)$. From Lemma 3.2.1, we see that the group generated by A and B cannot be discrete, as it contains a sequence of distinct elements converging to an element of $PSL(2, \mathbb{C})$. This contradicts the fact that Γ is discrete, so there cannot exist a non-trivial element $B \in \Gamma$ such that it shares exactly one fixed point in common with the loxodromic A.

3. There are two cases depending on whether C is a parabolic or a loxodromic. First, suppose C is a parabolic element. Then, we can conjugate Γ so that C fixes ∞ and assume that $C = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. As Γ is a non-elementary subgroup, there has to be another element $F \in \Gamma$ such that F fixes a point $x \in \partial\mathbb{H}^3$, where $x \neq \infty$. Suppose F is a loxodromic, it cannot fix ∞ by part 2 of this proposition, so F is the required loxodromic D. If F is a parabolic, we see that F is of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $c \neq 0$, as F does not fix ∞. It is clear that $C^n F$ cannot fix ∞ for any $n \in \mathbb{N}$. The trace of $C^n F$ is given by

$$\operatorname{tr}(C^n F) = a + nc + d = nc \pm 2$$

   For large enough n, we see that $\operatorname{tr}(C^n F) \notin [-2, 2]$, so $D = C^n F \in \Gamma$ is the required loxodromic.

Now, assume C is loxodromic. Again, we can conjugate Γ so that C fixes 0 and ∞ and assume that $C = \left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{smallmatrix}\right)$, where $|\rho| > 1$. As Γ is a discrete non-elementary group, from part 2 of this proposition we see that there exists an element $F \in \Gamma$ such that F fixes neither 0 nor ∞. If F is a loxodromic, then we see that it is the required loxodromic D. If F is a parabolic, then again we consider $C^n F$ and show that it is a loxodromic for sufficiently large n. We can assume that $F = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $c \neq 0$, and $a + d = \pm 2$. Then, we see that $C^n F$ cannot fix 0 or ∞, and $\mathrm{tr}(C^n F) = a\rho^n + d\rho^{-n}$. For large enough n, we see that $\mathrm{tr}(C^n F) \notin [-2, 2]$, so $D = C^n F \in \Gamma$ is the required loxodromic.

4. Consider a non-trivial element $A \in \Gamma$. By using part 3 of this proposition, we see that there exists a loxodromic B which shares no fixed points in common with A. If A is a loxodromic, then we are done. If not, we can apply part 3 of the proposition to B to obtain another loxodromic C which shares no fixed points in common with B. Hence, Γ must contain two loxodromics which have no common fixed points.

■

We will also need the following theorem on the convergence of non-elementary Kleinian groups. We state a corollary of the original version proved by Jorgensen and Klein [JK82], the proof of which can be found in [Mar07]. This theorem will be used in a crucial way in the proof of the Margulis lemma.

Theorem 3.3.1. *Suppose $\{\langle A_n, B_n \rangle\}$ is a sequence of non-elementary Kleinian groups without elliptics. Let A_n converge to A and B_n converge to B in $PSL(2, \mathbb{C})$. Then, $A \neq B$, and $\langle A, B \rangle$ is a non-elementary Kleinian group. Also, A and B share a fixed point if and only if A_n and B_n share a fixed point for all sufficiently large n.*

3.4 Margulis lemma and the thick-thin decomposition

We will now use the facts which we stated in previous sections to understand the Margulis lemma, which is a strong result on the structure of complete hyperbolic 3-manifolds. To state the result, we must first define a few terms.

Definition 3.4.1 (Rank 1 cusps, Rank 2 cusps and tubes around closed geodesics)**.** *Let Γ be an elementary Kleinian group without elliptics. Then depending on the structure of Γ elaborated in Proposition 3.3.1, we have the corresponding geometric pieces.*

- *Suppose Γ fixes only one point on $\partial\mathbb{H}^3$. We can conjugate Γ so that the fixed point is at ∞. Let H be the closed horoball about ∞ at height 1 in $\mathbb{H}^3$. That is,*

$$H = \{(x, y, t) \mid t \geq 1\}$$

  Then, we know from Proposition 3.3.1 that Γ is generated either by one or two parabolics which fix ∞. Suppose Γ is generated by one parabolic fixing ∞, then the quotient of the horball H/Γ is homeomorphic to $A \times [0, \infty)$, where A is an annulus. In this case, we say that H/Γ is a rank 1 cusp. If Γ is generated by two parabolics fixing ∞, then the quotient of the horoball H/Γ is homeomorphic to $T \times [0, \infty)$, where T is a torus. In this case, we say that H/Γ is a rank 2 cusp.

- *Suppose Γ fixes two points on $\partial\mathbb{H}^3$. We can conjugate Γ so that these two points are 0 and ∞. Then, by Proposition 3.3.1, we know that Γ is generated by a single loxodromic which leaves the geodesic from 0 to ∞ invariant. Let C_r be the closed solid cylinder of hyperbolic radius r around the geodesic from 0 to ∞. Then the quotient C_r/Γ is homeomorphic to a solid torus. We say that C_r/Γ is a tube of radius r around the core geodesic of the solid torus.*

Of these, rank 1 cusps have infinite volume while rank 2 cusps and tubes of radius r around geodesics are of finite volume.

The Margulis lemma gives us a decomposition of a complete hyperbolic manifold into a thick and a thin part which are defined in terms of injectivity radius. So, we shall next define these terms precisely.

Definition 3.4.2 (Injectivity radius)**.** *Let M be a complete hyperbolic manifold and let $x \in M$. Then the injectivity radius at x, denoted as $\mathrm{injrad}(x)$ is the supremal radius r for which a ball of radius r around x in M is embedded.*

Definition 3.4.3 (Thick and thin part)**.** *Let M be a complete hyperbolic 3-manifold. For $\epsilon > 0$, we define the ϵ-thin part of M, denoted by $M_{<\epsilon}$, by*

$$M_{<\epsilon} = \{x \in M \mid \mathrm{injrad}(x) < \epsilon/2\}$$

We define the ϵ-thick part of M to be the complement of the ϵ-thin part of M.

We are now in a position to state Margulis' thick-thin decomposition for complete hyperbolic 3-manifolds.

Theorem 3.4.1 (Thick-thin decomposition)**.** *There exists a universal constant ϵ_3, such that for any complete, orientable, hyperbolic 3-manifold M, if $0 < \epsilon \leq \epsilon_3$, then the ϵ-thin part of M consists of rank 1 cusps, rank 2 cusps and tubes around short geodesics of length at most ϵ.*

Remark 3.4.1. *Any number $\epsilon > 0$ for which the ϵ-thin part of M decomposes as per the conclusion of the thick-thin decomposition is called a Margulis number for M.*

The supremum of all constants ϵ_3 for which the conclusion of Theorem 3.4.1 holds is called the Margulis constant, and we denote it as ϵ_3. Currently, the best lower bound for ϵ_3 is 0.104, which is due to Meyerhoff [Mey87] and the best upper bound is 0.616 which is due to Weeks [Wee05].

The thick-thin decomposition for complete hyperbolic 3-manifolds follows from the Margulis lemma, which is actually a very general theorem about discrete groups acting on symmetric spaces. However, we shall use a restricted version which applies to Kleinian groups. The form of the Margulis lemma we are going to prove is due to Jorgensen and Marden and its proof is given in Marden's book [Mar07]. This version in fact holds for Kleinian groups with elliptics also, but we shall only state it for Kleinian groups without elliptics.

First, we establish some notation. Given a Kleinian group Γ, a point $x \in \mathbb{H}^3$, and a distance r, let $\Gamma(x,r)$ denote the set

$$\Gamma(x,r) = \{T \in \Gamma \mid d(x,Tx) < r\}$$

where $d(x,y)$ refers to the hyperbolic distance between the points x and y. Let $\langle\Gamma(x,r)\rangle$ denote the subgroup of Γ generated by $\Gamma(x,r)$. The Margulis lemma can then be stated as below.

Theorem 3.4.2 (Universal elementary neighbourhoods)**.** *There exists a universal constant $\epsilon_3 > 0$, such that for all $x \in \mathbb{H}^3$ and for any Kleinian group Γ without elliptics, the* *subgroup $\langle \Gamma(x, \epsilon_3) \rangle$ is elementary.*

Proof. We first show that for a fixed $x \in \mathbb{H}^3$, and for a fixed Kleinian group Γ without elliptics, there exists an $r > 0$, such that $\langle \Gamma(x, r) \rangle$ is elementary. Suppose this is not true, then for a sequence $r_n \to 0$, $\langle \Gamma(x, r_n) \rangle$ will be non-elementary for each $n \in \mathbb{N}$. Thus, we will obtain a sequence of distinct $T_n \in \Gamma(x, r_n)$ such that the distance $d(x, T_n x) < r_n$. Then, by Lemma 3.2.2, we see that the sequence T_n converges to an element $T \in PSL(2, \mathbb{C})$. Using Lemma 3.2.1, we see that this contradicts the fact that Γ is discrete. So, there exists a sufficiently small $r > 0$ such that $\langle \Gamma(x, r) \rangle$ is elementary.

We will show that there exists a fixed $r > 0$ such that the above result is true for any $x \in \mathbb{H}^3$ and for any Kleinian group $\Gamma \leq PSL(2, \mathbb{C})$, and this will prove the theorem. Suppose this is not true, then there must exists a sequence of points $x_n \in \mathbb{H}^3$, a sequence of distances $r_n \in \mathbb{R}_{>0}$, and a sequence of Kleinian groups $\Gamma_n \leq PSL(2, \mathbb{C})$, such that $\langle \Gamma_n(x_n, r_n) \rangle$ is not elementary. We first simplify the proof by conjugating all these groups to replace the different x_n's with a fixed $x \in \mathbb{H}^3$. For each x_n, let T_n be an element of $PSL(2, \mathbb{C})$ such that $T_n(x_n) = x$, for some fixed $x \in \mathbb{H}^3$. Then, we can conjugate the sequence of Kleinian groups Γ_n by T_n to work with a fixed $x \in \mathbb{H}^3$. As $\langle \Gamma_n(x_n, r_n) \rangle \leq \Gamma_n$ is a non-elementary subgroup, we see that $T_n \langle \Gamma_n(x_n, r_n) \rangle T_n^{-1} \leq T_n \Gamma_n T_n^{-1}$ is also a non-elementary subgroup. So, if we denote $T_n \Gamma_n T_n^{-1}$ as G_n, then $\langle G_n(x, r_n) \rangle = T_n \langle \Gamma_n(x_n, r_n) \rangle T_n^{-1}$. So, we shall henceforth work with the sequence of Kleinian groups G_n and the non-elementary subgroups $\langle G_n(x, r_n) \rangle$.

We shall now try and obtain elements A_n and B_n in $G_n(x, r_n)$, such that the subgroup $\langle A_n, B_n \rangle$ is non-elementary. As $\langle G_n(x, r_n) \rangle$ is non-elementary, we claim that the generating set $G_n(x, r_n)$ must contain at least two elements A_n and B_n which do not have any fixed points in common. As the subgroup $\langle G_n(x, r_n) \rangle$ is non-elementary, in particular, it is non-trivial, and hence $G_n(x, r_n)$ must contain a non-trivial element A_n. Whether A_n is parabolic or loxodromic, since $\langle G_n(x, r_n) \rangle$ fixes more than two points on $\partial \mathbb{H}^3$, the generating set $G_n(x, r_n)$ must consist of an element B_n which fixes a point other than the ones fixed by A_n. Note that B_n cannot be a loxodromic element which shares one of its fixed points with A_n, as then the subgroup $\langle A_n, B_n \rangle$ would not be discrete, due to the proof of the second part of

Proposition 3.3.2. So, either B_n is a parabolic which fixes a point not fixed by A_n, or B_n is a loxodromic which shares no fixed point in common with A_n. Thus the subgroup $\langle A_n, B_n \rangle$ is not elementary.

As $n \to \infty$, $r_n \to 0$, and since $d(x, A_n(x)) < r_n$ and $d(x, B_n(x)) < r_n$, we see that $A_n(x) \to x$, and $B_n(x) \to x$. So from Lemma 3.2.2, we see that there are subsequences of $\{A_n\}$ and $\{B_n\}$ which converge to elements A and B in $PSL(2, \mathbb{C})$. Then, from Theorem 3.3.1, we see that $\langle A, B \rangle$ is also non-elementary. As A_n and B_n do not have common fixed points for all $n \in \mathbb{N}$, A and B cannot share any fixed points in common, due to Theorem 3.3.1.

However, as $A_n, B_n \in G_n(x, r_n)$, and $r_n \to 0$, we see that $A_n(x) \to A(x) = x$, and similarly $B(x) = x$. This contradicts the previous assertion that A and B do not share common fixed points. This contradiction proves our theorem. ■

The proof of the thick-thin decomposition using the Margulis lemma uses a relation between the translation distance of isometries and the injectivity radius. This relation is given in the following lemma.

Lemma 3.4.1. *Let M be a complete, orientable, hyperbolic 3-manifold and let Γ be its holonomy group, that is $M \cong \mathbb{H}^3/\Gamma$. Consider $x \in M$ with a lift $\tilde{x}$ in $\mathbb{H}^3$. Then, we have*

$$\operatorname{injrad}(x) = \frac{1}{2} \inf_{T \neq id \in \Gamma} \{d(\tilde{x}, T\tilde{x})\} \tag{3.1}$$

Proof. The ball $B(x, r)$ is embedded in M if and only if $B(\tilde{x}, r)$ is disjoint from all the Γ-translates $A(B(\tilde{x}, r)) = B(A\tilde{x}, r)$, for all $A \in \Gamma$, as $\mathbb{H}^3 \to \mathbb{H}^3/\Gamma = M$ is a covering projection. The ball $B(\tilde{x}, r)$ is disjoint from all the Γ-translates $B(A\tilde{x}, r)$ if and only if $d(\tilde{x}, A\tilde{x}) \geq 2r$ for all $A \in \Gamma$. Thus, the supremal radius r for which $B(x, r)$ is embedded in M is equal to the infimum of the values $\frac{1}{2} d(\tilde{x}, A\tilde{x})$, where we vary over all $A \in \Gamma$. Hence, we have proved the required relation. ■

We are now in a position to prove Margulis' thick-thin decomposition for complete hyperbolic 3-manifolds. This proof is borrowed from the book by Benedetti and Petronio [BP12].

Proof. Let $x \in M$, and let $\pi\colon \mathbb{H}^3 \to \mathbb{H}^3/\Gamma = M$ be the covering projection. From the Margulis lemma (Theorem 3.4.2), we see that the subgroup $\langle \Gamma(\epsilon, x) \rangle$ is elementary for any $\epsilon < \epsilon_3$, where ϵ_3 is the Margulis constant. In the rest of this proof, we denote $\langle \Gamma(\epsilon, x) \rangle$ by Γ_ϵ. From Proposition 3.3.1, it is clear that Γ_ϵ is either generated by one or two independent parabolics which fix the same point on $\partial\mathbb{H}^3$, or it is generated by a single loxodromic element which fixes two points on $\partial\mathbb{H}^3$ and leaves the geodesic axis joining these two points invariant.

Let us first suppose that Γ_ϵ is either generated by one or two independent parabolics which fix the same point on $\partial\mathbb{H}^3$. We can conjugate the group Γ to ensure that this fixed point is $\infty \in \partial\mathbb{H}^3$. Let Γ_∞ be the subgroup of Γ consisting of all the parabolics in Γ which fix ∞. As Γ is discrete, we see that Γ_∞ must be generated by one or two independent parabolics fixing ∞. We shall consider the set

$$\tilde{L} = \{\tilde{y} \in \mathbb{H}^3 \mid \exists A \in \Gamma_\infty, A \neq \mathrm{id} \text{ such that } d(\tilde{y}, A\tilde{y}) \leq \epsilon\}$$

From the description of $\tilde{L}$, it is clear that $\tilde{L}$ is a horoball about ∞ in $\mathbb{H}^3$. As $\Gamma_\epsilon \leq \Gamma_\infty$, we see that $\tilde{x} \in \tilde{L}$, and hence $x \in \pi(\tilde{L})$. We shall consider the projection $\pi(\tilde{L}) \subset M$ and show that it is either a rank one or a rank two cusp. It is clear that $\tilde{L}/\Gamma_\infty$ is a rank one or a rank two cusp. We claim that $\pi(\tilde{L}) = \tilde{L}/\Gamma_\infty$. To prove this claim, we shall show that if there exists any $g \in \Gamma$ such that $g(\tilde{L}) \cap \tilde{L} \neq \varnothing$, then $g \in \Gamma_\infty$. Let $q \in g(\tilde{L}) \cap \tilde{L}$, such that $q = g(p)$, where $p \in \tilde{L}$. Then, by definition of $\tilde{L}$, there exist elements A and B in Γ_∞, such that

$$\begin{aligned} d(q, Bq) &\leq \epsilon \\ d(p, Ap) = d(q, gAg^{-1}q) &\leq \epsilon \end{aligned}$$

So, we see that B and gAg^{-1} both belong to $\langle \Gamma(p, \epsilon) \rangle$, and as $B \in \Gamma_\infty$, we see that $\langle \Gamma(p, \epsilon) \rangle$ is also an elementary group which fixes ∞. In particular, we have

$$\begin{aligned} gAg^{-1}(\infty) &= \infty \\ Ag^{-1}(\infty) &= g^{-1}(\infty) \end{aligned}$$

As ∞ is the only fixed point for A, we see that $g(\infty) = \infty$, and so g is a parabolic in Γ_∞. Note that g cannot be a loxodromic as then the group generated by g and other parabolics which fix ∞ will not be discrete, by the second part of Proposition 3.3.2. Hence, we have proved that $\pi(\tilde{L}) = \tilde{L}/\Gamma_\infty$, and hence x belongs to $\pi(\tilde{L})$, which is either a rank one cusp or a rank two cusp.

Now suppose that Γ_ϵ is generated by a loxodromic which fixes two points on $\partial\mathbb{H}^3$. We can conjugate Γ_ϵ so that these two points are 0 and ∞. We denote by l the geodesic joining 0 and ∞, which is the axis of all the loxodromics in Γ_ϵ. Let Γ_l be the subgroup of Γ consisting of all the loxodromics in Γ which fix l. As Γ is discrete, we see that Γ_l is generated by a single loxodromic which fixes l. We consider the following set

$$\tilde{N} = \{\tilde{y} \in \mathbb{H}^3 \mid \exists A \in \Gamma_l, A \neq \mathrm{id} \text{ such that } d(\tilde{y}, A\tilde{y}) \leq \epsilon\}$$

From the description of $\tilde{N}$, it is clear that $\tilde{N}$ is a cylinder of hyperbolic radius r around the geodesic l in $\mathbb{H}^3$, where r depends on ϵ. As $\Gamma_\epsilon \leq \Gamma_l$, we see that $\tilde{x} \in \tilde{N}$, and hence $x \in \pi(\tilde{N})$. We shall consider the projection $\pi(\tilde{N}) \subset M$ and show that it is a tube of radius r around the quotient of the geodesic l, which is a closed geodesic of length at most ϵ. It is clear that $\tilde{N}/\Gamma_l$ is a tube of radius r around l/Γ_l. We claim that $\pi(\tilde{N}) = \tilde{N}/\Gamma_l$. To this end, we claim that if there exists any $g \in \Gamma$ such that $g(\tilde{N}) \cap \tilde{N} \neq \varnothing$, then $g \in \Gamma_l$. The proof of this claim involves arguments which are similar to the parabolic case, so we skip them. Thus, we have proved that $\pi(\tilde{N}) = \tilde{N}/\Gamma_l$, and hence x belongs to $\pi(\tilde{N})$, which is a tube of radius r around a closed geodesic of length at most ϵ. ■

3.5 Implications

The thick-thin decomposition imposes strict conditions on the structure of complete hyperbolic 3-manifolds. In particular, we have the following classification for finite volume complete hyperbolic 3-manifolds.

