# In Search of Consistent Classical Theories of Gravity

**A Thesis**

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

Tushar Gopalka

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA. May, 2020

Supervisor: Dr. Shiraz Minwalla

# Certificate

This is to certify that this dissertation entitled ‘In Search of Consistent Classical Theories of Gravity ’ towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Tushar Gopalka at the Tata Institute of Fundamental Research, Mumbai under the supervision of Dr. Shiraz Minwalla, Senior Professor, Department of Physics , during the academic year 2019-2020.

Tushar Gopalka

Dr. Shiraz Minwalla

Tushar Gopalka

Committee:

Dr. Shiraz Minwalla

Dr. Sunil Mukhi

This thesis is dedicated to my parents, Rekha and Rajkumar

# Declaration

I hereby declare that the matter embodied in the report entitled ‘In Search of Consistent Classical Theories of Gravity ’, are the results of the work carried out by me at the Department of Physics, Tata Institute of Fundamental Research, Mumbai, under the supervision of Dr. Shiraz Minwalla and the same has not been submitted elsewhere for any other degree.

Tushar Gopalka

Dr. Shiraz Minwalla

Tushar Gopalka

# Permissions

This thesis is based on the following works done by the author along with supervisor and other collaborators:

- Classifying and constraining local four photon and four graviton S-matrices; S.D Chowdhury, A. Gadde, T. Gopalka, I. Haldar, L. Janagal, S. Minwalla; arxiv 1910.14392; Published in JHEP; DOI: 10.1007/JHEP02(2020)114.
- Classification of all 3 particle S-matrices quadratic in photons or gravitons; S. Chakraborty, S.D Chowdury, T. Gopalka, S. Kundu, S. Minwalla, A. Mishra; arxiv 2001.7117

# Acknowledgments

I would especially like to thank Dr. Shiraz Minwalla for presenting me with the opportunity to work on this project, and guiding me with care and support whenever necessary. I would also like to thank Dr. Sunil Mukhi for his constant support, and especially for teaching me several intangible lessons in physics and life. I would furthermore like to thank other mentors I have had such as Dr. Nabamita Banerjee, Dr. Sudarshan Ananth, and Dr. Ashoke Sen. This project would not have been completed without the help of Dr. Subham Dutta Chowdhury, a post-doc at TIFR, and hence I owe him a special note of thanks. I would also like to thank my other collaborators Dr. Abhijit Gadde, Dr. Lavneet Janagal, Indranil Haldar, Suman Kundu, Amiya Mishra and Dr. Somangshu Chakraborty for helping me on numerous occasions. I would certainly like to thank my parents for allowing me to pursue physics. I would especially like to thank my dear friends such as Shubhangi Khobragade, Devesh Verma, Madhav Sinha, Aniruddha Deshpande, Jaideep Mahajan, Prashant Dagale, Mrunmay Jagdale, Shirley Bharat Dixit and Shivali Dongre for helping me during the difficult moments of my life. I would also like to express my gratitude towards the people of India for their steady support of the basic sciences. The project is supported by the DST Inspire Fellowship awarded by the Government of India.

# Abstract

General Relativity has provided us a successful framework to describe gravity, and has passed numerous experimental as well as theoretical consistency conditions. However, general relativity can possibly be modified by a large class of high energy corrections. In this project, we try to constrain the space of all kinematically allowed classical gravitational theories based on certain consistency conditions. We explicitly construct the basis tree level S-matrices for four graviton and four photon scattering in all spacetime dimensions. From the space of possible S-matrices, the consistency condition used to rule out possible S-matrices is a conjecture, called as the Classical Regge Growth Conjecture (CRG conjecture). This conjecture puts a restriction on the growth of any classical (tree-level) S-matrix in the Regge limit. Assuming the CRG conjecture [1] to be true, we find that Einstein gravity is the unique classical theory of gravity in spacetime dimensions $D \leq 6$. In the latter part of the project, we classify all possible 3 point S-matrices quadratic in photons or gravitons [2].

# Contents

# List of Tables

# Chapter 1

# Introduction

General Relativity has been an extremely successful theory in describing gravity. However, it is well known that general relativity might only be a good enough approximation at low enough energies. In order to describe gravity at high energies, one possible modification to general relativity is to include higher derivative terms which are suppressed by some high energy scale. Such theories are known as higher derivative theories of gravity. In this project, we study classical four photon as well as four graviton S-matrices. One scenario in which such higher derivative terms are natural is the framework of string theory where the corrections are proportional to factors of $\alpha'$, which is the square of the string length scale. This project was partly motivated by a recent analysis ([6]), where they study the three point graviton S-matrix and use causality to exclude the Gauss-Bonnet and $(\text{Riemann})^3$ corrections to Einstein gravity.

## 1.1 Three graviton scattering

The case of three graviton scattering is particularly simple. It follows from kinematical considerations that the most general 3 graviton S-matrix is a linear combination of the two derivative structure ((4.1.17)), the four derivative structure ((4.1.18)) and the six derivative structure ((4.1.19)). In other words the most general 3 graviton S-matrix in any theory of gravity - classical or quantum - is specified by three real numbers.

In a classic paper: Camanho, Edelstein, Maldacena and Zhibeodov (CEMZ) [6] demonstrated that a classical theory with 3 gravitational S-matrices that include a non-zero admixture of the four derivative and six derivative three graviton structures *necessarily* violates causality unless its four graviton scattering amplitude include contributions from the exchange of poles of arbitrarily high spin. The constraints follow from a particular sign of the Shapiro-time delay which in turn corresponds to the sign of the phase shift in flat space. In AdS similar argument has been made in [7].

It thus follows from the results of [6] that the three graviton scattering amplitude is necessarily two derivative - i.e. that of the Einstein theory - in any causal classical theory of gravity whose four graviton S-matrices have exchange contributions that are bounded in spin. The uniqueness of graviton three point function has also been demonstrated using causality in conformal field theory [8, 9].

The importance of the above result should not be overstated since the three graviton S-matrix is specified by a finite number of parameters and hence is kinematically special. On the other hand four and higher point scattering amplitudes are specified by a finite number of functions of kinematical invariants ($s$ and $t$ in the case of four graviton scattering) and so an infinite number of real parameters. In this project we will focus on the study of four graviton scattering. We will attempt to cut down the space of allowed S-matrices by proposing a physical criterion that acceptable S-matrices must obey.

## 1.2 A conjectured bound on Regge scattering

In the classic analysis [6] were able to use the physical criteria (the requirement of causality) to constrain the parameters that appeared in the most general three point function. We will now attempt to do the same for the the four point function. More specifically, we will constrain classical theories using a conjectured bound of the Regge growth of classical scattering amplitudes that we now state

- **Classical Regge Growth Conjecture:** The S-matrix of a consistent classical theory never grows faster than $s^2$ at fixed $t$ - at all physical values of momenta and for every possible choice of the normalized polarization vector $\varepsilon_i$.

The first piece of evidence in favor of the CRG conjecture is that it is always obeyed by two derivative theories involving particles of spin no greater than two - theories that we independently expect to be consistent. The two derivative nature of interactions ensures that both contact contributions as well as $s$ and $u$ channel exchange contributions grow no faster than $s$ in the Regge limit. $t$ channel exchange graphs on the other hand grow no faster than $s^J$ where $J$ is the spin of the exchanged particle. Since we have assumed $J \leq 2$, all contributions obey the CRG conjecture.

The next piece of evidence in support of the conjecture described in this subsection is that it is obeyed by all classical string scattering amplitudes. Recall that, for instance the Type II string scattering amplitudes grow in the Regge limit like

$$s^{2+\frac{1}{2}\alpha' t} \tag{1.2.1}$$

As $t$ is negative at physical values of momenta it follows that this behavior obeys our conjecture. Note also that (1.2.1) reduces to $s^2$ in the limit $\alpha' \to 0$, matching with the fact that gravitational amplitudes, which grow like $\frac{s^2}{t}$ in the Regge limit, saturate the CRG bound.

The strongest evidence for the CRG conjecture follows from the observation that the CRG conjecture is tightly related to the chaos bound[10]. We pause to review how this works. Working in a large $N$ unitary CFT in $D \geq 2$ consider a four point $\langle OOOO \rangle$ where $O$ is a real scalar operator and the insertion points are taken to be

$$\left(\pm \sinh\left(\frac{\tau}{2}\right), \pm \cosh\left(\frac{\tau}{2}\right)\right).$$

All insertions are denoted by the doublet $(t,x)$; insertions all lie completely in the $(t,x)$ plane. The authors of [10] used the unitarity of the CFT to demonstrate that the growth of the connected correlator $\langle OOOO \rangle$ with boost times $\tau$ cannot be faster than $e^{\tau}$. This result holds in the large $N$ limit for boost times $\tau$ large compared to unity but small compared to $\ln N$.

Let us now study a situation in which the CFT under study has a bulk dual. Let the bulk field dual to the operator $O$ be denoted by $\phi$. Let us suppose that the fields $\phi$ in the bulk has a standard quadratic term and also has a have a local four point self interaction of the schematic form $\phi\partial...\partial\phi\partial...\partial\phi\partial...\partial\phi$. One can use the usual rules of AdS/CFT to directly compute the correlator $\langle OOOO \rangle$ of the previous paragraph, and so evaluate its growth with $\tau$. The authors of [11, 12] were able to carry through this computation for the most general bulk contact term using the classical bulk theory. They discovered the following interesting fact. Any bulk vertex which, in flat space, would give rise to an S-matrix that grows like $s^{m+1}$ in the Regge limit, turns out to give a contribution to $\langle OOOO \rangle$ that grows with boost time like $e^{m\tau}$. It follows that at least in AdS space, any bulk interaction associated with a flat-space S-matrix that grows faster than $s^2$ at fixed $t$ leads to a boundary correlator that violates a field theory theorem, and so must be classically inconsistent.

Although the results described above have been carefully verified only for scalar operator insertions, we feel it is likely that they will continue to hold for insertions of all spins. The tight connection between the CRG and the Chaos bound, is, in our opinion, striking evidence in favor of the CRG conjecture. Note that the CRG conjecture immediately implies the non existence of a consistent interacting theory of higher spin particles (of bounded spin) propagating in flat space. Let the highest spin in the theory be $J > 2$. As the spin $J$ particle is assumed to be interacting, there exists an S-matrix that receives contributions from spin $J$ exchange. In the $t$-channel this exchange contribution scales like $s^J$, violating the CRG conjecture. This argument has been used in [6] to rule out the possibility of spectrum of particles of spin $> 2$ that is bounded in spin.

## 1.3 Consequences of the CRG conjecture

### 1.3.1 The scattering of four identical scalars

For $D \geq 3$ the most general local contact scalar interaction term that obeys the CRG bound is given by

$$S = \int d^D x \left( a_1 \left(\phi^4\right) + a_2 \left(\phi^2 \partial_\mu \partial_\nu \phi \; \partial_\mu \partial_\nu \phi\right) + a_3 \left(\phi^2 \partial_\mu \partial_\nu \partial_\rho \phi \; \partial_\mu \partial_\nu \partial_\rho \phi\right)\right). \tag{1.3.1}$$

This implication of the CRG has been used effectively to compute the four point function of certain scalar operators in a theory with slightly broken higher spin symmetry [13].

### 1.3.2 The scattering of four identical gravitons

In $D \leq 6$ there are no contact four graviton Lagrangians consistent with the CRG conjecture. However, for $D \geq 7$ the unique such Lagrangian is the second Lovelock Lagrangian:

$$\chi_6 = \int \sqrt{-g} \left( \frac{1}{8} \, \delta^g_{[a} \delta^h_b \delta^i_c \delta^j_d \delta^k_e \delta^l_{f]} \, R_{ab}{}^{gh} R_{cd}{}^{ij} R_{ef}{}^{kl} \right). \tag{1.3.2}$$

The S-matrix that follows from this Lagrangian is proportional to

$$\left(\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4 \wedge p_1 \wedge p_2 \wedge p_3\right)^2. \tag{1.3.3}$$

## 1.4 Exchange contributions

Exchange diagrams in the $t$ channel generically scale like $s^J$ where $J$ is the largest number of symmetrized indices in the representation that labels the exchanged particle. This $s^J$ scaling holds independent of the nature of the external particles, and so applies equally to scalar, photon and graviton scattering. It follows that the exchange of particles with $J \geq 3$ generically violates the CRG conjecture. Exchange of particles with $J \leq 2$ never violates the CRG bound in the $t$ channel but may violate this conjecture in the $s$ and $u$ channels. As the contribution in the $s$ channel and $u$ channel is analytic in $t$, the violations in these channels mean violations at zero impact parameter. This is in contrast with the t-channel where the CRG bound for $J \geq 2$ is violated at finite impact parameter.

The behavior of scattering in the $s$ and $u$ channels is sensitive to the nature of the external scattering particles. In the case of four external scalars or four external photons it is easy to find examples of exchange contributions that do not violate the CRG bound. We have, for example, explicitly computed the contribution to four photon scattering from the exchange of a massive particle of arbitrary mass and demonstrated that its Regge growth is slower than $s^2$. In the case of external gravitons, on the other hand, we have shown by explicit computation that the exchange of massive scalars or massive spin two particles always leads to an S matrix that violates the CRG bound - and moreover violates it in a manner that cannot be cancelled by a compensating local contribution. The same is also true of exchange of a massless spin two gravition whenever the three gravition scattering amplitude deviates from the Einstein form. More generally we have demonstrated - under some hopefully reasonable assumptions - that *every* exchange contribution to four graviton scattering in $D \leq 6$ - other than graviton exchange with the Einstein three point scattering - violates the CRG bound in a way that cannot be compensated for by local counterterm contributions. This fact leads us to claim that atleast for spacetime dimensions $D \leq 6$, the only consistent classical gravitational S-matrix whose exchange poles are bounded in spin is the Einstein S-matrix.

# Chapter 2

# Generalities of four point S-matrices

In this section we review and discuss the general structural features of $2 \to 2$ S-matrices of four identical bosonic scalars, photons or gravitons in an arbitrary number of spacetime dimensions. Also, towards the end of the section, we study the Regge limit of local (polynomial) S-matrices on purely kinematical grounds.

## 2.1 Scattering data

### 2.1.1 Momenta

Consider the scattering of four massless particles in $D$-dimensional Minkowski space. Let $p_i^\mu$ be momentum of the $i^{\text{th}}$ particle. The masslessness of the scattering particles and momentum conservation means

$$p_i^2 = 0, \qquad \sum_{i=1}^{4} p_i^\mu = 0. \tag{2.1.1}$$

We use the mostly positive convention and define Mandelstam variables,

$$\begin{aligned}
s &:= -(p_1+p_2)^2 = -(p_3+p_4)^2 = -2p_1.p_2 = -2p_3.p_4 \\
t &:= -(p_1+p_3)^2 = -(p_2+p_4)^2 = -2p_1.p_3 = -2p_2.p_4 \\
u &:= -(p_1+p_4)^2 = -(p_2+p_3)^2 = -2p_1.p_4 = -2p_2.p_3.
\end{aligned} \tag{2.1.2}$$

(2.1.2) follows from (2.1.1). Note that momentum conservation gives us $s+t+u=0$.We need to make a specific choice of independent Mandelstam variables: take it to be $s$ and $t$.

### 2.1.2 Polarizations

The $2 \to 2$ S-matrix is a Lorentz invariant complex valued function of the momenta $p_i$, together with the data that specifies the internal or spin degree of freedom of each scattering particle.

#### Scalars

Scalar particles have no internal degrees of freedom, so $2 \to 2$ scalar S-matrices are functions only of momenta. For $D \geq 4$ Lorentz invariance ensures that S-matrices are, in fact, functions only of $s$ and $t$. For the case of $D = 3$, four scalar S-matrices can be either parity even or parity odd. Parity even S-matrices are simply a function of $s$ and $t$ as in higher dimensions. Parity odd S-matrices are given by $\varepsilon_{\mu\nu\rho} p_1^\mu p_2^\nu p_3^\rho$ times a second function of $s$ and $t$.

#### Photons

The internal degree of freedom of a photon may be taken to be its polarization vector $\varepsilon_i^\mu$. In the Lorentz gauge (which we use throughout this paper in order to preserve manifest Lorentz invariance),

$$\varepsilon_i \cdot p_i = 0 \tag{2.1.3}$$

The S-matrix must also be invariant under residual gauge transformations i.e. under the transformations,

$$\varepsilon_i^\mu \to \varepsilon_i^\mu + \zeta(p_i) p_i^\mu \tag{2.1.4}$$

We will sometimes use the notation

$$\zeta(p_i) = \zeta_i.$$

As $\zeta(p_i)$ is a completely arbitrary function of $p_i$ the four numbers $\zeta_i$ can be varied independently of each other. It follows that the requirement of gauge invariance is simply the condition that the S-matrix is separately invariant under each of the transformations

$$\varepsilon_i^\mu \to \varepsilon_i^\mu + \zeta_i p_i^\mu \tag{2.1.5}$$

separately for each $i$.

To summarize, a 4-photon S-matrix is a Lorentz invariant complex valued function $\mathscr{S}(p_i, \varepsilon_i)$, subject to the condition (2.1.3). It depends linearly on each of the four polarizations $\varepsilon_i^\mu$ and is separately invariant under each of the four shifts (2.1.5).

## Gravitons

The internal degrees of freedom of a graviton can be parameterized by its traceless symmetric polarization tensor $h_i^{\mu\nu}$. In Lorentz gauge,

$$h_i^{\mu\nu} p_i^\nu = 0 \tag{2.1.6}$$

As before, the S-matrix enjoys invariance under residual gauge transformations,

$$h_i^{\mu\nu} \to h_i^{\mu\nu} + \zeta_i^\mu p_i^\nu + \zeta_i^\nu p_i^\mu, \qquad \text{where} \quad \zeta_i \cdot p_i = 0. \tag{2.1.7}$$

Through most of this paper we will find it convenient to specialize to the special choice of polarization

$$h^i_{\mu\nu} = \varepsilon^i_\mu \varepsilon^i_\nu \qquad \text{where} \quad k_i \cdot \varepsilon_i = 0, \ \ \varepsilon_i \cdot \varepsilon_i = 0 \tag{2.1.8}$$

The gauge transformation parameter $\zeta_i^\mu = \zeta_i \varepsilon^i_\mu$, preserves the choice of the polarization (2.1.8) and induces the gauge transformations

$$\varepsilon_i^\mu \to \varepsilon_i^\mu + \zeta_i p_i^\mu. \tag{2.1.9}$$

These transformations are same as the ones in (2.1.4) [1].

In conclusion, with the choice (2.1.8), a 4-graviton S-matrix $\mathscr{S}(p_i, \varepsilon_i)$, like the photon, is a Lorentz invariant complex valued function $\mathscr{S}(p_i, \varepsilon_i)$ but this time one that is a bilinear function of each of the $\varepsilon_i$'s, subjected to the tracelessness condition $\varepsilon_i \cdot \varepsilon_i = 0$.

[1]The special choice (2.1.8) does not result in loss of generality. Let $S(\varepsilon)$ denote the S-matrix with a single special choice of polarization, $h_1^{\mu\nu} = \varepsilon^\mu \varepsilon^\nu$. Then the linear combination

$$S(u+v) - S(u) - S(v),$$

where $u$ and $v$ are orthogonal polarization vectors, yields the S-matrix for the choice of polarization $h_1^{\mu\nu} = u^\mu v^\nu + v^\mu u^\nu$ and this sort of polarizations form a basis for general symmetric traceless tensors $h_1^{\mu\nu}$. As the S-matrix is linear in $h_1^{\mu\nu}$, the S-matrix with the choice (2.1.8) carries the same information as the most general 4-graviton S-matrix.

### 2.1.3 Unconstrained polarizations

In the previous subsubsection we have expressed the S-matrix as shift invariant functions of the polarization vectors $\varepsilon_i$. It is possible to simultaneously 'solve' for the constraints on $\varepsilon_i$ (tracelessness) and the constraints on the S-matrix (shift or gauge invariance) and re express the S-matrix as a function of independent unconstrained variables as follows.

The momenta $p_i$ span a three dimensional subspace of $D$-dimensional Minkowski space. We refer to this subspace as the scattering 3-plane. The polarization vectors $\varepsilon_i$ can be decomposed into part transverse to the scattering plane $\varepsilon_i^{\perp}$ and part parallel to the scattering plane $\varepsilon_i^{||}$

$$\varepsilon_i = \varepsilon_i^{\perp} + \varepsilon_i^{||} \tag{2.1.10}$$

The condition $\varepsilon_i.p_i = \varepsilon_i^{||}.p_i = 0$ forces $\varepsilon_i^{||}$ to lie in a two dimensional subspace of the scattering plane. Moreover the constraint that S-matrices are invariant under the shifts $\varepsilon_i^{||} \to \varepsilon_i^{||} + p_i$ tells us that the S-matrix is a function only of one of the two free components of $\varepsilon_i^{||}$. It follows that for the purpose of evaluating gauge invariant S-matrices - the set of inequivalent vectors $\varepsilon_i^{||}$ may be parameterized by a single complex number $\alpha_i$. We choose the following (arbitrary) parameterization that obeys $\varepsilon_i^{||}.p_i = 0$.

$$\begin{aligned}
\varepsilon_1^{||} &= \alpha_1 \sqrt{\frac{st}{u}} \left(\frac{p_2}{s} - \frac{p_3}{t}\right) + a_1 p_1 \\
\varepsilon_2^{||} &= \alpha_2 \sqrt{\frac{st}{u}} \left(\frac{p_1}{s} - \frac{p_4}{t}\right) + a_2 p_2 \\
\varepsilon_3^{||} &= \alpha_3 \sqrt{\frac{st}{u}} \left(\frac{p_4}{s} - \frac{p_1}{t}\right) + a_3 p_3 \\
\varepsilon_4^{||} &= \alpha_4 \sqrt{\frac{st}{u}} \left(\frac{p_3}{s} - \frac{p_2}{t}\right) + a_4 p_4.
\end{aligned} \tag{2.1.11}$$

The numbers $a_i$ represent the freedom to shift $\varepsilon_i$ by gauge transformations; $a_i$ are redundancies of description and will not show up in any gauge invariant physical result. On the other hand the parameters $\alpha_i$ are physical. In particular

$$\varepsilon_i^{||}.(\varepsilon_i^{||})^* = |\alpha_i|^2 \tag{2.1.12}$$

With these definitions in place we can write

$$\varepsilon_i = \varepsilon_i^{\perp} + \varepsilon_i^{||}. \tag{2.1.13}$$

Equations (2.1.13) and (2.1.11) express the $D$ component vector $\varepsilon_i$ in terms of the $D-3$ component vector $\varepsilon_i^{\perp}$ and the single parameter $\alpha_i$ and the redundant variables $a_i$.

Unlike $\varepsilon_i$, the pair $(\varepsilon_i^{\perp}, \alpha_i)$ are unconstrained data in the case of photons. In the case of gravitons the data still has to obey the single constraint[2] - which is a consequence of the tracelessness of $\varepsilon_i$

$$\varepsilon_i^{\perp}.\varepsilon_i^{\perp} + \alpha_i^2 = 0 \tag{2.1.15}$$

This constraint can be used to solve for $\varepsilon_i^{\perp}.\varepsilon_i^{\perp}$ in terms of $\alpha_i^2$. In enumerating contraction structures we simply omit all terms containing factors of $\varepsilon_i^{\perp}.\varepsilon_i^{\perp}$. For counting purposes, therefore, $\varepsilon_i^{\perp}$ can effectively be treated as null.

The expressions (2.1.11) and (2.1.13) allow us to convert any Lorentz and gauge invariant expression parity even expression for a photon/graviton S-matrix, initially presented as a function of $\varepsilon_i$ and $p_i$, into an function of $(\varepsilon_i^{\perp}, \alpha_i)$ and $(s,t)$. This function is separately linear/bilinear in $(\varepsilon_i^{\perp}, \alpha_i)$. Note that the individual momenta $p_i$ enter into this reduced form of the S-matrix only through[3] $(s,t)$. We now move to discuss the structure of parity odd S-matrices.

There is a slight subtlety in the discussion of parity odd S-matrices, i.e. S-matrices constructed out of a single factor of the $D$-dimensional Levi-Civita tensor $\varepsilon$. The reason these structures are subtle is simply that $\varepsilon$ tensors in different numbers of dimensions have different numbers of indices and so do not simply map to one another. In order to resolve this subtlety it will prove convenient to formally regard the $\varepsilon$ tensor as one of the arguments of parity odd $S$-matrices. From this viewpoint, a parity odd S-matrix is a Lorentz and gauge invariant function of $p_i$, $\varepsilon_i$ and $\varepsilon$ that has the property that is linear in $\varepsilon$ (it also has the usual homogeneities in $\varepsilon_i$). We now define a $D-3$ dimensional $\widetilde{\varepsilon}^{D-3}$ tensor by the equation,

$$\widetilde{\varepsilon}^{D-3} = \varepsilon_{\mu_1 \ldots \mu_{D-3}\mu_{D-2}\mu_{D-1}\mu_D} p_1^{\mu_{D-2}} p_2^{\mu_{D-1}} p_3^{\mu_D}. \tag{2.1.16}$$

Note that $\widetilde{\varepsilon}^{D-3}$ is totally anti-symmetric under the $S_4$ permutation of particles. It has momentum

[2]It is sometimes also useful to view the polarizations $\varepsilon_i$ as normalized according to the condition

$$\varepsilon_i \cdot \varepsilon_i^* = 1 \qquad \Rightarrow \qquad |\varepsilon_i^{\perp}|^2 + |\alpha_i|^2 = 1. \tag{2.1.14}$$

Notice that $\varepsilon_i$ and $\varepsilon_i + p_i$ have the same norm, so this condition is gauge invariant. We will not need to impose this normalization condition in this paper.

[3]This follows from the fact $\varepsilon_i^{\perp}.p_j = 0$. While $\varepsilon_i^{\parallel}.p_j \neq 0$ the result of this dot product is given by $\alpha_i$ times an easily computed function of $(s,t)$.

degree three. Note that $\widetilde{\varepsilon}^{D-3}$ is proportional to $(stu)^{\frac{1}{2}}\varepsilon^{D-3}$, where $\varepsilon^{D-3}$ is a $D-3$ dimensional Levi-Civita tensor. For this reason it is sometimes useful to work with the 'normalized' tensor

$$N\left(\widetilde{\varepsilon}^{D-3}\right) = \frac{\widetilde{\varepsilon}_{\mu_1\ldots\mu_{D-3}}}{\sqrt{stu}} \tag{2.1.17}$$

$N(\widetilde{\varepsilon}^{D-3})$ is proportional to the Levi-Civita tensor in $D-3$ dimensions up to a sign[4]. The action of the permutation group on $\widetilde{\varepsilon}^{D-3}$ and $N(\widetilde{\varepsilon}^{D-3})$ is given by

$$P(\widetilde{\varepsilon}^{D-3}) = (-1)^{\mathrm{sgn}(P)}\widetilde{\varepsilon}^{D-3}, \quad P\left(N(\widetilde{\varepsilon}^{D-3})\right) = (-1)^{\mathrm{sgn}(P)}N(\widetilde{\varepsilon}^{D-3}) \tag{2.1.18}$$

where $P$ is an arbitrary permutation in $S_4$ of the momenta $p_1\ldots p_4$. In other words both $\widetilde{\varepsilon}^{D-3}$ and $N(\widetilde{\varepsilon}^{D-3})$ pick up a sign under every odd permutation (e.g. under single exchange permutations).

