Modular Invariance in Two-Dimensional Conformal Field Theories

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by

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Certificate

This is to certify that this dissertation entitled Modular Invariance in Two-Dimensional Conformal Field Theories towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents work carried out by Palash Singh at Indian Institute of Science Education and Research Pune under the supervision of Prof. Sunil Mukhi, Professor, Department of Physics during the academic year 2015-2020.

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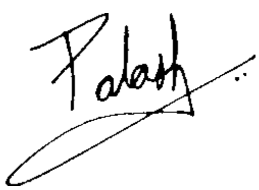
Declaration

I hereby declare that the matter embodied in the report entitled Modular Invariance in Two-Dimensional Conformal Field Theories are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research Pune, under the supervision of Prof. Sunil Mukhi and the same has not been submitted elsewhere for any other degree.



Palash Singh

Prof. Sunil Mukhi

Permissions

This thesis is based on the following work done by the author along with his supervisor and another collaborator:

- *Contour Integrals and the Modular $\mathcal{S}$ Matrix*, S. Mukhi, R. Poddar and **P. Singh**, JHEP, *to appear*, [1912.04298]
- *Rational CFT With Three Characters: The Quasi-Character Approach*, S. Mukhi, R. Poddar and **P. Singh**, JHEP 05 (2020) 003 [2002.01949]

This thesis is dedicated to my brother, Saahil and my parents, Surabhi and Jitendra.

“All that is gold does not glitter,

Not all those who wander are lost;

The old that is strong does not wither,

Deep roots are not reached by the frost."

- J.R.R. Tolkien

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Abstract

Modular invariance is the symmetry under the action of the modular group which two-dimensional conformal field theories enjoy. In this work, we will explore two disparate applications of the constraints implied by this symmetry by utilising the methods of modular bootstrap. We will first investigate the modular linear differential equation approach to the classification of rational conformal field theories. After a brief discussion of the known results, we will quickly review the original work in [1] which uses a contour-integral representation of RCFT characters to develop an algorithm to compute the modular $\mathcal{S}$ matrix. We will then present a detailed review of the original work in [2] on the classification of three-character RCFT. In this work, we conjectured several infinite families of quasi-characters in order 3 with Wronskian index, $\ell = 0$, studied their modular properties and used them to explicitly construct physical three-character rational conformal field theories with higher ℓ, in some sense mirroring the progress in the two-character case. We will then move on to the other application of modular bootstrap motivated from the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence and the search for pure gravity. After a brief review of the well-known Cardy formula, we will use a modular bootstrap equation implied by a generalisation of the modular $\mathcal{S}$ transformation, to derive a universal density of large spin Virasoro primaries, in the lightcone limit. On the way, we will also derive an upper bound on the twist gap for a generic CFT, as well as an upper bound on the mass of the lowest massive excitation of a theory of AdS_3 quantum gravity.

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Chapter 1

Introduction and Preliminaries

Conformal field theories (CFT) are distinguished quantum field theories which, apart from being Poincaré invariant, also enjoy conformal invariance. Such quantum field theories have found use in various areas of physics and mathematics. In string theory, the worldsheet description of a string is given by a CFT [3]. In statistical physics, the description of second order phase transition is also in terms of a CFT [4]. In addition to these, conformal field theories have also found use in condensed matter systems and topological quantum computing [5], as well as pure mathematics, in the guise of vertex operator algebras. In this chapter, we will give a quick review of two-dimensional conformal field theories, focusing on concepts and definitions which constitute the preliminary understanding for this work. A few good articles that review various aspects of two-dimensional conformal theories are [6, 7, 8, 9, 10].

Consider a d-dimensional metric $g_{\mu\nu}$, a *conformal transformation* is an invertible mapping of the coordinates ($x \to x'$) that leaves the components of the metric tensor invariant up to a scalar:

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \tag{1.1}$$

These transformations manifestly form a group and the subgroup which leaves the metric invariant is the Poincaré group. It is easy to see that these transformations preserve the angle between any two vectors.

This transformation has a different interpretation depending on whether the metric is dynamical or fixed in the background. For a dynamical metric, the transformation is just a diffeomorphism and is thus a gauge symmetry. Such diffeomorphisms can be undone by a *Weyl transformation*:

$$g'_{\mu\nu}(x) = \Omega(x) g_{\mu\nu}(x) \tag{1.2}$$

On the other hand, for fixed background metric, this transformation should be thought of as a physical symmetry which takes the spacetime point x to x'. Any theory which enjoys both diffeomorphism as well as Weyl invariance reduces to a conformally invariant theory, as soon as the background metric is fixed.

In $d \geq 3$-dimensions, the conformal group is finite dimensional and contains d translations, $d(d-1)/2$ rotations, 1 dilation (or scaling) and d special conformal transformations, which add up to a total of $(d+2)(d+1)/2$ transformations. An explicit computation of the algebra reveals that the generators of these transformations satisfy the $SO(d,2)$ algebra if $g_{\mu\nu}$ is Lorentzian, and $SO(d+1,1)$ algebra if it is Euclidean. It is important to note that this is the global conformal group and contains all transformations that are globally well-defined and connected to identity.

1.1 Two-Dimensional Conformal Field Theories

Conformal transformation and conformal group

We will now move on to the case of $d=2$ as it will be the primary focus of this work. In two dimensions, it is convenient to switch to complex coordinates $z = x^0 + ix^1, \bar{z} = x^0 - ix^1$. It turns out that the infinitesimal conformal transformations in two dimensions are mapped to the full set of analytic and anti-analytic functions [6]. This implies that the local conformal group in two dimensions is infinite dimensional and the global conformal group ($SO(3,1) \simeq SL(2,\mathbb{C})$) is a subgroup. One can obtain the generators of this local algebra by Laurent expanding the infinitesimal conformal transformation. On doing so, we obtain two copies of the Witt algebra, defined to be:

$$[l_n, l_m] = (n-m)l_{m+n} \quad \& \quad [\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{m+n} \quad \& \quad [l_n, \bar{l}_m] = 0 \tag{1.3}$$

Out of these infinite generators, $l_n, \bar{l}_n$ for $n \in \mathbb{Z}$, the ones corresponding to $n = 0, \pm 1$ are the global conformal group generators, since they are the only ones that are globally well-defined.

The fields $\varphi(z,\bar{z})$ having the simplest transformation under a conformal transformation ($z \to w(z)$) are called as *primary fields* and they transform as:

$$\varphi(z,\bar{z}) \to \varphi'(w,\bar{w}) = \left(\frac{\partial w}{\partial z}\right)^{-h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}} \varphi(z,\bar{z}) \tag{1.4}$$

where the *conformal dimensions*, h,$\bar{h}$ are defined to be the eigenvalues of l_0,$\bar{l}_0$.

Operator formalism

In the operator formalism of a QFT, we first distinguish a time direction and then evolve the states lying on the spatial slices by the Hamiltonian of the QFT. We start by defining our theory on an infinite spacetime cylinder endowed with the Minkowski metric. The time t on this cylinder goes from $-\infty$ to ∞ along the axis and the space is compactified with a coordinate x ranging from 0 to L, with the ends identified. We can now continue this to Euclidean space (τ, σ) and describe it by complex coordinates $(\omega, \bar{\omega})$, where $\omega = \tau + i\sigma$. We finally map the cylinder onto the Riemann sphere, $\mathbb{C} \cup \{\infty\}$ using the conformal map $z = e^{2\pi\omega/L}, \bar{z} = e^{2\pi\bar{\omega}/L}$.

This maps the constant τ slices on the cylinder to circles of constant radius on the plane. As $\tau \to -\infty, \infty$, we see that $z \to 0, \infty$, respectively. Thus the "infinite past" and the "infinite future" are situated at the origin of the plane and the point at infinity on the Riemann sphere, respectively. It is easy to show that the Hamiltonian on the plane is just the dilation operator and this approach to quantising a QFT is called as *radial quantisation*.

A field $\varphi(z, \bar{z})$ of dimensions $(h, \bar{h})$ can be mode expanded as:

$$\varphi(z, \bar{z}) = \sum_{m,n \in \mathbb{Z}} z^{-m-h} \bar{z}^{-n-\bar{h}} \varphi_{m,n} \ , \text{ where } \ \varphi_{m,n} = \oint \frac{dz}{2\pi i} z^{m+h-1} \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{n+\bar{h}-1} \varphi(z, \bar{z}) \tag{1.5}$$

In this formalism, time ordering in QFT manifests as radial ordering and can be expressed as:

$$\mathcal{R}\{\varphi_1(z)\varphi_2(w)\} = \begin{cases} \varphi_1(z)\varphi_2(w) & \text{if} \quad |z| > |w| \\ \varphi_2(w)\varphi_1(z) & \text{if} \quad |w| > |z| \end{cases} \tag{1.6}$$

In words, the operators at a larger distance from the origin are placed on the left.

A crucial feature of conformal field theories is that the classical energy momentum tensor is not just conserved, but is also traceless, i.e. $T^{\mu}_{\mu} = 0$. Tracelessness implies that $T_{z\bar{z}} = 0$ whereas conservation implies that T_{zz} is a completely holomorphic field, while $T_{\bar{z}\bar{z}}$ is completely anti-holomorphic. One can now study the implications of an infinitesimal conformal transformation $(z \to z + \epsilon(z))$ on a general field $\phi(z, \bar{z})$. The Ward identities of the CFT imply that:

$$\begin{aligned} \delta_\epsilon \phi(w, \bar{w}) &= -\frac{1}{2\pi i} \oint_w dz \epsilon(z) T(z) \phi(w, \bar{w}) + \frac{1}{2\pi i} \oint_{\bar{w}} d\bar{z} \bar{\epsilon}(\bar{z}) \bar{T}(\bar{z}) \phi(w, \bar{w}) \\ &= -\left[Q_\epsilon, \phi(w, \bar{w})\right] \end{aligned} \tag{1.7}$$

where $\oint_w$ denotes integration along a contour that goes around the point w (which is taken to be the origin by default). Thus we see that the conserved charge corresponding to conformal transformations can be written as:

$$Q_\epsilon = \frac{1}{2\pi i} \oint \left(dz\ \epsilon(z) T(z) + d\bar{z}\ \bar{\epsilon}(\bar{z}) \bar{T}(\bar{z}) \right) \tag{1.8}$$

Comparing the infinitesimal transformation of $\varphi(z, \bar{z})$ implied by the charge Q_ϵ and that from the definition of a primary field, one can derive the operator product expansion (OPE) between $T(z)$ and a primary field $\varphi(w, \bar{w})$ of weight $(h, \bar{h})$ to be:

$$\begin{aligned} T(z)\varphi(w,\bar{w}) &= \frac{h}{(z-w)^2}\varphi(w,\bar{w}) + \frac{1}{z-w}\partial_w \varphi(w,\bar{w}) + \cdots \\ \overline{T}(\bar{z})\varphi(w,\bar{w}) &= \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\varphi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}} \varphi(w,\bar{w}) + \cdots \end{aligned} \tag{1.9}$$

Here it is implicitly understood that $|z| > |w|$, and only the non-singular terms are written.

Let us now consider the OPE of the energy-momentum tensor with itself. Since $T(z)$ is a field with $(h, \bar{h}) = (2, 0)$, the scaling dimension on the LHS of the OPE will be 4 and therefore one would naively expect poles of order $1, 2, 3$, and 4 on the RHS. As it turns out, the requirement of unitarity as well as radial ordering eliminates the order 3 pole and we are left with:

$$\begin{aligned} T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \cdots \\ \overline{T}(\bar{z})\overline{T}(\bar{w}) &= \frac{\bar{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2\overline{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\partial_{\bar{w}} \bar{T}(\bar{w})}{\bar{z}-\bar{w}} + \cdots \end{aligned} \tag{1.10}$$

where $c, \bar{c}$ are called as the right- and left-moving central charges of the CFT. The term with an order 4 pole implies that the energy momentum tensor is not a primary field.

We will now find the local conformal group after the quantisation of the CFT. To do so, we Laurent expand the energy momentum tensor as follows:

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \quad \text{where } L_n = \frac{1}{2\pi i}\oint dz\ z^{n+1} T(z) \tag{1.11}$$

We similarly Laurent expand the infinitesimal conformal transformation $\epsilon(z) = \sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1}$. Substituting these two mode expansions in the definition of Q_ϵ we get $Q_\epsilon = \sum_n \epsilon_n L_n$. Thus we see that $L_n, \bar{L}_n$ are the generators of the local conformal transformations and they satisfy the following algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \tag{1.12}$$

This operator algebra is known as the *Virasoro algebra* and is the local conformal algebra in two dimensions. Note that this is a central extension of the Witt algebra. The sub-algebra generated by L_0, L_1, L_{-1} and their anti-holomorphic counterparts is $sl(2, \mathbb{C})$. One can now easily verify that the Hamiltonian is proportional to $L_0 + \bar{L}_0$, whereas the angular momentum is proportional to $L_0 - \bar{L}_0$.

Hilbert space and conformal families

To explore the Hilbert space of the CFT, we will first assume the existence of a unique vacuum. Using the $T\varphi$ OPE, we get the following commutation relation, where φ is a primary field:

$$[L_n, \varphi(w, \bar{w})] = h(n+1)w^n \varphi(w, \bar{w}) + w^{n+1}\partial\varphi(w, \bar{w}) \tag{1.13}$$

which implies that $L_0(\varphi(0)\left|0\right\rangle) = h(\varphi(0)\left|0\right\rangle)$. Thus we see that the state $\varphi(0)\left|0\right\rangle$ is an eigenstate of the Hamiltonian and will be denoted as $\left|h, \bar{h}\right\rangle$. Thus we see that primary fields act on the vacuum to create asymptotic states that are eigenvalues of the Hamiltonian. We can now easily check that:

$$[L_0, L_{-n}] = nL_{-n} \tag{1.14}$$

Thus we see that the generators L_{-n} and L_n for $n > 0$, increase and decrease the conformal dimension by n, respectively. Therefore we think of the *primary state* $\left|h, \bar{h}\right\rangle$ as the highest weight state with energy eigenvalue $h + \bar{h}$ and L_n, L_{-n} act as the lowering, raising operators, respectively for $n > 0$, i.e. $L_n\left|h, \bar{h}\right\rangle = 0$. The states we obtain after raising the highest weight states with L_{-n} are called as *descendant states*. A descendant state at level $(k, \bar{k})$ has the general form:

$$L_{-k_1}L_{-k_2}\dots L_{-k_N}\bar{L}_{-\bar{k}_1}\bar{L}_{-\bar{k}_2}\dots\bar{L}_{-\bar{k}_{\bar{N}}}\left|h\right\rangle \quad \text{with } 1 \leq k_1 \leq k_2 \leq \cdots \leq k_N \tag{1.15}$$

where $k_1 + k_2 + \cdots + k_N = k$, and similarly for $\{\bar{k}_i\}$. One can easily verify that these are also energy eigenstates with energy eigenvalue $h + k + \bar{h} + \bar{k}$.

The set containing a primary state along with all of its descendants is known as a *conformal family* or a Verma module, $\mathcal{V}_{c,h}$ labelled by the central charge and the relevant conformal dimensions. As the full conformal group is the direct product of subgroups describing the holomorphic (Γ) and anti-holomorphic ($\bar{\Gamma}$) transformations, the conformal families themselves are direct products of the representations of Γ and $\bar{\Gamma}$. The members of a conformal family transform among themselves under a conformal transformation, and the OPE of any member of a family with $T(z)$ is composed solely of other members from the same family. Therefore the full Hilbert space of a CFT is a sum over direct product of individual Verma modules:

$$\mathcal{H} = \bigoplus_{i,\bar{i}} V_{c,h_i} \times V_{\bar{c},\bar{h}_{\bar{i}}} \tag{1.16}$$

To end this review, we state the fact that a general correlation function in a CFT can be expressed in terms of the correlators of the primary fields in the CFT [1]. The computations are further simplified by using the OPE of various primary fields in the theory. Therefore the full CFT data consists of the central charges, conformal dimensions and all the OPE coefficients of the primary fields.

1.2 Rational Conformal Field Theories

For all $c < 1$ CFT (also known as the Virasoro minimal models), the sum in Eq. (1.16) is performed only over a finite number of representations of the Virasoro algebra [6]. These minimal models thus have a finite number of primary fields all of which have rational conformal dimensions. The Virasoro minimal models are particular examples of what are known as rational conformal field theories (RCFT). The Hilbert space of a rational CFT can be factorised into a sum over a finite number of representations of any extended symmetry algebra. In the case of Virasoro minimal models, the full symmetry algebra is just the Virasoro algebra, whereas more generally, Virasoro algebra exists as a sub-algebra in the full symmetry algebra.

Extended symmetry algebras in a CFT often occur due to the presence of conserved chiral fields in the theory and the corresponding algebra is referred to as the chiral algebra of the CFT. The prime examples of theories with an extended chiral symmetry are the Wess-Zumino-Witten (WZW) models, whose Hilbert spaces are made up of a finite number of integrable representations of a *Kac-Moody algebra*.

Recall that the Virasoro algebra is generated by a spin-2 conserved current, namely the energy-momentum tensor. Similarly, in WZW models, the chiral algebra is generated by spin-1 conserved currents $J^a(z)$ which transform under the adjoint representation of the underlying symmetry group algebra that characterises a WZW model [11]. The modes of these generators J^a_n satisfy the Kac-Moody algebra associated with the Lie algebra $\mathfrak{g}$ at level k given by:

$$\left[J^a_n, J^b_m\right] = if^{abc}J^c_{n+m} + km\delta^{ab}\delta_{n+m,0} \tag{1.17}$$

where f^{abc} are the structure constants of the underlying Lie algebra, $\mathfrak{g}$. The level, $k \in \mathbb{Z}_{\geq 0}$ of this

[1]This constitutes the seminal work by Belavin, Polyakov, and Zamalodchikov [6].

algebra is a constant, and such an algebra is called as an affine Lie algebra, $\mathfrak{g}_k$.

Since the full symmetry algebra of a WZW model is the Kac-Moody algebra, the Virasoro algebra must be a subgroup of this affine Lie algebra, in particular, one should be able to recover the energy-momentum tensor from the spin-1 conserved currents. This is accomplished by the Sugawara construction [12] which tells us that:

$$T(z) = \frac{1}{2\,(k+h^\vee)} \sum_a J^a(z) J^a(z) \quad \& \quad c = \frac{k\ \mathrm{dim}\mathfrak{g}}{k+h^\vee} \tag{1.18}$$

where $h^\vee$ is the dual Coxeter number defined by the quadratic Casimir in the adjoint representation of $\mathfrak{g}$, i.e. $f^{abc}f^{bcd} = 2h^\vee \delta^{ad}$ and c is the Virasoro central charge. The primary fields of a WZW model are the irreducible representations of the affine Lie group $\mathfrak{g}_k$. The conformal dimension of a primary corresponding to an irreducible representation $\mathcal{R}$ of $\mathfrak{g}_k$ is given as [13]:

$$h = \frac{c\,\ell(\mathcal{R})}{2\mathrm{dim}\mathcal{R}} \tag{1.19}$$

where $\ell(\mathcal{R})$ is the index of the irreducible representation $\mathcal{R}$.

1.3 Modular Invariance

A torus $\mathbb{T}$ equipped with a complex structure τ is equivalent to the complex plane $\mathbb{C}$ under the identification implied by the lattice Λ defined by two linearly independent vectors, $\omega_1, \omega_2 \in \mathbb{C}$:

$$\mathbb{T} = \mathbb{C}/\Lambda, \text{ where } \Lambda = \{n\,\omega_1 + m\,\omega_2 \mid n, m \in \mathbb{Z}\} \tag{1.20}$$

The modular parameter, τ is defined to be ratio of the two lattice vectors, i.e. $\tau = \omega_2/\omega_1$. The following transformation of the modular parameter:

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}, \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \tag{1.21}$$

takes a linear combination of old lattice vectors to give new lattice vectors ω_1', ω_2', but leaves the lattice, and hence the torus invariant. Such a transformation of τ is called as a *modular transformation* and the group of such transformations $PSL(2,\mathbb{Z})$ is called as the *modular group*, which we will denote as Γ. We work in the convention where $\mathrm{Im}(\tau) > 0$, i.e. τ lives on the upper half plane $\mathfrak{h}$. We will expand on this in the next chapter. The modular group has two generators $\mathcal{T}$

and $\mathcal{S}$ whose action on the modular parameter is given as:

$$\mathcal{T} : \tau \to \tau + 1, \quad \mathcal{S} : \tau \to -\frac{1}{\tau} \tag{1.22}$$

We will consider the subgroup for which $\mathcal{S}^2 = (\mathcal{S}\mathcal{T})^3 = \mathbb{1}$.

With this background, consider a general two-dimensional conformal field theory defined on a torus. The partition function of such a theory at temperature $\frac{1}{\beta}$, with a thermodynamic potential θ for the angular momentum J, can be written as

$$\mathcal{Z}(\beta, K) = Tr_{\mathcal{H}}\, e^{-\beta H + i\theta J} \tag{1.23}$$

where $\mathcal{H}$ is the full Hilbert space of the CFT. Recall that the Hamiltonian and the angular momentum of a CFT are given in terms of the Virasoro operators as:

$$H = 2\pi\left(L_0 - \frac{c}{24} + \bar{L}_0 - \frac{\bar{c}}{24}\right), \quad J = L_0 - \frac{c}{24} - \bar{L}_0 + \frac{\bar{c}}{24} \tag{1.24}$$

We can now redefine the modular parameter $\tau = \theta/2\pi + i\beta$ to get:

$$\mathcal{Z}(\tau, \bar{\tau}) = Tr_{\mathcal{H}}\left(q^{L_0 - \frac{c}{24}}\bar{q}^{L_0 - \frac{\bar{c}}{24}}\right) \ , \quad \text{where } q = e^{2\pi i\tau} \tag{1.25}$$

The torus partition function must now be remain invariant under the action of modular group on τ. Thus by defining the CFT on a torus we get additional constraints stemming from its invariance under modular transformations, or in other words, *modular invariance*.

Recall that the full Hilbert space of a 2d CFT is a direct sum of independent Verma modules $V_{c,h}$, labelled by the conformal dimension h, i.e. $\mathcal{H} = \bigoplus_{i,\bar{i}} V_{c,h_i} \times V_{\bar{c},\bar{h}_{\bar{i}}}$. This implies that the partition function can be written as the following sum:

$$\mathcal{Z}(\tau, \bar{\tau}) = \sum_{i,\bar{i}} M_{i\bar{i}}\; Tr_{V_{c,h_i}} q^{L_0 - \frac{c}{24}}\; Tr_{V_{\bar{c},\bar{h}_{\bar{i}}}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} = \sum_{i,\bar{i}} M_{i\bar{i}} \chi_i(\tau) \bar{\chi}_{\bar{i}}(\bar{\tau}) \tag{1.26}$$

where $M_{i\bar{i}}$ counts the multiplicity of each primary state, i.e. distinct primaries with the same trace over the Verma module. In this work, we will only consider diagonally-invariant theories in which case $M_{i\bar{i}} \equiv M_i \delta_{i\bar{i}}$. We thus define the "character" of a CFT to be:

$$\chi_i(\tau) = Tr_{V_{c,h_i}} q^{L_0 - \frac{c}{24}} = q^{-\frac{c}{24}} \sum_{n=0}^{\infty} d_n^{(i)} \langle h_i + n|\, q^{L_0}\, |h_i + n\rangle \tag{1.27}$$

where $d_n^{(i)}$ counts the degeneracy of the n^{th} state in the Verma module V_{c,h_i}. This definition can be trivially extended to the anti-holomorphic trace in the partition function.

This implies that the character of a CFT can be written as an integer q-series with a leading branch cut:

$$\chi_i(\tau) = q^{\alpha_i}\left(a_0^{(i)} + a_1^{(i)}q + a_2^{(i)}q^2 + \cdots\right) \ , \quad \text{where } \alpha_i = -\frac{c}{24} + h_i \tag{1.28}$$

The label $i = 0$ corresponds to the vacuum character and thus a unique vacuum implies $a_0^{(0)} = 1$ as well as $h_0 = 0$. Similarly $a_n^{(i)}$ are non-negative integers and can be interpreted as the degeneracies of secondary states under the full symmetry algebra.

Note that the characters themselves can have non-trivial modular transformation properties as modular invariance leaves only the full partition function invariant. It can easily be seen that under the modular $\mathcal{T}$ and $\mathcal{S}$ transformation the characters transform as:

$$\mathcal{T}: \quad \chi_i(\tau) \to \chi_i(\tau+1) = e^{2\pi i\alpha_i}\chi_\tau \ , \quad \mathcal{S}: \quad \chi_i(\tau) \to \chi_i\left(-\frac{1}{\tau}\right) = \sum_j \mathcal{S}_{ij}\chi_j(\tau) \tag{1.29}$$

where $\mathcal{S}_{ij}$ will be called as the *modular $\mathcal{S}$-matrix*. Thus the invariance of the partition function under the modular $\mathcal{T}$ transformation implies $c - \bar{c} \in 24\mathbb{Z}$ as well as $h - \bar{h} \in \mathbb{Z}$. Similarly the invariance of $\mathbb{Z}(\tau, \bar{\tau})$ under the modular $\mathcal{S}$ transformation implies $\mathcal{S}^\dagger M \mathcal{S} = M$, where $M = \text{diag}(M_i)$.