Theorem 3.5.1. *A complete hyperbolic 3-manifold M has finite volume if and only if M is closed (compact and without boundary) or M is cusped (it is homeomorphic to the interior of a compact manifold with torus boundary components).*

Thus, we see that the complement of hyperbolic knots in S^3 must have finite volume. The volume of a hyperbolic knot is an important invariant of the knot.

Chapter 4

Thurston's gluing equations

4.1 Introduction

A hyperbolic structure on an n-dimensional manifold M is given by the following data: each point of M has a neighbourhood which is homeomorphic to a ball neighbourhood in $\mathbb{H}^n$, and the transition maps are isometries of $\mathbb{H}^n$. The hyperbolic structure on M gives it a Riemannian metric by pull-back from the local homeomorphisms. We say that the hyperbolic structure on M is complete if this induced Riemannian metric is complete, that is, M is a complete metric space. In this case we say that M is a complete hyperbolic manifold. If M is a complete hyperbolic manifold, it has many nice properties. So, we are naturally interested in understanding the conditions under which M has a complete hyperbolic structure.

Thurston has developed an approach to determine whether a 3-manifold M has a complete hyperbolic structure and to find such a structure if it exists. This approach involves constructing M by gluing ideal tetrahedra, which are tetrahedra without their vertices. These ideal terahedra can be naturally realized as tetrahedra in $\mathbb{H}^3$ with their vertices on $\partial\mathbb{H}^3$. Each tetrahedron is assigned a parameter which determines its hyperbolic shape, and Thurston's equations are equations involving these parameters which determine whether these hyperbolic shapes glue together consistently to produce a complete hyperbolic structure on M.

Most of the exposition in this chapter will follow the online book by Jessica Purcell [Pur20]. Classic resources for these topics are Thurston's lecture notes [Thu80] and his book [TL97].

4.2 Ideal triangulations and edge parameters

4.2.1 Geometric ideal triangulation of a manifold

An ideal tetrahedron is a tetrahedron without its vertices. Ideal tetrahedra can be used as the building blocks of many complete cusped hyperbolic manifolds.

Definition 4.2.1 (Topological ideal triangulation). *A topological ideal triangulation of a manifold M is a realization of M as the quotient of a collection of ideal tetrahedra by face pairing homeomorphisms. Let $\widehat{M}$ be the cell complex obtained by gluing the collection of tetrahedra according to the face pairing homeomorphisms. Then, M is homeomorphic to the complement of the vertices in $\widehat{M}$.*

An ideal tetrahedron can be realised in $\mathbb{H}^3$ with its vertices on $\partial\mathbb{H}^3$, edges as geodesics in $\mathbb{H}^3$ joining these vertices and faces as geodesic ideal triangles.

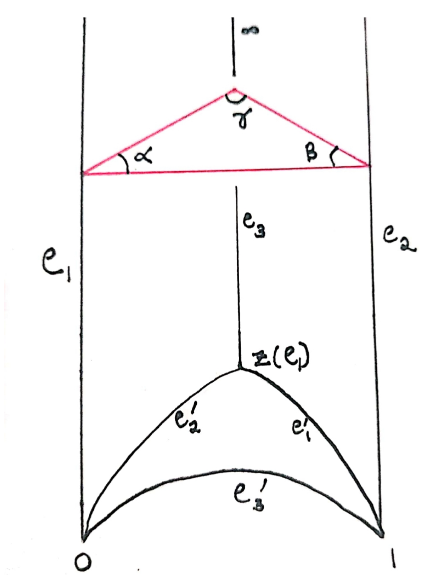


Figure 4.1: An ideal tetrahedron in $\mathbb{H}^3$

Definition 4.2.2. *A geometric ideal triangulation of M is a realization of M as a quotient of a collection of hyperbolic ideal tetrahedra by face pairing isometries such that the* *tetrahedra glue together consistently to give a complete hyperbolic structure on M.*

A geometric ideal triangulation of a complete hyperbolic manifold efficiently describes the hyperbolic structure on the manifold and can be used to obtain important geometric information about the manifold such as its volume.

4.2.2 Edge parameters of an ideal tetrahedron

In this section, we wish to understand how to parametrize the shape of a hyperbolic ideal tetrahedron. Consider the upper half space model of hyperbolic space and let the boundary plane be identified with the complex plane. Let e be an edge of the tetrahedron. Isometries of $\mathbb{H}^3$ can be used to put the vertices of e at 0 and ∞ on $\partial\mathbb{H}^3$ and ensure that a third vertex is at 1. We can also rotate and scale the tetrahedron using elliptic and loxodromic isometries to ensure that the fourth vertex is at some z, where z has positive imaginary part. The complex number z is said to be the edge parameter of the edge e. We denote the edge parameter of any edge by $z(e)$.

By bringing any ideal tetrahedron to the standard position with three of its vertices at 0, 1 and ∞ using isometries, we see that the set of edge parameters of a hyperbolic ideal tetrahedron uniquely determines its congruency class. Also, the edge parameter of any one edge is sufficient to determine all the other edge parameters of a hyperbolic ideal tetrahedron.

Proposition 4.2.1. *Consider an ideal tetrahedron and assume one of its vertices is at ∞. Let the vertical edges be labelled as e_1, e_2 and e_3 in the anti-clockwise direction (when looking from above). Label the non-vertical edge opposite to the edge e_i as e_i', as in Figure 4.1. Then the edge parameters satisfy the following relations:*

$$\begin{aligned} z(e_i) &= z(e_i') \\ z(e_2) &= \frac{1}{1 - z(e_1)} \\ z(e_3) &= \frac{z(e_1) - 1}{z(e_1)} \end{aligned}$$

Consider an ideal tetrahedron with one of its vertices at ∞. Then the link triangle of the vertex at ∞ is the Euclidean triangle which is given by the cross section of the ideal tetrahedron by a horosphere about ∞. The link triangle here is thus determined only up to Euclidean similarity. By bringing the ideal tetrahedron to standard position and fixing the three vertices at 0, 1 and ∞, we realise that the similarity class of the link triangle uniquely determines the congruency class of the ideal tetrahedron.

We will now describe how to go back and forth between these two descriptions of a hyperbolic ideal tetrahedron.

Proposition 4.2.2. *Suppose we know the similarity class of the link triangle of any one vertex of the ideal tetrahedron. Let the angles of the link triangle be α, β and γ. Then, the edge parameter of the edge e which has dihedral angle α is given by*

$$z(e) = \frac{sin(\gamma)}{sin(\beta)} \cdot e^{i\alpha}$$

Now suppose that we know the edge parameter of one of the edges of the ideal tetrahedron. Let this edge be labelled e_1 and let the other two edges be labelled as e_2 and e_3 in the anticlockwise sense. Then, if the angles α, β and γ correspond to the dihedral angles of the edges e_1, e_2 and e_3, then we see from Figure 4.1 that these angles are the arguments of the corresponding complex numbers, that is

$$\begin{aligned}
\alpha &= \arg(z(e_1)) \\
\beta &= \arg(z(e_2)) \\
\gamma &= \arg(z(e_3))
\end{aligned}$$

4.3 Gluing consistency equations

In the following sections, we will assume that M is homeomorphic to the interior of a compact manifold with torus boundary components unless stated otherwise. Using the edge parameters of the ideal tetrahedra it is possible to find out the conditions under which M admits a hyperbolic structure with respect to which each tetrahedron of the given topological ideal triangulation of M becomes a hyperbolic ideal tetrahedron. We arbitrarily assign a variable edge parameter to each edge of each ideal tetrahedron in the triangulation. The existence of a hyperbolic structure then imposes constraints on these edge parameters which we can solve to obtain the solution space of possible hyperbolic structures on M with respect to the given topological ideal triangulation. These equations are known as Thurston's gluing consistency equations.

Definition 4.3.1 (Hyperbolic structure)**.** *An n-dimensional manifold M is said to have a hyperbolic structure if each point of M has a neighbourhood isometric to a ball in $\mathbb{H}^n$ such that the transition maps are isometries of $\mathbb{H}^n$.*

The hyperbolic structure on M induces a Riemannian metric on M via pullback. We say that the hyperbolic structure on M is complete if the induced Riemannian metric on M is complete, or equivalently, M is a complete metric space with the metric induced by the Riemannian metric (this equivalence follows by the Hopf-Rinow theorem).

Definition 4.3.2. *M is said to have a hyperbolic structure (possibly incomplete) with respect to an ideal triangulation τ if each tetrahedron of the ideal triangulation is isometric to a hyperbolic ideal tetrahedron in the hyperbolic metric induced by the hyperbolic structure on M.*

Let τ be a topological ideal triangulation of M. Consider an edge e of τ. Let T_i be the ideal tetrahedra glued to the edge e in a cyclic order and let z_i be the edge parameter of the edge of T_i which is identified with e. Then the edge gluing consistency equation for the edge e is given by

$$\prod_{i=1}^{n} z_i = 1 \tag{4.1}$$

$$\sum_{i=1}^{n} \arg(z_i) = 2\pi \tag{4.2}$$

These two equations can be compactly expressed as a logarithmic equation.

$$\sum_{i=1}^{n} \log(z_i) = 2\pi i \tag{4.3}$$

Theorem 4.3.1 (Thurston's gluing consistency equations)**.** *M has a hyperbolic struc-*

ture with respect to the topological ideal triangulation τ if and only if the edge gluing consistency equations have a solution for each edge of τ.

Proof. Suppose the edge gluing consistency equations have a solution for each edge of τ. The edge parameters of such a solution will give a hyperbolic structure on the complement of the 1-skeleton of τ in M, which we shall denote by $\widehat{M}$. We shall prove that the hyperbolic structure extends to the 1-skeleton of τ. Consider an edge e of τ and let T_i be the tetrahedra which meet the edge e in a cyclic order and let e_i be the edge from T_i which is identified to e. Lift the tetrahedron T_1 to $\mathbb{H}^3$ such that the edge e_1 lifts to the edge $(0, \infty)$ in $\mathbb{H}^3$ and the tetrahedron is in standard position for the edge parameter $z(e_1)$. Then, lift the tetrahedron T_2 such that e_2 is identified to $(0, \infty)$ and the appropriate face of T_2 is identified with the face $(0, z(e_1), \infty)$ of T_1. We will ensure that the dihedral angle of T_2 at the edge $(0, \infty)$ is oriented in the anti-clockwise direction. So the vertices of T_2 will now be at 0, ∞, $z(e_1)$ and $z(e_1)z(e_2)$. Similarly, we can continue lifting all the tetrahedra from T_2 to T_n. The last vertex of T_n will be at the point $z(e_1)z(e_2)...z(e_n)$ which will actually be equal to 1, as the edge gluing equations are satisfied, ensuring that the lift of the tetrahedron T_n glues to T_1 consistently. The imaginary part of the edge gluing equation thus ensures that the dihedral angles around the edge e add up to 2π, so the tetrahedra are glued around the edge e for only one rotation. The real part of the equation ensures that there are no shearing singularities, that is, the final face of T_n glues consistently with the initial face of T_1. See Figure 4.2 and Figure 4.3. By lifting these tetrahedra in this way, we see that we can extend the hyperbolic structure on $\widehat{M}$ to the edge e. We can similarly extend the hyperbolic structure to all edges of τ and get a consistent hyperbolic structure on all of M.

Suppose M has a hyperbolic structure with respect to the ideal triangulation τ. Consider an edge e of this ideal triangulation. Any point on the edge has a neighbourhood isometric to a ball in $\mathbb{H}^3$, and the dihedral angles of the edges meeting e must add up to 2π at that point. Since we are given that each tetrahedron is isometric to a hyperbolic ideal tetrahedron under the induced hyperbolic metric, the dihedral angle will be constant at each point of the edge e, and hence the dihedral angles at the edge e will add up to 2π. This is equivalent to the existence of a solution to the imaginary part of the edge gluing equations. Also, as the faces of the ideal tetrahedra meeting e glue together without any shearing, the real part of the edge gluing equations also have a solution.

■

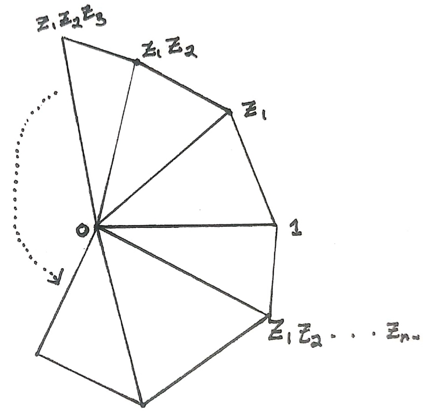


Figure 4.2: Proof of the edge gluing consistency equation

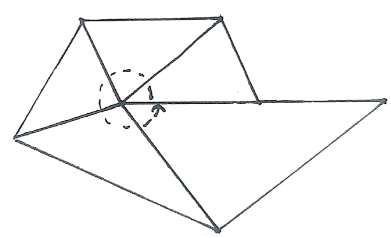


Figure 4.3: Shearing singularities while gluing tetrahedra around an edge

4.4 Gluing completeness equations

Given a topological ideal triangulation τ of M, we can assign edge parameters to each edge of each tetrahedron of τ and solve the gluing consistency equations to determine whether M has a hyperbolic structure with respect to τ. However, this hyperbolic structure may be incomplete. In fact, Mostow-Prasad rigidity guarantees that if a complete hyperbolic structure exists on M, it is unique up to isometry. So most solutions of the edge gluing equations will yield incomplete hyperbolic structures. Thurston's gluing completeness equations give additional constraints under which the solutions of the gluing consistency equations will yield a complete hyperbolic structure on M.

A hyperbolic structure on M induces a similarity structure on each cusp torus of M. A similarity structure on a torus is a $(\mathrm{Sim}(\mathbb{E}^2), \mathbb{E}^2)$ geometric structure on the torus, where $\mathrm{Sim}(\mathbb{E}^2)$ is the group of similarities of the Euclidean plane. In other words, each point on the torus has a neighbourhood which is homeomorphic to a neighbourhood in $\mathbb{E}^2$, and the transition maps are similarities of the Euclidean plane. Parabolic holonomy elements of the hyperbolic structure will correspond to Euclidean translations in the holonomy of the similarity structure, elliptic holonomy elements will correspond to Euclidean rotations, and loxodromic holonomy elements will correspond to the scaling holonomies of the similarity structure. We will state a criteria for the completeness of the hyperbolic structure on M in terms of the induced similarity structures on the cusp tori of M.

Theorem 4.4.1. *The hyperbolic structure on M is complete if and only if the induced similarity structure on each cusp torus is Euclidean.*

Proof. Suppose the induced similarity structure on one of the cusp tori is not Euclidean. That is, the holonomy group of the similarity structure consists not only of congruences, but also scaling transformations. These scaling transformations must be induced from loxodromic transformations in the holonomy group of the hyperbolic structure on M. If we lift this cusp to ∞ in $\mathbb{H}^3$, we see that the holonomy group of the hyperbolic structure on M must contain loxodromic transformations. Consider a non-trivial closed curve α on the cusp torus, such that the holonomy of α is a scaling transformation which corresponds to a loxodromic isometry of the form $x \mapsto \lambda x$, where $\lambda \in \mathbb{R}, \lambda > 1$. Let T_0 be a horospherical triangle in one of the tetrahedra corresponding to a cusp triangle which α meets and consider the segment of α in this triangle. T_0 meets the next tetrahedron along the curve α in a segment on their shared face; so, we can develop T_0 uniquely to get a horospherical triangle T_1 in the next tetrahedron and continue the segment of α in the triangle T_2. In this way, we can keep developing the horospherical triangles along the curve α and continuing the curve α in these triangles. As the holonomy $\rho(\alpha)$ is loxodromic, these horospherical triangles will not close up consistently, but will spiral out into the cusp in M. Let us denote the continuation of the curve α by $\widehat{\alpha}$; it will also spiral into the cusp in M. When seen in the fundamental domain for M in $\mathbb{H}^3$, the horospherical triangle $\rho(\alpha)(T_0)$ will not match up with the original horospherical triangle T_0, but it will be at a greater height in $\mathbb{H}^3$, where the ratio in heights is λ.

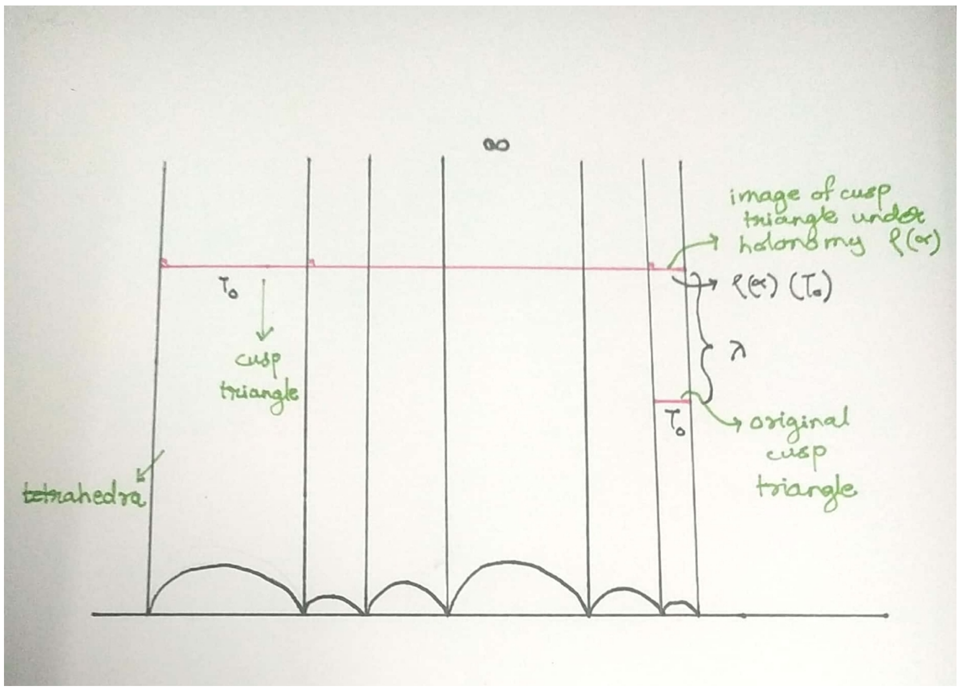


Figure 4.4: When the holonomy $\rho(\alpha)$ has non-trivial scaling, the horospherical triangles do not close up

Let x_0 be a point on the horospherical triangle T_0 in $\mathbb{H}^3$. Consider the sequence $(\rho(\alpha))^n(x_0)$ in $\mathbb{H}^3$ and let us label these points x_n. The distance between any x_n and x_{n+1} in the vertical direction in $\mathbb{H}^3$ is λ. Thus, if we consider the distance between x_n and x_{n+1} along the curve $\widehat{\alpha}$ in M and denote it by $d_{\widehat{\alpha}}(x_n, x_{n+1})$, we have

$$d_{\widehat{\alpha}}(x_n, x_{n+1}) = e^{-\lambda} d_{\widehat{\alpha}}(x_{n-1}, x_n)$$

Thus, the image of the sequence x_n in M is a Cauchy sequence in M. However, as the sequence x_n goes to ∞ in $\mathbb{H}^3$, the image of these points will not converge in M. Thus, M is not complete.