When $D$ is odd, we will choose to regard every parity odd S-matrix a function of $\widetilde{\varepsilon}^{D-3}$, $\varepsilon_i^{\perp}, \alpha_i$ and $(s,t)$; the function in question is linear in $\widetilde{\varepsilon}^{D-3}$ (it also has the usual homogeneities in $(\varepsilon_i^{\perp}, \alpha_i)$). When $D$ is even, on the other hand, we will choose to regard every parity odd S-matrix a function of $N(\widetilde{\varepsilon}^{D-3})$, $\varepsilon_i^{\perp}, \alpha_i$ and $(s,t)$; once again the function is linear in $N(\widetilde{\varepsilon}^{D-3})$ and has the usual homogeneities in $(\varepsilon_i^{\perp}, \alpha_i)$. [5] Besides being gauge invariant and Lorentz invariant, the S-matrices for four graviton or four photon scattering should be $S_4$ invariant. This constraint is imposed in two distinct steps and the reason for it is discussed in the subsection below.

## 2.2 Permutations: $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $S_3$

The permutation group $S_4$ has a special abelian subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by $(2143)$, $(3412)$ i.e. the subgroup of double transpositions[6]. The importance of this subgroup is that it leaves all the Mandelstam variables $s,t$ and $u$ invariant.

[4]Note that the LHS of (2.1.17) is precisely defined (unlike the $D-3$ Levi-Civita tensor which is precisely defined only once we specify an orientation in the $D-3$ plane orthogonal to the scattering plane).

[5]The reason we make this distinction between odd and even $D$ will become clearer later in this paper. Roughly speaking the reason goes as follows. We will see below that S-matrices can be expanded in a sort of Taylor Series in momenta. In every dimension the basis functions of this expansion for parity even S-matrices all have even powers of momenta. As far as parity odd S-matrices go, however, the basis functions are even in momenta when $D$ is even; the fact that $N(\widetilde{\varepsilon}^{D-3})$ is also even in powers of momenta makes (2.1.17) a natural building block of such S-matrices. On the other hand the building blocks for parity odd S-matrices in odd $D$ are odd in momenta; the fact that (2.1.16) is cubic in momenta makes it a natural building block for S-matrices in this case.

[6]We label an element of $S_4$ by the image of $(1234)$ under that element. The group $\mathbb{Z}_2 \times \mathbb{Z}_2$ consists of the two listed generators together with the identity permutation $(1234)$ and a fourth element $(4321)$.

Another feature of this subgroup is that it is normal[7]. As a result, the coset[8] $S_4/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ inherits the group structure of $S_4$. The coset group is easily identified. Every $S_4$ group element is $\mathbb{Z}_2 \times \mathbb{Z}_2$ equivalent to a unique element of the form $(abc4)$. It follows that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ 'gauge invariance' can be fixed by adopting the 'gauge fixing condition' that particle 4 is not permuted. This choice of gauge fixing clearly reveals the coset to be simply the $S_3$ that permutes particles 1, 2 and 3. Thus we conclude that

$$\frac{S_4}{(\mathbb{Z}_2 \times \mathbb{Z}_2)} = S_3. \tag{2.2.1}$$

It follows that the condition of $S_4$ invariance on the S-matrices can be imposed in two steps. In the first step, we impose only $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry on the gauge invariant functions of $\varepsilon_i^\mu$ and $p_i^\mu$ (with the necessary homogeneity properties in $\varepsilon_i^\mu$). We call the S-matrices thus obtained, *quasi-invariant* S-matrices. The coset group $S_3$ acts linearly on the space of quasi-invariant S-matrices. In order to obtain fully $S_4$ invariant S-matrices we must further project the space of quasi-invariant S-matrices down to its $S_3$ invariant subspace. The importance of the notion of quasi-invariant S-matrices lies in the fact that the multiplication of a quasi-invariant S-matrix by a function of $s,t$ leaves it quasi-invariant because $(s,t)$ are themselves individually invariant under $\mathbb{Z}_2 \times \mathbb{Z}_2$. It follows that the space of not-necessarily-polynomial quasi-invariant S-matrices forms a vector space over functions of $(s,t)$. This vector space is finite dimensional.

## 2.3 Local S-matrices and a module structure

Here, we define local S-matrices i.e S-matrices that are polynomial functions of $\varepsilon_i$ and $p_i$. In this section we turn our attention to such S-matrices. We focus, first on quasi-invariant S-matrices, postponing the task of enforcing $S_3$ invariance to later.

### 2.3.1 The local module

It is clear that the set of local S-matrices is closed under multiplication by any polynomial $p(s,t)$ and addition. This structure is reminiscent of the vector space except for one important difference. Polynomials of $(s,t)$ do not form a field but rather only a ring *i.e.* they do not have multiplicative inverse[9]. Consequently, the set of local quasi-invariant S-matrices forms a *module*, over the ring

[7]Recall that a subgroup $H \subset G$ is normal if it obeys the property that for any $h \in H$, $ghg^{-1} \in H$ for all $g \in G$. In other words, the normal subgroup is fixed by the adjoint action of the group.

[8]Here the coset is either by left action or by right action, both cosets are equivalent because subgroup is normal.

[9]This is a very important difference. In a genuine vector space if a vector $a$ is a multiple of a vector $b$ then it is also true that the vector $b$ is a multiple of the vector $a$. In a module, on the other hand, if $a$ equals a ring element times $b$ then it is usually not true that $b$ equals a ring element times $a$. In other words the notion of proportionality is inherently

of polynomials of $(s,t)$ and not a vector space. Viewed as a vector space over the field of complex numbers, the space of local quasi-invariant S-matrices is, of course, infinite dimensional. Viewed as a module, however, this space is 'finitely generated' as we now explain. We pause to introduce some (standard) mathematical terminology.

The elements of the form $r \cdot m$, where $m$ is a given element of the module and $r$ is any element of the ring, are said to form the span of $m$. We call the elements in the span of $m$, the descendants of $m$[10]. Sometimes we denote the descendant of $m$ in a more physical notation $r|m\rangle$. A subset $G = \{g_i\}$ of the module $M$ is said to generate $M$ if the smallest submodule which contains $G$ is $M$ itself. In other words, the union of spans of all descendants of $g_i$ is $M$ itself. A module $M$ is said to be finitely generated if it has a finite generator (i.e. a generator with a finite number of elements). A generator set $G$ is said to generate $M$ freely if the following condition holds,

$$\sum_i r_i \cdot g_i = 0 \qquad \text{iff} \qquad \text{all } r_i = 0. \tag{2.3.1}$$

In other words, every element of $m$ is a unique linear combination of $g_i$ over the ring. A module $M$ is a free module if there exists a $G$ that generates it freely. In this case the generator set $G$ is called the basis of $M$. A free module is the next best thing after a vector space. Understanding its structure is equivalent to understanding its basis elements. When the module is not free, one has to characterize the module by giving its generators and their relations[11].

We can find the generators of the module of local quasi-invariant S-matrices in following way. These S-matrices are obtained from local Lagrangians. We first look for a basis over complex numbers of local quasi-invariant S-matrices of the lowest degree. Next we again look for a basis over complex numbers of local quasi-invariant S-matrices of lowest degree that are not in the span of the previously chosen elements, and so on. This process terminates at a finite degree - intuitively because the gauge invariant field strengths built out of $\epsilon_i$ have a finite number of indices[12] - more on this later. It follows that the module of local quasi-invariant S-matrices is finitely generated. We

hierarchical in a ring. We elaborate on this below

[10]This is non-standard mathematical terminology but being physicists we connect well to the word "descendant". Note that the set of basis vectors of a conformal multiplet can be thought of as a module generated by the primary operator over the ring of polynomials in $P_\mu$. From this point of view, conformal descendants are descendants in our sense. Similarly, the Verma module can be thought of as the module generated by the primary operator over the ring of Virasoro creation operators.

[11]If the relations do not form a free module, then one has to characterize the relation module in the same way and so on. This is called the free resolution of a module.

[12]Once all these indices are contracted with momenta, the remaining momentum indices have to contract with each other yielding powers of $s,t$ and hence belonging to the span of lower degree structures.

will call this module, the local module for short and will label its generators as $E_J(p_i, \varepsilon_i)$ and the generator set as $L$.

As the local S-matrices are constrained by transversality and shift invariance of polarizations $\varepsilon_i$, and as these constraints involve the momenta $p_i$ that in turn define $s$, $t$ and $u$, it is much less clear that this module is freely generated, and, indeed we will find below that this is not always the case.

### 2.3.2 The bare module

As we have already explained in subsubsection 2.1.3, the equations (2.1.11) and (2.1.13) allow us to re-express any photon/graviton quasi-invariant S-matrix as a polynomial of $(\varepsilon_i^{\perp}, \alpha_i)$ that is simultaneously of degree one/two in each of these pairs of variables; the coefficients of this polynomial expressions are functions of $(s,t)$.

We now turn to a crucial point. If we start with a local quasi-invariant S-matrix, it is possible to show that the resulting expression, written as a polynomial of $(\varepsilon_i^{\perp}, \alpha_i)$, using (2.1.11) and (2.1.13), has coefficients that are polynomials (rather than generic functions) of $(s,t)$.

This motivates us to define a new module. We define the parity even part of the module of *bare* quasi-invariant photon/graviton S-matrices, or *bare* module for short, over the ring of polynomials of $(s,t)$, to be the set of parity even (i.e. $\widetilde{\varepsilon}^{D-3}$ independent) rotationally invariant and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant polynomials of $(\varepsilon_i^{\perp}, \alpha_i)$ and $(s,t)$ that are simultaneously of degree one/two in each of the pair of variables $(\varepsilon_i^{\perp}, \alpha_i)$[13]. Any basis of the vector space - over the field of complex numbers - of rotationally invariant polynomials of $(\varepsilon_i^{\perp}, \alpha_i)$ (subjected to the requirement of $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariance and appropriate homogeneity requirements) forms a generating set for this module[14]. Let us denote this generator set as $B$ and its elements as $e_I(\alpha_i, \varepsilon_i^{\perp})$. We sometimes call $e_I$ "index structures". Notice that our generators are all independent of $s$, $t$ and so, in particular, are of zero homogeneity in derivatives. As the variables $\varepsilon_i^{\perp}$ and $\alpha_i$ are completely unconstrained (in the case of photons) or obey only the momentum independent constraint 2.1.15 (in the case of gravitons), it is clear that this choice generates our module finitely and freely. This makes $B$ the basis of the bare module. The key point - made at the beginning of this subsection - is that the local module is a submodule

[13] In the case of the gravitational S-matrix, the variables $(\varepsilon_i^{\perp}, \alpha_i)$ are also constrained to obey (2.1.15).

[14] At the end of subsection (2.2) we had explained that the set of not necessarily local quasi-invariant S-matrices constitute a vector space over the field of functions of $s,t,u$. The generators of the bare module clearly also define a basis for this vector space.

of the free bare module.

The parity odd part of the bare module is defined in odd/even $D$ in a similar manner, to be set of rotationally invariant and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant polynomials of $(\varepsilon_i^{\perp}, \alpha_i)$ , $(s,t)$ and $\widetilde{\varepsilon}^{D-3}$ (resp. $N(\widetilde{\varepsilon}^{D-3})$) that are linear in $\widetilde{\varepsilon}^{D-3}$ (resp. $N(\widetilde{\varepsilon}^{D-3})$) and are also simultaneously of degree one or two (corresponding to photons and gravitons) in each of the pair of variables $(\varepsilon_i^{\perp}, \alpha_i)$. Basis elements are functions of $\widetilde{\varepsilon}^{D-3}$, (resp. $N(\widetilde{\varepsilon}^{D-3})$) $\alpha_i, \varepsilon_i^{\perp}$ only; there is no further dependence on $s$ and $t$. Note that these basis elements are of dimension zero in even $D$ but of dimension 3 in odd $D$.
As explained above, the local module generators are embedded in the bare module generators as follows:

$$E_J(p_i, \varepsilon_i) = \sum_{e_I \in B} p_{IJ}(s,t) e_I(\alpha_i, \varepsilon_i^{\perp}). \tag{2.3.2}$$

where $p_{IJ}(s,t)$ are polynomials[15] In our study of photon/graviton S-matrices later in this paper we encounter two cases. In the first case, (this holds for both photon and graviton scattering when $D \geq 5$), $|L| = |B|$. In the second case, which turns out to apply to both photon and graviton scattering in $D = 4$ and also photon scattering in $D = 3$, $|L| > |B|$. In this subsection we briefly discuss these two cases in turn.

Let us first consider the case $|L| = |B|$. In this case the local module is freely generated if and only if the equation

$$\sum_{E_J \in L} r^J(s,t) E_J(p_i, \varepsilon_i) = 0, \tag{2.3.3}$$

has no non-trivial solutions for polynomials $r^J(s,t)$. Plugging the expansion (2.3.2) into (2.3.3) and equating coefficients of $e_I$ we find that (2.3.3) turns into

$$\sum_J p_{IJ}(s,t) r^J(s,t) = 0. \tag{2.3.4}$$

For each value of $s$ and $t$, (2.3.4) is a set of $|B|$ linear equations for $|B|$ variables. This set of equations has non-trivial solutions if and only if

$$\mathrm{Det}\left[p_{IJ}(s,t)\right] = 0. \tag{2.3.5}$$

Equation (2.3.4) has no solutions unless (2.3.5) holds for every value of $s$ and $t$. Equation (2.3.5) is, of course, an extremely onerous condition, and we find that it is not met for the $p_{IJ}$ matrix that arises in the study of $S$ matrices with $D \geq 5$. It follows, as a consequence, that the local module is

[15]The remarkable fact is that $p_{IJ}(s,t)$ is simply a polynomial without any negative or fractional powers.

also freely generated for $D \geq 5$.

In the case that $|L| > |B|$ (which we encounter for $D = 4$ and also $D = 3$ for photons), on the other hand, it is very easy to see that $|L|$ cannot be freely generated[16]. Given that the local module is not freely generated in $D = 4$, it is important to discover the relations in this module.

## 2.4 Irreducible representations of $S_3$ and fusion rules

As we have explained above, the space of physical (hence $S_4$ invariant) S-matrices is the projection of the local module of S-matrices onto $S_3$ singlets. In this subsection we discuss the nature of this projection. As preparation for our discussion we first review elementary facts about $S_3$ representation theory. The permutation group of three elements is denoted by $S_3$, and has three irreducible representations: the one dimensional totally symmetric representation which we call $\mathbf{1_S}$, the one dimensional totally anti-symmetric representation which we call $\mathbf{1_A}$ and a two dimensional representation with mixed symmetry which we call $\mathbf{2_M}$. The subscript for $\mathbf{2_M}$ emphasizes the mixed symmetry. It is easy to decompose an representation of $S_3$, into the subspaces that transform, respectively, in the $\mathbf{1_S}$, the $\mathbf{1_A}$ and $\mathbf{2_M}$ representations. Complete symmetrization project onto the $\mathbf{1_S}$ subspace, complete anti symmetrization projects onto the $\mathbf{1_A}$ subspace and whatever is left over, *i.e.* the part that is annihilated by both complete symmetrization and complete anti symmetrization, transforms in the $\mathbf{2_M}$ representation.

In order to get some familiarity with these representations, let us first consider a 3 dimensional column vector whose elements are $q_1, q_2, q_3$ respectively. The permutation group has a natural action on this column vector; any given element $\sigma$ of $S_3$ maps this vector to the column with entries $q_{\sigma(1)}, q_{\sigma(2)}, q_{\sigma(3)}$[17]. This linear map is generated by a (unique) $3 \times 3$ matrix $M(\sigma)$ acting on acting on the column $(q_1, q_2, q_3)$. The collection of matrices $M(\sigma)$ yields a representation of $S_3$. We use the symbol $\mathbf{3}$ to denote this 'defining' representation of $S_3$. This representation is *not*

[16]The argument for this is the following. If $L$ were freely generated then the number of local quasi-invariant S-matrices of degree $d$ would grow like $|L|d$ at large $d$. This is larger than the number of bare quasi-invariant S-matrices, which grows like $|B|d$ at large $d$, contradicting the fact that the local module is a submodule of the bare module.

[17]For instance the $\mathbb{Z}_2$ element that flips one and two gives $(q_2, q_1, q_3)$.

irreducible but can be decomposed as[18]

$$\mathbf{3} = \mathbf{2_M} + \mathbf{1_S}. \tag{2.4.1}$$

As a second exercise let us study the 6 dimensional representation, $\mathbf{6}_{\text{left}}$ generated by the left action of $S_3$ onto itself. It is not difficult to demonstrate that

$$\mathbf{6}_{\text{left}} = \mathbf{1_S} + 2 \cdot \mathbf{2_M} + \mathbf{1_A}. \tag{2.4.2}$$

Of course the same decomposition also applies for the $\mathbf{6}_{\text{right}}$ representation generated by the right action of $S_3$ on itself.

As our next example consider the adjoint action of $S_3$ on itself $\sigma \to g^{-1}\sigma g$, which also yields a 6 dimensional representation $\mathbf{6}_{\text{adj}}$. The adjoint representation can be decomposed into the $\mathbf{1_S}$ (which acts on the identity element which is invariant under adjoint action) a $\mathbf{3}$ (which acts on the 3 exchange permutations $(213),(321),(132)$), and a $\mathbf{2_M}$ (which acts on the two cyclical permutations $(231)$ and $(312)$). In equations

$$\mathbf{6}_{\text{adj}} = \mathbf{1_S} + \mathbf{2_M} + \mathbf{3}. \tag{2.4.3}$$

Of course the $\mathbf{3}$ can itself be further decomposed using (2.4.1). As explained before, we obtain the physical S-matrices by projecting the quasi-invariant S-matrices onto $S_3$ singlets. This is explained in the next subsection.

## 2.5 Projecting onto $S_3$ singlets

We now return to a discussion of the local module and its $S_3$ projection. It follows from definitions that every bare and local quasi-invariant S-matrix, denoted as $m(p_i,\varepsilon_i)$ and $M(p_i,\varepsilon_i)$ respectively, can be expressed as

$$\begin{aligned} m(p_i,\varepsilon_i) &= \sum_{e_I \in B} p_I(s,t) e_I(\alpha_i, \varepsilon_i^{\perp}). \\ M(p_i,\varepsilon_i) &= \sum_{E_J \in L} P_J(s,t) E_J(p_i,\varepsilon_i). \end{aligned} \tag{2.5.1}$$

[18] To see why this is the case, note that the complete symmetrization the column $x_1,x_2,x_3$ yields a column whose elements are all equate to $2(x_1+x_2+x_3)$. This column is permutation invariant and so transforms in the one dimensional completely symmetric representation of $S_3$. Removing this column one is left with the action of $S_3$ on a column $(y_1,y_2,y_3)$ whose elements are subject to the constraint $y_1+y_2+y_3=0$, which generates the $\mathbf{2_M}$ representation.

where $p_I(s,t)$ and $P_J(s,t)$ are polynomials of $(s,t)$. The $S_4$ invariant local S-matrices are obtained by simply projecting the elements of the local module onto the trivial representation of $S_3$.

$$\boxed{\mathscr{S}(p_i,\varepsilon_i) = \sum_{\sigma\in S_3} M^{\sigma}(p_i,\varepsilon_i) = \sum_{E_J\in L}\sum_{\sigma\in S_3} P_J^{\sigma}(s,t)E_J^{\sigma}(p_i,\varepsilon_i).} \tag{2.5.2}$$

The superscript $\sigma$ denotes the action of $\sigma$ permutation. As the local module admits the action of $S_3$, its generators $E_J$'s can be decomposed into irreducible representations of $S_3$. Moreover the space of functions of $(s,t)$ can also be decomposed into irreducible representations of $S_3$. It follows from (2.5.2) that if a subset of $E_J$'s transforms in any given irreducible representation $\mathbf{R}$ of $S_3$ then the functions $P_J(s,t)$ must also transform in the same representation $\mathbf{R}$[19].

In order to understand the detailed structure of the projection of the local module onto $S_3$ invariants we need to understand the decomposition of the space of polynomials of $(s,t)$ into representations of $S_3$. This is done in detail in subsection (2.7) and (2.8) of [1].

## 2.6 Regge growth

Recall that the generators of the bare module are zero order in derivatives in the parity even sector, and also in the parity odd sector for even $D$. In these cases the generators are functions of $\alpha_i, \varepsilon_i^{\perp}$ but are not separately functions of $s$, $t$ and $u$. On the other hand when $D$ is odd, the parity odd generators are proportional to $\sqrt{stu}$ times functions of $\alpha_i, \varepsilon_i^{\perp}$. In order to deal uniformly with all cases below, we introduce the variable $a$; $a=0$ for parity even S-matrices in every $D$ and parity odd S-matrices in even $D$. $a=3$ for parity odd S-matrices in odd $D$.

We will now derive a lower bound for the Regge growth for local S-matrices at $2n+a$ order in derivatives. In order to do this we note that every such S-matrix is an $n^{\text{th}}$ order descendant of some bare generator. The generator in question might transform in the $\mathbf{1_S}$, the $\mathbf{1_A}$ or the $\mathbf{2_M}$ representation, or a linear combination of these. We take these cases up in turn.

Consider any bare generator, say $|e_{\mathbf{S}}\rangle$, that transforms in the $\mathbf{1_S}$. The most general $n^{\text{th}}$ level descendant of this generator that itself transforms in the $\mathbf{1_S}$ representation is given, for $n\geq 2$, by

$$\left(\sum_{k,m} a_{k,m}(stu)^k(s^2+t^2+u^2)^m\right)|e_{\mathbf{S}}\rangle \tag{2.6.1}$$

[19]This conclusion follows from the fusion rules of $S_3$ listed in the previous subsection.

where the sum runs over all terms with $3k+2m=n$. It is easy to convince oneself that all the S-matrices in (2.6.1) grow at least as fast as

$$s^{\left(2\left[\frac{n+2}{3}\right]+\frac{a}{3}\right)} \tag{2.6.2}$$

in the Regge limit[20]. where $[m]$ represents the largest integer no smaller than $m$.

Now consider a bare generator $|e_{\mathbf{A}}\rangle$ that transforms in the antisymmetric representation. The most general descendant at $2n+a$ order in derivatives is given by

$$\left(s^2u-u^2s+t^2s-t^2u-s^2t+u^2t\right)\left(\sum_{k,m} a_{k,m}(stu)^k(s^2+t^2+u^2)^m\right)|e_{\mathbf{A}}\rangle \tag{2.6.3}$$

where $3k+2m=n-3$. For $n=3$ and $n\geq 5$ all terms in (2.6.3) grow at least as fast in the Regge limit as[21],

$$s^{\left(2\left[\frac{n-1}{3}\right]+3+\frac{a}{3}\right)}. \tag{2.6.4}$$

Finally consider a bare generator multiplet that transforms in the $\mathbf{2_M}$ representation. Let the triplet of basis vectors

$$(|e^{(1)}_{\mathbf{M}}\rangle, |e^{(2)}_{\mathbf{M}}\rangle, |e^{(3)}_{\mathbf{M}}\rangle)$$

transform in the $\mathbf{2_M}$ representation in the symmetric basis. The most general $2n+a$ derivative descendant of these basis vectors is given either by

$$\left(\sum_{k,m} a_{k,m}(stu)^k(s^2+t^2+u^2)^m\right)\left(s|e^{(1)}_{\mathbf{M}}\rangle+t|e^{(2)}_{\mathbf{M}}\rangle+u|e^{(3)}_{\mathbf{M}}\rangle\right) \tag{2.6.5}$$

with $3k+2m=n-1$ or by

$$\left(\sum_{k,m} a_{k,m}(stu)^k(s^2+t^2+u^2)^m\right)\left(\left(t^2+u^2-2s^2\right)|e^{(1)}_{\mathbf{M}}\rangle+\left(u^2+s^2-2t^2\right)|e^{(2)}_{\mathbf{M}}\rangle+\left(s^2+t^2-2u^2\right)|e^{(3)}_{\mathbf{M}}\rangle\right) \tag{2.6.6}$$

with $3k+2m=n-2$. All the S-matrices in (2.6.5) and (2.6.6) grow at least as fast in the Regge

[20] When $n=3p$, we obtain the slowest growth when $a_{k,m}$ is non-zero only for $k=p$ and $m=0$. When $n=3p+1$ (and $p\geq 1$) the slowest growth is achieved when $a_{k,m}$ is non-zero only when $k=p-1$ and $m=2$. When $n=3p+2$ we get the slowest growth for the monomial with $k=p$ and $m=1$.

[21] When $n=3p$, we obtain the slowest growth when $a_{k,m}$ is non-zero only for $k=p-1$ and $m=0$. When $n=3p+1$ (and $p\geq 2$) the slowest growth is achieved when $a_{k,m}$ is non-zero only when $k=p-2$ and $m=2$. When $n=3p+2$ we get the slowest growth for the monomial with $k=p-1$ and $m=1$.

limit as[22]

$$
\begin{aligned}
& s^{\alpha+\frac{a}{3}} \\
& \alpha = 2p+1 \quad \text{when} \quad n = 3p \\
& \alpha = 2p+1 \quad \text{when} \quad n = 3p+1 \\
& \alpha = 2p+2 \quad \text{when} \quad n = 3p+2
\end{aligned}
\tag{2.6.7}
$$

Combining all the results above we conclude that every local S-matrix at $2n+a$ derivative order grows at least as fast in the Regge limit as

$$
\begin{aligned}
& s^{\alpha(n)+\frac{a}{3}} \\
& \alpha(n) = 2p \qquad \text{when} \quad n = 3p \\
& \alpha(n) = 2p+1 \quad \text{when} \quad n = 3p+1 \\
& \alpha(n) = 2p+2 \quad \text{when} \quad n = 3p+2
\end{aligned}
\tag{2.6.8}
$$

The bound in the first line in the first line in (2.6.8) is saturated by the state $(stu)^p|e_{\mathbf{S}}\rangle$, the bound in the second line is saturated by the state

$$
(stu)^p \left( s|e^{(1)}_{\mathbf{M}}\rangle + t|e^{(2)}_{\mathbf{M}}\rangle + u|e^{(3)}_{\mathbf{M}}\rangle \right)
$$

and the bound in the third line in (2.6.8) is saturated both by $(stu)^p(s^2+t^2+u^2)|e_{\mathbf{S}}\rangle$ and by

$$
(stu)^p \left( \left(t^2+u^2-2s^2\right)|e^{(1)}_{\mathbf{M}}\rangle + \left(u^2+s^2-2t^2\right)|e^{(2)}_{\mathbf{M}}\rangle + \left(s^2+t^2-2u^2\right)|e^{(3)}_{\mathbf{M}}\rangle \right).
$$

As we have mentioned in the introduction, we are particularly interested in local S-matrices that grow no faster than $s^2$ in the Regge limit. We end this section with a complete listing of all module elements that have this feature. For parity even S-matrices - or parity odd S-matrices in even $D$ these possibilities are

- At zero order in derivatives generators of the bare module in the $\mathbf{1_S}$ representation yield S-matrices that grow like $s^0$ in the Regge limit.

