Solvable Models and Mathematical Aspects of Conformal Field Theory

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

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April, 2020

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Certificate

This is to certify that this dissertation entitled Solvable Models and Mathematical Aspects of Conformal Field Theory towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Madhav Sinha at Australian National University under the supervision of Vladimir Bazhanov and Murray Batchelor, Professors at Department of Theoretical Physics, Australian National University, during the academic year 2019-2020.

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This thesis is dedicated to my parents and Dadi

Declaration

I hereby declare that the matter embodied in the report entitled Solvable Models and Mathematical Aspects of Conformal Field Theory are the results of the work carried out by me at the Department of Theoretical Physics, Australian National University, under the supervision of Vladimir Bazhanov and Murray Batchelor, and the same has not been submitted elsewhere for any other degree.

Madhav Sinha

Acknowledgments

I would firstly like to express my gratitude towards Prof. Vladimir Bazhanov, for guiding me very patiently and allowing me to explore topics of my interest. While working under him, I learnt many valuable lessons that I shall carry with me for the rest of my life. I am also grateful to Prof. Murray Batchelor for teaching me the basics, always being ready to discuss topics, and providing extremely useful references. I am also thankful to Prof. Sunil Mukhi for accepting to be Expert of this thesis and for his counsel throughout the year, particularly during the Mid-Year Evaluation. Prof. James Tener at MSI ANU also helped me to understand the key concepts required for this thesis.

This was a particularly hard year for me, and the thesis would not have been possible without the help of many people. I want to appreciate the work of Fire and Rescue Department in Australia for keeping everyone safe amid bushfires. The staff at Ursula Hall and the Research School of Physics were very kind and helpful. I would also like to thank the Australian National University for their Future Research Talent Scheme, which provided me a scholarship to write this thesis. I am in significant debt of Jay Poria, of the Future Research Talent Award team at ANU, for going out of the way to help me. I would also like to thank my teachers over the four years at IISER, as this thesis stands on their teachings. I made some great friends during my stay at Ursula Hall, in particular, the members of the soccer team, and I shall always treasure the memories we made. My batchmates, who were in ANU and the rest of the world, helped me by providing food, shelter, discussions, advice, support, laughter, and great conversations. I am forever indebted to them. Finally, this thesis is only possible because of the never-ending support of my parents, and I hope I can ever justify it.

Abstract

The thesis can be divided into two parts. The first part was on Solvable Models and the second on mathematical formulation of CFT. In the first part, we studied different solvable models and the methods used to solve them. We then focused on the 8-Vertex model and explored a novel technique of generating ansatz for the 8-vertex model. Through this technique, we managed to arrive at the two general solutions of the 8-Vertex model. However, despite significant efforts, a new solution could not be obtained. We then tried to study [10], which claims that all solutions to the 16-Vertex model can be expressed in terms of the two solutions of the 8-Vertex model, which we previously derived. The results of this paper could not be reproduced. However, we point out some ambiguities in it, and the techniques we used to try to reach the results claimed in it. The second part, on CFT, is based on [12]. Here, we study Conformal Transformation, Quantization of Symmetries, lifting Projective Representation to Unitary Representation(Bargmann's Theorem) and show that Virasoro Algebra is the non-trivial central extension of Witt Algebra.

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Chapter 1

Introduction

1.1 Solvable Models

Solvable lattice models in two dimensions give a deep insight into the behaviour of strongly correlated spin systems, as well as leading to beautiful mathematics. The most famous examples of such models include the "two-dimesional ice model" by Lieb ([1]) and Sutherland ([2]) and the "symmetric eight-vertex" model by Baxter ([3]). The 2D ice model was introduced to describe crystal lattices with hydrogen bonds, particularly the water ice. Even though the model is two-dimensional, it gives a remarkably good approximation for the residual entropy of the real (three-dimensional) water ice. Further, the exact solution of eight-vertex model [3] has for the first time demonstrated that the critical exponents for a statistical system at the point of the second order phase transition could be continuous numbers (and not just rational fractions as it was believed before).

The simplest lattice models of classical statistical mechanics describe interaction of fluctuating "integer spin" variables that are assigned to vertices, edges or faces of the lattice. Each spin configuration of the lattice is assigned the energy and the lattice is studied as a canonical ensemble. The model is called "solvable" if, at least, its free energy can be calculated exactly in the limit of an infinitely large lattice ([4]). (Strictly spaking, such models are defined by Baxter to be "exactly solvable", but not "rigorously solvable" as the calculation of free energy involved infinite dimensional matrix products, which were assumed to be convergent([6])).

1.1.1 Brief History

It was the solution of Onsager for 2-dimensional Ising Model[1] in 1944 ([7]), which is said to have attracted attention to this field, as it was the only solvable model which exhibited Phase Transition for many years.[2]. After this seminal work, Lieb and Sutherland found the solution to the six-Vertex model using Bethe Ansatz in 1960s. Baxter then showed how commuting transfer matrices could be used to solve the eight-vertex model, which used the famous star triangle or the Yang-Baxter equation. Yang-Baxter equation was not only limited to Vertex Models, but was successfully applied to solve other classes of models, such as face models. The commuting transfer matrices along with Q-operators, used by Baxter in his work, were further used by Bazhanov, Lukyanov and A.B. Zamolodchikov to study massive deformations of CFTs.

1.1.2 Vertex Models

In this thesis, we tried to solve and classify the solutions of eight-vertex model and later sixteen-vertex model. Here, by solving we refer to the conditions on local Boltzmann weights, such that the transfer matrices commute. All the calculations were performed on Mathematica. We first applied Perturbation Theory to eight-vertex model to determine the conditions on weights. In doing so we could rederive the two main class of solution in literature, however no new solutions could be found. Later, we tried to rederive the results found in [10]. According to the results in [10], all the solutions of sixteen-vertex models can be expressed in terms of the known solutions of eight-vertex model, which is a particularly surprising result, as even upon reducing the constraints, no new solution were obtained. An example was found which does not behave in the way as per the theory in [10], and it was explained why such case appeared. Despite significant efforts, the work in [10] could not be rederived.

[1] Zero-Field case

[2] In a letter to Casimir, Pauli writes that nothing much of interest happened except for Onsager's solution, when Casimir asked him regarding the developments in Theoretical Physics in Allied countries.[5]

1.1.3 Temperley-Lieb (TL) Algerba

Temperley-Lieb algebra is an associative algebra with its generators obeying certain rules. The Partition function of certain Face models in statistical mechanics obey the relations which are followed by the generators of TL algebra. The XXZ-Model is also related to TL algebra, as its Hamiltonian can be expressed in the terms of generators of this algebra. TL algebra and its relation with knot theory are explored in this thesis.

1.2 Mathematical Aspects of Conformal Field Theory

In most of physics textbooks, to derive conformal transformation, the concept of infinitesimal transformation is used. This concept is made more rigorous by defining Conformal Killing Fields and using local one parameter group corresponding to them. We then study how to transfer symmetry from Classical to Quantum space, after we have quantized. Then another attempt in this thesis is to understand how Virasoro Algebra is a Central Extension of Witt Algebra. These topics are studied mainly from [12], but some of the proofs are new, and the presentation is different, as we have tried to make the discussion more intuitive.

Chapter 2

Solvable Models

In this chapter, we study Vertex models on 2-dimensional square lattice and derive conditions on the local Boltzmann weights so that the transfer matrix corresponding to two different sets of Boltzmann weights commute. We will later look into Templerley-Lieb algebra and some interesting results related to it.

2.1 Vertex Models

In this case variables are assigned to links of the lattice and energy to each vertex - (x, y)

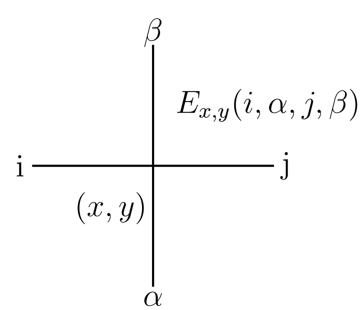

Figure 2.1: Site of a Vertex Model

In these models we allow the variables, such as α, β, i and j to only take the values ± 1. So consider the model on m x n lattice. The t^{th} row of this lattice is given below

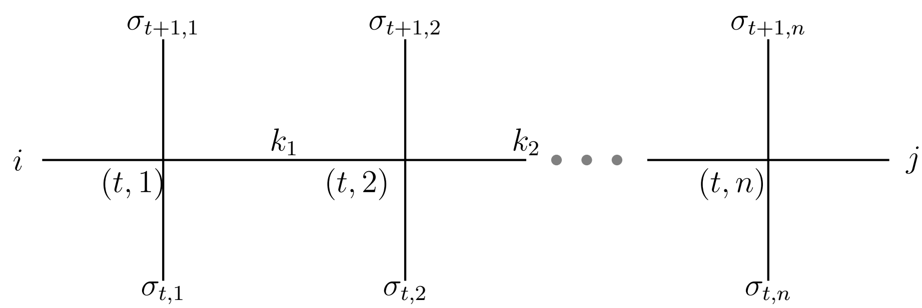

Figure 2.2: Row of a Vertex Model

In this model we apply periodic boundary condition in the horizontal direction, and set $j = i$. The set of variable on the lower row is denoted by $\sigma_t = \{\sigma_{t,1}, \sigma_{t,2}, ..., \sigma_{t,n}\}$ and similarly, σ_{t+1} for the upper row. The Partition function for this row

$$Z_{1\text{x}n}(\sigma_t, \sigma_{t+1}) = \sum_{i,k} e^{-\beta \sum_y E_{t,y}} \tag{2.1}$$

only depends on the variables σ_t and σ_{t+1} as we sum over all values of i and k's[1]. From the above equation we can define the transfer matrix [2], T, which has elements of the form

$$T_{\sigma_i}^{\sigma_j} = Z_{1\text{x}n}(\sigma_i, \sigma_j) \tag{2.2}$$

So the partition function for m x n lattice, with top row having spin variables σ_{m+1} and bottom row σ_1, can now be written as

$$Z_{m\text{x}n}(\sigma_1, \sigma_{m+1}) = \sum_{\sigma_2, \sigma_3, ... \sigma_m} T_{\sigma_1}^{\sigma_2} T_{\sigma_2}^{\sigma_3} .... T_{\sigma_{m-1}}^{\sigma_m} T_{\sigma_m}^{\sigma_{m+1}} = (T^m)_{\sigma_1}^{\sigma_{m+1}} \tag{2.3}$$

Now, we can apply periodic boundary condition over the vertical direction, i.e. set $\sigma_{m+1} = \sigma_1$ and then sum over all the values of σ_1, to get

$$Z_{m\text{x}n} = Tr(T^m) \tag{2.4}$$

[1]If another row is added to the top, then the variables on the new row's bottom should be same as the variables on the original row's top

[2]This matrix is a 2^n x 2^n dimensional matrix

if the Transfer matrix is diagonalizable, then for $m \to \infty$

$$Z_{mxn} = \lambda_0^m \tag{2.5}$$

where λ_0 is the largest eigenvalue of T.[3]

Now, let the energy $E_{x,y}(i, \alpha, j, \beta)$ be dependent on a parameter u, i.e.

$$E_{x,y}(i, \alpha, j, \beta) = E_u(i, \alpha, j, \beta)$$

So if two different vertices have the same value of (i, α, j, β) and u, then they have the same energy. So, $T(u)$ is the transfer matrix formed by all vertices in the row having the same u at each vertex.

Definition 2.1.1. *Integrability*
Let $T(u)$ be a collection of transfer matrices, corresponding to a statistical model dependent on a single variable, u in $\mathbb{C}$. A statistical mechanics model is integrable if there exists such a collection satisfying for $\forall$ u, u'

$$[T(u), T(u')] = 0 \tag{2.6}$$

The condition mentioned above helps in constructing the eigenvectors of the Transfer Matrix using T and Q-operators corresponding to the vertex model. Hence, it helps in finding the eigenvalue and thus solving the model. This construction can be found in Chapter 8 and 9 of [6] and Chapter 14 of [8].

2.2 8-Vertex Model and its solutions

8-Vertex model is a type of vertex model set on square lattice. As we had seen in the last section, each link is either assigned $\pm$ 1. For a general Vertex Model, there can be 16 types of vertices, depending on the value of the variable on the links surrounding it. However, 8 of them are disallowed by setting their energies to -∞ making their Boltzmann weight 0. The vertices with finite energy, with their Boltzmann weights are shown in Figure 2.3.

[3] As the Transfer Matrix is a square matrix with positive entries, the Perron-Frobenius theorem guarantees a real positive unique eigenvalue which is greater than the rest eigenvalues

[4]

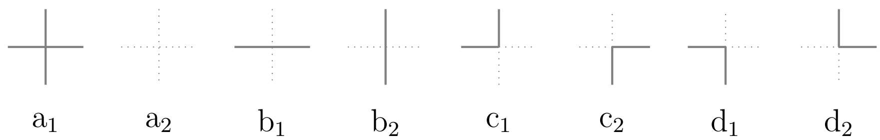

Figure 2.3: Boltzmann Weights for 8-Vertex Model

All these Boltzmann weights are a function of u. Let $T_n(u)$ be the transfer matrix for 8-vertex model, with exactly n sites in a row, with Boltzmann weights - $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$, and $T_n(u')$ similarly with n sites but with primed weights.

We want to find a set of functions $\{f_i\}$ which are invariants, that is if

$$f_i(a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2) = f_i(a_1', a_2', b_1', b_2', c_1', c_2', d_1', d_2') \tag{2.7}$$

then the Transfer Matrices, $T_n(u)$ and $T_n(u')$, commute for all n. For two site case, the Transfer Matrix is

$$T_2(u) = \begin{bmatrix} a_1^2 + b_2^2 & 0 & 0 & 2c_2 d_1 \\ 0 & a_1 b_1 + a_2 b_2 & c_1 c_2 + d_1 d_2 & 0 \\ 0 & c_1 c_2 + d_1 d_2 & a_1 b_1 + a_2 b_2 & 0 \\ 2c_1 d_2 & 0 & 0 & a_2^2 + b_1^2 \end{bmatrix} \tag{2.8}$$

$T_2(u')$ is same as $T_2(u)$, with just primed coefficients. The commutator, $[T_2(u), T_2(u')]$ is

$$\begin{bmatrix} -4c_1 c_2' d_1' d_2 + 4c_1' c_2 d_1 d_2 & 0 & 0 & \begin{array}{c} 2(-a_1'^2 a_2'^2 + b_1'^2 - b_2'^2) c_2 d_1 \\ -2(-a_1^2 + a_2^2 + b_1^2 - b_2^2) c_2' d_1' \end{array} \\ 0 & 0 & 0 & 0 \\ \begin{array}{c} 2(-a_1'^2 + a_2'^2 + b_1'^2 - b_2'^2) c_2 d_1 \\ -2(-a_1^2 a_2^2 + b_1^2 - b_2^2) c_2' d_1' \end{array} & 0 & 0 & -4c_1 c_2' d_1' d_2 + 4c_1' c_2 d_1 d_2 \end{bmatrix} \tag{2.9}$$

By setting this commutator to 0, we get the functions

$$f_{21}(c_1, c_2, d_1, d_2) = \frac{c_2 d_1}{c_1 d_2} \tag{2.10a}$$

[4] In the diagrams, dotted line represent link with -1 on it, and straight line signifies link carrying +1

$$f_{22}(a_1, a_2, b_1, b_2, c_2, d_1) = \frac{\left(-{a_1}^2 + {a_2}^2 + {b_1}^2 - {b_2}^2\right)}{c_2 d_1} \tag{2.10b}$$

to be the invariants. So, we can similarly calculate $T_n(u)$ and then set $[T_n(u), T_n(u')] = 0$, to find invariants for commutation of n-site Transfer Matrix. However, we want universal functions, which if are invariants then matrices for any n commute.