Now suppose the induced similarity structure on each cusp torus of M is Euclidean. This means that there are no scaling transformations in the holonomy group for any cusp torus, and the holonomy groups for each cusp torus consists only of Euclidean congruences. These congruences can be induced only by parabolic transformations in the holonomy group of the hyperbolic structure on M (since elliptic transformations do not act freely). Thus, if we consider any non-trivial closed curve on any cusp torus, its holonomy in the hyperbolic structure on M has to be parabolic. So, consider horospherical triangles around any cusp, they will close up consistently. For each cusp of M, choose horospherical triangles which close up around it. Delete the interior of the horoball bound by these horospheres from M and let the resultant compact submanifold of M with torus boundary components be called M_0. For any $t > 0$, let M_t be the compact manifold obtained by removing the interiors of horoballs which are at a distance t (towards the cusps) from the original choice of horoballs. Then the sets M_t for $t > 0$ satisfy the following properties:

1. For each $t > 0$, M_t is a compact submanifold of M with torus boundary components.

2. $\bigcup_{t>0} M_t = M$

3. M_{t+a} contains a neighbourhood of radius a around M_t.

Thus, any Cauchy sequence in M must lie entirely in M_t for sufficiently large t. As M_t is compact, the Cauchy sequence must converge in M_t. Hence, M is complete.

■

The ideal triangulation τ induces a triangulation on each cusp torus of M. Consider any cusp torus T of M and let the induced triangulation of T be $\bar{\tau}$. The edge parameters of the ideal edges corresponding to a solution of the edge gluing equations give rise to complex numbers corresponding to the vertices of each triangle of $\bar{\tau}$. These parameters can be used to formulate the condition given in the previous theorem into a set of equations known as the gluing completeness equations.

Definition 4.4.1. *Let α be a loop corresponding to the homotopy class $[\alpha] \in \pi_1(T)$. Then we can associate a complex number $H([\alpha])$ corresponding to each homotopy class of loops on T as follows:*

First orient the loop α. Homotope α such that it runs monotonically through each triangle of the triangulation, that is, it enters each triangle of the cusp triangulation through one side and exits through a different side, and cuts off one vertex in each triangle. Let z_i be the complex parameter corresponding to the vertex cut off by α in the triangle t_i. Assign a sign parameter ϵ_i which takes values based on the orientation of the curve α with respect to the vertex cut off by α in the triangle t_i. Set $\epsilon_i = 1$ if the vertex lies on the left side of α and $\epsilon_i = -1$ if the vertex lies on the right side of α. Then, we define the parameter $H([\alpha])$ as follows:

$$H([\alpha]) = \prod_{i=1}^{n} z_i^{\epsilon_i} \tag{4.4}$$

See an example worked out in Figure 4.5.

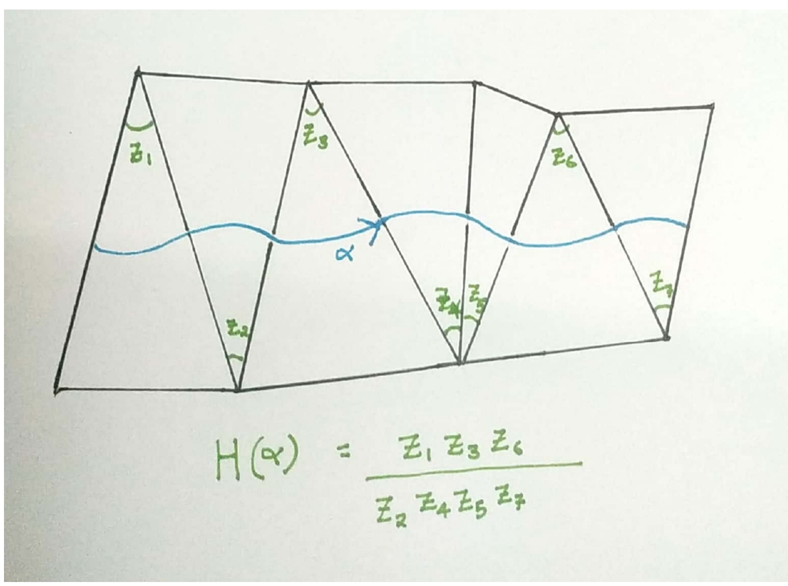


Figure 4.5: Calculating $H(\alpha)$

Proposition 4.4.1. *1. The map $H\colon \pi_1(T) \to \mathbb{C}^*$ is well defined, that is, it does not depend on the choice of loop in the homotopy class $[\alpha]$*

2. $H\colon \pi_1(T) \to \mathbb{C}^*$ is a group homomorphism.

$$H([\alpha] * [\beta]) = H([\alpha]) \cdot H([\beta])$$

Let $[\alpha]$ and $[\beta]$ be generators of $\pi_1(T)$, where T is a cusp torus of M. Then the completeness equations for the cusp with boundary T are

$$H([\alpha]) = H([\beta]) = 1 \tag{4.5}$$
$$\tag{4.6}$$

This equation can be written in the logarithmic form as

$$\log(H([\alpha])) = \log(H([\beta])) = 0 \tag{4.7}$$

Theorem 4.4.2. *Let z_i be edge parameters which solve the edge gluing consistency equations for M given the ideal triangulation τ. Then, the hyperbolic structure induced on M due to these parameters is complete if and only if the edge parameters z_i satisfy the gluing completeness equations for each cusp of M. Also, if the z_i solve the gluing completeness equations, the topological ideal triangulation τ with hyperbolic structure given by the edge parameters z_i is actually a geometric ideal triangulation of M.*

Proof. By Theorem 4.4.1, it is sufficient to prove that the induced similarity structure on each cusp torus is Euclidean if and only if the the edge parameters obtained as a solution of the edge gluing consistency equations satisfy the gluing completeness equations. We first show that if the gluing completeness equations are satisfied, the holonomy group for the similarity structure on each cusp torus consists only of translations and no scalings or rotations (rotations are induced by elliptic transformations which do not act freely). Let α_i and β_i be the generators for $\pi_1(T_i)$. From the completeness equations we have that $H(\alpha_i) = H(\beta_i) = 1$ for all i.We shall prove that the holonomy transformations $\rho(\alpha_i)$ and $\rho(\beta_i)$ are Euclidean translations using the complex parameters associated with the vertices of the cusp triangulation.

Let us focus our attention on the cusp torus T_1 and consider the generator α_1 for $\pi_1(T_1)$. We shall homotope α_1 and orient it as required in Definition 4.4.1. Let e_1 be an edge of the cusp triangulation which α_1 intersects. Let v be a vector of the same length as e_1 pointing in the direction of e_1 and oriented with respect to the orientation of α_1 as per the right hand rule. We shall consider the effect of the holonomy

$\rho(\alpha_1)$ on the vector v.

We can rotate v around a vertex of the edge e_1 and scale it appropriately to align it with the edge e_2. This rotation and scaling is the transformation given by multiplying the vector v by the complex parameter z_1 associated to the vertex. We rotate by the argument of z_1 and scale by the magnitude of z_1. Similarly, we can rotate and scale the vector along e_2 to align it with the edge e_3. As there is a path of edge vectors from the initial edge e_1 to the edge $\rho(\alpha_1)(e_1)$, we can keep track of how much we need to scale and rotate to go from e_1 to $\rho(\alpha_1)(e_1)$, which will give us the holonomy $\rho(\alpha_1)$. If we rotate anti-clockwise around a vertex, we multiply by the parameter of that vertex, and if we rotate clockwise around a vertex, we need to divide by the parameter of that vertex. The final rotation and scaling of the holonomy element $\rho(\alpha_1)$ is given by the product of all the complex parameters which we have multiplied and divided by to go from e_1 to $\rho(\alpha_1)(e_1)$. This is the same as the complex number $H(\alpha_1)$, defined in Definition 4.4.1. The holonomy $\rho(\alpha_1)$ is a Euclidean translation if and only if it does not rotate or scale, which happens if and only if $H(\alpha_1) = 1$.

The same argument will tell us that $H(\beta_1) = 1$ for the similarity structure induced on the cusp torus T_1 to be Euclidean. Similarly, we will obtain that $H(\alpha_i) = H(\beta_i) = 1$ for all i, since the similarity structure induced on each cusp torus must be Euclidean for the hyperbolic structure on M to be complete. All the steps in this proof are reversible, that is, any step is true if and only if the previous one is true; so, the converse is also proved. ■

4.5 Example of the figure eight knot complement

The figure eight knot complement is one of the simplest cusped hyperbolic manifolds and admits a geometric triangulation consisting of two regular ideal tetrahedra. A procedure developed by Menasco to obtain this standard triangulation of the figure eight knot complement is described in the first chapter of Purcell's book [Pur20]. In this section, we shall consider the standard topological ideal triangulation of the figure eight knot complement into two ideal tetrahedra and solve the edge gluing consistency and the gluing completeness equations for this triangulation. This example was first described by Thurston in his notes [Thu80].

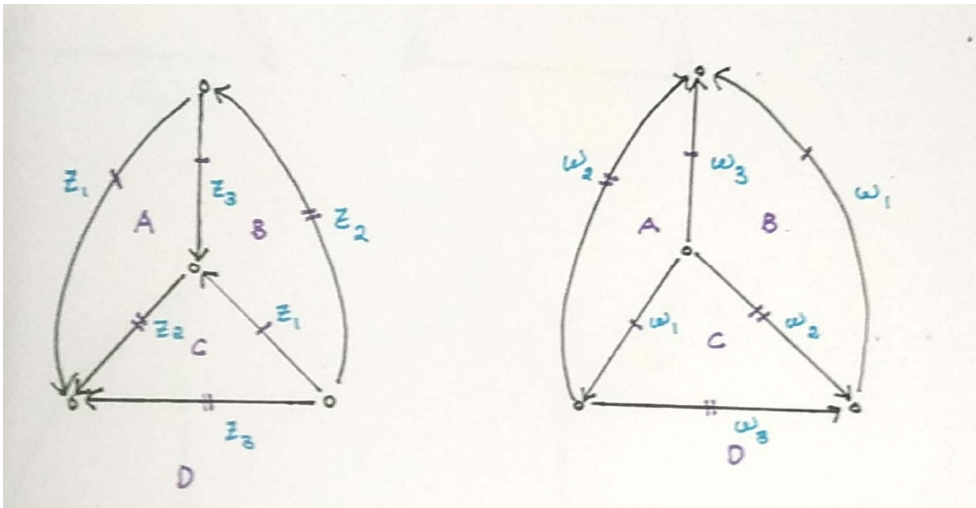


Figure 4.6: Standard ideal triangulation of the figure eight knot complement

The standard ideal triangulation of the figure eight knot complement is as shown in Figure 4.6. It consists of two ideal tetrahedra whose faces are identified pairwise. There are two edges in this triangulation and only one ideal vertex which corresponds to the cusp of the knot complement. We shall assign edge parameters to the edges of both the tetrahedra and solve Thurston's equations for this triangulation. Assign the edges of the first tetrahedron the edge parameters z_1, z_2, and z_3, and that of the second tetrahedron w_1, w_2, and w_3. Note that opposite edges of a tetrahedron must be assigned the same edge parameters and that the edge parameters of a tetrahedron must satisfy the relations of Proposition 4.2.1.

We first solve the edge gluing consistency equations for the two edges of the triangulation. For the edge with one tick mark, we have the equation

$$z_1^2 z_3 w_1^2 w_3 = 1 \tag{4.8}$$

For the edge with two tick marks, we have

$$z_2^2 z_3 w_2^2 w_3 = 1 \tag{4.9}$$

Substituting the relations between edge parameters from Proposition 4.2.1 into

Equation 4.8 and Equation 4.9, we get

$$z_1(z_1 - 1)w_1(w_1 - 1) = 1 \tag{4.10}$$

Solving for z_1 in terms of w_1, we get

$$z_1 = \frac{1 \pm \sqrt{1 + 4/(w_1(w_1 - 1))}}{2} \tag{4.11}$$

For the triangulation to have a hyperbolic structure, we need both z_1 and w_1 to have positive imaginary part. For each value of w_1, we get at most one solution for z_1 which has positive imaginary part. Also a solution for z_1 with positive imaginary part exists only if the discriminant $1 + 4/(w_1(w_1 - 1))$ is not a positive real number or zero. Thus, the solution space for w_1 is given in Figure 4.7. The value for z_1 is fixed given any value of w_1 from this solutions space, and then all the other edge parameters will be known. Thus, this solution space is a parameter space for the possible hyperbolic structures on the figure eight knot complement with the standard ideal triangulation.

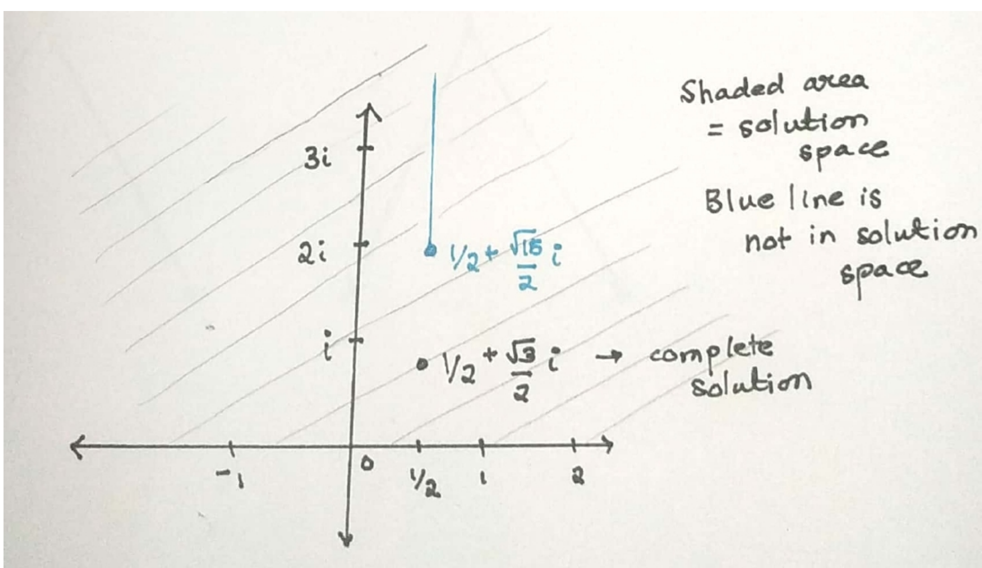
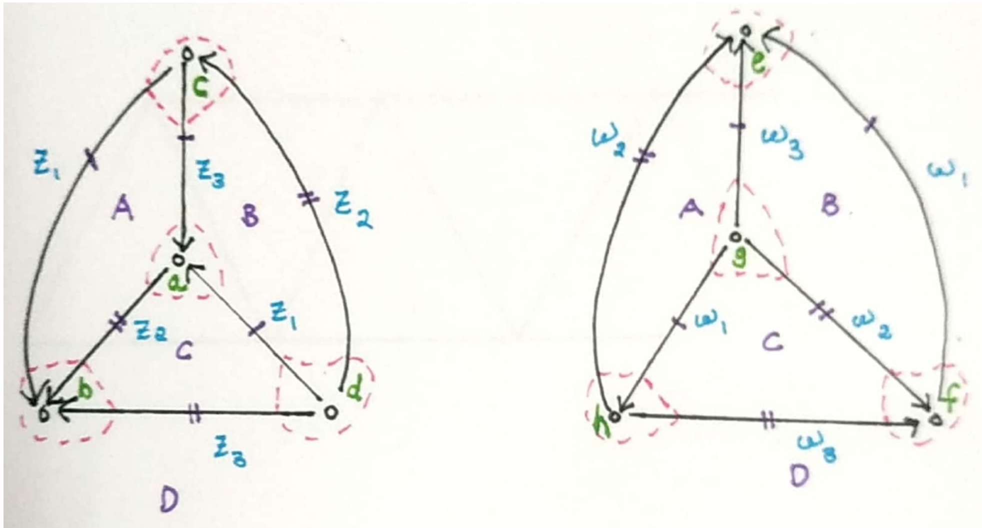
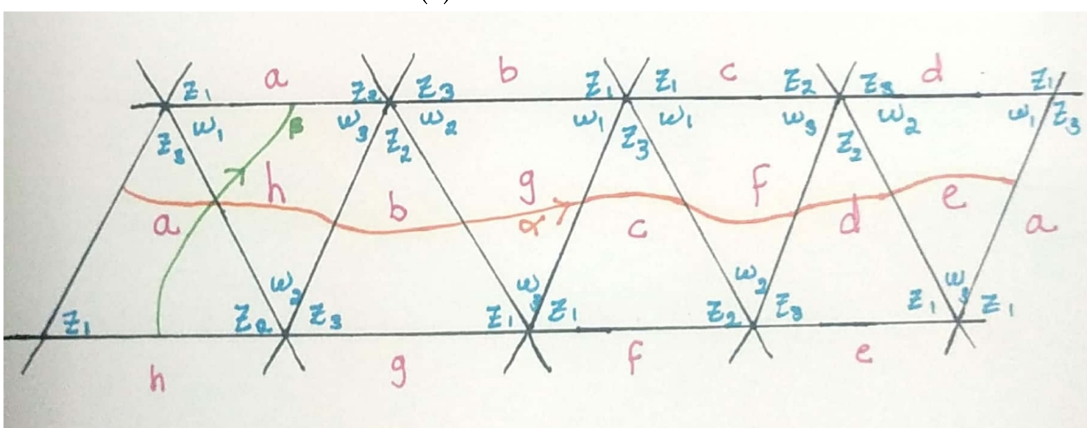


Figure 4.7: Solution space of w_1 for the edge gluing equations for the figure eight knot complement

All of these solutions except one will turn out to give incomplete hyperbolic struc-



(a)



(b)

Figure 4.8: Cusp triangulation for the figure eight knot complement

tures. To see this, we will solve Thurston's gluing completeness equations. We first obtain the cusp triangulation from the triangulation of the knot complement and assign each vertex of the triangulation a complex parameter which is the edge parameter of the ideal edge corresponding to that vertex. This cusp triangulation along with the complex parameters is shown in Figure 4.8b.

We then calculate the H map for the generators of the holonomy group of the cusp torus. In the case of the figure eight knot complement, we have two generators, α and β for the holonomy group of the cusp torus, which are depicted in figure Figure 4.8b in orange and green respectively. Then, we see that the value of the H

map for the generators is given by

$$H(\alpha) = (\frac{z_2 z_3}{w_2 w_3})^2 \tag{4.12}$$

$$H(\beta) = \frac{w_1}{z_2} \tag{4.13}$$

The completeness equations for the cusp are $H(\alpha) = H(\beta) = 1$. Solving these equations for the solutions of the edge gluing consistency equations, we get

$$z_1 = z_2 = z_3 = \frac{1+\sqrt{3}i}{2} \tag{4.14}$$

$$w_1 = w_2 = w_3 = \frac{1+\sqrt{3}i}{2} \tag{4.15}$$

Thus, the geometric triangulation of the figure eight knot complement consists of two regular ideal tetrahedra with all dihedral angles $\pi/3$.

4.6 Infinitely many geometric triangulations of the figure eight knot complement

In this section, we shall see that the figure eight knot complement has infinitely many geometric triangulations which are obtained from the standard triangulation by performing 2-3 Pachner moves. This method was first described by Blake Dadd and Aochen Duan in [DD16].

After performing a 2-3 Pachner move on the two tetrahedra in the initial geometric triangulation, if we set the edge parameters of the equatorial edges in the new triangulation to be the product of the equatorial edge parameters in the initial triangulation, and assign all the other edge parameters according to Proposition 4.2.1, then both the edge gluing consistency and the completeness equations are satisfied. The only issue that can occur is that the new triangulations may have edge parameters with negative or zero imaginary part. We will thus find the conditions required to obtain a triangulation where all the tetrahedra have edge parameters with positive imaginary part.