[22] When $n = 3p$, we obtain the slowest growth from the term in (2.6.5) with $k = p-1$ and $m = 1$. When $n = 3p+1$ the slowest growth comes from the term in (2.6.5) with $k = p$ and $m = 0$. When $n = 3p+2$ we get the slowest growth for the monomial in (2.6.6) with $k = p$ and $m = 0$.

- At the two derivative level we have S-matrices of the form (2.6.5) with $a_{k,m}$ non-zero only when $k = m = 0$. These S-matrices grow like $s$ in the Regge limit.
- At fourth order in derivatives we have S-matrices of the form (2.6.1) with $a_{k,m} = 0$ unless $k = 0, m = 1$. We also have S-matrices of the form (2.6.6) with $a_{k,m} = 0$ unless $k = m = 0$. These S-matrices grow like $s^2$ in the Regge limit.
- At six derivative order the unique such S-matrix is of the form (2.6.1) with $a_{k,m} = 0$ unless $k = 1$ and $m = 0$.

All other local S-matrices - in particular all S-matrices that are of 8 or higher order in derivatives - necessarily grow faster than $s^2$ in the Regge limit. The parity odd S-matrices in odd $D$, in particular all S-matrices of 7 or higher order in derivatives - grow faster than $s^2$ in the Regge limit. Hence, for the case of parity odd S-matrices in odd $D$, the only Module elements that grow no faster than $s^2$ in the Regge limit are

- At 3 derivative order we have generators of the bare module in the $\mathbf{1_S}$ representation. The corresponding S-matrices grow like $s$ in the Regge limit.
- At the five derivative level we have S-matrices of the form (2.6.5) with $a_{k,m}$ non-zero only when $k = m = 0$. These S-matrices grow like $s^2$ in the Regge limit.

# Chapter 3

# Generators of the bare module

In this chapter we will enumerate and explicitly construct $e_I(\alpha_i, \varepsilon_i^{\perp})$, the basis of the bare module defined in subsection (2.3.2), for four photon scattering in every spacetime dimension. We also keep track of the $S_3$ transformation properties of $e_I$.

## 3.1 Bare module structures for four photon scattering

In this subsection we explicitly construct the basis elements $e_I(\alpha_i, \varepsilon_i^{\perp})$ and thereby obtain their $S_3$ transformation properties for photons in $D \geq 5$. The complete detailed analysis for both photons and gravitons has been done in [1].

### 3.1.1 Enumeration

In this subsection we count the rank of the bare module, *i.e.* the number of linearly independent basis elements $e_I$. As explained in subsection 2.3.2, these are simply the set of $SO(D-3)$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant polynomials of $\alpha_i$ and $\varepsilon_i^{\perp}$ with the appropriate homogeneity properties. We separately enumerate parity odd and parity even generators of the bare module. Under $SO(D-3)$, the effective polarization for photon takes values in the space $\rho = (\mathtt{s} \oplus \mathtt{v})$ (for gravitons, $\rho = (\mathtt{s} \oplus \mathtt{v} \oplus \mathtt{t})$)[1]. Here $\mathtt{s}, \mathtt{v}, \mathtt{t}$ are scalar, vector and symmetric traceless tensor of $SO(D-3)$ respectively.

[1]The photon $S$ matrices are separately linear in each of $(\varepsilon_i^{\perp}, \alpha_i)$. Here $\varepsilon_i^{\perp}$ is the $\mathtt{v}$ while $\alpha_i$ is the $\mathtt{s}$. On the other hand gravitational S-matrices are quadratic separately in each of $(\varepsilon_i^{\perp}, \alpha_i)$; and are evaluated subject to the constraint $\varepsilon_i^{\perp} \cdot \varepsilon_i^{\perp} + \alpha_i^2 = 0$. The terms $\varepsilon_i^{\perp} \varepsilon_i^{\perp}$ is the $\mathtt{t}$ above (this term is effectively traceless as the constraint (2.1.15) allows us to trade its trace for $\alpha_i^2$), the terms $\alpha_i \varepsilon_i^{\perp}$ is the $\mathtt{v}$ and the terms $\alpha_i^2$ are the $\mathtt{s}$.

| photons | even | odd |
|---|---|---|
| $D \geq 8$ | 7 | 0 |
| $D = 7$ | 7 | 1 |
| $D = 6$ | 7 | 1 |
| $D = 5$ | 7 | 0 |
| $D = 4$ | 5 | 2 |
| $D = 3$ | 1 | 1 |

| gravitons | even | odd |
|---|---|---|
| $D \geq 8$ | 29 | 0 |
| $D = 7$ | 29 | 7 |
| $D = 6$ | 28 | 9 |
| $D = 5$ | 22 | 3 |
| $D = 4$ | 5 | 2 |
| $D = 3$ | - | - |

Table 3.1: Number of parity even and parity odd index structures for 4-photon and 4-graviton S-matrix as various dimensions.

The number of index structures is the number of singlets in

$$\rho^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2} : i.e. \quad (\mathtt{s}\oplus\mathtt{v})^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2} \text{ for photons}, \qquad (\mathtt{s}\oplus\mathtt{v}\oplus\mathtt{t})^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2} \text{ for gravitons} \tag{3.1.1}$$

where the notation $|_G$ stands for projection onto $G$ invariants.

In order to perform the necessary enumeration we use the formula

$$\rho^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2} = \rho^4 \ominus 3(S^2\rho \otimes \wedge^2\rho). \tag{3.1.2}$$

where $S^2\rho$ and $\wedge^2\rho$ stand for the symmetric and antisymmetric square of $\rho$ respectively. (3.1.2) was derived and employed in [3] to study a closely related problem, namely the enumeration of inequivalent tensor structures in CFT four point functions. The enumeration is listed in Table 3.1

### 3.1.2 Explicit listing of bare modules

In this subsection we explicitly construct the basis elements $e_I(\alpha_i, \epsilon_i^\perp)$ and thereby obtain their $S_3$ transformation properties. Our construction is motivated by construction of index structure for CFT four point functions in [4].

The parity even and odd generators for photons will be denoted by the letters $e$ and $o$ respectively. We will label the structures with the $S_3$ representation they transform under. For example, a parity even photon generator transforming in $\mathbf{3}$ will be denoted as $e_{\mathbf{3}}$. If there are multiple of them then we also include an arbitrarily assigned serial number in the subscript. We will only need to concern ourselves with $\mathbf{1}_S$, $\mathbf{1}_A$, $\mathbf{3}$ and $\mathbf{6}_{\text{left}} = \mathbf{3} \oplus \mathbf{3_A}$ representation. We will also include the space-time dimension in the superscript when it needs to be emphasized. This helps especially in the case of parity odd structures which crucially depend on space-time dimensions.

Let us first consider the case of parity even structures for photons. The structures will be labelled by the $S_3$ representation they transform under.

In this case $\rho^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2}$ has 7 distinct basis elements (see table 3.1). Keeping in mind that $\rho = \mathtt{s}+\mathtt{v}$, it follows that these 7 structures each have their origin in one (and only one) of the tensor products[2]

$$\mathtt{s}^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2}, \qquad \mathtt{s}^{\otimes 2}\mathtt{v}^{\otimes 2}|_{\mathbb{Z}_2\times\mathbb{Z}_2}, \qquad \mathtt{v}^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2}. \tag{3.1.3}$$

A slight generalization of the enumeration method described in the previous subsection allows us to separately enumerate the basis elements in each of these sectors. We find that there is one element in $\mathtt{s}^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2}$ and three each in $\mathtt{s}^{\otimes 2}\mathtt{v}^{\otimes 2}|_{\mathbb{Z}_2\times\mathbb{Z}_2}$ and $\mathtt{v}^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2}$.

It is easy to explicitly construct these basis elements. Consider, for example, the sector $\mathtt{v}^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2}$. The 3 basis elements in this sector are

$$e^{(1)}_{\mathbf{3},1} = (\varepsilon_1^\perp\cdot\varepsilon_2^\perp)(\varepsilon_3^\perp\cdot\varepsilon_4^\perp), \qquad e^{(2)}_{\mathbf{3},1} = (\varepsilon_3^\perp\cdot\varepsilon_1^\perp)(\varepsilon_2^\perp\cdot\varepsilon_4^\perp) \qquad e^{(3)}_{\mathbf{3},1} = (\varepsilon_2^\perp\cdot\varepsilon_3^\perp)(\varepsilon_1^\perp\cdot\varepsilon_4^\perp) \tag{3.1.4}$$

It is easy to check that these structures are $\mathbb{Z}_2\times\mathbb{Z}_2$ invariant as desired. Also, each of the three elements listed in (3.1.4) happens to be invariant under a single $\mathbb{Z}_2$ exchange transformation[3]; moreover the three elements are mapped to each other by the action of the cyclic elements of $S_3$ [4]. It follows from the discussion in the second last paragraph of subsection 2.4 that these elements transform in the **3** representation of $S_3$ defined in and around (2.4.1). The three structures that lie in the $\mathtt{s}^{\otimes 2}\mathtt{v}^{\otimes 2}|_{\mathbb{Z}_2\times\mathbb{Z}_2}$ sector are algebraically given by:

$$\begin{aligned} e^{(1)}_{\mathbf{3},2} &= \left(\varepsilon_1^\perp.\varepsilon_2^\perp\,\alpha_3\alpha_4 + \varepsilon_3^\perp.\varepsilon_4^\perp\,\alpha_1\alpha_2\right)\\ e^{(2)}_{\mathbf{3},2} &= \left(\varepsilon_1^\perp.\varepsilon_3^\perp\,\alpha_2\alpha_4 + \varepsilon_2^\perp.\varepsilon_4^\perp\,\alpha_1\alpha_3\right)\\ e^{(3)}_{\mathbf{3},2} &= \left(\varepsilon_2^\perp.\varepsilon_3^\perp\,\alpha_1\alpha_4 + \varepsilon_1^\perp.\varepsilon_4^\perp\,\alpha_2\alpha_3\right) \end{aligned} \tag{3.1.5}$$

As in the case of (3.1.4), the expressions in (3.1.5) are each invariant under a single $\mathbb{Z}_2$ exchange

[2]The tensor products $\mathtt{s}^{\otimes 3}\mathtt{v}$ and $\mathtt{v}^{\otimes 3}\mathtt{s}$ do not contribute as they contain no $SO(D-3)$ singlets.

[3]Viewed as elements of $S_3 = S_4/(\mathbb{Z}_2\times\mathbb{Z}_2)$ and working in the 'gauge' in which the fourth particle is fixed, the exchange elements that leave the three structures in (3.1.4) fixed respectively are (213), (321) and (132).

[4]See the discussion under (2.4.3) for a definition and listing of these cyclic elements.

element and are also mapped to each other by the action of $\mathbb{Z}_3$. It thus follows that the expressions in (3.1.5) - like those in (3.1.4) - transform in the **3** representation of $S_3$. Finally the corresponding expression for the single structure in $\mathtt{s}^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2}$ is simply $\alpha_1\alpha_2\alpha_3\alpha_4$, and clearly transforms in the $\mathbf{1_S}$ representation of $S_3$. We denote it as $e_{\mathbf{S}}$. Using (2.4.1) it follows that the seven parity even photon structures for $D \geq 7$ transform under $S_3$ as

$$7 = 3\cdot\mathbf{1_S} + 2\cdot\mathbf{2_M} \tag{3.1.6}$$

We now turn to a discussion of parity odd S-matrices in $D \geq 8$. Such S-matrices are linear in $\widetilde{\varepsilon}^{D-3}$ (see (2.1.16) for a definition). For $D \geq 8$ the number of free indices of $\widetilde{\varepsilon}^{D-3}$ (or $N(\widetilde{\varepsilon})^{D-3}$) tensor is $D-3$ which is $\geq 5$. As the only vectors available to contract with this tensor are the 4 $\varepsilon_i^{\perp}$, there are no parity odd S-matrices in $D \geq 8$. Let us now construct the parity odd structures for photon S-matrix in $D = 7$. The tensor $\widetilde{\varepsilon}^4$ has 4 free indices so it can be contracted with the 4 $\varepsilon_i^{\perp}$ in a unique way.

$$o_{\mathbf{S}}^{D=7} = \widetilde{\varepsilon}^4_{\mu\nu\rho\sigma}\varepsilon_1^{\perp\,\mu}\varepsilon_2^{\perp\,\nu}\varepsilon_3^{\perp\,\rho}\varepsilon_4^{\perp\,\sigma} = \varepsilon_{\alpha\beta\gamma\mu\nu\rho\sigma}\,p_1^{\alpha}\,p_2^{\beta}\,p_3^{\gamma}\varepsilon_1^{\perp\,\mu}\varepsilon_2^{\perp\,\nu}\varepsilon_3^{\perp\,\rho}\varepsilon_4^{\perp\,\sigma}. \tag{3.1.7}$$

Consequently there is a single parity odd structure in $\rho^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2}$ for the case of photons in seven dimensions. This S-matrix transforms in the $\mathbf{1}_S$ representation of $S_3$[5]. In equations

$$1 = 1\cdot\mathbf{1_S} \tag{3.1.8}$$

A complete construction of the bare module structures (both for parity odd as well as parity even sector) in every other dimension is given in [1]. It also has an explicit construction of the bare module structures for gravitons in all spacetime dimensions.

[5] In order to obtain the correct symmetry transformation property of this term it is important to permute the momenta that go into the definition of $\widetilde{\varepsilon}^4$ along with the factors of $\varepsilon_i^{\perp}$. In order to avoid errors it is best to express $\widetilde{\varepsilon}^4$ in term of $\varepsilon$ using (2.1.16) - as has been done on the RHS of (3.1.7) - before performing permutations.

# Chapter 4

# Local Lagrangian's and the local module

In this section we explore the space of inequivalent Lagrangians - and their connection with inequivalent four particle scattering for the case of scalars and gravitons. The case for photons has been discussed in [1]. In section (4.2) of this chapter, we also explain how to relate local Lagrangian's to the local module generators $E_J(p_i, \varepsilon_i)$, as defined in section (2.3.1).

## 4.1 Local Lagrangian's and S-matrices

The set of local four particle S-matrices is, of course, closely related to the set of all local quartic Lagrangians[1]. There is an obvious map from the set of local gauge invariant quartic vertices to the set of local 4 particle S-matrices. This map, however, is many to one. Two Lagrangians generate the same S-matrix if they differ only by total derivatives when evaluated on-shell (we will make this statement completely precise below). The map from equivalence classes of Lagrangians to S-matrices can also be inverted. Given polynomial S-matrix one can construct a local quartic Lagrangian vertex that is invariant under linearized gauge transformations that gives rise to that S-matrix[2]. There exists, in other words, a one to one map from the space of local equivalence classes of Lagrangians and local S-matrices; the classification of local S-matrices is the same as the classification of equivalence classes of local Lagrangians.

[1] $L(x)$ is a local Lagrangian if is is a function only of fields and their derivatives evaluated at $x$, subject to the restriction that the number of derivatives acting on any field is bounded from above (i.e. is finite).

[2] The map from S-matrices to Lagrangians played an important role in [12] for scalars.

### 4.1.1 Scalars

In this subsection we closely follow the analysis of [12]. Consider a theory of real massless scalars $\phi$ invariant under the $\mathbb{Z}_2$ transformations $\phi \to -\phi$. We wish to study the most general local action for this theory, retaining only those terms that affect four scalar scattering. The quadratic term in the Lagrangian is fixed to be:

$$S_2 = -\frac{1}{2}\int d^D x\, \partial_\mu \phi \partial^\mu \phi. \tag{4.1.1}$$

The most general local quartic interaction Lagrangian takes the form

$$S_4 = \int d^D x L_4, \qquad L_4 = \sum a_{m_1,m_2,m_3,m_4} \partial^{m_1}\phi\ \partial^{m_2}\phi\ \partial^{m_3}\phi\ \partial^{m_4}\phi \tag{4.1.2}$$

where the schematic summation in the last line of (4.1.2) runs over both the number of derivatives $m_i$ on the fields $\phi$ as well as the distinct ways of contracting the various derivative indices. A tree diagram computation using the action (4.1.2) yields a 4 scalar S-matrix. Two Lagrangians $L_4$ yield the same S-matrix if

- They differ by a total derivative.
- They can be related to each other by a field redefinition.

Consider a field redefinition of the schematic form

$$\begin{aligned} &\phi \to \phi + \delta\phi \\ &\delta\phi = \left(\sum b_{m_1,m_2,m_3} \partial^{m_1}\phi\ \partial^{m_2}\phi\ \partial^{m_3}\phi\right) \end{aligned} \tag{4.1.3}$$

Up to terms of sextic and higher order that we ignore, the field redefinition (4.1.3) shifts $L_4$ by

$$\delta L_4 = \partial^2\phi \left(\sum b_{m_1,m_2,m_3} \partial^{m_1}\phi\ \partial^{m_2}\phi\ \partial^{m_3}\phi\right) \tag{4.1.4}$$

It follows that the space of quartic terms $L_4$ may be divided up into equivalence classes. Two local quartic terms lie in the same equivalence class if they agree up to a total derivative when we set $\partial^2\phi = 0$[3].

[3]In the introduction to this subsection we mentioned that Lagrangians that are 'on-shell equivalent' generate the same S-matrices. In the context of the scalar theory we study in this subsubsection, the precise meaning of 'on-shell equivalent' is 'obeys the equation $\partial^2\phi = 0$. In the case of the photon/graviton studied in subsequent sub subsections, 'on-shell equivalent' means obeys the (free Maxwell) / (vacuum Einstein) equations respectively.

The map between equivalence classes of $L_4$ and four scalar S-matrices is one to one. To see this it is useful to move to momentum space. Let

$$\begin{aligned} \phi(x) &= \int \frac{d^d k}{(2\pi)^d} e^{ik.x}\, \widetilde{\phi}(k) \\ L_4 &= \int \prod_i \frac{d^d k}{(2\pi)^d} e^{i(\Sigma_j k_j x_j)}\, \widetilde{L}_4(k_1,k_2,k_3,k_4)\, \widetilde{\phi}(k_1)\widetilde{\phi}(k_2)\widetilde{\phi}(k_3)\widetilde{\phi}(k_4) \end{aligned} \tag{4.1.5}$$

It follows from the discussion above that $\widetilde{L}_4^1(k_1,k_2,k_3,k_4)$ and $\widetilde{L}_4^2(k_1,k_2,k_3,k_4)$ lie in the same equivalence class if and only if

$$\widetilde{L}_4^1(k_1,k_2,k_3,k_4) = \widetilde{L}_4^2(k_1,k_2,k_3,k_4) \quad \text{when} \tag{4.1.6}$$

$$\sum_{i=1}^4 k_i = 0, \quad \text{and} \quad k_i^2 = 0,\ i = 1\ldots 4 \tag{4.1.7}$$

But $\widetilde{L}_4(k_1,k_2,k_3,k_4)$, evaluated subject to (4.1.7), is precisely the tree level S-matrix evaluated using the Lagrangian $L$[4]. It follows that the equivalence classes of quartic Lagrangian terms are in fact labelled by their tree level S-matrix. Moreover any polynomial S-matrix $S(k_1,k_2,k_3,k_4)$ (defined on the space of momenta (4.1.7)) can be extended to a polynomial function of unconstrained variables $k_1$, $k_2$, $k_3$ and $k_4$ in many inequivalent ways. Choose any such extension, name it $\widetilde{L}_4(k_1,k_2,k_3,k_4)$. The equation (4.1.5) then may be viewed as a map from polynomial S-matrices to (any particular representative of) the equivalence classes of local Lagrangians. It follows that local 4 scalar S-matrices are in one to one correspondence with the equivalence classes $L_4$ described in this subsection.

### 4.1.2 Gravitons

In order to ensure diffeomorphism invariance, in this section we study gravitational Lagrangians constructed out of the Riemann tensor[5] and work order by order in powers of the Riemann tensor[6]. Before commencing our discussion we pause to define some terminology. Throughout this

[4]Up to a universal normalization factor.

[5]This covers almost all diffeomorphism invariant gravity Lagrangians. The exceptions to this rule are gravitational Chern Simons terms which we ignore in this subsubsection, but whose effects we account for later in this paper.

[6]We define an action to be of $m^{\text{th}}$ order in Riemann tensors if there is no manipulation that allows us to express the same action as an expression of higher orders in Riemann tensors in a local manner. For instance, we count an expression containing $[\nabla_\mu, \nabla_\nu]$ acting on $m$ explicit copies of the Riemann tensor as being of degree $m+1$ as the antisymmetric combination of derivatives can be replaced by a Riemann tensor. An expression that is of $m^{\text{th}}$ order in Riemann tensors does not contribute to $n$ point scattering amplitudes of gravitons for $n < m$. Terms of $m^{\text{th}}$ order typically do contribute to S-matrices for $m$ and higher point S-matrices. There are exceptions to this last rule; it is

subsubsection the symbol $H^{(n)}_{\mu\nu}[R_{\alpha\beta\gamma\delta}]$ will denote the most general local functional that is rank 2 symmetric tensor and that is $n^{\text{th}}$ order in the Riemann tensor, but with arbitrary powers of the metric and arbitrary numbers of symmetrized derivatives[7].

The unique diffeomorphism invariant action that is linear in Riemann tensors is, of course, the Einstein action

$$S_E = \int \sqrt{-g}R. \tag{4.1.8}$$

We can show that field redefinition

$$\delta g_{\mu\nu} = H^{(1)}_{\mu\nu}[R_{\alpha\beta\gamma\delta}] \tag{4.1.9}$$

may be used to cast the most general Lagrangian, quadratic in Riemann tensors, into the form

$$S = S_E + S_{GB} + \int \mathscr{O}(R_{\alpha\beta\gamma\delta})^3, \tag{4.1.10}$$

where,

$$\begin{aligned} S_{GB} &= \int \sqrt{-g}\ \delta^g_{[a}\delta^h_b\delta^i_c\delta^j_{d]}\ R_{ab}{}^{gh}R_{cd}{}^{ij} \\ &\propto \int \sqrt{-g}\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\right). \end{aligned} \tag{4.1.11}$$

and $\mathscr{O}(R_{\alpha\beta\gamma\delta})^3$ denotes all terms that are of cubic or higher order in the Riemann tensor[8]. In other words Einstein-Gauss-Bonnet is the most general action quadratic in the Riemann tensor up to total derivatives or terms that involve explicit factors of $R_{\mu\nu}$ and the Ricci scalar $R$[9].

When evaluated in a spacetime of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{4.1.12}$$

---

sometimes possible for an object to be of $m^{\text{th}}$ order in Riemann tensors but to contribute to S-matrices only at order $m+1$ or higher.

[7] One example of an allowed functional is

$$H^{(1)}_{\mu\nu} = aRg_{\mu\nu} + \nabla^2 Rg_{\mu\nu} + cR_{\mu\nu} + d\nabla^\alpha\nabla\beta R_{\alpha\mu\beta\nu}\dots$$

[8] In four dimensions the Gauss-Bonnet term vanishes identically.

[9] In particular the Einstein equations $R_{\mu\nu} = 0$ tell us that we need only work with Riemann tensor - terms containing Ricci tensor or Ricci scalar are effectively trivial.

it turns out that the Gauss-Bonnet term in (4.1.10) starts out at order $h^3$ (up to total derivatives). It follows, in other words, that - despite the appearance - the Gauss-Bonnet term does not modify the Einstein propagator but does contribute to three point scattering of gravitons. This term is, of course, topological in $D = 4$. We can show that field redefinitions of the form

$$\delta g_{\mu\nu} = H^{(2)}_{\mu\nu}[R_{\alpha\beta\gamma\delta}] \tag{4.1.13}$$

can be used to cast the most general cubic correction to the Einstein-Gauss-Bonnet action into the form

$$S = S_E + S_{GB} + a S^{(1)}_{R^3} + b\chi_6 + \int \sqrt{-g}\left(\mathscr{O}(R_{\alpha\beta\gamma\delta})^4\right) \tag{4.1.14}$$

$$\begin{aligned}
S^{(1)}_{R^3} &= \int \sqrt{-g}\left(R^{pqrs}R_{pq}{}^{tu}R_{rstu} + 2R^{pqrs}R_{p\ r}^{\ t\ u}R_{qtsu}\right)\\
\chi_6 &= \int \sqrt{-g}\left(\frac{1}{8}\,\delta^g_{[a}\delta^h_b\delta^i_c\delta^j_d\delta^k_e\delta^l_{f]}\,R_{ab}{}^{gh}R_{cd}{}^{ij}R_{ef}{}^{kl}\right)\\
&= \int \sqrt{-g}\Big(4R_{ab}{}^{cd}R_{cd}{}^{ef}R_{ef}{}^{ab} - 8R_{a\ b}^{\ c\ d}R_{c\ d}^{\ e\ f}R_{e\ f}^{\ a\ b} - 24R_{abcd}R^{abc}{}_{e}R^{de} + 3R_{abcd}R^{abcd}R\\
&\qquad + 24R_{abcd}R^{ac}R^{bd} + 16R_a{}^bR_b{}^cR_c{}^a - 12R_a{}^bR_b{}^aR + R^3\Big)
\end{aligned} \tag{4.1.15}$$

In other words, Einstein-Gauss-Bonnet corrected by two specific cubic terms is the most general action cubic in Riemann tensors - up to total derivatives and terms that vanish by the Einstein equations. When evaluated on the metric (4.1.12), the term $\chi_6$ starts out at order $h^4_{\mu\nu}$ (up to total derivatives). It follows in particular that this term does not contribute to three graviton scattering. $\chi_6$ is field redefinition equivalent to the simpler looking expression

$$S^{(2)}_{R^3} = \int \sqrt{-g}\left(R^{pqrs}R_{pq}{}^{tu}R_{rstu} - 2R^{pqrs}R_{p\ r}^{\ t\ u}R_{qtsu}\right) \tag{4.1.16}$$

(obtained by setting all terms involving $R_{\mu\nu}$ in $\chi_6$ to zero). The reason that we prefer to use $\chi_6$ rather than (4.1.16) in our action is the following; when evaluated on the configuration (4.1.12), the expression $S^{(2)}_{R^3}$ is of order $h^4_{\mu\nu}$ only on-shell; when evaluated off-shell this expression is of order $h^3_{\mu\nu}$. As a consequence, while the actions $\chi_6$ and $S^{(2)}_{R^3}$ lead to the same polynomial graviton 4 point function, this scattering amplitude has its source purely in contact terms in the case of $\chi_6$, but in the more complicated sum of contact and exchange diagrams (which are polynomial as on-shell 3 point functions vanish) in the case of $S^{(2)}_{R^3}$. Consequently $\chi_6$ is clearly dynamically simpler than $S^{(2)}_{R^3}$, even though it superficially looks more complicated. $\chi_6$ also has other interesting properties; it vanishes identically in less than six dimensions, and is topological in $d = 6$. In fact $\chi_6$ is sometimes called the '6 dimensional Euler density'. It is also the second in the sequence of Lovelock terms (the first is the Gauss-Bonnet term written above).

