Rational Conformal Field Theories: Classification and Applications

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by

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Certificate

This is to certify that this dissertation entitled Rational Conformal Field Theories: Classification and Applications towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Rahul Poddar at Indian Institute of Science Education and Research under the supervision of Prof. Sunil Mukhi, Department of Physics, during the academic year 2019-2020.

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This thesis is dedicated to my parents, Mahua and Gautam, for their constant support and encouragement.

Declaration

I hereby declare that the matter embodied in the report entitled Rational Conformal Field Theories: Classification and Applications are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Prof. Sunil Mukhi and the same has not been submitted elsewhere for any other degree.

Rahul Poddar

Prof. Sunil Mukhi

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Abstract

Rational Conformal Field Theories (RCFTs) are used to describe many physical systems including, but not limited to, the Fractional Quantum Hall effect. In that effect, classifying RCFTs is an interesting and tractable problem. In this thesis we first understand and develop the formalism of CFTs and specialise to WZW models to study examples of RCFTs. Using the techniques of Modular Invariance, we use the Modular Linear Differential Equations (MLDE) approach in our attempt of classification of RCFT. We note that the contour integral representation of certain characters lends itself very well to understand the monodromy properties of characters. This leads us to developing an algorithm to compute the modular $\mathcal{S}$-matrix for arbitrary number of characters, without specifying any chiral algebra. Subsequently we explore the space of solutions of the order 3 MLDE, and discover infinite families of solutions known as quasi-characters. Quasi-characters are solutions of the MLDE with integer but not necessarily positive q-expansion. The novel coset construction is used extensively to construct the families of solution. Using these quasi-characters we can construct new admissible characters. Following these original developments, we explore the application of RCFT in the Fractional Quantum Hall effect, after covering the basics of the Hall effect.

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Chapter 1

Two Dimensional Conformal Field Theories

Conformal field theories are quantum field theories with additional conformal symmetry. These theories explain a large number of physical phenomena and are used in a variety of places in physics. Physics at second order phase transitions, field theories on string theory world sheets, and probing the yet unknown theory of quantum gravity through the AdS/CFT correspondence.

Rational CFTs are an interesting subset of CFT, where the Hilbert space breaks up into a finite number of modules. These are useful once again, in many aspects of physics, for example, calculating critical exponents of statistical systems, realizing "stringy" examples of AdS/CFT. An interesting application is in the explanation of the Fractional Quantum Hall effect, and using these quantum Hall systems to achieve topological quantum computing.

This thesis contains a subset of the topics explored during my Master's project, which include: (i) the study of WZW models, (ii) the study of fields on AdS and their relation to CFT, (iii) Modular invariance, and modular differential equations for classification, (iv) computation of the modular $\mathcal{S}$-matrix for large subclasses of RCFTs, (v) the study of the quantum Hall effect and (vi) brief study of Chern-Simons theory and their relation to the Quantum Hall system.

We start by discussing properties of Quantum Field Theories in 2 dimensions which have Conformal invariance. Conformal transformations are spacetime transformations which preserve angles between vectors. Conformal transformations include scaling, rotations, translations, inversions about the unit sphere,"special" conformal transformations and their linear combinations. More precisely, the metric changes only up to a coordinate dependent factor. Quantum field theories invariant under such transformations are known as Conformal Field Theories (CFTs). In two dimensions, it has been proven that QFTs which are scale invariant are also conformally invariant. The following sections review the paper [1], and can be found in the textbook [2].

1.1 Conformal Invariance in two dimensions

Let us parametrize the 2D Euclidean plane in complex coordinates $z = x + iy$, $\bar{z} = x - iy$. Infinitesimal conformal transformations $x^\mu \to x^\mu + v^\mu$ are transformations which satisfy the conformal killing equation:

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \partial_\rho v^\rho \delta_{\mu\nu} \tag{1.1}$$

which imply that the transformation must satisfy the Cauchy-Riemann conditions:

$$\partial_1 v_1 = \partial_2 v_2 \quad \partial_1 v_2 = -\partial_2 v_1 \tag{1.2}$$

In other words, the transformations must be holomorphic. Similarly, the coordinate $\bar{z}$ transforms by an anti-holomorphic transformation. This shows that the local symmetry group is infinite dimensional, since there are infinitely many (anti-)holomorphic functions.

If we parametrize the local infinitesimal transformations $z \to z + v(z)$ via a Laurent expansion:

$$v(z) = \sum_{n \in \mathbb{Z}} c_n z^{n+1} \tag{1.3}$$

The associated vector fields are the generators of the local conformal transformation:

$$l_n = -z^{n+1}\partial_z \tag{1.4}$$

and its anti-holomorphic $(\bar{l}_n)$ counterpart. We can calculate the algebra of these generators to obtain

$$[l_n, l_m] = (n-m)l_{n+m} \quad [\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m} \quad [l_n, \bar{l}_m] = 0 \tag{1.5}$$

which is the direct sum of 2 independent Witt algebras, one for the holomorphic sector and one for the anti-holomorphic sector.

It is easy to verify that out of the infinite local generators, only three from each of the holomorphic and anti-holomorphic generators are well defined globally. By demanding well definedness at $z = 0$ and $z = \infty$, we see that the surviving generators are

$$l_{-1}, l_0, l_1, \bar{l}_{-1}, \bar{l}_0, \bar{l}_1 \tag{1.6}$$

These generators are the generators of the Mobius group $SL_2(\mathbb{C})/\mathbb{Z}_2$ given by

$$z \to \frac{az+b}{cz+d}, \text{ where } a, b, c, d \in \mathbb{C}, ab - cd \neq 0 \tag{1.7}$$

These 6 generators can be mapped to the 6 generators of the conformal group: 2 for translations, 2 for special conformal transformations, 1 rotation and 1 scaling. Using these generators, we can construct the dilations and rotation operators explicitly:

$$l_0 + \bar{l}_0; \quad l_0 - \bar{l}_0 \tag{1.8}$$

We shall see that the Witt algebra admits a central extension, which is called the Virasoro algebra and that the Virasoro algebra is the symmetry algebra of a CFT.

We can now define a primary field as a field which transforms under conformal transformations as

$$\phi(z, \bar{z}) \to \phi(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) \tag{1.9}$$

and the infinitesimal version is

$$\partial_\epsilon \phi = \epsilon(z)\partial_z \phi + h\partial_z \epsilon(z)\phi \tag{1.10}$$

Fields which obey the above transformations for only global transformations are called quasi-primary fields. The eigenvalue of operator l_0 is h, and is called the conformal dimension of the field. Eigenvalues of the dilation operator $D = l_0 + \bar{l}_0$ and the rotation operator $M = l_0 - \bar{l}_0$ are called scaling dimension Δ and spin s respectively.

$$\Delta = h + \bar{h}; \quad s = h - \bar{h} \tag{1.11}$$

1.1.1 The Stress Energy Tensor

In a quantum field theory, the conserved current due to Lorentz invariance is given by the stress energy tensor $T_{\mu\nu}$. In a CFT, one can show that the stress energy tensor is traceless, i.e. $T_{z\bar{z}} = 0$. Since it is a conserved current,

$$\partial_\mu T^{\mu\nu} = 0 \tag{1.12}$$

one can show that the diagonal components T_{zz} and $\bar{T}_{\bar{z}\bar{z}}$ are holomorphic and anti-holomorphic respectively. It is convenient to introduce the notation

$$T(z) \equiv T_{zz} \quad \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}} \tag{1.13}$$

Note that the stress tensor always has scaling dimension $D =$ number of dimensions. So for us, the stress tensor $T(z)$ has $D = 2 = h$, $\bar{h} = 0$. Thus the stress tensor is a spin 2 object.

The stress energy tensor changes under the following way , under a local infinitesimal conformal transformation $\epsilon(z)$:

$$\delta_\epsilon T(z) = \epsilon(z)T'(z) + 2\epsilon'(z)T(z) \tag{1.14}$$

Once we quantize the theory, we shall see that a new term will appear in the variation.

1.1.2 Radial Quantization

To quantize a theory, we need a proper notion of time, so that we can time order operators. Consider the theory defined on a cylinder, which is conformal to the plane. Let the coordinates on the cylinder be σ_0 and σ_1. Let the "space" direction σ_1 be the compact direction, taking values in $[1, 2\pi)$, and the time direction $\sigma_0 \in \mathbb{R}$. The map between the cylinder and plane reads

$$z = e^{\sigma_0 + i\sigma_1} \quad \bar{z} = e^{\sigma_0 - i\sigma_1} \tag{1.15}$$

The radial coordinate in the plane is identified with time in the cylinder. Thus, the far past on the cylinder is mapped to the origin, and the far future is mapped to the point ∞. The plane with the point ∞ is called the Riemann Sphere S^2. This allows us to use the powerful tools of complex analysis in the quantum theory.

Since the generator which generates radial scaling is the Dilatation operator, it is now identified with the Hamiltonian of the CFT on a cylinder. This is known as Radial Quantization. Thus, the equivalent operation of time ordering is called radial ordering

$$R\{\phi_1(z)\phi_2(w)\} = \begin{cases} \phi_1(z)\phi_2(w) & |z| > |w| \\ \phi_2(w)\phi_1(z) & |z| < |w| \end{cases} \tag{1.16}$$

1.1.3 Correlation Functions of two and three fields

Correlation functions are an important quantities to calculate in a QFT, since they are the observables in any QFT. Correlators of primary fields are severely constrained by conformal invariance. One point functions, or expectation values of single fields vanish, except for the Identity field. Two point functions can be determined exactly. Translation invariance implies the correlators is a function of the separation, rotation invariance implies that it depends only on the magnitude of separation. Using the rest of the conformal symmetries, we fix the full non-perturbative two point function to be

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle \propto \frac{\delta_{h_1,h_2}}{z_{12}^{2h_1}\,\bar{z}_{12}^{2\bar{h}_1}} \tag{1.17}$$

where $z_{12} = z_1 - z_2$ and the conformal dimensions of ϕ_1 and ϕ_2 are h_1 and h_2 respectively. Symmetry arguments can not fix the constant of proportionality.

Similar arguments can made for the three point function. The full three point function turns out to be

$$\begin{gathered} \langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3)\rangle \\ \propto \frac{1}{z_{12}^{h_1+h_2-h_3} z_{13}^{h_1+h_3-h_2} z_{23}^{h_3+h_2-h_1}} \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{13}^{\bar{h}_1+\bar{h}_3-\bar{h}_2} \bar{z}_{23}^{\bar{h}_3+\bar{h}_2-\bar{h}_1}} \end{gathered} \tag{1.18}$$

We shall see later that four point functions can not be fixed just by using symmetry arguments, and many techniques have been developed to solve them. We will explore some of those techniques later in this thesis.

1.1.4 Operator Product Expansion

Since $T(z)$ is the generator of infinitesimal transformations, the operator product $T(z)\phi(w) = \delta_\epsilon \phi(w)$. In a QFT, one can replace a product of two local operators with an infinite sum of local operators, known as the Operator Product Expansion (OPE):

$$\mathcal{O}_i(z)\mathcal{O}_j(w) = \sum_k C_{ijk}(z-w)\mathcal{O}_k(w) \tag{1.19}$$

in the limit where $z \to w$. Using the fact that a conserved current is given by $T(z)\epsilon(z)$, where $\epsilon(z)$ is a conformal killing vector, and symmetry transformations are generated by the associated

charge, $Q_\epsilon = \oint dz T(z)\epsilon(z)$ with an equal time commutator:

$$\delta_\epsilon \Phi(w) = [Q_\epsilon, \Phi(w)] \tag{1.20}$$

we can calculate the $T\phi$ OPE to be

$$T(z)\phi(w) = \frac{h}{(z-w)^2}\phi(w) + \frac{1}{z-w}\partial\phi(w) + \cdots \tag{1.21}$$

where $\phi(z)$ is a primary field and $\cdots$ represent non-singular terms in the limit $z \to w$.

One can compute the stress energy tensor for a number of QFTs, and then write a general OPE for the stress energy tensor with itself:

$$T(z)T(w) = \frac{1}{z-w}T'(w) + \frac{2}{(z-w)^2}T(w) + \frac{c}{2(z-w)^4} + \cdots \tag{1.22}$$

where the last term is an anomalous term, which shows that the stress tensor is not a primary field. We see that the anomalous term has a coefficient $c/2$. The parameter c is called the *central charge*, and captures information of the degrees of freedom of the system. Using the TT OPE we can calculate the infinitesimal variation of the quantum stress tensor to be:

$$\delta_\epsilon T(z) = \epsilon(z)T'(z) + 2\epsilon'(z)T(z) + \frac{c}{12}\epsilon'''(z) \tag{1.23}$$

1.1.5 Virasoro Algebra

Since the stress energy tensor $T(z)$ is the generator of conformal transformations, let us study its modes. Consider the Laurent expansion of $T(z)$

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{n-2} \tag{1.24}$$

or equivalently, the modes L_n can be written as

$$L_n = \oint dz z^{n+1} T(z) \tag{1.25}$$

The modes L_n are called the *Virasoro* generators. Computing their commutator using the TT OPE, we obtain the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{1.26}$$

which is the central extension to the Witt algebra, as alluded to earlier. Notice that the modes L_0, L_1 and L_{-1} recover the Witt algebra as the central term vanishes.

