EFFECT OF $\mathcal{N} = 2$ SUPERSYMMETRIC GAUSS-BONNET TERM ON BLACK HOLE ENTROPY

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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under the guidance of

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Certificate

This is to certify that this thesis entitled "Effect of $\mathcal{N} = 2$ Supersymmetric Gauss-Bonnet Term on Black Hole Entropy" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by Sukruti Bansal at IISER Pune, under the supervision of Dr. Nabamita Banerjee during the academic year 2014-2015.

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Declaration

I hereby declare that the matter embodied in the report entitled "Effect of $\mathcal{N} = 2$ Supersymmetric Gauss-Bonnet Term on Black Hole Entropy" are the results of the investigations carried out by me at the Department of Physics, Indian Institute of Science Education and Research Pune, under the supervision of Dr. Nabamita Banerjee and the same has not been submitted elsewhere for any other degree.

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Abstract

Black holes are solutions of classical Einstein gravity. They also have information about the quantum nature of the theory of gravity. Thus black holes provide a good laboratory to study some aspects of quantum gravity. In particular, the entropy of black hole is an important property to study. Entropy carries information about the degeneracy of the micro-states associated with a black hole. When the theory gets quantum corrections, the corresponding micro-state degeneracy changes and hence the black hole entropy also changes.

String theory is a promising candidate of quantum gravity. Hence, we study black hole solutions that appear as solutions in string theory. In particular we study supersymmetric and non-supersymmetric black hole solutions in supergravity theory (which is the low energy/classical limit of string theory). We only concentrate on $\mathcal{N} = 2$ supergravity theory. A new class of higher derivative terms has been recently found in this theory. Since these terms appear at higher order in derivatives, they are able to produce quantum effects in the corresponding black hole solutions.

In this thesis, we try to see if a special case of this higher derivative term, known as the Gauss-Bonnet term, affects the entropy of supersymmetric and non-supersymmetric black hole solutions or not. Our results signify that while the term does not change the entropy of supersymmetric black holes, it can potentially change the entropy of non-supersymmetric ones. We further aim to find the exact correction to its entropy.

Contents

Chapter 1

Introduction

1.1 The Big Picture

The ultimate goal of physics is to find a unified theory that could explain all the physical aspects of the universe. Such a hypothetical theory is called the theory of everything. There are four fundamental forces in nature – the electromagnetic force, the weak nuclear force, the strong nuclear force and the gravitational force. The first three forces – electromagnetism, weak and strong forces have been unified under the framework of the Standard Model. But there is yet to be found the theory that could unify these three forces with Einstein's gravity as well. String theory is a candidate for the theory of everything.

The first three forces describe the microscopic world of elementary particles, based on quantum mechanics, while the fourth force, i.e. the gravitational force, describes the dynamics of very massive objects over very large distances. In order to develop a theory of quantum gravity, it would be extremely helpful if we could study objects that need both quantum mechanics and general relativity for their description. Black holes are such objects. They need general relativity since they are extremely big and massive, exerting high gravitational force, and they need quantum mechanics since they emit radiation (known as Hawking radiation), which can be fully explained only by quantum mechanics. Hence, we study the entropy of black holes.

Our calculations have been carried out in the framework of supergravity, which is the low energy/large distance limit of string theory. Supergravity is a macroscopic and effective description of the microscopic description of strings in string theory. In the classical limit, when quantum effects are neglected, supergravity reduces to general relativity, thus providing a bridge between the non-experimentally verifiable string theory and the experimentally verifiable general relativity. Quantum corrections in string theory are effectively encoded in supergravity through new types of interactions between elementary particles (low energy/large distance approximation of strings), known as higher derivative couplings.

Since last several years many studies have been carried out to find the entropy of extremal black holes. The entropy of extremal supersymmetric black holes have been found to consistently take the same value through all the various methods used to compute them. Some of the methods used to compute the entropy of these black holes have been Wald's formalism, one particle irreducible (1PI) action formalism, Wilsonian effective action formalism, dimensional reduction of 5D theory to 4D, Sen's entropy function formalism, etc. This success has led us to ask if such a consistency of results can exist for the entropy of non-supersymmetric black holes as well.

Not much work has been done on the study of the entropy of non-supersymmetric black holes. Kraus and Larsen have given a general expression [4, 5] for the entropy of a class of extremal non-supersymmetric black holes obtained from the reduction of 5 dimensional theory to 4 dimensions, and having the near horizon geometry as $AdS_3 \times S^2$ in 5D. Sen calculated the entropy for non-supersymmetric black holes using his entropy function formalism [6] and found that his results disagree with those obtained from the expression given by Kraus and Larsen. He gave four possibilities to explain this discrepancy out of which he ruled out three. The only possibility that remains to explain the discrepancy is that there are some supersymmetric higher derivative terms missing in the 4D Lagrangian, which would otherwise come from the dimensional reduction of higher derivative invariants from 5 to 4 dimensions.

There are two classes of supersymmetric higher derivative invariants constructed in [7] and [8]. In this thesis we study the higher derivative invariant given in [8]. In chapters 3 and 4, which are largely based on [8, 9], we see the framework in which the new higher derivative term is derived and how it has been arrived at. In chapter 5 we see if the Gauss-Bonnet term (which is a curvature-squared term) obtained from the higher derivative term, contributes to the entropy function of extremal black holes or not.

1.2 Supersymmetry

Supersymmetry is a symmetry that acts on the fields of a theory and not on spacetime. It is like an internal symmetry but still different in nature as will be highlighted later. Supersymmetry transformation converts bosons and fermions into one another. This is analogous to how in the standard model, colour SU(3) group mixes the colour states of a given quark flavour among themselves and the weak SU(2) group mixes fields of weak doublets. Supersymmetry transformations happen via the algebras called superalgebras, which generalize the concept of Lie algebra to include algebraic systems that are defined by both commutators and anticommutators. The schematic form of the algebras is as follows:

$$\{Q, Q'\}_+ = X \qquad [X, X']_- = X'' \qquad [Q, X]_- = Q'' \tag{1.1}$$

Here X, X′ and X" denote the commuting (bosonic) part of the algebra while Q, Q′ and Q" denote the anticommuting (fermionic) part of the algebra. Since the supersymmetry generators, known as *supercharges*, convert particles with half integer spin to those with integer spin and vice versa, they have a spin half of their own, and thus form spinor representations of the Lorentz group in contrast to to the usual Lorentz scalars that generate symmetry transformations.

One feature of supersymmetry that markedly distinguishes it from the standard model is its intimate link to spacetime transformations, unlike the internal symmetries of the standard model. Transforming a field under supersymmetry twice successively gives back the same field at another spacetime position than the initial one (can be seen more precisely from the first equation of 3.12).

In supersymmetry, boson-fermion pairs of equal masses cancel each other's loop corrections exactly, thus making most of the UV divergences of conventional quantum field theory disappear and solving the hierarchy problem of the standard model.

1.3 Supergravity

The supersymmetric extension of general relativity is supergravity. On imposing invariance of a theory under local or gauged supersymmetric transformations, new fields enter into the theory which make it invariant under general coordinate transformations. Such a theory is called supergravity. Supergravity contains gauge fields for both spacetime translations P_a and supersymmetric transformations generated by Q_α. It contains the graviton described by Einstein gravity, along with its superpartner called the gravitino, predicted by supersymmetry.

1.4 Importance of Supersymmetry in Black Hole Entropy

Black holes, being classical solutions of Einstein gravity, appear in supergravity as well. The black holes that appear in supergravity have some special properties. A particular class of black holes, known as Bogolmonyi-Prasad-Sommerfield (BPS) black holes, satisfy an equality leading to unbroken supersymmetry in the spacetime near the black hole. Why this class of black holes has been studied most extensively for calculating black hole entropy so far, is because they can be used to calculate the entropy when the black hole constituents, viz. branes and strings are not interacting with each other. The advantage of studying BPS states is that the interaction free scenario that they describe, gives the value of the entropy which holds even when the couplings get turned on.

Chapter 2

Computation of Black Hole Entropy

2.1 Bekenstein-Hawking Entropy

The concept of the entropy of a black hole first arose when Bekenstein proposed it, on finding parallels between the area of the event horizon of a black hole and thermodynamic entropy. It was known that the horizon area of a black hole can only increase with time and the total horizon area of a black hole formed after the merging of two individual black holes, cannot be smaller than the sum of the horizon areas of the constituent black holes. This characteristic of the horizon area of a black hole showed similarities with the nature of entropy. The existence of black hole entropy could also explain why black holes do not violate the 2^{nd} law of thermodynamics, according to which the entropy of the universe can only increase (and not decrease by the loss of matter and entropy fallen into a black hole).

Hawking, in an attempt to disprove Bekenstein's proposal of black hole entropy, ended up proving that black holes do indeed have entropy, and derived the following relationship between black hole entropy and the area of the event horizon of a black hole:

$$S = \frac{k_b c^3}{G\hbar}\frac{A}{4} \tag{2.1}$$

where k_b is Boltzmann constant, c is the speed of light, A is the area of black hole horizon, G is Newton's gravity constant, and $\hbar$ is the Planck-Dirac constant $h/2\pi$.

2.2 Quantum Corrections to Black Hole Entropy

In the limit of large horizon size, i.e. small curvature, the value of black hole entropy comes out to be fairly accurate. However, in the case of large curvature and strong field strengths at the horizon, quantum corrections need to be added by adding higher derivative corrections to the effective action of string theory, i.e. supergravity. This thesis is about the computation of these quantum corrections and their effect on the entropy function of extremal black holes.

There exists a general formula for black hole entropy in the presence of higher derivative terms, known as Wald's formula. In the extremal limit, Wald's formula for spherically symmetric black holes is given by:

$$S_{BH} = -8\pi \int_H d\theta\, d\phi \, \frac{\delta \mathcal{S}}{\delta R_{rtrt}} \sqrt{-g_{rr}\, g_{tt}}\,, \tag{2.2}$$

where H denotes the horizon of the black hole and $\mathcal{S}$ denotes the action.

2.3 What kind of black holes are we going to study?

In this thesis, we would be looking at a special class of black holes in string theory, known as extremal black holes. The reason for studying them is that theirs is the only class of black holes for which the computation of statistical entropy has been possible so far, which allows its comparison with the Bekenstein Hawking entropy. An extremal black hole is a black hole having the minimal possible mass compatible with given charges and angular momentum. Also, it has zero temperature, and hence it cannot Hawking radiate. However, it has a finite horizon area, due to which it has a finite entropy.

The operational definition of a spherically symmetric extremal black hole in four dimensions, which is required in a general higher derivative theory of gravity, is quoted as the following postulate from [6]:

"In any generally covariant theory of gravity coupled to matter fields, the near horizon geometry of a spherically symmetric extremal black hole in four dimensions has $SO(2,1) \times SO(3)$ isometry."

An extremal black hole can be of two kinds:

i) Supersymmetric/BPS Black Hole: Such a black hole is invariant under a certain number of supersymmetry transformations. It obeys the attractor mechanism, which tells that the near horizon configuration of a black hole depends only on its electric and magnetic charges.

ii) Non-supersymmetric/Non-BPS Black Hole: Such a black hole is non-invariant under any supersymmetry transformation.

2.4 Sen's Entropy Function Formalism

Out of the various possible methods for calculating the entropy of black holes, as mentioned in section 1.1, one is Sen's entropy function formalism [6], which is the method we use. This formalism is applicable only for extremal black holes. Extremal black holes can be both supersymmetric and non-supersymmetric, but supersymmetric black holes are always extremal. Using Sen's entropy function, we get to find the entropy of both – supersymmetric as well as non-supersymmetric black holes, provided they are extremal.

A generic four dimensional spherically symmetric extremal black hole has $AdS_2 \times S^2$

near-horizon metric configuration with constant scalars and constant field strengths. The radii of AdS_2 and S^2 metrics and the values of the scalars and the field strengths are known as near horizon parameters, that depend on the particulars of the theory. Using the entropy function formalism, we find the values of these parameters in terms of the physical charges of the black hole.

This formalism assumes the existence of the full black hole solution. Then using the Lagrangian of the theory (that has the extremal black hole as a solution), it allows to construct a certain function, known as the entropy function, which on extremization with respect to the near-horizon parameters, gives fixed values to those parameters. The value of the entropy function at those extremized values of the parameters gives the entropy of the black hole. Below, we give an example of a four dimensional extremal Reissner-Nordstrom black hole.

2.4.1 Using Sen's Entropy Function Formalism for Reissner - Nordstrom Black Hole

The action of a four dimensional Reissner Nordstrom black hole in Maxwell-Einstein theory is given as

$$\mathcal{S} = \int d^4x \sqrt{-\det g}\, \mathcal{L} \quad \text{where} \quad \mathcal{L} = \frac{1}{16}\pi G_N R - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}\,. \tag{2.3}$$

The notations for Christoffel symbols and Riemann tensor that we use are as following:

$$\begin{aligned}
\Gamma^{\mu}_{\nu\rho} &= \frac{1}{2} g^{\mu\sigma}\left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}\right) \\
R^{\mu}_{\ \nu\rho\sigma} &= \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\tau\rho}\Gamma^{\tau}_{\nu\sigma} - \Gamma^{\mu}_{\tau\sigma}\Gamma^{\tau}_{\nu\rho} \\
R_{\nu\sigma} &= R^{\mu}_{\ \nu\mu\sigma}, \qquad R = g^{\nu\sigma}R_{\nu\sigma}\,.
\end{aligned} \tag{2.4}$$

The most general $AdS_2 \times S^2$ near-horizon background for an extremal Reissner Nordstrom black hole, which is $SO(2,1) \times SO(3)$ symmetric, is given as following:

$$\begin{aligned}
ds^2 &= v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \\
F_{rt} &= e, \qquad F_{\theta\phi} = \frac{p\sin\theta}{4\pi}\,.
\end{aligned} \tag{2.5}$$

Sen's entropy function is given by

$$\mathcal{E}(v_1, v_2, \vec{e}, \vec{q}, \vec{p}) = 2\pi\left(-\tfrac{1}{2}\,\vec{q}.\vec{e} - \int d\theta d\phi \sqrt{-\det g}\, \mathcal{L}\right) \tag{2.6}$$

In this case

$$\int d\theta d\phi \, \sqrt{-\det g}\, \mathcal{L} = 4\pi \, v_1 v_2 \, \left[\frac{1}{16\pi} G_N \, \left(-\frac{2}{v_1} + \frac{2}{v_2} \right) + \frac{1}{2} \, v_1^{-2} e^2 - \frac{1}{2} v_2^{-2} \left(\frac{p}{4\pi} \right)^2 \right] \tag{2.7}$$

Therefore

$$\mathcal{E}(v_1, v_2, \vec{e}, \vec{q}, \vec{p}) = 2\pi \left(-\tfrac{1}{2} \, \vec{q}.\vec{e} - \int d\theta d\phi \, \sqrt{-\det g} \, \mathcal{L} \right)$$

$$= 2\pi \left[-\tfrac{1}{2} q \, e - \frac{1}{4G_N} (2v_1 - 2v_2) - 2\pi \, \frac{v_2}{v_1} \, e^2 + 2\pi \, \frac{v_1}{v_2} \left(\frac{p}{4\pi} \right)^2 \right] \tag{2.8}$$

On extremezing the above function with respect to e, v_1 and v_2, we get

$$e = -\frac{q}{8\pi} \qquad v_1 = v_2 = G_N \, \frac{q^2 + 4p^2}{16\pi} \tag{2.9}$$

Substituting these extremized values of the near-horizon parameters into the entropy function $\mathcal{E}$, we get

$$S_{BH} \equiv \mathcal{E} = \frac{q^2 + 4p^2}{16} \tag{2.10}$$

Hence, we have seen a working example of Sen's entropy function formalism. We shall apply this formalism to calculate the entropy of extremal black holes that appear in $\mathcal{N} = 2$ supergravity, even in the presence of higher derivative corrections. Since $\mathcal{N} = 2$ supergravity is much more involved than the simple Einstein-Maxwell theory, the corresponding computations would be complicated. But the basic idea would be the same, that is, the near horizon ansatz for all fields will be dictated by $SO(2,1) \times SO(3)$ isometry and their evaluations will be solutions of some alegbraic equations.