In contrast to $\chi_6$, $S^{(1)}_{R^3}$ does contribute to the three point functions. In fact the Einstein term, the Gauss-Bonnet term and $S^{(1)}_{R^3}$ each contribute to three graviton scattering. It follows that the most general 3 graviton S-matrix is a linear sum of 3 independent structures. The Einstein action is quadratic in derivatives and leads to a 3 graviton S-matrix proportional to

$$\mathscr{A}^R = (\varepsilon_1.\varepsilon_2\varepsilon_3.p_1 + \varepsilon_1.\varepsilon_3\varepsilon_2.p_3 + \varepsilon_2.\varepsilon_3\varepsilon_1.p_2)^2 \tag{4.1.17}$$

The Gauss-Bonnet action is quartic in derivatives and leads to a 3 graviton S-matrix proportional to[10]

$$\mathscr{A}^{R^2} = (\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 \wedge p_1 \wedge p_2)^2 \tag{4.1.18}$$

The Riemann cube term is sextic in derivatives and leads to a 3 graviton S-matrix proportional to

$$\mathscr{A}^{R^3} = (\mathrm{Tr} F_1 F_2 F_3)^2 \,. \tag{4.1.19}$$

As the 3-graviton S-matrix is non vanishing, 4-graviton S-matrices that follow from the Lagrangian (4.1.14) have contributions from Feynman diagrams with a single graviton exchange. Such exchange diagrams lead to S-matrices that are not polynomial in $s$, $t$ and $u$ but instead have poles. We have explicitly evaluated the 4 graviton S-matrix that follows from the action (4.1.14). We discuss the S-matrices obtained from exchange diagrams in a future chapter. As we have mentioned above, $\chi_6$ does not contribute to 3 graviton scattering, but does contribute (polynomiallally) to four graviton scattering. The four graviton S-matrix that follows from this term is proportional to

$$\begin{aligned} T_1 &= (\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4 \wedge k_1 \wedge k_2 \wedge k_3)^2 \\ &\propto \delta^i_{[a}\delta^j_s\delta^k_d\delta^l_f\delta^m_g\delta^n_h\delta^p_{j]}\, \varepsilon^1_i\varepsilon^2_j\varepsilon^3_k\varepsilon^4_l p^1_m p^2_n p^3_p \varepsilon^{1a}\varepsilon^{2s}\varepsilon^{3d}\varepsilon^{4f}p^{1g}p^{2h}p^{3j}. \end{aligned} \tag{4.1.20}$$

Finally we turn to local Lagrangians of quartic or higher order in Riemann tensors. These terms, of course, do not contribute to 3 graviton scattering, but all give rise four graviton S-matrices that are polynomial in $\varepsilon_i$ and $k_i$. As above field redefinitions of the form

$$\delta g_{\mu\nu} = H^{(3)}_{\mu\nu}[R_{\alpha\beta\gamma\delta}] \tag{4.1.21}$$

[10] As remarked in the earlier footnote, this structure vanishes in $D = 4$ but a parity odd structure appears in its place.

can be used to simplify the most general quartic correction to the Einstein-Gauss-Bonnet-Riemann-cube action. Even up to the simplification afforded by the field redefinitions (4.1.21), however, the most general action that is quartic in Riemann tensors, turns out to be characterized by an infinite (rather than a finite, as was the case at quadratic and cubic order) number of parameters.

In this subsection, we thus demonstrate that the map from Lagrangians to S-matrices continues to be the obvious one. When evaluated on-shell, Lagrangians that differ only by total derivatives *or terms of order* $h^5$ *or higher* yield the same S-matrix[11]. There is also a complication in the reverse map: it is possible for local S-matrices to correspond to Lagrangians that are of lower than quadratic order in the Riemann tensor, as we have already seen in the example of the second Riemann cube term above.

## 4.2 Module generators and Lagrangians

Earlier in this section we presented a detailed discussion of the correspondence between S-matrices and Lagrangians (up to equivalences). Note that the correspondence described so far relates two structures, both of which are $S_4$ invariant. Lagrangians built out of identical bosonic fields are automatically $S_4$ invariant, while $S$ matrices are $S_4$ invariant by construction (see Section 2.2 for a detailed discussion).

In our general discussion about the structure of S-matrices we found it useful to regard $S_4$ invariant $S$ matrices as special members of a larger family of $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant 'quasi-invariant S-matrices (see subsection 2.3). Recall, in particular, that it is the space of quasi-invariant polynomial S-matrices (rather than the space of fully $S_4$ invariant polynomial $S$ matrices) that form a module. The space of physical (i.e. completely $S_4$ invariant) polynomial $S$ matrices is obtained by first enumerating the modules of quasi-invariant S-matrices and then projecting onto the subspace of $S_3$ singlets. As the module structure of local S-matrices plays a key role in their enumeration, it is somewhat unsatisfying to have Lagrangian structures 'dual' only to fully $S_4$ invariant S-matrices. In particular, recall that S-matrix modules are labelled by their generators which, in general, transform in non-trivial representations of $S_3$. In this brief subsection we describe a procedure that allows us to associate Lagrangians with generators of the local module even when the generators in question are not $S_3$ invariant. Any set of generators $M_a$ of the local module (that transform in some representation of $S_3$) is naturally associated with an infinite class of genuine ($S_3$ invariant)

[11]The stipulation about terms of higher order is necessary because the non-linearity of gravity makes it possible for two terms built out of four Riemann tensors, that are distinct even on-shell at the non-linear level, to agree at $\mathscr{O}(h^4)$.

S-matrices $S(M_a)$as follows. $S(M_a)$ is defined as the restriction of the span of $M_a$ to $S_3$ singlets, i.e. restriction to $S_3$ singlets of module elements of the form $r.M_a$ where $r$ is an element of the ring (i.e. is a polynomial of $s$ and $t$). In other words $S(M_a)$ are all the $S_3$ invariant descendants of the generators.

Similarly any Lagrangian $L$ can be associated with an infinite class of Lagrangians $C(L)$ defined as follows. $C(L)$ is defined as the set of Lagrangians obtained by taking derivatives the fields that appear in the Lagrangian and contracting the indices of these derivatives in pairs. We say that a Lagrangian $L$ is associated with the generators $M_a$ if the set of S-matrices obtained from the Lagrangians $C(L)$ coincide with $S(M_a)$. This association allows us to use Lagrangians to label generators (and more generally elements) of the local module. We will use this association in the next section. As an example consider the photon Lagrangian $\mathrm{Tr}(F^2)\mathrm{Tr}(F^2)$. The corresponding generators of the local Module are $\mathrm{Tr}(F_1F_2)\mathrm{Tr}(F_3F_4)$, $\mathrm{Tr}(F_1F_3)\mathrm{Tr}(F_2F_4)$ and $\mathrm{Tr}(F_1F_4)\mathrm{Tr}(F_3F_2)$; this set of generators transforms in the **3** of $S_3$.

For another example consider the photon Lagrangian term $F^{ab}\partial_a F^{\mu\nu}\partial_b F^{\nu\rho}F^{\rho\mu}$. In this case the generators corresponding to the given Lagrangian consist of the single element

$$\frac{1}{4}(F_1^{ab}\partial_a F_2^{\mu\nu}\partial_b F_3^{\nu\rho}F_4^{\rho\mu}+F_2^{ab}\partial_a F_1^{\mu\nu}\partial_b F_4^{\nu\rho}F_3^{\rho\mu}+F_3^{ab}\partial_a F_4^{\mu\nu}\partial_b F_1^{\nu\rho}F_2^{\rho\mu}+F_4^{ab}\partial_a F_3^{\mu\nu}\partial_b F_2^{\nu\rho}F_1^{\rho\mu}). \tag{4.2.1}$$

(4.2.1) had four terms rather than one because no single one of the terms above is $\mathbb{Z}_2\times\mathbb{Z}_2$ invariant. The resultant expression (4.2.1) transforms in the $\mathbf{1}_S$ of $S_3$.

# Chapter 5

# Explicit photon S-matrices and corresponding Lagrangians

The quasi-invariant polynomial S-matrices form a module, called the local module. In this section we completely characterize this module by specifying the generators $E_J(p_i, \varepsilon_i)$ for 4-photon S-matrices. We also present an explicit parameterization of the physical ($S_4$ invariant) S-matrices that are 'descendants' of these generators and thereby present an explicit parameterization of the most general allowed polynomial four photon S-matrix in every dimension. Finally we also present explicit expressions for the Lagrangians from which these S-matrices follow.

Before we dive into the analysis let us spare some time fixing up the notation and convention. In the case of photons, we denote the parity even generators of the local module as $E_{\mathbf{R}}$ and parity odd generators as $O_{\mathbf{R}}$. The subscript $\mathbf{R}$ is either $\mathbf{S}, \mathbf{A}$ or $\mathbf{3}$ denoting its $S_3$ representation $\mathbf{1_S}, \mathbf{1_A}$ or $\mathbf{3}$ respectively. When there are multiple generators transforming in the same representation are present, we assign them serial numbers which are also denoted in the subscript. For example, if there two symmetric parity even generators then they are denoted as $E_{\mathbf{S},1}$ and $E_{\mathbf{S},2}$. In the case when $\mathbf{R} = \mathbf{3}$ or $\mathbf{3_A}$, we use a superscript to denote the specific state of the three dimensional representation. By convention, we always choose $E_{\mathbf{3}}^{(1)}$ (or $O_{\mathbf{3}}^{(1)}$) to be invariant under the swap $1 \leftrightarrow 2$. The second and the third components are obtained by permuting $(234) \to (342)$. This means, the component $(2)$ is invariant under the swap $1 \leftrightarrow 3$ and the component $(3)$ is invariant under the swap $1 \leftrightarrow 4$. In one case we have to deal with the generator transforming in $\mathbf{3}_A$ representation. Recall that this is the representation obtained by acting on a state $(1)$ that is antisymmetric in the exchange of $1 \leftrightarrow 2$ by the cyclic permutation $(234) \to (342)$. Sometimes we also include the space-time dimension in the superscript when it needs to be emphasized, e.g. $E_{\mathbf{3}}^{D=4,(1)}$.

For gravitons, we have the same notation except that the letters $G$ and $H$ are used, instead of $E$ and $O$, to denote the parity even and parity odd local module generators. In all cases, the corresponding bare module generators are denoted by lower-case letters i.e. the parity even and parity odd bare module generators for photons are denoted as $e$ and $o$ respectively and for gravitons they are denoted as $g$ and $h$ respectively. In order to avoid excessive notation we use the photon notation $E, O$ and $e, o$ for scalars as well.

To construct the most general physical (i.e. $S_4$ invariant) S-matrix in the span of a quasi-invariant generator, say $E_{\mathbf{R}}$ we need to take the "inner product" with a general polynomial of $(s,t)$ that transforms in the same representation $\mathbf{R}$. For example, if $\mathbf{R} = \mathbf{S}$,

$$\mathscr{S} = \mathscr{F}^{E_{\mathbf{S}}}(t,u) E_{\mathbf{S}}. \tag{5.0.1}$$

where the function $\mathscr{F}^{E_{\mathbf{S}}}(t,u)$ is totally symmetric under $S_3$. When $\mathbf{R} = \mathbf{A}$,

$$\mathscr{S} = \mathscr{F}^{E_{\mathbf{A}}}(t,u) E_{\mathbf{A}}. \tag{5.0.2}$$

where the function $\mathscr{F}^{E_{\mathbf{A}}}(t,u)$ is totally antisymmetric under $S_3$.

The S-matrix is more involved for quasi-invariant structure that transforms in $\mathbf{3}$ representation of $S_3$. It is given by

$$\mathscr{S} = \mathscr{F}^{E_{\mathbf{3}}}(t,u) E_{\mathbf{3}}^{(1)} + \mathscr{F}^{E_{\mathbf{3}}}(u,s) E_{\mathbf{3}}^{(2)} + \mathscr{F}^{E_{\mathbf{3}}}(s,t) E_{\mathbf{3}}^{(3)}. \tag{5.0.3}$$

where $\mathscr{F}^{E_{\mathbf{3}}}(t,u)$ is a symmetric function in its two arguments (symmetry under the exchange of $t$ and $u$ is the same as the symmetry under the exchange of $1 \leftrightarrow 2$ which matches with the symmetry of $E_{\mathbf{3}}^{(1)}$ and so on). Sometimes, it helps use the shorthand

$$\mathscr{F}^{E_{\mathbf{3}}^{(1)}}(t,u) \equiv \mathscr{F}^{E_{\mathbf{3}}}(t,u), \qquad \mathscr{F}^{E_{\mathbf{3}}^{(2)}}(t,u) \equiv \mathscr{F}^{E_{\mathbf{3}}}(u,s), \qquad \mathscr{F}^{E_{\mathbf{3}}^{(1)}}(t,u) \equiv \mathscr{F}^{E_{\mathbf{3}}}(s,t). \tag{5.0.4}$$

So that the above S-matrix can be written as

$$\mathscr{S} = \sum_{i=1,2,3} \mathscr{F}^{E_{\mathbf{3}}^{(i)}}(t,u) E_{\mathbf{3}}^{(i)}. \tag{5.0.5}$$

The S-matrix corresponding to a generator transforming in $\mathbf{3_A}$ representation is also given by the equation (5.0.3) except that the function $\mathscr{F}$ is antisymmetric rather than symmetric in its two arguments. We will always label the function $\mathscr{F}$ by the quasi-invariant structure that it multiplies.

## 5.1 Scalar polynomial S-matrices and Lagrangians

As scalar S-matrices don't have any index structures, the local module and the bare module are identical (and so, in particular, are freely generated). In $D \geq 4$, they both are generated by a unique generator $E_{\mathbf{S}} = 1$ which is clearly $S_3$ invariant (i.e. transforms in the $\mathbf{1_S}$ representation of $S_3$). The Lagrangian corresponding to this generator is simply $\phi^4$.

### 5.1.1 Module generators and S-matrix partition functions

It is obvious why all four scalar scattering amplitudes in $D \geq 4$ are parity invariant. The reason for this is easy to understand. Four scalar scattering involves only 3 independent vectors (which can be chosen to be any three of the four scattering momenta). It follows that no $D \geq 4$ parity odd S-matrix exists as the number of free indices in the Levi-Civita tensor exceeds the number of independent vectors.

It is clear that the argument of the previous paragraph fails in $D = 3$ however. In this case we have the following parity odd structure which is a second generator of the local Module (the first generator continues to be unity)

$$O_{\mathbf{A}}^{D=3} = \varepsilon_{\mu\nu\rho} k_1^{\mu} k_2^{\nu} k_3^{\rho}. \tag{5.1.1}$$

The generator (5.1.1) is precisely $\widetilde{\varepsilon}$ in (2.1.16) for $D = 3$. The 'Lagrangian' associated with this generator (in the sense of subsection 4.2) is[1],

$$\varepsilon_{\mu\nu\rho} \partial_{\mu}\phi \partial_{\nu}\phi \partial_{\rho}\phi \phi. \tag{5.1.2}$$

For completeness we present a completely explicit parameterization of the most general 4 scalar S-matrix and associated Lagrangians. For $D \geq 4$, there is a unique quasi-invariant generator $E_{\mathbf{S}} = 1$. The general S-matrix is

$$\mathscr{S} = \mathscr{F}^{E_{\mathbf{S}}}(t,u) \tag{5.1.3}$$

where this function is completely symmetric under the exchange of $s,t,u$. Recall that $\mathscr{F}^{E_{\mathbf{S}}}(t,u)$ is

[1]The Lagrangian (5.1.2) vanishes for symmetry reasons; however its 'descendants' (Lagrangians obtained by taking derivatives of the four $\phi$ fields in (5.1.2) and contracting the indices in pairs) do not, in general, vanish. Consequently the Lagrangian (5.1.2) - while trivial as a functional - is non-trivial as the Lagrangian that labels module generators in the sense of subsection 4.2.

a polynomial in $t$ and $u$ and so can be expanded as a finite sum of the form

$$\mathscr{F}^{E_{\mathbf{S}}}(t,u) = \left(\mathscr{F}^{E_{\mathbf{S}}}\right)_{n,m} t^n u^m \tag{5.1.4}$$

It follows from the analysis of subsection 2.6 that the only S-matrices of the form (5.1.3) that grow no faster than $s^2$ in the Regge limit are

$$\mathscr{F}^{E_{\mathbf{S}}}(t,u)|_{<s^2} = a_0 + a_4(s^2+t^2+u^2) + a_6(stu). \tag{5.1.5}$$

The Lagrangian from which the S-matrix (5.1.5) follows is proportional to

$$L^{D\geq 4} = \sum_{m,n} \left(\mathscr{F}^{E_{\mathbf{S}}}\right)_{m,n} 2^{m+n} \left( \prod_{i=1}^{m} \prod_{j=1}^{n} \left(\partial_{\nu_j}\partial_{\mu_i}\phi\right) \phi \partial_{\mu_i}\phi \partial_{\nu_j}\phi \right). \tag{5.1.6}$$

We can extend the above analysis to the case of $D=3$, where the new content would be the parity odd sector. This has been done in detail in [1].

## 5.2 Construction of all parity even photon S-matrices for $D \geq 5$

In this subsection we begin this analysis by presenting an explicit construction of all parity even S-matrices in $D \geq 5$. It can be shown that the most general parity even gauge invariant Lagrangian can be obtained by taking linear combinations of pairs of contracted derivatives on the three 'generator' Lagrangians

$$\mathrm{Tr}(F^2)\mathrm{Tr}(F^2), \quad \mathrm{Tr}(F^4), \quad -F^{ab}\partial_a F^{\mu\nu}\partial_b F^{\nu\rho}F^{\rho\mu} \tag{5.2.1}$$

The generators of the local module dual to these Lagrangians (in the sense of section 4.2) are given by

$$\begin{aligned}
E^{(1)}_{\mathbf{3},1} &= 8\mathrm{Tr}(F_1F_2)\mathrm{Tr}(F_3F_4), \quad E^{(2)}_{\mathbf{3},1} = 8\mathrm{Tr}(F_1F_3)\mathrm{Tr}(F_2F_4), \quad E^{(3)}_{\mathbf{3},1} = 8\mathrm{Tr}(F_1F_4)\mathrm{Tr}(F_3F_2), \\
E^{(1)}_{\mathbf{3},2} &= 8\mathrm{Tr}(F_1F_3F_2F_4), \qquad E^{(2)}_{\mathbf{3},2} = 8\mathrm{Tr}(F_1F_2F_3F_4), \qquad E^{(3)}_{\mathbf{3},2} = 8\mathrm{Tr}(F_1F_3F_4F_2), \\
E_{\mathbf{S}} &\simeq -6F_1^{ab}\partial_a F_2^{\mu\nu}\partial_b F_3^{\nu\rho}F_4^{\rho\mu}|_{\mathbb{Z}_2\times\mathbb{Z}_2} \\
&= 6\left(-F_1^{ab}\partial_a F_2^{\mu\nu}\partial_b F_3^{\nu\rho}F_4^{\rho\mu} - F_2^{ab}\partial_a F_1^{\mu\nu}\partial_b F_4^{\nu\rho}F_3^{\rho\mu} - F_3^{ab}\partial_a F_4^{\mu\nu}\partial_b F_1^{\nu\rho}F_2^{\rho\mu} - F_4^{ab}\partial_a F_3^{\mu\nu}\partial_b F_2^{\nu\rho}F_1^{\rho\mu}\right).
\end{aligned} \tag{5.2.2}$$

Note that there are two four derivative generators in the **3** and one six derivative generators in the $\mathbf{1_S}$ of $S_3$. The second subscript on $E_{\mathbf{3}}$ is simply an arbitrarily assigned serial number. It turns out (as has been shown in [1]) that the above local module generators are freely generated for $D \geq 5$. However, the number of local module structures appear to be more than the counting of bare module structures for $D = 3$ and $D = 4$, which signifies that the local module is not freely generated.Hence, the S-matrix module is not completely specified by their generators (5.2.2); we also need to specify the relations obeyed within the modules generated by these generators. To proceed we express local module generators (5.2.2) in terms of the generators of the bare modules $e_I(\alpha_i, \varepsilon_i^{\perp})$ that were constructed in section 3.1.2. Explicitly, we have:

$$
\begin{aligned}
E^{(1)}_{\mathbf{3},1} &= -8s^2 e^{(1)}_{\mathbf{3},1} + 8s^2 e^{(1)}_{\mathbf{3},2} - 8s^2 e_{\mathbf{S}}, \qquad E^{(2)}_{\mathbf{3},1} = -8t^2 e^{(2)}_{\mathbf{3},1} + 8t^2 e^{(2)}_{\mathbf{3},2} - 8t^2 e_{\mathbf{S}}, \\
E^{(3)}_{\mathbf{3},1} &= -8u^2 e^{(3)}_{\mathbf{3},1} + 8u^2 e^{(3)}_{\mathbf{3},2} - 8u^2 e_{\mathbf{S}}, \\
\\
E^{(1)}_{\mathbf{3},2} &= -2(u^2 e^{(2)}_{\mathbf{3},1} + t^2 e^{(3)}_{\mathbf{3},1}) + 2(u(s-t) e^{(2)}_{\mathbf{3},2} + t(s-u) e^{(3)}_{\mathbf{3},2}) - 2(t^2+u^2) e_{\mathbf{S}}, \\
E^{(2)}_{\mathbf{3},2} &= -2(s^2 e^{(3)}_{\mathbf{3},1} + u^2 e^{(1)}_{\mathbf{3},1}) + 2(s(t-u) e^{(3)}_{\mathbf{3},2} + u(t-s) e^{(1)}_{\mathbf{3},2}) - 2(u^2+s^2) e_{\mathbf{S}}, \\
E^{(3)}_{\mathbf{3},2} &= -2(t^2 e^{(1)}_{\mathbf{3},1} + s^2 e^{(2)}_{\mathbf{3},1}) + 2(t(u-s) e^{(1)}_{\mathbf{3},2} + s(u-t) e^{(2)}_{\mathbf{3},2}) - 2(s^2+t^2) e_{\mathbf{S}}, \\
\\
E_{\mathbf{S}} &= 3\,stu\,(e^{(1)}_{\mathbf{3},2} + e^{(2)}_{\mathbf{3},2} + e^{(3)}_{\mathbf{3},2} - 2e_{\mathbf{S}}).
\end{aligned}
\tag{5.2.3}
$$

For completeness we present an explicit parameterization of the most general parity even S-matrix for four photon scattering in $D \geq 5$ and also of the Lagrangians that generate these S-matrices. The most general S-matrix is parametrized by two $\mathbb{Z}_2$ symmetric functions of $t$ and $u$, and one completely $s$, $t$, $u$ symmetric function. The expression is rather cumbersome, and hence is written explicitly in the next page. However, the essential point is that any parity even four photon local S-matrix in $D \geq 5$ can be written in the form of (5.2.4), and hence is characterized in terms of unique well-defined functions. These functions would be determined by the set of interaction vertices of the theory in which we evaluate the four photon S-matrix.

Explicitly we have:

$$
\begin{aligned}
\mathscr{S}^{D\geq 5}_{\text{even}} = 4\Bigg( & \mathscr{F}^{E_{\mathbf{3},1}}(t,u)\left(p^1_\mu\varepsilon^1_\nu - p^1_\nu\varepsilon^1_\mu\right)\left(p^2_\mu\varepsilon^2_\nu - p^2_\nu\varepsilon^2_\mu\right)\left(p^3_\alpha\varepsilon^3_\beta - p^3_\beta\varepsilon^3_\alpha\right)\left(p^4_\alpha\varepsilon^4_\beta - p^4_\beta\varepsilon^4_\alpha\right) \\
& +\mathscr{F}^{E_{\mathbf{3},1}}(u,s)\left(p^1_\mu\varepsilon^1_\nu - p^1_\nu\varepsilon^1_\mu\right)\left(p^3_\mu\varepsilon^3_\nu - p^3_\nu\varepsilon^3_\mu\right)\left(p^2_\alpha\varepsilon^2_\beta - p^2_\beta\varepsilon^2_\alpha\right)\left(p^4_\alpha\varepsilon^4_\beta - p^4_\beta\varepsilon^4_\alpha\right) \\
& +\mathscr{F}^{E_{\mathbf{3},1}}(s,t)\left(p^1_\mu\varepsilon^1_\nu - p^1_\nu\varepsilon^1_\mu\right)\left(p^4_\mu\varepsilon^4_\nu - p^4_\nu\varepsilon^4_\mu\right)\left(p^3_\alpha\varepsilon^3_\beta - p^3_\beta\varepsilon^3_\alpha\right)\left(p^2_\alpha\varepsilon^2_\beta - p^2_\beta\varepsilon^2_\alpha\right)\Bigg) \\
+4\Bigg( & \mathscr{F}^{E_{\mathbf{3},2}}(t,u)\left(p^1_\mu\varepsilon^1_\nu - p^1_\nu\varepsilon^1_\mu\right)(p^3_\nu\varepsilon^3_\alpha - p^3_\alpha\varepsilon^3_\nu)\left(p^2_\alpha\varepsilon^2_\beta - p^2_\beta\varepsilon^2_\alpha\right)\left(p^4_\beta\varepsilon^4_\mu - p^4_\mu\varepsilon^4_\beta\right) \\
& +\mathscr{F}^{E_{\mathbf{3},2}}(u,s)\left(p^1_\mu\varepsilon^1_\nu - p^1_\nu\varepsilon^1_\mu\right)(p^2_\nu\varepsilon^2_\alpha - p^2_\alpha\varepsilon^2_\nu)\left(p^3_\alpha\varepsilon^3_\beta - p^3_\beta\varepsilon^3_\alpha\right)\left(p^4_\beta\varepsilon^4_\mu - p^4_\mu\varepsilon^4_\beta\right) \\
& +\mathscr{F}^{E_{\mathbf{3},2}}(s,t)\left(p^1_\mu\varepsilon^1_\nu - p^1_\nu\varepsilon^1_\mu\right)(p^3_\nu\varepsilon^3_\alpha - p^3_\alpha\varepsilon^3_\nu)\left(p^4_\alpha\varepsilon^4_\beta - p^4_\beta\varepsilon^4_\alpha\right)\left(p^2_\beta\varepsilon^2_\mu - p^2_\mu\varepsilon^2_\beta\right)\Bigg) \\
& +\mathscr{F}^{E_{\mathbf{S}}}(t,u) \\
& \Big((p^1_a\varepsilon^1_b - p^1_b\varepsilon^1_a)\,p^2_a\left(p^2_\mu\varepsilon^2_\nu - p^2_\nu\varepsilon^2_\mu\right)p^3_b\,(p^3_\nu\varepsilon^3_\alpha - p^3_\alpha\varepsilon^3_\nu)\left(p^4_\alpha\varepsilon^4_\mu - p^4_\mu\varepsilon^4_\alpha\right) \\
& +(p^2_a\varepsilon^2_b - p^2_b\varepsilon^2_a)\,p^1_a\left(p^1_\mu\varepsilon^1_\nu - p^1_\nu\varepsilon^1_\mu\right)p^4_b\,(p^4_\nu\varepsilon^4_\alpha - p^4_\alpha\varepsilon^4_\nu)\left(p^3_\alpha\varepsilon^3_\mu - p^3_\mu\varepsilon^3_\alpha\right) \\
& +(p^3_a\varepsilon^3_b - p^3_b\varepsilon^3_a)\,p^4_a\left(p^4_\mu\varepsilon^4_\nu - p^4_\nu\varepsilon^4_\mu\right)p^1_b\,(p^1_\nu\varepsilon^1_\alpha - p^1_\alpha\varepsilon^1_\nu)\left(p^2_\alpha\varepsilon^2_\mu - p^2_\mu\varepsilon^2_\alpha\right) \\
& +(p^4_a\varepsilon^4_b - p^4_b\varepsilon^4_a)\,p^3_a\left(p^3_\mu\varepsilon^3_\nu - p^3_\nu\varepsilon^3_\mu\right)p^2_b\,(p^2_\nu\varepsilon^2_\alpha - p^2_\alpha\varepsilon^2_\nu)\left(p^1_\alpha\varepsilon^1_\mu - p^1_\mu\varepsilon^1_\alpha\right)\Big)\,.
\end{aligned} \tag{5.2.4}
$$

The functions $\mathscr{F}^{E_{\mathbf{3},1}}(t,u), \mathscr{F}^{E_{\mathbf{3},2}}(t,u)$ are each arbitrary functions that are symmetric in their two arguments. These functions with permuted arguments transform in the $\mathbf{3}$ of $S_3$. On the other hand $\mathscr{F}^{E_{\mathbf{S}}}(t,u)$ is a function that is completely symmetric under interchange of $s$, $t$ and $u$. Since, we are eventually interested in the Regge limit of S-matrices, and constrain the space of consistent theories by the CRG conjecture, we can see that the most general S-matrix of the form (5.2.4) that grows no faster than $s^2$ in the Regge limit is given by the four parameter set

$$
\mathscr{F}^{E_{\mathbf{3},1}}(t,u) = c_1, \quad \mathscr{F}^{E_{\mathbf{3},2}}(t,u) = c_2 + c_3(u+t), \quad \mathscr{F}^{E_{\mathbf{S}}}(t,u) = c_4. \tag{5.2.5}
$$

The S-matrices parameterized by $c_1$ and $c_2$ are both four derivative. The S-matrices parameterized by $c_3$ and $c_4$ are both 6 derivatives. All 4 S-matrices corresponding to $c_i$, $i = 1,2,3,4$ grow like $s^2$ in the Regge limit.