2.2.1 Linear Approximation

In this method, we do not solve the commutator equation for n sites in a row. We solve the commutator equation for n = 2, 3 and 4 site cases only, and as we will see later, this is sufficient.

For 3 and 4 sites cases, the matrices are of size 8 x 8 and 16 x 16 respectively. The commutator for 4 sites case has polynomials in primed and unprimed coefficients of total degree 8. So, straight away setting the commutator to 0 and then solving is difficult.

If the primed and unprimed coordinates were equal, then the transfer matrices would commute as they are the same. So we make the substitution that

$$a'_1 = a_1(1 + e\ w_1); \quad b'_1 = b_1(1 + e\ x_1); \quad c'_1 = c_1(1 + e\ y_1); \quad d'_1 = d_1(1 + e\ z_1) \tag{2.11a}$$

$$a'_2 = a_2(1 + e\ w_2); \quad b'_2 = b_2(1 + e\ x_2); \quad c'_2 = c_2(1 + e\ y_2); \quad d'_2 = d_2(1 + e\ z_2) \tag{2.11b}$$

Here e is an infinitesimal constant. While calculating the transfer matrix T'_n, this substitution (2.11) is made and terms of e, having power 2 or higher, are ignored. By ignoring the higher powers, we are effectively linearizing the system.

This greatly simplifies the commutator, and solving for it to be 0 is easy. However, we now have equation in unprimed coordinates and $\{w_i, x_i, y_i, z_i\}$, but we want invariants of the form (2.7). To go back to primed coordinates, there is no direct prescription and guesswork is required. However, the guesses can be made by looking at the solution of Zero field case(discussed below in (2.12)), which are much easier to find. Solving first for 2 and 3 sites in a row, then approaching the much harder four sites case, also helps.

2.2.1.1 Zero-Field Case

This corresponds to when

$$a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2 \tag{2.12}$$

and similarly for primed coordinates. Equation (2.6) was solved for different number of sites in a row with the coefficients obeying the Zero Field Condition.

Two Sites
For two sites, the commutator is 0, with just the Zero Field Condition. So no invariants are obtained in this case.

Three Sites
For three sites, the Linearizing method was not required and invariant obtained was

$$f_{31}(a_1, b_1, c_1, d_1) = \frac{(a_1^2 - 2a_1 b_1 + b_1^2 - c_1^2 - d_1^2)}{c_1 d_1} \tag{2.13}$$

Four Sites
Finding the relation for this system was harder. The commutator, $[T(a, b, c, d), T(a', b', c', d')]$, was calculated. As the commutator was 16 x 16 and involved high powers of the coefficients, it was hard to find conditions such that commutator is 0. So the substitutions, (2.11), were made and terms with power of e greater than 1 were ignored and thus the system got linearized. The conditions obtained by setting the Linearized Commutator to 0 were

$$w_1 + x_1 - y_1 - z_1 = 0 \tag{2.14a}$$

$$\left((w_1 + x_1)\left(a_1^2 + b_1^2 - c_1^2 - d_1^2\right) - 2\left(a_1^2 w_1 + b_1^2 x_1 - c_1^2 y_1 - d_1^2 z_1\right)\right) = 0 \tag{2.14b}$$

The two equations (2.14) correspond to

$$\frac{a_1 b_1}{c_1 d_1} = \frac{a_1' b_1'}{c_1' d_1'} \tag{2.15a}$$

$$\frac{(a_1^2 + b_1^2 - c_1^2 - d_1^2)}{a_1 b_1} = \frac{(a_1'^2 + b_1'^2 - c_1'^2 - d_1'^2)}{a_1' b_1'} \tag{2.15b}$$

i.e. if you substitute (2.11) in (2.15), and ignore powers of e higher than 1, then we exactly get equation (2.14). So, if our primed and unprimed equations satisfy (2.15), then the commutator made by linearizing the system would be zero.
Further, it can be checked that if we substitute (2.15) in the commutator of non-linearized system, even that is zero.
So the invariants obtained from the Four Sites case are

$$f_{41}(a_1, b_1, c_1, d_1) = \frac{a_1 b_1}{c_1 d_1} \tag{2.16a}$$

$$f_{42}(a_1, b_1, c_1, d_1) = \frac{(a_1^2 + b_1^2 - c_1^2 - d_1^2)}{a_1 b_1} \tag{2.16b}$$

If the Boltzmann weights are invariant under f_{41} and f_{41} (Equation (2.16)), then it can be checked that they are invariant under f_{31} (Equation (2.13)).

These invariants (2.16) are the only invariants needed for commutativity of transfer matrix, with not only 4 or 3, but any number of sites, as given in [8], with the condition (2.12) on weights. Baxter gave a parametrization, i.e. functions A, B, C and D[5], such that if

$$a_1 = A(u),\ a_1' = A(u') \tag{2.17a}$$

$$b_1 = B(u),\ b_1' = B(u') \tag{2.17b}$$

$$c_1 = C(u),\ c_1' = C(u') \tag{2.17c}$$

$$d_1 = D(u),\ d_1' = D(u') \tag{2.17d}$$

then, for i = 1,2

$$f_{4i}(a_1, b_1, c_1, d_1) = f_{4i}(a_1', b_1', c_1', d_1') \tag{2.18}$$

i.e. f_{41} and f_{42} are invariants. So Transfer Matrix, $T(u)$ formed with local Boltzmann Weights $A(u)$, $B(u)$, $C(u)$ and $D(u)$, are such that $[T(u), T(u')] = 0$. So, the 8-vertex model is integrable, according to Definition 2.1.1. This is the famous Baxter's solution to 8-Vertex model.

[5] These A, B, C and D are elliptic functions, and can be found in [6] and [8]

2.2.1.2 General Case

The general case was much difficult to solve, as the number of variables doubled. The prescription for linearizing, then solving and then trying to find the invariants, which were discussed in the previous subsection, were also used extensively here.

Two Sites

This case was discussed in Section 2.2, where we got the invariants as

$$f_{21}(c_1, c_2, d_1, d_2) = \frac{c_2 d_1}{c_1 d_2} \tag{2.19a}$$

$$f_{22}(a_1, a_2, b_1, b_2, c_2, d_1) = \frac{\left(-{a_1}^2 + {a_2}^2 + {b_1}^2 - {b_2}^2\right)}{c_2 d_1} \tag{2.19b}$$

Three Sites

The conditions obtained for the Transfer Matrices to commute are

$$f_{31} = \frac{c_2 d_1}{c_1 d_2} \tag{2.20a}$$

$$f_{32}(a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2) = \frac{\left(-{a_1}^3 + a_1 {b_1}^2 + {a_2}^2 b_2 - {b_2}^3 + (a_2 + b_1)(c_1 c_2 + d_1 d_2)\right)}{c_2 d_1 (a_1 + b_2)} \tag{2.20b}$$

$$f_{33}(a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2) = \frac{\left(-{a_2}^3 + a_2 {b_2}^2 + {a_1}^2 b_1 - {b_1}^3 + (a_1 + b_2)(c_1 c_2 + d_1 d_2)\right)}{c_2 d_1 (a_2 + b_1)} \tag{2.20c}$$

Four Sites

This case is overdetermined, on linearizing we are effectively solving for 6 variables, with 6 unknown. Without taking any conditions on the weights, only trivial solution is obtained. So, we have to impose conditions on weights, to obtain non-trivial solutions.

Free-Fermion Solution

The condition imposed on coefficients were

$$a_1a_2 + b_1b_2 - c_1c_2 - d_1d_2 = 0 \tag{2.21a}$$

$$a'_1a'_2 + b'_1b'_2 - c'_1c'_2 - d'_1d'_2 = 0 \tag{2.21b}$$

They were imposed as a lot of terms in commutators had such terms, so setting them to 0, simplified the commutator. After imposing these conditions, it was easier to find the invariants, which were

$$f_{41}(c_1, c_2, d_1, d_2) = \frac{c_2d_1}{c_1d_2} \tag{2.22a}$$

$$f_{42}(a_1, a_2, b_1, b_2, c_2, d_1) = \frac{(a_1b_2 + a_2b_1)}{c_2d_1} \tag{2.22b}$$

$$f_{43}(a_1, a_2, b_1, b_2, c_2, d_1) = \frac{-{a_1}^2 + {a_2}^2 + {b_1}^2 - {b_2}^2}{c_2d_1} \tag{2.22c}$$

Equation (2.21) is called the Free-Fermion condition on the Boltzmann weights. This is another solution for commutation of 8-vertex model Transfer Matrices, not just for 4 sites, but for any general n, as shown also in [9].

2.2.1.3 Conclusion

We managed to re-derive the two classes of solutions of the 8-Vertex Model when no weight is set to 0. It can be verified that these indeed are the solutions, by checking that weights for which invariants are given by Equation (2.15) and Equation (2.22), satisfy the Yang-Baxter relation, as shown in [8].

We wanted to see if a new solution for the 8-Vertex Model could be obtained by this method. However, despite significant efforts, no new solutions were obtained.

2.2.2 Yang-Baxter Equation for Vertex Models

Consider the vertex given in Figure(2.1). The Boltzmann weight corresponding to it is represented by

$$W(u)_{i\ \alpha}^{\beta\ j} = e^{-E_u(i,\alpha,j,\beta)}$$

So now we want to find the product of two transfer matrices with different u's. So,

$$(T(u')T(u))_{\sigma_i}^{\sigma_k} = \sum_{\sigma_j} T(u')_{\sigma_j}^{\sigma_k} T(u)_{\sigma_i}^{\sigma_j} \tag{2.23}$$

which gives the weight shown in Figure 2.4.

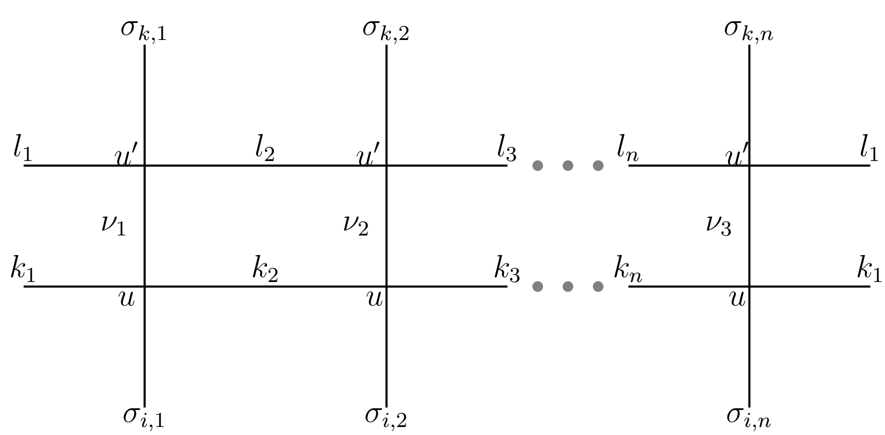

Figure 2.4: Two rows weight

Now, consider a pair of sites as in Figure 2.5, where

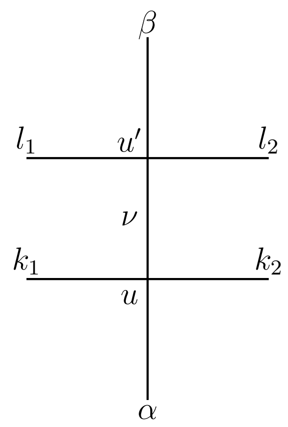

Figure 2.5: Two sites weight

$S(\alpha,\beta)_{l_1\ k_1}^{l_2\ k_2}$ is the weight for such an arrangement, with

$$S(\alpha,\beta)_{l_1\ k_1}^{l_2\ k_2} = \sum_{\nu} W(u')_{l_1\ \nu}^{\beta\ l_2}\ W(u)_{k_1\ \alpha}^{\nu\ k_2} \tag{2.24}$$

Now we can form a 4x4 matrix, $S(\alpha,\beta)$, with elements $S(\alpha,\beta)_{l_1\ k_1}^{l_2\ k_2}$, where the subscript elements (l_1,k_1) are used to label rows, and superscript elements (l_2,k_2) are used to label columns of the matrix. So, the weight of Figure 2.4 can be written as

$$(T(u')T(u))_{\sigma_i}^{\sigma_k} = Tr[S(\sigma_{i,1},\sigma_{k,1})S(\sigma_{i,2},\sigma_{k,2}).......S(\sigma_{i,n},\sigma_{k,n})] \tag{2.25}$$

and similarly we can write

$$(T(u)T(u'))_{\sigma_i}^{\sigma_k} = Tr[S'(\sigma_{i,1},\sigma_{k,1})S'(\sigma_{i,2},\sigma_{k,2}).......S'(\sigma_{i,n},\sigma_{k,n})] \tag{2.26}$$

where S' is the weight of Figure (2.5) after swapping u and u', i.e.

$$S'(\alpha,\beta)_{l_1\ k_1}^{l_2\ k_2} = \sum_{\nu} W(u)_{l_1\ \nu}^{\beta\ l_2}\ W(u')_{k_1\ \alpha}^{\nu\ k_2} \tag{2.27}$$

Note, if there is a matrix R such that

$$S(\alpha,\beta) = R\ S'(\alpha,\beta)\ R^{-1} \tag{2.28}$$

for all values of α and β, then by substituting (2.28) in (2.25) and then using the cyclic property of trace, we get

$$(T(u)T(u'))_{\sigma_i}^{\sigma_k} = (T(u')T(u))_{\sigma_i}^{\sigma_k} \tag{2.29}$$

Hence, the Transfer Matrices will commute if Equation (2.28) is satisfied. Here, R is the Intertwining or the R-Matrix, and we can write its elements as $R_{\mu'}^{\mu}{}_{\nu'}^{\nu}$. Then we can right multiply (2.28) with R, and expand S to get LHS and RHS as

$$(S(\alpha,\beta)\ R)_{l_0}^{l_2}{}_{k_0}^{k_2} = S(\alpha,\beta)_{l_1}^{l_2}{}_{k_1}^{k_2}\ R_{l_0}^{l_1}{}_{k_0}^{k_1} = W(u')_{l_1}^{\beta}{}_{\nu}^{l_2}\ W(u)_{k_1}^{\nu}{}_{\alpha}^{k_2}\ R_{l_0}^{l_1}{}_{k_0}^{k_1} \tag{2.30}$$

$$= (R\ S(\alpha,\beta))_{l_0}^{l_2}{}_{k_0}^{k_2} = R_{l_1}^{l_2}{}_{k_1}^{k_2}\ S'(\alpha,\beta)_{l_0}^{l_1}{}_{k_0}^{k_1} = R_{l_1}^{l_2}{}_{k_1}^{k_2}\ W(u)_{l_0}^{\beta}{}_{\nu}^{l_1}\ W(u')_{k_0}^{\nu}{}_{\alpha}^{k_1}$$

Let the elements of R matrix be the weight of Figure 2.6.