A similar analysis with $\bar{T}$ will give us the anti-holomorphic Virasoro algebra. Thus the full symmetry algebra is the direct sum of the holomorphic and anti-holomorphic Virasoro algebras.

1.2 Representations of the Virasoro Algebra and Null vectors

If we demand that the spectrum of a CFT must be bounded from below, we must define the notion of a ground state. Demanding that the stress energy tensor $T(z)\,|0\rangle$ is well defined globally, we get the conditions

$$L_m\,|0\rangle = 0 \quad \forall m \geq -1; \qquad \langle 0|\,L_m = 0 \quad \forall m \leq 1 \tag{1.27}$$

So the global symmetry generators L_0, L_1, L_{-1} annihilate the vacuum.

An asymptotic "in" state is created by a primary at the origin:

$$|\phi\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})\,|0\rangle \tag{1.28}$$

and similarly the out state is created by a primary at infinity:

$$\langle\phi| = \lim_{z,\bar{z}\to \infty} z^{2h}\bar{z}^{2\bar{h}}\,\langle 0|\,\phi(z,\bar{z}) \tag{1.29}$$

where the extra z and $\bar{z}$ factors arise from hermitian conjugation acting on a field $\phi^\dagger(z,\bar{z}) = \bar{z}^{-2h}z^{-2\bar{h}}\phi\left(\frac{1}{\bar{z}},\frac{1}{z}\right)$.

From the OPE of $T(z)$ with a primary field ϕ, we obtain:

$$L_0\,|\phi\rangle = h\,|\phi\rangle\,; \quad L_m\,|\phi\rangle = 0 \quad \forall m > 0 \tag{1.30}$$

which satisfies our condition for a "ground state." Note that the algebra of the Virasoro generators with L_0 is

$$[L_0, L_n] = -nL_n \tag{1.31}$$

and thus L_n's are lowering ($n > 0$) and raising ($n < 0$) operators.

Defining the primary state as the highest weight state, we can generate the rest of the spectrum by acting the raising operators on the primary state. This construction is known as the Verma module. The full spectrum of the CFT is the direct sum of all the holomorphic and anti-holomorphic Verma modules, generated from each primary in the theory.

The states created by the action of a raising operator on a primary state is called a descendant state. A k^{th} level descendant state, which can be created by acting the operator L_{-k} on a primary of dimension h, or $(L_{-1})^k$, or any other combination of raising operators whose indices add up to k, will have conformal dimension $h + k$. Clearly, the degeneracy at level k is given by the Partition of k, or the number of ways to sum positive integers to get k. This degeneracy grows exponentially with k.

1.2.1 Null Vectors in the Verma Modules

Do we know if these Verma modules are irreducible? If they are, then they will not contain any sub-modules or sub-representations. In a sub-representation, the L_0 operator will again bounded from below by some state $|\chi\rangle$, but will not be the primary which generated the module $|\phi\rangle$. Thus

$|\chi\rangle$ is a descendant of $|\phi\rangle$. If such a state $|\chi\rangle$ exists, then it must have zero norm. To show this, consider any descendant of $|\phi\rangle$

$$|\psi\rangle = L_{-k_1} \cdots L_{-k_l} |\phi\rangle \tag{1.32}$$

where k_i's are positive integers. Consider its inner product with $|\chi\rangle$,

$$\langle\psi|\chi\rangle = \langle h| L_{k_l} \cdots L_{k_1} |\chi\rangle = 0 \tag{1.33}$$

The first equality is true because $L_k^\dagger = L_{-k}$. The second equality is true because $|\chi\rangle$ is a primary state, so it will be annihilated by any lowering operator. Thus,

$$\langle\chi|\chi\rangle = 0 \tag{1.34}$$

since $|\chi\rangle$ is a special case of $|\psi\rangle$. Thus, this state is called a null vector, and since it is orthogonal with all other states in the Verma module, it is decoupled from the theory. After decoupling all such null vectors, the Verma module is now irreducible.

Existence of these null vectors allows us to write down differential equations for correlators. Also, it allows us to constrain the possible values of the parameters like the central charge and conformal dimensions of primaries. An example of this at work is the construction of the minimal models.

Victor Kac [3] determined that if a primary has a null vector at level $N = rs$, where r, s are positive integers, then

$$h_{r,s}(c) = \frac{c-1}{24} + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2 \tag{1.35}$$

where

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}} \tag{1.36}$$

Unitarity constraints tell us that unitary CFTs must have $c, h > 0$. For the values of $c \in [0, 1]$, one can show that two co-prime integers p, q, we have theories with finite number of primaries with central charge and conformal dimensions given by:

$$\begin{aligned} c &= 1 - 6\frac{(p-q)^2}{pq} \\ h_{n,m} &= \frac{(mp-nq)^2 - (p-q)^2}{4pq} \\ n \in 2, \cdots p-1, \quad m \in 2, \cdots, q-1; \quad & n, m \text{ are co-prime} \end{aligned} \tag{1.37}$$

There are an infinite set of CFTs with $c < 1$ for each co-prime pair of integers p, q, with a finite number of primary fields under the Virasoro algebra. These CFTs are known as minimal models.

1.3 Wess-Zumino-Witten Models

In the previous discussions we only considered theories with the Virasoro algebra as the local symmetry algebra. However, one can consider theories with extended symmetry algebras rather than just the Virasoro algebra. Just as we had defined primaries of the Virasoro algebra, we shall see that we can define primaries and Verma modules of these extended chiral algebras. If we define a character χ_i as the trace over all the states in a Verma module over a primary ϕ_i, and the partition as the full sum over states, then we see that the partition function can be expressed as

$$Z = \sum_i \chi_i \bar{\chi}_i$$

A Rational CFT (RCFT) is a CFT which has a finite number of primary fields under any local symmetry algebra. Thus, the sum over characters will be over a finite integer p, which are the number of distinct characters.

In this section we explore RCFTs which have more than just conformal symmetry, but also Lie group symmetry. Wess-Zumino-Witten Models, WZW Models for short are such CFTs. We shall see that unlike minimal models, which are RCFTs with only conformal symmetry and hence must have central charge less than 1, WZW models can only have central charge greater than 1. This is due to the fact, as we shall see, that the Lie group symmetry, manifest as the Kac-Moody algebra rearranges the infinite representations of the Virasoro algebra into finite Verma modules. [4, 5, 6, 2]

1.3.1 Nonlinear Sigma Model

Consider a field theory living on a two dimensional Euclidean (or Minkowski) plane. We would like the theory to be conformally invariant, and also invariant under other Lie algebras.

$$S_0 = \frac{1}{4\lambda^2} \int d^2z \, \mathrm{tr}\big(\partial_\mu g^{-1} \partial^\mu g\big) \tag{1.38}$$

The fields here are the map $g : S^2 \to G$, where G is a (semi-simple) Lie Group, and S^2 is the Riemann Sphere. Therefore, the field g is a unitary scalar matrix valued field on the Riemann Sphere. Due to the action being free, and the fields having length dimension zero, the action is scale invariant. We can parametrize the fields by exponentiating the generators t_a.

$$g = e^{i\phi^a t_a} \tag{1.39}$$

The Lagrangian thus turns out to be

$$\mathcal{L}_0 = \frac{1}{4\lambda^2} \mathrm{tr}\big(\partial_\mu g^{-1} \partial^\mu g\big) = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^b g_{ab}(\phi) \tag{1.40}$$

where

$$g_{ab} = \frac{1}{4\lambda^2}(2\,\mathrm{tr}(t_a t_b) + \phi^c\,\mathrm{tr}(t_a t_b t_c) + \phi^c\phi^d\,\mathrm{tr}(t_a t_b t_c t_d) + \dots) \tag{1.41}$$

is the metric of the group manifold, parametrized by the fields ϕ^a. The lagrangian (1.40) is the lagrangian of multiple scalar fields ($n^2 - 1$ for $SU(N)$) coupled to each other by the curvature of the manifold, captured in g_{ab}.

Symmetries of the Nonlinear Sigma Model

From the form of the lagrangian in terms of the multiplet ϕ^a and metric g_{ab} (1.40), it seems natural that isometries of the field space manifold are global symmetries of the theory.

To prove the claim, consider the general transformation $\delta\phi^a = \xi^a(\phi)$. An isometry of a space is a vector such that the Lie derivative of the metric along that vector should be zero. That is,

$$\mathfrak{L}_\xi g_{ab} = g_{ac}\partial_b\xi^c + g_{cb}\partial_a\xi^c + \xi^c\partial_c g_{ab} = 0 \tag{1.42}$$

Covariantizing the derivatives gives us Killing's equation $\nabla_{(a}\xi_{b)} = 0$. Therefore the variation of the Lagrangian under the above variation of ϕ is

$$\delta\mathcal{L}_0 = \frac{1}{2}\partial_\mu\xi^a\partial^\mu\phi^b g_{ab}(\phi) + \frac{1}{2}\partial_\mu\phi^a\partial^\mu\xi^b g_{ab}(\phi) + \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^b\xi^c\partial_c g_{ab}(\phi) \tag{1.43}$$

$$\partial_\mu\xi^a(\phi) = \partial_\mu\phi^c\partial_c\xi^a \tag{1.44}$$

$$\implies \delta\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^b(g_{ac}\partial_b\xi^c + g_{cb}\partial_a\xi^c + \xi^c\partial_c g_{ab}) = 0 \tag{1.45}$$

Thus, isometries of the manifold will make $\delta\mathcal{L}_0 = 0$, thus proving the claim.

Thus, we can see the explicit symmetry enforced by allowing the fields to be valued on a manifold. However, we can see some more enhanced symmetry in the initial form (1.38). It is straightforward to see that the transformation of the form $g \to h_L g h_R^{-1}$, where h_L, $h_R \in G$ leaves (1.38) invariant, by cyclicity of trace. Therefore the global symmetry group of the Nonlinear Sigma Model is $G \times G$.

Equations of Motion and Conserved Currents

Computing the equations of motion of the Nonlinear sigma model, we have

$$(\delta^a_c\partial_\mu + \Gamma^a{}_{bc}\partial_\mu\phi^b)\partial^\mu\phi^c = 0 \tag{1.46}$$

where $\Gamma^a{}_{bc}$ is the connection on the group manifold. Defining a covariant derivative D on the group manifold, we can rewrite this as

$$D^a{}_{\mu b}\partial^\mu\phi^b = 0 \tag{1.47}$$

The equations of motion are a nonlinear generalisation of the Klein Gordon equation, except one of the derivatives carries the connection in the target space group manifold.

A more useful form of the equations of motion are in terms of the fields g. This will expose a symmetry of the Lagrangian and help find its currents. This computation extensively uses $\partial(g^{-1}g) = 0 \implies \partial g^{-1} = -g^{-1}\partial g g^{-1}$, cyclicity of trace, and integration by parts to remove total derivative terms from the Lagrangian.

$$\delta\mathcal{L}_0 = \frac{1}{4\lambda^2}\operatorname{tr}\big(\partial_\mu \delta g^{-1}\partial^\mu g + \partial_\mu g^{-1}\partial^\mu \delta g\big) \tag{1.48}$$

$$= \frac{1}{2\lambda^2}\operatorname{tr}\big(g^{-1}\delta g(\partial^\mu(g^{-1}\partial_\mu g))\big) = 0 \tag{1.49}$$

Therefore the equations of motion are

$$\partial^\mu(g^{-1}\partial_\mu g) = 0 \tag{1.50}$$

This immediately tells us a conserved current of the theory is $j_\mu = g^{-1}\partial_\mu g$. This corresponds to the left action symmetry of the field $g \to h_L g$. We can obtain the other current from the equations of motion as well by

$$g\partial^\mu(g^{-1}\partial_\mu g)g^{-1} = \partial^\mu(\partial_\mu g g^{-1}) = 0 \tag{1.51}$$

Therefore the current for right action symmetry is $\widetilde{j}_\mu = \partial_\mu g g^{-1}$. These currents are conserved separately by the equations of motion, and thus, in $(z, \bar{z})$ coordinates,

$$\partial_z j_{\bar{z}} + \partial_{\bar{z}} j_z = 0 \tag{1.52}$$

$$\partial_z \widetilde{j}_{\bar{z}} + \partial_{\bar{z}} \widetilde{j}_z = 0 \tag{1.53}$$

Notice that the currents do not break into separately conserved holomorphic and antiholomorphic parts, which suggest that this is not a conformally invariant quantum field theory.

1.3.2 The Wess-Zumino Term

To make these models conformally invariant when quantized, Witten suggested to add the following term to the action, known as the Wess-Zumino term:

$$\Gamma(g) = \frac{1}{24\pi}\int_B d^3y \epsilon^{\alpha\beta\gamma}\operatorname{tr}\big(g^{-1}\partial_\alpha g g^{-1}\partial_\beta g g^{-1}\partial_\gamma g\big) \tag{1.54}$$

At first glance, it seems strange to add a three dimensional integral to a two dimensional action. The theory lives on S^2, the boundary of the ball B. The integrand can be written as a total derivative, so the integral is over S^2. We know that the second homotopy group of any Lie group $\pi_2(G) = \mathbb{1}$, that is, any map from $S^2 \to G$ can be continuously deformed to (homotopic to) identity. Therefore we can continue g inside the ball and thus $g : B \to G$. However, this extension of S^2 to B is not

unique topologically.