One of the implications of modular invariance of a CFT is the relation between the modular $\mathcal{S}$-matrix and fusion rules. Fusion rules of a CFT are selection rules that tell us which fields are allowed to appear in a given operator product expansion. The fusion rules are expressed as:

$$\phi_i \times \phi_j = \sum_k \mathcal{N}_{ij}^k \ \phi_k \tag{1.30}$$

Each CFT has a fixed set of fusion rules and thus each CFT falls within a fusion class. The fusion rule coefficients are given in terms of the modular $\mathcal{S}$ matrix by the Verlinde formula [14], only when $M = \mathbb{1}$:

$$\mathcal{N}_{ij}^k = \sum_n \frac{\mathcal{S}_i^n \mathcal{S}_j^n {\mathcal{S}^\dagger}_n^k}{\mathcal{S}_0^n} \tag{1.31}$$

This tells us that in the OPE of any two fields, one each from the ϕ_i and ϕ_j conformal families, fields belonging to the conformal family of ϕ_k can appear, whenever $\mathcal{N}_{ij}^k \neq 0$. Due to constraints on the length of this work, we will not describe this in any more detail.

Chapter 2

Modular Bootstrap

Modular bootstrap is a non-perturbative approach that constrains two-dimensional conformal field theories. Modular bootstrap aims to use the modular invariance of the partition function, discussed in the previous chapter, i.e.

$$\mathcal{Z}(\tau,\bar{\tau}) = \mathcal{Z}\left(\frac{a\tau+b}{c\tau+d}, \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) \ , \quad \text{where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \tag{2.1}$$

to find various constraints on the CFT. This has been extremely useful in the classification problem of RCFT as well as in the holographic applications of conformal field theories. In this chapter, we will review the application of modular invariance to the classification of RCFT proposed by Mathur, Mukhi, Sen in [15] and list some of the known results [16, 17, 18, 19, 20, 21]. A comprehensive review of the current understanding of this subject along with a list of sources is available in [22].

2.1 Modular Linear Differential Equation

The MMS classification [15] of RCFT relies heavily on the modular properties of the partition function. In this approach, one aims to classify all the RCFT with a given number of primaries with respect to the full symmetry algebra. One of the crucial aspects of this classification scheme is that it does not assume any particular chiral algebra and does not privilege the Virasoro algebra. As we shall see, the chiral algebra, the number of primary fields under it, and the modular $\mathcal{S}$ matrix are all outputs of this scheme.

Before we describe the scheme, it is useful to recall the following: the partition function is modular-invariant but is not a holomorphic function, whereas the characters are holomorphic functions but are not modular invariant. A crucial fact is that the set of characters, though not modular invariant individually, are modular covariant, i.e. the set of characters transforms as a vector valued modular form. This implies that the characters are solutions to a modular linear differential equation, that is both modular invariant and holomorphic.

The MMS classification scheme utilises this fact that the (anti-) holomorphic characters of an n-character RCFT satisfy an n^{th} order modular linear differential equation (MLDE) in the modular parameter $\tau(\bar{\tau})$. Since the characters transform into linear combination of themselves under modular transformations, the differential equation has to be modular invariant.

We will now briefly review modular forms, which are objects that have nice properties under modular transformations. A holomorphic modular form of weight $2k$ is a holomorphic function $f(\tau)$, defined on the upper half-plane, such that:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{2k} f(\tau) \tag{2.2}$$

It can shown that the Eisenstein series $E_4(\tau), E_6(\tau)$ generate the ring of holomorphic modular forms, i.e. a holomorphic modular form of arbitrary weight $2k$ can be a written as a string in E_4 and E_6. These two Eisenstein series can be expressed as:

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}\ , \quad E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n} \tag{2.3}$$

Interestingly, the Eisenstein series $E_2(\tau)$, defined as:

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} \tag{2.4}$$

does not transform as a modular form, but instead transforms as:

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) - \frac{6i}{\pi}\, c(c\tau+d) \tag{2.5}$$

However, this plays an important role in the MLDE by being the connection that we will use to define the covariant derivative on the moduli space.

It can easily be verified that the partial derivative of a modular form does not transform as a modular form. Therefore the above mentioned MLDE is in terms of a modular covariant derivative, whose action on a weight r modular form is defined to be:

$$\mathcal{D}_\tau^{(r)} = \partial_\tau - \frac{1}{6} i\pi r E_2(\tau) \tag{2.6}$$

The n^{th} order modular covariant derivative is defined to be:

$$\mathcal{D}_\tau^n = \mathcal{D}_\tau^{(r+2n-2)} \cdot \mathcal{D}_\tau^{(r+2n-4)} \cdot \dots \cdot \mathcal{D}_\tau^{(2)} \cdot \mathcal{D}_\tau^{(0)} \tag{2.7}$$

For brevity, we will now drop the superscript (r) as well the τ subscript from the modular covariant derivative.

With this machinery, we can now write the most general n^{th} order MLDE:

$$\left(\mathcal{D}^n + \sum_{k=0}^{n-1} \phi_{2(n-k)}(\tau)\mathcal{D}^k\right)\chi(\tau) = 0 \tag{2.8}$$

where $\phi_{2(n-k)}$ are meromorphic modular forms of weight $2(n-k)$. These functions can be expressed in terms of the n linearly independent solutions, χ_i using the Wronskian determinants as:

$$\phi_k = (-1)^{n-k}\frac{W_k}{W}\ , \quad W_k = \begin{vmatrix} \chi_0 & \chi_1 & \cdots & \chi_{n-1} \\ \mathcal{D}\chi_0 & \mathcal{D}\chi_1 & \cdots & \mathcal{D}\chi_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{D}^{k-1}\chi_0 & \mathcal{D}^{k-1}\chi_1 & \cdots & \mathcal{D}^{k-1}\chi_{n-1} \\ \mathcal{D}^{k+1}\chi_0 & \mathcal{D}^{k+1}\chi_1 & \cdots & \mathcal{D}^{k+1}\chi_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{D}^{n}\chi_0 & \mathcal{D}^{n}\chi_1 & \cdots & \mathcal{D}^{n}\chi_{n-1} \end{vmatrix} \tag{2.9}$$

with $W = W_n$. Since the individual characters are expected to be non-singular in the τ space, the number of poles of ϕ_k are determined by the number of zeroes of the Wronksian, W in the fundamental domain. The torus moduli space has two orbifold points, one at $\tau = i$ of order $\frac{1}{2}$, and another at $\tau = e^{\frac{2\pi i}{3}}$ of order $\frac{1}{3}$. Therefore the number of zeroes of W can be expressed as $\ell/6$ where $l \in \{0, 2, 3, 4, \dots\}$. The fractional number of zeroes indicates the fact that the moduli space has orbifold points.

The number of zeroes $\ell/6$ satisfies a relation with the exponents α_i of the characters. This relation is obtained by investigating the pole structure of W at infinity, i.e. $\tau \to i\infty$. In this limit, $\chi_i \sim e^{2\pi i\tau\alpha_i}$, where $\alpha_i = -c/24 + h_i$. Since the Wronskian is a sum of terms each of which has all of the χ_i multiplied, $W \sim e^{2\pi i\tau\sum_i\alpha_i}$. This implies that W has an order $-\sum_i\alpha_i$ pole as $\tau \to i\infty$. Notice that the each term of the above mentioned sum involves a total of $n(n-1)/2$ modular covariant derivatives. This implies that the Wronskian W is weight $n(n-1)$ modular form.

We now use a relation for *orders of vanishing* of a weight k modular form on the fundamental domain of a moduli space, which we will call as the Riemann-Roch relation. The relation [23, 24] states that for a weight-k modular form, $f(\tau)$ on Γ:

$$\sum_{P\in\,\mathfrak{h}/\Gamma} \frac{1}{n_P} ord_P(f) + ord_\infty(f) = \frac{k}{12} \tag{2.10}$$

where $ord_\infty(f)$ is the smallest integer n such that $a_n \neq 0$, where $f(z) = \sum_n a_n q^n$, and is called as the order of vanishing at infinity. Similarly $ord_P(f)$ is the order of vanishing of $f(\tau)$ at $P \in \mathfrak{h}/\Gamma$, i.e. the smallest integer $n \geq 0$ for which $\lim_{z\to P} \frac{f(z)}{z^n} \neq 0$. The integer n_P distinguishes between regular points in the fundamental domain and orbifold points. In our case, $n_P = 1$ for all points except the two orbifold points at $\tau = i, e^{2\pi i/3}$, where it takes the value $n_P = 2, 3$ respectively.

Therefore we see that the first term in Eq. (2.10) is just equal to the number of zeroes of W and is hence equal to $\ell/6$. Similarly, the order of vanishing at infinity for W is $\sum_i \alpha_i$, since $W \sim q^{\sum_i \alpha_i} + \mathcal{O}\left(q^{\sum_i \alpha_i + 1}\right)$. Thus we see that the order of vanishing at infinity is related to the order of the pole at infinity. Therefore for the Wronskian W, the Riemann-Roch relation tells us that

$$\sum_i \alpha_i + \frac{\ell}{6} = \frac{n(n-1)}{12} \tag{2.11}$$

For the rest of this work, we will refer to Eq. (2.11) as the *Riemann-Roch equation*, as it has been derived from the Riemann-Roch relation.

Now we can finally explain the classification strategy put forth by MMS. We start with a fixed number of primaries n and also fix the integer ℓ. This fixes the form of all the modular forms ϕ_k in terms of the Eisenstein series E_4 and E_6, as these two generate the full set of modular forms. This gives us an n^{th} order MLDE with undetermined parameters. We will proceed to solve this using the Frobenius method, i.e. we insert the following ansatz for the solutions:

$$\chi_i(\tau) = \sum_{n=0}^{\infty} a_n^{(i)} q^{\alpha_i + n} \tag{2.12}$$

We then solve the MLDE order by order in q, as every modular form can be expanded in an integer q-series, which is just the Fourier expansion of the modular forms. The zeroth order equation gives what is known as the indicial equation, which relates the parameters in the MLDE to the CFT exponents α_i. Since we want to interpret the solutions of the MLDE as characters of an RCFT, we will demand that the coefficients $a_n^{(i)}$ have to be non-negative integers.

The scheme looks for solutions of the MLDE which satisfy the following three criterion: (i) they transform as vector valued modular functions such that the partition function remains invariant, (ii) the ground state in the identity character is non-degenerate, which implies a unique vacuum, as any sensible QFT should have, and (iii) each character has non-negative integer degeneracies after suitably normalising the ground state, except for the identity character which is normalised to unity. A solution which satisfies these properties will be termed as an *admissible* character.

Before moving on, we will make a few quick comments on meromorphic CFT, i.e. one-character CFT. The partition function of these theories is just the mod-square of the identity character. Since for this case $\mathcal{S} = \mathbb{1}$ and $(\mathcal{S}\mathcal{T})^3 = \mathbb{1}$, the central charge has to be a multiple of 8 in order to preserve modular invariance. For $c = 8$, there is a unique character $\chi_{E_{8,1}}$, that is modular invariant up to a phase. The resulting CFT is the $E_{8,1}$ WZW model. For $c = 16$, there is again a unique character $\left(\chi_{E_{8,1}}\right)^2$ which corresponds to two different WZW models based on the algebras $E_8 \oplus E_8$ and Spin(32)/$\mathbb{Z}_2$. For $c = 24$ there are infinitely many characters of the form:

$$\chi(\tau) = j(\tau) + \mathcal{N} = q^{-1} + 744 + \mathcal{N} + 196884q + \cdots \tag{2.13}$$

where $j(\tau)$ is the Klein-j invariant, which is a modular invariant. One gets admissible characters for all $\mathcal{N} \geq -744$. It was shown in [25] that only 71 of these are consistent one-character CFT with $c = 24$. The case when $\mathcal{N} = -744$ is special as it corresponds to a CFT without any Kac-Moody symmetry (i.e. no spin-1 current). This $c = 24$ one-character CFT is a special one and is called as the Monster CFT, because of its relation to the Monster group, the largest sporadic simple group of order $\approx 8 \times 10^{53}$.

2.2 The Two-Character Case

We will now quickly review the known results in the two-character case, since our original work [2] generalises most of these results for the three-character case. The $\ell = 0$ characters of two-character CFT satisfy the following MLDE:

$$\left(\mathcal{D}^2 + \mu E_4\right)\chi = 0 \tag{2.14}$$

This equation is known as the MMS equation as Mathur, Mukhi, and Sen used it to classify all possible $\ell = 0$ two-character CFT in [15]. It turns out that the $\ell = 0, 2, 4$ second-order MLDE[1] give rise to a finite number of admissible characters, in the sense described in the previous section. If one actually relaxes the admissibility criteria to allow solutions of MLDE that have negative degeneracies of the state, i.e. negative integral coefficients in the q-series, one finds infinitely many solutions to the MLDE. Such solutions are called as *quasi-characters*, and it was shown in [20] that these can be used to construct admissible characters. It was also shown that these fall into various

[1]The $\ell = 1, 3, 5, \ldots$ cases can be ruled out to by studying the second-order MLDE [17].

infinite families, one for each fusion class.

The admissible characters occur within these families as special cases with all non-negative coefficients. By taking appropriate semi-definite linear combinations of quasi-characters within a particular fusion class, one can cancel the negative signs to generate admissible characters. If one starts from quasi-characters of Wronskian index $\ell = 0, 2, 4$, said linear combinations turn out to have $\ell = 6m, 6m+2, 6m+4$ respectively, where m is a positive integer. It has been shown by [20] that this classification for two-characters is complete, i.e. any admissible quasi-character with $\ell = 6m$ can be written as a linear combination of $\ell = 0$ quasi-characters, with similar results in the other cases. We shall now describe some relevant aspects as well as results of this classification.

It is well known that two-character CFT fall into 4 different fusion classes which were labelled as: $\mathcal{A}_1^{(1)}, \mathcal{A}_1^{(2)}, \mathcal{B}_2^{(1)}, \mathcal{A}_3^{(1)}$ by [26]. We will follow the notation of [20] and label them as A_1, Lee-Yang, A_2, D_4 classes respectively. Correspondingly, the infinite families of $\ell = 0$ quasi-character of order two fall into the following four sets:

$$\begin{aligned}
\text{Lee-Yang}: \quad & c = \frac{2}{5}(6k+1)\,, \quad k \neq 4 \bmod 5 \qquad \& \qquad A_1: \quad c = 6k+1 \\
A_2: \quad & c = 4k+2 \quad\;\;\;\;\;\;\; , \quad k \neq 2 \bmod 3 \qquad \& \qquad D_4: \quad c = 12k+4
\end{aligned} \tag{2.15}$$

where $k \in \mathbb{Z}_{\geq 0}$. Note that it is sufficient to specify only the central charge for each one of these, since one can use the Riemann-Roch equation to determine the conformal dimension $h = c/12 + 1/6$. Note that the cases with $k = 0, 1, 2, 3$ for Lee-Yang, $k = 0, 1$ for A_1 and A_2, and $k = 0$ for the D_4 series correspond to physical RCFT [15].

Note that the integer k takes only non-negative values, since the sub-series for negative k is the same as the for $k \geq 0$ under the exchange of the identity and the non-identity characters. To be precise, under this exchange we find $k \to -k-2$ for the Lee-Yang and A_2 class, and $k \to -k-1$ for the A_1 and D_4 class. It should be pointed out that the existence of these quasi-characters is not just an approximate statement that is true up to some large order of q, but has been rigorously proved by using recursion relations and other mathematical results as described in [20].

It was found out that the quasi-characters could be classified into two distinct types based on the asymptotic behaviour of the signs of the coefficients of the identity character. The type I quasi-characters have negative coefficients in the first few terms, but after some order all the coefficients are positive. On the other hand, type II quasi-characters have all negative coefficients, after some order, if the leading term is chosen to be positive. The non-identity characters always have all positive coefficients.

We will now describe the utility of quasi-characters, in particular of type I, in generating admissible characters of higher ℓ. We will pick a pair of quasi-characters with the same modular properties, one of which is an admissible characters with central charge c and the other one with central charge $c' = c + 24$. It is an observational fact that the latter one has a negative number as the coefficient of $q^{-c'/24+1}$ whereas the rest of them are positive, i.e.

$$\chi_0 = q^{-\frac{c}{24}}(1 + a_1 q + a_2 q^2 + \cdots) \quad \& \quad \chi_0' = q^{-\frac{c'}{24}}(1 - |b_1| q + b_2 q^2 + \cdots) \tag{2.16}$$

where $a_i, b_j \in \mathbb{Z}_{\geq 0}$. We will now add these identity characters as $\chi_0' + N\chi_0$ to get:

$$\widetilde{\chi}_0 = \chi_0' + N\chi_0 = q^{-\frac{c'}{24}}(1 + (N - |b_1|)q + (Na_1 + b_2)q^2 + \cdots) \tag{2.17}$$

Of course, we must also take the same linear combination of the non-identity characters as well, i.e. $\widetilde{\chi}_1 = \chi_1' + N\chi_1$. Note that because of the difference of 24 in the central charges, the leading term of χ_0 adds to the sub-leading term of χ_0'. Thus we see that for $N \geq |b_1|$ we obtain an admissible character. Additionally, this implies that the modular $\mathcal{T}$, $\mathcal{S}$ transformation for both these sets of characters, and the final admissible character is the same.

Now we will work out the central charge and conformal dimension of the new admissible characters $\widetilde{\chi}_i$. Since both χ_i and χ_i' are $\ell = 0$ characters, we know that $h = c/12 + 1/6$, which implies that one can write h' as $h' = h + 2$. The leading power of q in χ_1 can now be written as $q^{-\frac{c'}{24}+h'-1}$ whereas the leading power of q in χ_1' is $q^{-\frac{c'}{24}+h'}$. Therefore the leading power of q in $\widetilde{\chi}_1$ will be the one which is more singular and hence we see that $\widetilde{h} = h' - 1$. Therefore we can use $(\widetilde{c}, \widetilde{h}) = (c', h' - 1)$ to compute the Wronskian index for the new admissible character using the Riemann-Roch relation, Eq. (2.11) to get $\ell = 6$.

Therefore by appropriately adding two $\ell = 0$ quasi-characters, we end up with an admissible character with $\ell = 6$. We can generalise this by adding three or more characters, or by taking quasi-characters whose central charges differ by higher multiples of 24 to get admissible characters with higher ℓ. One can repeat this by starting with $\ell = 2, 4$ characters and repeat the above procedure to obtain admissible characters with $\ell = 6m + 2, 6m + 4$, respectively.

Chapter 3

Free Boson Conformal Field Theory

We will now work out an explicit example which displays many aspects of two-dimensional conformal field theories that we have reviewed till now. The example we will consider is the deceptively simple, massless free boson field theory in two dimensions, which has the action:

$$S = \frac{g}{2} \int d^2x \partial_\mu \phi \partial^\mu \phi \tag{3.1}$$

This is a conformal field theory, with the scaling dimension of ϕ equal to zero and is reviewed in detail in [27, 8, 3]. In this chapter, we will first quantise the theory and find its partition function. Then we will repeat the same for a free boson compactified on a circle. Finally, we will show that a special case of the compactified free boson is a rational conformal field theory.

3.1 Radial Quantisation of the Free Boson

Under radial quantisation, the boundary condition for the free boson field is:

$$\phi(t, x + L) = \phi(t, x) \tag{3.2}$$

where L is the circumference of the compactified spatial direction on the cylinder. The equation of motion of $\phi(t, x)$ is the familiar Klein-Gordon equation:

$$\left(\partial_t^2 - \partial_x^2\right) \phi(x, t) = 0 \tag{3.3}$$

Since the field ϕ is periodic in x, it cannot be a polynomial in x. The same is not true for the time variable and the field can have a linear term in t and still satisfy the equation of motion.

We will now Wick rotate the time direction to go to the Euclidean coordinates on the cylinder $(\tau = it, x)$. Next, we perform a change of coordinates on the cylinder $w = \tau + ix$ to go the complex coordinates $(w, \bar{w})$. Finally, we will map the cylinder onto the Riemann sphere by using the conformal map $z = e^{2\pi w/L}$. The equation of motion on the plane now becomes:

$$\partial_z \partial_{\bar{z}} \phi(z, \bar{z}) = 0 \implies \phi(z, \bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z}) \tag{3.4}$$

A general solution to this equation can be expressed as a sum of a holomorphic $\varphi(z)$ and an anti-holomorphic function $\bar{\varphi}(\bar{z})$.

We can now Laurent expand the (anti-) holomorphic functions about $(\bar{z})z = 0$ to get:

$$\varphi(z) = \frac{\varphi_0}{2} + \alpha \ \mathrm{ln}z + \frac{i}{\sqrt{4\pi g}} \sum_{n \neq 0} \frac{a_n}{n} z^{-n} \ , \quad \bar{\varphi}(\bar{z}) = \frac{\varphi_0}{2} + \bar{\alpha} \ \mathrm{ln}\bar{z} + \frac{i}{\sqrt{4\pi g}} \sum_{n \neq 0} \frac{\bar{a}_n}{n} \bar{z}^{-n} \tag{3.5}$$

where the origin of ln$z$ term is from the linear term in t. Note that taking z around the origin in the clockwise direction (and consequently $\bar{z}$ in the opposite direction) once will change $\phi(z, \bar{z})$ by a factor proportional to $(\alpha - \bar{\alpha})$. For the field to be single valued we will thus set $\alpha = \bar{\alpha}$, which is in agreement with the logarithmic term arising from the linear term in t in $\phi(x, t)$. We now have the mode expansion of ϕ in the $(z, \bar{z})$ coordinates, let us now work back to obtain its mode expansion in the (t, x) coordinates. After some calculations, we arrive at:

$$\phi(t, x) = \varphi_0 + \frac{4\pi i \ \alpha}{L} t + \frac{i}{\sqrt{4\pi g}} \sum_{n \neq 0} \frac{1}{n} \left\{ a_n e^{-\frac{2\pi i n}{L}(t+x)} + \bar{a}_n e^{-\frac{2\pi i n}{L}(t-x)} \right\} \tag{3.6}$$

One can now easily find the canonical momentum, $\Pi(t, x)$ of the field ϕ by using $\Pi = \partial\mathcal{L}/\partial\dot{\phi} = g\dot{\phi}$, where dot denotes derivative w.r.t. t:

$$\frac{\Pi(t, x)}{g} = \frac{4\pi i \ \alpha}{L} + \frac{2\pi}{L\sqrt{4\pi g}} \sum_{n \neq 0} \frac{1}{n} \left\{ a_n e^{-\frac{2\pi i n}{L}(t+x)} + \bar{a}_n e^{-\frac{2\pi i n}{L}(t-x)} \right\} \tag{3.7}$$

The Hamiltonian density is thus given as:

$$\begin{aligned} \mathcal{H} = \frac{\Pi^2}{2g} + \frac{1}{2} g\phi^2 = \frac{g}{2} \left(\frac{4\pi i \alpha}{L} \right)^2 &+ \frac{8\pi^2 i g \alpha}{L^2 \sqrt{4\pi g}} \sum_{n \neq 0} \left\{ a_n e^{\frac{-2\pi i n}{L}(t+x)} + \bar{a}_n e^{\frac{-2\pi i n}{L}(t-x)} \right\} \\ &+ g \left(\frac{2\pi}{L\sqrt{4\pi g}} \right)^2 \sum_{n,m \neq 0} \left\{ a_n a_m e^{\frac{-2\pi i}{L}(t+x)(n+m)} + \bar{a}_n \bar{a}_m e^{\frac{-2\pi i}{L}(t-x)(n+m)} \right\} \end{aligned} \tag{3.8}$$

To obtain the Hamiltonian of the CFT, we integrate the Hamiltonian density over the compactified space, i.e. $H = \int_0^L dx\mathcal{H}$. A useful tool in this computation is the following identity:

$$\int_0^L dx\; e^{\frac{2\pi i x(n+m)}{L}} = L\;\delta_{n+m,0} \tag{3.9}$$

Using this, the Hamiltonian of this CFT can be expressed as:

$$H = \frac{gL}{2}\left(\frac{4\pi i\,\alpha}{L}\right)^2 + \frac{2\pi}{L}\sum_{n>0}\{a_{-n}a_n + \bar{a}_{-n}\bar{a}_n\} \tag{3.10}$$

Each term in the sum is reminiscent of a harmonic oscillator and the total field looks like an infinite number of independent harmonic oscillators

To address this point, let us take a closer look at the Lagrangian of this CFT:

$$L = \frac{g}{2}\int_0^L dx\left(\dot{\phi}^2 - \phi'^2\right) \tag{3.11}$$

Consider the Fourier transform of the field in the spatial direction $\phi(t,x) = \sum_n e^{2\pi i n x/L}\phi_n(t)$. The Lagrangian in terms of the modes $\phi_n(t)$ then becomes:

$$L = \frac{gL}{2}\sum_n\left\{\dot{\phi}_{-n}\dot{\phi}_n - \left(\frac{2\pi n}{L}\right)^2\phi_{-n}\phi_n\right\} \tag{3.12}$$

This is the Lagrangian of an infinite number of harmonic oscillators, all with the same mass $m = gL$ and frequencies $\omega_n = \frac{2\pi|n|}{L}$ labelled by integers n. Note that the $n = 0$ mode is not a harmonic oscillator but rather a free particle, which is consistent with no $n = 0$ contribution to the sum in the Hamiltonian.