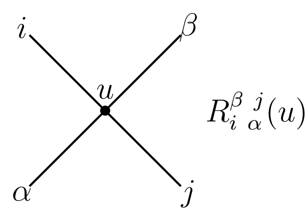

Figure 2.6: R-Matrix vertex

Then we can diagrammatically represent Equation (2.30) as in Figure 2.7.

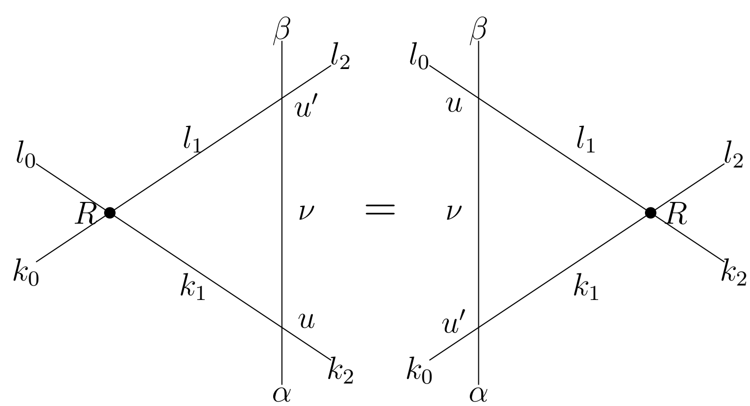

Figure 2.7: Yang-Baxter Relation for Vertex Model

A graphical proof for the commutation of Transfer Matrices, when weights satisfy Yang-Baxter Relation, can be found in [8].

2.3 Spectral Polynomial and Baxter's Equation

$\mathcal{L}$ and $\mathcal{L}^1$ are 2 x 2 matrices, with elements, operators in $\mathbb{C}^n$, which can themselves be represented as n x n matrices. So, $\mathcal{L}$ and $\mathcal{L}^1$ can also be interpreted as a $2n$ x $2n$ matrix, where elements are in $\mathbb{C}$. Elements of $\mathcal{L}$ are labelled as $\mathcal{L}^{i\ \alpha}_{j\ \beta}$, where $1 \leq i, j \leq n$ and $1 \leq \alpha, \beta \leq 2$. The Greek indices are used to label the block of the matrices and the Latin indices are for elements of the block. i and α are used to label elements of the column and j and β for elements of rows. Further for the case of $n = 2$, we can associate with $\mathcal{L}^{i\ \alpha}_{j\ \beta}$, the weight of the vertex, as in Figure 2.8.

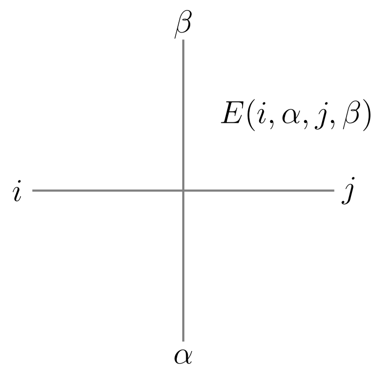

Figure 2.8: Corresponding Vertex

The Monodromy matrix, $\mathcal{T}$ is a 2 by 2 matrix, with elements of the form $\mathcal{T}^i_j$ where $1 \leq i, j \leq 2$. Each $\mathcal{T}^i_j$ are themselves a 2^n x 2^n matrix which has elements where $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $\beta = \{\beta_1, \beta_2, ...\beta_n\}$. $\mathcal{T}^{i,\alpha}_{j,\beta}$ corresponds to weight in Figure 2.9.

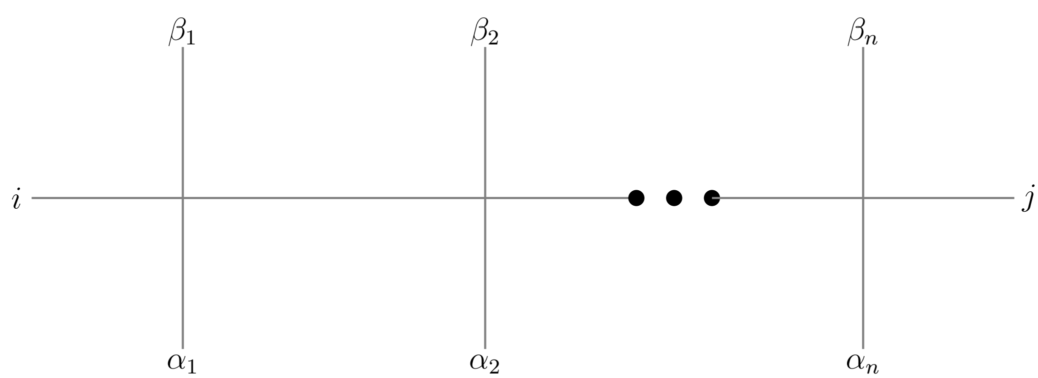

Figure 2.9: Weight corresponding to Monodromy Matrix

By taking trace of $\mathcal{T}$, we get the Transfer Matrix, T $= \mathcal{T}_1^1 + \mathcal{T}_2^2$, we obtain the row to row transfer matrix for Vertex model. It can be shown if

$$R(\mathcal{L} \otimes \mathcal{L}^1) = (\mathcal{L}^1 \otimes \mathcal{L})R \tag{2.31}$$

which is referred to as Baxter's Equation in [10], then

$$R(\mathcal{T} \otimes \mathcal{T}^1) = (\mathcal{T}^1 \otimes \mathcal{T})R \tag{2.32}$$

and if R is an invertible matrix, then the above equation implies

$$[T, T_1] = 0 \tag{2.33}$$

2.3.1 Vacuous Vector and Elliptic Curves

For this section we assume $\mathcal{L}$ to be a n x n matrix, but with elements 2 by 2 matrices. Let X and Y be n-dimensional column vectors and U and V be 2-dimensional column vectors, such that $U = (u, 1)$ and $V = (v, 1)$. X and Y are scaled such that $X_n = Y_n = 1$
Now, for a point (u, v) in $\mathbb{C}^2$, $X \otimes U$ is a vacuous vector of $\mathcal{L}$ iff there exist h and Y such that

$$\mathcal{L}(X \otimes U) = h(Y \otimes V) \tag{2.34a}$$

which in terms of matrix elements is

$$\mathcal{L}^{i\alpha}_{j\beta} X_i U_\alpha = h Y_j V_\beta \tag{2.34b}$$

Here, $h \in \mathbb{C}$. A vacuous vector, X, does not exist for any (u, v) in $\mathbb{C}^2$. To see this, let $\tilde{V} = (1, -v)$. By multiplying (2.34b) with vector $\tilde{V}$, we get

$$\tilde{V}^\beta \mathcal{L}^{i\alpha}_{j\beta} X_i U_\alpha = \tilde{V}^\beta h(u, v) Y_j V_\beta = 0 \tag{2.35}$$

Now defining a spectral matrix of $\mathcal{L}$, to be $L_\mathcal{L}(u, v)$, a n x n matrix where $(L_\mathcal{L}(u, v))^i_j = \tilde{V}^\beta \mathcal{L}^{i\alpha}_{j\beta} U_\alpha$, (2.35) becomes

$$L_\mathcal{L}(u, v) X(u, v) = 0 \tag{2.36}$$

and the above condition will be satisfied if and only if

$$P_{\mathcal{L}}(u,v) = det(L_{\mathcal{L}}(u,v)) = 0 \tag{2.37}$$

Now, note $P_{\mathcal{L}}(u,v)$, called the spectral polynomial, is formed by taking the determinant of spectral matrix, $L_{\mathcal{L}}$. So, vacuous vector for a given (u,v) will exist if and only if the spectral polynomial, $P_{\mathcal{L}}(u,v) = 0$.

$P_{\mathcal{L}}(u,v)$ is a n-degree polynomial in v, for a fixed u, and thus by fundamental theorem of algebra, the equation will have n roots counted upto multiplicity.

Now, consider the case when $\mathcal{L}$ is a 4 by 4 matrix. Then, in general, we will have two values of v, for each u such that $P(u,v) = 0$. Corresponding to the pair (u,v), there exists a single unique vacuous vector, $X(u,v)$.
I will try to illustrate this with an example. Consider the case of

$$\mathcal{L} = \begin{bmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{bmatrix}$$

then the spectral polynomial is

$$P_{\mathcal{L}}(u,v) = -cdu^2v^2 + (d^2 + c^2 - a^2 - b^2)uv + abu^2 + abv^2 - cd \tag{2.38}$$

which for general u, has 2 roots, v_1 and v_2, for a, b, c, d independent of each other. In this case, the unique vacuous vector $X(u, v_i) = (x(u, v_i), 1)$ has the form

$$x(u, v_i) = \frac{-d + cuv_i}{au - bv_i} \tag{2.39}$$

So, in general, v_1 and v_2 should be different, and the vacuous vector, $X(u, v_i)$ corresponding to these points are unique. We can similarly show that for a general 4 x 4 matrix, the spectral polynomial has 2 roots, v_1 and v_2, for a general u, and unique vacuous vector X, corresponding to each (u, v_i).

Hence corresponding to $\mathcal{L}$ and $\mathcal{L}'$, we can find polynomials $P_{\mathcal{L}}$ and $P_{\mathcal{L}'}$, where the form of these polynomials is

$$P_{\mathcal{L}}(u,v) = \sum_{i,j=0}^{n} a_{ij} u^i v^j = 0 \tag{2.40}$$

Corresponding to the condition in (2.40), we get a curve $\Gamma_{\mathcal{L}}$, i.e. $z = (u,v) \in \Gamma_{\mathcal{L}}$, iff $P_{\mathcal{L}}(u,v)$ $= 0$. We get a similar curve $\Gamma_{\mathcal{L}'}$ for $P_{\mathcal{L}'}(u,v)$. Note, due to form of (2.40), both these curves are elliptic curves for $n = 2$.
However, this is not true for all $\mathcal{L}$. If $\mathcal{L}$ is a 4 x 4 identity matrix. The spectral matrix, L_I, and the spectral polynomial, P_I have the form

$$L_I = \begin{bmatrix} u-v & 0 \\ 0 & u-v \end{bmatrix} \tag{2.41a}$$

$$P_I(u,v) = (u-v)^2 \tag{2.41b}$$

So, this has a single root for each u. When we put $u = v$, then L_I is a zero matrix, so there is no restriction on X for a point, as $L_I X = 0$, for any value of X. So, the uniqueness condition for the vacuous vector does not hold here. Such cases are not discussed in [10], as the paper considers only general case, where the elements of $\mathcal{L}$ are unrelated.

2.3.2 Using Vacuous Vector for Baxter's Equation

Let $\mathcal{L}$ and $\mathcal{L}'$ be 2 x 2 matrices, with elements being 2 x 2 matrices themselves (i.e. they are 4 x 4 effectively). The vacuous vectors corresponding to $\mathcal{L}$ and $\mathcal{L}'$ are $X(u,v) \otimes U$ and $X'(u',v') \otimes U'$, so

$$\mathcal{L}(X(u,v) \otimes U) = h(u,v)(Y(u.v) \otimes V) \tag{2.42a}$$

$$\mathcal{L}'(X'(u',v') \otimes U') = h'(u',v')(Y'(u',v') \otimes V') \tag{2.42b}$$

We say they satisfy Baxter's Equation if

$$R(\mathcal{L} \otimes \mathcal{L}^1) = (\mathcal{L}^1 \otimes \mathcal{L})R \tag{2.43}$$

where R is a 4 x 4 matrix with entries in $\mathbb{C}$. Treat $\mathcal{L}$ and $\mathcal{L}^1$ to be 2 x 2 matrices while tensoring them. So, $\mathcal{L} \otimes \mathcal{L}^1$ are 4 x 4 matrices, with entries being 2 x 2 matrices. So in computing $R(\mathcal{L} \otimes \mathcal{L}^1)$, treat $\mathcal{L} \otimes \mathcal{L}^1$ as 4x4 matrix, and so the product is well defined and again 4 x 4 matrices, with entries 2 x 2 matrices. Similarly we can compute $(\mathcal{L}^1 \otimes \mathcal{L})R$.

Let $\Lambda_1 = (\mathcal{L}^1 \otimes \mathcal{L})R$ and $\Lambda_2 = R(\mathcal{L} \otimes \mathcal{L}^1)$ and let Q_1 and Q_2 be the spectral polynomials corresponding to them, and Γ_{Λ_1} and Γ_{Λ_2} be the solution curve for

$$Q_i(u, w) = 0 \tag{2.44}$$

where $i \in \{1, 2\}$. As complex numbers are algebraically closed, for each u, there are 4 solutions $\{w_1, w_2, w_3, w_4\}$. Now consider the point u in context of $P_{\mathcal{L}}$. For $P_{\mathcal{L}}$, there are two points v_1 and v_2 such that

$$P_{\mathcal{L}}(u, v_i) = 0 \tag{2.45}$$

Similarly, for v_1 and v_2 there are points w_1, w_2, w_3 and w_4, such that

$$P_{\mathcal{L}'}(v_1, w_i) = 0 \text{ where } i \in \{1.2\} \tag{2.46a}$$

$$P_{\mathcal{L}'}(v_2, w_j) = 0 \text{ and } j \in \{3.4\} \tag{2.46b}$$

Now, using (2.42a) and (2.42b) it can be shown that[6]

$$\Lambda_1(X_1(u, w_j) \otimes U) = h(u, v_i)h'(v_i, w_j)(Y_1(u, w_j) \otimes W_j) \tag{2.47a}$$

$$X_1(u, w_j) = R^{-1}(X'(v_i, w_j) \otimes X(u, v_i)) \text{ and } Y_1(u, w_j) = Y'(v_i, w_j) \otimes Y(u, v_i) \tag{2.47b}$$

where if $i = 1$, $j \in$ {1,2} and if $i = 2$, $j \in$ {3,4}. Hence

$$Q_1(u, w_k) = 0, \text{where } k \in \{1, 2, 3, 4\} \tag{2.48}$$

Alternatively, we could have started from $\mathrm{P}_{\mathcal{L}'}$, finding $\tilde{v}_1$ and $\tilde{v}_2$ corresponding to u, and then using $P_{\mathcal{L}}$ to find roots corresponding to $\tilde{v}_1$ and $\tilde{v}_2$, say $\tilde{w}_1$, $\tilde{w}_2$, $\tilde{w}_3$ and $\tilde{w}_4$. Then by

[6]Note, X_1 is the vacuous vector upto a scaling, as the last coordinate of X_1 is not 1.

again using (2.42a) and (2.42b), the following can be verified

$$\Lambda_2(X_2(u, w_j) \otimes U) = h(\tilde{v}_i, w_j)h'(u, \tilde{v}_i)(Y_2(u, w_j) \otimes W_j) \tag{2.49a}$$

$$X_2(u, w_j) = X(\tilde{v}_i, w_j) \otimes X'(u, \tilde{v}_i) \text{ and } Y_2(u, w_j) = R(Y(\tilde{v}_i, w_j) \otimes Y(u, v_i)) \tag{2.49b}$$

Now, this implies

$$Q_2(u, \tilde{w}_k) = 0, \text{where } k \in \{1, 2, 3, 4\} \tag{2.50}$$

Now, due to Baxter's equation $Q_1(u, w)$ and $Q_2(u, w)$ are the same, say Q, so the sets $\{w_1, w_2, w_3, w_4\}$ and $\{\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4\}$ are the same.