By subtracting the Wess Zumino terms extended in topologically different balls, the integral is over only the interior of both balls, topologically S^3, since the boundary contribution cancels out. That is, $\Gamma_B - \Gamma_{B'} = \Gamma_{S^3}$. Since we are subtracting the second ball, the normal is reversed and becomes an outward normal for the entire S^3.

It turns out, for compact simple Lie groups the third homotopy group $\pi_3(G) = \mathbb{Z}$. In simple words, this means that compact simple Lie groups have an integer winding number for how many times one can wrap the three-sphere around it. Hence we can say that the topological sectors in which Γ is valued is labelled by the winding number. The infinitesimal variation of g in S^3 does not change value of Γ_{S^3}.

$$\delta\Gamma_{S^3}(g) = \frac{1}{8\pi}\int_{S^3} d^3y \epsilon^{\alpha\beta\gamma}\partial_\alpha \operatorname{tr}\big((g^{-1}\delta g)j_\beta j_\gamma\big) = 0 \tag{1.55}$$

This is zero because $\partial S^3 = \emptyset$. For brevity, we have denoted $g^{-1}\partial_\alpha g \equiv j_\alpha$, as the left current derived earlier is j_μ, which is j_α restricted to S^2. Since the infinitesimal variation of g in S^3 doesn't change Γ, it is a topological invariant, and the homotopy class remains unchanged. In this calculation we have used:

$$\begin{aligned}\epsilon^{\alpha\beta\gamma}\partial_\alpha j_\beta &= \epsilon^{\alpha\beta\gamma}(\partial_\alpha g^{-1}\partial_\beta g + g^{-1}\partial_\alpha\partial_\beta g)\\ &= -\epsilon^{\alpha\beta\gamma} j_\alpha j_\beta\end{aligned} \tag{1.56}$$

The Wess-Zumino term is in fact the second Chern number, which takes values $2\pi n$. Chern numbers are integrals of Chern classes, which are constructed from topologically invariant polynomials. Now we can add the Nonlinear sigma model with the Wess Zumino term Γ.

$$S[g] = S_0[g] + ik\Gamma[g] \tag{1.57}$$

This action, (1.57), is called the Wess-Zumino-Witten model. Consider the (Euclidean) path integral for the WZW model:

$$Z = \int [\mathcal{D}g] e^{-S_0[g] - ik\Gamma_B(g)} \tag{1.58}$$

On changing the extension of the fields to the ball to B', it is effectively adding an $ik\Gamma_{S^3}$ term to the exponential. To ensure the equations of motion of the theory do not change, we require this term to vanish. However, since $\Gamma_{S^3} = 2\pi n$, the extra piece in the path integral is now $e^{2\pi i n k}$. This vanishes for $k \in \mathbb{Z}$. Therefore, for well definedness of the quantum WZW model, k must take integer values. This integer k is known as the level of the WZW model. Thus we see that with integer k, the addition of the WZ term gives a well-defined QFT which we now go on to analyse.

1.3.3 Equations of Motion and Conserved Currents of the WZW Model

In this section we shall see that adding the Wess-Zumino term indeed makes this a conformally invariant CFT. Let us begin by studying its equations of motion. We have already computed the variation of the nonlinear sigma term in (1.49), and computed the variation of the Wess Zumino term in (1.55):

$$\delta S = \delta S_0 + ik\delta\Gamma \tag{1.59}$$

$$= \frac{1}{2\lambda^2}\int d^2z\,\mathrm{tr}\big(g^{-1}\delta g(\partial^\mu(g^{-1}\partial_\mu g))\big) - \frac{ik}{8\pi}\int d^2z\,\epsilon^{\mu\nu}\,\mathrm{tr}\big(g^{-1}\delta g\partial_\mu(g^{-1}\partial_\nu g)\big) = 0 \tag{1.60}$$

Setting the integrand to zero, we get:

$$\left(\frac{1}{2\lambda^2}+\frac{k}{8\pi}\right)\partial_z(g^{-1}\partial_{\bar z}g) + \left(\frac{1}{2\lambda^2}-\frac{k}{8\pi}\right)\partial_{\bar z}(g^{-1}\partial_z g) = 0 \tag{1.61}$$

What has happened is that the current has now broken up into its holomorphic and antiholomorphic parts, by the addition of the Wess-Zumino term. By setting

$$\lambda^2 = \frac{4\pi}{k} \tag{1.62}$$

the equations of motion become

$$\partial_z(g^{-1}\partial_{\bar z}g) = 0 \tag{1.63}$$

Thus, the left current only has a antiholomorphic part. Similarly the right current only has a holomorphic part. On computing the β function of the WZW Model, it turns out that this value of λ corresponds to the fixed point of the theory.

It is also important to note that this is true for $k > 0$. For $k < 0$, to ensure λ^2 is still positive, the RG fixed point is $\lambda^2 = -\frac{4\pi}{k}$, and the left current becomes holomorphic and right current becomes antiholomorphic.

We still have to show that the left current actually generates left action and the right current generates right action, as defined in 1.3.1. For this, let's consider the infinitesimal transformation

$$\begin{aligned} g \to h_L g h_R^{-1} &= (1+\omega)g(1-\bar\omega) \\ \implies \delta_L g = \omega g; &\quad \delta_R g = -g\bar\omega \end{aligned} \tag{1.64}$$

We know the general variation of the action as

$$\delta S = \frac{k}{16\pi}\int d^2z\,(\eta^{\mu\nu} - i\epsilon^{\mu\nu})\,\mathrm{tr}\big(g^{-1}\delta g\partial_\mu(g^{-1}\partial_\nu g)\big) \tag{1.65}$$

$$= \frac{k}{2\pi}\int d^2z\,\mathrm{tr}\big(g^{-1}\delta g\partial_z(g^{-1}\partial_{\bar z}g)\big) \tag{1.66}$$

so the right variation will be

$$\delta_R S = \frac{-k}{2\pi}\int d^2z\,\mathrm{tr}\big(\bar{\omega}\partial_z(g^{-1}\partial_{\bar{z}}g)\big) \tag{1.67}$$

and the left variation will be

$$\delta_L S = \frac{k}{2\pi}\int d^2z\,\mathrm{tr}\big(\omega\partial_{\bar{z}}(\partial_z g g^{-1})\big) \tag{1.68}$$

where we have used (1.51). Comparing with the general variation of an action

$$\delta S = -\int d^dx\omega\partial_\mu j^\mu \tag{1.69}$$

shows us that $j_{\bar{z}} = g^{-1}\partial_{\bar{z}}g$ generates right action, and $j_z = \partial_z g g^{-1}$ generates left action.

For the Nonlinear Sigma model we saw that the symmetry group was global $G \times G$. On addition of the Wess Zumino term to the action, we find the symmetry group is enhanced to local $G(z) \times G(\bar{z})$. To see that we can use the Polyakov Weigmann decomposition:

$$S(gh) = S(g) + S(h) - \frac{k}{2\pi}\int d^2z\,\mathrm{tr}\big(g^{-1}\partial_z g\partial_{\bar{z}}hh^{-1}\big) \tag{1.70}$$

$$S(gh^{-1}) = S(g) + S(h) + \frac{k}{2\pi}\int d^2z\,\mathrm{tr}\big(h^{-1}\partial_z h g^{-1}\partial_{\bar{z}}g\big) \tag{1.71}$$

Using these relations we can see that the following transformation is a symmetry of the WZW action.

$$g(z,\bar{z}) \to \Omega(z)g(z,\bar{z})\bar{\Omega}^{-1}(\bar{z}) \tag{1.72}$$

and the action changes as

$$\begin{aligned} S(g) &\to S(\Omega g\bar{\Omega}^{-1}) \\ &= S(\Omega g) + S(\bar{\Omega}) + \frac{k}{2\pi}\int d^2z\,\mathrm{tr}\big(\bar{\Omega}^{-1}\partial_z\bar{\Omega}g^{-1}\partial_{\bar{z}}g\big) \\ &= S(\Omega) + S(g) + S(\bar{\Omega}) - \frac{k}{2\pi}\int d^2z\,\mathrm{tr}\big(g^{-1}\partial_z g\partial_{\bar{z}}\Omega\Omega^{-1}\big) \\ &= S(\Omega) + S(g) + S(\bar{\Omega}) = S(g) \end{aligned} \tag{1.73}$$

The integrals vanish since $\Omega(z)$ and $\bar{\Omega}(\bar{z})$ are holomorphic and antiholomorphic respectively. Similarly $S(\Omega)$ and $S(\bar{\Omega})$ vanish if the holomorphic (antiholomorphic) property is preserved during the continuation into the ball B. Thus the global symmetry group has been extended into a local one!

1.3.4 Current Algebras and the Kac-Moody Algebra

To study the quantized WZW model, we need to look at correlation functions. To do this, we will see what possible constraints we can impose on them with whatever symmetries we have in the theory. In the previous section, we derived two conserved currents, the left and right holomorphic currents for left and right action invariance of the WZW action respectively. Let's normalize the right current as

$$J = -k\partial g g^{-1} = J^a t^a \tag{1.74}$$

Here $\partial \equiv \partial_z$ and $\bar{\partial} \equiv \partial_{\bar{z}}$. We have written the currents in terms of the generator representation of the group G such that

$$[t^a, t^b] = i f^{ab}{}_c t^c \tag{1.75}$$

Let us now look at the Operator Product Expansion (OPE) of the currents. These currents are of conformal dimension 1, so by dimensional analysis all the terms must be of dimension 2. In the OPE we only need to worry about the singular terms in the limit of the two operators coinciding, so we can write

$$J^a(z)J^b(w) \sim \sum_{p>0} \frac{X_p}{(z-w)^p} \tag{1.76}$$

To preserve dimensions on both sides, $\dim(X_p) = 2-p$. In a unitary CFT, the minimum dimension of any operator in the theory is 0, and that too only the identity operator. Therefore, $0 < p \leq 2$.

For $p = 2$, $\dim(X_2) = 0 \implies X_2 \propto \mathbb{1}$, and for $p = 1$, $\dim(X_1) = 1 \implies X_1 \propto J^a$, since currents are the only dimension 1 holomorphic objects in a CFT. If there were other dimension 1 holomorphic objects, say $A(z)$, then $\partial_\mu A^\mu = \bar{\partial} A(z) = 0$, implying that $A(z)$ is also a conserved current. However, we've not imposed any other symmetries in the theory, so these shouldn't exist.

Therefore,

$$J^a(z)J^b(w) \sim \frac{k^{ab}\mathbb{1}}{(z-w)^2} + \frac{i f^{abc} J^c(w)}{(z-w)} \tag{1.77}$$

where k^{ab} and f^{abc} are (suggestively named) constants to be fixed. To fix these constants, we shall use commutativity of OPEs (inside correlation functions) and associativity of OPEs. Commutativity, i.e. $J^a(z)J^b(w) = J^b(w)J^a(z)$ implies for the first term

$$k^{ab} = k^{ba} \tag{1.78}$$

since the denominator is symmetric under exchange of z and w, while the second term is antisymmetric, so

$$f^{abc} = -f^{bac} \tag{1.79}$$

Associativity of OPEs implies

$$k^{cb}f^{abd} = k^{bd}f^{cad} = k^{ad}f^{bcd} \tag{1.80}$$

$$f^{abd}f^{dce} + f^{bcd}f^{dae} + f^{cad}f^{dbe} = 0 \tag{1.81}$$

These relations directly imply that f^{abc} are structure constants for a Lie algebra (must be antisymmetric in their first 2 indices and must satisfy the Jacobi Identity (1.81). Also equation (1.80) tells us that k^{ab} that is an invariant 2 form of the algebra, which turns out to be the Killing form. The Killing form in the orthonormal basis of generators is $\propto \delta^{ab}$. It turns out that if you compute the OPE properly using canonical quantization, $k^{ab} = k\delta^{ab}$, where k is the level of the WZW model. So, the OPE turns out to be

$$J^a(z)J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{if^{abc}J^c(w)}{(z-w)} \tag{1.82}$$

Similarly, the right moving currents satisfy

$$\bar{J}^a(\bar{z})\bar{J}^b(\bar{w}) \sim \frac{k\delta^{ab}}{(\bar{z}-\bar{w})^2} + \frac{if^{abc}\bar{J}^c(\bar{w})}{(\bar{z}-\bar{w})} \tag{1.83}$$

These OPEs are called the *current algebra* of the theory.