Chapter 3

Conformal Supergravity[1]

3.1 Why go to conformal supergravity?

Our space is described in the framework of local Poincaré supergravity, but we are going to describe our theory in conformal supergravity or the superconformal formalism. Local Poincaré supergravity and its matter couplings are expressed in terms of large irreducible multiplet representations that transform non-linearly under local supersymmetry. On the other hand, superconformal formalism gets expressed in terms of minimal representations that transform linearly. Moreover, representations of the off-shell superconformal algebra are easier to find than their Poincaré counterparts. Therefore, we go to conformal supergravity. First let us see what is conformal gravity.

3.2 Conformal Gravity

Conformal gravity relies on the conformal group, the largest group of spacetime symmetries of a field theory. Poincaré group is a subroup of the conformal group. Increasing the size of the group from Poincaré to conformal, brings in more fields along with more constraints and hence symmetries. One of the symmetries present in the conformal group i.e. scale invariance, gives the same physics for all energy/mass scales. But we are trying to study quantum corrections to supergravity (which we will come to shortly, after bringing in supersymmetry) through higher derivative terms. Supergravity is an effective theory, i.e. it is a low energy limit of string theory, implying it need not give correct results in the high energy limit (where quantum corrections start playing a role). Therefore, we need to fix the scale gauge of our theory, thus breaking scale invariance. This is done using compensating multiplets.

The conformal group is the group of transformations SO(4,2). It is generated by translations P_a, Lorentz transformations M_{ab}, dilatations $\mathbb{D}$ and conformal boosts K_a or special conformal transformations. Note that the translations and Lorentz transformations of the conformal group, form the Poincaré group, due to which it is a subgroup of the conformal group. The transformation parameters and the gauge fields associated with the four transformation generators of the conformal group are listed in the table below.

[1]This chapter and the next one are substantially based on [8, 9]

generator	P^a	M^{ab}	$\mathbb{D}$	K^a
gauge fields	$e_\mu{}^a$	$\omega_\mu{}^{ab}$	b_μ	$f_\mu{}^a$
parameters	ξ^a	ε^{ab}	Λ_D	Λ_K^a

In the context of conformal gravity, the above mentioned generators are treated to be internal symmetry generators[2], acting on the corresponding (gauge) fields. Internal symmetry transformations act on the gauge fields corresponding to the symmetries of the group, which are the fields required to preserve the symmetry structure of the theory (through certain constraints), but these transformations do not act on spacetime coordinates. Spacetime transformations act on spacetime coordinates, and consequently on all fields too. Internal symmetry transformations mix the fields of the symmetry group and keep the results of the transformations within the domain of the fields, and not that of spacetime coordinates. In conformal gravity, we want to construct an action (which is a functional of fields) invariant under the action of the group. Therefore we allow the conformal generators to act only on the gauge fields of our theory to generate an internal symmetry group, as opposed to space-time transformations.

The infinitesimal transformations of the gauge fields associated with the four symmetry generators mentioned above can be obtained from the algebra of the SO(4,2) group as following:

$$\begin{aligned} \delta e_\mu{}^a &= \mathcal{D}_\mu \xi^a - \Lambda_{\mathrm{D}}\, e_\mu{}^a + \varepsilon^{ab} e_{\mu b} & \qquad \delta\omega_\mu{}^{ab} &= \mathcal{D}_\mu \varepsilon^{ab} + 2\,\Lambda_{\mathrm{K}}{}^{[a}\, e_\mu{}^{b]} + 2\xi^{[a}\, f_\mu{}^{b]} \\ \delta b_\mu &= \partial_\mu \Lambda_{\mathrm{D}} + \Lambda_{\mathrm{K}}{}^a\, e_{\mu a} - \xi^a f_{\mu a} & \qquad \delta f_\mu{}^a &= \mathcal{D}_\mu \Lambda_{\mathrm{K}}{}^a + \Lambda_{\mathrm{D}}\, f_\mu{}^a + \varepsilon^{ab} f_{\mu b} \end{aligned} \tag{3.1}$$

where $\mathcal{D}_\mu$ are the derivatives covariantized with respect to linearly transforming bosonic symmetries which, in this symmetry group, are dilatations and Lorentz transformations. An example of $\mathcal{D}_\mu$ acting on a parameter is

$$\mathcal{D}_\mu \xi^a = (\partial_\mu + b_\mu)\xi^a - \omega_\mu{}^{ab}\xi_b. \tag{3.2}$$

The full covariant derivative D_μ contains $\mathcal{D}_\mu$ along with the conformal boost connection $f_\mu{}^a$ which transforms non-linearly.

In a theory of gravity (constructed using general relativity), transformations are expected to be spacetime transformations or general coordinate transformations, allowing the theory to be local. Therefore, we need to be able to connect the internal symmetry transformations with the spacetime transformations. This is done using covariant curvatures, which are generated by the commutator of covariant derivatives D_μ as following:

$$[D_\mu, D_\nu](\cdot) = \big(-\tfrac{1}{2} R_{\mu\nu}{}^{ab}(M)\mathbb{M}_{ab} + R_{\mu\nu}{}^a(P)\mathbb{P}_a + R_{\mu\nu}(D)\mathbb{D} + R_{\mu\nu}{}^a(K)\mathbb{K}_a \big)(\cdot) \tag{3.3}$$

where $\cdot$ denotes a general covariant field.

The covariant curvatures corresponding to the four symmetry transformations of the

[2]We can do this in the corresponding tangent frame

conformal group are as following:

$$
\begin{aligned}
R(P)_{\mu\nu}{}^{a} &= 2\,\mathcal{D}_{[\mu}e_{\nu]}{}^{a} \\
R(M)_{\mu\nu}{}^{ab} &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\,\omega_{[\mu}{}^{ac}\,\omega_{\nu]c}{}^{b} - 4\,f_{[\mu}{}^{[a}\,e_{\nu]}{}^{b]} \\
R(D)_{\mu\nu} &= 2\,\partial_{[\mu}b_{\nu]} - 2\,f_{[\mu}{}^{a}\,e_{\nu]a} \\
R(K)_{\mu\nu}{}^{a} &= 2\,\mathcal{D}_{[\mu}f_{\nu]}{}^{a}
\end{aligned} \tag{3.4}
$$

Equating the general coordinate transformations to the covariant transformations of the conformal group, gives the following constraints on covariant curvatures, which are known as conventional constraints:

$$
R(P)_{\mu\nu}{}^{a} = 0\,, \qquad R(M)_{\mu\nu}{}^{ab}\,e_{b}{}^{\nu} = 0\,. \tag{3.5}
$$

The transformation rules 3.1 remain unaffected on imposing the conventional constraints 3.5 as these constraints are invariant under Lorentz transformations, dilatations and conformal boosts.

We now intend to construct two conformally invariant Lagrangians. We take a scalar field ϕ, which transforms under dilatations as

$$
\delta_{\mathrm{D}}\phi = w\Lambda_{\mathrm{D}}\phi \tag{3.6}
$$

where w is the Weyl weight of ϕ. Now we write down the various possible conformally covariant derivatives of ϕ that can appear in the Lagrangian, up till the fourth order.

$$
\begin{aligned}
D_\mu\phi &= \mathcal{D}_\mu\phi = \partial_\mu\phi - wb_\mu\phi\,, \\
D_\mu D_a\phi &= \mathcal{D}_\mu D_a\phi + w\,f_{\mu a}\,\phi\,, \\
D_\mu\Box_{\mathrm{c}}\,\phi &= \mathcal{D}_\mu\Box_{\mathrm{c}}\phi + 2(w-1)f_\mu{}^{a}D_a\phi\,, \\
\Box_{\mathrm{c}}\Box_{\mathrm{c}}\,\phi &= \mathcal{D}_a D^a\Box_{\mathrm{c}}\phi + (w+2)f\,\Box_{\mathrm{c}}\phi + 2(w-1)f_{\mu a}\,D^\mu D^a\phi\,,
\end{aligned} \tag{3.7}
$$

where $\Box_{\mathrm{c}} = D_\mu D^\mu$, the fully covariant d'Alembertian. The K-transformations of the derivatives above are as following:

$$
\begin{aligned}
\delta_{\mathrm{K}}D_a\phi &= -\,w\,\Lambda_{\mathrm{K}a}\,\phi\,, \\
\delta_{\mathrm{K}}D_\mu D_a\phi &= -\,(w+1)\left[\Lambda_{\mathrm{K}\mu}\,D_a + \Lambda_{\mathrm{K}a}\,D_\mu\right]\phi + e_{\mu a}\Lambda_{\mathrm{K}}{}^{b}\,D_b\phi\,, \\
\delta_{\mathrm{K}}\Box_{\mathrm{c}}\phi &= -\,2(w-1)\Lambda_{\mathrm{K}}{}^{a}\,D_a\phi\,, \\
\delta_{\mathrm{K}}D_\mu\Box_{\mathrm{c}}\phi &= -\,(w+2)\Lambda_{\mathrm{K}\mu}\,\Box_{\mathrm{c}}\phi - 2(w-1)\Lambda_{\mathrm{K}}{}^{a}\,D_\mu D_a\phi\,, \\
\delta_{\mathrm{K}}\Box_{\mathrm{c}}\Box_{\mathrm{c}}\phi &= -\,2(w-1)\Lambda_{\mathrm{K}}{}^{a}\,\Box_{\mathrm{c}}\,D_a\phi - 2(w+1)\Lambda_{\mathrm{K}}{}^{a}D_a\Box_{\mathrm{c}}\phi\,.
\end{aligned} \tag{3.8}
$$

Since we want conformally invariant transformations, we look for those transformations above which can somehow be set to 0. We find that the following two transformations can be set to 0 by giving particular values to the Weyl weights:

$$
\delta_{\mathrm{K}}\Box_{\mathrm{c}}\phi = 0 \qquad \text{(for } w=1\text{)} \tag{3.9a}
$$

$$
\begin{aligned}
\delta_{\mathrm{K}}\Box_{\mathrm{c}}\Box_{\mathrm{c}}\phi &= 2\,\Lambda_{\mathrm{K}}{}^{a}\left(\Box_{\mathrm{c}}\,D_a - D_a\Box_{\mathrm{c}}\right)\phi && \text{(for } w=0\text{)} \\
&= 2\,\Lambda_{\mathrm{K}}{}^{a}(D^b\left[D_b, D_a\right]\phi + \left[D_b, D_a\right]D^b\phi) && \text{(using Ricci identity and} \\
&= 0 && \text{curvature constraints)}
\end{aligned} \tag{3.9b}
$$

Now we multiply the above two derivatives with a similar scalar field ϕ' having the same Weyl weight as of ϕ, in order to bring kinetic terms into the Lagrangian, thus bringing in dynamical degrees of freedom. We get the following two conformally invariant Lagrangians upto total derivatives:

$$e^{-1}\mathcal{L} \propto \phi' \Box_c \phi = -\mathcal{D}^\mu \phi' \mathcal{D}_\mu \phi + f \phi' \phi \,, \qquad \text{(for } w = 1\text{)} \tag{3.10a}$$

$$e^{-1}\mathcal{L} \propto \phi' \Box_c \Box_c \phi = \mathcal{D}^2 \phi' \mathcal{D}^2 \phi + 2\,\mathcal{D}^\mu \phi' \left[2 f_{(\mu}{}^a e_{\nu)a} - f g_{\mu\nu}\right] \mathcal{D}^\nu \phi \,, \quad \text{(for } w = 0\text{)} \tag{3.10b}$$

Both the Lagrangians are symmetric in ϕ and ϕ'.

Out of the four gauge fields that have been used so far, viz. $e_\mu{}^a, \omega_\mu{}^{ab}, b_\mu$ and $f_\mu{}^a$, only $e_\mu{}^a$ and b_μ are independent, since $\omega_\mu{}^{ab}$ and $f_\mu{}^a$ depend on $e_\mu{}^a$ and b_μ as can be seen from their definitions in A.5. $e_\mu{}^a$ is invariant under K-transformation while b_μ transforms as

$$\delta_K b_\mu = \Lambda_{K\mu} \tag{3.11}$$

In order to keep the Lagrangians 3.10 conformally invariant, all the b_μ dependence in them (which is conformally non-invariant) cancels out.

3.2.1 Analysis of Lagrangian 3.10a

In Lagrangian 3.10a, since the dependence on b_μ, which is the gauge field of dilatations, cancels out, the Lagrangian is invariant under dilatations. This dilatational invariance gives us the freedom to give certain convenient values to its terms, through dilatations. The product $\phi'\phi$ can be made to transform under dilatations with a $\Lambda_{\mathbb{D}}$ such that it gets transformed to a constant. In such a case, the 2^{nd} term would be proportional to $f = f_\mu{}^\mu = \frac{1}{6}\mathcal{R}(e, b)$. Thus it would become proportional to the Ricci scalar, giving us the Einstein Hilbert term.

When ϕ' and ϕ are the same, setting their product constant tells us that that ϕ would be constant, making the kinetic term 0. When ϕ' and ϕ are not the same, their constant product allows to express ϕ' in terms of ϕ, thus enabling to write the kinetic term exclusively in terms of ϕ, proportional to $(\partial_\mu \phi)^2/\phi^2$. In this case the Lagrangian describes a scalar field coupled to Einstein gravity.

3.2.2 Analysis of Lagrangian 3.10b

The scalar fields in Lagrangian 3.10b, having weyl weight 0, cannot be given arbitrary values as in the case of 3.10a as they cannot dilate. Even though this Lagrangian is a four-derivative one, it does not have any curvature squared term in it, which is possible only by the presence of $f_\mu{}^a$ terms in $\Box_c \Box_c \phi$ as given in 3.7. Having set w=0 cancels the $f_\mu{}^a$ terms, due to which there are no curvature squared terms. Yet, this Lagrangian lays out the basic structure of a four derivative conformal Lagrangian, which will be used ahead to develop more involved higher derivatives, which would also end up yielding curvature squared terms.

3.3 $\mathcal{N}$= 2 conformal supergravity

One particular advantage of working in $\mathcal{N}$=2 supergravity is the liberty to close the superymmetry algebra off-shell, which is often not possible in higher extended supersymmetry. In the on-shell formulation, the algebra and transformation rules of the theory depend on the dynamical equations of the fields which vary from theory to theory. Using dynamical equations of motion is particularly more complicated in the case of higher derivative interactions, due to which it becomes even more necessary to close our algebra off-shell.

3.3.1 Weyl Multiplet - the gauge multiplet

Conformal supergravity allows to describe gravity as a gauge theory of the superconformal group. In order to build this theory, its gauge and matter structure are required. Weyl multiplet is the supermultiplet containing all the gauge fields of $\mathcal{N}$=2 superconformal algebra. The matter multiplet of conformal supergravity is the chiral multiplet, which would be presented shortly.