We will now canonically quantise this theory by using the equal-time commutation relation:

$$[\phi(t,x),\Pi(t,y)] = i\delta(x-y) \tag{3.13}$$

After some algebra, one can easily show that in terms of the plane coordinates we get:

$$\begin{aligned}[\phi(z,\bar{z}),\Pi(w,\bar{w})] = \frac{4\pi i g}{L}[\phi_0,\alpha] + \frac{i}{2L}\sum_{n,m\neq 0}\frac{1}{n}\Big\{&[a_n,a_m]z^{-n}w^{-m} + [a_n,\bar{a}_m]z^{-n}\bar{w}^{-m}\\ &+[\bar{a}_n,a_m]\bar{z}^{-n}w^{-m} + [\bar{a}_n,\bar{a}_m]\bar{z}^{-n}\bar{w}^{-m}\Big\}\end{aligned} \tag{3.14}$$

Since these are modes of an infinite number of harmonic oscillators, we postulate the following equal-time commutation relations, such that the above commutator becomes equal to $i\delta(x-y)$:

$$[a_n, a_m] = n\delta_{n+m,0} \qquad [\bar{a}_n, \bar{a}_m] = n\delta_{n+m,0} \qquad [\phi_0, \Pi_0] = i \tag{3.15}$$

where all the others are set to zero. Given this, the commutator of ϕ and Π reduces to:

$$[\phi(z,\bar{z}), \Pi(w,\bar{w})] = \frac{2\pi i g}{L}\left(2[\phi_0, \alpha] + \frac{1}{4\pi g}\sum_{n\neq 0}\left\{\left(\frac{w}{z}\right)^n + \left(\frac{\bar{w}}{\bar{z}}\right)^n\right\}\right) \tag{3.16}$$

Now notice that:

$$\begin{aligned} \frac{w}{z} = e^{2\pi i(y-x)/L} \implies \sum_{n\neq 0}\left(\frac{w}{z}\right)^n &= L\delta(x-y) - 1 \\ \implies [\phi(z,\bar{z}), \Pi(w,\bar{w})] &= \frac{4\pi i g}{L}[\phi_0, \alpha] + i\delta(x-y) - \frac{i}{L} \end{aligned} \tag{3.17}$$

For Eq. (3.16) to be the canonical commutation relation we infer that $[\phi_0, \alpha] = 1/4\pi g$. Note that in the entire space of modes, the only object which does not commute with ϕ_0 is the zero mode of the conjugate momentum Π_0. Therefore we conclude that imposing the canonical commutation relation constrains the α to be equal to $\Pi_0/4\pi g$.[1] Rewriting the Hamiltonian from Eq. (3.10) we get

$$H = \frac{1}{2gL}\Pi_0^2 + \frac{2\pi}{L}\sum_{n>0} a_{-n}a_n + \bar{a}_{-n}\bar{a}_n \tag{3.18}$$

One can now easily show that the mode operators a_{-n} and a_n, for $n > 0$, act as raising and lowering operators for the eigenstates of the Hamiltonian H. The anti-holomorphic follows through trivially.

From the mode expansion, it is obvious the $\varphi(z)$ is not a primary of the CFT, due to the logarithmic term. On the other hand, the first derivative of this field:

$$i\partial\varphi(z) = \frac{1}{\sqrt{2\pi g}}\sum_n a_n z^{-n-1}\,, \quad \text{where } a_0 = \bar{a}_0 = \frac{\Pi_0}{\sqrt{4\pi g}} \tag{3.19}$$

has the correct Laurent expansion for a $(h, \bar{h}) = (1, 0)$ primary.

Recall that the energy momentum tensor of this CFT can be expressed as:

$$T(z) = -2\pi g\{: \partial\phi(z)\partial\phi(z) :\} \tag{3.20}$$

[1]An interesting point to note here is that with this identification, the first two term of the mode expansion of $\phi(t,x)$ make physical sense. As we have already seen, the zero mode is a free particle and as such, should evolve as $\varphi_0(t) = \varphi_0 + \frac{\Pi_0}{m}t$, which are exactly the first two terms of Eq. (3.6).

where the dots $:\cdots:$ denote normal ordering. Using the mode expansion of $\varphi(z)$ we can write the energy-momentum tensor in terms of the modes a_n as:

$$T(z) = \frac{1}{2}\sum_{n,m} z^{-n-m-2}\{: a_n a_m :\} = \sum_n \left(\frac{1}{2}\sum_k \{: a_{n-k} a_k :\}\right) z^{-n-2} \tag{3.21}$$

where we have used a simple change of summation variable for the second equality. Thus we can identify the Virasoro modes with the modes of $\varphi(z)$ to get:

$$L_0 = \frac{1}{2}a_0^2 + \sum_{k>0}\{: a_{n-k} a_k :\} \quad \& \quad L_n = \frac{1}{2}\sum_k \{: a_{n-k} a_k :\} \quad \text{for } n \neq 0 \tag{3.22}$$

Using this we can finally conclude that the Hamiltonian of this CFT can just be written as:

$$H = \frac{2\pi}{L}\left(L_0 + \bar{L}_0\right) \tag{3.23}$$

which perfectly matches our expectations from radial quantisation. One can now use the TT OPE or the expression of the Virasoro modes in terms of a_n to compute the central charge of the CFT. It turns out that this is a left-right symmetric CFT with central charge equal to unity, i.e. $c = \bar{c} = 1$.

The partition function

We will now compute the modular invariant partition function of this $c = 1$ CFT. The partition function of the full theory is given by the trace:

$$\mathcal{Z} = Tr_{\mathcal{H}} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \tag{3.24}$$

where we have used the fact that the Fock space of this theory is divided into 4 independent regions, one each generated by $\varphi_0, \Pi_0, a_n, \bar{a}_n$. We will set $L = 1$ for convenience, as it can be achieved by a scaling of the spatial direction of the cylinder. The trace over the full Hilbert space can be expressed as:

$$\mathcal{Z} = q^{-\frac{1}{24}} \bar{q}^{-\frac{1}{24}} \int d\phi_0 \int d\Pi_0 \sum_m \sum_{\bar{m}} \langle\phi_0| \langle\Pi_0| \langle m| \langle\bar{m}| q^{L_0} \bar{q}^{\bar{L}_0} |\phi_0\rangle |\Pi_0\rangle |m\rangle |\bar{m}\rangle \tag{3.25}$$

Since ϕ_0 is a free particle, both itself and its conjugate momentum take continuous values and hence we integrate over these spaces. We will now simplify the RHS by focusing on a single integral/sum at a time while focusing on only the relevant contributions from $L_0 + \bar{L}_0$.

The trace over the φ_0 Hilbert space is trivial, as the Hamiltonian does not depend on φ_0. The relevant integrand for the integral over Π_0 is the zero mode contribution from both L_0 and $\bar{L}_0$, because of Eq. (3.19). The integral to be performed can be written as:

$$\int d\Pi_0 \langle \Pi_0 | \, q^{\frac{1}{2}a_0^2} \bar{q}^{\frac{1}{2}a_0^2} \, |\Pi_0\rangle = \int d\Pi_0 \langle \Pi_0 | \, e^{\pi i(\tau - \bar{\tau})a_0^2} \, |\Pi_0\rangle = \int d\Pi_0 e^{-\frac{\text{Im}\tau}{2g}\Pi_0^2} = \sqrt{\frac{2\pi g}{\text{Im}\tau}} \tag{3.26}$$

Next we will compute the trace over the $\mathcal{H}_a$ Hilbert space, which is represented in Eq. (3.25) as the sum over m. The relevant summand in this case is the contribution from the modes a_n, for $n \neq 0$ coming from L_0. Therefore the relevant sum we have is

$$\sum_m \langle m | \, q^{\sum_{k>0} a_{-k} a_k} \, |m\rangle = \sum_m \prod_{k=1}^{\infty} \langle m | \, q^{a_{-k} a_k} \, |m\rangle = \sum_m \prod_{k=1}^{\infty} \langle m | \, q^{n N_n} \, |m\rangle \tag{3.27}$$

where $N_n = \frac{a_{-n}}{\sqrt{n}} \frac{a_n}{\sqrt{n}}$ for $n > 0$ is the number operator, analogous to the harmonic oscillator. Now like the usual Fock space, the Hilbert space $\mathcal{H}_a$ will have states which have no excitation, one excitation and so on, therefore one can write

$$\sum_m \prod_{k=1}^{\infty} \langle m | \, q^{n N_n} \, |m\rangle = \prod_{k=1}^{\infty} \left(q^{0.k} + q^{1.k} + q^{2.k} + \cdots \right) = \prod_{k=1}^{\infty} \frac{1}{1 - q^k} \tag{3.28}$$

Similarly, the contribution from the $\mathcal{H}_{\bar{a}}$ Hilbert space, represented by the sum over $\bar{m}$, will give:

$$\sum_{\bar{m}} \langle \bar{m} | \, \bar{q}^{\sum_{\bar{k}>0} \bar{a}_{-\bar{k}} \bar{a}_{\bar{k}}} \, |\bar{m}\rangle = \prod_{\bar{k}=1}^{\infty} \frac{1}{1 - \bar{q}^{\bar{k}}} \tag{3.29}$$

Assembling everything together, we see that the full partition function is:

$$\mathcal{Z}_{\text{bos}}(\tau, \bar{\tau}) = \frac{\sqrt{2\pi g}}{(\text{Im}\tau)^{\frac{1}{2}}} q^{-\frac{1}{24}} \bar{q}^{-\frac{1}{24}} \prod_{k=1}^{\infty} \frac{1}{1 - q^k} \prod_{\bar{k}=1}^{\infty} \frac{1}{1 - \bar{q}^{\bar{k}}} = \frac{1}{(\text{Im}\tau)^{\frac{1}{2}}} \frac{1}{|\eta(\tau)|^2} \tag{3.30}$$

where we have ignored $\sqrt{2\pi g}$ term in the partition function, as we can ignore any constants in the partition function of any theory, and used the Dedekind eta function, defined as:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{k=1}^{\infty} (1 - q^k) \tag{3.31}$$

One can now easily check, using the well-known modular properties of the Dedekind η function, that this partition function is modular invariant, as it should be.

3.2 Compactified Boson Partition Function

We will now consider the case where we will restrict the free boson to a circle of radius R in the field space. This amounts to identifying the field ϕ with $\phi + 2\pi R$. This implies the following, more general boundary condition for the free boson:

$$\phi(t, x + L) = \phi(t, x) + 2\pi m R \;, \qquad m \in \mathbb{Z} \tag{3.32}$$

Notice that translating x by L is equivalent to taking $z \to e^{2\pi i} z$ and simultaneously $\bar{z} \to e^{-2\pi i}\bar{z}$. Under this transformation, mode expansion implies that the field $\phi(z, \bar{z})$ transforms as: [2]

$$\phi(z, \bar{z}) \to \phi(e^{2\pi i} z, e^{-2\pi i}\bar{z}) = \phi(z, \bar{z}) + 2\pi(\Pi_0 - \bar{\Pi}_0) \tag{3.33}$$

This implies that $\Pi_0 - \bar{\Pi}_0$ must be an integer multiple of the radius R. Without loss of generality, we can write:

$$\Pi_0 = \frac{mR}{2} + \widetilde{\Pi}_0 \;, \quad \bar{\Pi}_0 = -\frac{mR}{2} + \widetilde{\Pi}_0 \tag{3.34}$$

Note that $\widetilde{\Pi}_0$ contains information about the momentum of the zero mode (which is effectively a free particle). We know from quantum mechanics that the momentum eigenstate of a free particle is given as e^{ipx}. For this eigenstate to be well-defined on a circle of radius R, the momentum should be quantised in inverse radius, i.e. $p = \frac{n}{R}$, where $n \in \mathbb{Z}$. Therefore we can write:

$$\Pi_0 = \frac{mR}{2} + \frac{n}{R} \;, \quad \bar{\Pi}_0 = -\frac{mR}{2} + \frac{n}{R} \tag{3.35}$$

Substituting this back into the mode expansion and using the fact that $i\partial\varphi(z)$ (and its anti-holomorphic counterpart) are primary fields, we can identify a_0 with Π_0 (and $\bar{a}_0$ with $\bar{\Pi}_0$). Thus we can now write the zeroth Virasoro generators as:

$$L_0 = \frac{1}{2}a_0^2 + \sum_{n>0} a_{-n}a_n \;, \quad \bar{L}_0 = \frac{1}{2}\bar{a}_0^2 + \sum_{n>0} \bar{a}_{-n}\bar{a}_n \tag{3.36}$$

The integer m here measures the winding number of the boson, i.e. the number of times it has been wound around the field space circle, whereas n measures its momentum around the configuration space cylinder.

Let us now compute the partition function of the compactified boson. The trace over the $\mathcal{H}_a$ and $\mathcal{H}_{\bar{a}}$ Hilbert spaces will still be the same. Whereas the traces over ϕ_0 and Π_0 are no longer

[2] Note that we have set $g = 1/4\pi$ in this section for convenience.

integrals as they are now parameterised by two integers m, n, therefore the partition function of the compactified boson, $\mathcal{Z}_R$ can be expressed as:

$$\mathcal{Z}_R(\tau,\bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \sum_{n,m} q^{\frac{1}{2}\left(\frac{n}{R}+\frac{mR}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{n}{R}-\frac{mR}{2}\right)^2} \tag{3.37}$$

The summand can be simplified by expanding the squares and breaking up τ into its real and imaginary parameter:

$$\mathcal{Z}_R(\tau,\bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \sum_{m} e^{-\frac{\pi m^2 R^2}{2}\mathrm{Im}\tau} \sum_{n} e^{-\frac{2\pi n^2}{R^2}\mathrm{Im}\tau + 2\pi i n m\,\mathrm{Re}\tau} \tag{3.38}$$

Recall Poisson's resummation formula:

$$\sum_{n=-\infty}^{\infty} e^{-\pi a n^2 + 2\pi i n b} = a^{-\frac{1}{2}} \sum_{m=-\infty}^{\infty} e^{-\frac{\pi}{a}(m-b)^2} \tag{3.39}$$

In our case, $a = 2\mathrm{Im}\tau/R^2$ and $b = m\mathrm{Re}\tau$, therefore:

$$\mathcal{Z}_r(\tau,\bar{\tau}) = \frac{R}{\sqrt{2}} \frac{1}{|\eta(\tau)|^2} \sum_{m,m'} e^{-\frac{\pi R^2}{2\mathrm{Im}\tau}\left[m^2\mathrm{Im}\tau^2 + m'^2 + m^2\mathrm{Re}\tau^2 - 2mm'\mathrm{Re}\tau\right]} \tag{3.40}$$

It is trivial to show that $|m' - m\tau|^2 = m^2\mathrm{Im}\tau^2 + m'^2 + m^2\mathrm{Re}\tau^2 - 2mm'\mathrm{Re}\tau$ and hence the partition function can be expressed as:

$$\mathcal{Z}_R(\tau,\bar{\tau}) = \frac{R}{\sqrt{2}} \mathcal{Z}_{bos}(\tau,\bar{\tau}) \sum_{m,m'} e^{-\pi R^2 \frac{|m'-m\tau|^2}{2\mathrm{Im}\tau}} \tag{3.41}$$

where $\mathcal{Z}(\tau,\bar{\tau})$ is the partition of the free boson, Eq. (3.30). One can easily verify that this partition function is modular invariant as well.

3.3 Self Dual Boson as a Rational Conformal Field Theory

The partition function for the compactified boson has the following interesting duality, that is also known as the T duality in string theory [3]:

$$\mathcal{Z}_R(\tau,\bar{\tau}) = \mathcal{Z}_{\frac{2}{R}}(\tau,\bar{\tau}) \tag{3.42}$$

It is obvious that there exists a self dual point when $R^2 = 2$, let us investigate this special point a bit more deeply. The partition function at this value of R becomes:

$$\mathcal{Z}_{\sqrt{2}}(\tau,\bar{\tau}) = \frac{1}{|\eta(\tau)|^2}\sum_{n,m\in\mathbb{Z}} q^{\frac{1}{4}(n+m)^2}\bar{q}^{\frac{1}{4}(n-m)^2} = \frac{1}{|\eta(\tau)|^2}\sum_{\substack{k,l\in\mathbb{Z}\\ k-l\equiv 0(\text{mod } 2)}} q^{\frac{1}{4}k^2}\bar{q}^{\frac{1}{4}l^2} \tag{3.43}$$

where we have performed a simple change in the summation variable by defining $k = n+m, l = n-m$ which implies that $k - l = 2m$. Since the difference between k and l is an even number, it means that either both of them are even, or both of them are odd. We thus break the sum into these two sub parts.

$$\mathcal{Z}_{\sqrt{2}}(\tau,\bar{\tau}) = \frac{1}{|\eta(\tau)|^2}\sum_{\substack{k,l\in\mathbb{Z}\\ k-l\equiv 0(\text{mod } 2)\\ k=2a,\, l=2b}} q^{\frac{1}{4}k^2}\bar{q}^{\frac{1}{4}l^2} + \frac{1}{|\eta(\tau)|^2}\sum_{\substack{k,l\in\mathbb{Z}\\ k-l\equiv 1(\text{mod } 2)\\ k=2a+1,\, l=2b+1}} q^{\frac{1}{4}k^2}\bar{q}^{\frac{1}{4}l^2} \tag{3.44}$$

We now rewrite the sum with a and b as the summation variables instead of k and l to get:

$$\mathcal{Z}_{\sqrt{2}}(\tau,\bar{\tau}) = \frac{1}{|\eta(\tau)|^2}\sum_a q^{a^2}\sum_b q^{b^2} + \frac{1}{|\eta(\tau)|^2}\sum_a \bar{q}^{\left(a+\frac{1}{2}\right)^2}\sum_b \bar{q}^{\left(b+\frac{1}{2}\right)^2} \tag{3.45}$$

We have reduced the self dual partition function to a form where we write it using number theoretic objects called the Jacobi theta functions. The partition function can thus succinctly be written as:

$$\mathcal{Z}_{\sqrt{2}}(\tau,\bar{\tau}) = \frac{\theta_3(\tau)}{\eta(\tau)}\frac{\theta_3(\bar{\tau})}{\eta(\bar{\tau})} + \frac{\theta_2(\tau)}{\eta(\tau)}\frac{\theta_2(\bar{\tau})}{\eta(\bar{\tau})} \tag{3.46}$$

We first note that we have written the partition function of the self dual boson as the sum of two mod-squared quantities. This implies that the self dual boson is a rational CFT and there must be some underlying chiral algebra. The Virasoro algebra cannot be the full symmetry algebra of this theory because $c = 1$ for this CFT which implies that there are an infinite number of Virasoro primaries in this CFT. [3]

Recall that the identity character always has the ground state degeneracy equal to unity. We can therefore examine the $q \to 0$ limit of $\frac{\theta_3(\tau)}{\eta(\tau)}$ & $\frac{\theta_2(\tau)}{\eta(\tau)}$ to identify the identity and non-identity

[3] This can also be shown explicitly for the case at hand, i.e. self dual free boson.

characters:

$$\begin{aligned}
\chi_0(\tau) &= q^{-\frac{1}{24}} \frac{\sum_n q^{n^2}}{\prod_{k=1}^{\infty}(1-q^k)} = q^{-\frac{1}{24}}(1+3q+4q^2+\cdots) \\
\chi_1(\tau) &= q^{-\frac{1}{24}+\frac{1}{4}} \frac{\sum_n q^{n^2+n}}{\prod_{k=1}^{\infty}(1-q^k)} = q^{-\frac{1}{24}+\frac{1}{4}}(2+2q+6q^2+\cdots)
\end{aligned} \tag{3.47}$$

We can read-off the conformal dimension h of the non-identity primary from the leading exponent of χ_1 to be $\frac{1}{4}$.

Note that the degeneracy at the first level above vacuum is 3, which means that the CFT has 3 spin-1 currents, $J^a(z)$ where $a = 1, 2, 3$. Interestingly, the ground state degeneracy of χ_1 is 2 and hence the ground state of the non-identity primary has to be a doublet, which we will denote as $\left|\frac{1}{4}, i\right\rangle$, where $i = 1, 2$. From the mode expansion of $J^a(z)$, one can easily see that $J_0^a\left|0\right\rangle = 0$ for well-definedness of $J^a(z)\left|0\right\rangle$ as $z \to 0$. On the other hand, the zero mode J_0^a acts on $\left|\frac{1}{4}, i\right\rangle$ as a 2×2 matrix, i.e. as a doublet representation of the Lie group which forms the chiral algebra.

Let us now try to determine the chiral operator algebra in this CFT. Recall that we already know one of the three spin-1 currents: $J^3(z) = i\partial\varphi(z)$. It can be shown that the other two currents can be obtained using what are called as vertex operators of a CFT [7], defined as:

$$\mathcal{V}_\alpha(z) = \, : e^{i\alpha\varphi(z)} : \tag{3.48}$$

Using these, one can define the following objects:

$$J^1(z) = \frac{1}{2}\left(: e^{i\sqrt{2}\varphi(z)} : + : e^{-i\sqrt{2}\varphi(z)} :\right) \quad \& \quad J^2(z) = \frac{1}{2}\left(: e^{i\sqrt{2}\varphi(z)} : - : e^{-i\sqrt{2}\varphi(z)} :\right) \tag{3.49}$$

One can now compute the OPE of the three objects $J^a(z)$ to be:

$$J^a(z)J^b(w) = \frac{\delta^{ab}}{(z-w)^2} + \frac{i\epsilon^{abc}}{z-w}J^c(z) + \cdots \tag{3.50}$$

From this one can easily compute the algebra satisfied by the modes of the spin-1 currents to be:

$$\left[J_n^a, J_m^b\right] = i\epsilon^{abc}J_{n+m}^c + n\delta^{ab}\delta_{n+m,0} \tag{3.51}$$

which is the $SU(2)$ Kac-Moody algebra at level 1. Thus we conclude that the self dual boson CFT is the $SU(2)_1$ WZW model, which is a two-character CFT and corresponds to the $k = 0$ element of the A_1 series of quasi-characters in Eq. (2.15).

Chapter 4

Contour Integrals and the Modular $\mathcal{S}$-Matrix

In this original work based on "*Contour Integrals and the Modular $\mathcal{S}$-Matrix*" by Sunil Mukhi, Rahul Poddar, and **Palash Singh** available on arXiv:1912.04298, we investigate a conjecture [28] to describe the characters of a large class of RCFT in terms of contour integrals of Feigin-Fuchs type [29, 30]. Due to constraints on the size of this thesis, we will only provide a brief summary of the conjecture and the results. A more comprehensive and detailed treatment can be found in the original paper [1] or my co-author, Rahul Poddar's master's thesis.

We have reviewed the MLDE approach to the classification of admissible characters for RCFT along with its virtues. However, one drawback which the MLDE approach suffers from is that, because we solve the MLDE as a q-series, it is difficult to actually compute the modular transformation matrix $\mathcal{S}_{ij}$. Computations of this matrix, till now, have relied on being able to express the characters in closed form, in terms of special functions [31, 32, 33, 34]. Using contour manipulations on the contour-integral representation of RCFT characters, we provide a simple and elegant sum-over-paths algorithm to determine the modular $\mathcal{S}$-matrix for arbitrary number of characters. We also use the Verlinde formula [14], in conjunction with the modular $\mathcal{S}$-matrix, to obtain interesting constraints on the exponents of RCFT.

4.1 Contour-Integral Representation of Characters

The conjecture is inspired from the use of Feigin-Fuchs contour integrals to describe the sphere four-point functions of minimal models by [29, 30]. The idea of [28] was to replace the cross-ratio that appears in the sphere four-point function by the modular λ-function to describe characters of a conformal field theory. One of the special features of this contour integral representation is that it permits explicit calculation of the modular $\mathcal{T}$ and $\mathcal{S}$ matrices. Importantly, this does not rely on more intricate methods like expressing the characters as special functions, to calculate these matrices. The modular λ-function is defined as:

$$\lambda(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)} = 16q^{\frac{1}{2}}\left(1 - 8q^{\frac{1}{2}} + 44q + \cdots\right) \tag{4.1}$$

These contour integrals depend on an integer n and two rational parameters a, b. A given set of these parameters describe a family consisting of a total of $p = n + 1$ contour integrals. The proposal is that the characters of an $\ell = 0$ RCFT, as well as valid quasi-characters, can be expressed as the following set of integrals for all integers A such that $0 \leq A \leq n$:

$$\begin{aligned} J_A(a, b, \lambda) \equiv N_A\left(\lambda(1-\lambda)\right)^{\alpha} \int_1^{\infty} dt_n \cdots \int_1^{\infty} dt_{A+2} \int_1^{\infty} dt_{A+1} \int_0^{\lambda} dt_A \cdots \int_0^{\lambda} dt_2 \int_0^{\lambda} dt_1 \\ \times \prod_{i=1}^{A}\Big[t_i(1-t_i)(\lambda - t_i)\Big]^a \prod_{i=A+1}^{n}\Big[t_i(t_i - 1)(t_i - \lambda)\Big]^a \prod_{1 \leq k < i \leq n}(t_i - t_k)^{2\rho} \end{aligned} \tag{4.2}$$

where α is a function of a, ρ, n, and N_A are normalisation constants to be determined later. To avoid any overlapping contours, the individual contours are defined to have their imaginary parts ordered such that $\mathrm{Im}(t_i) > \mathrm{Im}(t_k) > 0$ for $i > k$. A very crucial feature of these contour integrals is that the prefactor and the integrand is the same for all the candidate characters, apart from normalisations. They differ only by the contours the integrands are being integrated over.

In terms of the modular λ-function, the modular $\mathcal{S}$ transformations can be expressed as $\lambda \to 1 - \lambda$. Since the prefactor remains invariant under this transformation, we find it convenient to define the "pure" integral $\widehat{J}_A$ to be the contour integrals J_A without the normalisation and the λ dependent prefactor. The modular $\mathcal{S}$-matrix can be defined in terms of integrals J_A as follows:

$$J_A(\lambda) = \sum_B \mathcal{S}_{AB}\ J_B(1-\lambda) \tag{4.3}$$

One can analogously define the auxiliary matrix $\widehat{\mathcal{S}}$, defined using the pure integrals $\widehat{J}_A$. This matrix is independent of the normalisation and is related to the modular $\mathcal{S}$-matrix as: $\mathcal{S} = N\widehat{\mathcal{S}}N^{-1}$ where $N = $diag$(N_A)$.