From these OPEs we can calculate the algebra obeyed by the modes of the currents. To obtain the modes, we Laurent expand the currents

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1} \leftrightarrow J^a_m = \oint \frac{dz}{2\pi i} J^a(z) z^m \tag{1.84}$$

To calculate the algebra, we find the commutator of the modes

$$\begin{aligned}
[J^a_m, J^b_n] &= \frac{1}{(2\pi i)^2}\left[\oint_{|z|>|w|} dzdw - \oint_{|z|<|w|} dzdw\right] z^m w^n J^a(z)J^b(w) \\
&= \frac{1}{(2\pi i)^2}\oint_0 dw\, w^n \oint_w dz\, z^m \left(\frac{k\delta^{ab}}{(z-w)^2} + \frac{if^{abc}J^c(w)}{(z-w)}\right) \\
&= \frac{1}{2\pi i}\oint_0 dw (m w^{m+n-1}k\delta^{ab} + w^{m+n} if^{abc}J^c(w)) \\
\implies [J^a_m, J^b_n] &= if^{abc}J^c_{m+n} + mk\delta^{ab}\delta_{m+n,0}
\end{aligned} \tag{1.85}$$

We have obtained the Kac-Moody algebra (1.85), denoted as $\hat{\mathfrak{g}}_k$. It is just the central extension of the Lie algebra of the group G, just as the Virasoro algebra is the central extension of the de Witt algebra. It is interesting to note that the zero modes realize just the algebra of G, just as the $m = 0, \pm 1$ generators of L_m of the Virasoro algebra realize the global conformal symmetry in a

CFT.

$$[J_0^a, J_0^b] = if^{abc}J_0^c \tag{1.86}$$

1.3.5 The Sugawara Construction

Until this point, we have assumed that the theory is a CFT, since we argued that this theory has separately conserved holomorphic and antiholomorphic currents. However, a (2-d) CFT must realize the Virasoro algebra in some way. The Sugawara construction helps us in obtaining the Virasoro algebra from the Kac-Moody algebra derived in the previous section.

To begin, we start with an ansatz for the energy momentum tensor, motivated by the same of the nonlinear sigma model.

$$T(z) = \gamma \ :J^aJ^a: \ (z) \tag{1.87}$$

This is the normal ordered product of holomorphic currents. We have yet to fix the value of γ, which we shall accomplish by using the OPE derived in the previous section. To work with normal ordering, we will define the normal ordered product to be the regular part of the OPE, i.e

$$:J^aJ^a: \ (w) = \frac{1}{2\pi i}\oint_w \frac{dz}{z-w} J^a(z)J^a(w) \tag{1.88}$$

As one can see, this integral precisely pulls out the nonzero regular part of the OPE. Similarly, we shall define the notation

$$\overbrace{AB} \tag{1.89}$$

as the only the singular terms in the OPE, known as a *contraction*.

We find that the JT contraction is

$$\begin{aligned}\overbrace{J^a(z)T}(w) &= \frac{2\gamma k J^a(w)}{(z-w)^2} - \gamma\frac{f^{abc}f^{cbd}}{(z-w)^2}J^d(w) \\ &= \frac{2\gamma(k+h^\vee)J^a(w)}{(z-w)^2}\end{aligned} \tag{1.90}$$

Here we have used the definition of the *dual Coxeter number* $h^\vee$: $f^{abc}f^{bcd} = 2h^\vee\delta^{ad}$. It is the quadratic Casimir in the adjoint representation. Now, using commutativity of the OPE

$$\begin{aligned}\overbrace{T(z)J^a}(w) &= \overbrace{J^a(w)T}(z) \\ &= \frac{2\gamma(k+h^\vee)J^a(z)}{(z-w)^2} \\ &= 2\gamma(k+h^\vee)\left(\frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}\right)\end{aligned} \tag{1.91}$$

This is exactly the OPE of the energy momentum operator with a Virasoro primary field, if γ takes the value

$$\gamma = \frac{1}{2(k+h^\vee)} \tag{1.92}$$

Now having fixed γ, we can write the energy momentum tensor in terms of Kac-Moody currents as

$$T(z) = \frac{1}{2(k+h^\vee)} : J^a J^a : (z) \tag{1.93}$$

Using this, we can also find a relation between the modes of the energy momentum tensor and the Kac-Moody modes:

$$T(z) = \sum_p L_p z^{-p-2} = \gamma \sum_{n,m} : J^a_n J^a_m : z^{-n-m-2} \tag{1.94}$$

$$\implies L_m = \gamma \sum_n : J^a_n J^a_{m-n} := \gamma \left(\sum_{n<0} J^a_n J^a_{m-n} + \sum_{n\geq 0} J^a_{m-n} J^a_n \right) \tag{1.95}$$

Since we know the commutation relations of J^a with J^b, we can use this relation to construct the commutation relations with the modes of the energy momentum tensor. Computing the commutator with L_m and J^a_n, we obtain:

$$[L_m, J^a_n] = -n J^a_{m+n} \tag{1.96}$$

Now we can compute the algebra of L_m's easily. This will also tell us if the theory is actually a CFT, since up to this point we have not explicitly shown that the WZW model is a CFT. We obtain:

$$[L_m, L_n] = (m-n)L_{m+n} + \gamma k \dim \mathfrak{g}\, m \left(\frac{m^2-1}{6} \right) \delta_{m+n,0}$$

Substituting the value of γ, we obtain

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \tag{1.97}$$

where the central charge c is given by

$$c = \frac{k \dim \mathfrak{g}}{k+h^\vee} \tag{1.98}$$

We have recovered the Virasoro algebra from the Kac-Moody algebra. Hence, we have shown that the WZW model is a CFT, with the energy momentum tensor having the Virasoro generators as its modes. This also confirms our ansatz of the energy momentum tensor as the normal ordered

product of the Kac-Moody currents. Now we also know the TT OPE is

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{1.99}$$

since this is the OPE of the energy momentum tensor of a CFT.

Compiling results, the full holomorphic algebra of the WZW model is thus

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{1.100}$$

$$[J^a_m, J^b_n] = if^{abc}J^c_{m+n} + mk\delta^{ab}\delta_{m+n,0} \tag{1.101}$$

$$[L_m, J^a_n] = -nJ^a_{m+n} \tag{1.102}$$

which we label as $\mathfrak{G}$. The complete algebra is thus $\mathfrak{G} \times \bar{\mathfrak{G}}$. The corresponding OPEs are

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{1.103}$$

$$J^a(z)J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{if^{abc}J^c(w)}{(z-w)} \tag{1.104}$$

$$T(z)J^a(w) \sim \frac{J^a}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} \tag{1.105}$$

Let us now focus on the central charge. We obtained the result

$$c = \frac{k \dim \mathfrak{g}}{k + h^\vee} \tag{1.106}$$

As we can see, the group plays an important role in defining the central charge. It can be shown

$$\operatorname{rank} \mathfrak{g} \leq c < \dim \mathfrak{g} \tag{1.107}$$

The lower bound is saturated for $k = 1$ in simply laced Lie algebras. Simply laced Lie algebras are algebras which have equal norm positive root vectors. These turn out to be the $su(n)$, $so(2n)$, and E_i Lie algebras. Therefore, the central charge takes integer values for these groups, which suggests that these are free theories of c bosons. This is known as the Frenkel-Kac-Segal construction. Similarly, free fermionic fields add half integer to the central charge. So, $c(so(n))_{k=1} = n/2$ suggests that the theory is of $n/2$ free bosons, or equivalently, n free fermions. This is Witten's famous non-abelian bosonization [4].

1.3.6 Representations of the Kac-Moody Algebra

The fields of the WZW model should be representations of the symmetry algebra of the theory, being the semi direct product of the Kac-Moody algebra and the Virasoro algebra $\mathfrak{G} \times \bar{\mathfrak{G}}$: (1.100), (1.101), (1.102). Using the State-Operator correspondence in radial quantization, we have the

asymptotic in-state $\phi(0,0)\,|0\rangle = |\phi\rangle$. A Primary state is a highest weight state of the algebra. Therefore, a Kac-Moody primary $|\phi\rangle$ and a Virasoro primary $|\psi\rangle$ are defined by

$$J_n^a\,|\phi\rangle = 0; \quad L_n\,|\psi\rangle = 0 \quad \text{for } n > 0 \tag{1.108}$$

A primary of the WZW model is a primary under both the Kac-Moody and the Virasoro algebras. From primaries, we can construct the rest of the fields in the theory, as descendants of the primary, using the raising operators of the algebra. In this case, the raising operators are L_{-n}, J^a_{-n}, for $n > 0$. Hence, the complete set of fields, as highest weight representations of $\mathfrak{G} \times \bar{\mathfrak{G}}$, are of the form

$$\left(\prod_{i=1}^{N} L_{-n_i} \prod_{j=1}^{\bar{N}} \bar{L}_{-\bar{n}_j} \prod_{k=1}^{M} J^{a_k}_{-m_k} \prod_{l=1}^{\bar{M}} \bar{J}^{\bar{b}_l}_{-\bar{m}_l}\right) \phi \tag{1.109}$$

where $n_i, \bar{n}_j, m_k, \bar{m}_l$ are arbitrary positive integers, and ϕ is a WZW primary. This is known as the Verma module of the primary ϕ, and is denoted as $[\phi]_{\mathfrak{G}}$. It is an infinite tower of states, with the highest weight being the primary ϕ, and all the descendants raised from it using the raising operators. All the fields are therefore in the set $\{A\} = \oplus_l [\phi_l]_{\mathfrak{G}}$, where l labels the primary. In general, the action of the modes on a general field can be calculated by

$$L_n A(z,\bar{z}) = \oint_z \frac{dw}{2\pi i} T(w)(w-z)^{n+1} A(z,\bar{z}) \tag{1.110}$$

$$J^a_n A(z,\bar{z}) = \oint_z \frac{dw}{2\pi i} J^a(w)(w-z)^{n} A(z,\bar{z}) \tag{1.111}$$

Using this, we see some notable descendant fields are $T(z) = L_{-2}\mathbb{1}$ and $J^a(z) = J^a_{-1}\mathbb{1}$.

The dimensions of the fields given by (1.109), calculated by using the commutation relations of L_0 with all the raising operators are

$$h^{\{n,m\}} = h + \sum_{i=1}^{N} n_i + \sum_{k=1}^{M} m_k \qquad \bar{h}^{\{\bar{n},\bar{m}\}} = \bar{h} + \sum_{j=1}^{\bar{N}} n_j + \sum_{l=1}^{\bar{M}} m_l \tag{1.112}$$

Using the Sugawara construction, we can see if Kac-Moody primaries have any special properties under the Virasoro algebra.

$$L_m\,|\phi\rangle = \gamma \left(\sum_{n<0} J^a_n J^a_{m-n}\,|\phi\rangle + \sum_{n\geq 0} J^a_{m-n} J^a_n\,|\phi\rangle\right) \tag{1.113}$$

For $m > 0, n < 0$; $m - n > 0$, so

$$L_m\,|\phi\rangle = \gamma J^a_m J^a_0\,|\phi\rangle = \gamma([J^a_m, J^a_0] + J^a_0 J^a_m)\,|\phi\rangle = \gamma i f^{aac} J^c_m\,|\phi\rangle = 0 \quad \forall\, m > 0 \tag{1.114}$$

Therefore a Kac-Moody primary is also a Virasoro primary. We know that the eigenvalue of the

L_0 is the dimension of the operator, so

$$L_0 |\phi\rangle = h |\phi\rangle \tag{1.115}$$

Since the zero modes of the Kac-Moody currents satisfy the Lie algebra of the group G, primaries of the Kac-Moody algebra should be representations of the Lie algebra $\mathfrak{g}$.

$$J_0^a |\phi\rangle = -t^a |\phi\rangle \tag{1.116}$$

where t^a will be the matrix corresponding to which representation $|\phi\rangle$ is. The minus sign is to ensure the operator algebra closes.

$$[J_0^a, J_0^b] |\phi\rangle = (-J^a t^b + J^b t^a) |\phi\rangle = (t^b t^a - t^a t^b) |\phi\rangle = -i f^{abc} t^c |\phi\rangle = i f^{abc} J_0^c |\phi\rangle \tag{1.117}$$

We have used the property $\mathcal{O}^a v_i = M_{ij}^a v_j \implies O^a O^b v_i = M_{ij}^b M_{jk}^a v_k$.

Similar to the expansion of the OPE of the energy momentum tensor with a Virasoro primary field, we can also expand the OPE of the current with a primary with its descendants, utilizing the Laurent expansion of the current in terms of its modes.

$$J^a(w)\phi(z) = \sum_n (z-w)^{-n-1} : J_n^a \phi : (z) \tag{1.118}$$

This fixes the OPE of J^a with a primary field

$$\begin{aligned} J_0^a |\phi\rangle &=: J_0^a \phi : (0,0) |0\rangle = -t^a \phi |0\rangle \\ \implies J_0^a \phi(z,\bar{z}) &= \oint \frac{dw}{2\pi i} J^a(w)\phi(z,\bar{z}) = -t^a \phi(z,\bar{z}) \end{aligned} \tag{1.119}$$

To satisfy this relation, we find that the $J^a\phi$ OPE must be

$$J^a(w)\phi(z,\bar{z}) \sim -\frac{t^a \phi(z,\bar{z})}{w-z} \tag{1.120}$$

The first non-singular term, the next term in the expression using (1.118) is

$$:J^a\phi:(z) = J_{-1}^a \phi \tag{1.121}$$

Notice, that using the Sugawara construction, J_{-1}^a is proportional to L_{-1}, when acting on a primary.

$$L_{-1}\phi = \gamma \left(\sum_{n<0} J_n^a J_{-n-1}^a + \sum_{n\geq 0} J_{-n-1}^a J_n^a \right) \phi = 2\gamma J_{-1}^a J_0^a \phi = -2\gamma t^a J_{-1}^a \phi \tag{1.122}$$

Consider now the primary spinless field $g(z,\bar{z})$ of dimension h, and its normal ordered product

with J. The field g is the fundamental representation of G, as it is an element of G.

$$:J^a(z)t^a g(z,\bar{z}):= \frac{1}{2\gamma}\partial_z g(z,\bar{z}) \tag{1.123}$$

Computing the dimension of g using the Sugawara construction, we get

$$\begin{aligned} L_0 g &= \gamma \left(2\sum_{n>0} J^a_{-n}J^a_n + J^a_0 J^a_0 \right) g \\ &= \gamma t^a t^a g = \gamma c_g g = hg \end{aligned}$$

where c_g is the quadratic casimir of the representation of g. Thus, after substituting for γ, we obtain

$$h = \frac{c_g}{2(k+h^\vee)} \tag{1.124}$$

Let us compute the dimension of $g \in SU(2)$, for the $SU(2)$ WZW model at $k = 1$. The quadratic casimir of $su(2)$ for the $j = \frac{1}{2}$ representation is $2j(j+1) = 3/2$. [1] Hence, the dimension of g will be

$$h = \frac{3/2}{2(1+2)} = \frac{1}{4} \tag{1.125}$$

We are in a position to make some comments on the properties of primaries in WZW models. Any primary ϕ which is a scalar with respect to the left and right symmetry groups G has $c_\phi = \bar{c}_\phi = 0$, which means $h = \bar{h} = 0$. Thus the primary ϕ is proportional to Identity. We also didn't make any use of any special property g might have as compared to any other primary ϕ in the theory. Hence, all the previous results holding true with c_g replaced by c_ϕ, the quadratic casimir of the representation of ϕ.