The superconformal group has four more generators added to the pre-existing four generators of the conformal group. All the generators of the superconformal group along with their corresponding gauge fields and parameters are listed in the table below:

generator	P_a	M_{ab}	$\mathbb{D}$	K_a	Q_i	S_i	$V_i{}^j$	$\mathbb{A}$
gauge fields	$e_\mu{}^a$	$\omega_\mu{}^{ab}$	b_μ	$f_\mu{}^a$	$\psi_\mu{}^i$	$\phi_\mu{}^i$	$\mathcal{V}_\mu{}^i{}_j$	A_μ
parameters	ξ^a	ε^{ab}	Λ_D	Λ_K^a	ϵ^i	η^i	$\Lambda_{\mathrm{SU(2)}}{}^i{}_j$	$\Lambda_{\mathrm{U(1)}}$

i) $Q^i(Q_i)$ are the Q supersymmetry generators of super Poincaré algebra and are hence physical. They satisfy the following algebra:

$$\begin{aligned} \{Q_i, \bar{Q}^j\} &= -2\delta^i_j(\gamma^a)P_a, \qquad \{Q_i, Q_j\} = 2\varepsilon_{ij}g(\Lambda), \\ [M_{ab}, \bar{Q}^i] &= -\frac{1}{2}\bar{Q}^i(\gamma_{ab}) \qquad [P_a, \bar{Q}^i] = -\frac{1}{2}\bar{Q}^i(\gamma_a) \end{aligned} \tag{3.12}$$

This algebra is invariant under the *R-symmetry group* U(2) $\simeq$ SU(2)xU(1), which commutes with the Lorentz group, rotating the supercharges, while leaving the supersymmetry algebra invariant.

ii) $S^i(S_i)$ are the new supercharges on which the commutator between $Q^i(Q_i)$ and K_a closes. They generate an auxiliary fermionic symmetry, S-supersymmetry, which is unphysical.

$$[K_a, Q^i] = -\gamma_a S^i \qquad \{S^i, S_j\} = \gamma^a K_a \delta^i_j \tag{3.13}$$

The anti-commutator between Q^i and S_j closes only on the inclusion of R-symmetry algebra U(2) [ref 70].

iii) $V_i{}^j$ are the generators of SU(2) symmetry. They are anti-hermitian and traceless, i.e. $\mathcal{V}_\mu{}^i{}_j = -\mathcal{V}_{\mu j}{}^i$ and $\mathcal{V}_\mu{}^i{}_i$.

iv) $\mathbb{A}$ is the generator of U(1) symmetry.

Due to the addition of Q and S supersymmetry, the transformation rules of gauge fields 3.1 and their associated curvatures 3.4 get modified by the addition of extra terms (see A.3 and A.6 for details). The off-shell degrees of freedom are reduced due to the gauge invariances brought in by superconformal gravity. In order to preserve the degrees of freedom of the theory, three auxiliary fields need to be added, which complete the Weyl supermultiplet.

The three fields are:

i) $T_{ab}{}^{ij}$ - an (anti-) selfdual tensor, anti-symmetric in both ab and ij indices, hence being a singlet under SU(2) transformations

ii) D - a real scalar field

iii) χ^i - a chiral spinor.

The addition of these fields balances fermionic and bosonic degrees of freedom, which is necessary in a supersymmetric theory.

$\omega_\mu{}^{ab}$ and $f_\mu{}^a$ depend on $e_\mu{}^a$ and b_μ while ϕ^i_μ depends on other gauge fields, as can be seen from their definitions in A.5. All the other gauge fields of the Weyl supermultiplet except these three are independent.

3.3.2 Chiral Multiplet - the matter multiplet

We now introduce the matter multiplets of our theory, that would act as compensators required for fixing the gauge of conformal supergravity in order to get physical results from it. First let us look at the compact form of a chiral multiplet, which is given by a chiral superfield. A superfield $\Phi(x^\mu, \theta^i, \bar{\theta}^i)$ is a Taylor expansion in terms of the Grassman coordinates or the θ coordinates. The coefficients of the θ coordinates are functions of both bosonic coordinates x^μ and fermionic coordinates θ^i, where $i = 1, 2...$ $\mathcal{N}$. The general form a chiral superfield (with fermionic indices suppressed) is given as following:

$$\Phi(x^\mu, \theta^i, \bar{\theta}^i) = A + \bar{\theta}^i \Psi_i + \frac{1}{2}\bar{\theta}^i\theta^j B_{ij} + \frac{1}{2}\varepsilon_{ij}\bar{\theta}^i\gamma^{ab}\theta^j F^-_{ab} + \frac{1}{3}[\varepsilon_{ij}\bar{\theta}^i\gamma^{ab}\theta^j]\bar{\theta}^{k\delta}(\gamma^{ab})^\epsilon_\delta\Lambda_{\kappa\varepsilon} + \frac{1}{12}[\epsilon_{ij}\bar{\theta}^i\gamma^{ab}\theta^j]^2 C \tag{3.14}$$

From this compact expression of a superfield, one can read off the components of the chiral multiplet, which are the coefficients of the θ coordinates in 3.14. Note that this expansion is always finite due to the anti-commuting properties of Grassman variables. Thus, the chiral supermultiplet in $\mathcal{N}=2$ supergravity has the following components:

(1) A: a complex scalar field with the lowest Weyl weight

(2) ψ_i: an SU(2) doublet fermion, one of the two components of Majorana spinor

(3) B_{ij}: a complex SU(2) triplet of scalars

(4) F^-_{ab}: a complex anti-selfdual Lorentz tensor

(5) Λ_i: another SU(2) doublet fermion, the other component of Majorana spinor

(6) C: a complex scalar field with the highest Weyl weight

The weyl and chiral weights of the multiplet components give the numerical coefficients of the components while undergoing dilatations and U(1) transformations respectively. For a chiral multiplet, weyl and chiral weights are related by c=$-w$. They are listed in the table below:

	A	Ψ_i	B_{ij}	F^-_{ab}	Λ_i	C
w	w	$w+\frac{1}{2}$	$w+1$	$w+1$	$w+\frac{3}{2}$	$w+2$
c	$-w$	$-w+\frac{1}{2}$	$-w+1$	$-w+1$	$-w+\frac{3}{2}$	$-w+2$

Table 3.1: The weyl and chiral weights of $\mathcal{N}=2$ chiral supermultiplet components with reference to the arbitrary weyl weight w of the lowest weyl weight component A

The transformations of the chiral supermultiplet components under Q- and S-supersymmetry in curved superspace, are given as following:

$$
\begin{aligned}
\delta A =& \bar{\epsilon}^i \Psi_i\,,\\
\delta \Psi_i =& 2\,\not{D} A \epsilon_i + B_{ij}\,\epsilon^j + \tfrac{1}{2}\gamma^{ab} F^-_{ab}\,\varepsilon_{ij}\epsilon^j + 2\,wA\,\eta_i\,,\\
\delta B_{ij} =& 2\,\bar{\epsilon}_{(i}\not{D}\Psi_{j)} - 2\,\bar{\epsilon}^k \Lambda_{(i}\,\varepsilon_{j)k} + 2(1-w)\,\bar{\eta}_{(i}\Psi_{j)}\,,\\
\delta F^-_{ab} =& \tfrac{1}{2}\varepsilon^{ij}\,\bar{\epsilon}_i \not{D}\gamma_{ab}\Psi_j + \tfrac{1}{2}\bar{\epsilon}^i\gamma_{ab}\Lambda_i - \tfrac{1}{2}(1+w)\,\varepsilon^{ij}\bar{\eta}_i\gamma_{ab}\Psi_j\,,\\
\delta \Lambda_i =& -\tfrac{1}{2}\gamma^{ab}\not{D}F^-_{ab}\epsilon_i - \not{D}B_{ij}\varepsilon^{jk}\epsilon_k + C\varepsilon_{ij}\,\epsilon^j + \tfrac{1}{4}\big(\not{D}A\,\gamma^{ab}T_{abij} + w\,A\,\not{D}\gamma^{ab}T_{abij}\big)\varepsilon^{jk}\epsilon_k\\
&- 3\,\gamma_a\varepsilon^{jk}\epsilon_k\,\bar{\chi}_{[i}\gamma^a\Psi_{j]} - (1+w)\,B_{ij}\varepsilon^{jk}\,\eta_k + \tfrac{1}{2}(1-w)\,\gamma^{ab}\,F^-_{ab}\eta_i\,,\\
\delta C =& -2\,\varepsilon^{ij}\bar{\epsilon}_i\not{D}\Lambda_j - 6\,\bar{\epsilon}_i\chi_j\,\varepsilon^{ik}\varepsilon^{jl}B_{kl}\\
&- \tfrac{1}{4}\varepsilon^{ij}\varepsilon^{kl}\big((w-1)\,\bar{\epsilon}_i\gamma^{ab}\not{D}T_{abjk}\Psi_l + \bar{\epsilon}_i\gamma^{ab}T_{abjk}\not{D}\Psi_l\big) + 2\,w\varepsilon^{ij}\bar{\eta}_i\Lambda_j\,.
\end{aligned}
\tag{3.15}
$$

The derivatives D in the equations above are the fully covariantized derivatives for each component. They include the derivatives $\mathcal{D}$, which are covariant with respect to all the linearly transforming bosonic symmetries, viz, dilatations, U(1) and SU(2) transformations, and Lorentz transformations. The special conformal transformations K_a, which are also bosonic, are not included in $\mathcal{D}$ as they transform non-linearly, but are included in D. D also includes the covariantizations with respect to fermionic symmetries. A noteworthy feature of the supersymmetry transformations in 3.15 is their linearity.

3.3.3 Vector Multiplet - a reduced chiral multiplet

As has already been mentioned in section 3.2, we need compensating multiplets to fix the gauge of the theory in order to break scale invariance. Chiral multiplets have the potential to act as compensators on being reduced to certain special kinds of multiplets, such as the vector multiplet. A vector multiplet is a special case of the chiral multiplet which is obtained when the chiral mutliplet has weyl weight 1 and obeys the following constraint:

$$\nabla_{ij}\mathcal{X} = \varepsilon_{ik}\varepsilon_{jl}\,\bar{\nabla}^{kl}\bar{\mathcal{X}}\ , \tag{3.16}$$

where $\mathcal{X}$ represents any component of the vector multiplet and ∇_{ij} is a 2nd order antisymmetric tangent space derivative in curved superspace (see appendix A.2). Written explicitly, the constraints are as following:

$$\begin{aligned}
&B_{ij} = \varepsilon_{ik}\varepsilon_{jl}B^{kl} && \longleftarrow \quad \text{reality constraint on } B_{ij}\\
&\Lambda_i = -\varepsilon_{ij}\partial\!\!\!/\psi^j && \\
&C = -2\partial^2\bar{A} && \\
&\partial_a F^{-ab} = \partial_a F^{+ab} && \longleftarrow \quad \text{Bianchi identity for } F^{ab} \text{ which}\\
& && \qquad\ \text{is now a physical field strength}
\end{aligned} \tag{3.17}$$

The last constraint shows most explicitly how the introduction of the vector multiplet makes the theory physical. The independent fields of the vector multiplet are denoted by $(X, \Omega_i, Y_{ij}, \mathcal{F}^-_{ab})$. They are connected to the components of the chiral supermultiplet as following:

$$\begin{aligned}
A|_{\text{vec}} &= X\ ,\\
\psi_i|_{\text{vec}} &= \Omega_i\ ,\\
B_{ij}|_{\text{vec}} &= Y_{ij} = \varepsilon_{ik}\varepsilon_{jl}\,Y^{kl}\ ,\\
F^-_{ab}|_{\text{vec}} &= \mathcal{F}^-_{ab} = F^-_{ab} + \tfrac{1}{4}\,(\bar{\psi}_\rho{}^i\gamma_{ab}\gamma^\rho\,\Omega^j + \bar{X}\,\bar{\psi}_\rho{}^i\gamma^{\rho\sigma}\gamma_{ab}\psi_\sigma{}^j - \bar{X}\,T_{ab}{}^{ij})\epsilon_{ij}\ ,\\
\Lambda_i|_{\text{vec}} &= -\varepsilon_{ij}\,D\!\!\!\!/\,\Omega^j\ ,\\
C|_{\text{vec}} &= -2\Box_c\bar{X} - \tfrac{1}{4}\mathcal{F}^+_{ab}\,T^{ab}{}_{ij}\epsilon^{ij} - 3\bar{\chi}_i\Omega^i\ ,
\end{aligned} \tag{3.18}$$

where $\mathcal{F}_{ab}$ is the supercovariantization of the abelian field strength W_μ, i.e. $\mathcal{F}_{ab} = 2e_a{}^{[\mu}e_b{}^{\nu]}\partial_\mu W_\nu$.

The weyl and chiral weights of the vector multiplet components are shown in the following table:

	X	Ω_i	W_μ	Y_{ij}
w	1	$\frac{3}{2}$	0	2
c	-1	$-\frac{1}{2}$	0	0

Table 3.2: The weyl and chiral weights of $\mathcal{N} = 2$ vector supermultiplet with reference to the arbitrary weyl weight w of the lowest weyl weight component X

The Q- and S-supersymmetry transformation rules for the vector multiplet components in superconformal background are as following:

$$\begin{aligned}
\delta X &= \bar{\epsilon}^i \Omega_i \, , \\
\delta \Omega_i &= 2 \not{D} X \epsilon_i + \tfrac{1}{2} \varepsilon_{ij} \hat{F}_{\mu\nu} \gamma^{\mu\nu} \epsilon^j + Y_{ij} \epsilon^j + 2\, X\, \eta_i \, , \\
\delta W_\mu &= \varepsilon^{ij} \bar{\epsilon}_i \left(\gamma_\mu \Omega_j + 2\, \psi_{\mu j}\, X \right) + \varepsilon_{ij} \bar{\epsilon}^i \left(\gamma_\mu \Omega^j + 2\, \psi_\mu{}^j\, \bar{X} \right) , \\
\delta Y_{ij} &= 2\, \bar{\epsilon}_{(i} \not{D} \Omega_{j)} + 2\, \varepsilon_{ik} \varepsilon_{jl}\, \bar{\epsilon}^{(k} \not{D} \Omega^{l)} \, .
\end{aligned} \tag{3.19}$$

Now that all the multiplets required for superspace and component calculus have been presented, along with their transformation rules under supersymmetry, we can go on to see the construction of the Lagrangians in superconformal formalism.

Chapter 4

A New Higher Derivative Term

In order to calculate quantum corrections to the entropy of a black hole, we need to add higher derivative terms to the Lagrangian of the black hole solution. These terms encode finer interactions between the strings and D-branes composing the black hole.

4.1 Previously found higher derivative terms

There have been quite many higher derivative terms found and studied previously, such as those containing the field strengths for supersymmetric gauge theories [10, 11, 12, 13, 14], those involving a chiral invariant having square of the Weyl tensor, by Bergshoeff, et al. [15], and those formed from tensor multiplets by de Wit, et al [16]. In the context of minimal Poincaré supergravity, an $\mathcal{R}^4$ term has been generated using full superspace integral [17]. Butter and Kuzenko found higher derivative Lagrangians in projective superspace in [18]. Bernard de Wit, et al found another class of higher derivatives [7] in conformal superspace using superconformal multiplet calculus. Their higher derivative vanishes identically and trivially in a supersymmetric background, along with its first derivative (with respect to the fields or the coupling constants) owing to the linear nature of the supersymmetry transformation of the multiplet used. Thus, their higher derivative invariant does not contribute to the entropy of supersymmetric black holes at all.

The higher derivative invariant that we are going to study, the one found by Butter, et al in [8] is constructed along similar lines as the one in [7], but this one does not vanish trivially for supersymmetric black holes.

4.2 Superconformal Lagrangian in Flat Superspace

4.2.1 General Form

We have already seen the construction of conformally invariant Lagranigans 3.10 with a scalar field ϕ. Now, we work in chiral superspace with a chiral supermultiplet instead of the field ϕ. Hence, we define the general form of a Lagrangian in chiral superspace, which is:

$$\mathcal{L} = \int \mathrm{d}^4\theta \, \Phi^2 \, . \tag{4.1}$$

Integrals of this kind are known as *chiral superspace integrals*. Due to the nature of Grassmanian coordinates, i.e. θ, an integration with respect to $\mathrm{d}^4\theta$, allows only the highest θ-component of the chiral superfield , i.e. C, to survive.