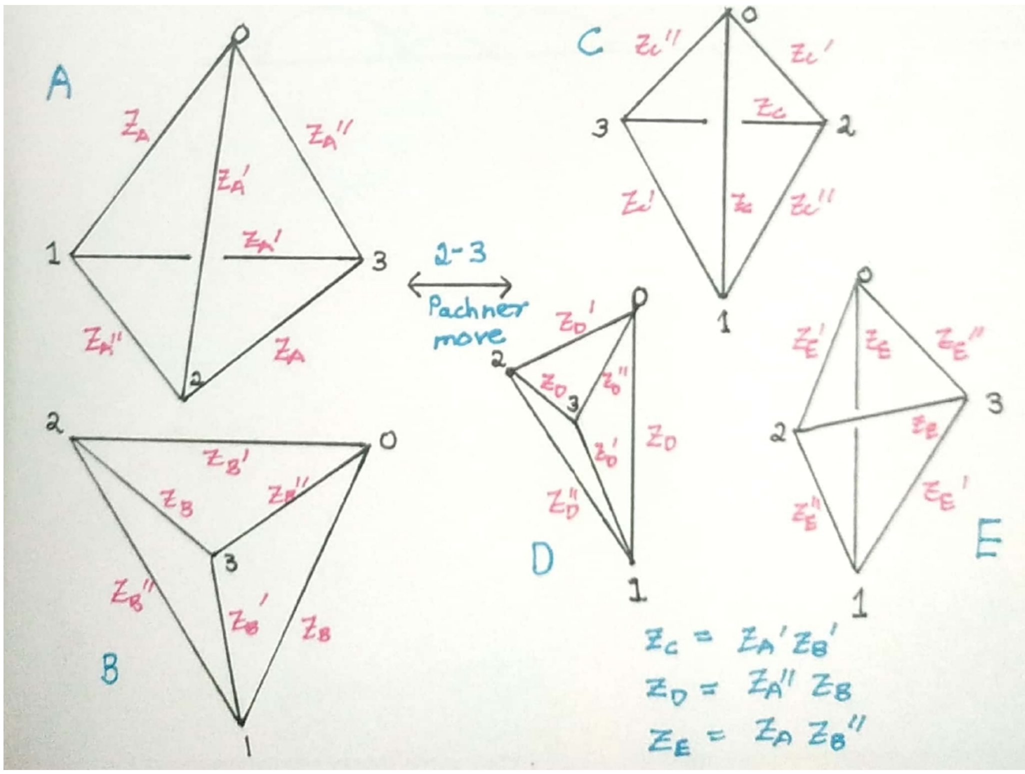


Figure 4.9: 2-3 Pachner move and edge parameters of new tetrahedra

Let τ be a geometric ideal triangulation of a cusped hyperbolic 3-manifold M. Suppose τ has two distinct tetrahedra A and B. Label the vertices of A and B with the numbers 0, 1, 2 and 3, and let the faces of the tetrahedra be labelled according to this labelling of the vertices. Suppose A and B are identified along two pairs of faces, that is $A123 = B230$ and $A012 = B013$. We will perform a 2-3 Pachner move along the face $A123 = B230$ to obtain three tetrahedra C, D and E which are relabelled as per Figure 4.9. We will see that the resulting triangulation is geometric, where the edge parameters of the equatorial edges set in the resulting triangulation are equal to the product of the equatorial edge parameters in the initial triangulation.

Theorem 4.6.1. *Let τ be a geometric ideal triangulation of a cusped hyperbolic 3-manifold M. Let A and B be two ideal tetrahedra of τ which satisfy the following conditions:*

1. *A and B are identified along two pairs of faces, that is $A123 = B230$ and $A012 = B013$*

2. *The edge parameters of A and B are equal, that is,* $z_A = z_B$

3. $Re(z_A) < 1$

Then a 2-3 Pachner move along the face $A123 = B230$ *produces a geometric triangulation of M. Suppose, we label the tetrahedra C, D and E produced by the 2-3 Pachner move as per Figure 4.9. Then, on relabelling the tetrahedron E as A and the tetrahedron D as B, they will again satisfy conditions 1, 2 and 3.*

The proof of this theorem can be found in [DD16]. We shall denote the figure eight knot complement as M_8. Let τ_2 be the standard triangulation of M_8 consisting of 2 ideal tetrahedra. We will repeatedly perform 2-3 Pachner moves on τ_2 and relabel the tetrahedra according to Theorem 4.6.1. In this process, we will obtain ideal triangulations of M_8 consisting of n tetrahedra for every $n \in \mathbb{N}$. We shall denote the ideal triangulation with n tetrahedra by τ_n.

Theorem 4.6.2. τ_n *is a geometric triangulation of the figure eight knot complement* M_8 *for every* $n \in \mathbb{N}, n \geq 2$.

Proof. We shall prove this theorem by induction. We know from the previous section that τ_2 is a geometric triangulation of M_8 which satisfies conditions 1, 2 and 3 of Theorem 4.6.1. Suppose τ_n is known to be a geometric triangulation of the M_8 satisfying conditions 1, 2 and 3 of Theorem 4.6.1, we shall show that τ_{n+1} is also a geometric triangulation of M_8 which satisfies the same conditions. But τ_{n+1} is obtained from τ_n by performing a 2-3 Pachner move along the shared face $A123 = B230$ of the two tetrahedra labelled A and B in τ_n; so, the previous claim is true due to Theorem 4.6.1. Thus, we can perform 2-3 Pachner moves repeatedly on τ_2 to obtain infinitely many geometric triangulations of the figure eight knot complement. ■

Chapter 5

Angle structures and the volume functional

5.1 Introduction

Thurston's gluing equations are non-linear and can be very difficult to solve except for the simplest of triangulations. So, Casson and Rivin came up with the approach of separating the equations into a linear and non-linear part. The linear part requires solving linear equations to find whether a triangulation supports an angle structure. An angle structure on a triangulation is the data of dihedral angles corresponding to the edges of all tetrahedra in the triangulation such that the angles satisfy certain constraints arising from Thurston's equations. Thus, angle structures effectively linearise Thurston's gluing equations.

Definition 5.1.1. *Let M be a 3-manifold and τ be an ideal triangulation of M. An angle structure on τ is an assignment of dihedral angles to the edges of each tetrahedron in τ which satisfies the following conditions:*

- *Each angle α is in the range $(0, \pi)$*
- *Opposite edges in a tetrahedron are assigned the same angle*
- *In each tetrahedron, the angles of the edges incident to an ideal vertex of the tetrahedron add up to π. Suppose these angles are α, β and γ. Then*

$$\alpha + \beta + \gamma = \pi$$

- *The angles of the edges which are identified to the same edge in the manifold add up to 2π. If these angles are α_i, then*

$$\sum_{i=1}^{n} \alpha_i = 2\pi$$

Thus, the existence of an angle structure for an ideal triangulation of a manifold is equivalent to the existence of a solution to the imaginary part of Thurston's edge gluing consistency equations. An angle structure is the solution to a system of finitely many linear equations and inequalities, so the existence of an angle structure is easy to verify computationally. The existence of an angle structure gives us a lot of information about the topology and geometry of the manifold, as we can see from the following theorem. This theorem is due to Casson, and a proof of this theorem can be found in Lackenby's paper [Lac00].

Theorem 5.1.1. *Let M be a compact orientable 3-manifold whose interior has an ideal triangulation admitting an angle structure. Then M has torus boundary components and M is irreducible, ∂-irreducible, atoroidal and anannular. Therefore, M admits a complete hyperbolic metric in its interior.*

In Casson and Rivin's approach, solving the non-linear part of Thurston's equations and finding the geometry of the ideal triangulation which realises the complete hyperbolic metric on the manifold is reduced to maximizing a certain volume functional over the space of all angle structures of the triangulation of M. This volume functional arises from the volume of the ideal tetrahedra in the triangulation. The following theorem was independently proved by both Casson and Rivin and its proof can be found in this excellent paper by Futer and Guéritaud [FG04].

Theorem 5.1.2. *Let M be a compact orientable 3-manifold with torus boundary components whose interior admits an ideal triangulation τ. Then if p is the critical point of the volume functional over the space of all angle structures on the triangulation, p corresponds to a complete hyperbolic metric on the interior of M and the interior of M has a geometric triangulation given by the angles of p.*

In this chapter, we will describe the program developed by Casson and Rivin in detail and prove Theorem 5.1.1 and Theorem 5.1.2. We will also calculate a formula for the volume of a hyperbolic ideal tetrahedron given its dihedral angles. Much of the exposition in this chapter is borrowed from that of the online book by Purcell [Pur20]. We have also followed the proofs originally given by Futer and Gueritaud in their paper [FG10].

5.2 Angle structures and hyperbolicity

In this section, we shall prove that the existence of an ideal triangulation on M which supports an angle structure implies that M is a complete hyperbolic manifold. Thurston's hyperbolization theorem (Corollary 1.3.1.1) will be a crucial tool in our proof of this theorem. Thus, we will show that the existence of an angle structure on a triangulation of M precludes the existence of essential spheres, disks, annuli, and tori. We will need the theory of normal surfaces in order to show this. We will define a combinatorial area for normal surfaces and relate it to the Euler characteristic of the surface to prove the non-existence of the aforementioned essential surfaces.

Definition 5.2.1 (Normal surfaces in an ideal polyhedral decomposition)**.** *Let M have a decomposition into ideal polyhedra. We describe when a properly embedded surface S is in normal form with respect to the polyhedral decomposition.*

Let P be obtained from an ideal polyhedron by truncating its ideal vertices to obtain 'boundary' faces. We call the new edges which bound these boundary faces boundary edges. Let D be a disk which is properly embedded in P, such that $\partial D \subset \partial P$ and consider the disk D in the truncation of P. Then D is said to be normal with respect to the polyhedron P if it satisfies the following properties:

- *The boundary of D is transverse to all the faces, boundary faces, edges and boundary edges of P.*
- *∂D does not lie entirely in any one face or boundary face of P.*
- *∂D meets each edge at most once.*
- *∂D meets each boundary face at most once.*
- *Let α be an arc of intersection of ∂D with a face of P. Then α cannot have both its endpoints on the same edge, on the same boundary edge, or on an adjacent edge and*

boundary edge of P.

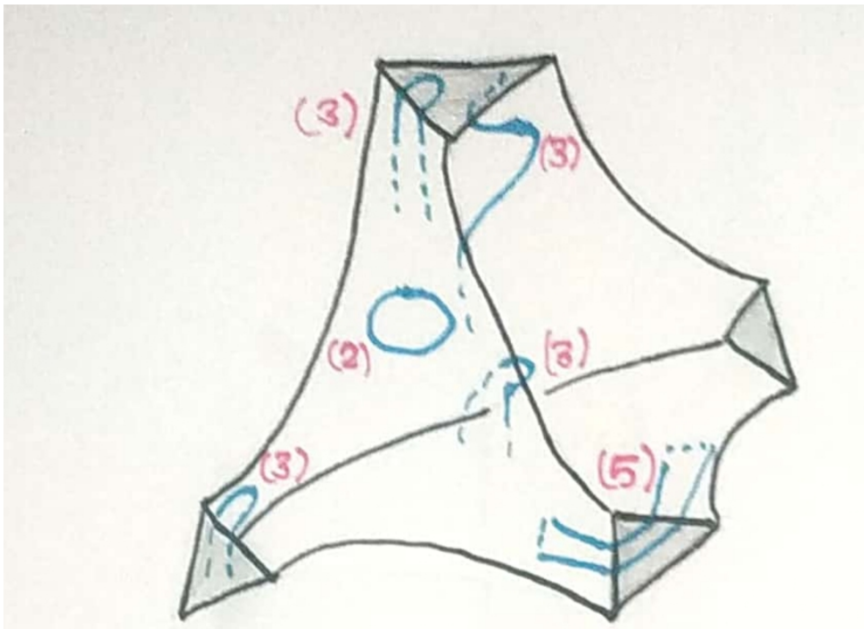


Figure 5.1: What is not allowed for a normal disk in a truncated tetrahedron

A properly embedded surface S in a manifold M is said to be normal with respect to an ideal polyhedral decomposition if it each intersects each polyhedron of the decomposition in normal disks.

Normal surfaces are easy to work with due to their simple combinatorial description. The following theorem due to Kneser will allow us to consider only normal surfaces in our proof of Theorem 5.1.1 using Thurston's hyperbolization theorem.

Theorem 5.2.1. *Let M be a 3-manifold admitting an ideal polyhedral decomposition. Then,*

- *If M has an essential sphere, then it must also contain an essential sphere in normal form.*
- *If M is irreducible and it has an essential disk, then it must also contain an essential disk in normal form.*
- *If M is irreducible and ∂-irreducible and it contains an essential surface S, then S can be isotoped in M to intersect the polyhedral decomposition in normal form.*

We need to characterise the essential surfaces in a triangulated manifold M which admits an angle structure using the combinatorial description of the surfaces in the normal form. So, we will define a quantity called combinatorial area which will relate to the Euler characteristic of the surface.

Definition 5.2.2 (Combinatorial area)**.** *Let M be a 3-manifold with an ideal triangulation τ. Suppose we have an angle structure on this triangulation. Suppose D is a normal disk with respect to a tetrahedron of τ such that the ideal edges which meet ∂D are assigned the angles α_1,..., α_n. Then the combinatorial area of D with respect to this angle structure is defined to be*

$$a(D) = \sum_{i=1}^{n}(\pi - \alpha_i) - 2\pi + \pi|\partial D \cap \partial M| \tag{5.1}$$

The combinatorial area of a normal surface S is defined to be the sum of the combinatorial areas of all the disks that make up S.

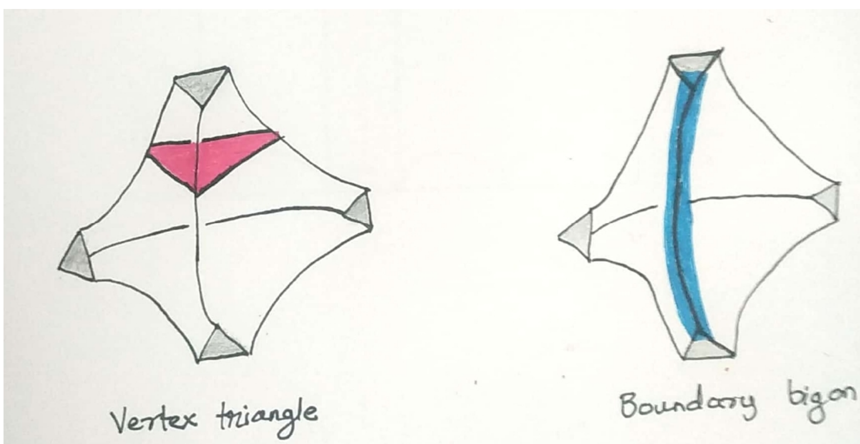


Figure 5.2: Vertex triangle and boundary bigon

We now find the combinatorial area of two important examples of normal disks. The first example, which is shown in Figure 5.2, is called a vertex triangle. Its combinatorial area is given by

$$a(D) = (\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) - 2\pi = \pi - (\alpha + \beta + \gamma) = 0$$

The combinatorial area of a vertex triangle is 0 as a vertex triangle is parallel to a truncated ideal vertex of the tetrahedron, and the dihedral angles around an ideal vertex sum to π. The second important example of a normal disk is called a boundary bigon, which is shown in Figure 5.2. The combinatorial area of a boundary bigon is given by

$$a(D) = 0 - 2\pi + \pi \cdot 2 = 0$$

Any boundary bigon thus has zero combinatorial area since a boundary bigon runs between two boundary faces of the tetrahedron and is parallel to an ideal edge of the tetrahedron.

Proposition 5.2.1. *Let M be a 3-manifold with an ideal triangulation τ which has an angle structure. Then the combinatorial area of every normal disk in M is non-negative. Moreover, the combinatorial area of a normal disk is zero if and only if the disk is a vertex triangle or a boundary bigon.*

Proof. We consider several possible cases for a normal disk D properly embedded in an ideal tetrahedron.

Case 1: Suppose ∂D meets at least two boundary faces of the ideal tetrahedron. Then, from the definition of combinatorial area, we get

$$a(D) \geq \sum_i (\pi - \alpha_i) \geq 0$$

Thus, $a(D)$ is non-negative in this case. Also, in this case, $a(D)$ is 0 if and only if ∂D does not meet any ideal edge and meets exactly two boundary faces of the ideal tetrahedron, so D must be a boundary bigon.

Case 2: Suppose ∂D meets exactly one boundary face of the ideal tetrahedron. Then, as ∂D cannot meet the ideal edges which are adjacent to the boundary face, it must meet two ideal edges on the opposite side of the boundary face. These edges cannot be opposite edges of the ideal tetrahedron. Suppose the dihedral angles associated to these edges are α and β, then the combinatorial area of D is given by

$$a(D) \geq (\pi - \alpha) + (\pi - \beta) - 2\pi + \pi = \pi - \alpha - \beta = \gamma > 0$$

Thus, $a(D)$ is strictly positive in this case.

Case 3: Suppose ∂D does not meet any boundary faces. Then, either D is a vertex triangle, or it is a quad separating two pairs of opposite edges. The combinatorial area of a vertex triangle is 0. The combinatorial area of a quad is strictly greater than 0 as

$$a(D) = 2(\pi - \alpha) + 2(\pi - \beta) - 2\pi = 2(\pi - \alpha - \beta) = 2\gamma > 0$$

Here, α and β are the dihedral angles assigned to the two pairs of opposite edges that the quad intersects, and γ is the remaining dihedral angle of the ideal tetrahedron.

Thus, we have shown that $a(D)$ is always non-negative and is 0 if and only if D is a boundary bigon or a vertex triangle. ■

We shall now state a Gauss-Bonnet type theorem involving combinatorial area which when combined with the previous proposition can be used to prove that any essential surface in M must have negative Euler characteristic.

Proposition 5.2.2 (Combinatorial Gauss-Bonnet). *If S is a normal surface in an ideal triangulation of M which has a given angle structure, then the combinatorial area of S satisfies*

$$a(S) = -2\pi\chi(S) \tag{5.2}$$

where χ is the Euler characteristic of S.

Proof. We know that the Euler characteristic of S is given by $\chi(S) = v - e + f$, where v, e and f are the number of vertices, edges and faces in a polygonal decomposition of S. The intersection of the ideal triangulation of M with S determines a polygonal decomposition of S. Here, f will be equal to the number of normal disks in the triangulation. The number of edges e in the polygonal decomposition corresponds to the number of intersections of S with the ideal faces of the triangulation. The number of vertices v of the polygonal decomposition corresponds to the number of intersections of the surface S with ideal edges of the triangulation. The intersections of S with boundary edges and boundary faces do not count as they cancel each other out. This is because each time S intersects a boundary face, it has to intersect a boundary edge bordering this face, and each time S intersects a boundary edge, it must intersect a boundary face which this edge borders. So the intersections of S with boundary faces of M are in one to one correspondence with the the intersections of S with boundary edges of M, and they cancel each other out.

The combinatorial area of S is then given by the sum of the combinatorial area of normal disks D which make up the surface S.

$$\begin{aligned} a(S) &= \sum_D a(D) \\ &= \sum_D (\sum_i (\pi - \alpha_i) + \pi|\partial D \cap \partial M| - 2\pi) \\ &= \pi \sum_D (\sum_i 1 + |\partial D \cap \partial M|) - \sum_D \sum_i \alpha_i - \sum_D 2\pi \end{aligned}$$

The last term in the above sum is equal to $2\pi f$, as the number of normal disks is equal to the number of faces in the polygonal decomposition of S. The middle term $\sum_D \sum_i \alpha_i$ is equal to the sum of the dihedral angles of all the ideal edges met by S. Therefore, this term is equal to $2\pi v$.