The modular $\mathcal{T}$-matrix can easily be computed by just looking at the leading power of q in the integrals [28]. It is easy to see that the $\tau \to i\infty$ limit is the same as $\lambda \to 0$ or $q \to 0$ limit and we have $J_A \sim q^{\frac{1}{2}(\alpha+\Delta_A)}$, where:

$$\alpha = \frac{1}{3}\left(-n_1(1+3a) - \rho\, n_1(n_1-1)\right)\ , \quad \Delta_A = A(1+2a) + \rho\, A(A-1) \tag{4.4}$$

This allows us to identify the CFT exponents with the parameters of the contour integrals:

$$c = -12\alpha\ , \quad h_i = \frac{1}{2}\Delta_A \tag{4.5}$$

where the index A takes the same p values as the index i. The Riemann-Roch equation Eq. (2.11) now implies that these contour integrals indeed describe characters of an $\ell = 0$ RCFT.

The fact that this proposal works for known theories is quite surprising. This is because a generic p-character CFT has discrete data consisting of the central charge and $p-1$ conformal dimensions, h_i. However, the contour integral manages to describe this data by using only an integer n and two real parameters a, ρ. There is no a priori reason why any choice of a, ρ, n should be able to describe p CFT exponents of any given theory.

It has been shown in [28] that for RCFT with ≤ 5 characters, the number of free parameters ≤ 4, and hence it is always possible to fit the CFT parameters with the parameters in J_A.[1] Since the CFT exponents for an $\ell = 0$ CFT with ≤ 5 characters completely determine the MLDE in the MMS classification, this proves that the contour integrals are the characters for these cases. Known examples with ≥ 6 characters for which this parameter matching is possible have been listed in [28, 1]. These include: (i) $SU(2)_k$ WZW model for all positive integer k, (ii) All $c < 1$ minimal models (unitary as well as non-unitary), (iii) $SU(N)_1$ WZW models for all N, and (iv) $SU(3)_2$ WZW model.

4.2 Computation of the Modular $\mathcal{S}$-Matrix

The $\hat{\mathcal{S}}$ matrix can be derived by manipulating the contours in a specific way, that involves the "unfolding" of the contours to express $\widehat{J}_A(\lambda)$ as a linear combination of $\widehat{J}_B(1-\lambda)$. The unfolding of a contour involves the following: (i) Deformation of the contour in the counter-clockwise direction such that the original contour is a part of a larger closed contour which encompasses the upper half plane. The integral over this larger contour is equal to zero since there are no singular points in the region. (ii) Expressing the original integral, which is over one of regions enclosed by the four singular points $0, \lambda, 1, \infty$ on the real line, as a sum of integrals over the remaining

[1]The actual proposal involves a more general class of integrals which have another set of variables identical to t_i and hence another integer n', in addition to n. Thus in total there are four parameters which can be matched to the CFT exponents.

three regions. This is because the integral over the contour tracing the infinity can be set to zero by a suitable choice of the parameters, which can later by continued analytically. (iii) Repeating this for the deformation of the contour in the clockwise direction and then taking a suitable linear combination of these two deformations of the original contour to cancel out the integral over one of the regions. If the original integral was over $(1, \infty)$, then we remove the $(\lambda, 0)$ piece, and vice versa. The integrals over the remaining two regions, $(0, -\infty)$ and $(1, \lambda)$ correspond to $\widehat{J}_A(1-\lambda)$ for $A = 0, 1$, modulo an overall sign.

Now note that the $\widehat{J}_A$ integral has A contour running from 0 to λ and $n - A$ contours running from 1 to ∞, which adds up to a total of $p = n + 1$ contours. The result of unfolding contour is that the original contour disappears and is replaced by two contours, one in each of the segments $(1, \lambda)$ and $(0, -\infty)$. If we carry this out $s < A$, number of times, sequentially on the A contours in the $(0, \lambda)$ region, we will be left with only $A - s$ of these contours, while the remaining s will be distributed in the regions $(0, -\infty)$ and $(1, \lambda)$. We introduce a positive integer m such that there are $\frac{s-m}{2}$ contours in th $(0, -\infty)$ region and $\frac{s+m}{2}$ in the $(1, \lambda)$ region. Clearly m takes values from the set $\{-s, -s+2, \ldots, s-2, s\}$. Thus for a fixed set of integers (s, m) we will denote the contour configuration as $\mathcal{V}_{s,m}$.

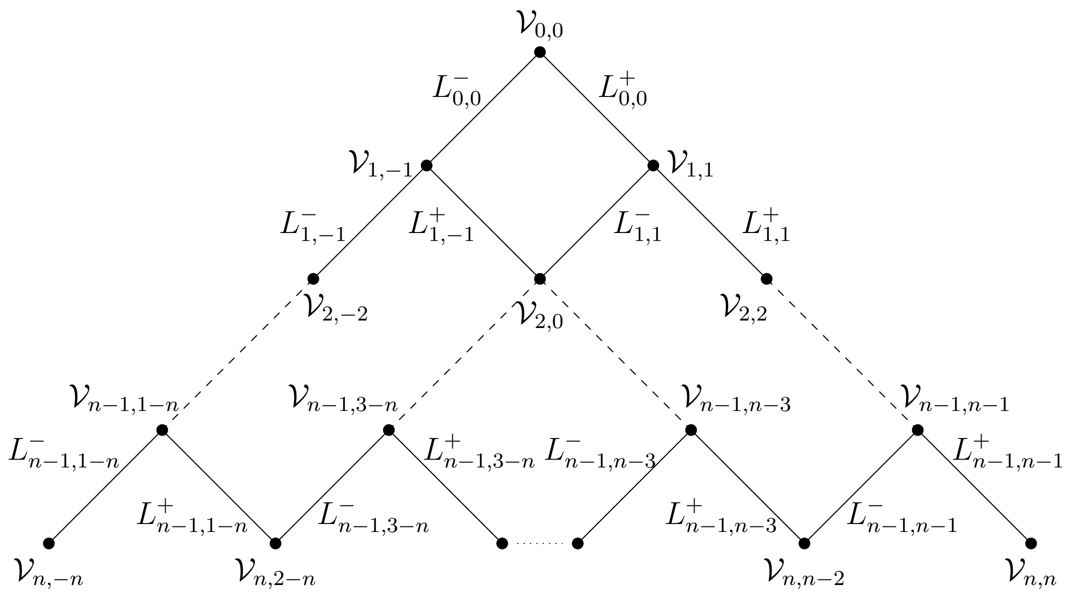
The unfolding of a single contour in the $(\lambda, 0)$ region in $\mathcal{V}_{s,m}$, with $s < A$, leads to $\mathcal{V}_{s+1,m-1}$ or $\mathcal{V}_{s+1,m+1}$, depending on whether the unfolded contour is replaced by a contour in the $(0, -\infty)$ region or the $(\lambda, 1)$ region, respectively. For $s \geq A$, there are no $0 \to \lambda$ contours left and hence one has to unfold the $1 \to \infty$ contours. In terms of these, the recursion relation between $\mathcal{V}_{s,m}$ and $\mathcal{V}_{s+1,m\pm1}$ has been derived in [1] and can be expressed as:

$$\begin{tikzcd} & \mathcal{V}_{s,m} \arrow[dl, dash, "L^-_{s,m}"'] \arrow[dr, dash, "L^+_{s,m}"] & \\ \mathcal{V}_{s+1,m-1} & & \mathcal{V}_{s+1,m+1} \end{tikzcd} \qquad \equiv \mathcal{V}_{s,m} = L^-_{s,m}\mathcal{V}_{s+1,m-1} + L^+_{s,m}\mathcal{V}_{s+1,m+1} \tag{4.6}$$

where we have also represented the unfolding procedure diagrammatically. The coefficients $L^\pm_{s,m}$ can be expressed as:

$$\begin{aligned} s < A: \quad & L^\pm_{s,m} = \frac{e^{i\pi\left(A - \frac{3s\pm m}{2} - 1\right)\rho}\ \mathfrak{s}(2a \mp a + (n + A - \left(\frac{s\pm m}{2}\right) - 1 \pm 1)\rho)}{\mathfrak{s}(2a + (n+m-1)\rho)} \\ s \geq A: \quad & L^\pm_{s,m} = \pm\frac{e^{i\pi\left(n - \frac{3s\pm m}{2} - 1\right)\rho}\ \mathfrak{s}(a + \left(\frac{s\pm m}{2}\right)\rho)}{\mathfrak{s}(2a + (n+m-1)\rho)} \end{aligned} \tag{4.7}$$

Now to compute the $\hat{\mathcal{S}}$ matrix for a $p = n+1$ character CFT, we need to perform n such unfolding. Identifying the integral $\widehat{J}_A(\lambda)$ with $\mathcal{V}_{0,0}$, the n unfolding can be represented as the following graph:



(4.8)

Notice that the vertices in the final row are the $\widehat{J}_A(1-\lambda)$ contour integrals with the identification:

$$\mathcal{V}_{n,2A-n} = (-1)^n \hat{J}_A(1-\lambda) \tag{4.9}$$

where the sign on the RHS comes from the change of variables $t_i \to 1 - t_i$. Thus the algorithm to compute the $\hat{\mathcal{S}}$ matrix is as follows:

1. Identify $J_A(\lambda)$ with $\mathcal{V}_{0,0}$, to compute the row of the $\hat{\mathcal{S}}$-matrix corresponding to $\widehat{J}_A(\lambda)$.

2. To compute the element $\hat{\mathcal{S}}_{A\,B}$, trace a path in the graph (4.8) starting from $\mathcal{V}_{0,0}$ to $\mathcal{V}_{n,2B-n}$ while successively multiplying the contribution from each link on the path.

3. Sum over the contributions from all such paths to obtain $\hat{\mathcal{S}}_{A\,B}$.

The normalisations N_A are defined by requiring that the lowest power of q in the contour integrals is the ground-state degeneracy D_A of the corresponding character. Thus it is sufficient to consider only the leading singular behaviour, in terms of q, of the contour integral to compute the normalisations N_A. It is easy to show that the integral obtained by considering only the leading power of λ can be expressed in terms of Selberg integrals, $S_m(\alpha, \beta, \gamma)$.[2] Using this, the

[2]Look at [35, 36] for the definition and properties of the Selberg integral.

normalisations N_A can be expressed as:

$$N_A = \frac{16^{-(\alpha+\Delta_A)}\, A!(n-A)!\, D_A}{\Theta_A(\rho)\, S_A\big(a+1, a+1, \rho\big)\ S_{n-A}\big(-1-3a-2\rho(n-1), a+1, \rho\big)} \tag{4.10}$$

Note that given the central charge and conformal dimensions of a CFT, all the terms in N_A can be explicitly calculated except the ground-state degeneracy D_A.

4.3 Conclusion and Results

Given the algorithm to compute the auxiliary $\hat{\mathcal{S}}$ matrix and the normalisations, one can compute the full modular $\mathcal{S}$-matrix. The only remaining detail are the degeneracies D_A which (among various other ways listed in [1]) can be computed by exploiting the symmetries and properties of the modular $\mathcal{S}$-matrix: $\mathcal{S}^2 = \mathbb{1}, \mathcal{S}^\dagger M \mathcal{S} = M$ where M =diag(M_i). General expressions for these in terms of a, ρ were obtained and explicitly listed for the two-, three-character case in [1].

We also used the Verlinde formula to compute the fusion rule coefficients after constructing the modular $\mathcal{S}$-matrix from our algorithm. Requiring that these coefficients are equal to 0 or 1, we reproduce the entire classification of $\ell = 0$ two-character RCFT, including quasi-characters [20]. This unfortunately proves to be somewhat intractable due to technical difficulties for three-character onward. However, it might be possible to overcome these technical issues with better computation or ingenuous approaches, since the procedure works in principle.

A technical limitation of this work is that we considered integrals with only a set of variables t_i rather than the most general case that has two sets t_i, τ_j. However, it should be possible to reproduce the current work for the general case, but was not done due to time constraints.

Though the conjecture of [28] is still unproven in general, we extend it to coset theories with three-, four-characters. We also add some evidence to the conjecture by explicitly matching the modular $\mathcal{S}$-matrix, computed using this conjecture, for the $SU(2)_k$ WZW models for all $k \leq 18$ with the known results. In addition to describing RCFT, we have also seen that the set of contour integrals can also describe quasi-characters. The contour integral representation, coupled with the MLDE approach and the study of the q-series, provides a useful way to approach the classification problem, as will be demonstrated in the next chapter which follows the the original work [2].

Chapter 5

Rational CFT With Three Characters: The Quasi-Character Approach

In this original work based on "*Rational CFT With Three Characters: The Quasi-Character Approach*" by Sunil Mukhi, Rahul Poddar, **Palash Singh** available on arXiv:2002.01949, we will explore the classification of three-character CFT using the modular linear differential equation approach. Some work on this subject is present in [16, 37] and the current understanding is listed in [22]. This chapter will review the original work [2] in detail.

We will conjecture several infinite families of quasi-characters and show in examples that these can be linearly combined to generate admissible characters with higher Wronskian index, ℓ, similar to how it was done in the two character case by [20]. We will also match some of these admissible characters to actual CFT constructed using Kervaire lattices [38, 39], following the methods of [21]. The original work done in this chapter crucially depends on the novel coset construction of [19]. We will see many parallels as well as stark contrasts with the two-character case along the way. As we will see, this work will crucially depend on the results of the previous sections, as we will use the modular $\mathcal{S}$ matrix obtained from the contour integral representation [1], to understand the modular properties of the conjectured quasi-characters.

5.1 Quasi-Characters in Order 3

The case of three-character CFT gives an infinite number of admissible characters at $\ell = 0$, in contrast with the finite number of CFT in the two-character case. Before this work, little was known about three-character CFT with $\ell > 0$, the starting point of this work was to look for $\ell = 0$

quasi-characters in order 3, i.e. quasi-character solutions in the third order $\ell = 0$ MLDE:

$$\left(\mathcal{D}^3 + \mu_1 E_4 \mathcal{D} + \mu_2 E_6\right) \chi(\tau) = 0 \tag{5.1}$$

Before going ahead, we would like to recall the Riemann-Roch equation for the three-character case:

$$\sum_{i=0}^{2} \alpha_i = -\frac{c}{8} + h_1 + h_2 = \frac{1}{2} - \frac{\ell}{6} \tag{5.2}$$

Fusion classes

Recall that the quasi-characters in order 2 fell into 4 distinct fusion classes. Similarly the quasi-characters of order 3 known so far, fall into 11 distinct fusion classes. It is possible, in general, that the number of primaries in a CFT are more than the number of character. This happens, for example, when the primary is complex, in which case both the primary and its complex conjugate give rise to the same character and we count such cases as non-unit multiplicities. In the three-character case, we have come across cases with three, four, five, or nine primary fields.

Let us define some notation for the fusion classes in order 3. The fusion algebra of a CFT having one or more self-conjugate fields, i.e. $\phi \times \phi = \mathbb{1}$ with n non-trivial primary fields will be denoted as $\mathcal{A}_n^{(m)}$. On the other hand, if there are mutually conjugate fields, i.e. $\phi \times \bar{\phi} = \mathbb{1}$ in a fusion algebra, it will be denoted as $\mathcal{B}_n^{(m)}$. The superscript (m) labels different fusion classes with the same number of primaries.

For three-character CFT with unit multiplicity, i.e. three primaries, the relevant fusion classes are $\mathcal{A}_2^{(m)}$. For those with one conjugate pair, the relevant fusion class will be $\mathcal{B}_3^{(m)}$. Similarly for those with two conjugate pairs, the relevant fusion class will be $\mathcal{B}_4^{(m)}$. The case with nine primaries is the $A_{2,1} \times A_{2,1}$ WZW model, as the $A_{2,1}$ theory has three primary fields $\mathbb{1}, \mathbf{3}, \bar{\mathbf{3}}$, where **3,** are conjugate pairs. Thus the tensor product has the following 9 primaries:

$$(\mathbb{1},\mathbb{1}), (\mathbb{1},\mathbf{3}), (\mathbb{1},\bar{\mathbf{3}}), (\mathbf{3},\mathbb{1}), (\mathbf{3},\mathbf{3}), (\mathbf{3},\bar{\mathbf{3}}), (\bar{\mathbf{3}},\mathbb{1}), (\bar{\mathbf{3}},\mathbf{3}), (\bar{\mathbf{3}},\bar{\mathbf{3}}) \tag{5.3}$$

The fusion rules for this theory can be computed by using the rule $(\phi_1, \phi_2) \times (\phi_1', \phi_2') = (\phi_1 \times \phi_1', \phi_2 \times \phi_2')$ together with the fusion rules for $A_{2,1}$: $\mathbf{3} \times \bar{\mathbf{3}} = \mathbb{1}, \mathbf{3} \times \mathbf{3} = \bar{\mathbf{3}}$. We will label this fusion class as $\mathcal{B}_8^{(1)}$.

To compute the fusion class of a given set of quasi-characters, one computes the fusion rule coefficients by using the Verlinde formula Eq. (1.31), given that we know the modular $\mathcal{S}$-matrix. Note that given the values of the central charge and the conformal dimensions, one can easily use the procedure described in the previous section or 1912.04298 to compute the modular $\mathcal{S}$-matrix. Recall that the Verlinde formula works only for unitary $\mathcal{S}$-matrix, i.e. for CFT with unit multiplicity. A general procedure to apply this to cases with non-unit multiplicity was given in [16] which expanded the modular $\mathcal{S}$-matrix to a larger matrix whose dimension is equal to the total number of primaries. This expanded matrix diagonalises the fusion rules and hence can be used in the Verlinde formula. We shall not review this procedure in the current work, but it was used to assign fusion classes to the conjectured quasi-character families.

Quasi-characters

Unlike the two-character case, where results from mathematical literature were borrowed to discover the quasi-character families for the $\ell = 0$ MLDE, no such result exists for the three-character case. We therefore use a completely different approach, based on the novel coset construction of [19]. A series of 15 coset pairs of three-character CFT were presented in [19] whose central charges add up to 24. The first theory of each such pair is a three-character WZW model, while the second one is a coset of one of the meromorphic theories, classified by Schellekens [25], by the first one.

In order 3, the coset of an $\ell = 0$ three-character CFT with a $c = 24$ meromorphic theory gives another $\ell = 0$ three-character CFT. This is in stark contrast with the two-character case, where the coset theory has $\ell = 2$. The novel coset relation now implies that the modular transformation of the theory and its coset dual are the same, and thus it is reasonable to guess that a three-character theory and its coset dual lie in a common family of quasi-characters. One can now linearly extrapolate from the central charge and conformal dimensions of these two theories and check if we obtain quasi-characters. This educated guess turns out to be correct, as we have verified in a large number of examples to quite high orders in the q-expansion. One can use this trick on all known three-character CFT and generate the quasi-character series listed in Table 5.1.

The first seven entries in Table 5.1 did not appear in [19] but three of them were found in [18], and the remaining four were guessed based on the logic mentioned above. The entries in the first column label each row and are well-known minimal or WZW models, with which we started the coset construction. These CFTs correspond to $k = 0$ in their own row and the $k = 1$ entry corresponds to their coset pair. We have then extrapolated to other values of k (both positive and negative), and found that in most cases, there are always some negative coefficients, thus these do give quasi-characters.

The previous chapter reported the algorithm we developed to compute the modular $\mathcal{S}$ matrix given c, h_1, h_2. Since quasi-characters are solutions to the same MLDE as admissible characters, one can similarly construct the modular $\mathcal{S}$-matrix. This can be used to compute the fusion rule

Label	c	h_1	h_2	Fusion Class	Remark
$\mathcal{M}_{2,7}$	$\frac{304}{7}k - \frac{68}{7}$	$\frac{20}{7}k - \frac{3}{7}$	$\frac{9}{7}k - \frac{2}{7}$	$\mathcal{A}_2^{(3)}$	$k \neq 4$ mod 7
$\mathcal{M}_{3,4}$	$23k + \frac{1}{2}$	$\frac{15}{8}k + \frac{1}{16}$	$k + \frac{1}{2}$	$\mathcal{A}_2^{(1)}$	
$\mathcal{M}_{2,5} \otimes \mathcal{M}_{2,5}$	$\frac{208}{5}k - \frac{44}{5}$	$\frac{14}{5}k - \frac{2}{5}$	$\frac{12}{5}k - \frac{1}{5}$	$\mathcal{A}_3^{(4)}$	$k \neq 3$ mod 5
$G_{2,1} \otimes G_{2,1}$	$\frac{64}{5}k + \frac{28}{5}$	$\frac{6}{5}k + \frac{2}{5}$	$\frac{2}{5}k + \frac{4}{5}$	$\mathcal{A}_3^{(4)}$	$k \neq 3$ mod 5
$F_{4,1} \otimes F_{4,1}$	$\frac{16}{5}k + \frac{52}{5}$	$\frac{4}{5}k + \frac{3}{5}$	$-\frac{2}{5}k + \frac{6}{5}$	$\mathcal{A}_3^{(4)}$	$k \neq 3$ mod 5
$E_{7,1} \otimes E_{7,1}$	$-4k + 14$	$\frac{1}{2}k + \frac{3}{4}$	$-k + \frac{3}{2}$	$\mathcal{A}_3^{(1)}$	
$A_{2,1} \otimes A_{2,1}$	$16k + 4$	$\frac{4}{3}k + \frac{1}{3}$	$\frac{2}{3}k + \frac{2}{3}$	$\mathcal{B}_8^{(1)}$	$k \neq 2$ mod 3
$A_{1,2}$	$21k + \frac{3}{2}$	$\frac{13}{8}k + \frac{3}{16}$	$k + \frac{1}{2}$	$\mathcal{A}_2^{(1)}$	
$C_{2,1}$	$19k + \frac{5}{2}$	$\frac{11}{8}k + \frac{5}{16}$	$k + \frac{1}{2}$	$\mathcal{A}_2^{(1)}$	
$A_{3,1}$	$18k + 3$	$\frac{5}{4}k + \frac{3}{8}$	$k + \frac{1}{2}$	$\mathcal{B}_3^{(4)}$	
$B_{3,1}$	$17k + \frac{7}{2}$	$\frac{9}{8}k + \frac{7}{16}$	$k + \frac{1}{2}$	$\mathcal{A}_2^{(1)}$	
$A_{4,1}$	$16k + 4$	$\frac{6}{5}k + \frac{2}{5}$	$\frac{4}{5}k + \frac{3}{5}$	$\mathcal{B}_4^{(1)}$	$k \neq 3$ mod 5
$B_{4,1}$	$15k + \frac{9}{2}$	$\frac{7}{8}k + \frac{9}{16}$	$k + \frac{1}{2}$	$\mathcal{A}_2^{(1)}$	
$D_{5,1}$	$14k + 5$	$\frac{3}{4}k + \frac{5}{8}$	$k + \frac{1}{2}$	$\mathcal{B}_3^{(4)}$	
$B_{5,1}$	$13k + \frac{11}{2}$	$\frac{5}{8}k + \frac{11}{16}$	$k + \frac{1}{2}$	$\mathcal{A}_2^{(1)}$	
$D_{6,1}$	$12k + 6$	$\frac{1}{2}k + \frac{3}{4}$	$k + \frac{1}{2}$	$\mathcal{A}_3^{(1)}$	
$B_{6,1}$	$11k + \frac{13}{2}$	$\frac{3}{8}k + \frac{13}{16}$	$k + \frac{1}{2}$	$\mathcal{A}_2^{(1)}$	
$D_{7,1}$	$10k + 7$	$\frac{1}{4}k + \frac{7}{8}$	$k + \frac{1}{2}$	$\mathcal{B}_3^{(4)}$	
$B_{8,1}$	$7k + \frac{17}{2}$	$-\frac{1}{8}k + \frac{17}{16}$	$k + \frac{1}{2}$	$\mathcal{A}_2^{(1)}$	
$D_{9,1}$	$6k + 9$	$-\frac{1}{4}k + \frac{9}{8}$	$k + \frac{1}{2}$	$\mathcal{B}_3^{(4)}$	
$D_{10,1}$	$4k + 10$	$\frac{1}{4}k + \frac{7}{8}$	$k + \frac{1}{2}$	$\mathcal{A}_3^{(1)}$	

Table 5.1: Quasi-character series from coset pairs

coefficients by employing the Verlinde formula. One finds that the quasi-characters lie in the same fusion class as the minimal/WZW model that was used to construct it. Note that the even though the fusion class is the same, the modular $\mathcal{S}$-matrix varies, albeit periodically, in a quasi-character family. The exceptions to this are the $B_{r,1}$ and $D_{r,1}$ series where the modular $\mathcal{S}$-matrix was actually the same for all members.

We will now extend the results of the previous table and present the full list of quasi-characters we have. To begin with, we noticed that the disparate models like the Ising model $\mathcal{M}_{3,4}$ and $A_{1,2}$ WZW model fall into the same fusion class as $B_{r,1}$ WZW model, and in fact behave as the $r = 0, 1$

entries of the latter one, respectively. More importantly, we found that the $B_{r,1}$ and $D_{r,1}$ WZW models could be extended to *all positive* values of r. For sufficiently large r though, the central charge $c > 24$ and there is no longer a coset CFT in the family and one instead gets a quasi-character. Similarly, on extending r in the above two cases to negative values, we get quasi-characters.

Interestingly, we found *additional* quasi-characters by considering half integer values of k for two of the families. However in the remaining examples we did not find quasi-characters for any fractional value of k. Thus our final list of families of quasi-characters is presented in the table 5.2.