4.2.2 Two-derivative Lagrangian in Flat Superspace

Once the Lagrangian has been expressed in terms of the chiral multiplet components, in this case, as $\sim C$, we need to gauge fix the chiral mutliplet, reducing it to a vector multiplet as shown in section 3.3.3, in order to reduce the anti-selfdual tensor F_{ab}^- to the field strength of the physical gauge field W_μ, and hence bring dynamical degrees of freedom into the theory. Following this multiplet reduction and other manipulations to conserve the weyl and chiral weight of the terms in the Lagrangian, leads to the following two-derivative Lagrangian [9]:

$$\mathcal{L} = -\tfrac{1}{2}\,C|_{\mathcal{X}^2} = 2\,X\Box\bar{X} - \tfrac{1}{2}\bar{\Omega}^i\,\overleftrightarrow{\not{\partial}}\,\Omega_i - \tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{4}\,Y_{ij}\,Y^{ij}\,. \tag{4.2}$$

Having shown above the general way for constructing Lagrangians in chiral superspace (for Lagrangian 4.2 quadratic in space-time derivatives, like 3.10a), we can now proceed to the construction of the higher derivative Lagrangian (like 3.10b). To do so, we need to introduce a new kind of multiplet, called the kinetic multiplet.

4.2.3 Four-derivative Lagrangian in Flat Superspace

The kinetic multiplet brings 2$^{\text{nd}}$ derivative terms into the Lagrangian, along with dynamical degrees of freedom, thus making it physical. The kinetic multiplet first appeared in [19], in the context of $\mathcal{N}$=1 tensor calculus, to construct the Lagrangian for kinetic terms. In $\mathcal{N}$=2 flat supergravity, the full superspace Lagrangian, which is the full superspace integral (i.e. involving both chiral and anti-chiral Grassman coordinates), is given up to total derivative terms as:

$$\int \mathrm{d}^4\theta\,\mathrm{d}^4\bar{\theta}\,\Phi'\,\bar{\Phi} = \int \mathrm{d}^4\theta\,\Phi'\,\big(\bar{D}^4\bar{\Phi}\big) = A'\,\Box\Box\bar{A} + \cdots \tag{4.3}$$

where $\bar{D}^4 = \frac{1}{48}\varepsilon_{ik}\varepsilon_{jl}\,\bar{D}^{ij}\,\bar{D}^{kl}$ (see appendix A.2) and A and A' are the lowest-θ components of the chiral superfields Φ and Φ', respectively (see A.7 for the product of two chiral multiplets).

4.3 Kinetic Multiplet

In $\mathcal{N}$=1 case where the kinetic multiplet was used for the first time, it was given by $\mathbb{T}(\bar{\Phi}) \sim \bar{D}^2\bar{\Phi}$, i.e. the 2$^{\text{nd}}$ derivative of the chiral multiplet, due to which it was called the kinetic multiplet. On generalising it to $\mathcal{N}$=2 supergravity, the kinetic multiplet becomes $\mathbb{T}(\bar{\Phi}) := -2\,\bar{D}^4\bar{\Phi}$ (being normalised conventionally). The flat-space components of the $\mathbb{T}(\bar{\Phi})$ are given in terms of the components of the anti-chiral multiplet $(\bar{\Phi})$

as following [8]:

$$\begin{aligned}
A|_{\mathbb{T}(\bar{\Phi})} &= \bar{C}\,, & \Psi_i|_{\mathbb{T}(\bar{\Phi})} &= -2\,\varepsilon_{ij}\,\partial\!\!\!/\Lambda^j\,,\\
B_{ij}|_{\mathbb{T}(\bar{\Phi})} &= -2\,\varepsilon_{ik}\varepsilon_{jl}\,\Box B^{kl}\,, & F^-_{ab}|_{\mathbb{T}(\bar{\Phi})} &= -4\big(\delta_a{}^{[c}\delta_b{}^{d]} - \tfrac{1}{2}\varepsilon_{ab}{}^{cd}\big)\partial_c\partial^e F^+_{ed}\,,\\
\Lambda_i|_{\mathbb{T}(\bar{\Phi})} &= 2\,\Box\,\partial\!\!\!/\Psi^j\varepsilon_{ij}\,, & C|_{\mathbb{T}(\bar{\Phi})} &= 4\,\Box\Box\bar{A}\,.
\end{aligned} \tag{4.4}$$

We want the kinetic multiplet $\mathbb{T}(\bar{\Phi}\,)$ to be that of a conformal primary superfield, for which it is required that its primary field, which is the lowest θ-component, i.e. $\bar{C}$, is conformally invariant and hence S-supersymmetric (see the 2nd relation of 3.13). For $\bar{C}$ to be invariant under S-supersymmetry, the weyl weight of the chiral multiplet $\bar{\Phi}$ should be 0 (as can be seen from the supersymmetry transformation rules 3.15). Therefore, the weyl weight of the chiral multiplet Φ (same as that of the antichiral multiplet $\bar{\Phi}$) in the $\mathbb{T}(\bar{\Phi})$ multiplet is 0. This implies that the weyl weight of the highest θ-component of Φ, i.e. C, is 0+2=2, which is the same as the weyl weight of $\bar{C}$. Therefore the weyl weight of $\mathbb{T}(\bar{\Phi})$ = weyl weight of its lowest θ-component, i.e. $\bar{C}$, is 2.

4.4 Superconformal Lagrangian in Curved Superspace

In order to generalize the flat superspace Lagrangian to curved superspace, it is required to substitute the flat superspace derivatves (∂_a, D_i, D^i) with the covariant derivatives $(\nabla_a, \nabla_i, \nabla^i)$ in curved superspace. These substitutions mainly take place firstly during the imposition of chirality constraint on the superfield of full superspace to make it a chiral superfield, and while imposing constraint 3.16 on the chiral multiplet for reducing it to a vector multiplet. The general form of full curved superspace integral, involving integrations over four chiral and four anti-chiral Grassman coordinates, is given as

$$\int d^4\theta\, d^4\bar{\theta}\, E\, \mathscr{L}. \tag{4.5}$$

for some superspace Lagrangian density $\mathscr{L}$. $E = \mathrm{Ber}(E_M{}^A)$, the measure factor, is the Berzinian (or superdeterminant) of the superspace vielbein.

The general form of a curved superspace Lagrangian in chiral superspace is

$$\mathcal{L}_{\rm ch} = \int d^4\theta\, \mathcal{E}\, \mathscr{L}_{\rm ch}\,, \tag{4.6}$$

where $\mathcal{E}$ is the measure of chiral superspace. $\mathscr{L}_{\rm ch}$ should be covariantly chiral, i.e. $\bar{\nabla}^i \mathscr{L}_{\rm ch} = 0$. The chiral superspace Lagrangian written in terms of the general full superspace Lagrangian $\mathscr{L}$ in expression 4.5 is

$$\int d^4\theta\, d^4\bar{\theta}\, E\, \mathscr{L} = \int d^4\theta\, \mathcal{E}\, \bar{\nabla}^4 \mathscr{L} \tag{4.7}$$

where the chiral projection operator $\bar{\nabla}^4 = \frac{1}{48}\varepsilon_{ik}\varepsilon_{jl}\bar{\nabla}^{kl}\bar{\nabla}^{ij}$ (see appendix A.2).

This is similar to obtaining chiral flat superspace Lagrangian from full flat superspace

Lagrangian as shown in equation 4.3. Comparing equation 4.6 with equation 4.7, we can see that $\mathscr{L}_{\text{ch}} = \bar{\nabla}^4 \mathscr{L}$. It should always be ensured that this Lagrangian is superconformally invariant, by checking that it is covariantly chiral and invariant under S-supersymmetry.

The form of $\mathcal{L}_{\text{ch}}$ in equation 4.6 was derived by in [21] by demanding superconformal invariance and the conservation of weyl and chiral weights of the terms used in the the equation. It is given as:

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{ch}} = {} & C - \varepsilon^{ij}\,\bar{\psi}_{\mu i}\gamma^{\mu}\Lambda_j - \tfrac{1}{8}\bar{\psi}_{\mu i}T_{ab\,jk}\gamma^{ab}\gamma^{\mu}\Psi_l\,\varepsilon^{ij}\varepsilon^{kl} - \tfrac{1}{16}A(T_{ab\,ij}\varepsilon^{ij})^2 \\ & - \tfrac{1}{2}\bar{\psi}_{\mu i}\gamma^{\mu\nu}\psi_{\nu j}\,B_{kl}\;\varepsilon^{ik}\varepsilon^{jl} + \varepsilon^{ij}\bar{\psi}_{\mu i}\psi_{\nu j}(F^{-\mu\nu} - \tfrac{1}{2}A\,T^{\mu\nu}{}_{kl}\,\varepsilon^{kl}) \\ & - \tfrac{1}{2}\varepsilon^{ij}\varepsilon^{kl}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu i}\psi_{\nu j}(\bar{\psi}_{\rho k}\gamma_{\sigma}\Psi_l + \bar{\psi}_{\rho k}\psi_{\sigma j}\,A) + \text{h.c.}\,. \end{aligned} \tag{4.8}$$

C has weyl weight $w = 2$, as mentioned at the end of section 4.3. The chiral superspace Lagrangian 4.8 is a general and important one, describing the couplings of a $w = 2$ chiral multiplet with the background superconformal fields. It would be used as the template for constructing all the other Lagrangians ahead in this thesis, by substituting different kinds of $w = 2$ chiral multiplets as per our requirements.

One requirement for the kind of chiral mutliplet to be considered in Lagrangian 4.8, stems from the fact that there are no kinetic terms for fields in it, making it unphysical. For this purpose, we need to introduce a kinetic multiplet, as was done in the case of flat superspace in equation 4.3.

4.4.1 Two-derivative Lagrangian in Curved Superspace

Now that we have the general form of a superconformal Lagrangian in curved superspace in terms of a chiral multiplet, we would try to arrive at the two-derivative Lagrangian in terms of the vector multiplet, which is the physical Lagrangian. The curved superspace generalisation of Lagrangian 4.1, is:

$$\int d^4\theta\,\mathcal{E}\,\Phi^2\,. \tag{4.9}$$

This Lagrangian has been expressed in terms of multiplet components in [21]. Following the method of [22], in order to write Lagrangian 4.9 in terms of multiplet components, the lowest component of a composite chiral multiplet, i.e. $A|_{\text{comp}}$ should be of the form

$$A|_{\text{comp}} = F(X^I)\,.$$

where $F(X^I)$ is a function of the scalar fields X^I belonging to the vector multiplet. F is required to be holomorphic, i.e. dependent only on X^I, in order to transform chirally. Also, supersymmetry invariance of Lagrangian 4.8 demands that its weyl weight be 2, which is possible only if F is homogeneous with degree 2 (as the weyl weight of the lowest θ-component of the vector multiplet, X^I is 1). F should have the following general form:

$$F(\lambda\,X^I) = \lambda^2\,F(X^I)\,,$$

for any complex $\lambda \neq 0$. F is called the *prepotential*. Its first and second derivatives with respect to X^I are written as F_I and F_{IJ}. Once the lowest θ-component of a composite chiral multiplet is known, all the remaining components can be obtained using the results of multiplet calculus (see A.8). Substituting these components into Lagrangian 4.8, while setting the fermionic terms to 0, gives the following Lagrangian:

$$\begin{aligned} e^{-1}\,\mathcal{L} \;=\;& \mathrm{i}\mathcal{D}^\mu F_I\,\mathcal{D}_\mu \bar{X}^I + \mathrm{i}F_I\,\bar{X}^I(\tfrac{1}{6}\mathcal{R}+D) - \tfrac{1}{8}\mathrm{i}F_{IJ}\,Y^I_{ij}Y^{Jij} \\ & +\tfrac{1}{4}\mathrm{i}F_{IJ}(F^{-I}_{ab} - \tfrac{1}{4}\bar{X}^I T^{ij}_{ab}\varepsilon_{ij})(F^{-Jab} - \tfrac{1}{4}\bar{X}^J T^{ijab}\varepsilon_{ij}) \\ & -\tfrac{1}{8}\mathrm{i}F_I(F^{+I}_{ab} - \tfrac{1}{4}X^I T_{abij}\varepsilon^{ij})T^{ab}_{ij}\varepsilon^{ij} - \tfrac{1}{32}\mathrm{i}F(T_{abij}\varepsilon^{ij})^2 + \text{h.c.} \end{aligned} \tag{4.10}$$

This two-derivative Lagrangian is the same as the one used by Sahoo and Sen in [23] in which they calculated the entropy of extremal black holes by adding a particular higher derivative term to this Lagrangian, which, for non-supersymmetric black holes, did not give the value predicted by Kraus and Larsen in [4, 5]. The new Gauss-Bonnet Lagrangian that we would see in the next chapter will need to be added to this two-derivative Lagrangian, to give the complete Lagrangian for a black hole solution.

4.5 A New Kinetic Multiplet

Butter, et al constructed a new kinetic multiplet in [8], in terms of the $\ln\phi$ supermultiplet, which does not transform linearly under supersymmetry transformations as can be seen from their covariant derivatives below.

$$\begin{aligned} D_\mu \ln\phi &= \mathcal{D}_\mu \ln\phi = \partial_\mu \ln\phi - w\, b_\mu\,, \\ D_\mu D_a \ln\phi &= \mathcal{D}_\mu D_a \ln\phi + w\, f_{\mu a}\,, \\ D_\mu \Box_{\mathrm{c}}\, \ln\phi &= \mathcal{D}_\mu \Box_{\mathrm{c}} \ln\phi - 2\, f_\mu{}^a D_a \ln\phi\,, \\ \Box_{\mathrm{c}}\Box_{\mathrm{c}}\, \ln\phi &= \mathcal{D}_a D^a \Box_{\mathrm{c}} \ln\phi + 2f\,\Box_{\mathrm{c}} \ln\phi - 2\, f_{\mu a}\, D^\mu D^a \ln\phi\,. \end{aligned} \tag{4.11}$$

For the conformally invariant Lagrangian that one demands, we need terms that are invariant under K-transformation of the covariant derivatvies. Therefore, we write down the special conformal transformations of the above derivatives below:

$$\begin{aligned} \delta_{\mathrm{K}} D_a \ln\phi &= -\,w\,\Lambda_{\mathrm{K}a}\,, \\ \delta_{\mathrm{K}} D_\mu D_a \ln\phi &= -\,\big[\Lambda_{\mathrm{K}\mu}\, D_a + \Lambda_{\mathrm{K}a}\, D_\mu\big] \ln\phi + e_{\mu a}\Lambda_{\mathrm{K}}{}^b\, D_b \ln\phi\,, \\ \delta_{\mathrm{K}} \Box_{\mathrm{c}} \ln\phi &= 2\,\Lambda_{\mathrm{K}}{}^a\, D_a \ln\phi\,, \\ \delta_{\mathrm{K}} D_\mu \Box_{\mathrm{c}} \ln\phi &= -\,2\,\Lambda_{\mathrm{K}\mu}\,\Box_{\mathrm{c}} \ln\phi + 2\,\Lambda_{\mathrm{K}}{}^a\, D_\mu D_a \ln\phi\,, \\ \delta_{\mathrm{K}} \Box_{\mathrm{c}}\Box_{\mathrm{c}} \ln\phi &= 2\Lambda_{\mathrm{K}}{}^a\,\Box_{\mathrm{c}}\, D_a \ln\phi - 2\,\Lambda_{\mathrm{K}}{}^a D_a \Box_{\mathrm{c}} \ln\phi = 0\,. \end{aligned} \tag{4.12}$$

Out of all of these conformally transformed derivatives, we see that only the last four-derivative term is conformally invariant (like Lagrangian 3.10b).