In general, the Verma module of a primary field would be given as the Virasoro and Kac-Moody descendants of the primary field. But, since we can express the Virasoro generators in terms of the Kac-Moody generators, the basis of the Verma module $[\phi]$ over a primary field ϕ is

$$\left(\prod_{i=1}^{n} J^{a_i}_{-l_i} \prod_{j=1}^{n'} \bar{J}^{b_j}_{-m_j} \right) \phi \qquad l_i, m_j > 0 \tag{1.126}$$

In other words, for WZW models, the representations $[\phi]$, which are representations of the semidirect product of Virasoro and Kac-Moody algebras, are the highest weight representations of the current algebra only.

[1] The factor of 2 comes because we have been working in the orthonormal basis of generators: $\mathrm{tr}\left(t^a t^b\right) = \delta^{ab}$. For SU(2), this means that $t^a = \frac{1}{\sqrt{2}}\sigma^a$, where σ^a are the Pauli matrices, satisfying $[\sigma^a, \sigma^b] = 2i\epsilon^{abc}\sigma^c$ and $\mathrm{tr}\left(\sigma^a\sigma^b\right) = 2\delta^{ab}$.

1.3.7 The Knizhnik-Zamolodchikov Equation and Correlation Functions

Using the OPE of the current and primary field,

$$J^a(w)g(z,\bar{z}) = \frac{t^a}{w-z}g(z,\bar{z}) + \cdots$$

we can obtain the Ward Identity:

$$\implies \langle J^a(w)g(z_1,\bar{z}_1)\cdots g(z_n,\bar{z}_n)\rangle = \sum_{i=1}^{n} \frac{t_i^a}{w-z_i}\langle g(z_1,\bar{z}_1)\cdots g(z_n,\bar{z}_n)\rangle \tag{1.127}$$

where t_i^a is the generator acting on the field $g(z_i,\bar{z}_i)$ inside the correlator. To use the results from the previous section, we multiply t_l^a on both sides of the equation, where $l \in \{1\ldots n\}$

$$t_l^a\langle J^a(w)g(z_1,\bar{z}_1)\cdots g(z_n,\bar{z}_n)\rangle = \left(\frac{c_g}{w-z_l} + \sum_{i=1;i\neq l}^{n}\frac{t_l^a t_i^a}{w-z_i}\right)\langle g(z_1,\bar{z}_1)\cdots g(z_n,\bar{z}_n)\rangle \tag{1.128}$$

Now, using relation (1.123), we can write this equivalently as

$$t_l^a\langle J^a(w)g(z_1,\bar{z}_1)\cdots g(z_n,\bar{z}_n)\rangle = \left(\frac{c_g}{w-z_l} + \kappa\partial_{z_l}\right)\langle g(z_1,\bar{z}_1)\cdots g(z_n,\bar{z}_n)\rangle \tag{1.129}$$

where $\kappa = 1/2\gamma = k + h^\vee$. Combining these two, in the limit $w \to z_l$, we get the Knizhnik-Zamolodchikov equation:

$$\left(\kappa\partial_{z_l} - \sum_{i=1;i\neq l}^{n}\frac{t_l^a t_i^a}{z_l - z_i}\right)\langle g(z_1,\bar{z}_1)\cdots g(z_n,\bar{z}_n)\rangle = 0 \tag{1.130}$$

This is a very powerful result, since now we can calculate correlation functions of the WZW model by finding the solutions to this differential equation, by imposing the required analytical properties.

1.4 Modular Invariance

Conformal Field theories defined on a torus have an additional symmetry, modular invariance. Modular Invariance further restricts objects, like the partition function and characters of the CFT, and hence is a useful tool in classifying RCFTs.

Tori are defined by defining a lattice Λ on $\mathbb{C}$ by the lattice vectors ω_1 and ω_2, and taking the quotient $\mathbb{C}/\Lambda$. The modular parameter is defined as $\tau = \omega_1/\omega_2$. Modular transformations are generated by

$$\mathcal{T} : \tau \to \tau + 1; \quad \mathcal{S} : \tau \to -\frac{1}{\tau} \tag{1.131}$$

which form the group $PSL_2(\mathbb{Z})$. The group $PSL_2(\mathbb{Z})$ acts on the modulus τ via a matrix:

$$\tau \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d} \tag{1.132}$$

such that $ad - bc = 1$.

The partition function of a CFT is defined as

$$Z(\tau, \bar{\tau}) = \mathrm{tr}\Big(q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}\Big) \tag{1.133}$$

where $q = e^{2\pi i \tau}$. The trace over each Verma module $[\phi_i]$ is called the character of the Verma module The partition function of a CFT must be modular invariant, i.e. independent of choice of modular parameter τ used to define the torus. This is because the properties of a CFT should not change by changing the torus, i.e. by choice of lattice vectors ω_1, ω_2, or their overall scaling, τ. This will put constraints on the operator content of the CFT.

Similarly, the trace over a single Verma module is called the character of the corresponding primary field.

$$\chi_i(\tau) = \mathrm{tr}_{[\phi_i]}\left(q^{L_0 - c/24}\right) = q^{-c/24 + h_i} \sum_n a_n^i q^n \tag{1.134}$$

where h_i is the conformal dimension of the primary and a_n^i is the degeneracy at the "energy" level n [2]. This means that $d(n) \in \mathbb{Z}_{\geq 0}$, and the characters can be expressed as an non-negative integer q-series. In terms of characters, one can thus write the partition function as

$$Z(\tau, \bar{\tau}) = \sum_{i=0}^{p-1} M_i \bar{\chi}_i(\bar{\tau}) \chi_i(\tau) \tag{1.135}$$

where there are p characters and M_i are the respective multiplicities of each of the characters. Clearly if the partition function is to remain invariant under modular transformations, then the characters must transform into specific linear combinations of themselves under the modular transformation, so as to keep the partition function invariant. Characters as defined in (1.134) are eigenfunctions of the $\mathcal{T}$ transformation

$$\mathcal{T}\chi_i(\tau) = \chi_i(\tau + 1) = e^{2\pi i (h_i - \frac{c}{24})} \chi(\tau) \tag{1.136}$$

Under the $\mathcal{S}$ transformation, things are more non-trivial, and will have to be explicitly computed.

1.4.1 Characters for the SU(2) WZW model

To discuss the characters for the SU(2)$_k$ WZW model, we will generalize the definition of characters to keep track of the SU(2) charge.

$$\chi_i(z, \tau) = \mathrm{tr}_{[\phi_i]}\left(q^{L_0 - \frac{c}{24}} y^{J_0^3}\right) \tag{1.137}$$

where $y = e^{2\pi i z}$, and z is the coordinate on the torus, and is known as the "chemical potential". Modular transformations act on them in the following way:

$$\begin{aligned} \mathcal{T}\chi_i(z,\tau) &= \chi_i(z,\tau+1) \\ \mathcal{S}\chi_i(z,\tau) &= e^{\frac{-kz^2}{2\tau}}\chi_i\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) \end{aligned} \tag{1.138}$$

Using what is known as the Kac-Weyl formula [7], which computes characters of Kac-Moody algebras, we obtain the following form of the characters

$$\chi_l(z,\tau) = \frac{\Theta^{(k+2)}_{2l+1}(z,\tau) - \Theta^{(k+2)}_{-2l-1}(z,\tau)}{\Theta^{(2)}_1(z,\tau) - \Theta^{(2)}_{-1}(z,\tau)} \tag{1.139}$$

where the theta functions are defined as

$$\Theta^{(k)}_m(z,\tau) \equiv \sum_{n\in\mathbb{Z}+\frac{m}{2k}} q^{kn^2}y^{kn} \tag{1.140}$$

Clearly, the characters are still an eigenfunction of the modular $\mathcal{T}$ transformation, picking up a phase $e^{2\pi i(h_i - \frac{c}{24})}$. For the SU(2)$_k$ WZW model, we have

$$\mathcal{T}\chi_l = e^{2\pi i\left(\frac{l(l+1)}{k+2} - \frac{k}{8(2+k)}\right)}\chi_l \tag{1.141}$$

We shall use the Poisson ressumation formula:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} g(m); \qquad g(m) = \int_{-\infty}^{\infty} dy\, f(y)e^{2\pi i m y} \tag{1.142}$$

where $g(m)$ is the Fourier transform of $f(n)$, to the Theta function, obtaining:

$$\Theta^{(k)}_m\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = \sqrt{\frac{-i\tau}{2k}}e^{\frac{2\pi i k z^2}{4\tau}}\sum_{m'=1-k}^{k} e^{-\frac{2\pi i m m'}{2k}}\Theta^{(k)}_{m'}(z,\tau) \tag{1.143}$$

using which we can obtain the $\mathcal{S}$ matrix for SU(2)$_k$ characters:

$$S_{ll'} = \sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi(2l+1)(2l'+1)}{k+2}\right) \tag{1.144}$$

Characters for other WZW models have also been calculated using the Kac-Weyl character formula as well. From here on, we shall only consider characters with $z = 0$, thus setting the y coordinate to 1 and removing it from the discussion. With no chemical potential, the characters no longer depend on the Cartan eigenvalues.

1.4.2 Modular Linear Differential Equations

Classifying RCFTs with a small number of characters is an incomplete but tractable problem. Many approaches have been taken for such a classification scheme, including fusion algebras [8, 9], representation theory of $\mathrm{SL}(2,\mathbb{Z})$ [10, 11, 12], and Hecke operators [13]. Here we shall focus on classification using the Modular Linear Differential Equation (MLDE) [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. To classify using MLDE, one does not need to provide the chiral algebra of the CFT, which is required in other approaches. Instead, one specifies some parameters in a p^{th} order differential equation, for a theory with p characters. The p independent solutions of the differential equation are the characters of the theory. These will have the chemical potential z set to 0 since that gives weight to the eigenvalues of the Cartan generators, which requires specification of a chiral symmetry algebra.

To understand the MLDE, we require what are known as modular forms. Weight $2r$-modular forms are defined as functions which transform under modular transformations (1.132) as

$$f'(\tau') = (c\tau + d)^{2r} f(\tau) \tag{1.145}$$

One can define a covariant derivative in the moduli (τ) space,

$$\mathcal{D}_\tau = \partial_\tau - \frac{i\pi}{6} r E_2(\tau) \tag{1.146}$$

on weight r-modular forms. Here $E_2(\tau)$ is the second Eisenstein series, and acts as the connection on the moduli space. This covariant derivative increases the weight of any modular form it acts on by 2.

Characters are holomorphic functions of the coordinate z, but are not modular invariant. As we saw before, they transform into linear combinations of themselves through modular transformations. Hence, we call them vector-valued modular forms, of weight 0. Thus characters will be the solutions of a p^{th} order differential equation, which itself is modular invariant.