The three functions $\mathscr{F}^{E_{\mathbf{3},1}}(t,u), \mathscr{F}^{E_{\mathbf{3},2}}(t,u)$ and $\mathscr{F}^{E_{\mathbf{S}}}(t,u)$ in (5.2.4) can be Taylor expanded in a manner completely analogous to (5.1.4). The Lagrangian that generates the S-matrix (5.2.4) is given by

$$
\begin{aligned}
L_{\text{even}}^{D\geq 5} = & \sum_{m,n} \left(\mathscr{F}^{E_{\mathbf{3},1}}\right)_{m,n} 2^{m+n} \left( \prod_{i=1}^{m} \prod_{j=1}^{n} \mathrm{Tr}\left(\partial_{\nu_j}\partial_{\mu_i} F F\right) \mathrm{Tr}\left(\partial_{\mu_i} F \partial_{\nu_j} F\right) \right) \\
& + \sum_{m,n} \left(\mathscr{F}^{E_{\mathbf{3},2}}\right)_{m,n} 2^{m+n} \left( \prod_{i=1}^{m} \prod_{j=1}^{n} \mathrm{Tr}\left(\partial_{\nu_j}\partial_{\mu_i} F \partial_{\mu_i} F F \partial_{\nu_j} F\right) \right) \\
& + \sum_{m,n} \left(\mathscr{F}^{E_{\mathbf{S}}}\right)_{m,n} 2^{m+n} \left( - \prod_{i=1}^{m} \prod_{j=1}^{n} \partial_{\mu_i}\partial_{\nu_j} F_{ab} \mathrm{Tr}\left(\partial_{\mu_i}\partial_a F \partial_{\nu_j}\partial_b F F\right) \right).
\end{aligned}
\tag{5.2.6}
$$

As mentioned above (5.2.4) and (5.2.6) describe the most general polynomial S-matrix (and corresponding local Lagrangian) for parity even four photon scattering in dimensions $D \geq 5$. In these dimensions the three functions label polynomial S-matrices in a one to one manner; every distinct choice of these functions yields a distinct S-matrix, and every polynomial S-matrix corresponds to some choice of these functions.

In fact the expressions (5.2.4) and (5.2.6) also apply to $D = 4$ and $D = 3$. In this case, however, the map between the functions $\mathscr{F}^{E_{\mathbf{3},1}}(t,u), \mathscr{F}^{E_{\mathbf{3},2}}(t,u)$ and $\mathscr{F}^{E_{\mathbf{S}}}(t,u)$ and polynomial S-matrices is many to one. While every S-matrix continues to correspond to some choice of these three functions, many different choices of these functions yield the same local S-matrix (this is another way of saying that the parity odd local S-matrix module in these dimensions is not freely generated but has relations). The situation for $D = 4$ and $D = 3$ has been explained in further detail in [1].

We now turn to a brief discussion of parity odd S-matrices, i.e. S-matrices that use a single copy of the Levi-Civita tensor. As this tensor has a different numbers of indices in different dimensions, the structure of the parity odd local module tends to be very specific to dimension. However there is one universal statement about parity odd S-matrices that is easy to make, namely that no such S-matrices exist for $D \geq 8$. This simple fact follows from the observation that in these the Levi-Civita tensor has 8 or more indices but only 7 independent vectors - three momenta and four polarizations - for these indices to contract with. It follows that all four photon S-matrices are parity even in $D \geq 8$ (this fact is also clear from Table 3.1). However, there are parity odd S-matrices in lower spacetime dimensions, and a complete classification of the S-matrices and the local Lagrangians from which they follow have been done in [1].

# Chapter 6

# Explicit graviton S-matrices and corresponding Lagrangians

We now turn to a study of Gravitational S-matrices and corresponding Lagrangians for the case of $D \geq 8$. As in the case of photon scattering, there are no parity odd gravitational S-matrices for $D \geq 8$. In the rest of this section we will provide a detailed description of the (automatically parity even) local S-matrix module in $D \geq 8$. A complete listing of the parity odd as well parity even S-matrices and their corresponding local Lagrangian's for lower spacetime dimensions is done exhaustively in [1].

## 6.1 Modules generated by Lagrangians with 8 or fewer derivatives

No gravitational Lagrangian that is linear or quadratic in $R_{\mu\nu\alpha\beta}$ produces a polynomial 4 graviton S-matrix (see subsubsection 4.1.2). $G_{\mathbf{S},1} \equiv \chi_6$ is the unique 3 Riemann Lagrangian that produces a polynomial S-matrix (see (4.1.15) and (4.1.20)). All other parity even Lagrangians that generate polynomial S-matrices can be written as the sum of products of derivatives of four Riemann tensors.

The simplest four Riemann Lagrangians are those with eight derivatives. These are constructed from contractions of four Riemann tensors (no derivatives). All inequivalent contractions of four Riemann tensors have been enumerated in [5]. Excluding those structures that involve $R$ and $R_{\mu\nu}$ and so can be removed by field redefinitions (see subsubsection 4.1.2), the authors of [5] find 7 inequivalent contractions in $D \geq 8$.

Five of the seven transform as $\mathbf{3}$ and are labeled as $G_{\mathbf{3},i}$, $i = 1,\ldots,5$. One generator transforms as $\mathbf{6}_{\text{left}}$, it is labeled as $G_{\mathbf{6}}$. It is convenient to decompose $\mathbf{6}_{\text{left}}$ into $\mathbf{3} \oplus \mathbf{3}_{\mathbf{A}}$. We label these pieces as $G_{\mathbf{3},6}$ and $G_{\mathbf{3}_{\mathbf{A}}}$. All these are listed in (6.2.4) and the associated Lagrangians are in (6.2.5). The remaining 8-derivative generator $G_{\mathbf{3},9}$ transforms in the $\mathbf{3}$. The Lagrangian associated to it is,

$$G_{\mathbf{3},9} = R_{pqrs}R_{ptru}R_{tvqw}R_{uvsw}. \tag{6.1.1}$$

Finally there is one additional subtlety that needs to be taken into account. Each of the seven generators $G_{\mathbf{3},1}\ldots G_{\mathbf{3},6}$ and $G_{\mathbf{3},9}$ have a single generator in the $\mathbf{1}_{\mathbf{S}}$ [1]. One linear combination of these seven $\mathbf{1}_{\mathbf{S}}$ structures is simply the third Lovelock term.

$$\begin{aligned}\chi_8 &= \varepsilon^{abcdefgh}\varepsilon^{\alpha\beta\gamma\delta\mu\nu\rho\sigma}R_{ab\alpha\beta}R_{cd\gamma\delta}R_{ef\mu\nu}R_{gh\rho\sigma} \\ &\propto \left(G_{\mathbf{3},1} + 2G_{\mathbf{3},2} + 16G_{\mathbf{3},3} + 32G_{\mathbf{3},4} + 8G_{\mathbf{3},5} - 16G_{\mathbf{3},6} - 64G_{\mathbf{3},9}\right)|_{\mathbf{S}}.\end{aligned} \tag{6.1.2}$$

When expanded to fourth order in fluctuations, $\chi_8$ and all its 'descendants' simply vanish on-shell. It follows that $\chi_8$ corresponds to no module element and plays no role in the discussion that follows. When studying S-matrices, therefore, one of the seven $\mathbf{1}_{\mathbf{S}}$ structures above - lets say the $\mathbf{1}_{\mathbf{S}}$ in $G_{\mathbf{3},9}$ - can be re-expressed as a linear combination of the other six, and so is not an independent module generator. As we remove the completely symmetric part from $G_{\mathbf{3},9}$, let us relabel it as $G_{\mathbf{2}_{\mathbf{M}}}$ to reflect its correct transformation properties. In the rest of this subsubsection we focus on the submodule - lets call it $M_8$ - of the local gravitational module that is generated by Lagrangians with at most 8 derivatives, i.e. the (independent terms in) descendants of $G_{\mathbf{S},1}$ plus $G_{\mathbf{3},1}\ldots G_{\mathbf{3},6}$, $G_{\mathbf{3}_{\mathbf{A}}}$ and $G_{\mathbf{2}_{\mathbf{M}}}$[2]. It turns out that the submodule of interest to this subsubsection is freely generated (the same holds true for the full local module). This statement - which is simply an unproved assertion at this stage - will effectively be demonstrated later in this subsection by comparison of the 'module' and plethystic partition functions.

One simple description of the sub module $M_8$ goes as follows. $G_{\mathbf{S},1}$ is clearly a generator of our submodule. The module generated by this 6 derivative element has exactly two 8 derivative 'descendants' which together transform in a single copy of in the $\mathbf{2}_{\mathbf{M}}$. This 8 derivative $\mathbf{2}_{\mathbf{M}}$ is a linear combination of the 8 eight derivative $\mathbf{2}_{\mathbf{M}}$'s present in the generators dual to the eight

[1] In each case this generator is simply the S-matrix that follows constructed from tree diagrams using the Lagrangians 'dual' the module elements above - i.e. the Lagrangians listed in [5].

[2] In the next subsubsection we will continue to describe the rest of the local module (the part of the module generated by terms with 10 or more derivatives).

derivative generators $G_{\mathbf{3},1}\dots G_{\mathbf{3},6}$, $G_{\mathbf{3_A}}$ and $G_{\mathbf{2_M}}$. The relation can be expressed as follows. Let us define $(r^{(1)},r^{(2)},r^{(3)})=(s,t,u)$. Then, for example[3],

$$r^{(i)}\,G_{\mathbf{S},1}=4\Big(-G^{(i)}_{\mathbf{3},1}-2G^{(i)}_{\mathbf{3},2}-16G^{(i)}_{\mathbf{3},3}+16G^{(i)}_{\mathbf{3},4}-2G^{(i)}_{\mathbf{3},5}+10G^{(i)}_{\mathbf{3},6}+16G^{(i)}_{\mathbf{2_M}}+(4G^{(i+1)}_{\mathbf{3_A}}-4G^{(i+2)}_{\mathbf{3_A}})\Big). \tag{6.1.3}$$

where $(i+1)$ and $(i+2)$ are defined cyclically; - for instance when $i=2$, $(i+1)=(3)$ and $(i+2)=(1)$. This means, the LHS of (6.1.3) (i.e. the 8 derivative descendant of $G_{\mathbf{S},1}$) together with the generators $G_{\mathbf{3},1}\dots G_{\mathbf{3},6}$ and $G_{\mathbf{3_A}}$ span the space of 8 derivative module elements. Note that $G_{\mathbf{2_M}}$ does not appear in this list of generators; we have (6.1.2) to eliminate the $\mathbf{1_S}$ part of $G_{\mathbf{3},9}$ and have used (6.1.3) to eliminate the $\mathbf{2_M}$ part of this generator.

## 6.2 The rest of the gravitational local submodule

In the previous section we have constructed the submodule of the local gravitational module that is generated by 6 and 8 derivative terms. As we have already accounted for the contribution of $G_{\mathbf{S},1}$, all remaining polynomial S-matrices are produced by Lagrangians quartic in the Riemann tensor. In order to capture the contribution of such terms to four graviton S-matrices, it is sufficient to linearize each of the four Riemann tensors and also to work on-shell:

$$R_{\mu\nu\rho\sigma}\propto(p_\mu\varepsilon_\nu-p_\nu\varepsilon_\mu)(p_\rho\varepsilon_\sigma-p_\sigma\varepsilon_\rho)\propto F_{\mu\nu}F_{\rho\sigma} \tag{6.2.1}$$

Note that the RHS of (6.2.1) is quadratic in $\varepsilon$ as expected. At fixed momentum the Riemann tensor is - formally- the second symmetric power of field strengths

$$R_{\mu\nu\rho\sigma}(p)=\frac{1}{2}F_{\mu\nu}(p)\otimes F_{\rho\sigma}(p). \tag{6.2.2}$$

One simple (but not necessarily exhaustive) way to construct polynomial graviton S-matrices is to take the second symmetric tensor power of photon S-matrices. The set of all polynomial gravitational S- matrices that can be constructed in this manner clearly form a submodule of the complete local gravitational module. The symmetric products of the three generators of the local photon module[4] are a special set of states within this submodule. The products of generators may be

[3] Each of $G_{\mathbf{3},1}\dots G_{\mathbf{3},5}$ transforms as $\mathbf{1_S}$ + $\mathbf{2_M}$, $G_{\mathbf{6}}$ transforms as $\mathbf{1_S}$ + $\mathbf{1_A}$ + $2\cdot\mathbf{2_M}$ and $G_{\mathbf{2_M}}$ transforms as $\mathbf{2_M}$. It follows that there are a total of 8 $\mathbf{2_M}$'s.

[4] Recall that the module of parity even polynomial photon S-matrices in $D\geq 5$ was generated by two four derivative generators $E_1$ and $E_2$ (both of which transform in the $\mathbf{3}$)and one six derivative generator $E_3$ (which transforms in the $\mathbf{1_S}$)

decomposed into familiar representations of $S_3$ as follows

$$
\begin{aligned}
&\frac{1}{16}S^2E_{\mathbf{3},1} = G_{\mathbf{3},1}\oplus G_{\mathbf{3},2}, \quad \frac{1}{16}S^2E_{\mathbf{3},2} = G_{\mathbf{3},3}\oplus G_{\mathbf{3},4},\\
&\frac{1}{16}E_{(\mathbf{3},1}\otimes E_{\mathbf{3},2)} = G_{\mathbf{3},5}\oplus G_{\mathbf{6}} = G_{\mathbf{3},5}\oplus G_{\mathbf{3},6}\oplus G_{\mathbf{3_A}},\\
&\frac{1}{16}E_{(\mathbf{3},1}\otimes E_{\mathbf{S})} = G_{\mathbf{3},7}, \quad \frac{1}{16}E_{(\mathbf{3},2}\otimes E_{\mathbf{S})} = G_{\mathbf{3},8}, \quad \frac{1}{16}S^2E_{\mathbf{S}} = G_{\mathbf{S},2}.
\end{aligned} \tag{6.2.3}
$$

where $S^2$ represents the symmetric square of an $S_3$ representation. The new (with 10 or higher number of derivatives) generators are $G_{\mathbf{3},7}, G_{\mathbf{3},8}$ and $G_{\mathbf{S},2}$. They are labeled by their $S_3$ transformation properties as per the convention.More explicitly the generators so obtained are given by

$$
\begin{aligned}
G^{(1)}_{\mathbf{3},1} &\equiv \frac{1}{16}E^{(1)}_{\mathbf{3},1}\otimes E^{(1)}_{\mathbf{3},1} = R^1_{abpq}R^2_{abpq}R^3_{cdrs}R^4_{cdrs}\\
G^{(1)}_{\mathbf{3},2} &\equiv \frac{1}{16}E^{(1)}_{\mathbf{3},1}\otimes E^{(2)}_{\mathbf{3},1}|_S = R^1_{abpq}R^3_{abrs}R^4_{cdpq}R^2_{cdrs} + R^1_{abpq}R^4_{abrs}R^3_{cdpq}R^2_{cdrs}\\
G^{(1)}_{\mathbf{3},3} &\equiv \frac{1}{16}E^{(1)}_{\mathbf{3},2}\otimes E^{(1)}_{\mathbf{3},2} = R^1_{abpq}R^2_{cdrs}R^3_{bcqr}R^4_{dasp}\\
G^{(1)}_{\mathbf{3},4} &\equiv \frac{1}{16}E^{(1)}_{\mathbf{3},2}\otimes E^{(2)}_{\mathbf{3},2}|_S = R^1_{abpq}R^3_{cdqr}R^4_{bcrs}R^2_{dasp} + R^1_{abpq}R^4_{cdqr}R^3_{bcrs}R^2_{dasp}\\
G^{(1)}_{\mathbf{3},5} &\equiv \frac{1}{16}E^{(1)}_{\mathbf{3},1}\otimes E^{(1)}_{\mathbf{3},2} = R^1_{abpq}R^2_{abrs}R^3_{cdqr}R^4_{cdsp}\\
G^{(1)}_{\mathbf{3},6} &\equiv \frac{1}{16}E^{(1)}_{\mathbf{3},1}\otimes E^{(2)}_{\mathbf{3},2}|_S = R^1_{abpq}R^2_{abqr}R^3_{cdrs}R^4_{cdsp} + R^1_{abpq}R^2_{abqr}R^4_{cdrs}R^3_{cdsp}\\
G^{(1)}_{\mathbf{3_A}} &\equiv \frac{1}{16}E^{(1)}_{\mathbf{3},1}\otimes E^{(2)}_{\mathbf{3},2}|_A = R^1_{abpq}R^2_{abqr}R^3_{cdrs}R^4_{cdsp} - R^1_{abpq}R^2_{abqr}R^4_{cdrs}R^3_{cdsp}\\
G^{(1)}_{\mathbf{3},7} &\equiv \frac{1}{16}E^{(1)}_{\mathbf{3},1}\otimes E_{\mathbf{S}}|_{\mathbb{Z}_2\otimes\mathbb{Z}_2} = R^1_{abpq}\partial_p R^2_{abrs}\partial_q R^3_{cdst}R^4_{cdtr}|_{\mathbb{Z}_2\otimes\mathbb{Z}_2}\\
G^{(1)}_{\mathbf{3},8} &\equiv \frac{1}{16}E^{(1)}_{\mathbf{3},2}\otimes E_{\mathbf{S}}|_{\mathbb{Z}_2\otimes\mathbb{Z}_2} = R^1_{abpq}\partial_p R^2_{cdrs}\partial_q R^3_{bcst}R^4_{datr}|_{\mathbb{Z}_2\otimes\mathbb{Z}_2}\\
G_{\mathbf{S},2} &\equiv \frac{1}{16}E_{\mathbf{S}}|_{\mathbb{Z}_2\otimes\mathbb{Z}_2}\otimes E_{\mathbf{S}}|_{\mathbb{Z}_2\otimes\mathbb{Z}_2} = R^1_{abpq}\partial_a\partial_p R^2_{cdrs}\partial_b\partial_q R^3_{dest}R^4_{eatr}|_{\mathbb{Z}_2\otimes\mathbb{Z}_2}.
\end{aligned} \tag{6.2.4}
$$

In (6.2.4) we have explicitly only listed the $(1)$ components of the generators that transform in $\mathbf{3}$ and in one case in $\mathbf{3_A}$.

The Lagrangians corresponding to all these generators are given by

$$
\begin{aligned}
G_{\mathbf{3},1} &= R_{abpq}R_{baqp}R_{cdrs}R_{dcsr} \\
G_{\mathbf{3},2} &= R_{pqrs}R_{pqtu}R_{tuvw}R_{rsvw} \\
G_{\mathbf{3},3} &= R_{pqrs}R_{ptru}R_{tvuw}R_{qvsw} \\
G_{\mathbf{3},4} &= -R_{pqrs}R_{ptuw}R_{tvws}R_{qvru} \\
G_{\mathbf{3},5} &= R_{pqrs}R_{pqtu}R_{rtvw}R_{suvw} \\
G_{\mathbf{6}} = G_{\mathbf{3},6} \oplus G_{\mathbf{3}_{\mathbf{A}}} &= R_{pqrs}R_{pqrt}R_{uvwt}R_{uvws} \\
G_{\mathbf{3},7} &= R_{pqab}\partial_a R_{qp\mu\nu}\partial_b R_{rs\nu\alpha}R_{sr\alpha\mu} \\
G_{\mathbf{3},8} &= R_{pqab}\partial_a R_{qr\mu\nu}\partial_b R_{rs\nu\alpha}R_{sp\alpha\mu} \\
G_{\mathbf{S},2} &= R_{abpq}\partial_p\partial_a R_{\mu\nu\beta\gamma}\partial_q\partial_b R_{\nu\alpha\gamma\delta}R_{\alpha\mu\delta\beta}.
\end{aligned}
\tag{6.2.5}
$$

Using `Mathematica` we have verified that this result has the following extension to the full local gravitational module for $D \geq 7$. *The set of module elements $G_{\mathbf{S},1}, G_{\mathbf{S},2}$ and $G_{\mathbf{3},1} \ldots G_{\mathbf{3},8}$ and $G_{\mathbf{3}_{\mathbf{A}}}$ are all independent generators of the the parity even part of the local gravitational module. In other words no one of these objects can be written as a linear sum over descendants of the others.* It will turn out - and we will proceed under the assumption that - the list of generators described above is exhaustive; i.e. that $G_{\mathbf{S},1}, G_{\mathbf{S},2}$ and $G_{\mathbf{3},1} \ldots G_{\mathbf{3},8}$ and $G_{\mathbf{3}_{\mathbf{A}}}$ generate the local gravitational module. This statement can be taken to be a guess at this stage, which will be verified by comparison with explicit results of plethystic counting below (see [1] for details). As the number of local generators matches the number of bare generators, it is of importance to know whether the stringent condition (2.3.5) is obeyed. It turns out it is not. It follows that $G_{\mathbf{S},1}, G_{\mathbf{S},2}$ and $G_{\mathbf{3},1} \ldots G_{\mathbf{3},8}$ and $G_{\mathbf{3}_{\mathbf{A}}}$ generate the local gravitational module freely.

## 6.3 Explicit listing of gravitational S-matrices in $D \geq 8$

When $D \geq 8$, the four gravity scattering is necessarily parity even. In these dimensions the generators of the local S-matrix module are $G_{\mathbf{S},1} = \chi_6$ (see (4.1.15) and (4.1.20)) along with $G_{\mathbf{3},1}, \ldots G_{\mathbf{3},8}$, $G_{\mathbf{3}_{\mathbf{A}}}$ and $G_{\mathbf{S},2}$ (see (6.2.4)). It is relatively straightforward to list the $S$ matrices generated by $G_{\mathbf{3},1}, \ldots G_{\mathbf{3},8}$, $G_{\mathbf{3}_{\mathbf{A}}}$, $G_{\mathbf{S},2}$ as well as the Lagrangian's that generate these S-matrices. It is also straightforward to list the S-matrices generated by $G_{\mathbf{S},1}$. In this subsection we first perform the simple part of our listing. We list the S-matrices generated by $G_{\mathbf{3},1}, \ldots G_{\mathbf{3},8}$, $G_{\mathbf{3}_{\mathbf{A}}}$, $G_{\mathbf{S},2}$.