The components of the $\ln\phi$ multiplet in terms of the components of the elementary chiral multiplet 3.3.2 are given as following:

$$
\begin{aligned}
\hat{A}|_{\ln\Phi} &= \ln A\,, \\
\hat{\Psi}_i|_{\ln\Phi} &= \frac{\Psi_i}{A}\,, \\
\hat{B}_{ij}|_{\ln\Phi} &= \frac{B_{ij}}{A} + \frac{1}{2A^2}\bar{\Psi}_{(i}\Psi_{j)}\,, \\
\hat{F}^-_{ab}|_{\ln\Phi} &= \frac{F^-_{ab}}{A} + \frac{1}{8A^2}\,\varepsilon^{ij}\bar{\Psi}_i\gamma_{ab}\Psi_j\,, \\
\hat{\Lambda}_i|_{\ln\Phi} &= \frac{\Lambda_i}{A} + \frac{1}{2A^2}\big(B_{ij}\varepsilon^{jk}\Psi_k + \tfrac{1}{2}F^-_{ab}\gamma^{ab}\Psi_i\big) + \frac{1}{24A^3}\gamma^{ab}\Psi_i\varepsilon^{jk}\bar{\Psi}_j\gamma_{ab}\Psi_k\,, \\
\hat{C}|_{\ln\Phi} &= \frac{C}{A} + \frac{1}{4A^2}\big(\varepsilon^{ik}\varepsilon^{jl}B_{ij}B_{kl} - 2F^{-ab}F^-_{ab} + 4\varepsilon^{ij}\bar{\Lambda}_i\Psi_j\big) \\
&\quad + \frac{1}{2A^3}\big(\varepsilon^{ik}\varepsilon^{jl}B_{ij}\bar{\Psi}_k\Psi_l - \tfrac{1}{2}\varepsilon^{kl}F^-_{ab}\bar{\Psi}_k\gamma^{ab}\Psi_l\big) - \frac{1}{32A^4}\varepsilon^{ij}\bar{\Psi}_i\gamma_{ab}\Psi_j\varepsilon^{kl}\bar{\Psi}_k\gamma^{ab}\Psi_l\,.
\end{aligned}
\tag{4.13}
$$

The Q- and S- supersymmetry transformation rules of the $\ln\Phi$ are as follows:

$$
\begin{aligned}
\delta\hat{A} &= \bar{\epsilon}^i\hat{\Psi}_i\,, \\
\delta\hat{\Psi}_i &= 2\,\not{D}\hat{A}\epsilon_i + \hat{B}_{ij}\,\epsilon^j + \tfrac{1}{2}\gamma^{ab}\hat{F}^-_{ab}\,\varepsilon_{ij}\epsilon^j + 2\,w\,\eta_i\,, \\
\delta\hat{B}_{ij} &= 2\,\bar{\epsilon}_{(i}\not{D}\hat{\Psi}_{j)} - 2\,\bar{\epsilon}^k\hat{\Lambda}_{(i}\,\varepsilon_{j)k} + 2\,\bar{\eta}_{(i}\hat{\Psi}_{j)}\,, \\
\delta\hat{F}^-_{ab} &= \tfrac{1}{2}\varepsilon^{ij}\,\bar{\epsilon}_i\not{D}\gamma_{ab}\hat{\Psi}_j + \tfrac{1}{2}\bar{\epsilon}^i\gamma_{ab}\hat{\Lambda}_i - \tfrac{1}{2}\,\varepsilon^{ij}\bar{\eta}_i\gamma_{ab}\hat{\Psi}_j\,, \\
\delta\hat{\Lambda}_i &= -\tfrac{1}{2}\gamma^{ab}\not{D}\hat{F}^-_{ab}\epsilon_i - \not{D}\hat{B}_{ij}\varepsilon^{jk}\epsilon_k + \hat{C}\varepsilon_{ij}\,\epsilon^j + \tfrac{1}{4}\big(\not{D}\hat{A}\,\gamma^{ab}T_{abij} + w\,\not{D}\gamma^{ab}T_{abij}\big)\varepsilon^{jk}\epsilon_k \\
&\quad - 3\,\gamma_a\varepsilon^{jk}\epsilon_k\,\bar{\chi}_{[i}\gamma^a\hat{\Psi}_{j]} - \hat{B}_{ij}\varepsilon^{jk}\,\eta_k + \tfrac{1}{2}\,\gamma^{ab}\,\hat{F}^-_{ab}\eta_i\,, \\
\delta\hat{C} &= -2\,\varepsilon^{ij}\bar{\epsilon}_i\not{D}\hat{\Lambda}_j - 6\,\bar{\epsilon}_i\chi_j\,\varepsilon^{ik}\varepsilon^{jl}\,\hat{B}_{kl} + \tfrac{1}{4}\varepsilon^{ij}\varepsilon^{kl}\big(\bar{\epsilon}_i\gamma^{ab}\not{D}T_{abjk}\hat{\Psi}_l - \bar{\epsilon}_i\gamma^{ab}T_{abjk}\not{D}\hat{\Psi}_l\big)\,.
\end{aligned}
\tag{4.14}
$$

4.6 Four-derivative Lagrangian in Curved Superspace

Taking the curved superspace generalisation of Lagrangian 4.3 in terms of the new kinetic multiplet, the four-derivative Lagrangian in curved superspace comes out to be[1]

$$
-\frac{1}{2}\int d^4\theta\,\mathcal{E}\,\Phi'\,\mathbb{T}(\ln\bar{\Phi})\,. \tag{4.15}
$$

where Φ' has weyl weight 0 and $\mathbb{T}(\ln\bar{\Phi})$ has weyl weight 2 as explained at the end of section 4.3.

[1]See [9] for more details on the derivation

The components of $\mathbb{T}(\ln\bar{\Phi})$ multiplet in terms of those of $\ln\bar{\Phi}$ are as given below:

$$
\begin{aligned}
A|_{\mathbb{T}(\ln\bar{\Phi})} =&\, \hat{\bar{C}}\,,\\
\Psi_i|_{\mathbb{T}(\ln\bar{\Phi})} =&\, -2\,\varepsilon_{ij}\not{D}\hat{\Lambda}^j - 6\;\varepsilon_{ik}\varepsilon_{jl}\chi^j\hat{B}^{kl} - \tfrac{1}{4}\varepsilon_{ij}\varepsilon_{kl}\,\gamma^{ab}T_{ab}{}^{jk}\,\overleftrightarrow{\not{D}}\,\hat{\Psi}^l\,,\\
B_{ij}|_{\mathbb{T}(\ln\bar{\Phi})} =&\, -2\,\varepsilon_{ik}\varepsilon_{jl}\big(\Box_{\rm c}+3\,D\big)\hat{B}^{kl} - 2\,\hat{F}^{+}_{ab}\,R(\mathcal{V})^{ab\,k}{}_i\,\varepsilon_{jk}\\
&- 6\,\varepsilon_{k(i}\,\bar{\chi}_{j)}\hat{\Lambda}^k + 3\,\varepsilon_{ik}\varepsilon_{jl}\,\hat{\bar{\Psi}}^{(k}\not{D}\chi^{l)}\,,\\
F^{-}_{ab}|_{\mathbb{T}(\ln\bar{\Phi})} =&\, -\big(\delta_a{}^{[c}\delta_b{}^{d]} - \tfrac{1}{2}\varepsilon_{ab}{}^{cd}\big)\big[4\,D_cD^e\hat{F}^{+}_{ed} + (D^e\hat{A}\,D_cT_{de}{}^{ij} + D_c\hat{A}\,D^eT_{ed}{}^{ij})\varepsilon_{ij} - wD_cD^eT_{ed}{}^{ij}\varepsilon_{ij}\big]\\
&+ \Box_{\rm c}\hat{A}\,T_{ab}{}^{ij}\varepsilon_{ij} - R(\mathcal{V})^{-}_{ab}{}^i{}_k\,\hat{B}^{jk}\,\varepsilon_{ij} + \tfrac{1}{8}T_{ab}{}^{ij}\,T_{cdij}\hat{F}^{+cd} - \varepsilon_{kl}\,\hat{\bar{\Psi}}^k\,\overleftrightarrow{\not{D}}\,R(Q)_{ab}{}^l\\
&- \tfrac{9}{4}\varepsilon_{ij}\,\hat{\bar{\Psi}}^i\gamma^c\gamma_{ab}D_c\chi^j + 3\,\varepsilon_{ij}\bar{\chi}^i\gamma_{ab}\not{D}\hat{\Psi}^j + \tfrac{3}{8}T_{ab}{}^{ij}\varepsilon_{ij}\bar{\chi}_k\hat{\Psi}^k\,,\\
\Lambda_i|_{\mathbb{T}(\ln\bar{\Phi})} =&\, 2\,\Box_{\rm c}\not{D}\hat{\Psi}^j\varepsilon_{ij} + \tfrac{1}{4}\gamma^c\gamma_{ab}(2\,D_cT^{ab}{}_{ij}\,\hat{\Lambda}^j + T^{ab}{}_{ij}\,D_c\hat{\Lambda}^j)\\
&- \tfrac{1}{2}\varepsilon_{ij}\big(R(\mathcal{V})_{ab}{}^j{}_k + 2{\rm i}\,R(A)_{ab}\delta^j{}_k\big)\,\gamma^c\gamma^{ab}D_c\hat{\Psi}^k\\
&+ \tfrac{1}{2}\,\varepsilon_{ij}\big(3\,D_bD - 4{\rm i}D^aR(A)_{ab} + \tfrac{1}{4}T_{bc}{}^{ij}\,\overleftrightarrow{D}_a\,T^{ac}{}_{ij}\big)\,\gamma^b\hat{\Psi}^j\\
&- 2\,\hat{F}^{+ab}\,\not{D}R(Q)_{ab\,i} + 6\,\varepsilon_{ij}D\,\not{D}\hat{\Psi}^j\\
&+ 3\,\varepsilon_{ij}\,\big(\not{D}\chi_k\,\hat{B}^{kj} + \not{D}\hat{A}\,\not{D}\chi^j\big)\\
&+ \tfrac{3}{2}\big(2\,\not{D}\hat{B}^{kj}\varepsilon_{ik} + \not{D}\hat{F}^{+}_{ab}\gamma^{ab}\,\delta_i^j + \tfrac{1}{4}\varepsilon_{kl}T_{ab}{}^{kl}\,\gamma^{ab}\,\not{D}\hat{A}\,\delta_i{}^j\big)\chi_j\\
&+ \tfrac{9}{4}\,(\bar{\chi}^l\gamma_a\chi_l)\,\varepsilon_{ij}\gamma^a\hat{\Psi}^j - \tfrac{9}{2}\,(\bar{\chi}_i\gamma_a\chi^k)\,\varepsilon_{kl}\gamma^a\hat{\Psi}^l\\
&- \tfrac{3}{2}w\,\varepsilon_{jk}\,D^aT_{ab}{}^{jk}\gamma^b\chi_i\,,\\
C|_{\mathbb{T}(\ln\bar{\Phi})} =&\, 4(\Box_{\rm c}+3D)\Box_{\rm c}\hat{A} + 6(D_aD)\,D^a\hat{A} - 16\,D^a\Big(R(D)^{+}_{ab}D^b\hat{A}\Big)\\
&- D^a(T_{abij}T^{cbij}D_c\hat{A}) - \tfrac{1}{2}D^a(T_{abij}T^{cbij})D_c\hat{A} - 9\,\bar{\chi}_j\gamma^a\chi^j\,D_a\hat{A}\\
&+ \tfrac{1}{2}D_aD^a(T_{bcij}\hat{F}^{bc+})\varepsilon^{ij} + 4\varepsilon^{ij}D_a\Big(D^bT_{bcij}\hat{F}^{ac+} + D^b\hat{F}^{+}_{bc}T^{ac}{}_{ij}\Big)\\
&- \tfrac{9}{2}\varepsilon^{jk}\bar{\chi}_j\gamma^{ab}\chi_k\hat{F}^{+}_{ab} + 9\bar{\chi}_j\chi_k\hat{B}^{jk} + \tfrac{1}{16}(T_{ab}{}^{ij}\varepsilon_{ij})^2\hat{\bar{C}}\\
&+ 6D^aD_a\bar{\chi}_j\hat{\Psi}^j + 3\bar{\chi}_j\not{D}\not{D}\hat{\Psi}^j + 3D_a(\bar{\chi}_j\gamma^a\not{D}\hat{\Psi}^j) + 9D\bar{\chi}_j\hat{\Psi}^j\\
&- 8D^a\bar{R}(Q)_{abj}D^b\hat{\Psi}^j + 6D_b\bar{\chi}_j\gamma^b\not{D}\hat{\Psi}^j\\
&+ \tfrac{3}{2}D^aT_{abij}\bar{\chi}^i\gamma^b\hat{\Psi}^j + 3D^a(T_{abij}\bar{\chi}^i\gamma^b\hat{\Psi}^j) + \tfrac{3}{2}D^a(T_{abij}\bar{\chi}^i)\gamma^b\hat{\Psi}^j\\
&+ 3\Big(\tfrac{1}{2}R(\mathcal{V})^{+}_{ab}{}^i{}_j - R(D)^{+}_{ab}\delta^i{}_j\Big)\bar{\chi}_i\gamma^{ab}\hat{\Psi}^j - 2R(\mathcal{V})^{+}_{ab}{}^i{}_j\bar{R}(Q)^{ab}{}_i\hat{\Psi}^j - \tfrac{1}{2}T^{ab}{}_{ij}\bar{R}(S)^{+}_{ab}{}^i\hat{\Psi}^j\\
&+ \tfrac{1}{8}\varepsilon^{ij}T_{abij}\Big(3\bar{\chi}_k\gamma^{ab}\hat{\Lambda}^k + 2\bar{R}(Q)^{ab}_k\hat{\Lambda}^k\Big)\\
&+ w\Big\{9\bar{\chi}_j\not{D}\chi^j - R(\mathcal{V})^{+}_{ab}{}^i{}_jR(\mathcal{V})^{ab+j}{}_i - 8R(D)^{+}_{ab}R(D)^{ab+}\\
&\qquad - D^aT_{abij}D_cT^{cbij} - D^a(T_{abij}D_cT^{cbij})\Big\}\,.
\end{aligned}
\tag{4.16}
$$

Substituting the composite multiplet $\Phi'\,\mathbb{T}(\ln\bar{\Phi})$ into the general expression of superconfomal Lagrangian in curved superspace i.e. 4.8, along with setting all the fermionic fields to 0, leaves only two terms in the Lagrangian.

$$e^{-1}\mathcal{L} = C - \tfrac{1}{16} A (T_{ab\,ij}\varepsilon^{ij})^2 \tag{4.17}$$

where C is

$$\begin{aligned} C|_{\mathbb{T}(\ln\bar\Phi)} =& \mathcal{D}_a V^a + \tfrac{1}{16}(T_{ab\,ij}\varepsilon^{ij})^2\,\hat{\bar{C}} \\ &+ w\big\{-2\,\mathcal{R}^{ab}\mathcal{R}_{ab} + \tfrac{2}{3}\mathcal{R}^2 - 6\,D^2 + 2\,R(A)^{ab}R(A)_{ab} - R(\mathcal{V})^{+\,i}_{ab\;\;j}R(\mathcal{V})^{ab+j}{}_i \\ &\quad + \tfrac{1}{128}T^{abij}T_{ab}{}^{kl}T^{cd}{}_{ij}T_{cdkl} + T^{acij}D_aD^bT_{bcij}\big\}\,, \end{aligned} \tag{4.18}$$

where V^a is

$$\begin{aligned} V^a =& 4\mathcal{D}^a\mathcal{D}^2\hat{\bar{A}} - 8\mathcal{R}^{ab}\mathcal{D}_b\hat{\bar{A}} + \tfrac{8}{3}\mathcal{R}\mathcal{D}^a\hat{\bar{A}} + 8D\,\mathcal{D}^a\hat{\bar{A}} - 8\mathrm{i}\,R(A)^{ab}\mathcal{D}_b\hat{\bar{A}} \\ &- 2T^{acij}T_{bcij}\mathcal{D}^b\hat{\bar{A}} + \tfrac{1}{2}\varepsilon^{ij}\mathcal{D}^aT_{bcij}\hat{\bar{F}}^{bc+} + 4\,\varepsilon^{ij}T^{ac}{}_{ij}\mathcal{D}^b\hat{\bar{F}}^+_{bc} \\ &+ w\big\{\tfrac{2}{3}\mathcal{D}^a\mathcal{R} - 4\,\mathcal{D}^aD - \mathcal{D}^b(T^{acij}T_{bcij})\big\}\,. \end{aligned} \tag{4.19}$$