Fusion Class	Theories	c	h_1	h_2	Remark
$\mathcal{A}_2^{(3)}$	$\mathcal{M}_{2,7}$	$\frac{304}{7}k - \frac{68}{7}$	$\frac{20}{7}k - \frac{3}{7}$	$\frac{18}{7}k - \frac{2}{7}$	$k \neq 4 \bmod 7$
	$\mathcal{M}_{2,5} \otimes \mathcal{M}_{2,5}$	$\frac{208}{5}k - \frac{44}{5}$	$\frac{14}{5}k - \frac{2}{5}$	$\frac{12}{5}k - \frac{1}{5}$	$k \neq 3 \bmod 5$
$\mathcal{A}_3^{(4)}$	$G_{2,1} \otimes G_{2,1}$	$\frac{64}{5}k + \frac{28}{5}$	$\frac{6}{5}k + \frac{2}{5}$	$\frac{2}{5}k + \frac{4}{5}$	$k \neq 3 \bmod 5$
	$F_{4,1} \otimes F_{4,1}$	$\frac{16}{5}k + \frac{52}{5}$	$\frac{4}{5}k + \frac{3}{5}$	$-\frac{2}{5}k + \frac{6}{5}$	$k \neq 3 \bmod 5$
$\mathcal{B}_4^{(1)}$	$A_{4,1}$	$16k + 4$	$\frac{6}{5}k + \frac{2}{5}$	$\frac{4}{5}k + \frac{3}{5}$	$k \in \frac{\mathbb{Z}}{2}, 2k \neq 1 \bmod 5$
$\mathcal{A}_2^{(1)}$	$B_{r,1}$	$(23-2r)k + \frac{2r+1}{2}$	$\frac{(15-2r)}{8}k + \frac{2r+1}{16}$	$k + \frac{1}{2}$	$r = 0, 1$ incl.
$\mathcal{A}_3^{(1)}$	$E_{7,1} \otimes E_{7,1}$	$-4k + 14$	$\frac{1}{2}k + \frac{3}{4}$	$-k + \frac{3}{2}$	
	$D_{r,1}$	$2(12-r)k + r$	$\frac{(8-r)}{4}k + \frac{r}{8}$	$k + \frac{1}{2}$	$r = 2 \bmod 4$
$\mathcal{B}_3^{(4)}$	$D_{r,1}$	$2(12-r)k + r$	$\frac{(8-r)}{4}k + \frac{r}{8}$	$k + \frac{1}{2}$	$r \neq 2 \bmod 4$
$\mathcal{B}_8^{(1)}$	$A_{2,1} \otimes A_{2,1}$	$16k + 4$	$\frac{4}{3}k + \frac{1}{3}$	$\frac{2}{3}k + \frac{2}{3}$	$k \in \frac{\mathbb{Z}}{2}, 2k \neq 1 \bmod 3$

Table 5.2: Infinitely many sets of quasi-character families for each fusion class. Here $n, r \in \mathbb{Z}$ subject to the restrictions above.

In the case of order 2, we could restrict quasi-character series to only $k > 0$ by suitably exchanging characters. However, the corresponding process is not possible in the case at hand, hence we keep both positive and negative values of k. It is worth mentioning that the values of c, h_1, h_2 for both the $B_{r,1}$ and $D_{r,1}$ families can be combined into a single row, but since they belong to different fusion classes we list them separately.

Let us now make a few comments about the "presentation" of the characters. It has been noted previously in [15] that the characters arising from an MLDE can be considered in multiple presentations. This is done by matching $-\frac{c}{24}$ to any one of the exponents, α_i corresponding to different solutions of the MLDE. However there is always an unique choice for the identity character, i.e. the most negative exponent can be identified by the identity exponent, α_0 which implies that the conformal dimensions will be positive. This was the choice we made in the two-character review earlier. In this table of quasi-character families of order 3, we have made a different choice. Since both r and k run over both positive and negative values, it is unnatural to keep changing the presentations within a given family. We thus fix a presentation such that the $r > 0, k = 1$ theories are exhibited in their *familiar* forms as minimal/WZW models. The rest of the members of the family may be in unitary or non-unitary presentations.

Another interesting property of this table is that the quasi-characters too satisfy a novel coset relation. Since the central charge for each entry in the table is of the form $Ak+B$, we will denote the corresponding quasi-characters as $\chi_i^{(A,B,k)}$. As originally defined in [19], the novel coset relation is a holomorphic bilinear relation with respect to a $c = 24$ meromorphic CFT:

$$\sum_{i=0}^{2} \chi_i^{(A,B,k=0)}(\tau)\chi_i^{(A,B,k=1)}(\tau) = j(\tau) + \mathcal{N} \tag{5.4}$$

for one of the integers $\mathcal{N}$ listed in [25]. A crucial property of these CFT is that their central charges add up to 24. We now notice that in every row of Table 5.2, one has $A + 2B = 24$. Therefore if we consider $\chi_i^{A,B,k}$ and $\chi_i^{A,B,1-k}$ for any k, this pair could potentially satisfy a bilinear relation of the form:

$$\sum_{i=0}^{2} \chi_i^{(A,B,k)}(\tau)\chi_i^{(A,B,1-k)}(\tau) = j(\tau) + \mathcal{N} \tag{5.5}$$

for some $\mathcal{N}$, not necessarily an integer. We have indeed verified that this is the case for large values of k, for all the listed families of quasi-characters up to high orders in q.

An important point to be mentioned here is that the MLDE approach does not uniquely determine the ground state degeneracy D_i for the quasi-characters, since modular invariance only tells us about $|D_i|^2$. For admissible characters, this of course has to be positive and there is no ambiguity. However the bilinear relation implies a non-trivial condition on the sign of D_iD_i' of the ground state degeneracies of each of these coset pairs, i.e. only for the correct choice of sign does the coset relation hold true. It is very satisfying to note that the sign implied by the modular $\mathcal{S}$-matrix matches exactly with that implied from the bilinear relation.

Modular properties

In this section we will apply the algorithm to compute the modular $\mathcal{S}$-matrix reported in the previous chapter to the conjectured families of the quasi-characters. We will explicitly write down only a few of the modular $\mathcal{S}$-matrix, but the rest can be determined easily by using the algorithm, or by simply using the explicit formula for the three-character modular $\mathcal{S}$-matrix given in [1].

Recall that the ground-state degeneracies D_i are positive integers only for admissible characters. For quasi-characters, these can in general be negative integers and this is reflected in the modular $\mathcal{S}$-matrix. Thus we will specify the sign of D_i when quoting the modular $\mathcal{S}$-matrix for members of each family. In the following, the sign is assumed to be positive unless mentioned otherwise. For the $A_{4,1}$ family, we have:

$$
\begin{aligned}
A_{4,1}\text{ family : }\quad \mathcal{S} &= \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{5-\sqrt{5}}{10} & \frac{-5-\sqrt{5}}{10} \\ \frac{1}{\sqrt{5}} & \frac{-5-\sqrt{5}}{10} & \frac{5-\sqrt{5}}{10} \end{pmatrix} \quad \begin{matrix} k=0 \\ 2k = 2,5 \bmod 10 \\ 2k = 7,10 \bmod 10 \quad (D_2<0) \end{matrix} \\
\mathcal{S} &= \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{5+\sqrt{5}}{10} & \frac{-5+\sqrt{5}}{10} \\ \frac{1}{\sqrt{5}} & \frac{-5+\sqrt{5}}{10} & \frac{5+\sqrt{5}}{10} \end{pmatrix} \quad \begin{matrix} 2k = 3,4 \bmod 10 \\ 2k = 8,9 \bmod 10 \quad (D_2<0) \end{matrix}
\end{aligned}
\tag{5.6}
$$

Similarly one can compute the modular $\mathcal{S}$-matrix for the $B_{r,1}$ and $D_{r,1}$ families to be:

$$
D_{r,1}\text{ family, all } r: \qquad \mathcal{S} = \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} \tag{5.7}
$$

and:

$$
B_{r,1}\text{ family, all } r: \qquad \mathcal{S} = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \tag{5.8}
$$

Here the pattern of the sign of the ground state degeneracies is somewhat complicated and we will not list them here. For example, in the $D_{3,1}$ family one find $D_2 > 0$ for $k = 0,1,2,3,4$ and $D_2 < 0$ for $k = 5,6,7,8$. The modular $\mathcal{S}$-matrix for the remaining quasi-character families can be found by using our algorithm.

5.2 Adding Quasi-Characters in Order 3

In this section we will demonstrate how to add quasi-characters to generate admissible characters with higher ℓ. We will also give a general formula for the Wronksian index of a sum of two quasi-characters.

Consider two sets of quasi-characters with central charges c, c' and conformal dimensions h_i, h_i':

$$\chi_i = q^{\alpha_i}\left(a_0^{(i)} + a_1^{(i)}q + a_2^{(i)}q^2 + \cdots\right) \quad , \quad \chi_i' = q^{\alpha_i'}\left({a'_0}^{(i)} + {a'_1}^{(i)}q + {a'_2}^{(i)}q^2 + \cdots\right) \tag{5.9}$$

where the exponents α_i, α_i' are related to the central charge and the conformal dimensions in the usual way, $\alpha_i = -c/24 + h_i$. Our goal is to take linear combination of the form:

$$\widetilde{\chi}_i = \chi_i' + N\chi_i = q^{\widetilde{\alpha}_i}\left(\widetilde{a}_0^{(i)} + \widetilde{a}_1^{(i)}q + \widetilde{a}_2^{(i)}q^2 + \cdots\right) \tag{5.10}$$

such that $\widetilde{\chi}_i$ must have the following two properties: (i) it must have a well-defined modular $\mathcal{S}$-matrix, (ii) it must have a series expansion in integer powers of q.

The first requirement implies that the χ_i, χ_i' lie in the same fusion class, moreover they must have the same modular $\mathcal{S}$-matrix. The second requirement implies that $c' - c$ must be an integer multiple of 24:

$$c' - c = 24m\,, \quad \text{where } m \in \mathbb{Z} \tag{5.11}$$

Since the central charges have the form $c = Ak + B$, these two requirements imply that $(k' - k)A$ is a multiple of 24.

We will now consider a few examples and work them out explicitly to demonstrate the above general ideas. To avoid cluttering, we will express the q-series only up to a few low orders, but the conclusions, if any, have been drawn by considering the q-series up to an order of around 1000, however the conclusions are still conjectural.

Example 1: $D_{r,1}$ series: $(r, k) = (5, 0), (-5, 1)$

The first example we will consider is from the $D_{r,1}$ family in Table 5.2. It can easily be verified that $c' - c = 24$ holds true for the pair $(r, k = 0)$ and $(-r, k = 1)$ for any r. [1] This family has

$$c = 2(12 - r)k + r\,, \quad h_1 = \frac{8 - r}{4}k + \frac{r}{8}\,, \quad h_2 = k + \frac{1}{2} \tag{5.12}$$

[1] Similarly one can verify that the pair $(r, k = 0)$ and $(-r - 1, k = 1)$ in the $B_{r,1}$ quasi-character family satisfies $c' - c = 24$.

We will choose the case where $r = 5$. The χ_i corresponding to $(r = 5, k = 0)$ are admissible characters and correspond to the $D_{5,1}$ WZW model and have $c = 5, h_1 = \frac{5}{8}, h_2 = \frac{1}{2}$. To this, we will add χ_i' with $c' = 29, h_1' = \frac{21}{8}, h_2' = \frac{3}{2}$, i.e. the set corresponding to $(r = -5, k = 1)$. It can easily be verified that after a suitable choice of sign of degeneracy (as dictated by the bilinear relation) the two sets have the same modular $\mathcal{S}$-matrix:

$$\mathcal{S} = \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} \tag{5.13}$$

With the appropriate choice of sign of degeneracy, as dictated by the modular $\mathcal{S}$-matrix, the q-series for the two sets are:

$$\begin{aligned}
&c = 5 : \\
&\quad \chi_0 = q^{-\frac{5}{24}}(1 + 45q + 310q^2 + 1555q^3 + \cdots) \\
&\quad \chi_1 = q^{\frac{5}{12}}(16 + 160q + 880q^2 + 3680q^3 + \cdots) \\
&\quad \chi_2 = q^{\frac{7}{24}}(10 + 130q + 712q^2 + 3130q^3 + \cdots) \\
&c' = 29 : \\
&\quad \chi_0' = q^{-\frac{29}{24}}(1 - 319q + 78590q^2 + 25022911q^3 + \cdots) \\
&\quad \chi_1' = q^{\frac{17}{12}}(3801088 + 397672448q + 17830903808q^2 + 486562070528q^3 + \cdots) \\
&\quad \chi_2' = q^{\frac{7}{24}}(-1624 + 1921192q + 235569088q^2 + 11440410216q^3 + \cdots)
\end{aligned} \tag{5.14}$$

If we now define the linear combination of these two sets to be $\widetilde{\chi}_i = \chi_i' + N\chi_i$ where N is an arbitrary integer. The new set $\widetilde{\chi}_i$ has the following q-series:

$$\begin{aligned}
\widetilde{\chi}_0 &= q^{-\frac{29}{24}}\big(1 + (-319 + N)q + 5(9N + 15718)q^2 + (310N + 25022911)q^3 + \cdots\big) \\
\widetilde{\chi}_1 &= q^{\frac{5}{12}}\Big(16N + (160N + 3801088)q + (880N + 397672448)q^2 \\
&\qquad + (3680N + 17830903808)q^3 + \cdots\Big) \\
\widetilde{\chi}_2 &= q^{\frac{7}{24}}\Big((10N - 1624) + (130N + 1921192)q + (712N + 235569088)q^2 \\
&\qquad + (3130N + 11440410216)q^3 + \cdots\Big)
\end{aligned} \tag{5.15}$$

Thus we see that the set $\widetilde{\chi}_i$ has a well-defined modular $\mathcal{S}$-matrix as well as sensible q-series. The exponents for the new set are $\widetilde{c} = 29, \widetilde{h}_1 = \frac{13}{8}, \widetilde{h}_2 = \frac{3}{2}$ from which one can verify that $\ell = 6$.

Example 2: $A_{2,1} \times A_{2,1}$ **series:** $k = 0, \frac{3}{2}$

The second example we will consider is from the $A_{2,1} \times A_{2,1}$ family of quasi-characters which has:

$$c = 16k + 4\ , \quad h_1 = \frac{4}{3}k + \frac{1}{3}\ , \quad h_2 = \frac{2}{3}k + \frac{2}{3} \tag{5.16}$$

where k can be an integer or a half-integer. For χ_i we will consider the $k = 0$ set which corresponds to the $A_{2,1} \times A_{2,1}$ WZW model and hence is an admissible character. To this we will add χ'_i which corresponds to the $k = \frac{3}{2}$ which is a quasi-character. The exponents for these cases are:

$$c = 4, h_1 = \frac{1}{3}, h_2 = \frac{2}{3} \quad \& \quad c' = 28, h'_1 = \frac{7}{3}, h'_2 = \frac{5}{3} \tag{5.17}$$

Both these sets have the same modular $\mathcal{S}$-matrix given by:

$$\mathcal{S} = \begin{pmatrix} \frac{1}{3} & \frac{4}{3} & \frac{4}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \tag{5.18}$$

The two sets have the following q-expansions:

$$\begin{aligned}
&c = 4 : \\
&\qquad \chi_0 = q^{-\frac{1}{6}}\Big(1 + 16q^2 + 98q^3 + \cdots\Big) \\
&\qquad \chi_1 = q^{\frac{1}{6}}\Big(3 + 33q + 150q^2 + 564q^3 + \cdots\Big) \\
&\qquad \chi_2 = q^{\frac{1}{2}}\Big(9 + 54q + 243q^2 + 828q^3 + \cdots\Big) \\
&c = 28 : \\
&\qquad \chi'_0 = q^{-\frac{7}{6}}\Big(1 - 77q + 64274q^2 + 14583702q^3 + \cdots\Big) \\
&\qquad \chi'_1 = q^{\frac{7}{6}}\Big(492075 + 63930384q + 3137548932q^2 + \cdots\Big) \\
&\qquad \chi'_2 = q^{\frac{1}{2}}\Big(5103 + 2924019q + 253103697q^2 + 10060647606q^3 + \cdots\Big)
\end{aligned} \tag{5.19}$$

We can now take the linear combination of χ_i and χ'_i by defining $\widetilde{\chi}_i = \chi_i + N\chi'_i$, which has the following q-series:

$$
\begin{aligned}
\widetilde{\chi}_0 &= q^{-\frac{7}{6}}\Big(1+(-77+N)q+(64274+16N)q^2+(14583702+98N)q^3+\cdots\Big)\\
\widetilde{\chi}_1 &= q^{\frac{1}{6}}\Big(3N+(492075+33N)q+(63930384+150N)q^2\\
&\qquad +(3137548932+564N)q^3+\cdots\Big)\\
\widetilde{\chi}_2 &= q^{\frac{1}{2}}\Big((5103+9N)+(2924019+54N)q+(253103697+243N)q^2\\
&\qquad +(10060647606+828N)q^3+\cdots\Big)
\end{aligned}
\tag{5.20}
$$

It is easy to verify that even in this case the Wronskian index of $\widetilde{\chi}_i$ is $\ell = 6$, which follows from the exponents $\widetilde{c} = 28, \widetilde{h}_1 = \frac{4}{3}, \widetilde{h}_2 = \frac{5}{3}$.

Wronskian index of a sum of two quasi-characters

In this subsection, we will generalise the observation that the sum of a set of quasi-characters (with the same Wronskian index $\ell = 0$) gives us a new set with $\ell = 6$, when the difference between the central charges was 24. Without loss of generality, we will consider cases where $c' - c = 24m$ for any positive integer m.

For the identity character we see that since $c' > c$, $\alpha_0' < \alpha_0 = \alpha_0' + m$ and therefore χ_0' is more singular than χ_0. In fact, on adding these two identity characters, the first term of χ_0 adds to the m^{th} term of χ_0' and therefore the new identity exponent $\widetilde{\alpha}_0$ is:

$$
\widetilde{\alpha}_0 = \alpha_0' = -\frac{c'}{24} = -\frac{c}{24} - m \tag{5.21}
$$

Next consider the first non-identity character, χ_1, χ_1'. The exponents can be written as:

$$
\alpha_1 = -\frac{c}{24} + h_1 = \widetilde{\alpha}_0 + h_1 + m\ , \quad \alpha_1' = \widetilde{\alpha}_0 + h_1' \tag{5.22}
$$

The final exponent $\widetilde{\alpha}_1$ corresponds to the more singular one out of these two, therefore we have

$$
\widetilde{\alpha}_1 = \widetilde{\alpha}_0 + \text{min}(h_1', h_1 + m) \tag{5.23}
$$

Following the exact same logic, we find that

$$
\widetilde{\alpha}_2 = \widetilde{\alpha}_0 + \text{min}(h_2', h_2 + m) \tag{5.24}
$$

Thus we get four different possibilities for the exponents $\widetilde{\alpha}_i$.

Let us first consider the case where $h_1' < h_1 + m$. Using the Riemann-Roch equation Eq. (5.2) and $c' - c = 24m$, we have:

$$-\frac{c}{8} + h_1 + h_2 = -\frac{c'}{8} + h_1' + h_2' \implies h_1' - h_1 + h_2' - h_2 = 3m \tag{5.25}$$

This in turn implies that:

$$h_1' - (h_1 + m) = m + (m + h_2) - h_2' < 0 \tag{5.26}$$

From the RHS, since $m > 0$, this implies that $h_2' > h_2 + 2m > h_2 + m$. Therefore we conclude that both h_1' and h_2' cannot simultaneously be smaller than $h_1 + m$ and $h_2 + m$ respectively, thus eliminating one of the four cases. Continuing with this case we see that:

$$\widetilde{\alpha}_0 = -\frac{c'}{24}\ , \quad \widetilde{\alpha}_1 = -\frac{c'}{24} + h_1'\ , \quad \widetilde{\alpha}_2 = -\frac{c'}{24} + h_2 + m \tag{5.27}$$

Substituting this in the Riemann-Roch equation we find that:

$$\begin{aligned}
-\frac{c'}{8} + h_1' + h_2 + m &= \frac{1}{2} - \frac{\ell}{6} \\
-\frac{c'}{8} + h_1' + h_2' + h_2 - h_2' + m &= \frac{1}{2} - \frac{\ell}{6} \\
\implies \frac{1}{2} + h_2 - h_2' + m &= \frac{1}{2} - \frac{\ell}{6} \\
\implies \ell &= 6(h_2' - h_2 - m)
\end{aligned} \tag{5.28}$$

Similarly, for the other case, i.e. $h_1' > h_1 + m$, we get that $\ell = 6(h_1' - h_1 - m)$. For the last case when both $h_i' > h_i + m$, the exponents simply can be written as:

$$\widetilde{\alpha}_0 = -\frac{c'}{24}\ , \quad \widetilde{\alpha}_1 = -\frac{c'}{24} + h_1 + m\ , \quad \widetilde{\alpha}_2 = -\frac{c'}{24} + h_2 + m \tag{5.29}$$

which after substituting in the Riemann-Roch equation implies that $\ell = 6m$. To summarise, on adding two quasi-characters of order 3 with $\ell = 0$, we get the following ℓ values:

$\min(h'_1, h_1 + m)$	$\min(h'_2, h_2 + m)$	ℓ
h'_1	$h_2 + m$	$6(h'_2 - h_2 - m)$
$h_1 + m$	h'_2	$6(h'_1 - h_1 - m)$
$h_1 + m$	$h_2 + m$	$6m$

Table 5.3: Possible values of the Wronskian index ℓ

5.3 New Admissible Characters with $\ell > 0$

In the previous section, we added a couple of sets of quasi-characters such that the sum had a well-defined modular $\mathcal{S}$-matrix and a sensible q-series. We found that for the examples we consider, the Wronskian index of the new set of was $\ell = 6$. Recall that we had an arbitrary integer N in the coefficients of the final q-series. We will now take a look at these examples and see if we can choose N such that we end up with admissible characters.

Before doing this, let us classify the types of solutions we have in our list of quasi-character families. In order 2, the full set of quasi-character was classified into two types labelled Type I and Type II, depending on the behaviour of the sign of coefficients of the identity character. In order 3, the situation is more complicated. We classify *each* of the three quasi-characters into the following behaviour:

- Behaviour A: The q-expansion has only non-negative coefficients.
- Behaviour B: Assuming that the ground state degeneracy has been chosen to be positive, there are only positive coefficients after a finite power of q.
- Behaviour C: Assuming that the ground state degeneracy has been chosen to be positive, there are only negative coefficients after a finite power of q.
- Behaviour D: The asymptotic sign of the coefficients does not stabilise but oscillates between ± 1.

One has to specify a behaviour for each χ_i in a set and hence the class is specified by a triplet. In this notation, admissible characters are denoted as the AAA class. We have found examples of class AAD, ABD, ACD, BBB, BBC and many more. Going back to the examples we have already considered, we note that one of χ_i, χ'_i was of class AAA in both the examples. We have verified up to high orders of q (~ 5000) that χ'_i lies in the BAC and BAA class in the first and the second example. To generate admissible characters, we consider special choices of the coefficient

N. Considering example 1, we see that the $\widetilde{\chi}_i$ becomes admissible for all $N \geq 319$. In example 2, we see that $\widetilde{\chi}_i$ becomes admissible for $N \geq 77$.

Cosets and three-character CFT with $\ell \geq 6$

Now that we have used quasi-characters to explicitly construct admissible characters with $\ell > 0$ by restricting N to specific ranges, we can ask the following question: For what values of N can we find a genuine CFT whose characters exactly match the ones we have constructed using quasi-characters?

To answer this question, we invoke the novel coset construction of [19]. This construction tells us that if we coset a meromorphic CFT of central charge $8r$ by a CFT with p characters and Wronskian index ℓ, the coset theory has Wronskian index:

$$\ell = p^2 + (2r-1)p - 6(n_1+n_2) - \ell \tag{5.30}$$

where n_1, n_2 are integers such that $h_i + \widetilde{h}_i = n_i$. It has been argued in [19] that the these integers are bounded below by 2. If assume these minimum values and consider a three-character $\ell = 0$ CFT as the theory we are coseting the meromorphic theory with, the Wronskian of the coset dual is given by:

$$\widetilde{\ell} = 6r - 18 \tag{5.31}$$

Thus we see that if we start with a 32-dimensional meromorphic CFT and coset it with a three-character $\ell = 0$ CFT, we will get a coset theory that has $\widetilde{\ell} = 6$.

We will now consider meromorphic CFT constructed using even self-dual lattices with a complete root systems as $c = 32$ meromorphic CFT and coset them with certain three-character $\ell = 0$ CFT. Finally we will match the characters of the coset theory with our admissible characters.

We start with the CFT of 32 free bosons defined on the Kervaire lattice with complete root system $A_1^8 A_3^8$ from [39, 38]. the corresponding level-1 Kac-Moody algebra has dimension 144 and the single character of this theory has the following q-series:

$$\chi = j^{\frac{1}{3}}(j - 848) = q^{-\frac{4}{3}}(1 + 144q + \cdots) \tag{5.32}$$

We will now take the quotient of this meromorphic CFT by the $A_{3,1}$ WZW model. The resulting coset theory is a three-character CFT with $c = 29, h_1 = \frac{13}{8}, h_2 = \frac{3}{2}$ and has Kac-Moody dimension 129.

Now we will compare this with our example 1 which gives us an infinite family (labelled by

N) of admissible characters that has the same set of CFT exponents. The last step is to note that in our example, the dimension of the Kac-Moody algebra (if it exists) would be $N - 319$. Thus we see that for the character corresponding to $N = 448$, the dimension of the Kac-Moody algebra matches that of the coset dual constructed using the CFT. One can now verify that the admissible characters with $N = 448$ in Example 1 are the characters of the coset theory constructed using the novel coset construction. To verify that this is indeed the case, we can evaluate the bilinear product of the admissible characters with those of the $A_{3,1}$ WZW model. The result, as expected, turns out to be $j^{\frac{1}{3}}(j - 848)$, which is exactly the character of the original $c = 32$ meromorphic theory.

As another example, we consider the CFT of 32 free bosons defined on the Kervaire lattice with the complete root system $A_2^2 A_{14}^2$ from from [39, 38]. The corresponding level-1 Kac-Moody algebra has dimension 464 and this meromorphic CFT has the character:

$$\chi = j^{\frac{1}{3}}(j - 528) = q^{-\frac{4}{3}}(1 + 464q + \cdots) \tag{5.33}$$

We can quotient this theory by the $A_{2,1} \times A_{2,1}$ WZW model to obtain a $\ell = 6$ three-character CFT with $c = 28, h_1 = \frac{5}{3}, h_2 = \frac{4}{3}$ with the a Kac-Moody algebra of dimension 448.