To construct the MLDE, let us consider the Wronskian: the determinant of all the characters and its modular derivatives. It is holomorphic since it is created from holomorphic objects, i.e. characters, and it is a modular form The "master" Wronskian is:

$$\det \begin{pmatrix} \chi_0 & \chi_1 & \cdots & \chi_{p-1} & \chi \\ \mathcal{D}\chi_0 & \mathcal{D}\chi_1 & \cdots & \mathcal{D}\chi_{p-1} & \mathcal{D}\chi \\ \vdots & \vdots & & \vdots & \vdots \\ \mathcal{D}^{p-1}\chi_0 & \mathcal{D}^{p-1}\chi_1 & \cdots & \mathcal{D}^{p-1}\chi_{p-1} & \mathcal{D}^{p-1}\chi \\ \mathcal{D}^{p}\chi_0 & \mathcal{D}^{p}\chi_1 & \cdots & \mathcal{D}^{p}\chi_{p} & \mathcal{D}^{p}\chi \end{pmatrix} = 0 \tag{1.147}$$

Here the last column χ is a linear combination of the characters $\chi_0, \cdots \chi_{p-1}$. Hence this determinant

is zero. If we expand the determinant about the last column, we get

$$\sum_{k=0}^{p}(-1)^{p-k}W_k\mathcal{D}^k\chi = 0 \tag{1.148}$$

where

$$W_k = \det\begin{pmatrix} \chi_0 & \chi_1 & \cdots & \chi_{p-1} \\ \mathcal{D}\chi_0 & \mathcal{D}\chi_1 & \cdots & \mathcal{D}\chi_{p-1} \\ \vdots & \vdots & & \vdots \\ \mathcal{D}^{k-1}\chi_0 & \mathcal{D}^{k-1}\chi_1 & \cdots & \mathcal{D}^{k-1}\chi_{p-1} \\ \mathcal{D}^{k+1}\chi_0 & \mathcal{D}^{k+1}\chi_1 & \cdots & \mathcal{D}^{k+1}\chi_{p-1} \\ \vdots & \vdots & & \vdots \\ \mathcal{D}^{p-1}\chi_0 & \mathcal{D}^{p-1}\chi_1 & \cdots & \mathcal{D}^{p-1}\chi_{p-1} \\ \mathcal{D}^{p}\chi_0 & \mathcal{D}^{p}\chi_1 & \cdots & \mathcal{D}^{p}\chi_{p-1} \end{pmatrix} \tag{1.149}$$

is the determinant with the row for the k^{th} derivative removed. Each W_k is a holomorphic modular form of weight $p(p+1)-2k$. If we write the differential equation in monic form,

$$\left(\mathcal{D}^p + \sum_{k=0}^{p-1}\phi_k\mathcal{D}^k\right)\chi = 0 \tag{1.150}$$

where $\phi_k = (-1)^{p-k}\frac{W_k}{W_p}$ are weight $2(p-k)$ modular forms. They are meromorphic functions, with poles at the location of the zeroes of W_n. The resulting differential equation is modular invariant. Even though the characters and thus the Wronskians are undetermined till now, we can determine ϕ_k's up to constants just by their modular properties, i.e. their modular weight and number of poles. Thus, by fixing the constants, we can solve this differential equation for the characters of an RCFT, without having to specify anything about a chiral algebra.

Zeroes of the Wronskian

The parameter τ is in the upper half plane, known as the moduli space. There are three special points in the moduli space, the ramification points $e^{\frac{1}{3}i\pi}$ and $e^{\frac{1}{2}i\pi}$. Here, modular forms can have zeroes of order $\frac{1}{3}$ and $\frac{1}{2}$ respectively. Otherwise, modular forms can have zeroes of integer order in the bulk of the moduli space. The Wronskian W_n can thus have total zeroes of order $\frac{\ell}{6}$, for all non-negative integers except $\ell = 1$. The integer ℓ is called the Wronskian index.

At the point $\tau = i\infty$, since the wronskian behaves as

$$W_n \sim e^{2\pi i\left(\sum_i h_i - \frac{pc}{24}\right)} \tag{1.151}$$

the wronskian has an order $\frac{pc}{24} - \sum_i h_i$ pole. Using the Riemann-Roch theorem which relates poles

and zeroes of a function, we have the following relation

$$\sum_i h_i - \frac{pc}{24} = \frac{\ell}{6} - \frac{p(p-1)}{12} \tag{1.152}$$

Solving the MLDE using the Frobenius method

First, we must specify the Wronskian index ℓ of the theory, and the number of characters p. The coefficients $\phi_k(\tau)$ will thus be meromorphic functions with maximum number of poles equal to ℓ. One can construct a modular form of weight $k = 4m + 6n$ with linear combinations of $E_4^m E_6^n$ including all possible values of m, n. Poles can be introduced by dividing by modular forms appropriately, whose zeroes are known. Therefore, one can specify the functions ϕ_k up to a finite number of constants μ_j from the linear combination.

To solve for the characters, we substitute the series expansion of the character:

$$\chi_i = q^{\alpha_i}(a_0^{(i)} + a_1^{(i)} q + a_2^{(i)} q^2 + \cdots) = q^{\alpha_i} \sum_{n=0}^{\infty} a_n^{(i)} q^n \tag{1.153}$$

where the exponents $\alpha_i = -\frac{c}{24} + h_i$. The modular covariant derivative $\mathcal{D}^n$ act on χ_i as follows:

$$\left(\partial_\tau - i\pi \frac{n(n-1)}{3}\right)\left(\partial_\tau - i\pi \frac{n(n-2)}{3}\right) \cdots \left(\partial_\tau - i\pi \frac{1}{3}\right) \partial_\tau \chi_i \tag{1.154}$$

Once the entire differential equation is written in terms of partial derivatives ∂_τ, one can construct a recurrence relation for the coefficients a_n. Using the indicial equation, one gets a relation between the constants μ_j and all the α_i, through the Riemann-Roch relation. Thus, the differential equation and thus the recurrence relation is completely specified by the exponents α_i.

In order to search for solutions which correspond to admissible characters for RCFT, the coefficients a_n must be integers. To do this, we set the first coefficient $a_0^{(i)}$ to be 1. For the identity character, this is the correct value, since the vacuum of a QFT is non-degenerate. If any of the $a_n^{(0)}$ are not integer, then the corresponding α_i do not correspond to an RCFT. For the other characters, it is not so straightforward, since the ground-state degeneracy for these need not be 1. Instead, we normalize $a_0^{(i)}$ to be 1, and then check for the stability of the denominator for large values of the expansion. Since the character is solved up to a multiplicative constant, one can multiply the LCM of the denominators and this should be (a factor of) the ground-state degeneracy of the character. If the denominator does not stabilize for large n, then the values of α_i once again do not correspond to any RCFT. In fact, in these cases the LCM of the denominator grows exponentially. One can use the recurrence relation to also construct diophantine equations, and use their solutions as a classification of admissible characters of possible RCFT with the specified values of ℓ and p [14, 30, 24, 25] These techniques were used extensively in our paper [32], which is briefly reviewed in chapter 4.

Chapter 2

Correlation Functions of CFT

2.1 Computing the 4-point function for the SU(N) WZW Model

In this chapter, we concern ourselves in calculating four point functions of CFTs. We start with the SU(N)$_k$ WZW model [6]. Because of the symmetry in the theory, we need to ensure that the correlation function is SU(N) invariant, since the theory is SU(N)$\times$ SU(N) invariant. We know that the SU(N) action on the field goes as $t^a g$, and therefore g^{-1} transforms as $g^{-1}t^a$, given t^a is hermitian. We know that g is in the "bifundamental" representation of SU(N)$\times$ SU(N), so can be written as the tensor product of fundamental states in both SU(N)'s.

$$g(z,\bar{z}) = \begin{pmatrix} \omega_1(z) \\ \vdots \\ \omega_N(z) \end{pmatrix} \otimes \begin{pmatrix} \omega_1(\bar{z}) & \cdots & \omega_N(\bar{z}) \end{pmatrix} \tag{2.1}$$

where ω_i is one of the states in the fundamental representation.

For a SU(N) invariant correlation function of g's, we require equal number of g's and g^{-1}'s in the correlation function, since each g will "fuse" with a g^{-1}. Until now we haven't been putting the group indices on the field $g(z,\bar{z})$, i.e, ${g_\alpha}^\beta(z,\bar{z})$, where α is the index which is transformed by the left holomorphic SU(N), and β is transformed by the right antiholomorphic SU(N). An example of an SU(N)$\times$ SU(N) invariant correlation function is

$$G^{\alpha_2\alpha_3\beta_1\beta_4}_{\alpha_1\alpha_1\beta_2\beta_3}(z_i,\bar{z}_i) = \left\langle {g_{\alpha_1}}^{\beta_1}(z_1\bar{z}_1){g^{-1}}^{\alpha_2}{}_{\beta_2}(z_2\bar{z}_2){g^{-1}}^{\alpha_3}{}_{\beta_3}(z_3\bar{z}_3){g_{\alpha_4}}^{\beta_4}(z_4\bar{z}_4)\right\rangle \tag{2.2}$$

since the fields in the correlator must fuse to form a singlet, and g is the N of SU(N), and hence g^{-1} is the $\bar{N}$ of SU(N) due to it being a unitary theory. Hence, the representations that will flow in the fusion are

$$N \otimes \bar{N} = \mathbb{1} \oplus (N^2 - 1) \tag{2.3}$$

That is, if a g and g^{-1} fuse, they form the identity and adjoint fields. If g_1 and g_2^{-1} fuse, then g_3^{-1}

and g_4 must fuse, both giving identity and adjoint fields. This is the s channel diagram. Similarly the t channel diagram is formed by fusing g_1 and g_3^{-1}, and fusing g_2^{-1} and g_4. To capture this, we decompose the full correlation function into its $SU(N)$ invariant matrices

$$I_1 = \delta^{\alpha_2}_{\alpha_1}\delta^{\alpha_3}_{\alpha_4} \quad I_2 = \delta^{\alpha_3}_{\alpha_1}\delta^{\alpha_2}_{\alpha_4} \tag{2.4}$$

$$\bar{I}_1 = \delta^{\beta_1}_{\beta_2}\delta^{\beta_3}_{\beta_4} \quad \bar{I}_2 = \delta^{\beta_1}_{\beta_3}\delta^{\beta_4}_{\beta_2} \tag{2.5}$$

The decomposition is thus

$$G^{\alpha_2\alpha_3\beta_1\beta_4}_{\alpha_1\alpha_1\beta_2\beta_3}(z,\bar{z}) = \sum_{A,B=1,2} I_A G_{AB}(z,\bar{z})\bar{I}_B \tag{2.6}$$

where the indices have been suppressed on the I's. Clearly, I_1's correspond to the s channel diagram, and I_2's correspond to the t channel diagram.

We know that up to three point functions are completely fixed by conformal invariance. Four point functions can be expressed in terms of the cross ratios

$$x = \frac{z_{12}z_{34}}{z_{14}z_{23}}, \quad \bar{x} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{14}\bar{z}_{23}}, \text{where } z_{ij} = z_i - z_j \tag{2.7}$$

as a single valued function of x and $\bar{x}$

$$G(z_i,\bar{z}_i) = \left\langle g(z_1,\bar{z}_1)g^{-1}(z_2,\bar{z}_2)g^{-1}(z_3,\bar{z}_3)g(z_4,\bar{z}_4)\right\rangle = (z_{14}z_{23}\bar{z}_{14}\bar{z}_{23})^{-2\Delta}G(x,\bar{x}) \tag{2.8}$$

In a CFT, we can decompose a correlation function of four fields into holomorphic (antiholomorphic) conformal blocks $\mathcal{F}$ ($\bar{\mathcal{F}}$).

$$G^{\alpha_2\alpha_3\beta_1\beta_4}_{\alpha_1\alpha_1\beta_2\beta_3}(z_i,\bar{z}_i) = \sum_p C^2_{1p}C^3_{4p}\mathcal{F}^{\alpha_2\alpha_3}_{\alpha_1\alpha_4}(p|z_i)\bar{\mathcal{F}}^{\beta_1\beta_4}_{\beta_2\beta_3}(p|\bar{z}_i) \tag{2.9}$$

where C^j_{ip}'s are the OPE coefficients. This is the s channel decomposition of the correlation function. And from (2.8), we can define the conformal blocks in terms of x

$$\mathcal{F}(z_i) = (z_{14}z_{23})^{-2\Delta}\mathcal{F}(x) \tag{2.10}$$

From (2.6), using the fact that the only SU(N) invariant way of fusing the fields is in terms of the invariant matrices,

$$\sum_{A,B=1,2} I_A G_{AB}(z,\bar{z})\bar{I}_B = \sum_p C^2_{1p}C^3_{4p}\mathcal{F}^{\alpha_2\alpha_3}_{\alpha_1\alpha_4}(p|z_i)\bar{\mathcal{F}}^{\beta_1\beta_4}_{\beta_2\beta_3}(p|\bar{z}_i) \tag{2.11}$$

$$= \sum_p C^2_{1p}C^3_{4p}\left(\sum_A I_A\mathcal{F}_A(p|z_i)\right)\left(\sum_B \bar{\mathcal{F}}_B(p|\bar{z}_i)\bar{I}_B\right) \tag{2.12}$$

again, suppressing the indices on I's. The left SU(N) invariant matrices are pulled out of the holomorphic block, and the right SU(N) invariant matrices are pulled out of the antiholomorphic block. Hence, we see that the conformal blocks also decompose by the SU(N) invariant matrices,

$$\mathcal{F}^{\alpha_2\alpha_3}_{\alpha_1\alpha_4}(x) = I_1\mathcal{F}_1(x) + I_2\mathcal{F}_2(x) \tag{2.13}$$

And we can express the decomposed correlator in terms of the decomposed conformal blocks as

$$\implies G_{AB} = \sum_p C^2_{1p}C^3_{4p}\mathcal{F}_A(p|z)\bar{\mathcal{F}}_B(p|\bar{z}) \tag{2.14}$$

Now applying the Knizhnik-Zamolodchikov equation to the the correlation function, and considering only the holomorphic part

$$\left(\kappa\partial_{z_i} + \sum_{i\neq j}\frac{t^a_i t^a_j}{z_i - z_j}\right)\mathcal{F}(z) = 0 \tag{2.15}$$

$$\implies \left(\kappa\partial_{z_i} + \sum_{i\neq j}\frac{t^a_i t^a_j}{z_i - z_j}\right)(z_{14}z_{23})^{-2\Delta}(I_1\mathcal{F}_1(x) + I_2\mathcal{F}_2(x)) = 0 \tag{2.16}$$