Substituting $C|_{\mathbb{T}(\ln\bar\Phi)}$ and $A|_{\mathbb{T}(\ln\bar\Phi)}$ from 4.16 in terms of the components of the $\ln\bar\Phi$ multiplet, we get the following higher derivative term, which is the main result of this chapter:

$$\begin{aligned} e^{-1}\mathcal{L} =& 4\,\mathcal{D}^2A'\,\mathcal{D}^2\hat{\bar{A}} + 8\,\mathcal{D}^aA'\,\big[\mathcal{R}_{ab} - \tfrac{1}{3}\mathcal{R}\,\eta_{ab}\big]\mathcal{D}^b\hat{\bar{A}} + C'\,\hat{\bar{C}} \\ &- \mathcal{D}^\mu B'_{ij}\,\mathcal{D}_\mu\hat{\bar{B}}^{ij} + (\tfrac{1}{6}\mathcal{R} + 2\,D)\,B'_{ij}\hat{\bar{B}}^{ij} \\ &- \big[\varepsilon^{ik}\,B'_{ij}\,\hat{\bar{F}}^{+\mu\nu}\,R(\mathcal{V})_{\mu\nu}{}^j{}_k + \varepsilon_{ik}\,\hat{\bar{B}}^{ij}\,F'^{-\mu\nu}R(\mathcal{V})_{\mu\nu j}{}^k\big] \\ &- 8\,D\,\mathcal{D}^\mu A'\,\mathcal{D}_\mu\hat{\bar{A}} + \big(8\,\mathrm{i}\,R(A)_{\mu\nu} + 2\,T_\mu{}^{cij}\,T_{\nu cij}\big)\mathcal{D}^\mu A'\,\mathcal{D}^\nu\hat{\bar{A}} \\ &- \big[\varepsilon^{ij}\mathcal{D}^\mu T_{bcij}\mathcal{D}_\mu A'\,\hat{\bar{F}}^{+bc} + \varepsilon_{ij}\mathcal{D}^\mu T_{bc}{}^{ij}\mathcal{D}_\mu\hat{\bar{A}}\,F'^{-bc}\big] \\ &- 4\big[\varepsilon^{ij}T^{\mu b}{}_{ij}\,\mathcal{D}_\mu A'\,\mathcal{D}^c\hat{\bar{F}}^+_{cb} + \varepsilon_{ij}T^{\mu bij}\,\mathcal{D}_\mu\hat{\bar{A}}\,\mathcal{D}^cF'^-_{cb}\big] \\ &+ 8\,\mathcal{D}_aF'^{-ab}\,\mathcal{D}^c\hat{\bar{F}}^+_{cb} + 4\,F'^{-ac}\,\hat{\bar{F}}^+_{bc}\,\mathcal{R}_a{}^b + \tfrac{1}{4}T_{ab}{}^{ij}\,T_{cdij}F'^{-ab}\hat{\bar{F}}^{+cd} \\ &+ w\,\Big\{-\tfrac{2}{3}\mathcal{D}^aA'\,\mathcal{D}_a\mathcal{R} + 4\mathcal{D}^aA'\,\mathcal{D}_aD - T^{acij}T_{bcij}\,\mathcal{D}^b\mathcal{D}_aA' \\ &\quad - 2\mathcal{D}^aF'^-_{ab}\,\mathcal{D}_cT^{cbij}\varepsilon_{ij} + \mathrm{i}\,F'^{-ab}R(A)^-_{ad}\,T_b{}^{dij}\varepsilon_{ij} + F^-_{ab}T^{abij}\varepsilon_{ij}(\tfrac{1}{12}\mathcal{R} - \tfrac{1}{2}D) \\ &\quad + A'\,\big[\tfrac{2}{3}\mathcal{R}^2 - 2\,\mathcal{R}^{ab}\mathcal{R}_{ab} - 6\,D^2 + 2\,R(A)^{ab}R(A)_{ab} - R(\mathcal{V})^{+abi}{}_j\,R(\mathcal{V})^{+j}_{ab\;\;i} \\ &\qquad + \tfrac{1}{128}T^{abij}T_{ab}{}^{kl}T^{cd}{}_{ij}T_{cdkl} + T^{acij}D_aD^bT_{bcij}\big]\Big\}\,. \end{aligned} \tag{4.20}$$

4.6.1 A closer look at the new four-derivative Lagrangian in Curved Superspace

In Lagrangian 4.15, i.e. $-\frac{1}{2}\int d^4\theta\,\mathcal{E}\,\Phi'\,\mathbb{T}(\ln\bar\Phi)$, the Φ' multiplet has $w = c = 0$ and also has no charges. The $\mathbb{T}(\ln\bar\Phi)$ multiplet has weyl weight $w = 2$ and chiral weight

$c = -2$, which on adding to the weyl and chiral weights of ε, which are -2 and 2 respectively, gives the total weyl and chiral weights of the Lagrangian $-\frac{1}{2}\int d^4\theta\, \mathbb{T}(\ln \bar{\Phi})$ to be 0. Also, $-\frac{1}{2}\int d^4\theta\, \mathbb{T}(\ln \bar{\Phi})$ is superconformally invariant. Therefore, it qualifies to form the four-derivative Lagrangian all by itself, without the necessity of bringing in Φ'. Now we ask the question as to why Φ' is brought in.

When Φ' is absent, we need to substitute only the components of $\mathbb{T}(\ln \bar{\Phi})$ into the expression 4.8. Doing so, along with setting all the fermionic terms to 0, gives a Lagrangian that has two parts. One part is the total derivative of 4.19, which on integrating over spacetime to get the action, vanishes, and hence does not contribute. The second part of the Lagrangian thus obtained, is only the last two lines of 4.20, which do not contain components of the $\ln \bar{\Phi}$ multiplet. These last two lines of the Lagrangian are the supersymmetric generalization of the Gauss-Bonnet term in general relativity. Thus we see that the Φ' multiplet is brought into the four-derivative Lagrangian in order to enable the components of $\ln \bar{\Phi}$ multiplet to contribute to the action.

Having got an overview of how the new supersymmetic higher derivative invariant came about, we now proceed to extract from it the particular term which we wish to analyse immediately, i.e. the new supersymmetric Gauss-Bonnet invariant, and see if it has any effect on the entropy of extremal black holes.

Chapter 5

Contribution of the New Gauss-Bonnet Invariant to the Entropy of Extremal Black Holes

The Gauss-Bonnet term which is one of the curvature squared terms in general relativity, has had the following issues associated with it:

1. In $\mathcal{N} = 2$ supersymmetry, it had never been constructed until Butter, et al found it as a part of their new higher derivative invariant in [8].
2. Sen calculated the entropy of BPS black holes in [24] using a Gauss Bonnet term multiplied to a dilaton field as the higher derivative term, without supersymmetrizing it. But puzzlingly his result matched that of Cardoso, et al [25, 26], which was based on the square of the Weyl tensor, which critically depended on its full supersymmetrization.

Now that we have the full class of Gauss-Bonnet invariant with us, we can check if it has any contribution to black hole entropy or not.

5.1 Reduction of the New Higher Derivative to Gauss-Bonnet Term

We want Lagrangian 4.20 to gives us the Gauss-Bonnet invariant exclusively. This can be done in two ways. One way is to not take the multiplet Φ' in the Lagrangian at all, as explained in section 4.6.1. Another way is to set the lowest θ-component of Φ', i.e. A' to be constant along with setting all the fermions of the theory to 0 consistently, which requires making them supersymmetric, lest they transform into something nonzero. This in turn leads to the supersymmetry invariance of the remaining multiplet components as well. Let us take a brief look at how this works.

Setting the fermion ψ'_i to 0, makes A' supersymmetry invariant (look at equations 3.15). Making ψ'_i invariant under supersymmetry, i.e. setting $\delta\psi'_i$=0, along with setting A' to be constant, implies the following for the terms in $\delta\psi'_i$:

1. In $\not{D}A'$, the partial derivative vanishes because A' is constant, and dilatational and chiral transformations vanish because $w = -c = 0$ (since Φ' is a $w = 0$ multiplet as mentioned in 4.6).

2. The last term in the transformation, i.e. $2wA'\eta_i$ vanishes, because w=0.

3. B'_{ij} and F'^{-}_{ab}, being symmetric and anti-symmetric respectively, vanish individually.

Following the same train of logic, it can be seen that all the components of Φ' multiplet vanish, leaving only the constant A'.

In the higher derivative term 4.20, all the terms except the last two lines involving the Gauss-Bonnet invariant, contain either a covariant derivative of A' or another component of Φ', both of which vanish under the mentioned conditions. Thus, we reduce the higher derivative to the full class of Gauss Bonnet invariant which can now be studied exclusively. The Gauss-Bonnet higher derivative invariant that we have is

$$\begin{aligned} e^{-1}\mathcal{L}_{\rm GB} =& A'\,\big[\tfrac{2}{3}\mathcal{R}^2 - 2\,\mathcal{R}^{ab}\mathcal{R}_{ab} - 6\,D^2 + 2\,R(A)^{ab}R(A)_{ab} - R(\mathcal{V})^{+abi}{}_j\,R(\mathcal{V})^{+\,j}_{ab\ \,i} \\ &+ \tfrac{1}{128}T^{abij}T_{ab}{}^{kl}T^{cd}{}_{ij}T_{cdkl} + T^{acij}D_aD^bT_{bcij}\big] \end{aligned} \tag{5.1}$$

The first two terms in it, viz. $\frac{2}{3}\mathcal{R}^2 - 2\,\mathcal{R}^{ab}\mathcal{R}_{ab}$ are derived in general relativity, while the remaining terms, which were previously unknown, come from supersymmetry.

5.2 The $AdS_2 \times S^2$ Background of Extremal Black Holes

We want to study the entropy of extremal (susy/non-susy) black holes that will appear in $\mathcal{N} = 2$ theory in the presence of the above mentioned supersymmetric Gauss-Bonnet term. We will use Sen's entropy function formalism (as discussed in section 2.4) for this purpose.

The near horizon $AdS_2 \times S^2$ background for extremal black holes given in [23] by Sen and Sahoo is as following:

$$\begin{aligned} & ds^2 = v_1(-r^2dt^2 + dr^2/r^2) + v_2(d\theta^2 + \sin^2\theta d\phi^2) \\ & F^I_{rt} = e_I, \quad F^I_{\theta\phi} = p^I\,\sin\theta, \quad X^I = x^I, \quad T^-_{rt} = v_1\,w \\ & D - \frac{1}{3}R = 0, \quad \mathcal{A}_\mu = 0, \quad \mathcal{V}^i{}_{j\mu} = 0, \quad W_\mu = 0. \end{aligned} \tag{5.2}$$

Here, e_I is the charge conjugate to the electric charge q_I of the black holes, and p^I is the magnetic charge of the black holes. v_1 and v_2 are scaling parameters of the AdS_2 and S_2 spacetimes respectively. F^I_{rt} and $F^I_{\theta\phi}$ are the only non-zero electromagnetic field strength tensors for this background. X^I are the scalar fields, the lowest-θ components of the vector multiplet in terms of which the entropy function is expressed. T^-_{rt} is the non-zero anti-self dual tensor, obeying the following general relation:

$$T^{-\mu\nu} = -\frac{i}{2}\,(\sqrt{-\det g})^{-1}\,\epsilon^{\mu\nu\rho\sigma}\,T^-_{\rho\sigma}\,. \tag{5.3}$$

The rest of the terms in the third line of 5.2 are the usual Weyl and vector multiplet components and gauge fields introduced in chapter 3.

5.3 The Gauss-Bonnet Invariant

We substitute the values for the various terms in expression 5.1 from the $AdS_2 \times S^2$ near horizon geometry 5.2. The Gauss-Bonnet term gets simplified as following:

$$
\begin{aligned}
e^{-1}\mathcal{L} &= A' \left[\tfrac{2}{3}\mathcal{R}^2 - 2\,\mathcal{R}^{ab}\mathcal{R}_{ab} - 6\,D^2 + 2\,R(A)^{ab}R(A)_{ab} - R(\mathcal{V})^{+abi}{}_j\,R(\mathcal{V})^{+\,j}_{ab\ \ i}\right.\\
&\quad \left. + \tfrac{1}{128}T^{abij}T_{ab}{}^{kl}T^{cd}{}_{ij}T_{cdkl} + T^{acij}D_aD^bT_{bcij}\right]\\
&\qquad\qquad (\text{setting the constant } A' \text{to 1, a convenient value})\\
&= \tfrac{2}{3}\mathcal{R}^2 - 2\,\mathcal{R}^{ab}\mathcal{R}_{ab} - 6\,D^2 + 2\,R(A)^{ab}R(A)_{ab} - R(\mathcal{V})^{+abi}{}_j\,R(\mathcal{V})^{+\,j}_{ab\ \ i}\\
&\quad + \tfrac{1}{128}T^{abij}T_{ab}{}^{kl}T^{cd}{}_{ij}T_{cdkl} + T^{acij}D_aD^bT_{bcij}\\
&\qquad\qquad (\because D = \mathcal{R}/3)\\
&= \tfrac{2}{3}\mathcal{R}^2 - 6\,(\mathcal{R}/3)^2 - 2\,\mathcal{R}^{ab}\mathcal{R}_{ab} + 2\,R(A)^{ab}R(A)_{ab} - R(\mathcal{V})^{+abi}{}_j\,R(\mathcal{V})^{+\,j}_{ab\ \ i}\\
&\quad + \tfrac{1}{128}T^{abij}T_{ab}{}^{kl}T^{cd}{}_{ij}T_{cdkl} + T^{acij}D_aD^bT_{bcij}\\
&\qquad\qquad (\because R(A)_{ab} = 0 \text{ and } R(\mathcal{V})^{+}_{ab} = 0)\\
&= \tfrac{2}{3}\mathcal{R}^2 - \tfrac{2}{3}\mathcal{R}^2 - 2\,\mathcal{R}^{ab}\mathcal{R}_{ab} + \tfrac{1}{128}T^{abij}T_{ab}{}^{kl}T^{cd}{}_{ij}T_{cdkl} + T^{acij}D_aD^bT_{bcij}\\
&= \qquad \mathcal{R}^{ab}\mathcal{R}_{ab} \quad + \quad \tfrac{1}{128}T^{abij}T_{ab}{}^{kl}T^{cd}{}_{ij}T_{cdkl} + T^{acij}D_aD^bT_{bcij}\\
&= -4\left(\frac{1}{v_1^2}+\frac{1}{v_2^2}\right) \quad + \quad \frac{1}{32}w^2\bar{w}^2 \qquad + \qquad 0\\
&= -4\left(\frac{1}{v_1^2}+\frac{1}{v_2^2}\right) + \frac{1}{32}w^2\bar{w}^2
\end{aligned}
\tag{5.4}
$$

On substituting this higher derivative term into Sen's entropy function [23], i.e.

$$
\mathcal{E}(v_1, v_2, w, \vec{x}, \vec{e}, \vec{q}, \vec{p}) = 2\pi\left(-\tfrac{1}{2}\,\vec{q}.\vec{e} - \int d\theta d\phi\,\sqrt{-\det g}\,\mathcal{L}\right), \tag{5.5}
$$

it's contribution to the entropy function of non-supersymmetric black holes comes out as

$$
\boxed{\mathcal{E}_{\mathrm{GB}} = 16\pi\left[\left(\frac{v_2}{v_1}+\frac{v_1}{v_2}\right) - \frac{1}{128}v_1v_2w^2\bar{w}^2\right]} \tag{5.6}
$$

The two-derivative entropy function calculated by substituting the two derivative Lagrangian 4.10 in expression 5.5 is

$$
\begin{aligned}
\mathcal{E} =& -\pi q_I e^I - \pi v_1 v_2\left[i\left(\frac{1}{v_1}-\frac{1}{v_2}\right)(x^I\bar{F}_I - \bar{x}^I F_I)\right.\\
& - \left\{\frac{i}{4}\frac{F_{IJ}}{{v_1}^2}\left(e^I - i\frac{v_1}{v_2}p^I - \frac{1}{2}\bar{x}^I v_1 w\right)\left(e^J - i\frac{v_1}{v_2}p^J - \frac{1}{2}\bar{x}^J v_1 w\right) + h.c.\right\}\\
& - \left\{\frac{i}{4}\frac{w\bar{F}_I}{v_1}\left(e^I - i\frac{v_1}{v_2}p^I - \frac{1}{2}\bar{x}^I v_1 w\right) + h.c.\right\}\\
& \left. + \left(\frac{i}{8}\bar{w}^2 F + h.c.\right)\right]
\end{aligned}
\tag{5.7}
$$