- The S-matrix corresponding to $G_{\mathbf{3},1}$ in (6.2.5) is specified by the polynomial $\mathscr{F}^{G_{3,1}}(t,u)$

which exhibits a $\mathbb{Z}_2$ symmetry ($t \leftrightarrow u$). In equations,

$$S^{G_{\mathbf{3},1}} = \frac{1}{4}\big(\mathscr{F}^{G_{\mathbf{3},1}}(t,u)\,\big[\big(p^1_p\varepsilon^1_q - p^1_q\varepsilon^1_p\big)\,\big(p^2_p\varepsilon^2_q - p^2_q\varepsilon^2_p\big)\,\big(p^3_r\varepsilon^3_s - p^3_s\varepsilon^3_r\big)\,\big(p^4_r\varepsilon^4_s - p^4_s\varepsilon^4_r\big)$$
$$\big(p^1_a\varepsilon^1_b - p^1_b\varepsilon^1_a\big)\,\big(p^2_a\varepsilon^2_b - p^2_b\varepsilon^2_a\big)\,\big(p^3_c\varepsilon^3_d - p^3_d\varepsilon^3_c\big)\,\big(p^4_c\varepsilon^4_d - p^4_d\varepsilon^4_c\big)\big]$$
$$+\mathscr{F}^{G_{\mathbf{3},1}}(s,u)\,[3\leftrightarrow 2] + \mathscr{F}^{G_{\mathbf{3},1}}(s,t)\,[2\leftrightarrow 4]\big). \tag{6.3.1}$$

The most general descendant which gives rise to S-matrix in (6.3.1) is given by,

$$L^{G_{\mathbf{3},1}} = \sum_{m,n}\left(\mathscr{F}^{G_{\mathbf{3},1}}\right)_{m,n} 2^{m+n}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\left(\partial_{\mu_i}\partial_{\nu_j}R_{abpq}\right)R_{baqp}\left(\partial^{\mu_i}R_{cdrs}\right)\left(\partial^{\nu_j}R_{dcsr}\right)\right). \tag{6.3.2}$$

We have defined the momenta polynomials as,

$$\mathscr{F}^{G_{\mathbf{3},1}}(t,u) \;\;=\;\; \sum_{m,n}\left(\mathscr{F}^{G_{\mathbf{3},1}}\right)_{m,n} t^m u^n. \tag{6.3.3}$$

In order to see the fact that Lagrangian (6.3.2) results in the S-matrix (6.3.1), we note that

$$R_{abpq}R_{baqp}R_{cdrs}R_{dcsr} \tag{6.3.4}$$

linearizes to give $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)$ plus permutations. Once linearized, it is clear that the structure has extra $\mathbb{Z}_2$ symmetry of 1 to 2 exchange. Upto permutations, the descendant Lagrangian (6.3.2) therefore linearizes to,

$$L^{G_{\mathbf{3},1}} = \frac{1}{16}\sum_{m,n}\left(\mathscr{F}^{G_{\mathbf{3},1}}\right)_{m,n} 2^{m+n}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\partial_{\mu_i}\partial_{\nu_j}\left(F^1_{ab}F^1_{pq}\right)F^2_{ba}F^2_{qp}\partial^{\mu_i}\left(F^3_{cd}F^3_{rs}\right)\partial^{\nu_j}\left(F^4_{dc}F^4_{sr}\right)\right). \tag{6.3.5}$$

- The S-matrix corresponding to $G_{\mathbf{3},2}$ in (6.2.5) is specified by the momenta polynomial $\mathscr{F}^{G_{\mathbf{3},2}}(s,u)$ which has the $\mathbb{Z}_2$ symmetry ($s \leftrightarrow u$). Explicitly it is given by

$$S^{G_{\mathbf{3},2}} = \frac{1}{4}\big(\mathscr{F}^{G_{\mathbf{3},2}}(s,u)\,\big[\big(p^1_p\varepsilon^1_q - p^1_q\varepsilon^1_p\big)\,\big(p^2_p\varepsilon^2_q - p^2_q\varepsilon^2_p\big)\,\big(p^3_v\varepsilon^3_w - p^3_w\varepsilon^3_v\big)\,\big(p^4_v\varepsilon^4_w - p^4_w\varepsilon^4_v\big)$$
$$\big(p^1_r\varepsilon^1_s - p^1_s\varepsilon^1_r\big)\,\big(p^4_r\varepsilon^4_s - p^4_s\varepsilon^4_r\big)\,\big(p^2_t\varepsilon^2_u - p^2_u\varepsilon^2_t\big)\,\big(p^3_t\varepsilon^3_u - p^3_u\varepsilon^3_t\big)\big]$$
$$+\mathscr{F}^{G_{\mathbf{3},2}}(t,u)\,[3\leftrightarrow 2] + \mathscr{F}^{G_{\mathbf{3},2}}(s,t)\,[3\leftrightarrow 4]\big). \tag{6.3.6}$$

The most general descendant is

$$L^{G_{\mathbf{3},2}} = \sum_{m,n} \left(\mathscr{F}^{G_{\mathbf{3},2}}\right)_{m,n} 2^{m+n} \left( \prod_{i=1}^{m} \prod_{j=1}^{n} \left(\partial_{\mu_i}\partial_{\nu_j} R_{pqrs}\right) \left(\partial^{\mu_i} R_{pqtu}\right) R_{tuvw} \left(\partial^{\nu_j} R_{rsvw}\right) \right). \quad (6.3.7)$$

That the descendant Lagrangian (6.3.7) generates the S-matrix (6.3.6) is easy to see; the Lagrangian

$$R_{pqrs}R_{pqtu}R_{tuvw}R_{rsvw} \quad (6.3.8)$$

linearizes to give $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)\mathrm{Tr}(F^1F^4)\mathrm{Tr}(F^2F^3)$ plus permutations. This structure has an obvious extra $\mathbb{Z}_2$ symmetry of 1 to 3 exchange. The descendant Lagrangian (6.3.7) then linearizes to give,

$$\frac{1}{16} \sum_{m,n} \left(\mathscr{F}^{G_{\mathbf{3},2}}\right)_{m,n} 2^{m+n} \left( \prod_{i=1}^{m} \prod_{j=1}^{n} \partial_{\mu_i}\partial_{\nu_j} \left(F^1_{pq}F^1_{rs}\right) \partial^{\mu_i} \left(F^2_{pq}F^2_{tu}\right) F^3_{tu}F^3_{vw} \partial^{\nu_j} \left(F^4_{rs}F^4_{vw}\right) \right). \quad (6.3.9)$$

plus permutations.

- The most general S-matrix corresponding $G_{\mathbf{3},3}$ in (6.2.5) is specified by the momenta polynomials $\mathscr{F}^{G_{\mathbf{3},3}}(s,u)$ which is symmetric under $(s \leftrightarrow u)$.

$$\begin{aligned} S^{G_{\mathbf{3},3}} = \frac{1}{4} \Big( & \mathscr{F}^{G_{\mathbf{3},3}}(s,u) \big[ \left(p^1_p\varepsilon^1_q - p^1_q\varepsilon^1_p\right) \left(p^2_p\varepsilon^2_t - p^2_t\varepsilon^2_p\right) \left(p^3_t\varepsilon^3_v - p^3_v\varepsilon^3_t\right) \left(p^4_q\varepsilon^4_v - p^4_v\varepsilon^4_q\right) \\ & \left(p^1_r\varepsilon^1_s - p^1_s\varepsilon^1_r\right) \left(p^2_r\varepsilon^2_u - p^2_u\varepsilon^2_r\right) \left(p^3_u\varepsilon^3_w - p^3_w\varepsilon^3_u\right) \left(p^4_s\varepsilon^4_w - p^4_w\varepsilon^4_s\right) \big] \\ & + \mathscr{F}^{G_{\mathbf{3},3}}(t,u)\,[3 \leftrightarrow 2] + \mathscr{F}^{G_{\mathbf{3},3}}(s,t)\,[3 \leftrightarrow 4] \Big). \end{aligned} \quad (6.3.10)$$

The most general descendant Lagrangian that gives rise to the S-matrix (6.3.10) is as follows

$$L^{G_{\mathbf{3},3}} = \sum_{m,n} \left(\mathscr{F}^{G_{\mathbf{3},3}}\right)_{m,n} 2^{m+n} \left( \prod_{i=1}^{m} \prod_{j=1}^{n} \left(\partial_{\nu_j}\partial_{\mu_i} R_{pqrs}\right) \left(\partial^{\mu_i} R_{ptru}\right) R_{tvuw} \left(\partial^{\nu_j} R_{qvsw}\right) \right). \quad (6.3.11)$$

In order to see the fact that Lagrangian (6.3.11) results in the S-matrix (6.3.10), we note that

$$R_{pqrs}R_{ptru}R_{tvuw}R_{qvsw} \quad (6.3.12)$$

linearizes to $\mathrm{Tr}(F^1F^2F^3F^4)\mathrm{Tr}(F^1F^2F^3F^4)$ plus permutations. This structure, like $\mathrm{Tr}(F^4)$ again has $\mathbb{Z}_2$ symmetry of $1 \leftrightarrow 3$, which manifests in the $u \leftrightarrow s$ symmetry of the momenta functions $\mathscr{F}^{G_{\mathbf{3},3}}(s,u)$. It follows therefore the Lagrangian (6.3.11) linearizes to

$$\frac{1}{16}\sum_{m,n}\left(\mathscr{F}^{G_{\mathbf{3},3}}\right)_{m.n}2^{m+n}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\partial_{\nu_j}\partial_{\mu_i}(F^1_{pq}F^1_{rs})\partial^{\mu_i}(F^2_{pt}F^2_{ru})F^3_{tv}F^3_{uw}\partial^{\nu_j}(F^4_{qv}F^4_{sw})\right) \quad (6.3.13)$$

plus permutations.

- The S-matrix corresponding to $G_{\mathbf{3},4}$ in (6.2.5) is specified by the momenta polynomials $\mathscr{F}^{G_{\mathbf{3},4}}(s,t)$ with the $\mathbb{Z}_2$ symmetry $(s \leftrightarrow t)$. The explicit S-matrix is as follows

$$\begin{aligned}S^{G_{\mathbf{3},4}} = \frac{1}{4}\big(&\mathscr{F}^{G_{\mathbf{3},4}}(s,t)\left[\left(p^1_p\varepsilon^1_q - p^1_q\varepsilon^1_p\right)\left(p^2_p\varepsilon^2_t - p^2_t\varepsilon^2_p\right)\left(p^3_t\varepsilon^3_v - p^3_v\varepsilon^3_t\right)\left(p^4_q\varepsilon^4_v - p^4_v\varepsilon^4_q\right)\right.\\ &\left.\left(p^1_r\varepsilon^1_s - p^1_s\varepsilon^1_r\right)\left(p^2_u\varepsilon^2_w - p^2_w\varepsilon^2_u\right)\left(p^3_w\varepsilon^3_s - p^3_s\varepsilon^3_w\right)\left(p^4_r\varepsilon^4_u - p^4_u\varepsilon^4_r\right)\right]\\ &+\mathscr{F}^{G_{\mathbf{3},4}}(s,u)\,[3\leftrightarrow 4] + \mathscr{F}^{G_{\mathbf{3},4}}(u,t)\,[2\leftrightarrow 4]\big).\end{aligned} \quad (6.3.14)$$

The most general descendant which gives rise to this S-matrix is given by

$$L^{G_{\mathbf{3},4}} = \sum_{m,n}\left(\mathscr{F}^{G_{\mathbf{3},4}}\right)_{m,n}2^{m+n}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\left(\partial_{\nu_j}\partial_{\mu_i}R_{pqrs}\right)\left(\partial^{\mu_i}R_{ptuw}\right)\left(\partial^{\nu_j}R_{tvws}\right)R_{qvru}\right). \quad (6.3.15)$$

In order to see the fact that Lagrangian (6.3.15) results in the S-matrix (6.3.14), we note that

$$R_{pqrs}R_{ptuw}R_{tvws}R_{qvru} \quad (6.3.16)$$

linearizes to $\mathrm{Tr}(F^1F^2F^3F^4)\mathrm{Tr}(F^1F^3F^2F^4)$ plus permutations. This structure has $\mathbb{Z}_2$ symmetry of $2 \leftrightarrow 3$ (and hence the $\mathbb{Z}_2$ symmetry of the momenta polynomials $\mathscr{F}^{G_{\mathbf{3},4}}(s,t)$). When linearized, the most general descendant Lagrangian (6.3.15) becomes,

$$\frac{1}{16}\sum_{m,n}\left(\mathscr{F}^{G_{\mathbf{3},4}}\right)_{m,n}2^{m+n}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\partial_{\nu_j}\partial_{\mu_i}(F^1_{pq}F^1_{rs})\partial^{\mu_i}(F^2_{pt}F^2_{uw})\partial^{\nu_j}(F^3_{tv}F^3_{ws})F^4_{qv}F^4_{ru}\right). \quad (6.3.17)$$

plus permutations.

- The S-matrix corresponding to $G_{\mathbf{3},5}$ in (6.2.5) is specified by momenta polynomials $\mathscr{F}^{G_{\mathbf{3},5}}(t,u)$

which has a $\mathbb{Z}_2$ symmetry in $(t \leftrightarrow u)$.

$$S^{G_{\mathbf{3},5}} = \frac{1}{4}\Big(\mathscr{F}^{G_{\mathbf{3},5}}(t,u)\left[\left(p_p^1\varepsilon_q^1 - p_q^1\varepsilon_p^1\right)\left(p_p^2\varepsilon_q^2 - p_q^2\varepsilon_p^2\right)\left(p_v^3\varepsilon_w^3 - p_w^3\varepsilon_v^3\right)\left(p_v^4\varepsilon_w^4 - p_w^4\varepsilon_v^4\right)\right.$$
$$\left.\left(p_r^1\varepsilon_s^1 - p_s^1\varepsilon_r^1\right)\left(p_t^2\varepsilon_u^2 - p_u^2\varepsilon_t^2\right)\left(p_r^3\varepsilon_t^3 - p_t^3\varepsilon_r^3\right)\left(p_s^4\varepsilon_u^4 - p_u^4\varepsilon_s^4\right)\right]$$
$$+\mathscr{F}^{G_{\mathbf{3},5}}(u,s)\left[2\leftrightarrow 3\right] + \mathscr{F}^{G_{\mathbf{3},5}}(s,t)\left[2\leftrightarrow 4\right]\Big) \tag{6.3.18}$$

The S-matrix (6.3.18) is produced by the general descendant Lagrangian

$$L^{G_{\mathbf{3},5}} = \sum_{m,n}\left(\mathscr{F}^{G_{\mathbf{3},5}}\right)_{m,n} 2^{m+n}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\left(\partial_{\nu_j}\partial_{\mu_i}R_{pqrs}\right)R_{pqtu}\left(\partial^{\mu_i}R_{rtvw}\right)\left(\partial^{\nu_i}R_{suvw}\right)\right). \tag{6.3.19}$$

In order to see the fact that Lagrangian (6.3.19) results in the S-matrix (6.3.18), we note that

$$R_{pqrs}R_{pqtu}R_{rtvw}R_{suvw} \tag{6.3.20}$$

linearizes to $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)\mathrm{Tr}(F^1F^3F^2F^4)$ plus permutation. The general descendant Lagrangian (6.3.19) therefore linearizes to

$$\frac{1}{16}\sum_{m,n}\left(\mathscr{F}^{G_{\mathbf{3},5}}\right)_{m,n} 2^{m+n}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\partial_{\nu_j}\partial_{\mu_i}\left(F_{pq}^1F_{rs}^1\right)F_{pq}^2F_{tu}^2\partial^{\mu_i}\left(F_{rt}^3F_{vw}^3\right)\partial^{\nu_i}\left(F_{su}^4F_{vw}^4\right)\right) \tag{6.3.21}$$

plus permutations.

- The most general S-matrix generated by $G_{\mathbf{6}} = G_{\mathbf{3},6} \oplus G_{\mathbf{3_A}}$ is specified by an arbitrary polynomial $\mathscr{F}^{G_{\mathbf{6}}}(t,u)$ with no symmetry restrictions. The corresponding S-matrix is

$$\begin{aligned}S^{G_{\mathbf{6}}} = \frac{1}{4}\mathscr{F}^{G_{\mathbf{6}}}(s,t)&\left[\left(p_p^1\varepsilon_q^1 - p_q^1\varepsilon_p^1\right)\left(p_p^2\varepsilon_q^2 - p_q^2\varepsilon_p^2\right)\left(p_u^3\varepsilon_v^3 - p_v^3\varepsilon_u^3\right)\left(p_u^4\varepsilon_v^4 - p_v^4\varepsilon_u^4\right)\right.\\ &\left.\left(p_r^1\varepsilon_s^1 - p_s^1\varepsilon_r^1\right)\left(p_r^2\varepsilon_t^2 - p_t^2\varepsilon_r^2\right)\left(p_w^3\varepsilon_t^3 - p_t^3\varepsilon_w^3\right)\left(p_w^4\varepsilon_s^4 - p_s^4\varepsilon_w^4\right)\right]\\ &+S_3 \text{ permutations (also act on } s,t,u).\end{aligned} \tag{6.3.22}$$

This S-matrix (6.3.22) is produced (up to proportionality) by the Lagrangian

$$L^{G_{\mathbf{6}}} = \sum_{m,n}\left(\mathscr{F}^{G_{\mathbf{6}}}\right)_{m.n} 2^{m+n}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\left(\partial_{\mu_i}\partial_{\nu_j}R_{pqrs}\right)\left(\partial^{\mu_i}R_{pqrt}\right)\left(\partial^{\nu_i}R_{uvwt}\right)R_{uvws}\right) \tag{6.3.23}$$

The fact that (6.3.23) yields the S-matrix (6.3.22) follows from the fact that

$$R_{pqrs}R_{pqrt}R_{uvwt}R_{uvws} \tag{6.3.24}$$

linearizes to $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)\mathrm{Tr}(F^1F^2F^3F^4)$ plus permutations (where the superscript, as usual, labels particles). It follows that the Lagrangian (6.3.23) linearizes to

$$\frac{1}{16}\sum_{m,n}\left(\mathscr{F}^{G_{\mathbf{6}}}\right)_{m.n}2^{m+n}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\partial_{\mu_i}\partial_{\nu_j}\left(F^1_{pq}F^1_{rs}\right)\partial^{\mu_i}\left(F^2_{pq}F^2_{rt}\right)\partial^{\nu_i}\left(F^3_{uv}F^3_{wt}\right)F^4_{uv}F^4_{ws}\right) \tag{6.3.25}$$

plus permutations. The replacement rule $\partial_\mu \to ik_\mu$ then turns (6.3.25) into (6.3.22).

- The S-matrix corresponding to $G_{\mathbf{3},7}$ in (6.2.5) is specified by $\mathscr{F}^{G_{\mathbf{3},7}}(t,u)$ which has $\mathbb{Z}_2$ symmetry of ($t \leftrightarrow u$). Explicitly,

$$\begin{aligned}
S^{G_{\mathbf{3},7}} = \frac{1}{16}\Big(&\mathscr{F}^{G_{\mathbf{3},7}}(t,u)\left(p^1_p\varepsilon^1_q - p^1_q\varepsilon^1_p\right)\left(p^2_p\varepsilon^2_q - p^2_q\varepsilon^2_p\right)\left(p^3_r\varepsilon^3_s - p^3_s\varepsilon^3_r\right)\left(p^4_r\varepsilon^4_s - p^4_s\varepsilon^4_r\right)\\
&+\mathscr{F}^{G_{\mathbf{3},7}}(s,u)\left(p^1_p\varepsilon^1_q - p^1_q\varepsilon^1_p\right)\left(p^3_p\varepsilon^3_q - p^3_q\varepsilon^3_p\right)\left(p^2_r\varepsilon^2_s - p^2_s\varepsilon^2_r\right)\left(p^4_r\varepsilon^4_s - p^4_s\varepsilon^4_r\right)\\
&+\mathscr{F}^{G_{\mathbf{3},7}}(t,s)\left(p^1_p\varepsilon^1_q - p^1_q\varepsilon^1_p\right)\left(p^4_p\varepsilon^4_q - p^4_q\varepsilon^4_p\right)\left(p^3_r\varepsilon^3_s - p^3_s\varepsilon^3_r\right)\left(p^2_r\varepsilon^2_s - p^2_s\varepsilon^2_r\right)\Big)\\
&\Big(\left(p^1_a\varepsilon^1_b - p^1_b\varepsilon^1_a\right)p^2_a\left(p^2_\mu\varepsilon^2_\nu - p^2_\nu\varepsilon^2_\mu\right)p^3_b\left(p^3_\nu\varepsilon^3_\alpha - p^3_\alpha\varepsilon^3_\nu\right)\left(p^4_\alpha\varepsilon^4_\mu - p^4_\mu\varepsilon^4_\alpha\right)\\
&+\left(p^2_a\varepsilon^2_b - p^2_b\varepsilon^2_a\right)p^1_a\left(p^1_\mu\varepsilon^1_\nu - p^1_\nu\varepsilon^1_\mu\right)p^4_b\left(p^4_\nu\varepsilon^4_\alpha - p^4_\alpha\varepsilon^4_\nu\right)\left(p^3_\alpha\varepsilon^3_\mu - p^3_\mu\varepsilon^3_\alpha\right)\\
&+\left(p^3_a\varepsilon^3_b - p^3_b\varepsilon^3_a\right)p^4_a\left(p^4_\mu\varepsilon^4_\nu - p^4_\nu\varepsilon^4_\mu\right)p^1_b\left(p^1_\nu\varepsilon^1_\alpha - p^1_\alpha\varepsilon^1_\nu\right)\left(p^2_\alpha\varepsilon^2_\mu - p^2_\mu\varepsilon^2_\alpha\right)\\
&+\left(p^4_a\varepsilon^4_b - p^4_b\varepsilon^4_a\right)p^3_a\left(p^3_\mu\varepsilon^3_\nu - p^3_\nu\varepsilon^3_\mu\right)p^2_b\left(p^2_\nu\varepsilon^2_\alpha - p^2_\alpha\varepsilon^2_\nu\right)\left(p^1_\alpha\varepsilon^1_\mu - p^1_\mu\varepsilon^1_\alpha\right)\Big)
\end{aligned} \tag{6.3.26}$$

The S-matrix (6.3.26) is generated by the descendant Lagrangian

$$L^{G_{\mathbf{3},7}} = -\sum_{m,n}\left(\mathscr{F}^{G_{\mathbf{3},7}}\right)_{m,n}2^{m+n}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\left(\partial_{\mu_i}\partial_{\nu_j}R_{pqab}\right)\left(\partial_a R_{qp\mu\nu}\right)\left(\partial_b\partial^{\mu_i}R_{rs\nu\alpha}\right)\left(\partial^{\nu_j}R_{sr\alpha\mu}\right)\right) \tag{6.3.27}$$

In order to see that S-matrix (6.3.26) is generated by (6.3.27), we note that

$$R_{pqab}\partial_a R_{qp\mu\nu}\partial_b R_{rs\nu\alpha}R_{sr\alpha\mu} \tag{6.3.28}$$

linearizes to give $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)F^1_{ab}\mathrm{Tr}(p^2_aF^2p^3_bF^3F^4)$ plus permutations. This structure again has only $\mathbb{Z}_2$ symmetry of $3 \leftrightarrow 4$. The descendant Lagrangian (6.3.27) then linearizes to give

$$-\frac{1}{16}\sum_{m,n}\left(\mathscr{F}^{G_{\mathbf{3},7}}\right)_{m,n}2^{m+n}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\partial_{\mu_i}\partial_{\nu_j}\left(F^1_{pq}F^1_{ab}\right)\partial_a F^2_{qp}F^2_{\mu\nu}\partial_b\partial^{\mu_i}\left(F^3_{rs}F^3_{\nu\alpha}\right)\partial^{\nu_j}\left(F^4_{sr}F^4_{\alpha\mu}\right)\right) \tag{6.3.29}$$

plus permutations.

- The S-matrix corresponding to $G_{\mathbf{3},8}$ in (6.2.5) is specified by the momenta functions $\mathscr{F}^{G_{\mathbf{3},8}}(s,u)$ which has the $\mathbb{Z}_2$ symmetry ($s \leftrightarrow u$). In equations,

$$\begin{aligned}
S^{G_{\mathbf{3},8}} = \frac{1}{16}\Big(&\mathscr{F}^{G_{\mathbf{3},8}}(s,u)\left(p^1_p\varepsilon^1_q - p^1_q\varepsilon^1_p\right)\left(p^2_q\varepsilon^2_r - p^2_r\varepsilon^2_q\right)\left(p^3_r\varepsilon^3_s - p^3_s\varepsilon^3_r\right)\left(p^4_s\varepsilon^4_p - p^4_p\varepsilon^4_s\right)\\
&\mathscr{F}^{G_{\mathbf{3},8}}(t,u)\left(p^1_p\varepsilon^1_q - p^1_q\varepsilon^1_p\right)\left(p^3_q\varepsilon^3_r - p^3_r\varepsilon^3_q\right)\left(p^2_r\varepsilon^2_s - p^2_s\varepsilon^2_r\right)\left(p^4_s\varepsilon^4_p - p^4_p\varepsilon^4_s\right)\\
+&\mathscr{F}^{G_{\mathbf{3},8}}(t,s)\left(p^1_p\varepsilon^1_q - p^1_q\varepsilon^1_p\right)\left(p^3_q\varepsilon^3_r - p^3_r\varepsilon^3_q\right)\left(p^4_r\varepsilon^4_s - p^4_s\varepsilon^4_r\right)\left(p^2_s\varepsilon^2_p - p^2_p\varepsilon^2_s\right)\Big)\\
&\Big(\left(p^1_a\varepsilon^1_b - p^1_b\varepsilon^1_a\right)p^2_a\left(p^2_\mu\varepsilon^2_\nu - p^2_\nu\varepsilon^2_\mu\right)p^3_b\left(p^3_\nu\varepsilon^3_\alpha - p^3_\alpha\varepsilon^3_\nu\right)\left(p^4_\alpha\varepsilon^4_\mu - p^4_\mu\varepsilon^4_\alpha\right)\\
&+\left(p^2_a\varepsilon^2_b - p^2_b\varepsilon^2_a\right)p^1_a\left(p^1_\mu\varepsilon^1_\nu - p^1_\nu\varepsilon^1_\mu\right)p^4_b\left(p^4_\nu\varepsilon^4_\alpha - p^4_\alpha\varepsilon^4_\nu\right)\left(p^3_\alpha\varepsilon^3_\mu - p^3_\mu\varepsilon^3_\alpha\right)\\
&+\left(p^3_a\varepsilon^3_b - p^3_b\varepsilon^3_a\right)p^4_a\left(p^4_\mu\varepsilon^4_\nu - p^4_\nu\varepsilon^4_\mu\right)p^1_b\left(p^1_\nu\varepsilon^1_\alpha - p^1_\alpha\varepsilon^1_\nu\right)\left(p^2_\alpha\varepsilon^2_\mu - p^2_\mu\varepsilon^2_\alpha\right)\\
&+\left(p^4_a\varepsilon^4_b - p^4_b\varepsilon^4_a\right)p^3_a\left(p^3_\mu\varepsilon^3_\nu - p^3_\nu\varepsilon^3_\mu\right)p^2_b\left(p^2_\nu\varepsilon^2_\alpha - p^2_\alpha\varepsilon^2_\nu\right)\left(p^1_\alpha\varepsilon^1_\mu - p^1_\mu\varepsilon^1_\alpha\right)\Big)
\end{aligned} \tag{6.3.30}$$

The descendant is of the general form

$$L^{G_{\mathbf{3},8}} = -\sum_{m,n}\left(\mathscr{F}^{G_{\mathbf{3},8}}\right)_{m,n}2^{m+n}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\left(\partial_{\mu_i}\partial_{\nu_j}R_{pqab}\right)\left(\partial_a\partial^{\mu_i}R_{qr\mu\nu}\right)\left(\partial_b R_{rs\nu\alpha}\right)\left(\partial^{\nu_j}R_{sp\alpha\mu}\right)\right) \tag{6.3.31}$$

Reader can convince himself/herself that the descendant Lagrangian (6.3.31) gives rise to the S-matrix (6.3.30) by noting that

$$R_{pqab}\partial_a R_{qr\mu\nu}\partial_b R_{rs\nu\alpha}R_{sp\alpha\mu} \tag{6.3.32}$$

linearizes to give $\mathrm{Tr}(F^1F^2F^3F^4)F^1_{ab}\mathrm{Tr}(p^2_aF^2p^3_bF^3F^4)$ plus permutations. This structure has neither $\mathbb{Z}_2\times\mathbb{Z}_2$, which although is preserved by the first trace but broken by the $F\mathrm{Tr}(\ldots)$ part, nor it has $S^3$ which is preserved by the $F\mathrm{Tr}(...)$ part but broken by the $\mathrm{Tr}(F^4)$ part. Only $\mathbb{Z}_2$ is preserved, that is just $2 \leftrightarrow 4$ flip symmetry. Consequently the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetrization had to be done explicitly in (6.3.30). The descendant Lagrangian (6.3.31) then linearizes to

$$-\frac{1}{16}\sum_{m,n}\left(\mathscr{F}^{G_{\mathbf{3},8}}\right)_{m,n} 2^{m+n}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\partial_{\mu_i}\partial_{\nu_j}\left(F^1_{pq}F^1_{ab}\right)\partial_a\partial^{\mu_i}\left(F^2_{qr}F^2_{\mu\nu}\right)\partial_b\left(F^3_{rs}F^3_{\nu\alpha}\right)\partial^{\nu_j}\left(F^4_{sp}F^4_{\alpha\mu}\right)\right)|_{\mathbb{Z}_2\times\mathbb{Z}_2}$$

- The S-matrix corresponding to $G_{\mathbf{S},2}$ in (6.2.5) is given by the momenta polynomial $\mathscr{F}^{G_{\mathbf{S},2}}(s,t)$ (6.3.33) which is fully symmetric in $s,t$ and $u$. The explicit expression for the S-matrix is given by,

$$\begin{aligned} S^{G_{\mathbf{S},2}} = \frac{1}{16}\left(\mathscr{F}^{G_{\mathbf{S},2}}(s,t)\right) \times \\ \Big[\left(p^1_a\varepsilon^1_b - p^1_b\varepsilon^1_a\right)p^2_a\left(p^2_\mu\varepsilon^2_\nu - p^2_\nu\varepsilon^2_\mu\right)p^3_b\left(p^3_\nu\varepsilon^3_\alpha - p^3_\alpha\varepsilon^3_\nu\right)\left(p^4_\alpha\varepsilon^4_\mu - p^4_\mu\varepsilon^4_\alpha\right) \\ \left(p^1_p\varepsilon^1_q - p^1_q\varepsilon^1_p\right)p^2_p\left(p^2_\beta\varepsilon^2_\gamma - p^2_\gamma\varepsilon^2_\beta\right)p^3_q\left(p^3_\gamma\varepsilon^3_\delta - p^3_\delta\varepsilon^3_\gamma\right)\left(p^4_\delta\varepsilon^4_\beta - p^4_\beta\varepsilon^4_\delta\right) \\ +(1\leftrightarrow 2)+(1\leftrightarrow 3)+(1\leftrightarrow 4)\Big] \end{aligned} \tag{6.3.34}$$

The most general descendant Lagrangian giving rise to (6.3.34) is

$$L^{G_{\mathbf{S},2}} = \sum_{m,n}\left(\mathscr{F}^{G_{\mathbf{S},2}}\right)_{m,n}2^{m+n}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\left(\partial_{\mu_i}\partial_{\nu_j}R_{abpq}\right)\left(\partial^{\mu_i}\partial_p\partial_a R_{\mu\nu\beta\gamma}\right)\left(\partial^{\nu_j}\partial_q\partial_b R_{\nu\alpha\gamma\delta}\right)R_{\alpha\mu\delta\beta}\right) \tag{6.3.35}$$

It is easy to see that the descendant Lagrangian (6.3.35) generates the S-matrix (6.3.34). Consider the Lagrangian

$$R_{abpq}\partial_p\partial_a R_{\mu\nu\beta\gamma}\partial_q\partial_b R_{\nu\alpha\gamma\delta}R_{\alpha\mu\delta\beta} \tag{6.3.36}$$

which linearizes to give $F^1_{pq}\mathrm{Tr}(p^2_pF^2p^3_qF^3F^4)F^1_{ab}\mathrm{Tr}(p^2_aF^2p^3_bF^3F^4)$ plus permutations. This structure has $S^3$ symmetry, because $2,3,4$ can be permuted and the structure remains invariant. The descendant Lagrangian (6.3.35) linearizes to give,

$$\frac{1}{16}\sum_{m,n}\left(\mathscr{F}^{G_{\mathbf{S},2}}\right)_{m,n}2^{m+n}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\partial_{\mu_i}\partial_{\nu_j}(F^1_{ab}F^1_{pq})\partial^{\mu_i}\partial_p\partial_a(F^2_{\mu\nu}F^2_{\beta\gamma})\partial^{\nu_j}\partial_q\partial_b(F^3_{\nu\alpha}F^3_{\gamma\delta})F^4_{\alpha\mu}F^4_{\delta\beta}\right) \tag{6.3.37}$$

plus permutations.