Considering the $i = 1$ case, and converting the z derivative to an x derivative we have

$$\left(\frac{-2h}{z_{14}} + \left(\frac{x}{z_{12}} - \frac{x}{z_{14}}\right)\kappa\partial_x + \sum_{j=2}^{4}\frac{t^a_1 t^a_j}{z_1 - z_j}\right)(I_1\mathcal{F}_1(x) + I_2\mathcal{F}_2(x)) = 0 \tag{2.17}$$

Due to conformal invariance, we are allowed to fix 3 out of 4 points on the plane as:

$$z_1 = x \quad z_2 = 0 \quad z_3 = 1 \quad z_4 = \infty \tag{2.18}$$

Then, the differential equation reduces to

$$\left(\kappa\partial_x + \frac{t^a_1 t^a_2}{x} + \frac{t^a_1 t^a_3}{x-1}\right)(I_1\mathcal{F}_1(x) + I_2\mathcal{F}_2(x)) = 0 \tag{2.19}$$

Using the form of I_1 and I_2 and properties of t^a's, it can be shown that the differential equation simplifies to

$$-\kappa\partial_x\mathcal{F}_1 = \frac{N^2-1}{N}\frac{\mathcal{F}_1}{x} - \frac{\mathcal{F}_1}{N(x-1)} + \frac{\mathcal{F}_2}{x} \tag{2.20}$$

$$-\kappa\partial_x\mathcal{F}_2 = \frac{N^2-1}{N}\frac{\mathcal{F}_2}{x-1} - \frac{\mathcal{F}_2}{Nx} + \frac{\mathcal{F}_1}{x-1} \tag{2.21}$$

Eliminating $\mathcal{F}_2$ in terms of $\mathcal{F}_1$, and combining the two equations, we get a second order

differential equation:

$$\left[x\kappa^2\partial_x^2+\left(\kappa^2+\kappa\frac{N^2-2}{N}\left(1+\frac{x}{x-1}\right)\right)\partial_x\right.$$
$$\left.+\left(\frac{\kappa}{N(x-1)^2}-\frac{1}{x-1}-\left(\frac{N^2-1}{N(x-1)}-\frac{1}{Nx}\right)\left(\frac{x}{N(x-1)}-\frac{N^2-1}{N}\right)\right)\right]\mathcal{F}_1=0 \quad (2.22)$$

solutions to which can be shown to be:

$$\mathcal{F}_1^{(-)}=x^{-2\Delta}(1-x)^{h-2\Delta}{}_2F_1\left(\frac{1}{\kappa},-\frac{1}{\kappa},1-\frac{N}{\kappa};x\right) \quad (2.23)$$

$$\mathcal{F}_1^{(+)}=x^{h-2\Delta}(1-x)^{h-2\Delta}{}_2F_1\left(\frac{N-1}{\kappa},\frac{N+1}{\kappa},1+\frac{N}{\kappa};x\right) \quad (2.24)$$

By using the relation between $\mathcal{F}_1$ and $\mathcal{F}_2$, we also obtain

$$\mathcal{F}_2^{(-)}=\frac{1}{k}x^{1-2\Delta}(1-x)^{h-2\Delta}{}_2F_1\left(1+\frac{1}{\kappa},1-\frac{1}{\kappa},2-\frac{N}{\kappa};x\right) \quad (2.25)$$

$$\mathcal{F}_2^{(+)}=-Nx^{h-2\Delta}(1-x)^{h-2\Delta}{}_2F_1\left(\frac{N-1}{\kappa},\frac{N+1}{\kappa},\frac{N}{\kappa};x\right) \quad (2.26)$$

where ${}_2F_1$ is the hypergeometric function. These are known as the "current blocks". Therefore, the conformal blocks of this 4 point function are

$$\mathcal{F}^-=\begin{pmatrix}\mathcal{F}_1^-\\ \mathcal{F}_2^-\end{pmatrix}=\mathcal{F}_{\mathbb{1}} \qquad \mathcal{F}^+=\begin{pmatrix}\mathcal{F}_1^+\\ \mathcal{F}_2^+\end{pmatrix}=\mathcal{F}_{\Phi} \quad (2.27)$$

as the identity and adjoint conformal blocks respectively.

Crossing symmetry will help us fix the full correlator, not just the conformal blocks:

$$G(x,\bar{x})=G(1-x,1-\bar{x}) \quad (2.28)$$

which switches the s channel and t channel partial waves. We know that I_1's are the s channel blocks and I_2's are the t channel, so crossing symmetry implies $1\leftrightarrow 2$ in G_{AB}, which can be written as $A\to 3-A$.

If we write the correlation function as

$$G_{AB}=\sum_{p,q=+,-}U_{pq}\mathcal{F}^p\bar{\mathcal{F}}^q \quad (2.29)$$

then, by demanding that for small x, the correlation function must be single valued, we can say that the off diagonal terms $U_{+-}=U_{-+}=0$. This is because of the fractional powers of x in the current blocks will be multi-valued around the origin unless they're powers of $|x|$, which is only possible by multiplying $\mathcal{F}^p$ with its conjugate $\bar{\mathcal{F}}^p$.

Using the identity

$$
\begin{aligned}
{}_2F_1\left(a,b,c;x\right) = &\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1\left(a,b,a+b-c+1,1-x\right)\\
&+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}\,{}_2F_1\left(c-a,c-b,c-a-b+1;1-x\right)
\end{aligned}
$$

we can show that the blocks transform as

$$\mathcal{F}_A^{(p)} = \sum C_q^p \mathcal{F}_{3-A}^{(q)} \tag{2.30}$$

By applying crossing symmetry twice, we obtain the following relations as a consistency check

$$C_-^- = C_+^+ \tag{2.31}$$

$$C_-^+ C_+^- + C_-^- C_+^+ = 1 \tag{2.32}$$

Evaluating the necessary coefficients, one gets

$$C_-^- = N\frac{\Gamma\left(\frac{N}{\kappa}\right)\Gamma\left(-\frac{N}{\kappa}\right)}{\Gamma\left(\frac{1}{\kappa}\right)\Gamma\left(-\frac{1}{\kappa}\right)} \tag{2.33}$$

$$C_-^+ = -N\frac{\Gamma^2\left(\frac{N}{\kappa}\right)}{\Gamma\left(\frac{N+1}{\kappa}\right)\Gamma\left(\frac{N-1}{\kappa}\right)} \tag{2.34}$$

Combining everything, we get

$$U_{++} = \frac{C_-^{-2}-1}{C_-^{+2}}U_{--} \tag{2.35}$$

with U_{--} set to 1. Hence we finally obtain the correlation function

$$G_{AB}(x,\bar{x}) = \mathcal{F}_A^{(-)}(x)\bar{\mathcal{F}}_B^{(-)}(\bar{x}) + U_{++}\mathcal{F}_A^{(+)}(x)\bar{\mathcal{F}}_B^{(+)}(\bar{x}) \tag{2.36}$$

For $k = 1$, C_-^- is 1, so the second term goes to zero, which suggests there is no adjoint field for $k = 1$.

2.2 Using the Wronskian to Calculate the 4-Point Function

We can also use Wronskians to calculate correlation functions of four fields, by using the behaviour of the conformal blocks near poles using physical arguments [33]. Correlation functions are crossing symmetric, but are not holomorphic, while conformal blocks are holomorphic, but are not crossing symmetric. In fact they transform into each other. It is easily seen that Wronskians made out of conformal blocks will therefore be both crossing symmetric and holomorphic. We will

define W_k as the Wronskian with the k^{th} derivative removed.

$$W_k \equiv \det \begin{pmatrix} f_1 & \cdots & f_n \\ \partial f_1 & \cdots & f_n \\ \vdots & & \vdots \\ \partial^{k-1} f_1 & \cdots & \partial^{k-1} f_n \\ \partial^{k+1} f_1 & \cdots & \partial^{k+1} f_n \\ \vdots & & \vdots \\ \partial^n f_1 & \cdots & \partial^n f_n \end{pmatrix} \tag{2.37}$$

where f_i are the conformal blocks, where there are n conformal families participating in the fusion rules, and the derivative ∂ is with respect to one of the points in the correlation function, say z_1. We can construct another determinant, by adding a column of some linear combination $f = \sum_i a_i f_i$ to the full Wronskian with no removed derivatives. This is zero because f is not independent of f_i. We can use this to create an equation of the blocks, by expanding about this last column.

$$\det \begin{pmatrix} f_1 & \cdots & f_n & f \\ \partial f_1 & \cdots & \partial f_n & \partial f \\ \vdots & & \vdots & \vdots \\ \partial^n f_1 & \cdots & \partial^n f_n & \partial^n f \end{pmatrix} = \sum_{k=0}^{n} (-1)^{n-k} W_k \partial^k f = 0 \tag{2.38}$$

We can write this in monic form, and choose $f = f_\alpha$,

$$\partial^n f_\alpha + \sum_{k=0}^{n-1} \psi_k \partial^k f_\alpha = 0 \tag{2.39}$$

where the ψ_k's will need to be determined by the behaviour of the blocks, and in turn Wronskians, since

$$\psi_k = (-1)^{n-k} \frac{W_k}{W_n} \tag{2.40}$$

The ψ_k's have following pole structure as $z_1 \to z_a$ (where a labels $2, 3, 4$),

$$\psi_k \sim \frac{1}{z_{1a}^{n-k}} \tag{2.41}$$

Due to inversion symmetry, number of poles and number of zeroes must be equal on the plane. Hence, as $z_1 \to \infty$

$$\psi_k \sim \frac{1}{z_1^{3(n-k)}} \tag{2.42}$$

for a 4 point function. We can determine the ψ_k's exactly when there are only a few conformal blocks. In particular, we will be dealing with the fusion rules of the type $\phi_A \otimes \phi_A = \phi_B \oplus \phi_C$. Hence, we will have two blocks, one for ϕ_B and ϕ_C each. The leading singularity for these blocks in the coincidence limits are

$$f_1 \sim \frac{1}{z_{1a}^{2h_A}} \quad f_2 \sim \frac{1}{z_{1a}^{2h_A - h_B}} \tag{2.43}$$

and the behaviour of fields, and therefore blocks at infinity is fixed by conformal invariance to be

$$f_1 \sim \frac{1}{z_1^{2h_A}} \quad f_2 \sim \frac{1}{z_1^{2h_A}} \tag{2.44}$$

However, in WZW models we have multicomponent fields, such as the fundamental primary field $g(z, \bar{z})$, which has N components in SU(N) WZW models. In such theories, a primary may not flow in the fusion, as we saw while calculating the 4 point function in the previous section, due to group theoretic or other (for CFTs like the Baby monster CFT, which is not a WZW model but still is a theory with multicomponent primaries) reasons. Since we will be computing the correlation function (2.2), we will be interested only in the fusion $g \otimes g^{-1} = \mathbb{1} + \Phi$. Since there will be two blocks, the relevant differential equation is

$$\partial^2 f_i + \psi_1 \partial f_i + \psi_0 f_i = 0 \tag{2.45}$$

The blocks will behave as such, in the coincidence limit $z_1 \to z_2$

$$f_1 \sim \frac{g_1(z_2, z_3, z_4)}{z_{12}^{2\Delta - n_1}} \quad f_2 \sim \frac{g_2(z_2, z_3, z_4)}{z_{12}^{2\Delta - h - n_2}} \tag{2.46}$$

where $\Delta = L_0 g$, $h = L_0 \Phi$, and n_1, n_2 are the level of the descendants flowing in the fusion. The other coincident limits will therefore be simply $z_1 \to z_a$, $a = 3, 4$:

$$f_1 \sim \frac{\widetilde{g}_1(z_2, z_3, z_4)}{z_{1a}^{2\Delta}} \quad f_2 \sim \frac{\widetilde{g}_2(z_2, z_3, z_4)}{z_{1a}^{2\Delta - h}} \tag{2.47}$$

since definite representations have been chosen to flow in the s channel (12→34).

From this, we can calculate the behavior of the relevant Wronskians

$$W_2 = f_1 \partial f_2 - f_2 \partial f_1 \sim \begin{cases} g_1 g_2 (h + n_2 - n_1) z_{12}^{-4\Delta + h + m - 1} & z_1 \to z_2 \\ \widetilde{g}_1 \widetilde{g}_2 \, h \, z_{1a}^{-4\Delta + h - 1} & z_1 \to z_a \\ z_1^{-4\Delta - 2} & z_1 \to \infty \end{cases} \tag{2.48}$$

where $m = n_1 + n_2$. Since number of poles is equal to number of zeroes, using the infinity and

pole behavior of W_2, we get the following relation

$$h = \frac{8\Delta + 1 - m}{3} \tag{2.49}$$

which we can use to eliminate h from the equations.