We claim that the sum $\sum_i 1 + |\partial D \cap \partial M|$ is equal to the number of edges e of D which lie in the ideal faces of the ideal triangulation, which we denote as $e(D)$. To see this, we must orient the boundary ∂D. Then, we see that each edge lying in an ideal face is in one-one correspondence with its initial vertex. If the initial vertex lies on an ideal edge, it is counted in the sum $\sum_i 1$, and if the initial vertex lies on a boundary edge it is counted in the term $|\partial D \cap \partial M|$. This does not count edges which lie on boundary faces as the term $|\partial D \cap \partial M|$ only accounts for one edge for each boundary face, which is the edge that exits the face and has its initial vertex on a boundary edge of that face.

Thus, the term $\sum_D (\sum_i 1 + |\partial D \cap \partial M|)$ in the summation above is equal to $\sum_D e(D)$, which counts each edge exactly twice, so its value is exactly $2e$. Substituting these values in the calculation of $a(S)$, we get

$$a(S) = 2\pi e - 2\pi v - 2\pi f = -2\pi\chi(S)$$

■

We now have all the tools to prove Theorem 5.1.1.

Proof. Suppose M has an essential sphere. Then, by Theorem 5.2.1, M must contain an essential sphere S which is in normal form with respect to the ideal triangulation of M. From Proposition 5.2.1, we see that the combinatorial area of S must be non-negative. However, Proposition 5.2.2 tells us that $a(S) = -4\pi$. This is a contradiction. So, M must be irreducible. Similarly, we can prove that M does not contain an essential disk; so M is ∂-irreducible.

The boundary of M is made up of the boundary faces of the triangulation. The boundary can be pushed in slightly to obtain a boundary parallel surface which is homeomorphic to the boundary. This surface is thus made of vertex triangles, which have zero combinatorial area. Thus, ∂M is a closed surface in the orientable manifold M which has zero combinatorial area. So, ∂M must be a disjoint union of tori.

Suppose M contains an essential torus T. Then by Theorem 5.2.1, the torus T can be isotoped such that it is in normal form with respect to the triangulation of M. Proposition 5.2.2 then tells us that $a(T) = 0$. Thus, from Proposition 5.2.1, we see that T must be composed of vertex triangles and boundary bigons. As T is a closed surface properly embedded in M, it does not meet the boundary of M, and hence all the normal disks of T must be vertex triangles. However, as vertex triangles are parallel to boundary faces of the triangulation, we see that the torus T is parallel to a component of ∂M. Thus, T is boundary parallel, which contradicts the assumption that T is essential. Thus, M does not contain any essential tori.

Now, suppose C is an essential annulus in M, then it can be isotoped to normal form by Theorem 5.2.1. Then, from Proposition 5.2.2, we see that $a(C) = 0$. So, from Proposition 5.2.1, we see that C must be must be composed of vertex triangles and boundary bigons. As C is a properly embedded surface which has non-empty boundary, it must contain at least one boundary bigon. This bigon must be glued to another boundary bigon in an adjacent tetrahedron. This is becuase the common edge of these normal disks shared by these tetrahedra runs between two boundary faces, and the normal disk in the adjacent tetrahedron is thus restricted to be a boundary bigon due to the restrictions imposed on normal disks by Definition 5.2.1. So, all the normal disks which make up C are boundary bigons, and they are forced to encircle an edge of the ideal triangulation due to the way boundary bigons glue to each other. Then, C is compressible which contradicts the assumption that C is essential. Thus, M does not contain any essential annuli.

Hence, by Thurston's hyperbolization theorem (Corollary 1.3.1.1) we see that M must admit a complete hyperbolic structure. ■

5.3 Volume of an ideal tetrahedron

In this section, we will obtain a formula for the volume of a hyperbolic ideal tetrahedron given its dihedral angles. This formula involves the Lobachevsky function Λ, which is related to the dilogarithm function.

Definition 5.3.1. *Define the Lobachevsky function $\Lambda \colon \mathbb{R} \to \mathbb{R}$ by*

$$\Lambda(\theta) = -\int_0^\theta \log|2\sin u| du \tag{5.3}$$

Theorem 5.3.1. *The volume of a hyperbolic ideal tetrahedron T with dihedral angles α, β and γ is given by*

$$\mathrm{vol}(T) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$$

We first need to understand the properties of the Lobachevsky function by relating it to the dilogarithm function.

Proposition 5.3.1. *The Lobachevsky function Λ satisfies the following properties:*

- *It is well defined and continuous on $\mathbb{R}$*
- *It satisfies a series expansion given by*

$$\Lambda(\theta) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n^2}$$

- Λ *is an odd function, that is,* $\Lambda(-\theta) = -\Lambda(\theta)$

- *It is a periodic function with period* π, *that is,* $\Lambda(\theta + \pi) = \Lambda(\theta)$

- *It satisfies the identity* $\Lambda(2\theta) = 2\Lambda(\theta) + 2\Lambda(\theta + \pi/2)$

- $\Lambda(0) = \Lambda(\pi/2) = \Lambda(\pi) = 0$

To calculate the volume of an ideal tetrahedron in hyperbolic space, we will first divide it into simpler pieces called orthosimplices. Let the ideal tetrahedron have one vertex at ∞ in the upper half space model of $\mathbb{H}^3$. The other three vertices then lie on the circular boundary of a hemisphere on the boundary complex plane of $\mathbb{H}^3$, where we are viewing $\mathbb{H}^3$ as $\mathbb{H}^3 = \{(x, y, t) \mid (x, y) \in \mathbb{C}, t \in \mathbb{R}, t > 0\}$. We can use isometries of $\mathbb{H}^3$ to scale and shift this hemisphere such that it has its centre at 0 and has Euclidean radius 1, so that the circular boundary of the hemisphere now corresponds to the unit circle in $\mathbb{C}$. Let the three vertices now lie at p, q and r on the unit circle.

Let us first drop a perpendicular from ∞ to the hemisphere; this will be a vertical ray from $(0, 0, 1) \in \mathbb{H}^3$ to ∞. There are two cases to consider, depending on whether the point $(0, 0, 1)$ lies in the interior or exterior of the ideal triangle which makes up the base of our ideal tetrahedron.

Let us first consider the case where $(0, 0, 1)$ lies in the interior of the ideal triangle which forms the base of the ideal tetrahedron. We will draw perpendicular arcs from $(0, 0, 1)$ to the three edges of the ideal triangle. We will also draw arc joining the point $(0, 0, 1)$ to the points p, q and r. Finally, we will cone all these arcs to ∞ - this will divide the ideal tetrahedron into six simpler pieces called orthoschemes.

Lemma 5.3.1. *All the six tetrahedra obtained by the procedure mentioned above have the following properties:*

- *They have two finite vertices and two ideal vertices*

- *Three of their dihedral angles are* $\pi/2$ *and the other three are* ζ, ζ *and* $\pi/2 - \zeta$, *for some* $\zeta \in (0, \pi/2)$

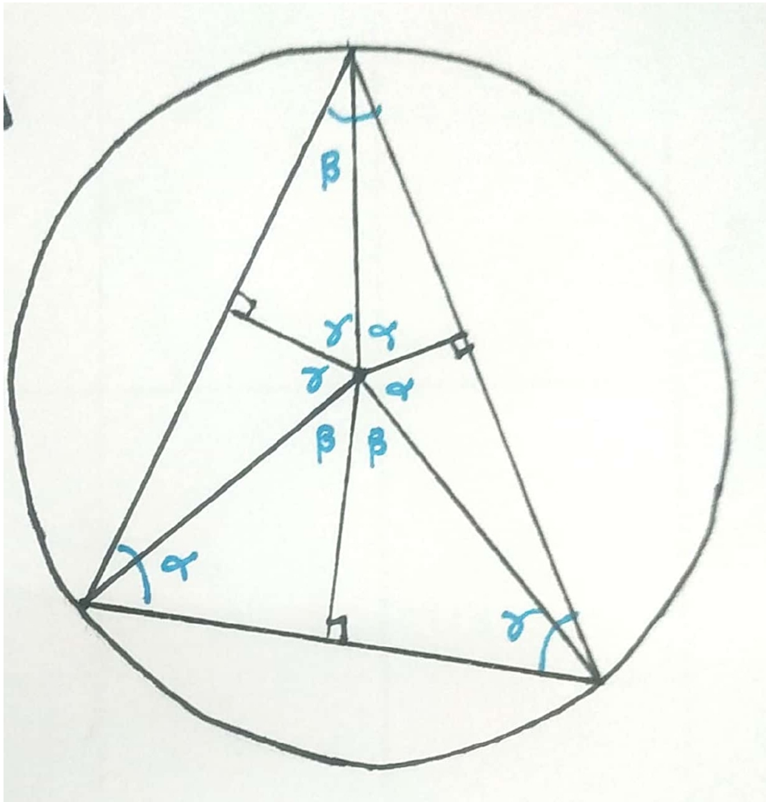


Figure 5.3: Dividing an ideal tetrahedron into orthoschemes

An orthoscheme is depicted in Figure 5.4. Let $S(\zeta)$ denote an orthoscheme with angle ζ which is of the form described by the previous lemma. The volume of an orthoscheme can be easily calculated by integration.

Proposition 5.3.2 (Volume of an orthoscheme)**.** *The volume of an orthoscheme with angle ζ is given by*

$$\mathrm{vol}(S(\zeta)) = \frac{1}{2}\Lambda(\zeta) \tag{5.4}$$

Using this formula and the division of an ideal tetrahedron into orthoschemes, depicted in Figure 5.3, we can easily prove the formula for the volume of an ideal tetrahedron.

Corollary 5.3.1.1. *Among all the hyperbolic ideal tetrahedra, the regular ideal tetrahedron with all its dihedral angles $\pi/3$ has the maximum volume.*

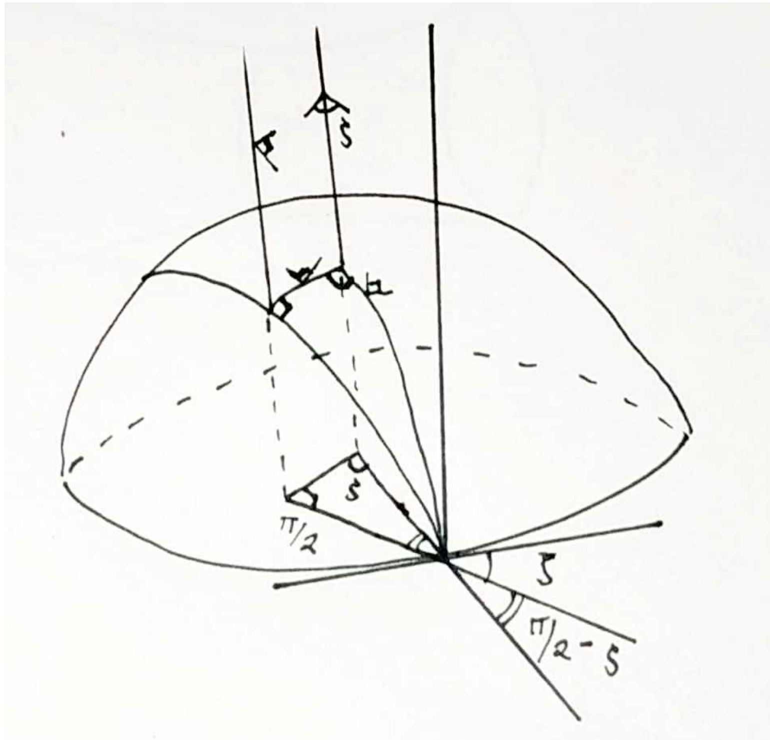


Figure 5.4: An orthoscheme

The volume of the regular ideal tetrahedron is $3\Lambda(\pi/3)$ and we denote this quantity as v_{tet}.

5.4 The volume functional and the space of angle structures

Given an angle structure on an ideal triangulation τ of a manifold M, we can calculate the edge parameters corresponding to the dihedral angles, using Proposition 4.2.2. The dihedral angles of an angle structure satisfy constraints which are equivalent to the corresponding edge parameters satisfying the imaginary part of Thurston's edge gluing equations. In this section, we will define a volume functional on the space of all angle structures of a triangulation. We will show that an interior maxima of this functional on this space corresponds to a set of edge parameters which solves both the edge gluing consistency and the gluing completeness equations. Thus, we will obtain a geometric ideal triangulation of M corresponding to the triangulation τ and explicitly compute the complete hyperbolic structure on M.

We will first understand the properties of the parameter space of angle structures on the triangulation τ, which we denote by $\mathcal{A}(\tau)$.

Lemma 5.4.1. *If $\mathcal{A}(\tau) \neq \emptyset$, then $\mathcal{A}(\tau)$ is a convex, bounded, finite-sided open polytope in $(0,\pi)^{3n} \subset \mathbb{R}^{3n}$, where n is the number of tetrahedra in the ideal triangulation τ.*

Proof. Any angle structure in $\mathcal{A}(\tau)$ assigns to each tetrahedron three dihedral angles between 0 and π which correspond to the three pairs of opposite edges in the tetrahedron. So, we need a total of $3n$ co-ordinates to describe an angle structure. Hence, it is clear that $\mathcal{A}(\tau) \subset [0,\pi]^{3n} \subset \mathbb{R}^{3n}$. The equations which define an angle structure are linear equations whose solution set is an affine subspace of $\mathbb{R}^{3n}$. The intersection of this affine subspace with $[0,\pi]^{3n}$ is a bounded, convex, open, finite sided polytope in $\mathbb{R}^{3n}$. ■

Definition 5.4.1 (Volume functional on $\mathcal{A}(\tau)$)**.** *The volume functional $\nu\colon \mathcal{A}(\tau) \to \mathbb{R}$ is defined as*

$$\nu(a_1, ..., a_{3n}) = \sum_{i=1}^{3n} \Lambda(a_i) \tag{5.5}$$

Given the dihedral angles of an angle structure, the volume functional gives as output the sum of the hyperbolic volumes of all the tetrahedra in the triangulation with respect to that angle structure.

We will now state a few important properties of the volume functional.

Proposition 5.4.1. *Let τ be an ideal triangulation of M consisting of n ideal tetrahedra. The volume functional $\nu\colon \mathcal{A}(\tau) \to \mathbb{R}$ satisfies the following properties:*

- *$\nu\colon \mathcal{A}(\tau) \to \mathbb{R}$ is strictly concave on $\mathcal{A}(\tau)$.*

- *Consider $a = (a_1, ..., a_{3n}) \in \mathcal{A}(\tau)$ and let $w = (w_1, .., w_{3n})$ belong to the tangent space at $a \in \mathcal{A}(\tau)$, that is, $w \in T_a(\mathcal{A}(\tau))$. Then the directional derivatives of ν in* *the direction w satisfy the relations*

$$\frac{\partial \nu}{\partial w} = \sum_{i=1}^{3n} -w_i \log(\sin a_i) \tag{5.6}$$

$$\frac{\partial^2 \nu}{\partial w^2} < 0 \tag{5.7}$$

Proof. As w is a tangent vector to $\mathcal{A}(\tau)$ at the point a, w must satisfy $w_{i(T)} + w_{j(T)} + w_{k(T)} = 0$, where $i(T)$, $j(T)$ and $k(T)$ are the co-ordinates of w corresponding to distinct dihedral angles in a tetrahedron T of τ. This is because the angle structure a satisfies $a_{i(T)} + a_{j(T)} + a_{k(T)} = \pi$. Using the properties of the Lobachevsky function (Proposition 5.3.1), we can calculate the directional derivatives of the volume functional at a in the direction of w.

$$\begin{aligned}
\frac{\partial \nu}{\partial w} &= \sum_{i=1}^{3n} -w_i \log|2\sin a_i| \\
&= -\log 2 \sum_{T\in\tau} (w_{i(T)} + w_{j(T)} + w_{k(T)}) + \sum_{i=1}^{3n} -w_i \log|\sin a_i| \\
&= \sum_{i=1}^{3n} -w_i \log \sin a_i
\end{aligned}$$

The last step follows as $w_{i(T)} + w_{j(T)} + w_{k(T)} = 0$ for each tetrahedron $T \in \tau$. Also, as $a_i \in (0, \pi)$, we know that $\sin a_i > 0$.

We will now calculate the second derivative of the volume functional in the direction of w. We know that $a_{i(T)} + a_{j(T)} + a_{k(T)} = \pi$, so at least two of $a_{i(T)}$, $a_{j(T)}$ and $a_{k(T)}$ must be strictly less than $\pi/2$. Let us assume without loss of generality that for each tetrahedron T, $a_{i(T)}$ and $a_{j(T)}$ are strictly less than $\pi/2$. Then, we see that the second derivative is given by

$$\frac{\partial^2 \nu}{\partial w^2} = \sum_{i=1}^{3n} -w_i^2 \cot a_i$$

Now, as $a_{k(T)} = \pi - a_{i(T)} - a_{j(T)}$ and $w_{k(T)} = w_{i(T)} - w_{j(T)}$, from some trigonometric manipulations, we get

$$\begin{aligned} w_{k(T)}^2 \cot a_{k(T)} &= (w_{i(T)} + w_{j(T)})^2 \cot\left(\pi - a_{i(T)} - a_{j(T)}\right) \\ &= -(w_{i(T)} + w_{j(T)})^2 \frac{\cot a_{i(T)} \cot a_{j(T)} - 1}{\cot a_{i(T)} + \cot a_{j(T)}} \end{aligned}$$

Thus, we can rewrite the second derivative of the volume functional ν as

$$\begin{aligned} -\frac{\partial^2 \nu}{\partial w^2} &= \sum_{T \in \tau} \left(w_{i(T)}^2 \cot a_{i(T)} + w_{j(T)}^2 \cot a_{j(T)} - (w_{i(T)} + w_{j(T)})^2 \frac{\cot a_{i(T)} \cot a_{j(T)} - 1}{\cot a_{i(T)} + \cot a_{j(T)}} \right) \\ &= \sum_{T \in \tau} \frac{(w_{i(T)} + w_{j(T)})^2 + (w_{i(T)} \cot a_{i(T)} - w_{j(T)} \cot a_{j(T)})^2}{\cot a_{i(T)} + \cot a_{j(T)}} \end{aligned}$$

The denominator of the last term is strictly positive as both $a_{i(T)}$ and $a_{j(T)}$ belong to $(0, \pi/2)$ for all $T \in \tau$. The numerator of the last term is a sum of two squares, hence it must be non-negative. If the numerator is zero, then we must have $w_{i(T)} = -w_{j(T)}$ and $\cot a_{i(T)} = -\cot a_{j(T)}$. But as $a_{i(T)}, a_{j(T)} \in (0, \pi/2)$, this is not possible. Thus, both the numerator and denominator of each term in the last summation are strictly positive, so we see that $\partial^2 \nu / \partial w^2$ is strictly negative. Hence, the volume functional ν is strictly concave on the space of angle structures $\mathcal{A}(\tau)$. ■

5.5 Leading trailing deformations

In order to prove Theorem 5.1.2, we shall define special vectors called leading-trailing deformations given an angle structure $a \in \mathcal{A}(\tau)$ and show that these vectors lie in the tangent space $T_a(\mathcal{A}(\tau))$. We shall take the directional derivative of the volume functional ν along these vectors and relate them with quantities which arise from Thurston's gluing equations. This proof of Theorem 5.1.2 follows from the ideas of Futer and Gueritaud [FG10].