Comparing this with the $N = 525$ case of the Example 2 we discussed, we see that our example has the correct dimension of the Kac-Moody algebra and N lies within the range of admissibility. To match the CFT exponents, however, one has to switch $\widetilde{h}_1$ and $\widetilde{h}_2$. Once again to verify that our admissible character gives the character of the coset theory constructed using the lattice, we evaluate the bilinear product of our admissible characters with the characters of the $A_{2,1} \times A_{2,1}$ WZW model. The result, as expected, turns out to be $j^{\frac{1}{3}}(j - 528)$, which is exactly the character of the $c = 32$ meromorphic CFT we started out with.

To summarise, we have proposed a large collection of families of quasi-characters with vanishing Wronskian index ($\ell = 0$) and provided evidence for their existence. We also demonstrated that linear combinations of these quasi-characters can be used to construct admissible characters with higher ℓ. In particular, we provided explicit examples that correspond to actual $\ell = 6$ CFT. On the way, we also reviewed the modular properties of these quasi-character families, using the contour-integral representation from the previous chapter. We also noted many examples of the bilinear relation among the quasi-character families.

Chapter 6

Holography & Modular Bootstrap

Till now we have focused on application of modular bootstrap that severely constraints rational conformal field theories. We will now switch to the application of modular bootstrap which deals with more general properties of conformal field theories. The focus here is on making approximate statements about generic conformal field theories, rather than classifying RCFT. In particular, we will consider irrational CFT and look for possible holographic application

The $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence states that a theory of quantum gravity in $2+1$ dimensions with negative cosmological constant has its dynamics encoded by a dual two-dimensional conformal field theory on the boundary of the spacetime [40, 41, 42, 43]. TASI lectures on AdS/CFT [44] are a good starting point to study this in detail. One of the most well studied aspects of this duality is its semi-classical limit, which is the case when the AdS radius, $\ell >> G_N$, where G_N is the Gravitational constant in three dimensions. By using the $\mathrm{AdS}_3/\mathrm{CFT}_2$ dictionary, which we will expand on in a later section, this limit translates to $c + \bar{c} >> 1$ where $c, \bar{c}$ are the right- and left-moving central charges of the dual CFT.

We will be focusing on the case in which the bulk theory of quantum gravity is pure, i.e. the only fields in this theory are the spin-2 gravitons. On the dual CFT side, this translates to saying that the only conserved current in the theory is the energy-momentum tensor. This implies that the Virasoro algebra is the full symmetry algebra of the CFT, since the presence of chiral spin-1 fields in boundary theory would be translated to gauge bosons in the bulk.

In this chapter, we will first look at some aspects of $2+1$ dimensional gravity [45, 46], in particular gravity with $\Lambda < 0$. We will then investigate the BTZ black hole and its spectrum [47, 48], along with some holographic results. After this brief review, we shall use modular invariance to compute the asymptotic density of states in a CFT at high energies, given by the well-known Cardy formula [49] which, as we verify, matches the Bekenstein-Hawking entropy of the BTZ black hole [50]. We will then use the medium temperature expansion to obtain constraints on the derivatives of partition function. Using these constraints, we will follow [51] in deriving an upper on the scaling dimension of the lowest non-trivial primary operator, which translates into an upper bound on the mass of the lowest massive excitation. Finally, we will investigate the modular $\mathcal{S}$ transformation and its generalisations in the lightcone limit, while following [52] to obtain modular bootstrap equations from the invariance of the partition function. This will in turn give us an upper bound on the twist gap of a generic CFT as well as the universal density of large spin Virasoro primaries, which could be considered as a generalisation of the Cardy formula [53, 54].

6.1 Gravity in $2+1$ Dimensions and the BTZ Black Hole

Three-dimensional gravity is described by the following action:

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{-g}(R-2\Lambda) \tag{6.1}$$

This has been the subject of much study, since after the initial works of [55, 56], three-dimensional gravity has served as a playground for exploring the connections between classical and quantum gravity. One reason for doing so is the simple observation that the number of independent components of the Riemann tensor in three-dimensions is equal to zero. Another important feature is that three is the lowest number of dimensions in which Einstein's gravity makes sense, as the Einstein tensor, $G_{\mu\nu}$ vanishes identically in any lower dimension.

Since we will be considering the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, let us consider only the $\Lambda < 0$ case of the above action. The maximally symmetric solution of the Einstein's equations with $\Lambda = -1/\ell^2 < 0$ is known as the *anti-de Sitter spacetime* AdS_3. The line element of AdS_3 can be expressed as:

$$ds^2_{AdS_3} = -\left(1+\frac{r^2}{\ell^2}\right)dt^2 + \left(1+\frac{r^2}{\ell^2}\right)^{-1}dr^2 + r^2 d\theta^2 \tag{6.2}$$

where ℓ is known as the AdS radius. The coordinates here vary as $-\infty < t < \infty, 0 \leq r < \infty, 0 \leq \theta \leq 2\pi$.

Let us now investigate some properties of gravity in three dimensions:

Matter curves space-time only locally: This is implied by the fact the Riemann tensor $R_{\mu\nu\alpha\beta}$ vanishes in empty space-time, i.e. a region without any matter source. This implies that every object moves independently of any other object that is separated from it by vacuum.

No propagating degrees of freedom: This follows from the fact that the Weyl tensor, which carries information about that part of the curvature that is not determined by the local matter distribution, vanishes identically in three dimensions. This implies that outside matter distributions, there are no gravitational waves and in particular, there are no propagating degrees of freedom.

No Newtonian limit: It is well known that the Newtonian gravitational law in two space dimensions predicts a logarithmic dependence on the radial distance. Whereas the Newtonian limit of three-dimensional Einstein's gravity predicts that free particles propagate unhindered.

A straightforward implication of the above properties of three-dimensional gravity is that the light which escapes from the surface of a matter distribution in a $2+1$ dimensional spacetime, escapes all the way to infinity. This naïvely implies that black holes in three-dimensions are not possible.

It was thus a surprise when Bañados, Teitelboim and Zanelli found a black hole solution in $2+1$ dimensional gravity [47]. This was termed as the BTZ black hole and has the following line element:

$$ds^2_{BTZ} = -N^2 dt^2 + N^{-2} dr^2 + r^2 \left(N^\phi dt + d\phi\right)^2 \tag{6.3}$$

where $-\infty < t < \infty, 0 \leq r < \infty, 0 \leq \theta \leq 2\pi$, and

$$N^2 = -8GM + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\,, \quad N^\phi = -\frac{J}{2r^2} \tag{6.4}$$

The parameters M, J in the line element can be interpreted as the mass and the angular momentum of the black hole. The BTZ black hole is asymptotically anti-de Sitter and curiously has no curvature singularity at its center. It is still a black hole as it has an event horizon, r_+ and an inner horizon, r_- (only in the rotating case). These horizons occur at the values of radii where N^2 vanishes, i.e.

$$r_\pm = \ell \left[4GM\left(1 \pm \sqrt{1 - \left(\frac{J}{M\ell}\right)^2}\right)\right]^{\frac{1}{2}} \tag{6.5}$$

It is clear that in order for these horizons to exist, i.e. for $r_\pm$ to lie in the given range, one must have $M > 0, |J| < M\ell$.

One encounters naked singularities for all negative values of M, thus we exclude them as valid solutions which implies that they will not appear in the physical spectrum of the corresponding theory of quantum gravity. The single exception is the BTZ black hole with $M = -1/8G, J = 0$ where the singularity disappears, hence even though there is no horizon, nothing needs to be hidden. Therefore this turns out to be a valid solution and in fact corresponds to the pure anti-de Sitter geometry Eq. (6.2).

Therefore the BTZ black hole spectrum can be summarised as follows: the vacuum is the global

AdS$_3$ spacetime, which corresponds to the $M = -1/8G, J = 0$ BTZ black hole. The first excited state is the lightest BTZ black hole, i.e. $M = J = 0$ BTZ black hole, which appears after a mass gap of $1/8G$. The rest of the spectrum is continuous as the mass, M and the spin, J vary over continuous values subject to $|J| < M\ell$.

The asymptotic symmetries of the anti-de Sitter spacetime can be investigated by using the Poisson bracket algebra implied by the diffeomorphisms that preserve the asymptotic AdS nature [57]. On doing so, one finds that the asymptotic symmetry algebra of AdS$_3$ can be expressed as a pair of Virasoro algebras (the symmetry algebra of a 2d CFT) with central charges:

$$c = \bar{c} = \frac{3}{2\,G\sqrt{|\Lambda|}} = \frac{3\ell}{2G} \tag{6.6}$$

It can now be shown that the relation between the parameters of the BTZ black hole and the boundary CFT can be expressed as follows [50]:

$$M = \frac{L_0 + \bar{L}_0}{\ell} - \frac{1}{8G}\,, \quad J = L_0 - \bar{L}_0 \tag{6.7}$$

where the additive constant in the mass is such that the $M = -1/8G, J = 0$ BTZ black hole corresponds to the vacuum state of the Hilbert space generated by the boundary Virasoro generators, which we will refer to as the boundary CFT.

From these relations, it is clear that the mass gap of $1/8G$, explicitly visible in the above relation, translates to a gap in the scaling dimension, $\Delta_{gap} = 1/8G = c/12$ in the boundary CFT. Similarly, Eq. (6.7) implies that the smallest non-zero value of the twist, $2t = 2\text{min}(h, \bar{h})$ of the boundary CFT (that corresponds to $M = J = 0$ BTZ black hole in the bulk) is equal to $1/16G = c/24$, which implies that $h, \bar{>} c/24$. Note that we reach these conclusions by analysing a pure theory of gravity in AdS$_3$, and have merely expressed the results in terms of the central charge of the corresponding boundary CFT. In the later sections of this chapter, we will attempt to obtain constraints on the scaling dimension and the twist purely from a CFT standpoint, and then compare them with the results implied from pure gravity.

6.2 Density of States from Modular Invariance

In this section we will explore one of the most important consequences of modular invariance, the asymptotic density of states in a CFT at high energies. Recall the partition function of a CFT Eq.

(1.25) and $\tau = (K + i\beta)/2\pi$:

$$\mathcal{Z}(\beta) = \int dh \int d\bar{h}\ \rho(h,\bar{h})\, q^{-\frac{c}{24}+h} \bar{q}^{-\frac{\bar{c}}{24}+\bar{h}} \tag{6.8}$$

where $\rho(h,\bar{h})$ is the density of states. We will now consider the case $K = 0$ and hence the modular parameter is equal to the inverse temperature, up to some multiplicative constants. In this case, the nome becomes $q = e^{-\beta}$. After a modular $\mathcal{S}$ transformation, this becomes $q' = e^{-\frac{4\pi^2}{\beta}}$. Therefore the new inverse temperature of the system is $\beta' = 4\pi^2/\beta$. Modular invariance now tells us that the partition function should remain invariant under a modular $\mathcal{S}$ transformation, i.e.

$$\mathcal{Z}(\beta) = \mathcal{Z}\left(\frac{4\pi^2}{\beta}\right) \tag{6.9}$$

Written in this form, modular invariance is implying that the theory at low temperatures $\beta \to \infty$ is equal to the theory at a corresponding high temperature $\beta \to 0$. This manifestation of modular invariance is also known as the high-low temperature duality.

We will now consider the high temperature limit, i.e. $\beta \to 0$, in this limit, the RHS of the equation diverges. In fact, one finds that the vacuum term, $\sim e^{\frac{4\pi^2}{\beta}\frac{c}{24}}$, dominates the contribution from the rest of the spectrum. Note that the partition function, as expressed above, is the Laplace transform of the density of states $\rho(h,\bar{h})$. Therefore to find the asymptotic density of states, we consider the inverse Laplace transform of the partition function:

$$\rho(h,\bar{h}) = \int_{-i\infty}^{i\infty} d\tau\ e^{-2\pi i\tau\left(h-\frac{c}{24}\right)} \int_{-i\infty}^{i\infty} d\bar{\tau}\ e^{-2\pi i\bar{\tau}\left(\bar{h}-\frac{\bar{c}}{24}\right)} \mathcal{Z}\left(\frac{-1}{\tau}, \frac{-1}{\bar{\tau}}\right) \tag{6.10}$$

Now in the $\beta \to 0$, or equivalently $\tau, \bar{\tau} \to 0$ limit, we have:

$$\rho(h,\bar{h})\big|_{\beta\to 0} = \int_{-i\infty}^{i\infty} d\tau\ e^{-2\pi i\tau\left(h-\frac{c}{24}\right)+\frac{2\pi i}{\tau}\frac{c}{24}} \int_{-i\infty}^{i\infty} d\bar{\tau}\ e^{-2\pi i\bar{\tau}\left(\bar{h}-\frac{\bar{c}}{24}\right)+\frac{2\pi i}{\bar{\tau}}\frac{\bar{c}}{24}} \tag{6.11}$$

We will now use the saddle point approximation to solve both of these integrals as their integrand is of the form $e^{-f(\tau)}$ where $f(\tau) = 2\pi i\tau\left(h - \frac{c}{24}\right) - \frac{2\pi i}{\tau}\frac{c}{24}$. To do so, we expand the function around its minima τ_0, i.e. $\tau = \tau_0 + \tilde{\tau}$, where $|\tilde{\tau}| < |\tau_0|$, such that $f'(\tau_0) = 0$. One can use the final equality to find the value of the *saddle point* as:

$$\tau_0 = i\sqrt{\frac{c/24}{h - c/24}} \tag{6.12}$$

This means that after using the saddle point approximation, the first integral becomes:

$$\int_{-i\infty}^{i\infty} d\tau \; e^{-2\pi i\tau\left(h-\frac{c}{24}\right)+\frac{2\pi i}{\tau}\frac{c}{24}} \approx e^{4\pi\sqrt{\frac{c}{24}\left(h-\frac{c}{24}\right)}} \tag{6.13}$$

This holds trivially for the other integral as well, therefore the asymptotic density of states at high energies turns out to be:

$$\rho(h,\bar{h}) = e^{4\pi\sqrt{\frac{c}{24}\left(h-\frac{c}{24}\right)}+4\pi\sqrt{\frac{\bar{c}}{24}\left(\bar{h}-\frac{\bar{c}}{24}\right)}} \tag{6.14}$$

This formula is known as the *Cardy formula* and one can read off the entropy of a CFT at high energies as:

$$S = 4\pi\sqrt{\frac{c}{24}\left(h-\frac{c}{24}\right)} + 4\pi\sqrt{\frac{\bar{c}}{24}\left(\bar{h}-\frac{\bar{c}}{24}\right)} \tag{6.15}$$

In the semi-classical limit, $c >> 1$, one can now easily compute the area, A of a BTZ black hole Eq. (6.3) as:

$$A = \int d\phi \; r = \pi\ell\sqrt{16GM\left(1+\sqrt{1-\left(\frac{J}{M\ell}\right)^2}\right)} \tag{6.16}$$

Using this one can compute the Bekenstein-Hawking entropy of the BTZ black hole to obtain:

$$S = \frac{A}{4G} = \pi\sqrt{\frac{\ell}{2G}}\left[\sqrt{M\ell+J}+\sqrt{M\ell-J}\right] \tag{6.17}$$

which exactly matches the Cardy formula Eq. (6.15) we computed purely from the CFT side, after using Eq. (6.7). This is an evidence for the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence [50] and gives us a way of probing near-horizon microstates of black holes in AdS spacetime.

6.3 Bound on Lowest Massive Excitation

One of the major drawbacks of the Cardy formula is that it alone is not enough to falsify the consistency of a candidate spectrum. This is because it deals with the behaviour of densities at sufficiently high energies, but does not specify the threshold beyond which the asymptotic behaviour kicks in. We thus want to devise a method that considers a finite number of energy levels in a candidate spectrum of a CFT and probes its consistency.

Medium temperature expansion

Recall the action of the modular $\mathcal{S}$ transformation on the modular parameter $\tau \to -1/\tau$. It is obvious that the orbifold point $\tau = i$ is a fixed point under this transformation. This point in the moduli space corresponds to a square torus. The inverse temperature β corresponding to this point is 2π and hence can be though of as lying exactly between high-temperature and low-temperature regimes. Equivalently the path integral over this square torus can be thought of as lying between large- and small-complex structure limits of the moduli space of the torus.

We now perform an expansion of the modular parameter around this special point and probe the implications of modular invariance. To do so, consider a parameter s such that:

$$\tau = ie^{s} \qquad \Longrightarrow \qquad \mathcal{S} : s \to -s \tag{6.18}$$

This is trivially extended to $\bar{\tau}$ by defining a corresponding $\bar{s}$. The modular $\mathcal{S}$ transformation now implies the following:

$$\mathcal{Z}(ie^{s}, -ie^{\bar{s}}) = \mathcal{Z}(ie^{-s}, -ie^{-\bar{s}}) \tag{6.19}$$

Taking the small s limit is what we call the medium complex structure expansion, or medium temperature expansion when s is purely real.

It is straightforward to see that the first s-derivative of the partition function at $s = 0$ is zero. It is trivial to see that this holds true for any odd order derivative of s at $s = 0$. Analogously the same holds true for derivatives with respect to $\bar{s}$ at $\bar{s} = 0$. Thus we have the following equation:

$$\left(\frac{\partial}{\partial s}\right)^{N_R} \left(\frac{\partial}{\partial \bar{s}}\right)^{N_L} \mathcal{Z}(ie^{s}, -ie^{\bar{s}})\Bigg|_{s=0} = 0 \quad \text{for } N_R + N_L \text{ odd} \tag{6.20}$$

Thus for purely real s or equivalently, purely imaginary complex structure $\tau = i\beta/2\pi$, we get:

$$\left(\beta\frac{\partial}{\partial \beta}\right)^{p} \mathcal{Z}(\beta)\Bigg|_{\beta=2\pi} = 0 \quad \text{for } p \text{ odd} \tag{6.21}$$

We will now make a few remarks about the medium temperature expansion: firstly, the constraint for every odd p is independent of the previous ones and hence this provides us with an infinite set of constraints of the partition function. Secondly, this expansion contains complementary information from the high- and low-temperature expansions. The invariance of the partition function under $\mathcal{S}$ transformation is completely invisible in the low-temperature expansion, whereas in the medium temperature expansion, this invariance is manifest.

Warm-up case

In this section we show that every unitary CFT (with $c, \bar{c} > 1$) has a local operator whose scaling dimension Δ is bounded above by $\Delta_+^{(warm-up)}$ where:

$$\Delta_+^{(warm-up)} = \frac{c+\bar{c}}{12} + \frac{3}{2\pi} \tag{6.22}$$

Note that this bound holds for *any* non-vacuum state with no distinction between primaries and descendants. We know that the state corresponding to the energy momentum tensor is the level-2 descendant above the identity, therefore our bound will be meaningful only for $\Delta_+^{(warm-up)} < 2$ which implies that $c + \bar{c} > 24 - \frac{18}{\pi} \approx 18.270$. We will make use of only the $p = 1$ and $p = 3$ constraints provided by Eq. (6.21) to obtain this bound.

We start with the partition function:

$$\mathcal{Z}(\beta) = \sum_{n=0}^{\infty} a_n e^{-\beta E_n} \tag{6.23}$$

where E_1 will be the lowest energy excitation in the full spectrum of the CFT, E_0 is the vacuum energy, and a_n is the degeneracy of the energy level with energy E_n. The $p = 1$ constraint in this language can be written as:

$$\sum_{n=1}^{\infty} a_n E_n e^{-2\pi \widetilde{E}_n} = -a_0 E_0 \tag{6.24}$$

where $\widetilde{E}_n = E_n - E_0$. Similarly the $p = 3$ constraint can be written as:

$$\sum_{n=1}^{\infty} a_n E_n I(E_n) e^{-2\pi \widetilde{E}_n} = -a_0 E_0 I(E_0) \tag{6.25}$$

where $I(E) = 1 - 6\pi E + 4\pi^2 E^2$. After substituting the RHS of the $p = 1$ constraint into the RHS of the $p = 3$ constraint and rearranging we get:

$$\sum_{n=1}^{\infty} a_n E_n (I(E_n) - I(E_0)) e^{-2\pi \widetilde{E}_n} = 0 \tag{6.26}$$

We now analyze the function $I(E)$ to try and reach a contradiction. Note that $I(E)$ is an upward parabola with a minima at $E = \frac{3}{2\pi}$ and zeroes at $E = \pi(3 \pm \sqrt{5})$. Thus we see that $I(E)$ is a symmetric function about $E = \frac{3}{2\pi}$ and hence it attains the value $I(E_0)$ at $E = E_0, \frac{3}{2\pi} - E_0$. Since $I(E_0)$ is a positive number (because $E_0 < 0$) it implies that $\frac{3}{2\pi} - E_0 > 3 + \sqrt{5}$ and hence $I(E)$

is monotonically increasing for all $E > \frac{3}{2\pi} - E_0$. Recall that we have ordered all energies in an increasing order: $E_0 < E_1 < E_2 < \cdots$ and hence if $E_1 > \frac{3}{2\pi} - E_0$ this implies that $I(E_n) > I(E_0)$ for all positive n.

Coming back to our constraint equation (6.26), we see that if we consider $E_1 > \frac{3}{2\pi} - E_0$ then every term in the constraint is positive and thus we reach a contradiction. Since the vacuum energy of an CFT is equal to $-(c+\bar{c})/24$ and the scaling dimension $\Delta = E - E_0$ we see that the modular invariance implies the existence of a state with scaling dimension Δ bounded above by $\Delta_+^{(warm-up)} = \frac{c+\bar{c}}{12} + \frac{3}{2\pi}$.

Bound on the lowest non-trivial primary operator

In this section we will derive an upper bound on the scaling dimension of non-trivial primary fields in a CFT. This will generalise the previous derivation in two ways: (i) the bound will be for a primary state in the theory rather than a general state, (ii) the bound will work for $c + \bar{c} > 18.270$. We will consider a general two-dimensional conformal field theory subject to the following mild conditions: (i) unitarity, (ii) discrete operator spectrum, (iii) $c, \bar{c} > 1$, and (iv) absence of any chiral algebra. The last condition implies that there are no primary fields with $h \neq 0, \bar{h} = 0$ or vice versa.

The partition function of any such CFT can be written as a sum over Virasoro characters:

$$\chi_0(\tau) = (1-q)\frac{q^{-\frac{c-1}{24}}}{\eta(q)}\ , \quad \chi_i(\tau) = \frac{q^{-\frac{\bar{c}-1}{24}+h_i}}{\eta(q)}, \quad i \neq 0 \tag{6.27}$$

Recall that the partition function can thus be written as:

$$\mathcal{Z}(\tau,\bar{\tau}) = \sum_{i,\bar{i}} M_{i\bar{i}}\chi_i(\tau)\chi_{\bar{i}}(\bar{\tau}) = \frac{q^{-\frac{c-1}{24}}\bar{q}^{-\frac{\bar{c}-1}{24}}}{|\eta(i\beta/2\pi)|^2}\left((1-q)(1-\bar{q}) + \sum_{i,\bar{i}} q^{h_i}\bar{q}^{\bar{h}_{\bar{i}}}\right) \tag{6.28}$$

Taking the modular parameter to be purely imaginary $\tau = \frac{i\beta}{2\pi}$ we get:

$$\mathcal{Z}(\beta) = \frac{e^{-\left(E_0+\frac{1}{12}\right)\beta}}{|\eta(i\beta/2\pi)|^2}\left(\left(1-e^{-\beta}\right)^2 + \sum_i e^{-\beta\Delta_i}\right) \equiv G(\beta)\left(V(\beta) + N(\beta)\right) \tag{6.29}$$

where $E_0 = -\frac{c+\bar{c}}{24}$.