A central claim of [10] is

Claim All polynomials $Q(u, w)$ which correspond to solution of Baxter's Equation $(R, \mathcal{L}, \mathcal{L}')$ are reducible, i.e.

$$Q(u, w) = Q_\alpha(u, w).Q_\beta(u, w) \tag{2.51}$$

where $Q_\alpha(u, w)$ and $Q_\beta(u, w)$ are polynomials in u and w.

• Just to perform basic checks, we calculated the spectral polynomial corresponding to Baxter's Eight Vertex(i.e. Equation (2.14)) and Free Fermion solution(i.e. Equation (2.22)) and found that the spectral polynomial corresponding to them actually reduced.

• To prove this claim, the arrangement of roots are considered in [10], and we discuss this argument next.

For a general point u, it has all four different roots, w_1, w_2, w_3 and w_4. Now, the arrangement of these roots can only be in the two different ways, shown in Figure 2.10.

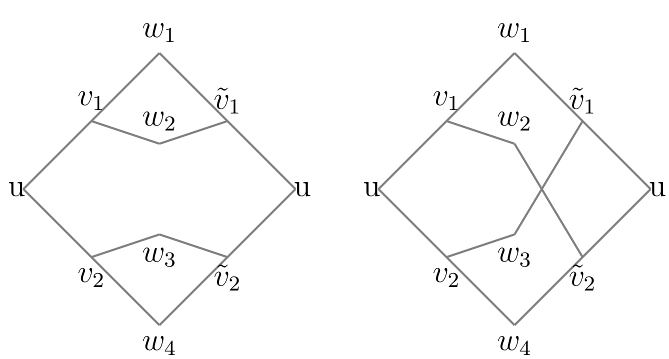

Figure 2.10: Arrangement of roots

Let us assume that the arrangement of roots is in the form of first diagram(on the left) in Figure 2.10.

Now, as Λ_1 and Λ_2 are same, their vacuous vectors (X_1 and X_2 in (2.47b) and (2.49b)) should be equal upto a constant. So lets take points (u, w_1) and (u, w_2). The vacuous vector for (u, w_1) are $R^{-1}(X'(v_1, w_1) \otimes X(u, v_1))$ and $X(\tilde{v}_1, w_1) \otimes X'(u, \tilde{v}_1)$, which implies

$$R(X(\tilde{v}_1, w_1) \otimes X'(u, \tilde{v}_1)) = g(u, w)(X'(v_1, w_1) \otimes X(u, v_1)) \tag{2.52}$$

If we start with (u, w_2), we similarly get

$$R(X(\tilde{v}_1, w_2) \otimes X'(u, \tilde{v}_1)) = g_1(u, w)(X'(v_1, w_2) \otimes X(u, v_1)) \tag{2.53}$$

From the two equations above, we note that for the point $(x'(u, \tilde{v}_1), x(u, v_1))$ [7], there are two vacuous vector, $X(\tilde{v}_1, w_1)$ and $X(\tilde{v}_2, w_1)$,[8] which are different due to them being vacuous vector for different points, and this contradicts the uniqueness of vacuous vector for a point. So, only the second diagram is allowed. Here we have assumed that the spectral polynomial corresponding to R, P_R, has unique vacuous vector for each solution.

However, consider the special case when $\mathcal{L} = \mathcal{L}'$. Then it is seen that Baxter's Equation (Equation (2.43)) is satisfied when R is identity matrix. As we saw in (2.41), the spectral polynomial for R has the form

$$P_R(u, v) = (u - v)^2 \tag{2.54}$$

and there is no condition for uniqueness of vacuous vectors. Hence the first diagram is not ruled out. In fact, the roots are actually found to be arranged in the form of diagram 1 when $\mathcal{L} = \mathcal{L}'$.

However, in [10], it is claimed that such an arrangement of roots is "impossible", for solution of Baxter's Equation, and the argument used was the one we presented before.

In the proof for the claim given in [10], it is extensively used that solutions have roots arranged in the form of Diagram 2. Hence, we could not reproduce the proof given in [10] for the claim, that all polynomials which correspond to Baxter's solutions are reducible. So, we tried to prove this computationally.

[7] x is the first component of X and x' of X'

[8] Note the similarity between Equation 2.52 and 2.53 and Equation 2.42a

2.3.3 Checking Reducibility

We consider two polynomials P and P_1, which are the spectral polynomial for $\mathcal{L}$ and $\mathcal{L}'$ respectively. $\mathcal{L}$ and $\mathcal{L}'$ satisfy the Baxter's Equation (2.31). On these polynomials, we impose the condition that $\mathcal{L}$ and $\mathcal{L}'$ satisfy Baxter's Equation by calculating Q_1 and Q_2 (2.44) and setting them equal. However, we do not know the form of R matrix. So, we look at two methods of calculating Q_1 and Q_2 in Appendix A and compare them.
So, we calculated Q_1 and Q_2 using resultant method, [9] and set

$$Q_1(u, w) = Q_2(u, w) \tag{2.55}$$

to obtain conditions on the coefficients of P_1 and P_2. We wanted to see after imposing these conditions on coefficients whether or not the polynomial Q reduced. However, the relations which we obtained from (2.55) were very complex and the computation time became very long. So, Krichever's claim, that $Q(u, w)$ is reducible could not be verified computationally.

[9] discussed in Appendix

2.4 Temperley Lieb Algebra

2.4.1 Algebra

Definition 2.4.1. *Let A be an R-module, where R is a Commutative Ring with identity, and A is an Abelian group(+). This structure is said to be associative R-Algebra if*

1. *Bilinearity There is an R-bilinear map* $\star : A \ X \ A \to A$ *i.e.* $\forall \ x_1, x_2, y_1 \in A$ *and* $r \in R$, $(rx_1 + x_2) \star (y_1) = r(x_1 \star y_1) + x_2 \star y_1$ *and similarly for* $(y_1) \star (rx_1 + x_2)$
2. *Associativity* $\forall \ a_1, \ a_2, \ a_3 \in A, \ (a_1 \star a_2) \star a_3 = a_1 \star (a_2 \star a_3)$
3. *Unitality There exists 1 in A, such that* $\forall \ a \in A, \ a \star 1 = 1 \star a = a$

The algebra is said to be finitely generated, if there exist finitely many elements (generators) $a_1, \ a_2, \ a_3, .. \ a_k$*, such that every element can be represented as a polynomial in them with coefficients from R.*

2.4.2 Temperley-Lieb Algebra

Definition 2.4.2. *Let δ be an element of a ring R. $TL_n(\delta)$, an associative R-algebra is called Temperley-Lieb Algebra iff it is generated by elements U_1, U_2, ... U_{n-1} and their inverses which satisfy the following relations -*

$$U_i^2 = \delta \ U_i \tag{2.56a}$$

$$U_i \ U_{i\pm1} \ U_i = U_i \tag{2.56b}$$

$$U_i \ U_j = U_j \ U_i \ if \ \mid i - j \mid \geq 2 \tag{2.56c}$$

We can use diagrams to represent Temperley-Lieb Algebra. The image of generators U_1, U_2, are shown in figure 2.11.

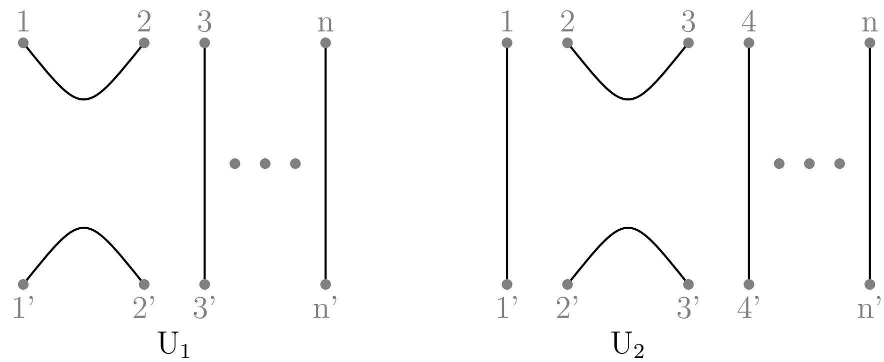

Figure 2.11: Generators of Temperley-Lieb Algebra

The multiplication of two diagrams, interpreted in terms of diagrams is represented by just the concatenation of two diagrams, whereby if a closed curve is produced, then it is just removed by multiplying with δ.
It can be checked that the diagrams of generator follow the relations for generators, given in (2.56). For example, U_2^2 is shown in Figure 2.12.

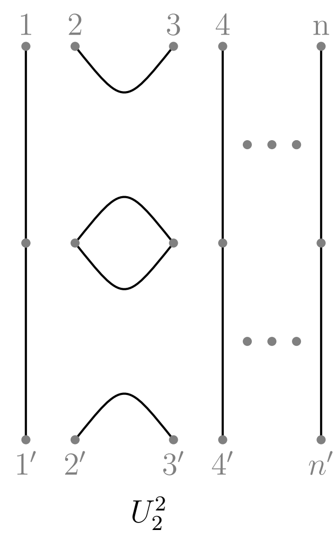

Figure 2.12: Product of Generators

This diagram is δ U_2, after we remove the loop in middle. Hence, we have verified that the diagrams do follow equation (2.56a). The other equations in (2.56) can also be verified for diagrams.

2.5 Temperley-Lieb Algebra and Knot Theory

In this section, we will consider the Temperley-Lieb Algebra, over the Ring $\tilde{\mathbb{Z}}$[A, A^{-1}].[10] For this Temperley-Lieb Algebra, we take δ = -A^2 - A^{-2}. In this section, a map from the Braid group to Temperley Lieb Algebra will be discussed.

2.5.1 Braid Group

Definition 2.5.1. *The Braid Group of N strands(B_n) has N - 1 generators σ_i (1 $\leq i \leq$ N-1). These generators have the following relations*

1. $\sigma_i\ \sigma_j\ = \sigma_j\ \sigma_i$, *if* $\mid i - j \mid\ \geq 2$

2. $\sigma_i\ \sigma_{i+1}\ \sigma_i\ = \sigma_{i+1}\ \sigma_i\ \sigma_{i+1}$

Here σ_i represents twisting of i^{th} and $i+1^{th}$ strings.

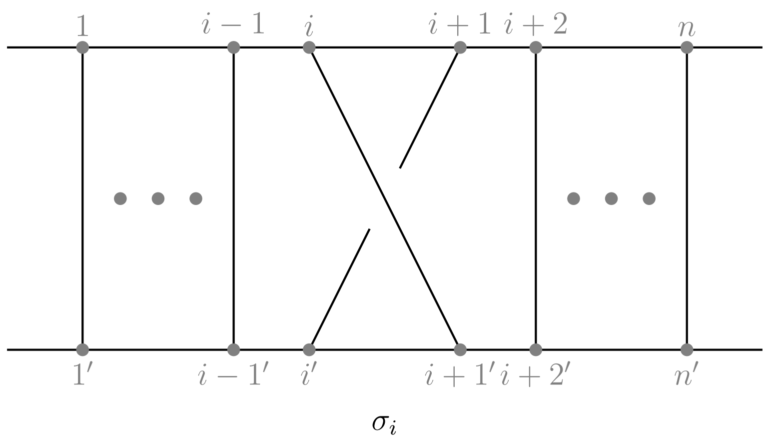

Figure 2.13: Generator of Braid Group

The product of two generators is by concatenating them vertically. The above mentioned relations can be verified by drawing the generators and then concatenating them with the strings being allowed to move but not cut across each other.

[10]This ring is made of elements of the form P/Q, $Q \neq 0$, where P and Q belong to $\mathbb{Z}[A,\ A^{-1}]$(Laurent Polynomial)

Alexander's Theorem states that any knot can be transformed into a closed braid. Hence, Braids are of significant importance in Knot Theory.

2.5.2 The Representation of Braid Group

In this section the representation of Braid group in Temperley-Lieb Algebra will be discussed. So any element of the Temperley-Lieb Algebra Consider the map $\rho : B_n \to \mathrm{End}(\mathrm{TL}_n)$, such that

$$\rho(\sigma_i) \coloneqq A.U_i + A^{-1}.id \tag{2.57a}$$

$$\rho(\sigma_i^{-1}) \coloneqq A^{-1}.U_i + A.id \tag{2.57b}$$

They act by multiplication, that is for example $\rho(\sigma_i)(a_1U_1 + a_2U_2) = (A.U_i + A^{-1}.id)(a_1U_1 + a_2U_2)$.
And to find the action of a complex element, we can break it down into the product of generators, and then use

$$\rho(\sigma_i\sigma_j) \coloneqq \rho(\sigma_i)\rho(\sigma_j)$$

However, what has to be checked is that if the map is well-defined, that is

$$\rho(id_B) = \rho(\sigma_i\sigma_i^{-1}) = \rho(\sigma_i)\rho(\sigma_i^{-1}) \stackrel{?}{=} id \tag{2.58a}$$

$$\rho(\sigma_i)\rho(\sigma_j) \stackrel{?}{=} \rho(\sigma_j)\rho(\sigma_i)\text{where } |i-j| \geq 2 \tag{2.58b}$$

$$\rho(\sigma_i)\rho(\sigma_{i+1})\rho(\sigma_i) \stackrel{?}{=} \rho(\sigma_{i+1})\rho(\sigma_i)\rho(\sigma_{i+1}) \tag{2.58c}$$

if the above equations are true.
This is checked in An Invariant of Isotopy, Louis Kauffman and is just simple calculation. An important fact used in the calculations is that the δ for Temperley-Lieb algebra is -A^2 - A^{-2} here.
Thus we have shown that the map ρ is a representation of the Braid group into Temperley-Lieb Algebra. The map ρ is a very canonical map. It can be shown to be originating from the smoothing of knots. So, consider the Trefoil knot shown in Figure 2.14.