- Finally we turn to the specification of the S-matrices descended from $G_{\mathbf{S},1}$. If we are interested in specifying only the S-matrix - and not the Lagrangian that gives rise to this S-matrix - this job is easily done. In addition to the S-matrices already listed in this appendix we have one additional contribution specified by $\mathscr{F}^{G_{\mathbf{S},1}}$, a fully symmetric polynomial of $s,t,u$ that is otherwise unconstrained. The S-matrix is given by

$$S^{G_{\mathbf{S},1}}_{D\geq 7} = 24(3\mathscr{F}^{G_{\mathbf{S},1}}(t,u)\varepsilon^{ijklmnp}\varepsilon^{asdfghj}\varepsilon^1_i\varepsilon^2_j\varepsilon^3_k\varepsilon^4_l p^1_m p^2_n p^3_p \varepsilon^1_a\varepsilon^2_s\varepsilon^3_d\varepsilon^4_f p^1_g p^2_h p^3_j) \tag{6.3.38}$$

At the level of S-matrices we have now completed our listings. The most general sum of (6.3.38), (6.3.22), (6.3.18), (6.3.10), (6.3.14), (6.3.1), (6.3.6), (6.3.30), (6.3.26), (6.3.34) gives the most general local S-matrix for gravitational scattering in $D \geq 8$. There is a slight subtlety in constructing the Lagrangian corresponding to the S-matrix in 6.3.38. This is because of the fact that this S-matrix arises from a Lagrangian cubic in the Riemann curvature tensor, and hence the Mandelstam variables s,t,u are not well defined. This problem has been discussed with appropriate details in section 6 of [1].

To summarize the section, we have explicitly constructed all independent parity even four graviton S-matrices for spacetime dimensions $D \geq 8$. Also, we have provided an exhaustive listing of the most general descendant Lagrangian corresponding to the basis of the four graviton S-matrices.

# Chapter 7

# Exchange contributions

The most general classical (i.e. tree level) four particle S-matrix that follows from a local Lagrangian is given as the sum of two kinds of terms. These are

- Local S-matrices (i.e. S-matrices that are polynomials in the variables $\varepsilon_i, k_i$). These S-matrices, which have their origin in local contact type interactions in the Lagrangian.
- Pole terms that come from the exchange of an intermediate particle.

Consider for instance, a four graviton (gggg) scattering amplitude. Consider the pole contribution to this amplitude from the exchange of a particle $P$ of mass $m$ that transforms in the representation $\mathscr{P}$ of the massive Lorentz little group $SO(D-1)$. The most important thing about this amplitude is that the residue of its pole is completely fixed by the on-shell three particle S-matrix $ggP$[1]. It follows that the most general S-matrix that comes from a local Lagrangian is characterized by the masses and spins of the exchange particles $P$ together with the three point $(ggP)$ couplings - in addition to the data that specifies polynomial S-matrices.

In order to complete the classification of polynomial (e.g. 4 graviton) S-matrices presented earlier in this project into a complete classification of all S-matrices that could possibly originate

[1]Let $s=-(k_1+k_2)^2$ denote the exchange momentum. The full exchange diagram involves an intermediate off-shell $P$ particle of squared mass $s$- and so is completely specified only once we are given a 'generalized' three point amplitude in which the gravitons are on-shell but the particle $P$ is off-shell. However all off-shell extensions of the same on-shell amplitude agree when $s=m^2$. Moreover these three point amplitudes are polynomial in momenta, so the difference between the numerators exchange diagram built out of any two distinct off-shell extensions of the same on-shell 3 point function contains at least one factor of $(s-m^2)$. This overall propagator cancels the pole originating from the exchange propagator, and we are left with a polynomial S-matrix.

in local Lagrangians, all we need to do is to work out all possible $ggP$ couplings, and stitch two of these couplings together through the propagator for the particle $P$. Every element in this program is straightforward to carry through. It is easy to list the representations $\mathscr{P}$ of $P$ that can have nonzero on-shell three point functions with our scattering particles. For instance, scalar $P$ scattering can be nonzero only if $P$ transforms in the traceless symmetric representation with an even number of indices. It is also not difficult to enumerate the most general kinematically allowed on-shell three point functions. The spin $P$ propagator is given simply by the projection - in index space - onto the representation space $\mathscr{P}$ (in the $D-1$ dimensional space orthogonal to $k_1+k_2$ ) divided by $s-m^2$. Sewing these elements together allows us to explicitly construct the most general pole contributions to S-matrices. In this project we will not systematically carry through the program outlined in the previous paragraph; we leave this exercise for future work. Detailed study for some of the cases has been studied in the paper [1]. Here, we compute and present results for the most general graviton exchange contribution to four graviton scattering.

The main focus of our discussion in this section is the Regge growth of exchange contributions to S-matrices. As in the case of contact interactions discussed earlier in this project, we are particularly interested in classifying those exchange contributions to 4 particle scattering that grow no faster than $s^2$ in the Regge limit. [2] It is very easy to see that the exchange of a massive particle of spin $J$ [3] in the $t$ channel yields a contribution to scattering that cannot grow faster than $s^J$. [4] Moreover we expect that this inequality is generically saturated - i.e. that spin $J$ exchange in the $t$ channel will grow like $s^J$ in the Regge limit. We thus expect that the exchange of spin $J$ particles with $J>2$ will always violate the CRG conjecture. We should note that this violation is non polynomial in $t$ and so cannot be cancelled by a local counterterm. This discussion applies equally well to the scattering of scalars, photons and gravitons.

[2]More generally we would like to classify those exchange contributions which grow no faster than $s^2$ in the Regge limit after being combined with suitable polynomial S-matrices of the sort we have enumerated earlier in this paper. Any such combination of an exchange S-matrix plus a 'local counterterm subtraction' reflects an addition to four particle scattering that is not ruled out by the CRG conjecture.

[3]Any massive exchanged particle transforms under some representation of the little group $SO(D-1)$. There representations can be labelled by Young Tableaux. We say that a particle has spin $J$ if the length of the largest row in the Young Tableaux labelling that particle is $J$.

[4]The reason for this is as follows. In the $t$ channel, the scattering particles are grouped into those with momenta $k_1,\ k_3$ and those with momenta $k_2,\ k_4$. Contraction of momenta within a group - e.g. the dot products $k_1.k_3$ - produces factor of $t$ but never of $s$. Moreover the unique contraction of momenta between two groups - which happens through the propagator of the exchanged particle - is $(k_1-k_3).(k_2-k_4)$. If the exchanged particle has no more than $J$ symmetrized indices, there cannot be more than $J$ factors of $(k_1-k_3).(k_2-k_4)$, simply because the original three point function between two scattering particles and the exchanged particle could not have had any vector - in this case $k_1-k_2$ - contract with more than $J$ indices of the exchanged particle.

Let us now turn to the exchange of particles with spin $\leq 2$. The $t$ channel contributions to such exchange processes are always consistent with the CRG conjecture. However the question of whether the $s$ and $u$ channel contributions to exchange contributions violates the CRG conjecture depends on the nature of the external particle. In Appendix L of [1] we demonstrate that spin zero and spin 2 exchange contributions to four scalar and four photon scattering are both consistent with the CRG conjecture even in the $s$ and $u$ channels. In four graviton scattering all possible (non Einstein) exchange of massless gravitons, massive scalars and massive spin 2 particles violate the CRG conjecture in a way that cannot be fixed by a local counterterm.

Why did the sample low spin exchange contributions that we have explicitly computed violate the CRG conjecture for the case of external gravitons? The key point here is that three point $ggP$ S-matrices appear always to be generated by Lagrangian couplings of (derivatives of) two factors of the Riemann tensor to the particle $P$; consequently the three point couplings are always at least 4 derivative order in derivatives. [5] Assuming this to be the case, in subsection 7.2, we have given an argument that demonstrates that such exchange contributions *always* violate the CRG conjecture in a way that cannot be canceled by local counterterms, at least in $D \leq 6$. The argument of subsection (7.2) applies to every exchange contribution including those that we have not explicitly computed.

## 7.1 4 Graviton scattering from graviton exchange

In this section we construct and study all possible graviton exchange contributions to four point scattering amplitudes of gravitons. As we have explained above, the pole contributions to these exchange diagrams is given by sewing on-shell three point functions through graviton propagators. The kinematically allowed on-shell 3 point functions for gravitons have been listed in (4.1.17) , (4.1.18), (4.1.19).

[5] The assumption of this section - namely that $ggP$ couplings are always 4 derivative or higher - can, easily be verified by algebraic means - i.e. by simply constructing all gauge invariant $ggP$ three point scattering amplitudes. This has been proven in a subsequent work [2].

For the convenience of the reader we reproduce the relevant expressions here[6].

$$A^R = (\varepsilon_1.\varepsilon_2\varepsilon_3.p_1 + \varepsilon_1.\varepsilon_3\varepsilon_2.p_3 + \varepsilon_2.\varepsilon_3\varepsilon_1.p_2)^2 \tag{7.1.1}$$

$$A^{R^2} = (\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3 \wedge k^1 \wedge k^2)^2 \tag{7.1.2}$$

$$A^{R^3} = F^1_{ab}F^2_{ab}F^2_{cd}F^3_{cd}F^3_{ef}F^1_{ef} + \text{perm.} \tag{7.1.3}$$

The graviton propagator is simple; it is

$$G_{\mu\nu,\rho\sigma} = \frac{1}{k^2}\left(\frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma}) - \frac{1}{D-2}\,\eta_{\mu\nu}\eta_{\rho\sigma}\right). \tag{7.1.4}$$

We will now use (7.1.1) and (7.1.4) to obtain the graviton exchange contribution for the four point functions in a theory whose (ggg) three point function is given by

$$A = \alpha_R A^R + \alpha_{R^2} A^{R^2} + \alpha_{R^3} A^{R^3}. \tag{7.1.5}$$

The general exchange S-matrix takes the form

$$\mathscr{A} = \alpha_R^2 \mathscr{A}_{R-R} + \alpha_R\alpha_{R^2}\mathscr{A}_{R-R^2} + \alpha_R\alpha_{R^3}\mathscr{A}_{R-R^3} + \alpha_{R^2}^2\mathscr{A}_{R^2-R^2} + \alpha_{R^2}\alpha_{R^3}\mathscr{A}_{R^2-R^3} + \alpha_{R^3}^2\mathscr{A}_{R^3-R^3} \tag{7.1.6}$$

Note that the structures $A^{R^2}$ and $A^{R^3}$ are gauge invariant off-shell (i.e. without using $k_i^2 = 0$). It follows that exchange diagrams that sew two of these vertices together - i.e. $\mathscr{A}_{R^2-R^2}$, $\mathscr{A}_{R^2-R^3}$ and $\mathscr{A}_{R^3-R^3}$ - are automatically gauge invariant separately in each channel. In other words these three amplitudes are can be evaluated using the same sewing process utilized when the exchanged particle is not a graviton but another particle[7][8].

[6]The $R^2$ and $R^3$ three point functions are sometimes quoted as

$$A^{R^2} = 2(\varepsilon_1.\varepsilon_2\varepsilon_3.p_1 + \varepsilon_1.\varepsilon_3\varepsilon_2.p_3 + \varepsilon_2.\varepsilon_3\varepsilon_1.p_2)(\varepsilon_1.p_2\varepsilon_2.p_3\varepsilon_3.p_1)$$

and

$$A^{R^3} = 6(\varepsilon_1.p_2\varepsilon_2.p_3\varepsilon_3.p_1)^2$$

On-shell these expressions agree with those listed in (7.1.1). However the form of the expressions in (7.1.1) has the added advantage that they are off-shell gauge invariant (i.e. gauge invariant without needing to use the conditions $k_i^2 = 0$.

[7]In particular each of the amplitudes $\mathscr{A}_{R^2-R^2}$, $\mathscr{A}_{R^2-R^3}$ and $\mathscr{A}_{R^3-R^3}$ can be decomposed admits a gauge invariant decomposition into a piece that has an $s$ pole, a piece that has a $t$ pole and a piece that has a $u$ pole.

[8] On the other hand exchange pieces that involve one or two $R$ vertices (i.e. terms proportional to one or two powers of $\alpha_R$) cannot, in general, be decomposed as described above in a gauge invariant manner. This term is best written in the form $\frac{B}{stu}$ where $B$ is a gauge invariant polynomial.

The sewing process is easily performed in each channel: we find

$$\mathscr{A}_{R^2-R^2} = \begin{cases} S^{G_{\mathbf{6}}} + S^{G_{\mathbf{3},1}}, \\ \mathscr{F}^{G_{\mathbf{6}}}(t,u) = \frac{1}{8s}, \mathscr{F}^{G_{\mathbf{3},1}}(t,u) = \frac{-D}{32(D-2)s} \end{cases}$$

$$\mathscr{A}_{R^2-R^3} = \begin{cases} S^{G_{\mathbf{3},6}} + S^{G_{\mathbf{3},5}} + S^{G_{\mathbf{3},1}} + S^{G_{\mathbf{3},2}} + S^{G_{\mathbf{3},7}} + S^{G_{\mathbf{3}\mathbf{A}}}, \\ \mathscr{F}^{G_{\mathbf{3},6}}(t,u) = \frac{3}{8},\ \mathscr{F}^{G_{\mathbf{3},5}}(t,u) = \frac{-3}{16},\ \mathscr{F}^{G_{\mathbf{3},1}}(t,u) = \frac{-3(D+2)}{64(D-2)}, \\ \mathscr{F}^{G_{\mathbf{3},2}}(t,u) = \frac{3}{32},\ \mathscr{F}^{G_{\mathbf{3},7}}(t,u) = \frac{3}{2s},\ \mathscr{F}^{G_{\mathbf{3}\mathbf{A}}}(t,u) = \frac{-12(t-u)}{64s}. \end{cases}$$

$$\mathscr{A}_{R^3-R^3} = \begin{cases} S^{G_{\mathbf{3},6}} + S^{G_{\mathbf{3},5}} + S^{G_{\mathbf{3},1}} + S^{G_{\mathbf{3},2}} + S^{G_{\mathbf{3},7}} + S^{G_{\mathbf{3}\mathbf{A}}}, \\ \mathscr{F}^{G_{\mathbf{3},6}}(t,u) = \frac{9s}{32}, \mathscr{F}^{G_{\mathbf{3},5}}(t,u) = \frac{-9s}{32}, \mathscr{F}^{G_{\mathbf{3},1}}(t,u) = \frac{-9(t^2+u^2+Dut)}{64(D-2)s}, \mathscr{F}^{G_{\mathbf{3},2}}(t,u) = \frac{-9s}{128}, \\ \mathscr{F}^{G_{\mathbf{3},7}}(t,u) = \frac{-9}{4},\ \mathscr{F}^{G_{\mathbf{3}\mathbf{A}}}(t,u) = \frac{-9(t-u)}{32}. \end{cases}$$

These S-matrices (7.1.7) are formally generated by the non local Lagrangians

$$(stu)\mathscr{A}_{R^2-R^2} \propto \left( -\frac{D}{4(D-2)} \nabla_\mu \nabla_\nu R_{pqrs} R_{pqrs} \nabla^\mu R_{abcd} \nabla^\nu R_{abcd} + 2\nabla_\mu \nabla_\nu R_{pqrs} R_{pqrt} \nabla^\mu R_{uvwt} \nabla^\nu R_{uvws} \right) \tag{7.1.7}$$

$$\begin{aligned} &(stu)\mathscr{A}_{R^2-R^3} \\ &\propto \Big( 6\ \nabla_x \nabla_y \nabla_\mu R_{pqrs} \nabla^\mu R_{pqrt} \nabla^x R_{uvwt} \nabla^y R_{uvws} - 12\ \nabla_x \nabla_y \nabla_\mu R_{pqrs} R_{pqrt} \nabla^x \nabla^\mu R_{uvwt} \nabla^y R_{uvws} \\ &+ 6\ \nabla_x \nabla_y \nabla^\mu R_{pqrs} R_{pqtu} \nabla^x \nabla_\mu R_{rtvw} \nabla^y R_{suvw} + \frac{3(D+2)}{2(D-2)} \nabla_x \nabla_y \nabla^\mu R_{pqrs} R_{pqrs} \nabla^x \nabla_\mu R_{pqrs} \nabla^y R_{pqrs} \\ &+ \frac{3}{2}\ \nabla_x \nabla_y \nabla^\mu R_{pqrs} \nabla_\mu R_{pqtu} \nabla^x R_{tuvw} \nabla^y R_{rsvw} + 12\ \nabla_x \nabla_y R_{\mu\nu ab} \nabla_a R_{\nu\mu mn} \nabla_b \nabla^x R_{\alpha\beta np} \nabla^y R_{\beta\alpha pm} \Big) \end{aligned} \tag{7.1.8}$$

$$
\begin{aligned}
&(stu)\mathscr{A}_{R^3-R^3} \\
&\propto \Bigg( -36\nabla^a\nabla^b\nabla^\mu\nabla^\nu R_{pqrs}\nabla_\mu R_{pqrt}\nabla_a\nabla_\nu R_{uvwt}\nabla_b R_{uvws} \\
&-18\nabla^a\nabla^b\nabla^\mu\nabla^\nu R_{pqrs}R_{pqtu}\nabla_a\nabla_\mu\nabla_\nu R_{rtvw}\nabla_b R_{suvw} - 18\nabla^b\nabla^a\nabla^\mu\nabla^\nu R_{pqrs}R_{pqtu}\nabla_a\nabla_\mu R_{rtvw}\nabla_\nu\nabla_b R_{suvw} \\
&-\frac{9}{D-2}\nabla^a\nabla^b\nabla^\mu\nabla^\nu R_{\alpha\beta cd}R_{\alpha\beta cd}\nabla_a R_{pqrs}\nabla_b\nabla_\mu\nabla_\nu R_{pqrs} \\
&-\frac{9D}{2(D-2)}\nabla^a\nabla^b\nabla^\mu\nabla^\nu R_{\alpha\beta cd}R_{\alpha\beta cd}\nabla_a\nabla_\mu R_{pqrs}\nabla_\nu\nabla_b R_{pqrs} + \frac{9}{2}\nabla_b\nabla_a\nabla^\mu\nabla^\nu R_{pqrs}\nabla_\mu\nabla_\nu R_{pqtu}\nabla^b R_{tuvw}\nabla^a R_{rsvw} \\
&-72\nabla_l\nabla_k\nabla_\gamma R_{\mu\nu ab}\nabla_a R_{\nu\mu mn}\nabla^\gamma\nabla_b\nabla^k R_{\alpha\beta np}\nabla^l R_{\beta\alpha pm}\Bigg)
\end{aligned}
\tag{7.1.9}
$$

We now turn to the evaluation of the remaining three amplitudes; $\mathscr{A}_{R-R}$, $\mathscr{A}_{R-R^2}$ and $\mathscr{A}_{R-R^3}$. These amplitudes are distinguished by the fact that they sew diagrams including at least one copy of the amplitude $A^R$, which is gauge invariant on-shell but not off-shell. As the exchange diagram includes an off-shell propagator, the corresponding diagrams are not gauge invariant. Note that this complication is a direct consequence of the fact that the exchanged particle is, itself, a graviton - rather than some completely different particle. This is why the three point functions are not automatically gauge invariant when the exchanged particle is off-shell. It follows that the three exchange diagrams discussed in this paragraph cannot be computed simply by sewing the corresponding three point functions with the graviton propagator. In order to recover gauge invariance we must also add in the contribution of contact 4 point terms from the Einstein action (in the case of $\mathscr{A}_{R-R}$), the contribution of the contact term of the Gauss-Bonnet action (in the case of $\mathscr{A}_{R-R^2}$) and the contribution of the contact term from Riemann cube action (in the case of $\mathscr{A}_{R-R^3}$)[9]. A direct computation of four graviton tree level scattering matrix starting with the Lagrangian

$$
S = \int \sqrt{g}\,\left(\alpha_R R + \alpha_{R^2}(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) + \alpha_{R^3}R_{\mu\nu\rho\sigma}R^{\mu\nu ab}R_{ab}{}^{\rho\sigma}\right), \tag{7.1.10}
$$

is algebraically intensive. We use a different method.

We first note that the full gauge invariant result for each of $\mathscr{A}_{R-R}$, $\mathscr{A}_{R-R^2}$ and $\mathscr{A}_{R-R^3}$ is given

[9]For example, consider the $R-R^2$ exchange which occurs at $\mathscr{O}(\alpha_R\alpha_{R^2})$. There is a polynomial contribution to this exchange diagram from the Gauss bonnet term to fourth order in perturbation. Together the exchange diagram and the contact piece are gauge invariant.

by the sum of a term with an $s$ pole, a term with a $t$ pole, a term with a $u$ pole and a polynomial contact term. Every such S-matrix can be manipulated into the form

$$\frac{\sum_i \beta_i S_i}{stu} \tag{7.1.11}$$

Here $S_i$ are the most general local gauge invariant S-matrices at 8 derivative order (in the case of $\mathscr{A}_{R-R}$), 10 derivative order (in the case of $\mathscr{A}_{R-R^2}$), and 12 derivative order (in the case of $\mathscr{A}_{R-R^3}$). Recall that we have already explicitly constructed a basis of all such local S-matrices in section 6 above. $\beta_i$ are the as yet unknown constant coefficients of these basis structures.

In order to determine the as yet unknown constants $\beta_i$ we now impose the following conditions. The expression (7.1.11) is a meromorphic function $\varepsilon_i$ and $k_i$. Holding $\varepsilon_i$ and all other components of $k_i$ constant, for a moment, we note that (7.1.11) has a pole in the variable $s$. We impose the condition that the residue of this pole is the residue of the s channel exchange diagram obtained by sewing the relevant 3 point functions through the graviton propagator (this residue is gauge invariant, because it only samples the 3 point functions when all participating particles are on-shell). This condition unambiguously determines all $\beta_i$ coefficients in the case of the amplitudes $\mathscr{A}_{R-R}$ and $\mathscr{A}_{R-R^2}$. The fact that these two amplitudes are unambiguously determined by their poles is easy to understand. These amplitudes are, respectively, of homogeneity 2 and 4 in derivatives. An ambiguity in these amplitudes would be a gauge invariant 2 or 4 derivative polynomial S-matrix, and we have demonstrated above that no such S-matrix exists.