Adding the Gauss Bonnet term 5.6 to the pre-existing two derivative entropy function 5.7, we get the full entropy function for extremal black holes to be

$$\begin{aligned}\mathcal{E} = &- \pi q_I e^I - \pi v_1 v_2 \Big[i \left(\frac{1}{v_1} - \frac{1}{v_2} \right) (x^I \bar{F}_I - \bar{x}^I F_I) \\ &- \left\{ \frac{i}{4} \frac{F_{IJ}}{v_1{}^2} \left(e^I - i\frac{v_1}{v_2} p^I - \frac{1}{2} \bar{x}^I v_1 w \right) \left(e^J - i\frac{v_1}{v_2} p^J - \frac{1}{2} \bar{x}^J v_1 w \right) + h.c. \right\} \\ &- \left\{ \frac{i}{4} \frac{w \bar{F}_I}{v_1} \left(e^I - i\frac{v_1}{v_2} p^I - \frac{1}{2} \bar{x}^I v_1 w \right) + h.c. \right\} \\ &+ \left(\frac{i}{8} \varpi^2 F + h.c. \right) - 16(v_1^2 + v_2^2) + \frac{1}{8} w^2 \varpi^2 \Big] \end{aligned} \tag{5.8}$$

5.4 Contribution to the Entropy of Supersymmetric Extremal Black Holes

Sen's entropy function formalism requires entropy function 5.8 to be extremized with respect to the near horizon parameters, i.e. obey the following extremization equations:

$$\frac{\partial \mathcal{E}}{\partial v_i} = 0, \qquad \frac{\partial \mathcal{E}}{\partial x^I} = 0, \qquad \frac{\partial \mathcal{E}}{\partial w} = 0, \qquad \frac{\partial \mathcal{E}}{\partial e^I} = 0\,, \tag{5.9}$$

For extremal supersymmetric black holes, these equations are satisfied by a set of equations known as supersymmetric attractors. The supersymmetric attractors for the two derivative entropy function 5.7 as given in [23] are as following:

$$\begin{aligned} &v_1 = v_2 = \frac{16}{\varpi w} \\ &e^I - i\frac{v_1}{v_2} p^I - \frac{1}{2} \bar{x}^I v_1 w = 0 \\ &\frac{\bar{F}_I}{\varpi} - \frac{F_I}{w} = -\frac{i}{4} q_I \end{aligned} \tag{5.10}$$

It can be checked easily that the first equation of 5.10 sets the Gauss-Bonnet term 5.6 to zero. Therefore, the Gauss Bonnet term does not contribute to the entropy of supersymmetric black holes.

5.5 Contribution to the Entropy of Extremal Black Holes in the STU Model

Now we will calculate the contribution of our newly found Gauss-Bonnet term to the entropy functions of extremal black holes in the STU model, that have already been solved for in [23] using another higher derivative term. Following is the prepotential (introduced in section 4.4.1) of the black holes that we are going to consider:

$$F(X^0, X^1, X^2, X^3, \hat{A}) = -\frac{X^1 X^2 X^3}{X^0} - C\hat{A}\,\frac{X^1}{X^0} \tag{5.11}$$

where C is a constant and $\hat{A} = T^{-\mu\nu}T^{-}_{\mu\nu}$. For this class of extremal STU black holes, Sen has redefined the electric and magnetic charges as following:

$$\begin{aligned} &Q_1 = q_2, \quad Q_2 = -p^1, \quad Q_3 = q_3, \quad Q_4 = q_0 \\ &P_1 = p^3, \quad P_2 = p^0, \quad P_3 = p^2, \quad P_4 = q_1 \end{aligned} \tag{5.12}$$

The entropy of a special class of these black holes, for which P.Q=0, has already been calculated by Sen and Sahoo in [23] for both supersymmetric and non-supersymmetric cases. Let us see whether or not the Gauss-Bonnet term 5.6 contributes to the entropies of these black holes.

5.5.1 Entropy of Supersymmetric STU Black Holes

For supersymmetric black holes, the definiteness condition for the electric and magnetic charges dictates that $Q^2P^2 > (Q.P)^2$. For the special class of black holes that we are considering, it is required that $Q^2 > 0$ and $P^2 > 0$. The representative element taken in [23] is

$$P_1 = P_3 = P_0, \quad Q_2 = Q_4 = -Q_0, \quad P_2 = P_4 = Q_1 = Q_3 = 0, \quad Q_0, P_0 > 0 \tag{5.13}$$

where, by the assignations 5.12, the non-zero q_I and p^I take the values:

$$p^1 = Q_0, \quad p^2 = P_0, \quad p^3 = P_0, \quad q_0 = -Q_0 \tag{5.14}$$

Due to the scale invariance of this theory, the parameter w is gauge fixed by setting $w = 1$. The extremized values of the near-horizon parameters of this black hole are given in [23], which have been verified by us. They are:

$$\begin{aligned} &x^0 = \frac{1}{8}\sqrt{P_0^2 + 256C}, \quad x^1 = \frac{i}{8}Q_0, \quad x^2 = \frac{i}{8}P_0, \quad x^3 = \frac{i}{8}P_0, \\ &e^0 = \sqrt{P_0^2 + 256C}, \quad e^1 = e^2 = e^3 = 0\,, \quad w = 1\,, \\ &v_1 = 16, \quad v_2 = 16 \end{aligned} \tag{5.15}$$

The leading order solution for black hole entropy is calculated by setting the higher derivative term in the entropy function to 0. The solution 5.15 for entropy holds for the two-derivative entropy function 5.7, thus giving the leading order solution for 5.8.

Now we need to check whether the Gauss-Bonnet term 5.6 takes any contributing value on the leading order solution or not. For this, we substitute the values of the near-horizon parameters 5.15, into our Gauss Bonnet term 5.6. It can be seen explicitly what value it takes in the following steps:

$$\begin{aligned}\mathcal{E}_{\mathrm{GB}}|_{\mathrm{BPS}} &= 16\pi\left[\left(\frac{v_2}{v_1}+\frac{v_1}{v_2}\right)-\frac{1}{128}v_1 v_2 w^2 \bar{w}^2\right] \\ &= 16\pi\left[\left(\frac{16}{16}+\frac{16}{16}\right)-\frac{1}{128}(16)(16)(1)^2(1)^2\right] \\ &= 0\end{aligned} \tag{5.16}$$

Therefore, now we see for the specific case of a class of extremal STU BPS black holes that the Gauss-Bonnet term 5.6 does not contribute to their entropy.

5.5.2 Entropy of Non-supersymmetric STU Black Holes

For non-supersymmetric black holes, the definiteness condition for the electric and magnetic charges dictates that $Q^2P^2 < (Q.P)^2$. For the special class of black holes that we are considering, it is required that $Q^2 < 0$ and $P^2 > 0$. The representative element taken by Sahoo and Sen in [23] is

$$P_1 = P_3 = P_0, \quad Q_2 = -Q_4 = Q_0, \quad P_2 = P_4 = Q_1 = Q_3 = 0, \quad Q_0, P_0 > 0 \tag{5.17}$$

where, by the assignations 5.12, the non-zero q_I and p^I take the values:

$$p^1 = Q_0, \quad p^2 = P_0, \quad p^3 = P_0, \quad q_0 = Q_0 \tag{5.18}$$

w is gauge fixed by setting $w = \frac{1}{2}$. It has been found convenient to rescale the entropy function by expressing it in terms of real variables, using the following change of variables:

$$x^0 = P_0 y^0, \quad x^1 = iQ_0 y^1, \quad x^2 = iP_0 y^2, \quad x^3 = iP_0 y^3, \quad e^0 = P_0 \tilde{e}^0 \tag{5.19}$$

All of the newly introduced variables y^0, y^1, y^2, y^3 and $\tilde{e}^0$ are real. After this change of variables, the leading order entropy function 5.7 comes out to be as

$$\begin{aligned}\mathcal{E} = \pi Q_0 P_0 \Bigg[&-\tilde{e}^0 - \frac{v_1}{v_2}\frac{y^1+y^2+y^3}{y^0} + \left\{\frac{v_1}{y^0} - \frac{\tilde{e}^0}{(y^0)^2}\right\}(y^1y^2+y^2y^3+y^1y^3) \\ &+ \left\{-\frac{(\tilde{e}^0)^2 v_2}{v_1 (y^0)^3} + \frac{\tilde{e}^0 v_2}{(y^0)^2} + 8\frac{(v_2-v_1)}{y^0} - \frac{v_1 v_2}{2y^0}\right\} y^1y^2y^3\Bigg]\end{aligned} \tag{5.20}$$

After extremizing this entropy function with respect to the near-horizon parameters, the values taken by the parameters, as found in [23] and verified by us, are

$$v_1 = 16, \quad v_2 = 16, \quad \tilde{e}^0 = -1, \quad y^0 = \frac{1}{8}, \quad y^1 = \frac{1}{8}, \quad y^2 = \frac{1}{8}, \quad y^3 = \frac{1}{8} \tag{5.21}$$

Now again we check whether the Gauss-Bonnet term 5.6 takes any contributing value on this leading order solution or not. Substituting the values of the near-horizon parameters 5.21 into the Gauss-Bonnet term 5.6, we find

$$\begin{aligned}
\mathcal{E}_{\mathrm{GB}}|_{\mathrm{non-BPS}} &= 16\pi\left[\left(\frac{v_2}{v_1}+\frac{v_1}{v_2}\right)-\frac{1}{128}v_1v_2w^2\bar{w}^2\right] \\
&= 16\pi\left[\left(\frac{16}{16}+\frac{16}{16}\right)-\frac{1}{128}(16)(16)\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2\right] \\
&= 16\pi\left(2-\frac{1}{8}\right) \\
&= 16\pi\left(\frac{15}{8}\right) \\
&= 30\pi \neq 0
\end{aligned} \tag{5.22}$$

This shows us that the supersymmetric Guass-Bonnet invariant found by de Wit, et al in [8] *does* contribute to the entropy of extremal non-supersymmetric black holes.

This result tells us that the Gauss-Bonnet term could end up being quite interesting on behalf of supersymmetric invariance, for telling us more about the entropy of non-supersymmetric black holes.

Chapter 6

Conclusions and Discussions

We had begun with the question of whether the new Gauss-Bonnet term found by Butter, et al [8] has any contribution in the entropy of extremal black holes. Now we have seen that the new supersymmetric Gauss-Bonnet term does not contribute to the entropy of extremal supersymmetric black holes, just like all the other higher derivative terms found so far. But the term is non-zero for extremal non-supersymmetric black-holes and hence can potentially contribute to their entropy.

Next, we have to find out the correction to the entropy of non-supersymmetric black holes in the presence of this Gauss-Bonnet term. For this purpose, we need to find the corrected near horizon geometry. Since we treat the term perturbatively, the form of the near horizon geometry will remain unchanged, but all the parameters will get corrections due to the new term. We aim to find the higher order corrections to the leading order solution, using entropy function formalism. After that, we should apply the same procedure all over again for the complete higher derivative Lagrangian, to find the net contribution to the entropy of non-supersymmetric black holes. It would be interesting to see if the entropy matches the value predicted by Kraus and Larsen for non-supersymmetric black holes, which is so far a mystery.

We know that fully BPS black holes do not receive any quantum corrections from the supersymmetric higher derivative invariants, while non-BPS black holes do. These two case are at the two extreme ends of BPS black holes. Once the effect of higher derivative corrections on these two cases has been seen, one could also try to explore the possible answers to the same questions for intermediate cases, i.e. for quarter-BPS and half-BPS black holes. This means whether breaking quarter or half of supersymmetry would be enough to get quantum corrections or not, and why. Several other such questions regarding the impact of the higher derivative corrections on black hole entropy and how to play around with the higher derivative terms, remain open.

Appendix A

A.1 Superconformal Covariant Derivative

The derivative covariant under linearly transforming bosonic symmteries, is given as

$$\mathcal{D}_\mu \epsilon^i = \big(\partial_\mu - \tfrac{1}{4}\omega_\mu{}^{cd}\,\gamma_{cd} + \tfrac{1}{2}\,b_\mu + \tfrac{1}{2}\mathrm{i}\,A_\mu\big)\epsilon^i + \tfrac{1}{2}\,\mathcal{V}_\mu{}^i{}_j\,\epsilon^j\,. \tag{A.1}$$

The full superconformally covariant derivative is given by D_μ which, in addition to $\mathcal{D}_\mu$, also contains derivatives with respect to fermionic symmetries and superconformal boost (which transforms non-linearly).

A.2 Tangent Space Derivatives in Superspace

The tangent space derivatives in flat superspace are as following:

$$\partial_a = \frac{\partial}{\partial x^a}\,, \qquad D_{\alpha i} = \frac{\partial}{\partial \theta^{\alpha i}} + \mathrm{i}(\sigma^a)_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha}{}_i\frac{\partial}{\partial x^a}\,, \quad \bar D^{\dot\alpha i} = \frac{\partial}{\partial \bar\theta_{\dot\alpha i}} + \mathrm{i}(\bar\sigma^a)^{\dot\alpha\alpha}\,\theta_\alpha{}^i\frac{\partial}{\partial x^a}\,. \tag{A.2}$$

The curved superspace extensions of these derivatives are notated by ∇_a, $\nabla_{\alpha i}$ and $\bar\nabla_{\dot\alpha i}$ respectively.