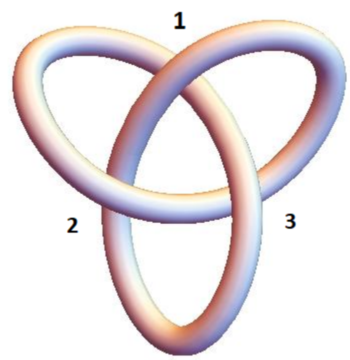

Figure 2.14: Trefoil Knot - drawn using Inkscape

Here 1, 2 and 3 are crossing. Bracket Polynomial is a polynomial associated with each knot. The prescription to calculate it is to go to each crossing and resolving it by replacing with the two diagrams given in Figure 2.15, called the horizontal and vertical smoothing, which are multiplied by A and A^{-1} respectively.

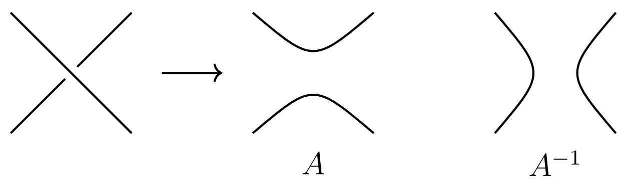

Figure 2.15: Smoothing of Crossing

The resolving of crossing for the braid group σ_i, is shown in Figure 2.16.

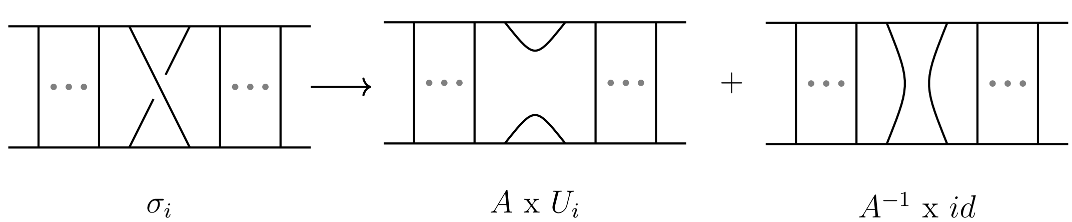

Figure 2.16: Smoothing of a Braid Generator

This is exactly the map (2.57a). This map gives a way of calculating the Bracket polynomial of the closure of a braid.

Chapter 3

Mathematical Aspects of CFT

3.1 Conformal Transformations

Definition 3.1.1. *Conformal Transformation*
Let U and U' be open subsets[1] *of M and M' respectively, where M and M' are semi-Riemannian Manifold. Let $\phi : U \to U'$ be a differentiable map, then it is a conformal transformation iff $\exists$ a differentiable map $\Omega : U \to \mathbb{R}_+$* [2] *such that $\forall\ p \in U$,*

$$(\phi^* g')(p) = \Omega^2(p)\ g(p) \tag{3.1}$$

where g and g' are the metric (symmetric bilinear forms) on M and M', and ϕ^* is the pullback of ϕ.

This above definition can be restated as, ϕ is conformal iff $\forall\ p \in U$

$$g'_{ij}(\phi(p))\ \partial_\mu \phi^i(p)\ \partial_\nu \phi^j(p) = \Omega^2(p) g_{\mu\nu}(p) \tag{3.2}$$

Here ϕ^i is the i^{th} component of $\psi_{U'} \circ \phi \circ \psi_U^{-1}$, where ψ_U and $\psi_{U'}$ are homeomorphisms from U and U' respectively, to Euclidean Space[3].

[1] Any subset U should be regarded as an open subset from now on
[2] when $\Omega^2 = 1$, ϕ preserves distance locally
[3] (U, ψ_U) and $(U', \psi_{U'})$ lie in atlas of M and M' respectively

From now, we will only be dealing with subsets[4] of $\mathbb{R}^{p,q}$. A vector field, X on $U \subseteq \mathbb{R}^{p,q}$, is a map $X : U \to T(U)$, i.e. to each $p \in U$, it assigns $X_p \in T_p(U)$. If X is a smooth vector field[5], we have a local one-parameter group, $\phi : W \subset \mathbb{R} \text{ x } U \to U$, corresponding to it. which is such that

$$\frac{d}{dt}(\phi(t,a)) = X_{\phi(t,a)}$$

i.e. for any $f \in \mathcal{C}^\infty_{\phi(t,a)}(U)$,

$$X_{\phi(t,a)}f = \frac{d}{dz}\bigg|_{z=t} f(\phi(z,a)) \tag{3.3}$$

The local one parameter group also has property that $\phi(0,x) = x$ and for a fixed t, $\phi(t,x)$, also represented as ϕ_t, is a diffeomorphism between open subsets of U.[6] Note, ϕ_t is a map between two open subsets of manifold. For such functions, we have already defined the condition in (3.1.1), for them to be conformal.

Definition 3.1.2. *Conformal Killing Field*
Let X be a vector field on $U \subseteq \mathbb{R}^{p,q}$, and ϕ be the local one parameter group corresponding to it. If there exists a neighbourhood around 0, say $(-\epsilon_1, \epsilon_2)$, such that $\forall$ t in $(-\epsilon_1, \epsilon_2)$, ϕ_t is conformal, then X is called a Conformal Killing Field.

Here, observe that ϕ_0 is $id : U \to U$ and as ϕ is a one parameter group[7], ϕ_t evolves continuously from id as we move from 0 to t, i.e. for $t \to 0$, ϕ_t and id differ infinitesimally.

Theorem 1. Let $U \subseteq \mathbb{R}^{p,q}$ and X [8] be a Conformal Killing Field on it, then there is a smooth function $\kappa : U \to \mathbb{R}$ such that $\forall$ $p \in U$

$$X_{\mu,\nu}(p) + X_{\nu,\mu}(p) = g_{\mu\nu}\kappa(p) \tag{3.4}$$

Proof. As X is a conformal killing field, ϕ_t are conformal maps, for t in an open interval around 0. For each ϕ_t we have a map Ω_t, which is the scaling map that appears in (3.1.1). Then it can be checked that $\frac{d}{dt}{\Omega_t}^2|_{t=0}$ is the function κ which was needed (3.4). $\square$

[4]So we will not have to worry about different charts and transition maps between them. The metric, g is also constant now.

[5]$X_p = \sum_\mu X^\mu(p)\partial_\mu$, if X^μ are smooth functions then X is a smooth vector field

[6]A discussion on one-parameter group can be found in [11]

[7]for a fixed x, $\phi(t,x)$ is a smooth curve in U

[8]$\mathrm{X} = X^\mu\partial_\mu$ and $X_\mu = g_{\mu\nu}\mathrm{X}^\nu$ where $g_{\mu\nu}$ is the metric tensor on $\mathbb{R}^{p,q}$, $X_{\mu,\nu} = \partial_\nu X_\mu$

If for any $\kappa : U \to \mathbb{R}$, there is a vector field X, such that (3.4) holds true, then κ is called Conformal Killing Factor.
Now, we have shown that to each conformal killing field, X, we have a corresponding conformal killing factor. Next, we give a restriction on the form of κ, so that we can find all the conformal killing fields for a given space.

Theorem 2. *Let $\kappa : U \subseteq \mathbb{R}^{p,q} \to \mathbb{R}$ be a smooth function, where $p + q = n$. If κ is a conformal killing factor, then $\forall\ p \in U$*

$$(n-2)\kappa_{,\mu\nu}(p) + g_{\mu\nu}\Delta_g \kappa(p) = 0 \tag{3.5}$$

where $\Delta_g = g^{kl}\partial_k\partial_l$ and $\kappa_{,\mu\nu} = \partial_\mu\partial_\nu\kappa$

Proof. This can be proven by using the definition of Conformal Killing Factor and switching of order of partial derivatives acting on X_μ.[9] □

Now, this gives us a constraint on the form of conformal killing factor, and so also on conformal killing field. We will next examine the solutions of (3.5). But still we do not know whether a function which satisfies (3.5) is a conformal killing factor.

It can be shown, that for each κ that satisfies (3.5), we can actually find a vector field, X, such that (3.4) is satisfied. So, we effectively prove that any function which satisfies (3.5), is actually a conformal killing factor, i.e,

Theorem 3. *Let $\kappa : U \subseteq \mathbb{R}^{p,q} \to \mathbb{R}$ be a smooth function and $p + q = n$. If it satisfies, Equation (3.5), then it is a conformal killing factor.*

This theorem is basically the converse of the implication given in Theorem 2. We will show this for the case when $n = 2$(i.e. $p = 1$, $q = 1$ and $p = 2$, $q = 0$) in the Appendix.

[9]which can be done as X_μ's are $\mathcal{C}^\infty$ functions

3.2 Quantization of Symmetries

3.2.1 Preliminaries

Let us assume we have a Hilbert Space($\mathbb{H}$) over $\mathbb{C}$.

Definition 3.2.1. *Projective Space of* $\mathbb{H}$
The projective space corresponding to this Hilbert space is $\mathbb{P}(\mathbb{H})$*(or just* $\mathbb{P}$*), is the set*

$$(\mathbb{H} \setminus 0)/\sim, \textit{ where } x \sim y \textit{ iff } x = ay, \textit{ for some } a \in \mathbb{C} \setminus 0 \tag{3.6}$$

The projective space is the space of rays of Hilbert space, where quantum mechanics happens. Consider any two elements in $\mathbb{P}$, say [x] and [y]. To define a distance between these elements, take any $a \in [x]$ and $b \in [y]$, then

$$\delta([x]\ , [y]) = \langle \frac{a}{||a||}, \frac{b}{||b||} \rangle^2 \tag{3.7}$$

δ,[10] which gives the transition probability between two points, serves as the metric for $\mathbb{P}$, and hence defines a topology on it. An automorphism of $\mathbb{P}$ is a bijective map, say T, which conserves the transition probability, i.e.

$$\delta(T([x]), T([y])) = \delta([x], [y]);\ \forall\ [x], [y] \in \mathbb{P} \tag{3.8}$$

The set of all automorphism over $\mathbb{P}$ is written as Aut($\mathbb{P}$), forms a group where composition of maps is the group multiplication.

Definition 3.2.2. *Unitary and Anti-Unitary operator*
Unitary (or anti-Unitary) are $\mathbb{C}$*-Linear (or* $\mathbb{R}$*-Linear) and bijective operators on* $\mathbb{H}$*, such that for unitary operators* (U)[11]

$$\langle U(x), U(y) \rangle = \langle x, y \rangle \textit{ and } U(cx + y) = c\, U(x) + U(y) \tag{3.9a}$$

[10] Here $\langle\ ,\ \rangle$ is the inner product of $\mathbb{H}$
[11] $c \in \mathbb{C}$, $r \in \mathbb{R}$ and x, $y \in \mathbb{H}$

and for anti-Unitary operators (U')

$$\langle U'(x), U'(y)\rangle = \langle y, x\rangle,\ U(rx+y) = rU(x) + U(y),\ \textit{and}\ U'(ix) = -iU'(x) \tag{3.9b}$$

$U(\mathbb{H})$ is the group consisting of all unitary transformations over $\mathbb{H}$, with composition as group multiplication. A topological group is a group, on which we define a topology, such that the group multiplication and inverse functions are continuous maps.[12] $U(\mathbb{H})$ can be made into a topological group, as done in [12]. The set of Anti-Unitary maps does not form a group over composition as U'^2 is a unitary map, for U' Anti-Unitary.

There is a canonical map, $\hat{\gamma}$ from $U(\mathbb{H})$ to Aut($\mathbb{P}$). Consider $U \in$ U($\mathbb{H}$), then $\hat{\gamma}(U) \in$ Aut($\mathbb{P}$), where

$$\hat{\gamma}(U)([x]) = \gamma(U(x)), \tag{3.10}$$

where γ is the quotient map from $\mathbb{H} \setminus 0$ to $\mathbb{P}$. For this map to be well defined, it has to be checked that $\gamma(U(y)) = \gamma(U(x))$ is same for all $y \in [x]$. For $\hat{\gamma}(U)$ to belong to Aut($\mathbb{P}$), it also has to be checked that the map conserves transition probability. We define $U(\mathbb{P}) := \hat{\gamma}(U(\mathbb{H}))$.[13], with the following topology

$$W \subseteq U(\mathbb{P}) \text{ is open iff } (\hat{\gamma})^{-1}(W) \text{ is open in } U(\mathbb{H}) \tag{3.11}$$

Note, if U' is an anti-Unitary operator, even then $\hat{\gamma}(U')$ is defined(in the same way as in 3.10), and it also belongs to Aut($\mathbb{P}$). It can be shown that $U(\mathbb{P})$ is a topological groups with respect to the topology defined above.

Definition 3.2.3. *Unitary and Projective Representation*
A map T is a Unitary Representation (or Projective Representation), if T is a continuous homomorphism from G[14] $\rightarrow U(\mathbb{H})(or\ \ U(\mathbb{P}))$.

[12]If the topology on the group G, makes it a differentiable manifold, with group multiplication and inverse differentiable functions, then the topological group G is also a Lie Group
[13]$U(\mathbb{P})$ is a subset of Aut($\mathbb{P}$)
[14]where G is a topological group

3.2.2 Quantization

Suppose the classical phase space of a system is Y. We have a hamiltonian corresponding to this system H. Suppose we have a trajectory in the phase space, say $(\hat{q}(t), \hat{p}(t))$ which satisfies the equations of motion. A symmetry is then a transformation $T \in \mathrm{Aut}(Y)$, such that $(T_1(\hat{q}(t)), T_2(\hat{p}(t)))$ also satisfies the equations of motion. So we say after such a transformation the "physics" remains unchanged.
We can also have symmetries arising from groups. For example say we are in $\mathbb{R}^2$ and so the phase space is $\mathbb{R}^2$ x $\mathbb{R}^2$. Suppose the system has rotational symmetry, then corresponding to every angle, we can find an automorphism of $\mathbb{R}^2$ which is a symmetry. The map

$$(x, y) \to (g^{-1}x, g^{-1}y) \tag{3.12a}$$

where

$$g = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \tag{3.12b}$$

is a symmetry. The group generating these rotational symmetries for each angle is $SO(2)$. So symmetries such as rotational symmetries can be represented using the group homomorphism

$$\tau : G \to \mathrm{Aut}(Y) \quad ^{15} \tag{3.13}$$

Instead of studying each symmetry individually, it is more convenient to study the group G. In classical space we have observables which satisfy Poisson Bracket relations. The procedure of Quantization is basically finding a Hilbert Space, $\mathbb{H}$, where observables in classical space become operators, and the poisson bracket relations followed by classical observables translate to commutation relation followed by corresponding operators in Hilbert space.
It is expected on carrying out this procedure, the symmetries in the classical phase space, τ, correspond to now

$$T : G \to U(\mathbb{P}) \tag{3.14}$$

a projective representation of G. This is called quantization of symmetry. [16]. In [12], Schottenloher writes that the existence of (3.14), for a classical symmetry is an assumption regarding the procedure of quantization. It is needed, as while quantizing, we want classical symmetries to also manifest in the space of Quantum Mechanical system, $\mathbb{P}$.