In what follows, M will be homeomorphic to the interior of a compact orientable 3-manifold with torus boundary components and τ will be an ideal triangulation of M consisting of n ideal tetrahedra.

Definition 5.5.1 (Leading-trailing deformations)**.** *Let C be a cusp of M and consider the triangulation of the cusp torus induced by the ideal triangulation τ of M. Let α be an oriented curve on the cusp torus. We can isotope α such that it runs monotonically through each triangle in the cusp triangulation, such that α enters the triangle from one side, leaves from another side, and cuts off exactly one vertex of the triangle. Let $\alpha_1,...,\alpha_k$ be the oriented segments of α in the triangles t_i of the cusp triangulation, and suppose the curve α intersects the cusp triangulation in distinct triangle t_i. In the triangle t_i, we will call the vertex opposite the side from which α enters the leading vertex, and the vertex opposite the side from which α exits the trailing vertex.*

For an angle structure $a \in \mathcal{A}(\tau)$, let every vertex of a triangle of the cusp triangulation inherit the same dihedral angle as the ideal edge corresponding to that vertex. Thus each vertex of each triangle of the cusp triangulation corresponds to a co-ordinate of $\mathcal{A}(\tau) \subset \mathbb{R}^{3n}$. Similarly, given $a \in \mathcal{A}(\tau)$, each vertex of each triangle of the cusp triangulation corresponds to a co-ordinate of the tangent space $T_a\left(\mathcal{A}(\tau)\right) \subset \mathbb{R}^{3n}$.

We shall define a vector $w(\alpha_i) \in \mathbb{R}^{3n}$ corresponding to the segment α_i of α. We set the co-ordinate of $w(\alpha_i)$ corresponding to the leading vertex of t_i to be 1, and the co-ordinate corresponding to the trailing vertex of t_i to be -1. We will set all other co-ordinates of $w(\alpha_i)$ to be 0. We will define the leading trailing deformation $w(\alpha)$ corresponding to α by

$$w(\alpha) = \sum_i w(\alpha_i)$$

We shall now prove that the leading trailing deformations of certain special curves which arise from Thurston's equations belong to the tangent space of $\mathcal{A}(\tau)$.

Lemma 5.5.1. *Let θ be a curve on the cusp torus which encircles a vertex of the cusp triangulation. Let ζ be a curve on the cusp torus which corresponds to a generator of its fundamental group. Then the leading trailing deformations $w(\theta)$ and $w(\zeta)$ both belong to the tangent space $T_a\left(\mathcal{A}(\tau)\right)$, for any $a \in \mathcal{A}(\tau)$*

Proof. Let $f_T(a) = a_{i(T)} + a_{j(T)} + a_{k(T)}$ denote the sum of dihedral angles of the tetrahedron T for the angle structure a. Let $g_e(a) = \sum_i a_{e_i}$ denote the sum of dihedral angles assigned by the angle structure a to the edges e_i which are identified with the edge e of τ. We see from the definition of an angle structure that $\mathcal{A}(\tau)$ is a submanifold of $\mathbb{R}^{3n}$ which is cut out by the linear equations $f_T(a) = \pi$ and

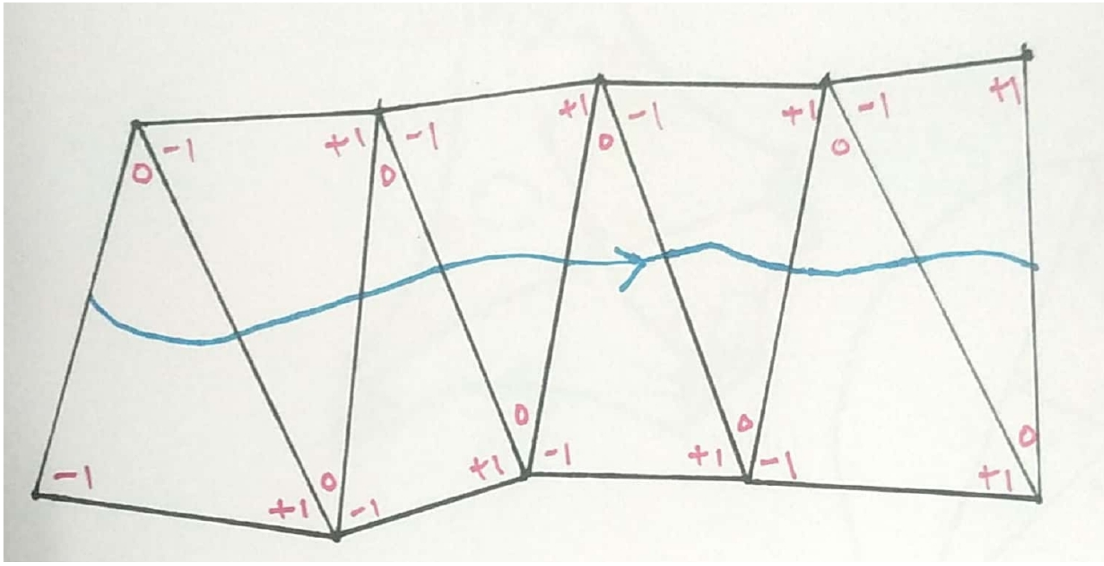
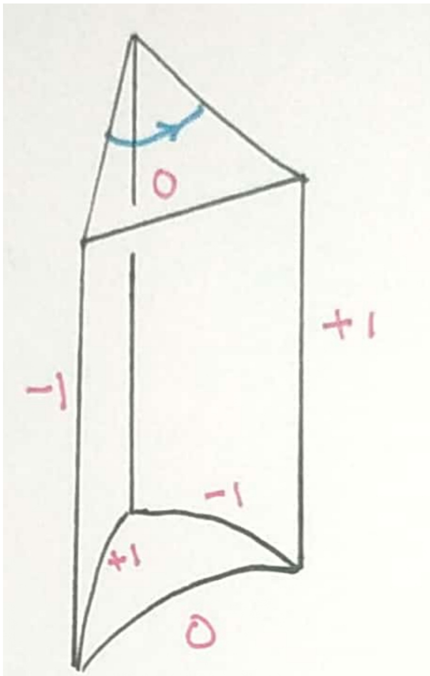


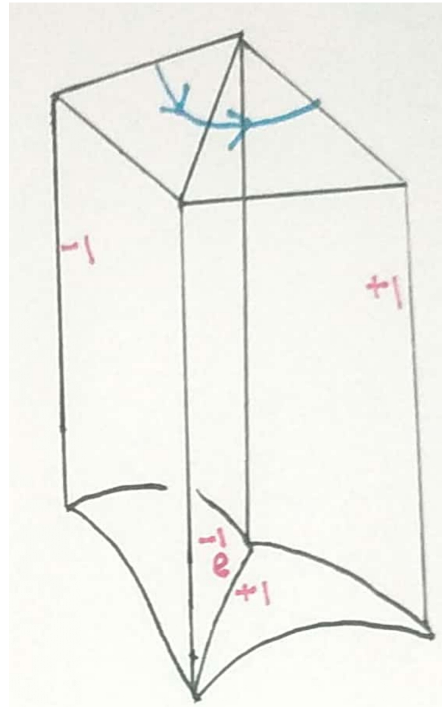
Figure 5.5: Calculating the leading trailing deformation vector of the blue curve

$g_e(a) = 2\pi$ for all tetrahedra T and edges e of the triangulation τ. Thus, to prove that a vector is tangent to $\mathcal{A}(\tau)$ at the point a, we need to show that it is perpendicular to the gradient vectors $\nabla f_T(a)$ and $\nabla g_e(a)$ for all tetrahedra T and edges e in the triangulation τ. By using an argument involving the Euler characteristic of M, we see that the number of tetrahedra in τ is equal to the number of edges in τ. So, we need to check $2n$ conditions to show that a vector is tangent to $\mathcal{A}(\tau)$ at a point a. We shall use this approach to prove that the leading trailing deformations of the special curves mentioned in the theorem are tangent to the space of angle structures.

Let α be one of the curves mentioned in the theorem. We can orient α and homotope it such that it satisfies the conditions imposed in Definition 4.4.1, and ensure that α intersects each cusp triangle at most once. We see that ∇f_T is the vector $(0, 0, ..., 1, 1, 1, 0, 0, ..., 0)$ where the only non-zero coordinates are $i(T)$, $j(T)$ and $k(T)$ corresponding to the tetrahedron T. The tetrahedron T contributes four triangles to the cusp triangulations, and $\nabla f_T \cdot w(\alpha)$ depends on how the curve α runs through the cusp triangulation. Suppose no segment of α meets any of the four cusp triangles, then $w(\alpha)$ has zeroes in the coordinates corresponding to the non-zero entries of ∇f_T, so $\nabla f_T \cdot w(\alpha) = 0$ in this case. Now suppose that some segment α_t of α passes through some cusp triangle t of the tetrahedron T. Then



(a) Assigning the co-ordinates of $w(\alpha_t)$



(b) When the non-vertical edge of T which has $+1$ in $w(\alpha_t)$ is identified to the edge e

Figure 5.6

in the vector $w(\alpha_t)$, the leading corner of t will be assigned 1, the trailing corner will be assigned -1, and the other corner of t will be assigned 0. Thus, in $w(\alpha_t)$, the three co-ordinates corresponding to the three 1's in ∇f_T are assigned the co-ordinates $+1$, -1 and 0, so the dot product $\nabla f_T \cdot w(\alpha_t) = 0$. Since $w(\alpha) = \sum_t w(\alpha_t)$, we see that $\nabla f_T \cdot w(\alpha) = 0$. Hence, the leading-trailing deformation for the curve α is orthogonal to the gradient vector ∇f_T for each tetrahedron T in the triangulation τ.

We shall now show that $\nabla g_e \cdot w(\alpha) = 0$ for each edge e of the triangulation τ. We shall consider the contributions to $\nabla g_e \cdot w(\alpha)$ coming from each segment α_t of the curve α and show that these contributions add up to 0.

Let the segment α_t pass through the cusp triangle t in the tetrahedron T. Thus, $w(\alpha_t)$ will assign $+1$ to the leading corner, -1 to the trailing corner and 0 to the remaining corner of t, and the coordinates of $w(\alpha_t)$ corresponding to these corners will be assigned the respective values, as shown in Figure 5.6a. If none of the edges of the tetrahedron T are identified to the edge e, then $\nabla g_e \cdot w(\alpha_t) = 0$. Now, suppose the edges of T which are assigned 0 by $w(\alpha_t)$ are identified with the edge e, then ∇g_e will have $+1$ or $+2$ in that co-ordinate, depending on whether one or two edges in the pair are identified to e. However, $\nabla g_e \cdot w(\alpha_t)$ will still contribute 0 to $\nabla g_e \cdot w(\alpha)$, as $w(\alpha_t)$ has 0 in the corresponding co-ordinates.

Now, consider the case where either one or both the edges of T which are assigned $+1$ by $w(\alpha_t)$ are identified to e. Then, $w(\alpha_t)$ will contribute $+1$ or $+2$ to $\nabla g_e \cdot w(\alpha)$. We will show that these positive contributions are cancelled by corresponding equal negative contributions coming from other segments.

First, consider the case when the non-vertical edge of T which is assigned $+1$ by $w(\alpha_t)$ is identified to the edge e. This will contribute $+1$ to the dot product $\nabla g_e \cdot w(\alpha)$. Consider the previous segment of α which lies in a cusp triangle t_{-1} in the tetrahedron T_{-1}. We denote this segment as $\alpha_{t_{-1}}$. As the segment exits the cusp triangle t_{-1} and enters the cusp triangle t, it will assign -1 to the trailing corner of the cusp triangle t_{-1}. We see from the Figure 5.6b that this trailing corner corresponds to a vertical edge of the tetrahedron T_{-1}, which is opposite to the edge e. Hence, $w(\alpha_{t_{-1}})$ contributes -1 to $\nabla g_e \cdot w(\alpha)$, cancelling the $+1$ contribution from $\nabla g_e \cdot w(\alpha_t)$.

Now, let us consider the case where the vertical edge of T which is assigned $+1$ by $w(\alpha_t)$ is identified with the edge e. Again, this will contribute $+1$ to the dot product $\nabla g_e \cdot w(\alpha)$. Let α_t, α_{t+1},..., α_{t+r} be the maximal collection of consecutive segments of α which follow after α_t and belong to cusp triangles adjacent to the edge e. If the curve α encircles a vertex, then we see that $r = 1$, that is, there can be only two consecutive segments in cusp triangles adjacent to the edge e. Suppose α is a generator for the homology of the cusp torus. Then, $r \geq 1$ and the segments α_t,..., α_{t+r} cannot encircle the edge e.

From Figure 5.7, we see that the contribution from the segments α_{t+1} up to α_{t+r-1} to the dot product $\nabla g_e \cdot w(\alpha)$ is 0, since $w(\alpha_{t+k})$ has 0 in the coordinate corresponding to the edge e for $1 \leq k \leq r-1$. As the segment α_{t+r} exits the cusp triangle, we

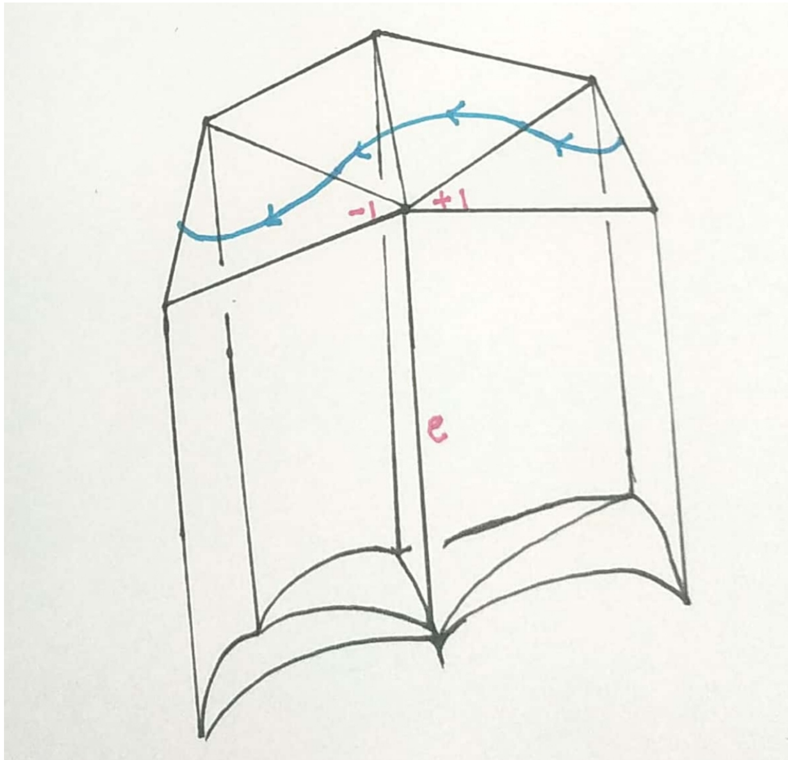


Figure 5.7: When the vertical edge of T which has $+1$ in $w(\alpha_t)$ is identified to the edge e

see that it will assign -1 to the trailing corner of the cusp triangle which actually corresponds to the edge e of the triangulation. Thus, $\nabla g_e \cdot w(\alpha_{t+r})$ contributes -1 to $\nabla g_e \cdot w(\alpha)$ and cancels the $+1$ contribution coming from $\nabla g_e \cdot w(\alpha_t)$.

Thus, we see that each $+1$ contribution by a segment of α to $\nabla g_e \cdot w(\alpha)$ is cancelled by a -1 contribution from another segment. As the segment which offers the cancelling contribution is distinct for different $+1$ contributions, we deduce that $\nabla g_e \cdot w(\alpha) \leq 0$. Similarly, we can also show that $\nabla g_e \cdot w(\alpha) \geq 0$. Therefore, $\nabla g_e \cdot w(\alpha) = 0$ for each edge e of the triangulation. ■

We shall now relate the directional derivative of the volume functional ν along leading trailing deformation vectors to the real part of Thurston's gluing equations.

Lemma 5.5.2. *Let $a \in \mathcal{A}(\tau)$ be any angle structure on the ideal triangulation τ of M. Let σ be one of the curves θ or ζ from Lemma 5.5.1 and let $w(\sigma) \in T_a(\mathcal{A}(\tau))$ be the leading trailing deformation. Let $H(\sigma)$ be the complex number associated with α according to*

Definition 4.4.1. Then,

$$\frac{\partial \nu}{\partial w(\sigma)} = \mathrm{Re}(\log H(\sigma))$$

Proof. Let σ_1,..., σ_k be the segments of the curve σ in the cusp triangle t_1,...t_k. We label the angles of the triangle t_i as α_i, β_i and γ_i in the clockwise order such that the angle α_i is cut off by σ_i. From the Definition 4.4.1, we know

$$\mathrm{Re}(\log H(\sigma)) = \sum_i \epsilon_i(\log |z(\sigma_i)|)$$

where $z(\sigma_i)$ is the edge parameter associated with the edge with the dihedral angle σ_i, and $\epsilon_i = +1$ if the vertex with angle α_i is to the left of the curve σ_i and $\epsilon_i = -1$ if the vertex with angle α_i is to the right of the curve σ_i. From the assignments of $+1$ and -1 in the definition of $w(\sigma_i)$, we see that when the vertex with angle α_i is to the left of σ_i, the vector $w(\sigma_i)$ has $+1$ in the coordinate corresponding to β_i and -1 in the coordinate corresponding to γ_i. When the vertex with angle α_i is to the right of σ_i, the vector $w(\sigma_i)$ has -1 in the coordinate corresponding to β_i and $+1$ in the coordinate corresponding to γ_i. Then from Proposition 5.4.1, we see that

$$\begin{aligned}
\frac{\partial \nu}{\partial w} &= \sum_{k=1}^{3n} -w_k \log \sin a_k \\
&= \sum_i -\epsilon_i \log \sin \beta_i + \epsilon_i \log \sin \gamma_i \\
&= \sum_i \epsilon_i \log \left(\frac{\sin \gamma_i}{\sin \beta_i} \right) \\
&= \sum_i \epsilon_i (\log |z(\alpha_i)|)
\end{aligned}$$

The last step follows from Proposition 4.2.2. Thus, we have proved the required relation. ∎

We are now in a position to prove Theorem 5.1.2. We shall show that any critical point of the volume functional corresponds to a solution of the gluing equations. As the volume functional ν is strictly concave on $\mathcal{A}(\tau)$, a critical point of the volume functional must be the global maximum of the function on $\mathcal{A}(\tau)$.

Proof. Given any $a \in \mathcal{A}(\tau)$, we can obtain the edge parameters corresponding to the dihedral angles of a using Proposition 4.2.2. Now suppose $a \in \mathcal{A}(\tau)$ is a critical point of the volume functional ν. Then, we shall show that the edge parameters corresponding to a satisfy Thurston's edge gluing consistency and gluing completeness equations. These edge parameters will then describe a complete hyperbolic structure on M.

We shall consider the real part and imaginary part of these equations separately. The definition of an angle structure ensures that the imaginary part of the edge gluing equations are satisfied. Now, as a is a critical point for ν on $\mathcal{A}(\tau)$, the directional derivative of ν should be zero in any direction. In particular, let θ be a curve on the cusp torus which encircles a vertex corresponding to an edge e of the triangulation. Then, as $\partial\nu/\partial w(\theta) = 0$, using Lemma 5.5.2, we see that $\mathrm{Re}(\log H(\theta)) = 0$. So, the real part of the edge gluing equation is also satisfied for each edge e of the triangulation.