We will now define a convenient set of polynomials to ease our calculations:

$$\left(\beta\frac{\partial}{\partial\beta}\right)^p \frac{e^{-z\beta}}{|\eta(i\beta/2\pi)|^2}\bigg|_{\beta=2\pi} = \frac{(-1)^p e^{-2\pi z}}{|\eta(i)|^2} f_p(z) \tag{6.30}$$

The first few of these polynomials in terms of $r = \frac{\eta''(i)}{\eta(i)} \approx 0.0120528$ are given as:

$$\begin{aligned} & f_0(z) = 1 \quad , \quad f_1(z) = 2\pi z - \frac{1}{2} \quad , \quad f_2(z) = (2\pi z)^2 - 2(2\pi z) + \frac{7}{8} + r \\ & f_3(z) = (2\pi z)^3 - \frac{9}{2}(2\pi z)^2 + \left(\frac{41}{8} + 6r\right) 2\pi z - \frac{17}{16} - 3r \end{aligned} \tag{6.31}$$

Now consider the p^{th} derivative of the partition function at $\beta = 2\pi$ using the operator $\beta\partial_\beta$. The derivative of the first term $G(\beta)V(\beta)$ would be given by:

$$\begin{aligned} \left(\beta\frac{\partial}{\partial\beta}\right)^p G(\beta)V(\beta)\bigg|_{\beta=2\pi} &= \left(\beta\frac{\partial}{\partial\beta}\right)^p \left[\frac{e^{-\widehat{E}_0\beta}}{|\eta(i\beta/2\pi)|^2} - 2\frac{e^{-(\widehat{E}_0+1)\beta}}{|\eta(i\beta/2\pi)|^2} + \frac{e^{-(\widehat{E}_0+2)\beta}}{|\eta(i\beta/2\pi)|^2}\right]\Bigg|_{\beta=2\pi} \\ &\equiv \frac{(-1)^p e^{-2\pi\widehat{E}_0}}{|\eta(i)|^2} b_p(\widehat{E}_0) \end{aligned} \tag{6.32}$$

where $b_p(x) = f_p(x) - 2e^{-2\pi} f_p(x+1) + e^{-4\pi} f_p(x+2)$ and $\widehat{E}_0 = E_0 + \frac{1}{12}$. Similarly the derivative of the second term $G(\beta)N(\beta)$ is:

$$\begin{aligned} \left(\beta\frac{\partial}{\partial\beta}\right)^p G(\beta)N(\beta)\bigg|_{\beta=2\pi} &= \sum_i \left(\beta\frac{\partial}{\partial\beta}\right)^p \frac{e^{-\left(\widehat{E}_0+\Delta_i\right)\beta}}{|\eta(i\beta/2\pi)|^2}\Bigg|_{\beta=2\pi} \\ &= \frac{(-1)^p e^{-2\pi\widehat{E}_0}}{|\eta(i)|^2} \sum_i e^{-2\pi\Delta_i} \; f_p\left(\widehat{E}_0 + \Delta_i\right) \end{aligned} \tag{6.33}$$

Recall that for odd integer values of p we have

$$\left(\beta\frac{\partial}{\partial\beta}\right)^p \mathcal{Z}(\beta)\bigg|_{\beta=2\pi} = 0 \implies \frac{(-1)^p e^{-2\pi\widehat{E}_0}}{|\eta(i)|^2} \left[b_p(\widehat{E}_0) + \sum_i e^{-2\pi\Delta_i} \; f_p\left(\widehat{E}_0 + \Delta_i\right)\right] = 0 \tag{6.34}$$

Thus the constraint we obtain for odd integer values of p on the partition function is:

$$\implies \sum_i e^{-2\pi\Delta_i} f_p(\widehat{E}_0 + \Delta_i) = -b_p(\widehat{E}_0) \tag{6.35}$$

Using the $p = 1, 3$ constraints, we can write

$$\sum_i e^{-2\pi\Delta_i} \left(f_3(\widehat{E}_0 + \Delta_i) b_1(\widehat{E}_0) - f_1(\widehat{E}_0 + \Delta_i) b_3(\widehat{E}_0)\right) = 0 \tag{6.36}$$

Since the polynomial is cubic in Δ, it has either one real root and two imaginary roots, or all three real roots. We will denote the largest real root Δ_+. Note that the polynomial goes to $+\infty$ as $\Delta \to \infty$, which implies that the polynomial takes strictly positive values for all $\Delta > \Delta_+$. Also note that the polynomial is positive for $\Delta = 0$. Since we have ordered the scaling dimensions in an increasing order: $\Delta_0 = 0 < \Delta_1 < \Delta_2 < \cdots$, if $\Delta_1 > \Delta_+$, then the exponential in Eq. (6.36) is positive and the polynomial will only take positive values. This leads to a contradiction as the LHS of Eq. (6.36) is a sum of positive terms and hence cannot vanish identically.

Thus we conclude that a general two-dimensional conformal field theory with $c, \bar{c} > 1$, with no chiral algebra and a discrete operator spectrum, there exists a non-trivial primary operator with scaling dimension bounded above by Δ_+.

We would now like to write down an expression for the bound Δ_+. We will do this in the large central charge limit, $c + \bar{c} >> 1$, as this is amenable to semi-classical holographic interpretations. Since we are in the large central charge limit, we will expand Δ_+ as a Laurent series in $c + \bar{c}$:

$$\Delta_+ = \sum_{n=-\infty}^{\infty} \delta_n \left(\frac{c+\bar{c}}{24}\right)^n \tag{6.37}$$

The defining equation for Δ_+ is Eq. (6.36) or equivalently:

$$f_3(\widehat{E}_0 + \Delta_+) b_1(\widehat{E}_0) - f_1(\widehat{E}_0 + \Delta_+) b_3(\widehat{E}_0) = 0 \tag{6.38}$$

Note that in the $c + \bar{c} \to \infty$ limit, the above equation has to be satisfied order by order in $c + \bar{c}$. Let us examine the behaviour of the two terms in the constraint equation for $\Delta_+ \sim c + \bar{c}^r$, where $r > 0$:

$$f_1(\widehat{E}_0 + \Delta_+) b_3(\widehat{E}_0) \sim c + \bar{c}^{r+3} \ , \quad f_3(\widehat{E}_0 + \Delta_+) b_1(\widehat{E}_0) \sim c + \bar{c}^{3r+1} \tag{6.39}$$

Therefore we see that the only value of r for which the constraint can be satisfied is $r = 1$. For $r \leq 0$, the dominant power of $c + \bar{c}$ in both $f_1(\widehat{E}_0 + \Delta_+)$ and $f_3(\widehat{E}_0 + \Delta_+)$ from Eq. (6.31) is unity and hence the constraint can be always be satisfied with a suitable choice of Laurent modes. Therefore the final Laurent expansion for the cut-off Δ_+ we have is:

$$\Delta_+ = \sum_{n=-\infty}^{1} \delta_n \left(\frac{c+\bar{c}}{24}\right)^n \tag{6.40}$$

We will now input this expansion into the constraint and solve it for the Laurent modes order by order. At leading order, the constraint equation implies:

$$-\frac{\pi^4 \left(1-e^{-2\pi}\right)^2}{20736}\left(\delta_1^3-3\delta_1^2+2\delta_1\right)c+\bar{c}^4+\mathcal{O}\left(c+\bar{c}^3\right)=0 \tag{6.41}$$

Thus we see that the constraint is satisfied to leading order in $c+\bar{c}$ for $\delta_1=0,1,2$. We choose the $\delta_1=2$ solution as it will corresponds to the largest solution. Repeating this for the next order after fixing $\delta_1=2$, we get $\delta_0=0.473695+\mathcal{O}\left(10^{-7}\right)$. Therefore for large total central charge, the upper bound on the scaling dimension of the first non-trivial primary is:

$$\Delta_1 \leq \Delta_+ = \frac{c+\bar{c}}{12}+0.473695+\mathcal{O}\left(c+\bar{c}^{-1}\right) \tag{6.42}$$

Here comes holography

We now turn to the holographic interpretation of our results in the previous section. We have derived a completely universal bound for two-dimensional conformal field theories subject to some mild conditions. The AdS/CFT correspondence connects this bound to gravitational physics in anti-de Sitter spacetime and enables us to derive a non-trivial, non-asymptotic constraint on the mass of excitation in the spectrum of quantum gravity with $\Lambda<0$.

Recall that the relation between the central charge and the cosmological constant [57] in the context of the $\mathrm{AdS}_3/\mathrm{CFT}_2$, given in Eq. (6.6), predates the original proposal by [40]. Beyond this, we now wish to match the spectrum of primary operators in the boundary CFT with massive states in the bulk. A primary state of the CFT must be thought of as corresponding to a massive state at rest in the bulk with respect to the global time coordinate of AdS, since its energy cannot be lowered by the action of boost generators. Recall that the Hamiltonian operator in a 2d CFT is proportional to the dilation operator. This is why the global time coordinate of AdS has been chosen specifically, as the dilation operator in these coordinates is proportional to ∂_τ, where τ is the global time. Following this logic we have:

$$E^{(rest)}=\frac{\Delta}{\ell} \tag{6.43}$$

where $E^{(rest)}$ is the rest energy of an object in the bulk of AdS and Δ is the scaling dimension of a primary operator. Note that this corresponds to the mass relation in Eq. (6.7).

We now turn to the bulk interpretation of descendants of primary states. The descendants obtained by acting $L_{-n},\bar{L}_{-n}$ with $n\geq 2$ on the primary states correspond to quadrapole and higher modes of the metric. In 2+1 dimensions, these are localised at spatial infinity and can be thought of as boundary gravitons or boundary metric excitations. On the other hand, states obtained by acting $L_{-1}, \bar{L}_{-1}$ excites the dipole modes of the metric which, in a theory of gravity, is equivalent to adding linear/angular momentum to a bulk state. Therefore, these operators boost the massive object in the bulk to a state of motion with higher energy. Naturally, these transformations are pure gauge when applied to the vacuum, but are not pure gauge when applied to a state with a massive object in the bulk. We thus have the interpretation that primary states of the CFT correspond to massive states at rest in the bulk, with no boundary graviton excitations. While their descendants correspond to states either in a non-zero state of motion, or with some boundary excitations, or both.

We now turn to the interpretation of our bound in the holographic context. Interpreting a primary state of the CFT as a massive bulk state, we see that the bound on scaling dimension of the CFT translates to a bound on the lowest massive excitation in the bulk. Therefore we see that the $M = -1/8G$ BTZ black hole corresponds to the vacuum state of the CFT whereas those with $M \geq 0$ correspond to the primary states in the CFT. Thus every consistent theory of quantum gravity with $\Lambda = -\ell^{-2} < 0$ must necessarily contain a massive state in the bulk (with no boundary gravitons excited), with center-of-mass energy equal to:

$$M_1 \leq M_+ \equiv \frac{\Delta_+}{\ell}\bigg|_{c+\bar{c}=\frac{3\ell}{G}} \tag{6.44}$$

This is true, of course, for only those values of the AdS radius which correspond to $c, \bar{c} > 1$. Therefore the lowest massive bulk excitation in the flat space limit ($\ell \to \infty$) becomes:

$$M_1 < \frac{1}{4G} \tag{6.45}$$

Since we have already seen that quantum gravity with $\Lambda < 0$ has a mass gap of $1/8G$, we see that our result is consistent with the bound from pure gravity via holography. It is still an open problem to exactly match this bound from a purely CFT computation.

6.4 Lightcone Modular Bootstrap: Modular S Transformation

We will now consider the large c limit of generic CFT, i.e. CFT dual to pure gravity. These are necessarily irrational and a measure of the *irrationality* of these CFT is given by its twist gap. The *twist*, $2t$ of an operator is defined as:

$$2\,t = \Delta - |j| = 2\,\min(h,\bar{h}) \tag{6.46}$$

Twist gap of a CFT is defined to be the twist of the lowest twist non-trivial operator. Thus, theories with conserved currents, i.e. with a chiral operator, such as rational CFT, necessarily have a vanishing twist gap.

In this section, we will use the technique of lightcone modular bootstrap to prove an upper bound on the twist gap of any unitary CFT and to compute the universal-Cardy like growth for the large spin states. We will consider purely imaginary modular parameters $\tau,\bar{\tau}$, such that:

$$\tau = \frac{i\beta}{2\pi}\ ,\ \bar{\tau} = -\frac{i\bar{\beta}}{2\pi} \qquad \Longrightarrow \qquad q = e^{-\beta}\ ,\ \bar{q} = e^{-\bar{\beta}} \tag{6.47}$$

where $\beta,\bar{\beta}$ are independent positive numbers. The lightcone limit is defined to the limit where we will take $\beta \to 0$ while keeping $\bar{\beta}$ fixed. We will denote the modular $\mathcal{S}$ transformed variables with a prime. Thus, the modular bootstrap equation implied by the modular $\mathcal{S}$ transformation is given by the invariance of the partition function, $\mathcal{Z}(\tau,\bar{\tau})$ of the CFT.

Twist gap

We will study 2d CFT with finite twist gap, $2\ t_{gap} > 0$, i.e. CFT with no conserved currents. Under this assumption, the torus partition function can be expressed as:

$$\mathcal{Z}(\tau,\bar{\tau}) = \chi_0(\tau)\bar{\chi}_0(\bar{\tau}) + \sum_{h,\bar{h}\geq t_{gap}} n_{h,\bar{h}}\ \chi_h(\tau)\bar{\chi}_{\bar{h}}(\bar{\tau}) \tag{6.48}$$

where $n_{h,\bar{h}} \in \mathbb{N}$ denotes the number of Virasoro primaries with conformal dimensions $(h,\bar{h})$. One can also express the partition function as the following integral over the *half-twist*, t:

$$\mathcal{Z}(\tau,\bar{\tau}) = \sum_{j\in\mathbb{Z}} \int_0^\infty dt\ \rho_j(t)\ \chi_h(\tau)\bar{\chi}_{\bar{h}}(\bar{\tau}) \tag{6.49}$$

where we change the dummy variable from $h,\bar{h}$ to j,t. The density of Virasoro primaries, $\rho_j(t)$ is defined to be the discrete sum of Virasoro primaries with spin j, i.e.

$$\rho_j(t) = \sum_{\mathcal{O}\text{ with spin }j} \delta(t - t_{\mathcal{O}}) \tag{6.50}$$

where $t_{\mathcal{O}} = \min\,(h_{\mathcal{O}}, \bar{h}_{\mathcal{O}})$.

The partition function in the dual channel, i.e. the modular $\mathcal{S}$ transformed partition function can be written as:

$$\begin{aligned}\mathcal{Z}(\tau', \bar{\tau}') &= \sum_{j\in\mathbb{Z}} \int_0^\infty dt\ \rho_j(t)\ \chi_h(\tau')\bar{\chi}_{\bar{h}}(\bar{\tau}') \\ &= \chi_0(\tau')\bar{\chi}_0(\bar{\tau}') + \sum_{j\in\mathbb{Z}} \int_{t_{gap}}^\infty dt\ \rho_j(t)\ \chi_h(\tau')\bar{\chi}_{\bar{h}}(\bar{\tau}')\end{aligned} \tag{6.51}$$

where the final line follows from the definition of twist gap and the requirement of a non-degenerate vacuum of a CFT. Since these are Virasoro characters, we can substitute the explicit expression of the characters in terms of $\beta, \bar{\beta}$. However, with foresight, we will only substitute the holomorphic Virasoro characters so as to make the β dependence explicit. After said substitution, we obtain:

$$\mathcal{Z}(\tau', \bar{\tau}') = \frac{e^{\frac{4\pi^2}{\beta}\gamma}}{\eta(\tau')} \left[(1 - e^{-\frac{4\pi^2}{\beta}})\bar{\chi}_0(\bar{\tau}') + \sum_{j\in\mathbb{Z}} \int_{t_{gap}}^\infty dt\ \rho_j(t)\ e^{-\frac{4\pi^2}{\beta}h}\bar{\chi}_{\bar{h}}(\bar{\tau}') \right] , \quad \gamma \equiv \frac{c-1}{24} \tag{6.52}$$

In the lightcone limit, $\beta \to 0$, with $\bar{\beta}$ fixed, we see that the vacuum term in the dual channel is divergent and dominates the contribution from the rest of the spectrum. Thus in the lightcone limit, the partition function in the dual channel can be written as:

$$\mathcal{Z}(\tau', \bar{\tau}') = \frac{e^{\frac{4\pi^2}{\beta}\gamma} e^{\frac{4\pi^2}{\bar{\beta}}\gamma}}{\eta(\tau')\eta(\bar{\tau}')} \left[\left(1 - e^{-\frac{4\pi^2}{\beta}}\right) \left(1 - e^{-\frac{4\pi^2}{\bar{\beta}}}\right) + \cdots \right] \tag{6.53}$$

The invariance of the partition function under modular transformations implies that the partition function in the dual channel $\mathcal{Z}(\tau', \bar{\tau}')$ is equal to $\mathcal{Z}(\tau, \bar{\tau})$. This implies that:

$$\begin{aligned}&\frac{e^{\gamma\beta}e^{\gamma\bar{\beta}}}{\eta(q)\eta(\bar{q})} \left[(1 - e^{-\beta})(1 - e^{-\bar{\beta}}) + \sum_{j\in\mathbb{Z}} \int_{t_{gap}}^\infty dt\ \rho_j(t)\ e^{-\beta h}e^{-\bar{\beta}\bar{h}} \right] \\ =&\frac{e^{\frac{4\pi^2}{\beta}\gamma} e^{\frac{4\pi^2}{\bar{\beta}}\gamma}}{\eta(\tau')\eta(\bar{\tau}')} \left[\left(1 - e^{-\frac{4\pi^2}{\beta}}\right) \left(1 - e^{-\frac{4\pi^2}{\bar{\beta}}}\right) + \cdots \right]\end{aligned} \tag{6.54}$$

One can now use the transformation of the Dedekind eta function:

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \sqrt{-i(c\tau + d)}\ \eta(\tau) \tag{6.55}$$

to obtain the bootstrap equation in the lightcone limit:

$$\begin{aligned}
&e^{\gamma\beta}e^{\gamma\bar{\beta}}(1-e^{-\beta})(1-e^{-\bar{\beta}})+\sum_{j\in\mathbb{Z}}\int_{t_{gap}}^{\infty}dt\ \rho_j(t)\ e^{-\beta(h-\gamma)}e^{-\bar{\beta}\left(\bar{h}-\gamma\right)} \\
=\ \ &\frac{2\pi}{\sqrt{\beta\bar{\beta}}}\ e^{\frac{4\pi^2}{\beta}\gamma}e^{\frac{4\pi^2}{\bar{\beta}}\gamma}\left[\left(1-e^{-\frac{4\pi^2}{\beta}}\right)\left(1-e^{-\frac{4\pi^2}{\bar{\beta}}}\right)+\cdots\right]
\end{aligned}\tag{6.56}$$

Note that the RHS diverges in the $\beta\to 0$ limit, but no single term diverges on the LHS. Therefore the divergence on the RHS has to be reproduced by an infinite number of terms on the LHS. Since no single term diverges on the LHS, we will ignore the vacuum term on the LHS. The remaining sum over j can be broken into two regions, $j\geq 0$ and $j<0$. Since $j=h-\bar{h}$, the former region implies that $t=\bar{h}$, whereas the latter one implies that $t=h$. Therefore the LHS can be written as:

$$\begin{aligned}
LHS&=\sum_{j=0}^{\infty}\int_{t_{gap}}^{\infty}d\bar{h}\ \rho_j(\bar{h})\ e^{-\beta h-\bar{\beta}\bar{h}+(\beta+\bar{\beta})\gamma}+\sum_{j=-\infty}^{-1}\int_{t_{gap}}^{\infty}dh\ \rho_j(h)\ e^{-\beta h-\bar{\beta}\bar{h}+(\beta+\bar{\beta})\gamma} \\
&=\sum_{j=0}^{\infty}e^{-\beta j}\int_{t_{gap}}^{\infty}d\bar{h}\ \rho_j(\bar{h})\ e^{-(\beta+\bar{\beta})\left(\bar{h}-\gamma\right)}+\sum_{j=-\infty}^{-1}e^{\bar{\beta}j}\int_{t_{gap}}^{\infty}dh\ \rho_j(h)\ e^{-(\beta+\bar{\beta})(h-\gamma)}
\end{aligned}\tag{6.57}$$

In the $\beta\to 0$ limit (with $\bar{\beta}$ finite), the second term in the equation above as finite, as there is an unhindered exponential fall-off due to sum on j. Therefore we will drop this term as well and hence the modular bootstrap equation in the lightcone limit becomes:

$$\sum_{j=0}^{\infty}e^{-\beta j}\int_{t_{gap}}^{\infty}d\bar{h}\ \rho_j(t)\ e^{-\bar{\beta}\left(\bar{h}-\gamma\right)}=\frac{2\pi}{\sqrt{\beta\bar{\beta}}}\ e^{\frac{4\pi^2}{\beta}\gamma}e^{\frac{4\pi^2}{\bar{\beta}}\gamma}\left(1-e^{-\frac{4\pi^2}{\beta}}\right)\left(1-e^{-\frac{4\pi^2}{\bar{\beta}}}\right)+\cdots\tag{6.58}$$

We will show that the twist gap is bounded from above by using proof by contradiction. The claim is that the twist cannot be larger than $\frac{c-1}{12}$. Let us assume the contrary, i.e $2\ t_{gap}>\frac{c-1}{12}$ and multiply the modular bootstrap equation Eq. (6.58) by $e^{\bar{\beta}(t_{gap}-\gamma)}$ to obtain:

$$\begin{aligned}
&\sum_{j=0}^{\infty}e^{-\beta j}\int_{t_{gap}}^{\infty}d\bar{h}\ \rho_j(t)\ e^{-\bar{\beta}\left(\bar{h}-t_{gap}\right)} \\
=\ \ &\frac{2\pi}{\sqrt{\beta\bar{\beta}}}\ e^{\frac{4\pi^2}{\beta}\gamma}e^{\frac{4\pi^2}{\bar{\beta}}\gamma+\bar{\beta}(t_{gap}-\gamma)}\left(1-e^{-\frac{4\pi^2}{\beta}}\right)\left(1-e^{-\frac{4\pi^2}{\bar{\beta}}}\right)+\cdots
\end{aligned}\tag{6.59}$$

Now let us consider the $\bar{\beta}$ derivative of this expression, the LHS becomes:

$$\partial_{\bar{\beta}}\,\text{LHS} = -\sum_{j=0}^{\infty} e^{-\beta j} \int_{t_{gap}}^{\infty} d\bar{h}\ (\bar{h} - t_{gap})\ \rho_j(t)\ e^{-\bar{\beta}\left(\bar{h}-t_{gap}\right)} \tag{6.60}$$

which is a strictly negative expression. For brevity, we shall denote all the β dependent factors on the RHS as A, which is a positive constant with respect to $\bar{\beta}$. Therefore the derivative of the RHS becomes:

$$\partial_{\bar{\beta}}\,\text{RHS} = \frac{A}{\sqrt{\bar{\beta}}}\ e^{\frac{4\pi^2}{\bar{\beta}}\gamma + \bar{\beta}(t_{gap}-\gamma)} \left(t_{gap} - \gamma - \frac{1}{2\bar{\beta}} \left(\frac{8\pi^2}{\bar{\beta}} + \frac{8\pi^2}{\bar{\beta}}\gamma \right) \right) \tag{6.61}$$

Since $t_{gap} > \gamma$ and the other terms in the bracket have a vanishing contribution in the $\bar{\beta} \to \infty$ limit, this expression will become positive for large enough $\bar{\beta}$, which is a contradiction. Therefore we conclude that for an unitary 2d CFT, with a fixed central charge c, the twist gap is bounded above by $2\gamma = \frac{c-1}{12}$, which is a weaker bound than that implied by the BTZ black hole spectrum, i.e. $c/24$.

Extended Cardy formula

In this subsection, we will investigate the universal spectrum of large spin Virasoro primaries for CFT with non-zero twist gap. Additionally, we shall show that this universal density of primaries satisfies the modular bootstrap equation implied by the modular $\mathcal{S}$ transformation Eq. (6.58).

As we have already seen, the vacuum contribution in the dual channel, in the lightcone limit, dominates the contribution from the other primaries, therefore we can approximate the partition function in the dual channel as:

$$\mathcal{Z}(\tau', \bar{\tau}') \approx \chi_0(\tau')\bar{\chi}_0(\bar{\tau}') \tag{6.62}$$

Recall that the partition function can be written as a sum over conformal dimensions Eq. (6.48) with a discrete density of states $n_{h,h}$. This can be promoted to an integral over $h, \bar{h}$ with a density of states, $\rho(h, \bar{h})$ defined as a discrete sum of delta functions. Therefore one can express the partition function as:

$$\mathcal{Z}(\tau, \bar{\tau}) = \int dh\ d\bar{h}\ \rho(h, \bar{h})\ \chi_h(\tau)\bar{\chi}_{\bar{h}}(\bar{\tau}) \tag{6.63}$$

The modular $\mathcal{S}$-matrix (which is an infinite dimensional matrix since we are considering irrational CFT) in this case is defined as:

$$\chi_{h'}(\tau') = \int dh\ \mathcal{S}(h', h)\ \chi_h(\tau) \tag{6.64}$$

This can be trivially extended to the anti-holomorphic characters as well. This implies that

$$\chi_0(\tau')\bar{\chi}_0(\bar{\tau}') = \int dh\, d\bar{h}\; \mathcal{S}(0,h)\mathcal{S}(0,\bar{h})\; \chi_h(\tau)\bar{\chi}_{\bar{h}}(\bar{\tau}) \tag{6.65}$$

Using the invariance of the partition function under the modular $\mathcal{S}$-transformation as well as equations (6.62), (6.63) and (6.65), we see that the universal density of states can be expressed as:

$$\rho(h,\bar{h}) = \mathcal{S}(0,h)\mathcal{S}(0,\bar{h}) \tag{6.66}$$

Now it has been shown in [58] that $\mathcal{S}(0,h)$ or $c > 1$ CFT can be expressed as:

$$\mathcal{S}(0,h) = \sqrt{\frac{2}{h-\gamma}}\left[\cosh\left(4\pi\sqrt{\gamma(h-\gamma)}\right) - \cosh\left(4\pi\sqrt{(\gamma-1)(h-\gamma)}\right)\right] \tag{6.67}$$

We will now consider the large spin limit, $j >> c$ with a non-zero twist gap $t_{gap} > 0$, i.e. $\bar{h} > t_{gap}$. To obtain the large spin behaviour of the universal density of states, let us examine the $h, \gamma \to \infty$ limit of $\mathcal{S}(0,h)$:

$$\begin{aligned} \mathcal{S}(0,h\to\infty) &= \sqrt{\frac{2}{h-\gamma}}\left[\cosh\left(4\pi\sqrt{\gamma(h-\gamma)}\right) - \cosh\left(4\pi\sqrt{(\gamma-1)(h-\gamma)}\right)\right] \\ &\approx \frac{e^{4\pi\sqrt{\gamma\,(h-\gamma)}}}{\sqrt{2(h-\gamma)}} \end{aligned} \tag{6.68}$$

Therefore we see that in the large spin limit, the universal density of Virasoro primaries, which we will now denote by $\rho_j(\bar{h})$, for a generic 2d CFT with a finite twist gap $2t_{gap} = \frac{c-1}{12}$ can be expressed as:

$$\begin{aligned} \rho_j(\bar{h}) = &\frac{e^{4\pi\sqrt{\gamma(\bar{h}+j-\gamma)}}}{\sqrt{\left(\bar{h}+j-\gamma\right)\,\left(\bar{h}-\gamma\right)}}\,\Theta\left(\bar{h}-\gamma\right) \\ &\times\left[\cosh\left(4\pi\sqrt{\gamma(\bar{h}-\gamma)}\right) - \cosh\left(4\pi\sqrt{(\gamma-1)(\bar{h}-\gamma)}\right)\right] \end{aligned} \tag{6.69}$$

Note that we have expressed this equation in terms of $\bar{h}, j$ instead of $h, \bar{h}$. To see that this generalises the Cardy formula, consider the large $\bar{h}$ limit as well to obtain:

$$\rho_j(\bar{h}\to\infty) \longrightarrow e^{4\pi\sqrt{\gamma(h-\gamma)}+4\pi\sqrt{\gamma(\bar{h}-\gamma)}} \tag{6.70}$$

which is the Cardy formula for a generic CFT. Thus we see that Eq. (6.69) extends the range of applicability of Cardy formula $h, \bar{h} >> c$.