[15] the inverse sign in (3.12a) is so that the mapping from SO(2) to Aut(Y) is a group homomorphism
[16] The existence of T corresponding to a given τ is an assumption while quantizing

However, we do not want to deal with projective representation. We are interested in "lifting" projective representation to unitary representation, as the theory of the latter is well developed[17].

Note, each $T \in U(\mathbb{P})$ has a pre-image, say U in $U(\mathbb{H})$, such that $\hat{\gamma}(U) = T$ [18]. U is called a lift of T here. Now, we get a bit audacious, and ask if we can lift the entire projective representation, i.e. $T : G \to U(\mathbb{P})$, to a unitary representation, i.e. $S : G \to U(\mathbb{H})$, such that $\hat{\gamma} \circ S = T$. Constructing such a lift is complicated, as $\hat{\gamma}$ is not invertible, so to each $T(g) \in U(\mathbb{P})$, there are multiple U such that $\hat{\gamma}(U) = T(g)$. Even if we could lift each $T(g)$ to a Unitary operator, say $S(g)$ we have to do it in such a way that S is a continuous homomorphism from $G \to U(\mathbb{H})$, i.e. S is a unitary representation. This is a hard task.
We need a few more mathematical structures before we attempt to address this question.

3.2.3 Group Extension

Definition 3.2.4. *Extension of a Group by Abelian Group*
Extension of a group G, by an ablelian group A, is given by an exact sequence

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1 \tag{3.15}$$

This sequence of groups is connected by group homomorphisms, and as the sequence is exact, the image of a map is the kernel of next map in the sequence. So Ker(i) $= id_A$, which implies $A \cong i(A)$. Also $\pi(E) = G$, as Kernel of the last map in the sequence is G. So π is a surjection.
This extension is central iff

$$\forall x \in i(A), \forall y \in E, x * y = y * x \tag{3.16}$$

i.e. $i(A) \subseteq$ center(E)

[17]Unitary representations can be broken down into a sum of irreducible representations
[18]As $U(\mathbb{P})$ was defined to be $\hat{\gamma}(U(\mathbb{H}))$

It can further be shown that the sequence

$$1 \longrightarrow U(1) \xrightarrow{i} U(\mathbb{H}) \xrightarrow{\hat{\gamma}} U(\mathbb{P}) \longrightarrow 1 \tag{3.17}$$

is in fact a central extension of $U(\mathbb{P})$ by $U(1)$, where $i(x) = x\ id_{\mathbb{H}}$ and $\hat{\gamma}$ is the map described in (3.10).

Looking back at eq(3.14), we now have the added structure of G.

$$\begin{tikzcd} & & & G \arrow[dl, "S??"'] \arrow[d, "T"] & \\ 1 \arrow[r] & U(1) \arrow[r] & U(\mathbb{H}) \arrow[r, "\hat{\gamma}"'] & U(\mathbb{P}) \arrow[r] & 1 \end{tikzcd} \tag{3.18}$$

The question is that whether there exists a map S, a unitary representation, called the lift of projective representation, T, such that the diagram commutes. Generating this lift is only possible for certain G, which will be discussed at length in this chapter.
However, it can be shown that we can construct a group E for G, such that we have a central extension of G from $U(1)$ and that the following diagram commutes.

$$\begin{tikzcd} 1 \arrow[r] & U(1) \arrow[r, "i"] \arrow[d, "id"] & E \arrow[r, "pr_2"] \arrow[d, "T' = pr_1"] & G \arrow[r] \arrow[d, "T"] & 1 \\ 1 \arrow[r] & U(1) \arrow[r] & U(\mathbb{H}) \arrow[r, "\hat{\gamma}"] & U(\mathbb{P}) \arrow[r] & 1 \end{tikzcd} \tag{3.19}$$

Here the group E is a subgroup of $U(\mathbb{H})$ x G such that

$$(U, g) \in \text{E iff } T(g) = \hat{\gamma}(U) \tag{3.20}$$

where $U \in$ U($\mathbb{H}$) and $g \in$ G and

$$i : U(1) \rightarrow E : i(x) = (xid_{\mathbb{H}}, id_G) \tag{3.21}$$

pr_1 and pr_2 are projection map from E to $U(\mathbb{H})$ and G respectively. It is easy to check that E is in fact a subgroup and that the diagram commutes. This particular construction will be again used in the proof of Bargmann's Theorem. The next few sections are regarding when we can construct a continuous homomorphism, σ: $G \rightarrow E$, such that $T' \circ \sigma$ is the map S, the lift of T, as discussed in (3.18).

3.2.4 Central Extension and Cohomology Group - Classification of Central Extensions

Definition 3.2.5. *Equivalence of Extensions*
Let there be two extension of G by the group A,

$$
\begin{gathered}
1 \longrightarrow A \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1 \\
\\
1 \longrightarrow A \xrightarrow{i'} E' \xrightarrow{\pi'} G \longrightarrow 1
\end{gathered}
\tag{3.22}
$$

They are equivalent if there exists a group isomporphism, $\phi : E \to E'$, such that the following diagram commutes.

$$
\begin{tikzcd}
1 \arrow[r] & A \arrow[r, "i"] \arrow[d, "id"] & E \arrow[r, "\pi"] \arrow[d, "\phi"] & G \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & A \arrow[r, "i'"] & E' \arrow[r, "\pi'"] & G \arrow[r] & 1
\end{tikzcd}
\tag{3.23}
$$

It can be shown that this equivalence is actually an Equivalence relation. [19]

Definition 3.2.6. *Splitting of Extensions*
An exact sequence

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$$

splits if there exists a homomorphism $\sigma : G \to E$, such that $\pi \circ \sigma = id_G$.

As π is surjective, due to the sequence being exact, it is always possible to find maps τ, such that $\pi \circ \tau = id_G$. However, they may not necessarily be homomorphisms nor unique(as

[19] Symmetric is using $\phi = id_E$, reflexive is by using ϕ^{-1} as the the isomorphism required, and for transitivity, the isomorphism is the composition of ϕ_1 and ϕ_2, where ϕ_1 and ϕ_2 are isomorphisms coming from equivalence of extensions

π may not be injective).

A Trivial Extension of G by the group A, is the sequence

$$1 \longrightarrow A \xrightarrow{i} A \text{ x } G \xrightarrow{pr_2} G \longrightarrow 1 \tag{3.24}$$

where $i(a) = (a, 1)$ and $pr_2(a, g) = g$

Theorem 4. A central extension splits if and only if it is equivalent to the Trivial Extension.

Proof. ($\Rightarrow$): If a central extension splits, then the map

$$\phi : A \text{ x } G \to E, \;\; \phi(a, g) = i(a)\ \sigma(g)$$

is the isomorphism, like in Defintion (3.2.5), needed to show the equivalence with Trivial Extension. Here, σ is the homomorphism coming from the splitting of the extension - (3.2.6),
($\Leftarrow$) : If a central extension is equivalent to the trivial extension then we have an isomorphism $\phi : A \text{ x } G \to E$, as in Defintion (3.2.5). Then the following map

$$\sigma(g) = \phi(1_A, g) \tag{3.25}$$

is the homomorphism needed for showing that the central extension splits. □

As stated just after Definition 3.2.6, for an exact sequence, we can always construct a map
$\tau : G \to E$, such that $\pi \circ \tau = id_G$, with the added condition that $\tau(1_G) = 1_E$. From now we will use τ_g to denote $\tau(g)$
Let the sequence

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1 \tag{3.26}$$

be a central extension. Then we can define a map $\omega : G \text{ x } G \rightarrow A$ by,

$$\omega(g_1, g_2) = \tau_{g_1}\tau_{g_2}\tau^{-1}_{g_1 g_2} \tag{3.27}$$

Here $\tau^{-1}_{g_1 g_2}$ is the inverse of $\tau_{g_1 g_2}$ in E. Note $\omega(g_1, g_2) \in E$, but as $\pi(\omega(g_1, g_2)) = 1_E$, it is an element of $i(A) \cong A$, both due to the sequence being exact. It can be shown that

$$\omega(1, 1) = 1 \tag{3.28a}$$

$$\omega(g_1, g_2)\ \omega(g_1 g_2, g_3) = \omega(g_1, g_2 g_3)\ \omega(g_2, g_3); \forall g_1, g_2, g_3 \in G \tag{3.28b}$$

The second property (3.28b) requires the extension to be central.

Definition 3.2.7. *A map $\omega : G \ X \ G \rightarrow A$, which satisfies the above mentioned properties (3.28) is called 2-cocycle on G with values in A.*

Now, given a cocycle on G with values in A, we can define a central extension of G by A. Consider $A \text{ x}_\omega G$, which is basically the set of cartesian product of A and G, which has multiplication defined by

$$(a_1, g_1) \cdot (a_2, g_2) = (\omega(g_1, g_2)a_1 a_2, g_1 g_2) \forall a_1, a_2 \in A \text{ and } g_1, g_2 \in G. \tag{3.29}$$

It can be shown this is a group. Then the following sequence is a central extension.

$$1 \longrightarrow A \xrightarrow{i} A \text{ x}_\omega G \xrightarrow{pr_2} G \longrightarrow 1 \tag{3.30a}$$

$$\text{where } i(a) = (a, 1) \text{ and } pr_2(a, g) = g \tag{3.30b}$$

Let there be a central extension of G by group A, say (3.26). Corresponding to π, we can construct τ, such that such that $\pi \circ \tau = id_G$ and $\tau(1_G) = 1_E$, where this τ is not unique, as π is not injective. Corresponding to a particular τ we can construct a 2-cocyle ω, using the scheme in Equation 3.27. Corresponding to this ω, we can construct a central extension, as shown above. In the next theorem, it will be shown that the newly constructed central extension from ω, (3.30), and the original central extension - (3.26) are equivalent.

Theorem 5. The central extension of G by group A, i.e. (3.26), is equivalent to the central extension constructed using ω, that is, there exists an isomorphism ϕ, such that the following diagram commutes.

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r, "i"] \arrow[d, "id"] & E \arrow[r, "\pi"] \arrow[d, "\phi"] & G \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & A \arrow[r, "i'"] & A \text{ x}_\omega G \arrow[r, "\pi'"] & G \arrow[r] & 1
\end{tikzcd} \tag{3.31}$$

Proof. The proof follows by construction of the map ϕ. It can be shown that the map

$$\phi : E \to A \text{ x}_\omega G; \phi(e) = (e * \tau^{-1}_{\pi(e)}, \pi(e)) \tag{3.32a}$$

$$\phi' : A \text{ x}_\omega G \to E; \phi'(a, g) = i(a)\tau_g \tag{3.32b}$$

ϕ and ϕ' are both homomorphisms and inverses of each other, and the diagram commutes. Thus, ϕ is the isomorphism which was needed. □

So, corresponding to a central extension, i.e. (3.15), we can construct several map τ, each of which corresponds to ω, a 2-cocylce, as constructed in (3.27). We can construct a central extension from this ω, as shown in (3.30). In the above theorems we have proven that all the extension constructed using ω are equivalent to the original extension and thus to each other too due to transitivity.

Definition 3.2.8. *Second Cohomology Group of G with coefficients in A*
This group - $H_2(G, A)$ is defined by

$$H_2(G, A) = \{\omega : G \ X \ G \to A \mid \omega \textit{ is a 2-cocyle}\}/\sim \tag{3.33}$$

where $\omega_1 \sim \omega_2$ iff there exists a map $\lambda : G \to A$, such that for all g_1, g_2 in G

$$\lambda(g_1 g_2) = \omega_1(g_1, g_2)\ \omega_2(g_1, g_2)^{-1}\lambda(g_1)\lambda(g_2) \tag{3.34}$$

Here, the group multiplication is defined by pointwise multiplication in A. This group is abelian as A is abelian. It is trivial to check that (3.33) is indeed an equivalence relation.

Theorem 6. There is a bijective map between the second cohomology group - $H_2(G, A)$ and the equivalence classes[20] of central extension of G by A.

$$\{\omega : G \text{ x } G \to A \mid \omega \text{ is a 2-cocyle}\}/\sim \longleftrightarrow \{\ 1 \longrightarrow A \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1\ \}/\sim' \tag{3.35}$$

Proof. We have a way of mapping 2-cocyles to central extension, as shown in (3.30). This map can be used to construct a map from the LHS to RHS of the equation above.
The map from LHS to the RHS is

$$f([\omega_1]) = [\ 1 \longrightarrow A \xrightarrow{i} A \text{ x}_\omega G \xrightarrow{pr_2} G \longrightarrow 1\ ]$$

It has to be checked that this map is well defined. That is, if $\omega_1 \sim \omega_2$, then the central extensions corresponding to them will be equivalent. So, the aim is to construct an isomorphism ϕ, such that the following diagram commutes.

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "i"] \arrow[d, "id"] & A \text{ x}_{\omega_1} G \arrow[r, "\pi"] \arrow[d, "\phi"] & G \arrow[r] \arrow[d, "id"] & 1 \\ 1 \arrow[r] & A \arrow[r, "i'"] & A \text{ x}_{\omega_2} G \arrow[r, "\pi'"] & G \arrow[r] & 1 \end{tikzcd} \tag{3.36}$$

Now, as $\omega_1 \sim \omega_2$, we have $\lambda : G \to A$, such that

$$\lambda(g_1 g_2) = \omega_1(g_1, g_2)\ \omega_2(g_1, g_2)^{-1}\lambda(g_1)\lambda(g_2)$$

Then, it can be checked that the map

$$\phi : A \text{ x}_{\omega_1} G \to A \text{ x}_{\omega_2} G;\ \phi(a, g) = (\lambda(g)a, g)$$

is the isomorphism needed. Hence, the map is well defined.
So, here we have shown a map from LHS to RHS. Now, we will show that this map is injective, i.e. if $f([\omega_1]) = f([\omega_2])$, then $[\omega_1] = [\omega_2]$, i.e. $\omega_1 \sim \omega_2$. We have to basically show that if there is an isomorphism ϕ, such that the diagram above (3.36) commutes for any given ω_1 and ω_2, then $\omega_1 \sim \omega_2$, that is there exists λ such that equation (3.34) is satisfied.
Now,

[20] Here in an equivalence class, the members are all the equivalent central extension

$$\phi : A \text{ x}_{\omega_1} G \to A \text{ x}_{\omega_2} G; \phi(a,g) = (\phi_1(a,g), \phi_2(a,g))$$

Here ϕ_1: $A \text{ x}_{\omega_1} G \to$ A. It can be shown that ϕ_1(a,g)a^{-1} is independent of A. So let us define

$$\lambda(g) = \phi_1(a,g)a^{-1} \tag{3.37}$$

It can be checked that this is the λ [21], for which Equation (3.34) is satisfied.