Consider a cusp of the manifold M and let ζ_1 and ζ_2 be the homology generators of the cusp torus. Then, we see that $\partial\nu/\partial w(\zeta_1) = 0$ and $\partial\nu/\partial w(\zeta_2) = 0$. Therefore, we get

$$\mathrm{Re}(\log H(\zeta_1)) = \mathrm{Re}(\log H(\zeta_2)) = 0$$

Therefore, the real part of each of the gluing completeness equations is also satisfied.

As the angle structure a satisfies the edge gluing equations, M must have a hyperbolic structure which is possibly incomplete. So, the cusp tori will all inherit at least a similarity structure. Consider the developing image of a fundamental domain for a cusp torus of M; it will be a quadrilateral as the cusp torus has an affine structure. Now, as the real parts of the gluing completeness equations are satisfied, we see that the holonomy elements $\rho(\zeta_1)$ and $\rho(\zeta_2)$ do not scale. Thus, the developing image of a fundamental domain for the cusp torus is a quadrilateral which has opposite sides of equal length, which must hence be a parallelogram. Hence, we see that the holonomy elements $\rho(\zeta_1)$ and $\rho(\zeta_2)$ have trivial rotation, that is, the imaginary part of each of the gluing completeness equations is also satisfied. ■

5.6 Converse of the theorem

The converse of Theorem 5.1.2 is also true and can be proved easily using Schlafli's formula for hyperbolic ideal tetrahedra. This proof of the converse was given by Ken Chan in his honours thesis with Craig Hodgson [CH02].

Theorem 5.6.1. *Let M be the interior of a compact orientable 3-manifold with torus boundary components. Suppose M is a finite volume complete hyperbolic 3-manifold. Let τ be a geometric ideal triangulation of M. Let $a \in \mathcal{A}(\tau)$ be the angle structure on the ideal triangulation given by the dihedral angles of the geometric triangulation. Then a is a critical point of the volume functional $\nu\colon \mathcal{A}(\tau) \to \mathbb{R}$ and hence it is the global maximum of the volume functional on $\mathcal{A}(\tau)$*

Schlafli's formula for ideal tetrahedra gives us the variation in the volume of an ideal tetrahedron given the distance between horospheres centred at its ideal vertices and the variation in its dihedral angles. In this form, this formula was proved by Milnor [Mil94].

Theorem 5.6.2 (Schlafli's formula for ideal tetrahedra)**.** *Let T be a hyperbolic ideal tetrahedron. Number the ideal vertices of T and let e_{ij} denote the edge running between the ideal vertices i and j and let θ_{ij} be the dihedral angle of the edge e_{ij}. Choose horospheres $H_1,\ldots,H_4$ centred at the corresponding ideal vertices of T. Let $l(e_{ij})$ be the signed distance between the horospheres H_i and H_j along the edge e_{ij}, where the distance is negative if the horospheres have non-trivial intersection. Then, the variation in the volume of T is given by the formula*

$$d\,\mathrm{vol}(T) = \frac{1}{2}\sum_{i,j} l(e_{ij})d\theta_{ij} \tag{5.8}$$

We shall now prove Theorem 5.6.1 using Schlafli's variational formula.

Proof. We know that M is a finite volume complete hyperbolic manifold which has a geometric triangulation τ given by the angle structure a. As the structure on M is complete, for each cusp of M, we can make a choice of horospherical triangle in each ideal tetrahedron meeting the cusp such that the triangles close up consistently to form a horospherical cusp torus. Using this choice of horospheres, we can define the edge lengths $l(e_{ij})$ in each tetrahedron such that edges which are identified to the same edge of τ have the same edge length.

$$\begin{aligned} dv(\tau, a) &= \sum_{T \in \tau} d \operatorname{vol} T \\ &= \sum_{T \in \tau} \sum_{i,j} l(e_{ij}) d\theta_{ij} \end{aligned}$$

We can split the summation differently to get

$$\begin{aligned} dv(\tau, a) &= \sum_{\substack{\{e \mid e \text{ is an} \\ \text{edge of } \tau\}}} \sum_{\substack{\{i \mid e_i \text{ is} \\ \text{glued to } e\}}} l(e_i) d\theta_i \\ &= \sum_{\substack{\{e \mid e \text{ is an} \\ \text{edge of } \tau\}}} l(e) \sum_{\substack{\{i \mid e_i \text{ is} \\ \text{glued to } e\}}} d\theta_i \\ &= 0 \end{aligned}$$

The last step follows as the sum of dihedral angles of the edges identified to an edge e of τ add up to 2π, so the variation $\sum_i d\theta_i$ is 0. Therefore, we see that the the complete hyperbolic structure a is a critical point for the volume functional ν, and it must correspond to the maximum of the volume functional ν, as ν is strictly concave on $\mathcal{A}(\tau)$. ■

5.7 Applications

Casson and Rivin's programme has been successfully applied to prove that certain families of 3-manifolds have a complete hyperbolic metric and to find geometric triangulations corresponding to the complete hyperbolic metric on these manifolds. The basic steps of this programme are as follows:

1. We first consider some ideal triangulation τ of the manifold M, preferably one which arises naturally from the description of the 3-manifold.

2. We then show that the space of angle structures $\mathcal{A}(\tau) \neq \varnothing$ by finding an angle structure on τ.

3. As $\overline{\mathcal{A}(\tau)}$ is a compact set, the volume functional ν will achieve a global maximum on it. We have to show that this maximum occurs in the interior of $\overline{\mathcal{A}(\tau)}$. For this, we can show that angle structures on the boundary of $\overline{\mathcal{A}(\tau)}$ cannot maximize the volume functional.

This program has been successfully applied by Futer and Gueritaud [FG04] to find geometric triangulations of 2-bridge knot complements and once-punctured torus bundles.

Corollary 5.7.0.1 (Lower bounds on volume using angle structures)**.** *Suppose we know that M has an ideal triangulation τ and that there exists a critical point of the volume functional ν in the interior of $\overline{\mathcal{A}(\tau)}$. Let $q \in \overline{\mathcal{A}(\tau)}$ be any other angle structure. Then we have the following lower bound for the volume of M.*

$$\nu(q) \leq \mathrm{vol}(M)$$

Equality holds in the above equation if and only if q is the critical point of ν in the interior of $\overline{\mathcal{A}(\tau)}$.

This corollary has been recently used by Champanerkar and Purcell to obtain lower bounds on the volume of weaving knot complements [CKP16].

We end this section with an open conjecture about angle structures.

Conjecture 5.7.1 (Casson's conjecture)**.** *Suppose M is a finite volume complete cusped hyperbolic 3-manifold with torus boundary components. Let τ be any topological ideal triangulation of M. Then if $\mathcal{A}(\tau) \neq \varnothing$, then the maximum value of the volume functional on $\overline{\mathcal{A}(\tau)}$ is at most the hyperbolic volume of M.*

If Casson's conjecture is true, we will be able to easily obtain good lower bounds for the hyperbolic volume of cusped hyperbolic manifolds.

Chapter 6

Geometric triangulations of constant curvature manifolds

We have been studying geometric triangulations of cusped hyperbolic 3-manifolds. In this chapter, we shall work in the setting of constant curvature compact manifolds and understand how geometric triangulations of such manifolds are related by Pachner moves. This work is due to Tejas Kalelkar and Advait Phanse and we have borrowed the exposition of this material from their papers [KP19a][KP19b].

6.1 Geometric triangulations of constant curvature manifolds and geometric Pachner moves

We first define geometric triangulations of Riemannian manifolds precisely.

Definition 6.1.1. *A geometric triangulation of a Riemannian manifold M is a triangulation of M where each simplex of the triangulation is a totally geodesic disk in the Riemannian metric on M.*

If a Pachner move on a geometric triangulation produces a new triangulation which is again geometric, we call it a geometric Pachner move. We wish to show that geometric triangulations of constant curvature Riemannian manifolds are related by geometric Pachner moves. Similar results are known in the case of PL triangulations and smooth triangulations. Pachner proved in his paper [Pac91] that PL triangulations of PL homeomorphic manifolds are related by Pachner moves, which are local combinatorial moves which produce new PL triangulations from a given PL triangulation. Whitehead [Whi40] proved that smooth triangulations of diffeomorphic manifolds are related by smooth Pachner moves.

Theorem 6.1.1 (Kalelkar, Phanse)**.** *Let K_1 and K_2 be geometric simplicial triangulations of a compact constant curvature manifold M with a (possibly empty) subcomplex L with $|L| \supset \partial M$. When M is spherical, we assume that the diameter of the star of each simplex is less than π. Then, for some $s \in \mathbb{N}$, the s-th derived subdivisions $\beta^s K_1$ and $\beta^s K_2$ are related by geometric Pachner moves which keep $\beta^s L$ fixed.*

For manifolds of dimension 2 and 3, derived subdivisions can be realised by geometric Pachner moves. So we have the following corollary

Corollary 6.1.1.1. *Let K_1 and K_2 be geometric simplicial triangulations of a closed constant curvature manifold M, where the dimension of M is* 2 *or* 3*. When M is spherical we assume that the diameter of the star of each simplex is less than π. Then K_1 is related to K_2 by geometric Pachner moves.*

For cusped hyperbolic manifolds, it is unknown whether a common geometric subdivision exists for any two geometric simplicial ideal triangulations. This is because the simplices of one triangulation may spiral into the cusp and possibly intersect the simplices of another triangulation infinitely many times. We shall discuss this problem in more detail in the next chapter. But once we have a common geometric subdivision, we can again relate the two geometric triangulations to the common subdivision via geometric Pachner moves, up to derived subdivisions.

Theorem 6.1.2 (Kalelkar, Phanse)**.** *Let K_1 and K_2 be geometric simplicial ideal triangulations of a cusped hyperbolic manifold which have a common geometric subdivision. Then, for some $s \in \mathbb{N}$, the s-th derived subdivisions $\beta^s K_1$ and $\beta^s K_2$ are related by geometric Pachner moves.*

6.2 Upper bound on Pachner moves relating geometric triangulations

Given two simplicial triangulations, we wish to understand when they represent homeomorphic manifolds. Pachner has shown that if two finite simplicial triangulations of a manifold have a common subdivision, then they must be related by a finite number of Pachner moves. For any two simplicial triangulations with n_1 and n_2 simplices, there exists a function $F\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that the two triangulations are related by less than $F(n_1, n_2)$ Pachner moves. This is true because there are only finitely many triangulations for a given fixed number of simplices. However, this function F may not be computable. The existence of a computable function F is equivalent to the existence of an algorithm to recognize whether two triangulations represent homeomorphic manifolds [CL14]. Mijatovic has given such explicit functions for a large class of 3-manifolds in his papers [Mij04] [Mij05]. These upper bounds are in terms of towers of exponentials in the number of tetrahedra in the triangulations.

In this section, we restrict our attention to the special case of geometrically triangulated constant curvature manifolds and try to solve the homeomorphism problem for them by obtaining an explicit bound on the number of Pachner moves (not necessarily geometric) required to related two different geometric simplicial triangulations of a constant curvature manifold. The main theorem proved by Kalelkar and Phanse [KP19a] and the important corollaries that follow from it are reproduced below.

Theorem 6.2.1 (Kalelkar, Phanse)**.** *Let M be a closed spherical, Eulcidean or hyperbolic n-manifold with geometric triangulations K_1 and K_2. Let K_1 and K_2 have p and q many n-simplices respectively with the length of edges bounded above by λ and let $inj(M)$ be the injectivity radius of M. When M is spherical, we require $\Lambda \leq \pi/2$. Then the 2^{n+1}-th barycentric subdivisions of K_1 and K_2 are related by less than $2^n(n+1)!^{4+3m'}pq(p+q)$ Pachner moves which do not remove common vertices. When $n \leq 4$, then K_1 and K_2 are directly related by $2^n(n+1)!^{4+3m}pq(p+q)$ Pachner moves which do not remove common vertices. Here $m' = max(2^{n+1}, m)$ and m is an integer greater than $\mu \log(\Lambda/inj(M))$, where μ is as follows:*

1. *When M is Euclidean, $\mu = n + 1$*
2. *. When M is spherical, $\mu = 2n + 1$*
3. *When M is hyperbolic, $\mu = n \cosh^{n-1} \Lambda + 1$*

For certain low dimensional spherical manifolds and for hyperbolic manifolds of dimension greater than 3, whether 2 manifolds are isometric is entirely determined by whether they are homeomorphic. So, we have an algorithm to check isometry of these classes of manifolds.

Corollary 6.2.1.1 (Kalelkar, Phanse)**.** *Let (M, K_M) and (N, K_N) be geometrically triangulated closed hyperbolic manifolds of dimension at least 3 or closed spherical manifolds of dimension at most 6 and edge length at most $\pi/2$. Then M is isometric to N if and only if the 2^{n+1}-th barycentric subdivisions of K_M and K_N are related by less than $2^n(n+1)!^{4+3m'}pq(p+q)$ Pachner moves followed by a simplicial isomorphism, with m', p and q as defined in Theorem 6.2.1.*

We can express the injectivity radius in terms of the diameter of M and the volume of M, so we just need an upper diameter bound and a lower volume bound on M. These bounds can be obtained if we know a lower bound and an upper bound on the lengths of the edges of the triangulation. So we have a bound on the number of Pachner moves required to relate two geometric simplicial triangulations entirely in terms of bounds on the lengths of edges and the number of simplices in both the triangulations.

Chapter 7

Geometric ideal triangulations of cusped hyperbolic manifolds

7.1 Introduction

In this chapter, we wish to investigate whether the simplexes of two geometric ideal triangulations of an orientable complete finite volume cusped hyperbolic manifold can intersect each other infinitely many times. To go from one geometric ideal triangulation of a manifold to another via geometric Pachner moves, we would like both the geometric ideal triangulations to intersect 'nicely', that is, we would like the polytopal complex formed by the intersection of the two triangulations to have only finitely many polytopes. However, since cusped hyperbolic manifolds are non-compact, we cannot a priori rule out the case that two ideal simplexes intersect each other infinitely many times. For example, there are two topological ideal triangulations of the 3-punctured sphere which intersect each other infinitely many times as one of the simplexes spirals infinitely many times as it enters the cusps. This is shown in Figure 7.1. We wish to understand if such a phenomenon can occur when the triangulations are geometric ideal triangulations. The result we prove in this chapter, which is original work, answers this question in the negative.

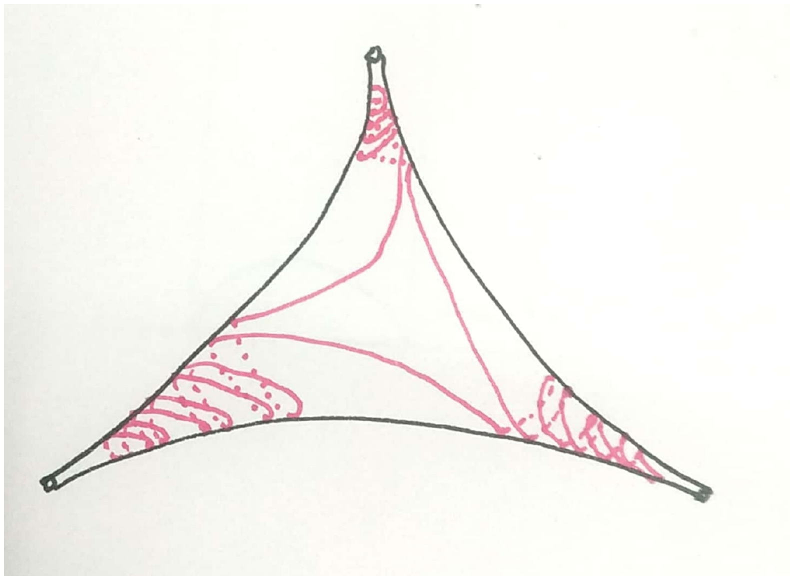


Figure 7.1: Two ideal triangulations of the 3-punctured sphere which intersect each other infinitely many times

7.2 Background

Let M be an orientable complete finite volume hyperbolic n-dimensional manifold (where n is 2 or 3). Then, we know that M is either closed (compact and without boundary) or cusped (homeomorphic to the interior of a compact manifold with circle or torus boundary components). The thick-thin decomposition also tell us that the cusp must be homeomorphic to $S^1 \times [0, \infty)$ when $n = 2$, and $T \times [0, \infty)$ when $n = 3$, where $T = S^1 \times S^1$. In fact, the cusp is isometric to the quotient of a horoball about ∞ in $\mathbb{H}^2$ or $\mathbb{H}^3$ by a discrete subgroup of parabolic isometries which is isomorphic to $\mathbb{Z}$ when $n = 2$, and $\mathbb{Z}^2$ when $n = 3$. This discrete subgroup is generated by parabolic isometries which act as linearly independent translations on each horosphere about ∞. Thus, if we denote the horoball by H and the discrete subgroup by Γ, then the cusp is isometric to H/Γ.

A geometric triangulation of a hyperbolic manifold is a triangulation of the manifold by geometric simplexes - simplexes in which each face is a totally geodesic submanifold. A singular triangulation which is homeomorphic to the manifold upon removing the vertices is called an ideal triangulation of the manifold. A geometric ideal triangulation of a cusped hyperbolic manifold is thus an ideal triangulation of the manifold where each simplex is totally geodesic.

7.3 Intersection of geometric ideal triangulations

For the rest of this chapter, M will refer to a complete orientable finite volume cusped hyperbolic manifold of dimension 2 or 3, unless mentioned otherwise. The main result we prove in this chapter is the following:

Theorem 7.3.1. *Let M be a complete orientable finite volume cusped hyperbolic manifold of dimension 2 or 3. Let τ_1 and τ_2 be two distinct geometric ideal triangulations of M. Then the polytopal complex which is formed by the intersection of τ_1 and τ_2 has only finitely many polytopes.*

For the sake of clarity, we will first define a few terms that we use throughout the chapter.

Definition 7.3.1. *Let M be an n-manifold (where n is 2 or 3) homeomorphic to the interior of a compact manifold $\hat{M}$ with circle or torus boundary components. Then the intersection of a collar neighbourhood of a boundary component of $\hat{M}$ with M is said to be a cusp neighbourhood of M. Thus, a cusp neighbourhood of M is homeomorphic to $S^1 \times (0, \infty)$, when $n = 2$, and to $T \times (0, \infty)$, when $n = 3$. The boundary of the cusp neighbourhood corresponds to $S^1 \times \{0\}$, when $n = 2$, and to $T \times \{0\}$, when $n = 3$. A cusp C of M is a collection of cusp neighbourhoods about a particular boundary component of $\hat{M}$, and any cusp neighbourhood in this collection is called a neighbourhood of the cusp C.*

Definition 7.3.2. *Let M be a complete orientable finite volume cusped hyperbolic n-manifold (where n is 2 or 3). We say that a geodesic ray $\gamma \colon [0, \infty) \to M$ converges to a cusp C of M, if given any neighbourhood of the cusp C, there exists some $t_0 \in [0, \infty)$, such that for $t > t_0$, $\gamma(t)$ lies within the given neighbourhood of the cusp. We say a geodesic converges to a cusp C of M if one end of it is a geodesic ray which converges to C.*

Note that each end of a geodesic ideal edge of a geometric ideal triangulation must converge to some cusp of M. This is because the manifold M is homeomorphic to the complement of the vertices in the cell complex K obtained by identifying the simplexes of the singular triangulation, and the cusp neighbourhoods thus correspond to neighbourhoods of the vertices of K.