The 2$^{\text{nd}}$ order anti-symmetric tangent space derivatives in flat superspace are given as:

$$D_{ij} := -D_{\alpha(i}D^\alpha{}_{j)}\,, \qquad D_{\alpha\beta} := -\varepsilon^{ij}D_{(\alpha i}D_{\beta)j} \tag{A.3}$$

The 2$^{\text{nd}}$ order anti-symmetric tangent space derivatives in curved superspace are given as:

$$\nabla_{ij} := -\nabla_{\alpha(i}\nabla^\alpha{}_{j)}\,, \qquad \nabla_{\alpha\beta} := -\varepsilon^{ij}\nabla_{(\alpha i}\nabla_{\beta)j} \tag{A.4}$$

A.3 Transformation of the independent Weyl gauge fields in superconformal group

The transformation rules of the Weyl multiplet under Q-supersymmetry, S-supersymmetry and conformal boosts are as following:

$$
\begin{aligned}
\delta e_\mu{}^a &= \bar\epsilon^i\,\gamma^a\psi_{\mu i} + \bar\epsilon_i\,\gamma^a\psi_\mu{}^i\,,\\
\delta\psi_\mu{}^i &= 2\,\mathcal{D}_\mu\epsilon^i - \tfrac{1}{8}T_{ab}{}^{ij}\gamma^{ab}\gamma_\mu\epsilon_j - \gamma_\mu\eta^i\\
\delta b_\mu &= \tfrac{1}{2}\bar\epsilon^i\phi_{\mu i} - \tfrac{3}{4}\bar\epsilon^i\gamma_\mu\chi_i - \tfrac{1}{2}\bar\eta^i\psi_{\mu i} + \text{h.c.} + \Lambda_{\mathrm{K}}^a e_{\mu a}\,,\\
\delta A_\mu &= \tfrac{1}{2}\mathrm{i}\bar\epsilon^i\phi_{\mu i} + \tfrac{3}{4}\mathrm{i}\bar\epsilon^i\gamma_\mu\,\chi_i + \tfrac{1}{2}\mathrm{i}\bar\eta^i\psi_{\mu i} + \text{h.c.}\,,\\
\delta\mathcal{V}_\mu{}^i{}_j &= 2\,\bar\epsilon_j\phi_\mu{}^i - 3\bar\epsilon_j\gamma_\mu\,\chi^i + 2\bar\eta_j\,\psi_\mu{}^i - (\text{h.c.}\ ;\ \text{traceless})\,,\\
\delta T_{ab}{}^{ij} &= 8\,\bar\epsilon^{[i}R(Q)_{ab}{}^{j]}\,,\\
\delta\chi^i &= -\tfrac{1}{12}\gamma^{ab}\,\not{D}T_{ab}{}^{ij}\,\epsilon_j + \tfrac{1}{6}R(\mathcal{V})_{\mu\nu}{}^i{}_j\gamma^{\mu\nu}\epsilon^j - \tfrac{1}{3}\mathrm{i}R_{\mu\nu}(A)\gamma^{\mu\nu}\epsilon^i + D\epsilon^i + \tfrac{1}{12}\gamma_{ab}T^{abij}\eta_j\,,\\
\delta D &= \bar\epsilon^i\,\not{D}\chi_i + \bar\epsilon_i\,\not{D}\chi^i\,.
\end{aligned}
\tag{A.5}
$$

A.4 Covariant Curvatures

The covariant curvatures of all possible transformations in the superconformal group are as following:

$$
\begin{aligned}
R(P)_{\mu\nu}{}^a &= 2\,\partial_{[\mu}\,e_{\nu]}{}^a + 2\,b_{[\mu}\,e_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}\,e_{\nu]b} - \tfrac{1}{2}(\bar\psi_{[\mu}{}^i\gamma^a\psi_{\nu]i} + \text{h.c.})\,,\\
R(Q)_{\mu\nu}{}^i &= 2\,\mathcal{D}_{[\mu}\psi_{\nu]}{}^i - \gamma_{[\mu}\phi_{\nu]}{}^i - \tfrac{1}{8}\,T^{abij}\,\gamma_{ab}\,\gamma_{[\mu}\psi_{\nu]j}\,,\\
R(A)_{\mu\nu} &= 2\,\partial_{[\mu}A_{\nu]} - \mathrm{i}\left(\tfrac{1}{2}\bar\psi_{[\mu}{}^i\phi_{\nu]i} + \tfrac{3}{4}\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_i - \text{h.c.}\right)\,,\\
R(\mathcal{V})_{\mu\nu}{}^i{}_j &= 2\,\partial_{[\mu}\mathcal{V}_{\nu]}{}^i{}_j + \mathcal{V}_{[\mu}{}^i{}_k\,\mathcal{V}_{\nu]}{}^k{}_j + 2(\bar\psi_{[\mu}{}^i\,\phi_{\nu]j} - \bar\psi_{[\mu j}\,\phi_{\nu]}{}^i) - 3(\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_j - \bar\psi_{[\mu j}\gamma_{\nu]}\chi^i)\\
&\quad - \delta_j{}^i(\bar\psi_{[\mu}{}^k\,\phi_{\nu]k} - \bar\psi_{[\mu k}\,\phi_{\nu]}{}^k) + \tfrac{3}{2}\delta_j{}^i(\bar\psi_{[\mu}{}^k\gamma_{\nu]}\chi_k - \bar\psi_{[\mu k}\gamma_{\nu]}\chi^k)\,,\\
R(M)_{\mu\nu}{}^{ab} &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\,\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b - 4f_{[\mu}{}^{[a}e_{\nu]}{}^{b]} + \tfrac{1}{2}(\bar\psi_{[\mu}{}^i\,\gamma^{ab}\,\phi_{\nu]i} + \text{h.c.})\\
&\quad + (\tfrac{1}{4}\bar\psi_\mu{}^i\,\psi_\nu{}^j\,T^{ab}{}_{ij} - \tfrac{3}{4}\bar\psi_{[\mu}{}^i\,\gamma_{\nu]}\,\gamma^{ab}\chi_i - \bar\psi_{[\mu}{}^i\,\gamma_{\nu]}\,R(Q)^{ab}{}_i + \text{h.c.})\,,\\
R(D)_{\mu\nu} &= 2\,\partial_{[\mu}b_{\nu]} - 2f_{[\mu}{}^a e_{\nu]a} - \tfrac{1}{2}\bar\psi_{[\mu}{}^i\phi_{\nu]i} + \tfrac{3}{4}\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_i - \tfrac{1}{2}\bar\psi_{[\mu i}\phi_{\nu]}{}^i + \tfrac{3}{4}\bar\psi_{[\mu i}\gamma_{\nu]}\chi^i\,,\\
R(S)_{\mu\nu}{}^i &= 2\,\mathcal{D}_{[\mu}\phi_{\nu]}{}^i - 2f_{[\mu}{}^a\gamma_a\psi_{\nu]}{}^i - \tfrac{1}{8}\not{D}T_{ab}{}^{ij}\gamma^{ab}\gamma_{[\mu}\psi_{\nu]\,j} - \tfrac{3}{2}\gamma_a\psi_{[\mu}{}^i\,\bar\psi_{\nu]}{}^j\gamma^a\chi_j\\
&\quad + \tfrac{1}{4}R(\mathcal{V})_{ab}{}^i{}_j\gamma^{ab}\gamma_{[\mu}\psi_{\nu]}{}^j + \tfrac{1}{2}\mathrm{i}R(A)_{ab}\gamma^{ab}\gamma_{[\mu}\psi_{\nu]}{}^i\,,\\
R(K)_{\mu\nu}{}^a &= 2\,\mathcal{D}_{[\mu}f_{\nu]}{}^a - \tfrac{1}{4}\big(\bar\phi_{[\mu}{}^i\gamma^a\phi_{\nu]i} + \bar\phi_{[\mu i}\gamma^a\phi_{\nu]}{}^i\big)\\
&\quad + \tfrac{1}{4}\big(\bar\psi_\mu{}^i D_b T^{ba}{}_{ij}\psi_\nu{}^j - 3\,e_{[\mu}{}^a\psi_{\nu]}{}^i\not{D}\chi_i + \tfrac{3}{2}D\,\bar\psi_{[\mu}{}^i\gamma^a\psi_{\nu]j} - 4\,\bar\psi_{[\mu}{}^i\gamma_{\nu]}D_bR(Q)^{ba}{}_i + \text{h.c.}\big)\,.
\end{aligned}
\tag{A.6}
$$

A.5 Definition of the non-independent Weyl gauge fields in superconformal group

On imposing the conventional constraints

$$
\begin{aligned}
R(P)_{\mu\nu}{}^{a} = 0\,, \quad \gamma^{\mu} R(Q)_{\mu\nu}{}^{i} + \tfrac{3}{2}\gamma_{\nu}\chi^{i} = 0\,, \\
e^{\nu}{}_{b}\, R(M)_{\mu\nu a}{}^{b} - \mathrm{i}\tilde{R}(A)_{\mu a} + \tfrac{1}{8} T_{abij} T_{\mu}{}^{bij} - \tfrac{3}{2} D\, e_{\mu a} = 0\,.
\end{aligned} \tag{A.7}
$$

the gauge fields $\omega_{\mu}{}^{ab}$, $f_{\mu}{}^{a}$ and $\phi_{\mu}{}^{i}$ get modified by the addition of some extra terms in their definitions and are given as following:

$$
\begin{aligned}
\omega_{\mu}{}^{ab} &= -2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_{\sigma}e_{\nu}{}^{c} - 2e_{\mu}{}^{[a}e^{b]\nu}b_{\nu} - \tfrac{1}{4}(2\bar{\psi}_{\mu}^{i}\gamma^{[a}\psi_{i}^{b]} + \bar{\psi}^{ai}\gamma_{\mu}\psi_{i}^{b} + \text{h.c.}) \\
f_{\mu}{}^{a} &= \tfrac{1}{2}R(\omega,e)_{\mu}{}^{a} - \tfrac{1}{4}\big(D + \tfrac{1}{3}R(\omega,e)\big)e_{\mu}{}^{a} - \tfrac{1}{2}\mathrm{i}\tilde{R}(A)_{\mu}{}^{a} + \tfrac{1}{16}T_{\mu b}{}^{ij}T^{ab}{}_{ij} \\
\phi_{\mu}{}^{i} &= \tfrac{1}{2}\left(\gamma^{\rho\sigma}\gamma_{\mu} - \tfrac{1}{3}\gamma_{\mu}\gamma^{\rho\sigma}\right)\left(\mathcal{D}_{\rho}\psi_{\sigma}{}^{i} - \tfrac{1}{16}T^{abij}\gamma_{ab}\gamma_{\rho}\psi_{\sigma j} + \tfrac{1}{4}\gamma_{\rho\sigma}\chi^{i}\right)
\end{aligned} \tag{A.8}
$$

A.6 Transformation of the non-independent Weyl gauge fields in superconformal group

The transformation of the superconformal gauge fields $\omega_{\mu}{}^{ab}$, $f_{\mu}{}^{a}$ and $\phi_{\mu}{}^{i}$ under the superconformal group are as following:

$$
\begin{aligned}
\delta\omega_{\mu}{}^{ab} =& -\tfrac{1}{2}\bar{\epsilon}^{i}\gamma^{ab}\phi_{\mu i} - \tfrac{1}{2}\bar{\epsilon}^{i}\psi_{\mu}{}^{j}\,T^{ab}{}_{ij} + \tfrac{3}{4}\bar{\epsilon}^{i}\gamma_{\mu}\gamma^{ab}\chi_{i} \\
&+ \bar{\epsilon}^{i}\gamma_{\mu}\,R(Q)^{ab}{}_{i} - \tfrac{1}{2}\bar{\eta}^{i}\gamma^{ab}\psi_{\mu i} + \text{h.c.} + 2\,\Lambda_{\mathrm{K}}{}^{[a}e_{\mu}{}^{b]}\,, \\
\delta\phi_{\mu}{}^{i} =& -2\,f_{\mu}{}^{a}\gamma_{a}\epsilon^{i} + \tfrac{1}{4}R(\mathcal{V})_{ab}{}^{i}{}_{j}\gamma^{ab}\gamma_{\mu}\epsilon^{j} + \tfrac{1}{2}\mathrm{i}R(A)_{ab}\gamma^{ab}\gamma_{\mu}\epsilon^{i} - \tfrac{1}{8}\not{D}T^{ab\,ij}\gamma_{ab}\gamma_{\mu}\epsilon_{j} \\
&+ \tfrac{3}{2}[(\bar{\chi}_{j}\gamma^{a}\epsilon^{j})\gamma_{a}\psi_{\mu}{}^{i} - (\bar{\chi}_{j}\gamma^{a}\psi_{\mu}{}^{j})\gamma_{a}\epsilon^{i}] + 2\,\mathcal{D}_{\mu}\eta^{i} + \Lambda_{\mathrm{K}}{}^{a}\gamma_{a}\psi_{\mu}{}^{i}\,, \\
\delta f_{\mu}{}^{a} =& -\tfrac{1}{2}\bar{\epsilon}^{i}\psi_{\mu}{}^{i}\,D_{b}T^{ba}{}_{ij} - \tfrac{3}{4}e_{\mu}{}^{a}\bar{\epsilon}^{i}\not{D}\chi_{i} - \tfrac{3}{4}\bar{\epsilon}^{i}\gamma^{a}\psi_{\mu i}\,D \\
&+ \bar{\epsilon}^{i}\gamma_{\mu}\,D_{b}R(Q)^{ba}{}_{i} + \tfrac{1}{2}\bar{\eta}^{i}\gamma^{a}\phi_{\mu i} + \text{h.c.} + \mathcal{D}_{\mu}\Lambda_{\mathrm{K}}{}^{a}\,.
\end{aligned} \tag{A.9}
$$

A.7 Product of two Chiral Multiplets

The product of two chiral multiplets is given as:

$$
\begin{aligned}
&\big(A, \Psi_{i}, B_{ij}, F_{ab}^{-}, \Lambda_{i}, C\big) \otimes \big(a, \psi_{i}, b_{ij}, f_{ab}^{-}, \lambda_{i}, c\big) = \\
&\quad \big(A\,a\,,\; A\,\psi_{i} + a\,\Psi_{i},\; A\,b_{ij} + a\,B_{ij} - \bar{\Psi}_{(i}\psi_{j)}\,, \\
&\quad A\,f_{ab}^{-} + a\,F_{ab}^{-} - \tfrac{1}{4}\varepsilon^{ij}\bar{\Psi}_{i}\gamma_{ab}\psi_{j}\,, \\
&\quad A\,\lambda_{i} + a\,\Lambda_{i} - \tfrac{1}{2}\varepsilon^{kl}(B_{ik}\,\psi_{l} + b_{ik}\,\Psi_{l}) - \tfrac{1}{4}(F_{ab}^{-}\gamma^{ab}\psi_{i} + f_{ab}^{-}\gamma^{ab}\Psi_{i})\,, \\
&\quad A\,c + a\,C - \tfrac{1}{2}\varepsilon^{ik}\varepsilon^{jl}B_{ij}\,b_{kl} + F_{ab}^{-}f^{-ab} + \varepsilon^{ij}(\bar{\Psi}_{i}\lambda_{j} + \bar{\psi}_{i}\Lambda_{j})\big)\,.
\end{aligned} \tag{A.10}
$$

A.8 Composite Chiral Multiplet

A composite chiral multiplet $\mathcal{G}(\Phi)$ written in terms of the components of another chiral multiplet Φ is given as follwing:

$$
\begin{aligned}
A|_{\mathcal{G}} &= \mathcal{G}(A)\,,\\
\Psi_i|_{\mathcal{G}} &= \mathcal{G}(A)_I\,\Psi_i{}^I\,,\\
B_{ij}|_{\mathcal{G}} &= \mathcal{G}(A)_I\,B_{ij}{}^I - \tfrac{1}{2}\mathcal{G}(A)_{IJ}\,\bar\Psi_{(i}{}^I\Psi_{j)}{}^J\,,\\
F_{ab}^-|_{\mathcal{G}} &= \mathcal{G}(A)_I\,F_{ab}^{-I} - \tfrac{1}{8}\mathcal{G}(A)_{IJ}\,\varepsilon^{ij}\bar\Psi_i{}^I\gamma_{ab}\Psi_j{}^J\,,\\
\Lambda_i|_{\mathcal{G}} &= \mathcal{G}(A)_I\,\Lambda_i{}^I - \tfrac{1}{2}\mathcal{G}(A)_{IJ}\big[B_{ij}{}^I\varepsilon^{jk}\Psi_k{}^J + \tfrac{1}{2}F_{ab}^{-I}\gamma^{ab}\Psi_k{}^J\big]\\
&\quad + \tfrac{1}{48}\mathcal{G}(A)_{IJK}\,\gamma^{ab}\Psi_i{}^I\,\varepsilon^{jk}\bar\Psi_j{}^J\gamma_{ab}\Psi_k{}^K\,,\\
C|_{\mathcal{G}} &= \mathcal{G}(A)_I\,C^I - \tfrac{1}{4}\mathcal{G}(A)_{IJ}\big[B_{ij}{}^I B_{kl}{}^J\,\varepsilon^{ik}\varepsilon^{jl} - 2\,F_{ab}^{-I}F^{-abJ} + 4\,\varepsilon^{ik}\bar\Lambda_i{}^I\Psi_j{}^J\big]\,,\\
&\quad + \tfrac{1}{4}\mathcal{G}(A)_{IJK}\big[\varepsilon^{ik}\varepsilon^{jl}B_{ij}{}^I\Psi_k{}^J\Psi_l{}^K - \tfrac{1}{2}\varepsilon^{kl}\bar\Psi_k{}^I F_{ab}^{-J}\gamma^{ab}\Psi_l{}^K\big]\\
&\quad + \tfrac{1}{192}\mathcal{G}(A)_{IJKL}\,\varepsilon^{ij}\bar\Psi_i{}^I\gamma_{ab}\Psi_j{}^J\,\varepsilon^{kl}\bar\Psi_k{}^K\gamma_{ab}\Psi_l{}^L\,.
\end{aligned}
\tag{A.11}
$$

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