Surjectivity is guaranteed as corresponding to a central extension, we can generate a co-cycle, as shown in the scheme given in (3.27). □

3.2.5 Extension of Lie Algebras

A Finite-Dimensional Real or Complex Lie Algebra - $\mathfrak{g}$ is a vector space of finite dimensions over Real or Complex field, endowed with a map, called Lie Bracket,

$$[\ ,\ ] : \mathfrak{g} \text{ x } \mathfrak{g} \to \mathfrak{g}$$

which is bilinear, skew-symmetric and follows the Jacobi Identity. The Lie Algebra is abelian iff the Lie Bracket is identically 0.
A Linear Map - ψ between Lie Algebras $\mathfrak{g}$ and $\mathfrak{h}$ is a Lie Algebra homomorphism iff

$$[\psi(X), \psi(Y)] = \psi([X,Y])$$

where the Bracket in LHS is on $\mathfrak{h}$ and for RHS on $\mathfrak{g}$.

Definition 3.2.9. *Let $\mathfrak{g}$ be a Lie Algebra and $\mathfrak{a}$ be an abelian Lie Algerbra, both over field K. Extension of $\mathfrak{g}$ by $\mathfrak{a}$ is given by an exact sequence*

$$0 \longrightarrow \mathfrak{a} \xrightarrow{i} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0 \tag{3.38}$$

where i and π are Lie Algebra Homomorphisms such that the image of a map is the kernel of next map. This extension is central iff

$$[X,Y] = 0$$

[21] λ(g) $= \phi_1(a,g)a^{-1} = \phi_1(\omega_1(g,1), g)\ \omega_2^{-1}(g,1)$, which is independent of a

for any $X \in i(\mathfrak{a})$ and $Y \in \mathfrak{h}$.

We can note now itself that $\mathfrak{h} \cong \mathfrak{g} \oplus \mathfrak{a}$ at least as vector spaces(may not be true as Lie Algebras). This comes from the above sequence being exact and that a finite dimensional vector space can be written as the direct sum of kernel (i.e. $i(\mathfrak{a}) \cong \mathfrak{a}$) and image (i.e. $\mathfrak{g}$) of a map (i.e. π) originating from it. The map

$$\phi : \mathfrak{g} \oplus \mathfrak{a} \to \mathfrak{h};\ \phi(X, Y) = \beta(X) + Y \tag{3.39}$$

is the vector space isomorphism between $\mathfrak{g} \oplus \mathfrak{a}$ and $\mathfrak{h}$, with

$$\phi^{-1} : \mathfrak{h} \to \mathfrak{g} \oplus \mathfrak{a};\ \phi^{-1}(Z) = (\pi(Z), Z - \beta(\pi(Z)))$$

Now for an exact sequence of Lie Algebra homomorphisms, say (3.38), corresponding to π, we can always construct a linear map $\beta : \mathfrak{g} \to \mathfrak{h}$, such that $\pi \circ \beta = id_{\mathfrak{g}}$. However it may not be a Lie Algebra homomorphism.[22]

Definition 3.2.10. *Splitting of central extension*
A central extension, say (3.38), splits if β, which is the map as described above, is a Lie algebra homomorphism.

Now, we can transport the definition of equivalence from last subsection, i.e. Definition 3.2.5, to here. For Lie Algebras, the same definition is there for equivalence of extensions, just ϕ should be Lie Algebra Isomorphism. It can be shown using similar maps as in Theorem (4) that a central extension is equivalent to trivial extension iff the central extension splits. Here, the Trivial Extension is

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\ i\ } \mathfrak{g} \oplus \mathfrak{a} \xrightarrow{\ pr_1\ } \mathfrak{g} \longrightarrow 0 \tag{3.40}$$

where the Lie Bracket on $\mathfrak{g} \oplus \mathfrak{a}$ is [23]

$$[(X_1, Y_1), (X_2, Y_2)]_{\mathfrak{g} \oplus \mathfrak{a}} = [X_1, X_2]_{\mathfrak{g}} \tag{3.41}$$

[22] Note the similarities between β and τ from previous subsection.
[23] $X_1, X_2 \in \mathfrak{g}$ and $Y_1, Y_2 \in \mathfrak{a}$

If $\mathfrak{g}$ and $\mathfrak{a}$ are Lie Algebras, then on their direct sum

$$[(X_1, Y_1), (X_2, Y_2)]_{\mathfrak{g}\oplus\mathfrak{a}} = ([X_1, X_2]_{\mathfrak{g}}, [Y_1, Y_2]_{\mathfrak{a}}) \tag{3.42}$$

this is the canonical Lie Bracket. The above Lie Bracket reduces to Equation(3.41) as $\mathfrak{a}$ is an abelian Lie Algebra.

For every central extension, we have map β, such that $\pi \circ \beta = id_{\mathfrak{g}}$, if β is a Lie Algebra Homomorphism,then the sequence also splits. Corresponding to each β, we can define a map, called the 2-cocycle in $\mathfrak{g}$ with values in $\mathfrak{a}$, $\theta : \mathfrak{g}$ x $\mathfrak{g} \to \mathfrak{a}$

$$\theta(X_1, X_2) = [\beta(X_1), \beta(X_2)] - \beta([X_1, X_2]) \tag{3.43}$$

$\beta : \mathfrak{g} \to \mathfrak{h}$, so the RHS lies in $\mathfrak{h}$, but note that $\pi(\theta(X_1,X_2)) = 0$. Hence, $\theta(X_1,X_2)$ lies in $i(\mathfrak{a})$ which is identified to $\mathfrak{a}$, as they are isomorphic. So $\theta(X_1,X_2)$ lies in $\mathfrak{a}$.

It can be checked that θ is bilinear and alternating, and satisfies this property which is somewhat like the Jacobi identity

$$\theta(X_1, [X_2, X_3]) + \theta(X_2, [X_3, X_1]) + \theta(X_3, [X_1, X_2]) = 0 \tag{3.44}$$

Definition 3.2.11. *Any map $\theta : \mathfrak{g}\ X\ \mathfrak{g} \to \mathfrak{a}$ which is bilinear and alternating, and satisfies (3.44) is called a 2-cocycle of $\mathfrak{g}$ with values in $\mathfrak{a}$*

Now, corresponding to a cocycle, we can define the following Lie Bracket on $\mathfrak{g}\oplus\mathfrak{a}$

$$[(X_1, Y_1), (X_2, Y_2)]_{\theta} = [X_1, X_2]_{\mathfrak{g}} + \theta(X_1, X_2) \tag{3.45}$$

which is a Lie Algebra that we denote by $\mathfrak{g} \oplus_\theta \mathfrak{a}$. Now, $\mathfrak{g} \oplus_\theta \mathfrak{a}$ gives a central extension of $\mathfrak{g}$, which is of the same form as (3.40), but with the Lie Bracket on direct sum of the form given above instead of Equation (3.41).[24] It can be shown that this generated central extension

[24]Note for a splitting map $\theta = 0$, as β is a Lie Algebra Homomorphism. So (3.45) and (3.41) agree.

and the original central extension are equivalent, i.e.

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{a} \arrow[r, "i"] \arrow[d, "id"] & \mathfrak{g} \oplus_\theta \mathfrak{a} \arrow[r, "pr_1"] \arrow[d, "\phi"] & \mathfrak{g} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \mathfrak{a} \arrow[r, "i"] & \mathfrak{h} \arrow[r, "\pi"] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd} \tag{3.46}$$

Here $\phi(X, Y) = \beta(X) + Y$. This result is similar to Theorem 5, with even the isomorphism map having the same structure[25]. From the commutation of

Theorem 7. A central extension of the form

$$0 \longrightarrow \mathfrak{a} \xrightarrow{i} \mathfrak{g} \oplus_\theta \mathfrak{a} \xrightarrow{pr_1} \mathfrak{g} \longrightarrow 0 \tag{3.47}$$

splits iff $\exists$ a homomorphism $\mu : \mathfrak{g} \to \mathfrak{a}$, such that

$$\tilde{\mu}(X_1, X_2) = \theta(X_1, X_2) \tag{3.48}$$

where $\tilde{\mu}(X_1,X_2) = \mu([X_1,X_2])$.

Proof. ($\Rightarrow$) If the extension splits, then we have $\sigma : \mathfrak{g} \to \mathfrak{g} \oplus_\theta \mathfrak{a}$, of the form $\sigma(x) = (\lambda(x), \mu(x)))$. where λ and μ are homomorphisms from $\mathfrak{g}$ to $\mathfrak{g}$ and $\mathfrak{a}$ respectively[26] It can be checked that this μ satisfies the condition (3.48).

($\Leftarrow$) If we have a $\mu(x)$ satisfying (3.48) then the splitting map is $\sigma(x) = x + \mu(x)$

$\square$

[25] The multiplication in group becomes addition in Lie Algebra and τ and β are the inverses of π
[26] λ(x) = x, as $pr_1(\sigma(x)) = \lambda(x) = x$.

Definition 3.2.12. *Second cohomology group of* $\mathfrak{g}$ *with values in* $\mathfrak{a}$
Now, the space of alternating functions on $\mathfrak{g}$ X $\mathfrak{g}$ *forms a linear space, which is*

$$\Lambda^2(\mathfrak{g}, \mathfrak{a}) = \{\theta : \mathfrak{g} \ X \ \mathfrak{g} \to \mathfrak{a} \mid \theta \textit{ is bilinear and alternating}\}$$ [27]

A linear subspace of the above space is

$$C^2(\mathfrak{g}, \mathfrak{a}) = \{\theta : \mathfrak{g} \ X \ \mathfrak{g} \to \mathfrak{a} \ |\theta \in \Lambda^2(\mathfrak{g}, \mathfrak{a}) \textit{ and satisfies } (3.44)\}$$

Further

$$B^2(\mathfrak{g}, \mathfrak{a}) = \{\theta : \mathfrak{g} \ X \ \mathfrak{g} \to \mathfrak{a} \ |\theta \in \ C^2(\mathfrak{g}, \mathfrak{a}) \textit{ and } \exists \ \tilde{\mu} \textit{ such that } \theta = \tilde{\mu}\}$$

Then the second cohomology group of $\mathfrak{g}$ *with values in* $\mathfrak{a}$ *is given by*

$$H^2(\mathfrak{g}, \mathfrak{a}) = C^2(\mathfrak{g}, \mathfrak{a})/B^2(\mathfrak{g}, \mathfrak{a})$$

Theorem 8. The set of central extension of $\mathfrak{g}$ by $\mathfrak{a}$ upto equivalence is in bijection with $H^2(\mathfrak{g}, \mathfrak{a})$

The above mentioned theorem is similar to Theorem 6. The proof is also similar, it just involves converting the homomorphisms used in the earlier proof in the form of Lie Algebra homomorphisms.

[27] This resembles alternating tensors particularly when $\mathfrak{g}$ is $\mathbb{R}$

3.2.6 Bargmann's Theorem

Bargmann's theorem is an attempt to find the conditions when we can lift projective representation to a unitary representation for the symmetry group G, that is finding the map S in (3.18).

Theorem 9. Bargmann's Theorem
Let G be a finite dimensional connected and simply connected Lie Group such that $H^2(\mathrm{Lie}(G), \mathbb{R}) = 0$, then it can be shown that corresponding to any continuous homomorphism, $T : G \to \mathrm{U}(\mathbb{P})$, there is a continuous homomorphism, $S : G \to U(\mathbb{H})$, such that $\hat{\gamma} \circ S = T$. That is, any continuous projective representation from G can be lifted to a continuous unitary representation from G.

The condition $H^2(Lie(G), \mathbb{R}) = 0$ is satisfied by many Lie Algebras which are of interest in Physics. Any finite dimensional semi-simple Lie Algebra over any field satisfies this condition. Before we begin the sketch of the proof, we need few more results.

Let G and H be Finite Dimensional Lie groups and $\phi : G \to H$ is a Lie Group Homomorphism. Then there exist $Lie(G)$ and $Lie(H)$, which are Lie Algebras corresponding to G and H[28], and a unique Lie Algebra Homomorphism, $d\phi : Lie(G) \to Lie(H)$.[29]
So further for a central extension

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1 \tag{3.49}$$

we have a sequence

$$0 \longrightarrow Lie(A) \xrightarrow{di} Lie(E) \xrightarrow{d\pi} Lie(G) \longrightarrow 0 \tag{3.50}$$

which can be shown to be a Lie Algebra Central Extension of Lie(G) by Lie(A). [30].
Further, if G is a connected and simply connected Lie Groups, then for a Lie Algebra Homormorphism $\phi : Lie(G) \to Lie(H)$, there exists a unique Lie Group Homomorphism $int(\phi) : G \to H$ such that $d(int(\phi)) = \phi$, as shown in Theorem 3.27 of [15]. So if $d\pi$ in

[28] Tangent space at identity
[29] as shown in Corollary 7.10 of [11].
[30] to show sequence is exact, use $Ker(Lie(H)) = Lie(Ker(H))$

(3.50) splits, we have $\sigma : Lie(G) \to Lie(E)$. If G is connected and simply connected, the map $int(\sigma) : G \to E$, is the splitting map for π in (3.49). So if (3.50) splits, then (3.49) also splits, when G is connected and simply connected.