Lemma 7.3.1. *Consider a finite set S of geodesic rays converging to a cusp C of M. Then there exists an $\epsilon > 0$ such that these geodesic rays intersect the ϵ-thin neighbourhood of the cusp C in parallel geodesic rays which converge to the cusp C.*

Proof. From the thick thin decomposition, we see that the ϵ_n-thin neighbourhood of the cusp C is the quotient of a horoball in $\mathbb{H}^n$ by parabolic isometries (where n is 2 or 3 and ϵ_n is the corresponding Margulis constant). Lift this cusp neighbourhood to a horoball H about ∞ in $\mathbb{H}^n$. Let $\gamma\colon [0, \infty) \to M$ be one of the geodesic rays in the set S under consideration. We claim that γ intersects the ϵ_n-thin neighbourhood of the cusp C in components which are either finite length geodesic segments with endpoints on the cusp boundary or geodesic rays which converge to the cusp. If a component of γ is perpendicular to the horospherical boundary of the cusp, it must lift to a vertical geodesic ray in $\mathbb{H}^n$ perpendicular to the horospherical boundary of H. If a component of γ is not perpendicular to the horospherical boundary of the cusp, it will lift to a segment of a semicircular geodesic in $\mathbb{H}^n$, that is, it must be a finite length geodesic segment with endpoints on the cusp boundary.

Also, we claim that γ can only intersect the cusp neighbourhood in a finite number of components. As γ converges to the cusp C, there exists a $t_0 > 0$, such that $\gamma(t)$ lies in the cusp neighbourhood for $t > t_0$. So the component of intersection of γ given by $\gamma|_{(t_0,\infty)}$ will lift to a vertical geodesic ray perpendicular to the horospherical boundary of H. As the remaining portion of γ given by $\gamma|_{[0,t_0]}$ is of finite length, γ will intersect the cusp neighbourhood only in finitely many geodesic segments, which lift to a finite number of semicircular geodesic segments in the horoball H. It is clear that these segments only penetrate the cusp neighbourhood up to a finite depth, that is the semicircular geodesic segments all lie below a certain height in $\mathbb{H}^n$.

Thus, we can choose an $\epsilon_\gamma < \epsilon_n$ small enough such that the ϵ_γ-thin neighbourhood of the cusp C avoids all the segments of γ, that is, it lifts to a horoball in $\mathbb{H}^n$ which is high enough that it avoids the semicircular geodesic segments in the lift of γ. Let $\epsilon_S = \min\{\epsilon_\gamma \mid \gamma \in S\}$. Then, the finite set of geodesic rays will intersect the ϵ_S-thin neighbourhood of the cusp C in parallel geodesic rays which converge to the cusp C. ∎

Corollary 7.3.1.1. *Let τ_1 and τ_2 be two distinct geometric ideal triangulations of M. Then, there exists an ϵ (which depends on τ_1 and τ_2), such that the geodesic ideal edges of τ_1 and τ_2 will intersect any cusp neighbourhood contained in the ϵ-thin part of M only in parallel geodesic rays converging to the cusp.*

Lemma 7.3.2. *Let ϵ be a Margulis number for M, that is, the ϵ-thin part of M decomposes as per the thick-thin decomposition. Then, the intersection of any geodesic ideal edge of a geometric ideal triangulation of M with the complement of the ϵ-thin part of M is a finite collection of geodesic segments each of finite length.*

Proof. Without loss of generality, we can lift the geodesic ideal edge to a vertical geodesic in $\mathbb{H}^n$ such that its endpoints are at 0 and ∞. The ϵ-thin cusp neighbourhood in M will lift to disjoint horoballs about 0 and ∞ in $\mathbb{H}^n$. Suppose these horoballs intersect the geodesic at heights a and b, where $a < b$. Then, the length of the geodesic ideal edge which lies in the complement of the ϵ-thin part of M is less than the length of the lifted geodesic between the height a and b. This is true because the quotient map is a local isometry, and hence preserves the length of geodesics which do not close up in the quotient. A simple calculation shows that the length of the lifted geodesic between height a and b is $\ln(b/a)$, which is finite. So, the intersection of the geodesic ideal edge with the complement of the ϵ-thin part of M is also of finite length, and hence it must consist of a finite collection of geodesic segments each of finite length. ■

Lemma 7.3.3. *Two finite length geodesic segments in a compact Riemannian manifold cannot intersect infinitely many times. In particular, if ϵ is a Margulis number for M, then any two finite length geodesic segments in the complement of the ϵ-thin part of M cannot intersect infinitely many times.*

Proof. Let $\alpha\colon [0,1] \to M_c$ and $\beta\colon [0,1] \to M_c$ be two finite length geodesic segments in the complement of the ϵ-thin part of M, which is a compact manifold M_c. Suppose they intersect each other infinitely many times. For each intersection point, let s_n be a point in the interval which maps to that point under α, and similarly let t_n be a point in the interval which maps to it under β. Now, since α and β intersect each other in infinitely many points, we will have infinitely many distinct preimages s_n and t_n. As the interval is compact, any infinite set must have a limit point. Let s be a limit point of the set s_n and pass to a subsequence of s_n which converges to s, which we denote again by s_n. Let t_n be such that $\alpha(s_n) = \beta(t_n)$. Again, there exists a subsequence of t_n which must converge to a limit t. So, finally, we pass to a subsequence of both s_n and t_n such that $s_n \to s$ and $t_n \to t$ and $\alpha(s_n) = \beta(t_n)$.

Now, due to continuity of α and β, we will have $\alpha(s) = \beta(t)$. Let us denote this point in M as x. Thus, x is also a point of intersection of α and β. Now, as M_c is a compact Riemannian manifold, the global convexity radius of M_c is positive and is less than or equal to half of the global injectivity radius of M_c (this follows from results by Dibble and Klingenberg and is proved in Lemma 3.12 in [KP19a]). Denote this global convexity radius by $r(M)$. Then a ball of radius $r(M)$ around x must be strongly convex, that is, any two points in this ball can be joined by a unique geodesic segment which lies entirely within the ball. Denote this ball by B. Then we see that $\alpha^{-1}(B)$ contains a small neighbourhood $(s-\delta, s+\delta)$ of the point s, and similarly $\beta^{-1}(B)$ contains a small neighbourhood $(t-\delta, t+\delta)$ of the point t. Choose a k large enough so that $s_k \in (s-\delta, s+\delta)$ and $t_k \in (t-\delta, t+\delta)$, so that $\alpha(s_k) = \beta(t_k) = x_0 \in B$. Now, $\alpha|_{[s_k,s]}$ and $\beta|_{[t_k,t]}$ are two distinct geodesic segments joining x_0 and x, as if they were to coincide, α and β would have to be the same geodesic, due to Picard's theorem. Thus, $\alpha|_{[s_k,s]}$ and $\beta|_{[t_k,t]}$ are two distinct geodesic segments joining x_0 and x, which lie entirely in the ball B, contradicting the fact that B is a strongly convex ball. So, two finite length geodesic segments in a compact Riemannian manifold cannot intersect each other infinitely many times, proving the desired result. ■

Combining the results from all the previous lemmas, we have shown

Proposition 7.3.1. *Let τ_1 and τ_2 be two distinct geodesic ideal triangulations of M. Then, two geodesic ideal edges of τ_1 and τ_2 cannot intersect infinitely many times.*

This is sufficient to prove Theorem 7.3.1 for the case when M is 2-dimensional, because if two ideal triangulations of a surface intersect infinitely many times, there must be two ideal edges which intersect infinitely many times. The same argument can be used to prove the following corollary to Lemma 7.3.3.

Corollary 7.3.1.2. *Let τ_1 and τ_2 be two geometric triangulations of a compact Riemannian manifold M, where M is a 2-dimensional manifold. Then, τ_1 and τ_2 cannot intersect each* *other in infinitely many polygons.*

To prove the theorem for the case when M is a 3-manifold, we need to do a little more work.

Lemma 7.3.4. *Let M be a complete orientable finite volume cusped hyperbolic 3-manifold. Let τ_1 and τ_2 be two distinct geometric ideal triangulations of M. Then, there exists an $\epsilon > 0$, such that τ_1 and τ_2 cannot intersect each other infinitely many times in the ϵ-thin part of M.*

Proof. By Corollary 7.3.1.1, we can choose an ϵ small enough such that $M_{<\epsilon}$ will be a disjoint union of distinct cusp neighbourhoods such that ideal edges of both τ_1 and τ_2 intersect them only in parallel geodesic rays converging to the cusp. Lift the ϵ-thin neighbourhood of one of these cusps to a horoball about ∞ in $\mathbb{H}^3$ and lift the tetrahedra meeting this cusp to vertical ideal tetrahedra in $\mathbb{H}^3$. Then, we can also make sure that ϵ is small enough so that the ϵ-thin cusp neighbourhood lifts to a horoball in $\mathbb{H}^3$ which is high enough that it avoids all the bottom faces of the vertical tetrahedra. Thus, we can choose ϵ to make sure that this property holds for all cusps of M.

Consider the boundary torus of the ϵ-thin cusp neighbourhood of one of the cusps in M. It has a Euclidean structure induced by the hyperbolic structure on M, as it is the quotient of a horosphere in $\mathbb{H}^3$. Thus, the boundary torus has Euclidean triangulations $\overline{\tau_1}$ and $\overline{\tau_2}$ which are induced by the triangulations τ_1 and τ_2. The intersection of the ideal tetrahedra with the cusp neighbourhood is entirely determined by the triangulation induced on the boundary torus. The triangulation τ_i intersects the cusp neighbourhood in a cell complex, where each cell is the product of a triangle of $\overline{\tau_i}$ with $[0, \infty)$. Thus, the tetrahedra of τ_1 and τ_2 intersect each other in the cusp in the same number of components as the triangles of $\overline{\tau_1}$ and $\overline{\tau_2}$ intersect each other. However, as the boundary torus is a compact Euclidean 2-manifold, by Corollary 7.3.1.2, $\overline{\tau_1}$ and $\overline{\tau_2}$ can only intersect each other in finitely many polygons. Thus, τ_1 and τ_2 intersect each other only finitely many times in the ϵ-thin neighbourhood of each cusp in M. Hence, τ_1 and τ_2 cannot intersect each other infinitely many times in the ϵ-thin part of M. ∎

Lemma 7.3.5. *Let M be a complete orientable finite volume cusped hyperbolic 3-manifold.*

Let τ_1 and τ_2 be two distinct geometric ideal triangulations of M. Then, τ_1 and τ_2 cannot intersect each other infinitely many times in the thick part M_c, the complement of the ϵ-thin part of M.

Proof. The triangulations of M induce a cell structure on M_c in which each cell is a truncated tetrahedron. If, τ_1 and τ_2 intersect each other infinitely many times in M_c, then there must be two truncated tetrahedra which intersect each other infinitely many times, and hence there must be two two truncated triangular faces C and D which intersect each other in infinitely many connected components. Let X be a truncated triangle and let $\phi\colon X \to C$ and $\psi\colon X \to D$ be the maps induced by the triangulation of the manifold M. We will follow a strategy similar to the one adopted in the proof of Lemma 7.3.3 to end up with a contradiction.

Choose a point x_n corresponding to each connected component of the intersection of C and D. Consider the preimage of these points under ϕ and ψ and call them s_n and t_n respectively. Since we have infinitely many distinct preimage points, they must have a limit point, as X is compact. As in the proof of Lemma 7.3.3, we can pass to a subsequence of both s_n and t_n such that $s_n \to s$ and $t_n \to t$ and $\phi(s_n) = \psi(t_n)$. Now, due to continuity of ϕ and ψ, we will have $\phi(s) = \psi(t)$. Let us denote this point in M as x. Thus, x is also a point of intersection of C and D.

Consider a ball $B(x, r)$ around x of radius r less than the global convexity radius $r(M)$ (which is positive as M is compact). We can assume that x does not belong to the same connected component as any other x_n that lies in this ball, because, if it does, we can consider a ball of smaller radius around x such that this property is satisfied. Now, as C and D are truncated geodesic faces, they are totally geodesic submanifolds, and $B(x, r) \cap C$ and $B(x, r) \cap D$ are disks of radius r around x contained in the faces C and D. Also, r is less than the global convexity radius of both C and D as they are both totally geodesic. Thus, $B(x, r) \cap C$ and $B(x, r) \cap D$ are convex disks. Consider an intersection point x_0 which belongs to the sequence x_n and lies in the ball $B(x, r)$ (such an x_0 exists as $x_n \to x$). Thus, x_0 belongs to both $B(x, r) \cap C$ and $B(x, r) \cap D$. Now, there exists a geodesic segment joining x_0 to x in the face C which lies in the disk $B(x, r) \cap C$ and another geodesic segment joining x_0 to x in the face D which lies in the disk $B(x, r) \cap D$. These two geodesic segments are distinct, otherwise the segment would be a subset of a component of intersection of C and D, but we have chosen x_0 and x from distinct connected components. The existence of two distinct geodesic segments joining x_0 and x within the ball $B(x,r)$ contradicts the fact that the ball is strongly convex. Hence, the two truncated faces C and D cannot intersect in infinitely many connected components, and therefore, neither can the truncated triangulations. ■

From Lemma 7.3.4 and Lemma 7.3.5, it clearly follows that two distinct geometric ideal triangulations of a complete finite volume orientable cusped hyperbolic 3-manifold M cannot intersect infinitely many times. Following the proof of Theorem 6.1.2 in [KP19b], we can show that two distinct geometric ideal triangulations of a complete finite volume orientable cusped hyperbolic 3-manifold M are related by geometric Pachner moves.

Chapter 8

Conclusion

We have studied the rich interplay between hyperbolic geometry, knot theory and 3-manifolds in this thesis. In the last chapter, we have proved that there are only finitely many polytopes in the intersection of two geometric ideal triangulations of a cusped hyperbolic 3-manifold. Thus, any two geometric ideal triangulations of a cusped hyperbolic 3-manifold have a common geometric subdivision with a finite number of polytopes. Thus, geometric triangulations of cusped hyperbolic 3-manifolds are related by geometric Pachner moves. In the future, we would like to extend this result and obtain an upper bound on the number of polytopes in the intersection of two geometric ideal triangulations of a cusped hyperbolic 3-manifold. We would like to use this bound to obtain an upper bound on the number of Pachner moves required to go between two geometric ideal triangulations of a cusped hyperbolic 3-manifold, following the technique developed in [KP19a].

Bibliography

[Whi40] J. H. C. Whitehead. "On C1-Complexes." In: *Annals of Mathematics* 41(4) (Mar. 1940), pp. 809–824. DOI: 10.2307/1968861.

[Mil62] J. Milnor. "A Unique Decomposition Theorem for 3-Manifolds". In: *American Journal of Mathematics* 84.1 (1962), p. 1. DOI: 10.2307/2372800.

[Gro69] Jonathan L. Gross. "A Unique Decomposition Theorem for 3-Manifolds with Connected Boundary". In: *Transactions of the American Mathematical Society* 142 (1969), p. 191. DOI: 10.2307/1995352.

[JS79] William Jaco and Peter B. Shalen. "Seifert Fibered Spaces In 3-Manifolds". In: *Geometric Topology* (1979), 91–99. DOI: 10.1016/b978-0-12-158860-1.50013-7.

[Thu80] William P Thurston. *Geometry and topology of three-manifolds*. 1980. URL: http://library.msri.org/books/gt3m/.

[JK82] Troels Jørgensen and Peter Klein. "Algebraic Convergence Of Finitely Generated Kleinian Groups". In: *The Quarterly Journal of Mathematics* 33.3 (1982), 325–332. DOI: 10.1093/qmath/33.3.325.

[Mey87] Robert Meyerhoff. "A Lower Bound for the Volume of Hyperbolic 3-Manifolds". In: *Canadian Journal of Mathematics* 39.5 (Jan. 1987), 1038–1056. DOI: 10.4153/cjm-1987-053-6.

[Pac91] Udo Pachner. "P.L. Homeomorphic Manifolds are Equivalent by Elementary Shellings". In: *European Journal of Combinatorics* 12.2 (1991), 129–145. DOI: 10.1016/s0195-6698(13)80080-7.

[Mil94] John W. Milnor. *John Milnor collected papers*. Vol. 1. Publish or Perish., 1994.

[TL97] William P. Thurston and Silvio Levy. *Three-dimensional geometry and topology*. Princeton University Press, 1997.

[Lic99] W B R Lickorish. “Simplicial moves on complexes and manifolds”. In: *Proceedings of the Kirbyfest* (1999). DOI: 10.2140/gtm.1999.2.299.

[Lac00] Marc Lackenby. “Word hyperbolic Dehn surgery”. In: *Inventiones Mathematicae* 140.2 (Jan. 2000), 243–282. DOI: 10.1007/s002220000047.

[CH02] Ken Chan and Craig Hodgson. “Constructing hyperbolic 3-manifolds using ideal triangulations”. MA thesis. University of Melbourne, 2002.

[FG04] David Futer and Francois Guéritaud. “On canonical triangulations of once-punctured torus bundles and two-bridge link complements”. In: *Geometry and Topology* 10 (July 2004). DOI: 10.2140/gt.2006.10.1239.

[Mij04] Aleksandar Mijatović. “Triangulations of Seifert fibred manifolds”. In: *Mathematische Annalen* 330.2 (2004), 235–273. DOI: 10.1007/s00208-004-0547-9.

[Mij05] Aleksandar Mijatović. “Triangulations of fibre-free Haken 3-manifolds”. In: *Pacific Journal of Mathematics* 219.1 (Jan. 2005), 139–186. DOI: 10.2140/pjm.2005.219.139.

[Wee05] Jeff Weeks. “Computation of Hyperbolic Structures in Knot Theory”. In: *Handbook of Knot Theory* (2005), 461–480. DOI: 10.1016/b978-044451452-3/50011-3.

[Hat07] Allen Hatcher. *Notes on Basic 3-Manifold Topology*. 2007. URL: https://pi.math.cornell.edu/~hatcher/3M/3Mdownloads.html.

[Mar07] Albert Marden. *Outer circles: an introduction to hyperbolic 3-manifolds*. Cambridge University Press, 2007.

[FG10] David Futer and François Guéritaud. “From angled triangulations to hyperbolic structures”. In: (Apr. 2010). DOI: 10.1090/conm/541/10683.

[BP12] R. Benedetti and Carlo Petronio. *Lectures on hyperbolic geometry*. World Publishing Corporation, 2012.

[CL14] Alexander Coward and Marc Lackenby. “An upper bound on Reidemeister moves”. In: *American Journal of Mathematics* 136.4 (2014), 1023–1066. DOI: 10.1353/ajm.2014.0027.

[CKP16] Abhijit Champanerkar, Ilya Kofman, and Jessica Purcell. "Volume bounds for weaving knots". In: *Algebraic and Geometric Topology* 16.6 (2016), 3301–3323. DOI: 10.2140/agt.2016.16.3301.

[DD16] Blake Dadd and Aochen Duan. "Constructing infinitely many geometric triangulations of the figure eight knot complement". In: *Proceedings of the American Mathematical Society* 144.10 (June 2016), 4545–4555. DOI: 10.1090/proc/13076.

[Mar16] Bruno Martelli. *An Introduction to Geometric Topology*. 2016. arXiv: 1610.02592 [math.GT].

[MY18] Hitoshi Murakami and Yoshiyuki Yokota. *Volume conjecture for knots*. Springer., 2018.

[KP19a] Tejas Kalelkar and Advait Phanse. *An upper bound on Pachner moves relating geometric triangulations*. 2019. arXiv: 1902.02163 [math.GT].

[KP19b] Tejas Kalelkar and Advait Phanse. *Geometric moves relate geometric triangulations*. 2019. arXiv: 1907.02643 [math.GT].

[Pur20] Jessica S. Purcell. *Hyperbolic Geometry and Knot Theory*. 2020. URL: http://users.monash.edu/~jpurcell/book/HypKnotTheory.pdf.