Now let us verify that this universal density of large spin primaries satisfies our bootstrap equation Eq. (6.58). We will reproduce the divergence on the RHS with the infinite number of terms on the LHS. We begin by substituting $\rho_j(\bar{h})$ into the RHS of Eq. (6.58), to obtain:

$$\begin{aligned}\sum_{j=0}^{\infty} e^{-\beta j} \int_{\gamma}^{\infty} d\bar{h}\; \mathcal{S}(0,h)\mathcal{S}(0,\bar{h})\; e^{-\bar{\beta}(\bar{h}-\gamma)} \\ = \frac{2\pi}{\sqrt{\beta\bar{\beta}}} e^{\frac{4\pi^2}{\beta}\gamma} e^{\frac{4\pi^2}{\beta}\gamma} \left(1-e^{-\frac{4\pi^2}{\beta}}\right)\left(1-e^{-\frac{4\pi^2}{\beta}}\right)\end{aligned} \tag{6.71}$$

It can be shown in the small β limit that the sum over j on the LHS can be approximated by an integral over j, up to a term that is β-independent, refer to Appendix A of [52]. This is done by using the Abel-Plana formula which states that for a well-behaved function $f(z)$, the following relation is true:

$$\begin{aligned}\sum_{k=0}^{\infty} f(k) = \int_0^{\infty} dz\; f(z) + \frac{1}{2} f(0) + i\int_0^{\infty} dz\; \frac{f(iz)-f(-iz)}{e^{2\pi z}-1} \\ + \lim_{\Lambda\to\infty}\left(\frac{1}{2}f(\Lambda) + i\int_0^{\infty} dz\; \frac{f(\Lambda-iz)-f(\Lambda+iz)}{e^{2\pi z}-1}\right)\end{aligned} \tag{6.72}$$

By choosing the summand of Eq. (6.58) as $f(j)$, we see that all the other terms do not diverge in the $j\to\infty$ limit. Thus we can write:

$$\begin{aligned}\int_0^{\infty} dj\; e^{-\beta j} \int_{\gamma}^{\infty} d\bar{h}\; \mathcal{S}(0,\bar{h}+j)\mathcal{S}(0,\bar{h})\; e^{-\bar{\beta}(\bar{h}-\gamma)} \\ = \frac{2\pi}{\sqrt{\beta\bar{\beta}}} e^{\frac{4\pi^2}{\beta}\gamma} e^{\frac{4\pi^2}{\beta}\gamma} \left(1-e^{-\frac{4\pi^2}{\beta}}\right)\left(1-e^{-\frac{4\pi^2}{\beta}}\right)\end{aligned} \tag{6.73}$$

Let us focus on the LHS now:

$$\begin{aligned}\int_0^{\infty} dj\; e^{-\beta j} \int_{\gamma}^{\infty} d\bar{h}\; \mathcal{S}(0,\bar{h}+j)\mathcal{S}(0,\bar{h})\; e^{-\bar{\beta}(\bar{h}-\gamma)} \\ = \int_{\gamma}^{\infty} d\bar{h}\; \mathcal{S}(0,\bar{h})\; e^{-\bar{\beta}(\bar{h}-\gamma)} \int_0^{\infty} dj\; \mathcal{S}(0,\bar{h}+j) e^{-\beta j}\end{aligned} \tag{6.74}$$

We can now perform the substitution $z = j + \bar{h}$ to obtain:

$$\begin{aligned}\int_{\gamma}^{\infty} d\bar{h}\ \mathcal{S}(0,\bar{h})\ e^{-\bar{\beta}(\bar{h}-\gamma)} \int_{0}^{\infty} dj\ \mathcal{S}(0,\bar{h}+j)e^{-\beta j}& \\ &= \int_{\gamma}^{\infty} d\bar{h}\ \mathcal{S}(0,\bar{h})\ e^{-\bar{\beta}(\bar{h}-\gamma)} \int_{\gamma}^{\infty} dz\ \mathcal{S}(0,z)e^{-\beta(z-\gamma)}e^{\beta(\bar{h}-\gamma)} \\ &= \int_{\gamma}^{\infty} d\bar{h}\ \mathcal{S}(0,\bar{h})\ e^{-\bar{\beta}(\bar{h}-\gamma)} \int_{\gamma}^{\infty} dz\ \mathcal{S}(0,z)e^{-\beta(z-\gamma)}\end{aligned} \tag{6.75}$$

where we have added the integral over z from $\gamma \to \bar{h}$, in the second line, because it is finite in the $\beta \to 0$ limit. Similarly in this limit, the exponential $e^{\beta(\bar{h}-\gamma)}$ goes to 1. Now we recall the defining equation for the modular $\mathcal{S}$ matrix Eq. (6.65). For $h' = 0$ we get:

$$e^{\frac{4\pi^2}{\bar{\beta}}\gamma}\left(1 - e^{-\frac{4\pi^2}{\bar{\beta}}}\right) = \sqrt{\frac{\bar{\beta}}{2\pi}} \int_{\gamma}^{\infty} d\bar{h}\ \mathcal{S}(0,\bar{h})\ e^{-\bar{\beta}(\bar{h}-\gamma)} \tag{6.76}$$

Note that the integrals over $\bar{h}$ and z in Eq. (6.75) is of this form, and hence we can use Eq. (6.76) to obtain:

$$\int_{\gamma}^{\infty} d\bar{h}\ \mathcal{S}(0,\bar{h})\ e^{-\bar{\beta}(\bar{h}-\gamma)} \int_{\gamma}^{\infty} dz\ \mathcal{S}(0,z)e^{-\beta(z-\gamma)} = \frac{2\pi}{\sqrt{\beta\bar{\beta}}} e^{\frac{4\pi^2}{\beta}\gamma} e^{\frac{4\pi^2}{\bar{\beta}}\gamma}\left(1 - e^{-\frac{4\pi^2}{\beta}}\right)\left(1 - e^{-\frac{4\pi^2}{\bar{\beta}}}\right) \tag{6.77}$$

which matches exactly with the RHS of Eq. (6.58).

Let us make a few comments on the corrections to this universal density of large spin Virasoro primaries Eq. (6.69). Firstly, there might be error terms because we are approximating a discrete spectrum by a continuous density of states (Cardy formula suffers from the same issue). Secondly, there are contributions from the lowest twist, non-vacuum primary operators. Lastly there might be vacuum contributions from elements of the modular group $PSL(2,\mathbb{Z})$ other than the modular $\mathcal{S}$ transformation.

6.5 Lightcone Modular Bootstrap: Generalised Transformations

We will now attempt to address the last correction to $\rho_j(\bar{h})$. Instead of considering the modular bootstrap equation implied from the modular $\mathcal{S}$ transformation we consider a more general set of transformations. To do so, we will add a rational real part to the modular parameter. In particular, we have the following:

$$\tau = i\frac{\beta}{2\pi} + \frac{r}{s}\ , \quad \bar{\tau} = -i\frac{\bar{\beta}}{2\pi} + \frac{r}{s} \implies q = e^{-\beta + 2\pi i \frac{r}{s}}\ , \quad \bar{q} = e^{-\bar{\beta} - 2\pi i \frac{r}{s}} \tag{6.78}$$

where r, s are co-prime integers with s positive. We will again consider the lightcone limit $\beta \to 0$ at fixed $\bar{\beta}$ and perform the following modular transformation on $\tau, \bar{\tau}$:

$$\begin{pmatrix} a & b \\ s & -r \end{pmatrix} \in SL(2, \mathbb{Z})\ , \quad -ar - bs = 1 \tag{6.79}$$

which gives:

$$\tau' = \frac{2\pi i}{s^2\beta} + \frac{a}{s}\ , \ \bar{\tau}' = -\frac{2\pi i}{\bar{\beta}s^2} + \frac{a}{s} \implies q' = e^{-\frac{4\pi^2}{s^2\beta} + 2\pi i \frac{a}{s}}\ , \ \bar{q}' = e^{-\frac{4\pi^2}{s^2\bar{\beta}} - 2\pi i \frac{a}{s}} \tag{6.80}$$

Now modular invariance dictates that the partition function remains unchanged under this transformation. Following exactly the same steps as before, since the modular transformation we are considering is a generalisation of the modular $\mathcal{S}$ transformation, it is easy to obtain:

$$\begin{aligned} &\sum_{j=0}^{\infty} e^{-\left(\beta - 2\pi i \frac{r}{s}\right) j} \int_{t_{gap}}^{\infty} d\bar{h}\ \rho_j(\bar{h})\ e^{-\bar{\beta}\left(\bar{h} - \gamma\right)} \\ &= \frac{2\pi}{s\sqrt{\beta\bar{\beta}}}\ e^{\frac{4\pi^2}{s^2\beta}\gamma} e^{\frac{4\pi^2}{s^2\bar{\beta}}\gamma} \left(1 - e^{-\frac{4\pi^2}{s^2\beta} - 2\pi i \frac{\left(r^{-1}\right)_s}{s}}\right) \left(1 - e^{-\frac{4\pi^2}{s^2\bar{\beta}} - 2\pi i \frac{\left(r^{-1}\right)_s}{s}}\right) + \cdots \end{aligned} \tag{6.81}$$

where we have used Eq. (6.55) for the present case and replaced a by $-\left(r^{-1}\right)_s$, since $ar \equiv -1(\text{mod})s$. We will henceforth refer to this equation as the generalised modular bootstrap equation. Note that we obtain Eq. (6.58) by setting $r = 0, s = 1$, as expected. This can be interpreted as the modular bootstrap equation associated to the more general $PSL(2, \mathbb{Z})$ elements Eq. (6.79).

It is obvious that the generalised density of states in Eq. (6.69) does not satisfy the above modular crossing equation for generic values of r, s. This is because Eq. (6.69) is independent of the integer r which appears in the exponential of the generalised bootstrap equation. This implies that the density of large spin states must receive additional universal contributions that are sub-leading to Eq. (6.69). The solution to Eq. (6.81) involves the Kloosterman sum, defined as:

$$S(j, J; s) = \sum_{\substack{r:\gcd(r,s)=1 \\ 0 \le r < s}} exp\left(2\pi i \frac{rj + \left(r^{-1}\right)_s J}{s}\right) \tag{6.82}$$

The Kloosterman sum satisfies the following identities, which will prove to be very useful later on:

$$\sum_{j=0}^{\mathrm{lcm}(s,s')-1} S(j,J;s)e^{-2\pi i\frac{rj}{s}} = s\, e^{2\pi i\frac{\left(r^{-1}\right)_s J}{s}}\delta_{s,s'}\ , \quad S(j,J;1)=1 \tag{6.83}$$

For brevity, we will also define the following functions for $i=0,1$:

$$d_n(h,s) = \sqrt{\frac{2}{s(h-\gamma)}}\cosh\left(\frac{4\pi}{s}\sqrt{\left(\gamma-\delta_{n,1}\right)\left(h-\gamma\right)}\right)\Theta\left(h-\gamma\right) \tag{6.84}$$

It is clear that the modular $\mathcal{S}$ kernel, $\mathcal{S}(0,h)$ in the previous section is equal to $d_0(h,1)-d_1(h,1)$.

We claim that the the universal density of large spin states that satisfies the generalised modular bootstrap equation Eq. (6.81) is given by:

$$\rho_j(\bar{h}) = \sum_{s'=1}\rho_{j,s'}(\bar{h}+j,\bar{h})\ , \quad j>>c \tag{6.85}$$

where:

$$\begin{aligned}\rho_{j,s}(h,\bar{h}) = &\ S(j,0;s)d_0(h,s)d_0(\bar{h},s) - S(j,-1;s)d_0(h,s)d_1(\bar{h},s)\\ &-S(j,1;s)d_1(h,s)d_0(\bar{h},s) + S(j,0;s)d_1(h,s)d_1(\bar{h},s)\end{aligned} \tag{6.86}$$

Note that the $s'=1$ term is the extended Cardy formula Eq. (6.69), which grows as $e^{4\pi\sqrt{\gamma j}}$ in the large j limit. The other $s'\neq 0$ terms grow as $e^{\frac{4\pi}{s}\sqrt{\gamma j}}$, and are hence sub-leading corrections to Eq. (6.69).

We will now show that the density of states given in Eq. (6.85) satisfies Eq. (6.81). In particular, we will show that for the generalised modular bootstrap equation labelled by s, only the $\rho_{j,s'=s}$ term produces the leading order divergence of the RHS of Eq. (6.81) in the lightcone limit. We now substitute $\rho_j(\bar{h})$ in the LHS of Eq. (6.81) to obtain:

$$\begin{aligned}&\sum_{j=0}^{\infty} e^{-\left(\beta-2\pi i\frac{r}{s}\right)j}\int_{t_{gap}}^{\infty} d\bar{h}\ \rho_j(\bar{h})\ e^{-\bar{\beta}\left(\bar{h}-\gamma\right)}\\ =&\sum_{s'}\sum_{j=0}^{\infty} e^{2\pi i\frac{rj}{s}}e^{-\beta j}\int_{t_{gap}}^{\infty} d\bar{h}\ e^{-\bar{\beta}\left(\bar{h}-\gamma\right)}\\ &\times\left[S(j,0;s')d_0(h,s')d_0(\bar{h},s') - S(j,-1;s')d_0(h,s')d_1(\bar{h},s')\right.\\ &\left.-S(j,1;s')d_1(h,s')d_0(\bar{h},s') + S(j,0;s')d_1(h,s')d_1(\bar{h},s')\right]\end{aligned} \tag{6.87}$$

We will restrict ourselves to the first term for now. Rewriting $j = \tilde{j} + k\ \text{lcm}(s, s')$ where $k \geq 0, 0 \leq \tilde{j} \leq \text{lcm}(s, s') - 1$ we get:

$$\sum_{s'} \sum_{j=0}^{\infty} e^{2\pi i \frac{rj}{s}} e^{-\beta j} \int_{t_{gap}}^{\infty} d\bar{h}\ e^{-\bar{\beta}\left(\bar{h}-\gamma\right)} S(j, 0; s') d_0(h, s') d_0(\bar{h}, s') = \sum_{s'} \sum_{\tilde{j}=0}^{\text{lcm}(s,s')-1} e^{2\pi i \frac{r\tilde{j}}{s}} S(\tilde{j}, 0; s') \sum_{k=0}^{\infty} e^{-(\tilde{j}+k\ \text{lcm}(s,s'))\beta} \int_{t_{gap}}^{\infty} d\bar{h}\ d_0(\bar{h} + \tilde{j} + k\ \text{lcm}(s, s'), s') d_0(\bar{h}, s') e^{-\left(\bar{h}-\gamma\right)} \tag{6.88}$$

We will now use the Abel-Plana formula to convert the sum over k into an integral over k and perform the following change of variable: $k \to k - \frac{\tilde{j}}{\text{lcm}(s,s')}$. This change of variable will replace $\tilde{j} + k\ \text{lcm}(s, s')$ by $k\ \text{lcm}(s, s')$. Hence the RHS of the above equation becomes:

$$\begin{aligned} &\sum_{s'} \sum_{\tilde{j}=0}^{\text{lcm}(s,s')-1} e^{2\pi i \frac{r\tilde{j}}{s}} S(\tilde{j}, 0; s') \int_{0}^{\infty} dk e^{-(\tilde{j}+k\ \text{lcm}(s,s'))\beta} \\ &\qquad \times \int_{t_{gap}}^{\infty} d\bar{h}\ d_0(\bar{h} + \tilde{j} + k\ \text{lcm}(s, s'), s') d_0(\bar{h}, s') e^{-\left(\bar{h}-\gamma\right)} \\ &= \sum_{s'} \left[\sum_{\tilde{j}=0}^{\text{lcm}(s,s')-1} e^{2\pi i \frac{r\tilde{j}}{s}} S(\tilde{j}, 0; s') \right] \\ &\qquad \int_{0}^{\infty} dk \int_{t_{gap}}^{\infty} d\bar{h}\ d_0(\bar{h} + k\ \text{lcm}(s, s'), s') d_0(\bar{h}, s') e^{-\beta\ k\ \text{lcm}(s,s')} e^{-\bar{\beta}\left(\bar{h}-\gamma\right)} \end{aligned} \tag{6.89}$$

where the integral over k from 0 to $\tilde{j}/\text{lcm}(s, s')$ part has been ignored, since it is finite in the $\beta \to 0$ limit.

We will now use the identity of the Kloosterman sum Eq. (6.83) to simplify the bracketed sum over $\tilde{j}$ to obtain:

$$\begin{aligned} &s \sum_{s'} \delta_{s,s'} \int_{0}^{\infty} dk \int_{t_{gap}}^{\infty} d\bar{h}\ d_0(\bar{h} + k\ \text{lcm}(s, s'), s') d_0(\bar{h}, s') e^{-\beta\ k\ \text{lcm}(s,s')} e^{-\bar{\beta}\left(\bar{h}-\gamma\right)} \\ &\qquad = s \int_{0}^{\infty} dk \int_{t_{gap}}^{\infty} d\bar{h}\ d_0(\bar{h} + ks, s) d_0(\bar{h}, s) e^{-\beta\ k\ s} e^{-\bar{\beta}\left(\bar{h}-\gamma\right)} \end{aligned} \tag{6.90}$$

We can now redefine $z = \bar{h} + ks$ to obtain:

$$\begin{aligned} &s \int_0^\infty dk \int_{t_{gap}}^\infty d\bar{h}\; d_0(\bar{h}+ks, s) d_0(\bar{h}, s) e^{-\beta\, k\, s} e^{-\bar{\beta}\left(\bar{h}-\gamma\right)} \\ &= \int_{t_{gap}}^\infty d\bar{h}\; d_0(\bar{h}, s) e^{-\bar{\beta}\left(\bar{h}-\gamma\right)} \int_{\bar{h}}^\infty dz\; d_0(z, s) e^{-\beta(z-\bar{h})} e^{\beta(\bar{h}-\gamma)} \end{aligned} \tag{6.91}$$

Since $\bar{h} > \gamma$, we can, once again, take the lower limit of the integral over z to be γ by ignoring the integral from $\gamma \to \bar{h}$, as it is finite in the lightcone limit. Similarly, we can replace $e^{-\beta(k-\bar{h})}$ by $e^{-\beta(k-\gamma)}$ in the $\beta \to 0$ limit. Recall Eq. (6.76), since $\mathcal{S}(0, h) = d_0(h, 1) + d_1(h, 1)$, we have:

$$\int_\gamma^\infty dh\; d_n(h, 1) e^{-\beta(h-\gamma)} = \sqrt{\frac{2\pi}{\beta}}\; e^{\frac{4\pi^2}{\beta}\gamma} e^{-\frac{4\pi^2}{\beta}\delta_{n,1}} \tag{6.92}$$

Using the change of variables $h \to h' = \frac{h-\gamma}{s^2} + \gamma$, one can easily show that:

$$\int_\gamma^\infty dh\; d_n(h, s) e^{-\beta(h-\gamma)} = \sqrt{\frac{2\pi}{s\beta}}\; e^{\frac{4\pi^2}{s^2\beta}\gamma} e^{-\frac{4\pi^2}{s^2\beta}\delta_{n,1}} \tag{6.93}$$

Using all of this, Eq. (6.91) can be simplified to give:

$$\begin{aligned} &\int_{t_{gap}}^\infty d\bar{h}\; d_0(\bar{h}, s) e^{-\bar{\beta}\left(\bar{h}-\gamma\right)} \int_{\bar{h}}^\infty dz\; d_0(z, s) e^{-\beta(z-\bar{h})} e^{\beta(\bar{h}-\gamma)} \\ &\qquad\qquad = \int_\gamma^\infty d\bar{h}\; d_0(\bar{h}, s) e^{-\bar{\beta}\left(\bar{h}-\gamma\right)} \int_\gamma^\infty dz\; d_0(z, s) e^{-\beta(z-\bar{h})} \\ &\qquad\qquad = \frac{2\pi}{s\sqrt{\beta\bar{\beta}}}\; e^{\frac{4\pi^2}{s^2\beta}\gamma} e^{\frac{4\pi^2}{s^2\bar{\beta}}\gamma} \end{aligned} \tag{6.94}$$

which is exactly the first piece of the divergent term on the RHS in the generalised modular bootstrap equation Eq. (6.81). We can now repeat this for the remaining three terms in Eq. (6.87). The evaluation of the fourth term is identical to the one we have performed, with the only difference being that $d_1(h, s)$ is integrated in the last step instead of $d_0(h, s)$, which trivially reproduces the last term in Eq. (6.81). The only considerable difference between the evaluation of the first and the second or third term is that, while there was no exponential term after summing over $\tilde{j}$ in Eq. (6.91) for the first term, the other cases get an exponential $\exp\left\{\pm 2\pi i \frac{\left(r^{-1}\right)_s J}{s}\right\}$ as a global factor, with $+/-$ for the third/second term. It is easy to see that just the $\rho_{j,s'=s}$ reproduces the divergent vacuum contribution in the dual channel in Eq. (6.81). Therefore we have verified that the universal density of large spin states Eq. (6.85) satisfies the general modular bootstrap equation Eq. (6.81).

Chapter 7

Conclusions and Future Directions

The main premise of this work was to utilise modular invariance to gain insight into the properties of two-dimensional conformal field theories. The aim of the first part of this work was to attempt the classification of three-character CFT by using the classification scheme proposed in [15]. We started by reviewing some results from the complete classification of two-character rational conformal field theories using quasi-characters [20].

Attempting to improve on the limitation of the MLDE approach in describing the properties of the characters under the modular $\mathcal{S}$-matrix, we investigated a conjectured contour-integral representation of $\ell = 0$ RCFT characters [28]. We were able to use this representation to develop a simple sum-over-paths algorithm to compute the modular $\mathcal{S}$-matrix. We verified our result with many examples, thus giving some evidence for the original conjecture.

Equipped with the properties of the characters of a three-character CFT under the modular $\mathcal{S}$ transformation, we turned to the classification problem of the three-character CFT from the MLDE approach. Using the novel construction of [19] we conjectured an infinite series of families of quasi-characters in the three-character case. We noted that, just like the two-character case, these quasi-character families could be segregated into various fusion classes based on the their modular properties, computed using the contour integral representation.

We then moved on to taking linear combinations of pairs of quasi-characters to obtain admissible characters in order 3, which manifestly had a higher Wronskian index, ℓ. On the way, we provided a general derivation for the Wronskian index of a sum of two quasi-characters of order 3. This sum was taken in such a way that the two sets have the same modular $\mathcal{S}$-matrix and that their central charges differed by an integer multiple of 24. As the final step, we constructed, for both the examples, an explicit physical RCFT whose characters were given by the admissible characters we generated using quasi-characters. This construction was carried out by considering $c = 32$ meromorphic CFT defined using even self-dual lattices with a complete root system and coseting them with certain $\ell = 0$ three-character CFT.

We finally turned to the application of modular bootstrap which deals with the general properties of a CFT, in particular irrational CFT. Motivated by the $\text{AdS}_3/\text{CFT}_2$ duality and the search for pure AdS_3 quantum gravity, we first reviewed aspects of AdS gravity in $2+1$ dimensions and some crucial properties of the BTZ black hole. After this, we considered $c > 1$ CFT with the Virasoro algebra as the full symmetry algebra. We first derived a universal density of high energy states and hence the entropy, better known as the Cardy formula, and showed that it matches with the Bekenstein-Hawking entropy of the BTZ black hole. We then utilised the methods of medium-temperature expansion to derive an upper bound on the scaling dimension of the lowest non-trivial primary operator. In the holographic interpretation, this translated into an upper bound on the mass of the lowest massive bulk excitation in a theory of quantum gravity with $\Lambda < 0$.

We then moved on to the lightcone limit of a CFT on a torus and used it in conjunction with the modular $\mathcal{S}$ invariance of the partition function to obtain a bootstrap equation in the lightcone limit. We noticed that the vacuum term in the dual channel diverges and dominates the contribution from the rest of the spectrum. By a simple manipulation of the modular crossing equation, we derived an upper bound on the twist gap of any unitary CFT and a universal density of large spin primaries, generalising the Cardy formula, and showed that it solves the bootstrap equation. Finally, we considered a more general class of modular transformations that generalised the modular $\mathcal{S}$ transformation. Repeating the previous analysis, we first derived a bootstrap equation in the lightcone limit, and then found a universal density of large spin primaries that solves the bootstrap equation. The leading contribution of this turned out to be the generalised Cardy formula whereas contributions from other elements of the modular group were sub-leading to it.

We will now note some future directions from this work. Firstly, we have provided strong evidence for the conjectured contour-integral representation of RCFT characters, but it is still unproven and can be investigated further. Secondly, our work towards the classification of three-character CFT is an important first step in the direction, but no claim for its completeness has been made. One could also, with sufficiently novel methods, generalise the quasi-character approach to higher number of characters.

The bound on the lowest massive excitation which we have proved is by no means the best bound obtainable. Restricting this bound to match the bound implied from black hole physics is a very interesting direction to pursue. In the context of lightcone modular bootstrap, we are working on deriving a better bound on the twist gap using the generalised modular bootstrap equation. Additionally, we are currently pursuing the general results from lightcone modular bootstrap that can be interpreted holographically, with the hope that it sheds light on pure AdS_3 quantum gravity.

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