On the other hand the amplitude $\mathscr{A}_{R-R^3}$ is of homogeneity 6 in derivatives, and so is determined by its poles only up to the addition of the unique 6 derivative local gauge invariant 4 graviton S-matrix (6.3.38). Algebraically we do indeed find that $\beta_i$ are determined only up to this ambiguity[10]. In reporting our answer below we make an arbitrary choice to fix this ambiguity. Our final results are

[10]The fact that $\beta_i$ are determined only up to this ambiguity is very natural from a Lagrangian viewpoint. While the Lagrangians that gave rise to the Einstein and Gauss-Bonnet 3 point functions were unique, the Lagrangians that give rise to the $R^3$ 3 point function have a one parameter ambiguity, parameterized by the coefficient of the second Lovelock terms - which is an $R^3$ term whose contribution to the 3 graviton S-matrix vanishes. It is thus clear that the 4 point function that follows from the exchange of such a vertex has a contribution from the 2nd Lovelock term with an undetermined coefficient.

$$\mathscr{A}_{R-R} = \begin{cases} S^{G_{\mathbf{3},6}} + S^{G_{\mathbf{3},5}} + S^{G_{\mathbf{3},3}} + S^{G_{\mathbf{3},4}} + S^{G_{\mathbf{3},1}} + S^{G_{\mathbf{3},2}}, \\ \mathscr{F}^{G_{\mathbf{3},6}}(s,t) = \frac{-1}{64stu},\ \mathscr{F}^{G_{\mathbf{3},5}}(t,u) = \frac{-1}{32stu},\ \mathscr{F}^{G_{\mathbf{3},3}}(t,u) = \frac{-1}{32stu}, \\ \mathscr{F}^{G_{\mathbf{3},4}}(t,u) = \frac{-1}{8stu}, \mathscr{F}^{G_{\mathbf{3},1}}(t,u) = \frac{1}{256stu}, \mathscr{F}^{G_{\mathbf{3},2}}(t,u) = \frac{1}{128stu}, \end{cases}$$

$$\mathscr{A}_{R-R^2} = \begin{cases} S^{G_{\mathbf{3},6}} + S^{G_{\mathbf{3},5}} + S^{G_{\mathbf{3},3}} + S^{G_{\mathbf{3},4}} + S^{G_{\mathbf{3},8}} + S^{G_{\mathbf{3},7}} + S^{G_{\mathbf{3}_\mathbf{A}}}, \\ \mathscr{F}^{G_{\mathbf{3},6}}(s,t) = \frac{1}{32tu},\ \mathscr{F}^{G_{\mathbf{3},5}}(t,u) = \frac{-1}{16tu},\ \mathscr{F}^{G_{\mathbf{3},3}}(t,u) = \frac{1}{4tu}, \\ \mathscr{F}^{G_{\mathbf{3},4}}(t,u) = \frac{1}{4tu}, \mathscr{F}^{G_{\mathbf{3},8}}(t,u) = \frac{1}{stu}, \mathscr{F}^{G_{\mathbf{3},7}}(t,u) = \frac{-1}{4stu}, \mathscr{F}^{G_{\mathbf{3}_\mathbf{A}}}(t,u) = \frac{-(t-u)}{32stu} \end{cases}$$

$$\mathscr{A}_{R-R^3} = \begin{cases} S^{G_{\mathbf{3},6}} + S^{G_{\mathbf{3},5}} + S^{G_{\mathbf{3},3}} + S^{G_{\mathbf{3},4}} + S^{G_{\mathbf{3},1}} + S^{G_{\mathbf{3},2}} + S^{G_{\mathbf{3},8}} + S^{G_{\mathbf{S},2}} + S^{G_{\mathbf{3}_\mathbf{A}}}, \\ \mathscr{F}^{G_{\mathbf{3},6}}(t,u) = \frac{-3s^2+2t^2+2u^2}{128stu},\ \mathscr{F}^{G_{\mathbf{3},5}}(t,u) = \frac{3}{64s},\ \mathscr{F}^{G_{\mathbf{3},3}}(t,u) = \frac{-2tu-10s^2}{64stu}, \\ \mathscr{F}^{G_{\mathbf{3},4}}(t,u) = \frac{5}{16s}, \mathscr{F}^{G_{\mathbf{3},1}}(t,u) = \frac{3tu+s^2}{512stu}, \mathscr{F}^{G_{\mathbf{3},2}}(t,u) = \frac{-tu-s^2}{256stu}, \mathscr{F}^{G_{\mathbf{3},8}}(t,u) = \frac{-1}{tu}, \\ \mathscr{F}^{G_{\mathbf{S},2}}(t,u) = \frac{-1}{3stu}, \mathscr{F}^{G_{\mathbf{3}_\mathbf{A}}}(t,u) = \frac{-s(t-u)}{128stu}. \end{cases}$$

$$(7.1.12)$$

The non-local effective Lagrangians that generate these amplitudes are

$$\begin{aligned} (stu)\mathscr{A}_{R-R} \propto \Bigg( & \frac{1}{32}(R_{pqrs}R_{pqrs})^2 - \frac{1}{2}R_{pqrs}R_{pqrt}R_{uvws}R_{uvwt} + \frac{1}{16}R_{pqrs}R_{pqtu}R_{tuvw}R_{rsvw} \\ & - \frac{1}{4}R_{pqrs}R_{pqtu}R_{rtvw}R_{suvw} - R_{pqrs}R_{ptru}R_{tvws}R_{qvuw} + \frac{1}{2}R_{pqrs}R_{ptru}R_{tvuw}R_{qvsw} \Bigg) \end{aligned}$$

$$\begin{aligned} (stu)\mathscr{A}_{R-R^2} \propto \Bigg( & -2\ (\nabla_\mu R_{pqrs}R_{pqrt}\nabla^\mu R_{uvwt}R_{uvws}) + 2(\nabla_\mu R_{pqrs}R_{pqtu}\nabla^\mu R_{rtvw}R_{suvw}) \\ & - 8\ (\nabla_\mu R_{pqrs}\nabla^\mu R_{ptru}R_{tvuw}R_{qvsw}) - 8\ (\nabla_\mu R_{pqrs}\nabla^\mu R_{ptuw}R_{tvws}R_{qvru}) \\ & - 8R_{\alpha\beta ab}\nabla_a R_{\beta\gamma cd}\nabla_b R_{\gamma\delta de}R_{\delta\alpha ec} + 2R_{\alpha\beta ab}\nabla_a R_{\beta\alpha cd}\nabla_b R_{\gamma\delta de}R_{\delta\gamma ec} \Bigg) \end{aligned} \qquad (7.1.13)$$

| Exchange | Regge behavior (large $s$, fixed $t$) | Regge behavior after subtraction |
| --- | --- | --- |
| $\mathscr{A}_{R-R}$ | $s^2/t$ | - |
| $\mathscr{A}_{R-R^2}$ | $s^2$ | - |
| $\mathscr{A}_{R-R^3}$ | $s^2 t$ | - |
| $\mathscr{A}_{R^2-R^2}$ | $s^3$ | - |
| $\mathscr{A}_{R^2-R^3}$ | $s^4$ | $s^3 t$ |
| $\mathscr{A}_{R^3-R^3}$ | $s^5$ | $s^4 t$ |

Table 7.1: Regge behavior of exchange diagrams

$$
\begin{aligned}
(stu)\mathscr{A}_{R-R^3} \propto \Bigg( & (-\nabla^\mu\nabla^\nu R_{pqrs}\nabla_\mu\nabla_\nu R_{pqrt}R_{uvwt}R_{uvws} + 2\nabla^\mu\nabla^\nu R_{pqrs}R_{pqrt}\nabla_\mu\nabla_\nu R_{uvwt}R_{uvws} \\
& + \nabla^\mu\nabla^\nu R_{pqrs}\nabla_\nu R_{pqrt}\nabla_\mu R_{uvwt}R_{uvws}) + \frac{3}{2}\nabla^\mu\nabla^\nu R_{pqrs}R_{pqtu}\nabla^\mu R_{rtvw}\nabla^\nu R_{suvw} \\
& - 10\nabla_\mu\nabla_\nu R_{pqrs}\nabla^\mu\nabla_\nu R_{ptru}R_{tvuw}R_{qvsw} - 11\nabla_\mu\nabla_\nu R_{pqrs}\nabla_\nu R_{ptru}R_{tvuw}\nabla^\mu R_{qvsw} \\
& + 10\nabla_\mu\nabla_\nu R_{pqrs}\nabla_\mu R_{ptuw}\nabla_\nu R_{tvws}R_{qvru} + \frac{1}{8}\nabla^\mu\nabla^\nu R_{abcd}R_{abcd}R_{pqrs}\nabla_\mu\nabla_\nu R_{pqrs} \\
& + \frac{5}{16}\nabla^\mu\nabla^\nu R_{abcd}R_{abcd}\nabla_\nu R_{pqrs}\nabla_\mu R_{pqrs} - \frac{1}{4}\nabla_\mu\nabla_\nu R_{pqrs}\nabla^\mu\nabla^\nu R_{pqtu}R_{tuvw}R_{rsvw} \\
& - \frac{3}{8}\nabla_\mu\nabla_\nu R_{pqrs}\nabla^\nu R_{pqtu}R_{tuvw}\nabla^\mu R_{rsvw} - 32\nabla_\mu R_{\alpha\beta ab}\nabla^\mu\nabla_a R_{\beta\gamma cd}\nabla_b R_{\gamma\delta de}R_{\delta\alpha ec} \\
& - \frac{8}{3}R_{abcd}\nabla_a\nabla_c R_{\alpha\beta\gamma\delta}\nabla_b\nabla_d R_{\beta\mu\delta\nu}R_{\mu\alpha\nu\gamma}\Bigg)
\end{aligned} \tag{7.1.14}
$$

While it is not manifest from the expressions above, we have checked that all scattering amplitudes involving the Gauss-Bonnet 3 point function vanishes for $D = 4$ , as expected (recall the Gauss-Bonnet Lagrangian is topological in 4 dimensions; in particular its contribution to 3 graviton scattering vanishes).

## Regge growth

The Regge behavior of the amplitudes constructed in this section is easily determined[11]. In every case the '$t$ channel contributions' (i.e. the terms in the S-matrix that are non polynomial in $t$ when expressed as functions of particle momenta and $\varepsilon_i$) grow no faster than $s^2$, consistent with the fact that we are studying the exchange of a spin 2 particle. All other contributions to the $S$ matrix are analytic in $t$. It follows from dimensional analysis that these remaining contributions can grow no faster than $s$ (in the case of $\mathscr{A}_{R-R}$), or $s^2$ (in the case of $\mathscr{A}_{R-R^2}$). Dimensional analysis would have allowed $\mathscr{A}_{R-R^3}$ $s^3$ growth but we find that the amplitude actually grows more slowly like $s^2 t$.

In the case of $\mathscr{A}_{R^2-R^2}$ the sum of $s$ and $u$ channel exchanges gives rise to an S-matrix that is 6th order in derivatives and grows like $s^3$ - and so faster than $s^2$ - in the Regge limit. It is easy to see that this faster than $s^2$ growth cannot be canceled by a local counter-term. This can be seen in two equivalent ways. First, in our exhaustive classification of local counter-terms earlier in this paper there is only one S-matrix that is of sixth order in derivatives, and this S-matrix grows like $s^2 t$ rather than like $s^3$ in the Regge limit. Equivalently, we have explicitly constructed the Lagrangian that gives rise to the $\mathscr{A}_{R^2-R^2}$ S-matrix (see (7.1.7)) and it simply is not local, even in the Regge limit.

In the case of $\mathscr{A}_{R^3-R^3}$ the S-matrix is 10th order in derivatives and grows like $s^5$ - and so considerably faster than $s^2$ - in the Regge limit. This growth can be slightly ameliorated by counter-term subtractions. The explicit S-matrix for this term is listed in (7.1.7). Notice that in (7.1.7) the functions $\mathscr{F}$ are all polynomials. It follows that all the contributions from these functions can be canceled by local counter-terms. The only piece in $\mathscr{A}_{R^3-R^3}$ that cannot be cancelled by a local counter-term is the part of the S-matrix parameterized by $\mathscr{F}^{G_{3,1}}(t,u) = \frac{-9(t^2+u^2+Dut)}{64(D-2)s}$. After a further local counter-term subtraction we are left with $\mathscr{F}^{G_{3,1}}(t,u) \propto \frac{tu}{t+u}$. In both the $u$ and the $s$ channels the subtracted $\mathscr{F}^{G_{3,1}}(t,u)$ is now proportional to $t$ in the Regge limit, resulting in a (maximally subtracted ) scattering amplitude that scales like $s^4 t$ in the Regge limit.

Finally, in case of $\mathscr{A}_{R^2-R^3}$ the explicit S-matrix (see (7.1.7)) is of 8 derivative order and grows like $s^4$. Once again counter-term subtractions can be used to reduce this growth down to $s^3 t$. In particular, the contribution to this S-matrix from $\mathscr{F}^{G_{3,7}}(t,u) = \frac{3}{2s}$ - a term which clearly cannot be cancelled by a local counter-term - grows like $s^3 t$. It follows that local counter-terms cannot be

[11]As usual one obtains the Regge behavior by explicitly decomposing the polarizations into transverse and parallel components using (2.1.11) and (2.1.13) and evaluating the resulting S-matrix at large $s$, keeping t fixed.

used to further reduce the Regge growth of this S-matrix.

Our final results are summarized in Table 7.1. Plugging the results of Table 7.1 into (7.1.6) we conclude that the only graviton exchange contributions (7.1.6) that grow no faster than $s^2$ in the Regge limit are those with $\alpha_{R^2} = \alpha_{R^3} = 0$.[12][13] Hence, this computation reproduces the CEMZ conclusion i.e adding just the GB and/or the $(Riemann)^3$ correction to Einstein gravity is not allowed. However, the reason for excluding such higher derivative corrections is different: CEMZ used the physical criteria of causality, whereas here we assume the CRG conjecture to be true (which we eventually expect to be proved by physical constraints such as causality, boundedness of energy, and other classical constraints).

## 7.2 Exchange contribution to gravitational scattering and Regge growth

Above we have computed the contribution to graviton scattering from the exchange of the graviton, with the Gauss-Bonnet and $(Riemann)^3$ term added to the Einstein-Hilbert action. Additionally, in [1] we have also computed the contribution to 4-graviton S-matrix from the exchange of a massive scalar, as well as massive spin-2 particle. In each case we have seen that the exchange contributions grow faster than $s^2$ in the Regge limit, and also that this growth cannot be sufficiently tamed (i.e. brought down to growth like $s^2$ or slower) by the subtraction of local counter-term contributions. In this section we argue (under a plausible but not completely justified assumption) that this feature is general: it applies to the contribution to four graviton scattering from the exchange of any particle, atleast for the case when spacetime dimension $D \leq 6$.

### 7.2.1 Regge growth of general exchange contributions

The contribution to gravitational scattering of the exchange of a massive particle of any spin takes the form

$$\mathscr{S} = \frac{|\alpha_1\rangle}{s-m^2} + \frac{|\alpha_2\rangle}{t-m^2} + \frac{|\alpha_3\rangle}{u-m^2}. \tag{7.2.1}$$

Here $|\alpha_i\rangle$ are elements of the local Module (this follows from the fact that 3 point functions are local and gauge invariant).

[12] While $\mathscr{A}_{R-R}$,$\mathscr{A}_{R-R^2}$ and $\mathscr{A}_{R-R^3}$ grow like $s^2$ the remaining 3 amplitudes grow faster than $s^2$. The fact that the coefficient of $\mathscr{A}_{R^2-R^2}$ must vanish forces $\alpha_{R^2}$ to vanish. The fact that the coefficient of $\mathscr{A}_{R^3R^3}$ must also vanish forces $\alpha_{R^3}$ to vanish.

[13] Also note that the six derivative contact term ambiguity that one encounters in $\mathscr{A}_{R-R^3}$, scales as $stu$ and hence is Regge allowed.

Let us suppose that $|\alpha_i\rangle$ are of $2n^{\rm th}$ order in derivatives. It follows that

$$(s-m^2)(t-m^2)(u-m^2)\mathscr{S} \tag{7.2.2}$$

is local, and of degree $2n+4$ in derivatives. We note for later use that the part of (7.2.2) that is of order $2n+4$ in derivative is given by

$$stu\mathscr{S} \tag{7.2.3}$$

It follows that the growth of (7.2.2) is no slower that $s^{\alpha(n+2)+\frac{a}{3}}$ where $\alpha(n)$ is listed in (2.6.8). It then follows that the growth of $\mathscr{S}$ in the Regge limit is at least as fast as $s^{\alpha(n-1)+\frac{a}{3}}$. In the special case that $a=0$ the exchange contributions always grow faster than $s^2$ whenever $n-1>3$, i.e. for $n>4$.

Let us now focus on the borderline 'dangerous' case $n=4$. In this case we obtain an exchange S-matrix $\mathscr{S}$ that grows like $s^2$ in the Regge limit only when the quantity in (7.2.3) is of the form

$$3(stu)^2|g_{\bf S}\rangle$$

where $|g_{\bf S}\rangle$ is a symmetric generator of the bare module (see subsection 2.6) [14]. When this is the case the S-matrix is given by

$$\mathscr{S}=3(stu)|g_{\bf S}\rangle \tag{7.2.4}$$

Comparing (7.2.4) and (7.2.1) we conclude that an exchange S-matrix can have the 'dangerous' $s^2$ growth only if and only if the module elements $|\alpha_i\rangle$ take the form

$$|\alpha_1\rangle = s(stu)|g_{\bf S}\rangle, \quad |\alpha_2\rangle = t(stu)|g_{\bf S}\rangle, \quad |\alpha_3\rangle = u(stu)|g_{\bf S}\rangle. \tag{7.2.5}$$

### 7.2.2 Structure of $|\alpha_i\rangle$ for the case of gravitational scattering.

Let us now specialize to the special case of exchange contributions to four graviton scattering by a particle of general spin, $\chi_{mn...}$. We expect - and assume - that the three point function between $\chi_{mn..}$ and two gravitons to take the schematic form

$$\chi_{mn..}R_{abcd}R_{efgh} \tag{7.2.6}$$

[14] The factor of 3 is inserted for later algebraic convenience.

where all indices are appropriately contracted and the three point functions may also involve extra derivatives[15]. The contribution of the exchange of $\chi_{mn..}$ to four graviton scattering thus leads to an S-matrix of the form (7.2.1) with $|\alpha_i\rangle$ given by descendants of four Riemann structures, i.e. by elements of the module described in subsubsection 6.1.

We have argued in the previous subsubsection that such a contribution can grow like $s^2$ or slower only $|\alpha_i\rangle$ are 8 derivative objects (i.e. are linear combinations of the four Riemann generators described in subsubsection 6.1) and additionally if (7.2.5) holds. However the only multiplet of four R structures that is of the form (7.2.5) are the descendants of $|G_{\mathbf{S},1}\rangle$ (see subsubsection 6.1, in particular see (6.1.3))[16]. We conclude that the only possible exchange contribution that grows no faster than $s^2$ in the Regge limit is one proportional to the S-matrix from $G_{\mathbf{S},1}$. It follows, in particular, that all exchange contributions to graviton scattering in $D \leq 6$ grow faster than $s^2$ in the Regge limit.

Note that while we have not been able to rule out the possibility of an exchange contribution proportional to $G_{\mathbf{S},1}$ in $D \geq 7$, it is entirely possible that such a term is never actually generated[17]. We leave the careful investigation of this point to the future.

### 7.2.3 Counter-term cancellation

Say we have an exchange contribution that grows faster than $s^2$ in the Regge limit. In this section we investigate whether its growth can be cancelled by a local counter-term.

Let us once again focus on S-matrices of the form (7.2.1), and focus on the part of $|\alpha_i\rangle$ that is of 8th order in derivatives. We have just argued that all such terms grow faster than $s^2$ in $D \leq 6$. The denominator in (7.2.1) turns the 8 derivative numerator into a six derivative S matrix. It is immediately clear in $D \leq 6$ that this six derivative term cannot be cancelled by local counter-terms, simply because we have carefully enumerated all available counter-terms earlier in this paper, and all these counter-terms are of 8 or higher order in derivatives when $D \leq 6$.

[15] Note that a coupling of the schematic form $\chi_{mn..}R_{abcd}$ induces mixing between $\chi_{mn..}$ and the graviton at quadratic order. Such couplings are eliminated by field redefinitions. The lowest order couplings that survive after field redefinitions render the Lagrangian diagonal at quadratic level are those of the form (7.2.6).

[16] The fact that no other multiplet of four Riemann structures are of the form (7.2.5) follows from the fact that the module $M_8$ described in subsubsection 6.1 is freely generated. Had another relation like (6.1.3) existed, there would have been null states in $M_8$ - of exactly the same form as the null states of $M_8'$.

[17] Such a term can only be generated $|\alpha_1\rangle = s|G_{\mathbf{S},1}\rangle$ is a sum of 'perfect squares'; it is entirely possible that this is not the case. We hope to address this issue in the future

It follows that it is impossible to use local counter-terms to cancel the offending large $s$ behavior of exchange diagrams unless the 8 derivative part of $|\alpha_i\rangle$ vanish. It seems extremely unlikely that this can happen unless $|\alpha_i\rangle$ itself vanishes[18].

### Cancellation between exchange diagrams

The reader may wonder whether the offending Regge behavior in exchange contributions to gravitational scattering can cancel between themselves. Could, for instance, the contribution from the exchange of a particle at mass $m_1$ in some representation cancel offending part from the exchange of a particle of mass $m_2$ in the same representation? We believe this cannot happen for the reasons we now describe. When the particle exchanged lies in a representation with four or more symmetrized Lorentz indices, it is kinematically obvious that cancellation cannot sufficiently improve Regge behavior. This is because the exchange of such particles lead to violation of $s^2$ growth even in the $t$ channel. The violating contribution in this channel scales like

$$\frac{s^l}{t-m^2}.$$

As the functional form of this amplitude is a function of $t$ with complicated $m^2$ dependence, it is obvious that the Regge growths of particles of different mass cannot cancel each other.

When the particle exchanged lies in a representation with three or fewer symmetrized indices, the faster than $s^2$ Regge growth appears in the $s$ and $u$ channels. The dependence of these violations on $m^2$ are relatively simple. Even though this is the case, two different exchange contributions cannot cancel against each other, simply because each exchange contribution is a perfect square; contributions that are proportional to each other are all of the same sign, and so can only add and never cancel. The positivity demanded above follows from the requirement that all exchange particles have the right sign kinetic term, and that all three point couplings are real - these are both constraints that any sensible classical theory should clearly have.

[18]Exchange contributions are not homogeneous in derivatives. An $|\alpha_i\rangle$ that, for instance, starts out at 10th order in derivatives also has a piece at $8th$ order in derivatives obtained by Taylor expanding the answer in $m^2$.

# Chapter 8

# Conclusion and Discussion

The principal technical accomplishment of this project (which is a part of [1]) is the detailed classification of all polynomial four photon and four graviton S-matrices[1]. The basis of constraining the space of classical S-matrices is using the CRG conjecture. This project has not presented any details as to how the conjecture can be proved, and is left to future work. It is suspected that we should be able to prove the CRG conjecture using classical constraints such as Causality, boundedness of energy and stability. One important point to note is that the CRG conjecture is violated by exchange contributions using the three point scattering amplitudes from the Gauss-Bonnet and three Riemann terms (see (4.1.15)). In other words the CRG conjecture, in addition to constraining four graviton scattering, also gives an alternate derivation of the results for three graviton scattering obtained in [6].

An important assumption made in this project was that the coupling of two gravitons to a massive particle $P$, transforming in a general irrep of $SO(D-1)$ is of the form below:

$$\int \sqrt{-g}\,(RRP) \tag{8.0.1}$$

This is an important assumption and was **proven in a subsequent paper** (see [2] for details). In [2] have demonstrated that every graviton-graviton-$P$ 3 particle S-matrix is generated by a Lagrangian of the form (8.0.1) and so is of atleast fourth order in derivatives. It follows immediately from this observation that every two derivative theory of gravity interacting with other fields admits a

[1] Somewhat unrelated to the main theme of the project, our classification of polynomial S-matrices can be thought of the classification of counter-terms that contribute to four photon and four photon scattering. We thank R. Loganayagam for this observation

consistent truncation to Einstein gravity at cubic order in amplitudes. It seems very likely that this result continues on to arbitrary order. A specification of all 3 point graviton-graviton-$P$ S-matrices completely specifies the Lagrangian (8.0.1). Expanding (8.0.1) in powers of the metric fluctuation $h$ then also specifies a class of (graviton)$^n$ P couplings for $n \geq 3$ (these couplings are tied to the given graviton-graviton-$P$ couplings by diffeomorphism invariance). Of course (graviton)$^n$ P couplings are not uniquely determined by three particle S-matrix data. For instance in the case $n = 4$ we could have additional couplings generated by Lagrangians of the schematic form

$$\int \sqrt{-g}\,(RRRS)\,. \tag{8.0.2}$$

However every such Lagrangian is of 6 or higher order in derivatives. This discussion can be continued. Once we have fixed (graviton)$^3$ P scattering, the new data in (graviton)$^4$ P scattering appears likely to lie at 8 and higher order in derivatives and so on. In particular it seems extremely likely to us that any two derivative theory of gravity coupling to any number of additional fields, just on kinematical grounds, always admits a consistent truncation to Einstein gravity at the full non-linear level.

Finally it would be useful to 'sew' two identical copies of each of the graviton-graviton-$P$ three point functions, classified in [2] through a $P$ propagator in order to compute the explicit form for all kinematically allowed $P$ exchange contributions to four graviton scattering. Conceptually, these contributions are the scattering analogues of conformal blocks. Simple examples of these blocks were constructed in [1]. It would be useful to have explicit expressions for these blocks for the most general case.

In order to complete a classification of classical theories of gravity it is important that we are able to generalize our analysis to the study of 5 and higher point scattering amplitudes as well. Such a study might require a generalization of the CRG conjecture to higher point scattering, a result that would be easiest to obtain once (and if) we are able to prove the CRG conjecture for four particle scattering.

Once several issues (some of which have been stated here) are cleared up, it may be possible to begin a meaningful study of the utterly fascinating possibility that string theory is the unique consistent classical extension of Einstein gravity (as stated in Conjecture 1 of the paper [1]).

# Bibliography


[1] S. D. Chowdhury, A. Gadde, T. Gopalka, I. Halder, L. Janagal and S. Minwalla, JHEP **2002**, 114 (2020) doi:10.1007/JHEP02(2020)114 [arXiv:1910.14392 [hep-th]].

[2] S. Chakraborty, S. D. Chowdhury, T. Gopalka, S. Kundu, S. Minwalla and A. Mishra, arXiv:2001.07117 [hep-th].

[3] P. Kravchuk and D. Simmons-Duffin, JHEP **1802**, 096 (2018) doi:10.1007/JHEP02(2018)096 [arXiv:1612.08987 [hep-th]].

[4] M. S. Costa, J. Penedones, D. Poland and S. Rychkov, JHEP **1111**, 154 (2011) doi:10.1007/JHEP11(2011)154 [arXiv:1109.6321 [hep-th]].

[5] S. A. Fulling, R. C. King, B. G. Wybourne and C. J. Cummins, Class. Quant. Grav. **9**, 1151 (1992). doi:10.1088/0264-9381/9/5/003

[6] X. O. Camanho, J. D. Edelstein, J. Maldacena and A. Zhiboedov, JHEP **1602**, 020 (2016) doi:10.1007/JHEP02(2016)020 [arXiv:1407.5597 [hep-th]].

[7] M. Kulaxizi, A. Parnachev and A. Zhiboedov, JHEP **1806**, 121 (2018) doi:10.1007/JHEP06(2018)121 [arXiv:1705.02934 [hep-th]].

[8] N. Afkhami-Jeddi, T. Hartman, S. Kundu and A. Tajdini, JHEP **1712**, 049 (2017) doi:10.1007/JHEP12(2017)049 [arXiv:1610.09378 [hep-th]].

[9] N. Afkhami-Jeddi, T. Hartman, S. Kundu and A. Tajdini, JHEP **1903**, 201 (2019) doi:10.1007/JHEP03(2019)201 [arXiv:1709.03597 [hep-th]].

[10] J. Maldacena, S. H. Shenker and D. Stanford, JHEP **1608**, 106 (2016) doi:10.1007/JHEP08(2016)106 [arXiv:1503.01409 [hep-th]].

[11] L. Cornalba, arXiv:0710.5480 [hep-th].

[12] I. Heemskerk, J. Penedones, J. Polchinski and J. Sully, JHEP **0910**, 079 (2009) doi:10.1088/1126-6708/2009/10/079 [arXiv:0907.0151 [hep-th]].

[13] G. J. Turiaci and A. Zhiboedov, JHEP **1810**, 034 (2018) doi:10.1007/JHEP10(2018)034 [arXiv:1802.04390 [hep-th]].