Proof. [31] A sketch of the proof will be presented in this thesis. This sketch is based on the proof in [12], but we have not discussed continuity here. So, consider the central extension which we discussed in (3.19)

$$\begin{tikzcd} 1 \arrow[r] & U(1) \arrow[r, "i"] \arrow[d, "id"] & E \arrow[r, "pr_2"] \arrow[d, "pr_1"] & G \arrow[r] \arrow[d, "T"] & 1 \\ 1 \arrow[r] & U(1) \arrow[r] & U(\mathbb{H}) \arrow[r, "\hat{\gamma}"] & U(\mathbb{P}) \arrow[r] & 1 \end{tikzcd} \tag{3.51}$$

It is showed in [12] that the group E, which has the definition as given in (3.20), is in fact a Lie Group. Now, the top row in the chain of Equation (3.51) is

$$1 \longrightarrow U(1) \xrightarrow{i} E \xrightarrow{pr_2} G \longrightarrow 1 \tag{3.52}$$

is a central extension of Lie Groups. The sequence formed by the Lie Algebra of these Lie Groups is [32]

$$0 \longrightarrow \mathbb{R} \xrightarrow{di} Lie(E) \xrightarrow{d(pr_2)} Lie(G) \longrightarrow 0 \tag{3.53}$$

which is a central extension of Lie Algebras. Now, as $H^2(Lie(G), \mathbb{R}) = 0$, it implies that $d(pr_2) : Lie(E) \to Lie(G)$ given above splits, and hence so does the original map $pr_2 : E \to G$ in (3.52). Let the splitting map be $\sigma : G \to E$, i.e. $pr_2 \circ \sigma = id_G$.

$$\begin{tikzcd} & & E \arrow[d, "pr_1"] & G \arrow[l, "\sigma"'] \arrow[d, "T"] \arrow[dl, "S"] & \\ 1 \arrow[r] & U(1) \arrow[r] & U(\mathbb{H}) \arrow[r, "\hat{\gamma}"'] & U(\mathbb{P}) \arrow[r] & 1 \end{tikzcd} \tag{3.54}$$

So, now we have a map $S : G \to U(\mathbb{H}) = pr_1 \circ \sigma$. Now from the commutation of the chain in Equation (3.51), we have $T \circ pr_2 = \hat{\gamma} \circ pr_1$. So, $\hat{\gamma} \circ S = \hat{\gamma} \circ pr_1 \circ \sigma = T \circ pr_2 \circ \sigma = T$. So, we have managed to construct a unitary representation, $S : G \to U(\mathbb{H})$, corresponding to the given projective representation, $T : G \to U(\mathbb{P})$. $\square$

[31] Another proof can be found in [14]
[32] $Lie(U(1)) = \mathbb{R}$

3.3 Central Extension of Witt Algebra - Virasoro Algebra

Holomorphic maps with non-vanishing derivatives are conformal transformation in $\mathbb{R}^{2,0}$ as discussed in Appendix B.2. We can write infinitesimal conformal transformation as [33]

$$f(z) = z + \sum_n a_n z^n \tag{3.55}$$

such a transformation is generated by the vector field

$$X(z) = \sum_n a_n z^{n+1} \frac{d}{dz} \tag{3.56}$$

Now consider elements of the form

$$W = \{l_n = -z^{n+1} \frac{d}{dz} \mid n \in \mathbb{Z}\} \tag{3.57}$$

These are the generators of the transformations in (3.55).

Definition 3.3.1. *Witt Algebra*
The linear span of elements of W over $\mathbb{C}$ is called Witt Algebra, $\mathfrak{W}$, which is a Lie Algebra with Lie Bracket

$$[l_n, l_m]_{\mathfrak{W}} = (n-m) l_{n+m} \tag{3.58}$$

Now we will examine a central extension of $\mathfrak{W}$ by $\mathbb{C}$.
Consider a bilinear map $\omega : \mathfrak{W} \text{ x } \mathfrak{W} \to \mathbb{C}$ defined by

$$\omega(l_n, l_m) = \delta_{n+m} \frac{n}{12}(n^2 - 1) \tag{3.59}$$

It can be shown that ω is in fact a 2-cocyle, hence it gives rise to a central extension of $\mathfrak{W}$ by $\mathbb{C}$.[34]
It can also be shown that the extension is not trivial as ω does not split, as there does not exist a homomorphism $\mu : \mathfrak{W} \to \mathbb{C}$ such that $\omega(X_1, X_2) = \mu([X_1, X_2]_{\mathfrak{W}})$ (refer to Theorem

[33] here we are talking about local conformal transformation
[34] For calculations check Theorem 5.1 of [12]

([7]). To show this let us assume there exists such a μ, then

$$\omega(l_n, l_{-n}) = \mu([l_n, l_{-n}]_{\mathfrak{W}}) \tag{3.60}$$

Then from (3.59)

$$\omega(l_n, l_{-n}) = \frac{n}{12}(n^2 - 1) = \mu([l_n, l_{-n}]_{\mathfrak{W}}) \tag{3.61}$$

And using the Lie Bracket relation (3.58)

$$\mu([l_n, l_{-n}]_{\mathfrak{W}}) = 2n\mu(l_0) \tag{3.62}$$

which implies

$$\mu(l_0) = \frac{1}{24}(n^2 - 1) \tag{3.63}$$

In Equation 3.63, RHS depends on n, but not LHS, hence the contradiction. Hence, the assumption, (3.60), is wrong, and so we have a non-trivial Central Extension of Lie Algebras

$$0 \longrightarrow \mathbb{C} \xrightarrow{\bar{i}} \mathfrak{W} \oplus_\omega \mathbb{C} \xrightarrow{pr_1} \mathcal{W} \longrightarrow 0 \tag{3.64}$$

Definition 3.3.2. *Virasoro Algeebra([18])*
The space $\mathfrak{W} \oplus_\omega \mathbb{C}Z$ [35] is called the Virasoro Algebra, $\mathfrak{V}$. It is a Lie Algebra over $\mathbb{C}$ and it can be shown that the Lie Bracket is given by

$$[l_n, l_m]_{\mathfrak{V}} = (n - m)l_{n+m} + \delta_{n+m}\frac{n}{12}(n^2 - 1)Z \tag{3.65a}$$

$$[l_n, Z]_{\mathfrak{V}} = 0 \tag{3.65b}$$

[36]The above relations can be verified by comparing to (3.45), here we have ω instead of θ

$$[(X_1, Y_1), (X_2, Y_2)]_{\mathfrak{V}} = [X_1, X_2]_{\mathfrak{W}} + \omega(X_1, X_2)Z \tag{3.66}$$

For $X_1 = l_n$, $X_2 = l_m$ and $Y_1 = Y_2 = 0$, Equation (3.66) reduces to (3.65a) by using Equation (3.59). In Equation (3.65b), the second term in commutator has only complex part, so $X_2 = 0$, for which the RHS of Equation (3.66) is 0, verifying (3.65b).

[35] Z is just an index attached to complex numbers here
[36] Here we use the notation l_n for $(l_n, 0)$. Similarly $(X, 0)$ or $(0, Y) \in \mathfrak{W} \oplus_\omega \mathbb{C}Z$, are written as X and Y respectively.

We are interested in Witt Algebra, as it is the Lie Algebra corresponding to holomorphic transformation(which are conformal transformations). So, if we have a classical system which is invariant under such transformations, then upon quantizing we have a projective transformation $T : \mathfrak{W} \to U(\mathbb{P})$, as discussed in Section 3.2.2. Now, we are interested in lifting this projective transformation, to a unitary transformation, due to reasons again mentioned in Section 3.2.2. However, this lift cannot be generated, as $H^2(\mathfrak{W}, \mathbb{C}) \neq 0$. But, what can be done is to get a unitary representation $\mathrm{S} : \mathfrak{V} \to U(\mathbb{H})$ corresponding to T. Hence, Virasoro Algebra occupies a central role when we quantize conformal symmetry in two dimensions.[37]

[37] A discussion on this can be found in [17]

Appendices

Appendix A

Calculating Q

$Q_1(u, w)$ is the spectral polynomial corresponding to Λ_1. $Q_1(u, w) = 0$ if and only if we can find a vacuous vector corresponding to Λ_1 However, as we showed earlier in (2.47b and 2.49b), we can always generate vacuous vectors if

$$P(u, v) = 0 \text{ and } P_1(v, w) = 0 \tag{A.1}$$

So, we want to determine a condition on u and w, such that if u and w satisfy the given condition, then we can always find a v, such that (A.1) is satisfied. This condition is the polynomial $Q_1(u, w)$. There are two ways which will be discussed here, they are by finding roots and by finding resultant.

A.1 Finding roots

We can solve $P(u, v) = 0$, for an arbitrary u, that is write P in the form

$$P(u, v) = (a_{02} + a_{12}u + a_{22}u^2)v^2 + (a_{01} + a_{11}u + a_{21}u^2)v + (a_{00} + a_{10}u + a_{20}u^2)$$

and then solve for the roots of the quadratic equation. We will get two roots in general, which are in general function of u, say $v_1(u)$ and $v_2(u)$ [1].
Then we take the other polynomial $P_1(v, w)$ and now solve for $P_1(v_1(u), w) = 0$, to find two roots $w_1(u)$ and $w_2(u)$. We repeat the procedure for the other root of P, $v_2(u)$, to again get two roots, $\mathrm{w}_3(v)$ and $\mathrm{w}_4(v)$.

Now note, if both the roots for P and P_1 are dependent on the other variable, then this procedure tells us that there will for a given u, be only at the maximum 4 values of w, such that the condition A.1 can be satisfied, as for any other w, neither ($v_1(u)$,w) nor ($v_2(u)$,w) satisfies $P_1 = 0$, and only $v_1(u)$ and $v_2(u)$ satisfy P(u,v) = 0.
So the function

$$Q_1(u, w) = (w - w_1(u))(w - w_2(u))(w - w_3(u))(w - w_4(u)) = 0$$

gives the condition, on u and w, that we can find v, such that the triplet $\{u, v, w\}$ satisfy (A.1)
However, this condition works only when the roots of both polynomial depend on the other variable, that is v_1 and v_2 are dependent on u, and similarly for the other polynomial.
For example, consider the case when the polynomial are

$$P(u, v) = uv(a_{11} + a_{21}(u + v) + a_{22}u^2v^2) \tag{A.2a}$$

$$P_1(u, v) = uv(b_{11} + b_{21}(u + v) + b_{22}u^2v^2) \tag{A.2b}$$

Then $(u, 0)$ and $(0, w)$ are roots for both. So, for any pair of u and w, we can always use 0, to form the triplet $\{u, 0, w\}$ which satisfies condition (A.1). Hence, there is no condition on u and w, that is $Q_1(u, w) = 0$. However, upon following the prescription given in this section, we get non-zero Q_1.

[1] Not always true. There can be a case when there is always a trivial root, say 0. This will be discussed in detail

A.2 Resultant

Consider two univariate polynomials, where all the variables and constants are in $\mathbb{C}$

$$R_1(x) = a_n x^n + ...a_1 x + a_0$$

$$R_2(x) = b_m x^m + ...b_1 x + b_0$$

Then the resultant of the polynomials is

$$Resultant[R_1, R_2, x] = c\ \Pi(\alpha_i - \beta_j) \tag{A.4}$$

It is given by where the product is of the pairwise difference of the roots of R_1 (α_i) and R_2 (β_j), where the roots are repeated according to multiplicity.
Resultant being zero implies that the two polynomials share a root. Hence, setting the resultant to 0, gives condition on the coefficients such that the two equation share a root.
So the following function

$$Q_1(u, w) = Resultant[P(u, v), P_1(v, w), v] = 0 \tag{A.5}$$

Gives us the condition on u and w such that we can find a v, so that condition (A.1) is satisfied.
The discrepancy which arose while calculating Q_1, which arose for polynomials in (A.2) does not come here. As 0 is a root for both P and P_1, we will get a 0 term in the product in (A.4). Hence Q_1 is 0, as we expected.
It can be proven that resultant is equal to determinant of Sylvester Matrix, which is formed from the coefficient of the polynomial. For example for the equations

$$R_1(x) = a_2 x^2 + a_1 x + a_0$$

$$R_2(x) = b_2 x^2 + b_1 x + b_0$$

The Sylvester Matrix is

$$\begin{bmatrix} a_0 & 0 & 0 & b_0 & 0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & 0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 \\ 0 & a_2 & a_1 & 0 & b_2 & b_1 \\ 0 & 0 & a_2 & 0 & 0 & b_2 \end{bmatrix} \tag{A.7}$$

As it is a determinant of a matrix, Q_1 by this method is much faster to calculate as compared to calculating Q_1 by the prescription given in previous section. Even the form of Q_1 is much simpler when found using resultant.

Appendix B

Conformal Transformations for n = 2

B.1 Case 1 : n = 2 (p = 1, q = 1)

$\phi = (\phi_1,\, \phi_2) : U \subseteq \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$ is conformal iff

$$\partial_1 \phi_1^2 > \partial_1 \phi_2^2 \tag{B.1a}$$

$$\partial_1 \phi_1 = \partial_2 \phi_2, \partial_2 \phi_1 = \partial_1 \phi_2 \ (1) \text{ or } \partial_1 \phi_1 = -\partial_2 \phi_2, \partial_2 \phi_1 = -\partial_1 \phi_2 \ (2) \tag{B.1b}$$

These conditions can be derived by plugging $\phi = (\phi_1, \phi_2)$ into (3.2). These are general conformal transformations, which may not be connected with any conformal killing field at all. In this case (3.5) just reduces to $\Delta_g\ \kappa = 0$[1]. All solutions of this equation are of the form

$$\kappa(x, y) = f(x + y) + g(x - y)$$

where f and g are smooth real maps. So such a κ can be shown to be the conformal killing factor corresponding to the conformal killing field, $X = (X^1, X^2)$ of the form

$$(X^1(x, y), X^2(x, y)) = (F(x + y) + G(x - y), F(x + y) - G(x - y))$$

where F and G are the integrals of $\frac{1}{2}f$ and $\frac{1}{2}g$. Hence, this proves Theorem 3 for $n = 2(p = 1, q = 1)$ case.

[1]by expanding this equation observe that it has the same form as wave equation with c = 1

The one parameter group corresponding to the Killing field of above type, which are infinitesimal conformal transformations, satisfy the (1) condition in (B.1b). Hence, ϕ which satisfies (B.1a) and (2) in (B.1b) are conformal transformation, but not infinitesimal ones.

B.2 Case 2: n = 2 (p = 2, q = 0)

$\phi = (\phi_1,\, \phi_2) : \mathrm{U} \subseteq \mathbb{R}^{2,0} \to \mathbb{R}^{2,0}$ is conformal iff

$$\partial_1 \phi_1^2 + \partial_1 \phi_2^2 = \partial_2 \phi_1^2 + \partial_2 \phi_2^2 = \Omega^2 \tag{B.2a}$$

$$\partial_1 \phi_1 \partial_2 \phi_1 + \partial_1 \phi_2 \partial_1 \phi_2 = 0 \tag{B.2b}$$

again derived by plugging $\phi = (\phi_1, \phi_2)$ into (3.2). If we map $\mathbb{R}^{2,0}$ to $\mathbb{C}$, then $\phi = \phi_1 + i\phi_2$ which satisfy such conditions in (B.2), are holomorphic and antiholomorphic functions only, with non-zero derivative. Again, these are all conformal transformations, which may not be arising out of conformal killing fields.

As in Case 1, again condition (3.5) reduces to $\Delta_g\ \kappa = 0$ as $n = 2$. So consider the vector field $X = (X^1, X^2)$ where

$$(X^1(x,y), X^2(x,y)) = (\int_0^x \frac{\kappa(z,y)}{2} dz, \int_0^y \frac{\kappa(x,z)}{2} dz) \tag{B.3}$$

Then for any κ which satisfies $\Delta_g\ \kappa = 0$, in Euclidean 2 dimensional plane, $X = (X^1, X^2)$ of the the form in (B.3) satisfies (3.4). So every κ satisfying $\Delta_g\ \kappa = 0$, in Euclidean 2 dimensional plane is a conformal killing factor. Hence, we have finally concluded the proof of the Theorem 3 for $n = 2$. The one parameter group corresponding to X of the form (B.3), which are the infinitesimal conformal transformation, are exactly holomorphic functions with non-zero derivative.

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