We can now calculate the other relevant Wronskians,

$$W_1 = \partial W_2 \tag{2.50}$$

$$W_0 = \partial f_1 \partial^2 f_2 - \partial f_2 \partial^2 f_1 \sim \begin{cases} g_1 g_2 (n_1 - 2\Delta)(h + n_2 - 2\Delta)(h + n_2 - n_1) z_{12}^{-4\Delta + h + m - 3} & z_1 \to z_2 \\ 2\widetilde{g}_1 \widetilde{g}_2 \Delta h (2\Delta - h) z_{1a}^{-4\Delta + h - 3} & z_1 \to z_a \end{cases} \tag{2.51}$$

Here (2.50) is called Abel's relation. Keeping track of the arbitrary z_2, z_a dependences and all the coefficients, we can calculate the ψ_k's by using the limits calculated for W_k's

$$\psi_1 = -\frac{W_1}{W_2} = \frac{2}{3}(2\Delta + 1 - m)\frac{1}{z_{12}} + \frac{2}{3}\left(2\Delta + 1 + \frac{m}{2}\right)\left(\frac{1}{z_{13}} + \frac{1}{z_{14}}\right) \tag{2.52}$$

$$\psi_0 = -\frac{1}{3}(2\Delta - m + n_2)(2\Delta + 1 - m + 3n_2)\frac{1}{z_{12}^2} - \frac{2}{3}\Delta(2\Delta + 1 - m)\left(\frac{1}{z_{13}^2} + \frac{1}{z_{14}^2}\right) + \frac{Q}{z_{12}z_{13}} + \frac{Q}{z_{12}z_{14}} + \frac{R}{z_{13}z_{14}} \tag{2.53}$$

Here, for ψ_0, in the last three terms, we've used the symmetry in switching z_3 and z_4, and the fact that ψ_0 dimensionally goes as powers of z_1^{-2}. To fix the values of Q and R in ψ_0, we have to impose that in the limit $z_1 \to z_3$, $z_2 \to z_4$, the solution is $f = z_{13}^{-2\Delta} z_{24}^{-2\Delta}$. Doing so, we obtain:

$$\begin{aligned} Q &= \frac{4\Delta}{3}\left(2\Delta + 1 - \frac{3m}{2} + 2n_2\right) - \frac{1}{3}(m - n_2)(1 + 3n_2 - m) \\ R &= \frac{4\Delta}{3}(2\Delta + 1 + m - 2n_2) + \frac{1}{3}(m - n_2)(1 + 3n_2 - m) \end{aligned} \tag{2.54}$$

This completely determines the differential equation we are to solve, to precisely obtain the conformal blocks. We have obtained $m + 1$ differential equations, where $n_2 \in \{0, \cdots, m\}$. Now, we can set the points $z_1 = x, z_2 = \infty, z_3 = 1, z_4 = 0$, and we finally obtain

$$\partial^2 f + \frac{2}{3}\left(2\Delta + 1 + \frac{m}{2}\right)\left(\frac{1}{x-1} + \frac{1}{x}\right)\partial f + \left[-\frac{2}{3}\Delta(2\Delta + 1 - m)\left(\frac{1}{(x-1)^2} + \frac{1}{x^2}\right) + \left(\frac{4\Delta}{3}(2\Delta + 1 + m - 2n_2) + \frac{1}{3}(m - n_2)(1 + 3n_2 - m)\right)\frac{1}{x(x-1)}\right] f = 0 \tag{2.55}$$

whose solutions are

$$f_1^{(n)}(x) = x^{-2\Delta}(1-x)^{-2\Delta}{}_2F_1\left(\frac{1}{3}(1-4\Delta-m+3n), m-4\Delta-n, \frac{2}{3}(1-4\Delta)+\frac{m}{3}, x\right) \tag{2.56}$$

$$f_2^{(n)}(x) = \mathcal{N}x^{-2\Delta+h}(1-x)^{-2\Delta}{}_2F_1\left(\frac{1}{3}(1+2\Delta-m)+n, \frac{1-4\Delta+2m}{3}-n, \frac{4}{3}(1-4\Delta)-\frac{m}{3}, x\right) \tag{2.57}$$

where the normalization $\mathcal{N}$ is determined by crossing symmetry as we saw before.

Now let's match our result with what we obtained using the KZ equation. Specializing to the SU(2)$_1$ case, where $N=2$, $k=1$. Here, $\Delta=\frac{1}{4}$, $h=\frac{2}{3}$, $\kappa=3$ and $m=1$. We see that from the KZ equation, the Identity conformal block is the only surviving block, and is

$$\mathcal{F}_{\mathbb{1}} = \begin{pmatrix} \sqrt{\frac{1-x}{x}} \\ \sqrt{\frac{x}{1-x}} \end{pmatrix} \tag{2.58}$$

and hence the correlation function is

$$\begin{aligned} G(x,\bar{x}) &= \sum_{A,B=1,2} I_A G_{AB} \bar{I}_B \\ &= I_1\bar{I}_1\sqrt{\frac{1-x}{x}}\sqrt{\frac{1-\bar{x}}{\bar{x}}} + I_1\bar{I}_2\sqrt{\frac{1-x}{x}}\sqrt{\frac{\bar{x}}{1-\bar{x}}} \\ &\quad + I_2\bar{I}_1\sqrt{\frac{x}{1-x}}\sqrt{\frac{1-\bar{x}}{\bar{x}}} + I_2\bar{I}_2\sqrt{\frac{x}{1-x}}\sqrt{\frac{\bar{x}}{1-\bar{x}}} \end{aligned} \tag{2.59}$$

Using the Wronskian method detailed in this section, we obtain the only surviving block is the Identity block as the solution (2.56)

$$\mathcal{F'}_{\mathbb{1}} = \begin{pmatrix} \sqrt{\frac{1-x}{x}} + \sqrt{\frac{x}{1-x}} \\ \sqrt{\frac{1-x}{x}} - \sqrt{\frac{x}{1-x}} \end{pmatrix} \tag{2.60}$$

so, the correlation function is

$$\begin{aligned} G(x,\bar{x}) &= \sum_{A,B=+,-} I_A G_{AB} \bar{I}_B \\ &= I_+\bar{I}_+(a+b)(\bar{a}+\bar{b}) + I_+\bar{I}_-(a+b)(\bar{a}-\bar{b}) + I_-\bar{I}_+(a-b)(\bar{a}+\bar{b}) + I_-\bar{I}_-(a-b)(\bar{a}-\bar{b}) \end{aligned} \tag{2.61}$$

where

$$I_{\pm} = I_1 \pm I_2; \qquad a = \sqrt{\frac{1-x}{x}};\ b = \frac{1}{a} \tag{2.62}$$

This matches with the result obtained using the KZ method, by employing the change of basis (2.62).

In the KZ method, we found the two solutions of a 2$^{\text{nd}}$ order differential equation, with the two solutions being the identity and adjoint current blocks. The other current block was found by using the relation between the two, derived from the KZ equation. But, in the Wronskian method, by using the behaviour of the expected identity and adjoint blocks in coincident limits, we found two *different* differential equations, one for each n_2 (level of the secondary flowing in the fusion). The two solutions obtained per differential equation are the two components(current blocks) of the same conformal block.

2.3 Dotsenko-Fateev Construction and Correlation functions of Minimal Models

Here we briefly go through the method used by Dotsenko and Fateev [34] to construct 4 point correlators of Minimal models, since we focused on WZW models in the previous sections of this chapter. As will come in handy later, we study the expectation value of vertex operators of the free boson CFT, but it will be modified with a "background charge". The free boson theory has central charge $c = 1$. The vertex operator we are interested in is

$$V_\alpha(z) = :e^{i\sqrt{2}\alpha\phi(z)}: \tag{2.63}$$

with conformal dimension $h = \alpha^2$ and $\bar{h} = 0$. Since the propagator for a free boson is $\langle\phi(z)\phi(w)\rangle = -\log(z-w)$, and using the easy to show result,

$$\left\langle :e^{A_1}: \cdots :e^{A_n}: \right\rangle = e^{\sum_{i<j}\langle 0|A_iA_j|0\rangle} \tag{2.64}$$

the correlator of vertex operators is

$$\langle V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(z_n)\rangle = \prod_{i<j}(z_i - z_j)^{2\alpha_i\alpha_j} \tag{2.65}$$

But, the free boson action has a shift symmetry $\phi \to \phi + a$, for any constant a, which the correlator also must obey. This leads to the neutrality condition: $\sum_i \alpha_i = 0$, otherwise the above correlator is 0. However, by suitable changing the free boson action, one can introduce a "background charge density" $-2\alpha_0$, which potentially makes the theory non-unitary because the action is no longer real. However, it modifies the central charge, conformal dimension of the vertex operator and the

neutrality condition:

$$\begin{aligned} c &= 1 - 24\alpha_0 \\ h_\alpha &= \alpha^2 - 2\alpha_0\alpha \\ \sum_i \alpha_i &= 2\alpha_0 \end{aligned} \tag{2.66}$$

For certain values of α_0 which correspond to making the central charge that of a minimal model, the theory becomes unitary. Notice that the vertex operators $V_\alpha = V_{2\alpha_0-\alpha}$ have the same conformal dimension. Screening operators are operators with conformal dimension 0 but with charge. These vertex operators have conformal dimension 1:

$$V_{\alpha_\pm}, \quad \text{where} \quad \alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1} \tag{2.67}$$

such that their integral has conformal dimension 0. Thus the screening operators are

$$Q_\pm = \oint dz\, V_\pm \tag{2.68}$$

with charge $\alpha_\pm$. Note that $\alpha_+ - \alpha_- = 2\alpha_0$ and $\alpha_+\alpha_- = -1$. On inserting these screening operators in correlators, we can enforce the neutrality to condition to be satisfied, but will not change its conformal properties. With these screening operators inserted in correlators, it is possible to show that admissible charges α are

$$\alpha_{r,s} = \frac{1}{2}(1-r)\alpha_+ + \frac{1}{2}(1-s)\alpha_- \tag{2.69}$$

and thus the conjugation $\alpha \to 2\alpha_0 - \alpha$ symmetry becomes $r, s \to -r, -s$, and the conformal dimensions of these operators are

$$h_{r,s}(c) = \frac{1}{4}(r\alpha_+ + s\alpha_0)^2 - \alpha_0^2 \tag{2.70}$$

which is Kac's formula for primary conformal dimensions of minimal models.

Using Wick's theorem, we can show that the correlation functions with the appropriate number

of screening operators to neutralize the charge will be the integrals:

$$\begin{aligned}
&\langle\phi_{\alpha_1}(0)\phi_{\alpha_2}(z)\phi_{\alpha_3}(1)\phi_{\alpha_4}(\infty)\rangle = \left\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)Q_-^{n_1}Q_+^{n_2}\right\rangle \\
\sim& z^{2\alpha_1\alpha_2}(1-z)^{2\alpha_3\alpha_2}\int_{\mathcal{C}_A}\prod_{i=1}^{n_1}dt_i\int_{\mathcal{C}_{A'}}\prod_{j=1}^{n_2}d\tau_j \\
&\times\prod_{i=1}^{n_1}t_i^a(1-t_i)^b(t_i-z)^c\prod_{j=1}^{n_2}\tau_j^{a'}(1-\tau_j)^{b'}(z-\tau_j)^{c'} \\
&\times\prod_{1\le k<i\le n_1}(t_i-t_k)^g\prod_{1\le l<j\le n_2}(\tau_j-\tau_l)^{g'}\prod_{i,j}(t_i-\tau_j)^{-2}
\end{aligned} \tag{2.71}$$

where

$$\begin{gathered}
a = 2\alpha_+\alpha_0 \quad b = 2\alpha_+\alpha_3 \quad c = 2\alpha_+\alpha_2 \\
a' = 2\alpha_-\alpha_0 \quad b' = 2\alpha_-\alpha_3 \quad c' = 2\alpha_-\alpha_2 \\
g = 2\alpha_+^2 \quad g' = 2\alpha_-^2
\end{gathered}$$

where different contours represent different conformal blocks. We will see that these integrals will be used to represent characters of various RCFT, increasing their scope from representations of minimal model correlators.

Chapter 3

Contour Integral Representation of Characters and the Modular S-matrix

This chapter is a review of the content of the original work presented in the paper [35] with my guide Prof. Sunil Mukhi and collaborator Palash Singh.

The modular $\mathcal{S}$ transformation is the transformation $\tau \to -\frac{1}{\tau}$. Since characters transform into linear combinations of themselves under modular transformations, the modular $\mathcal{S}$ transformation can be expressed as a $p \times p$ matrix. Using Verlinde's formula [36], we can calculate the fusion rules using the $\mathcal{S}$-matrix, which dictate which fusion channels will appear in correlation functions, which are the observables one computes for a QFT. The properties of the $\mathcal{S}$-matrix also provides powerful constraints on possible allowed values of the critical exponents $\alpha_i = -\frac{c}{24} + h_i$ of the theory. Thus it can also be used in classifying RCFT. Hence determining the $\mathcal{S}$-matrix is an important task.

The modular $\mathcal{S}$-matrix has been computed for minimal models and WZW models using properties of the Theta functions using Poisson resummation, as we saw in section 1.4.1. However, this requires knowledge of the symmetry algebra of the CFT. In the MLDE approach, one only provides the number of characters p and Wronskian index ℓ in an attempt to classify all possible RCFT with these values, and will include more than just minimal models and WZW models, like coset theories. But the series solution of the characters does not lend itself to computing the $\mathcal{S}$-matrix for these characters.

However, in the $p = 2$ case, it was possible to convert the differential equation into the hypergeometric differential equation and thus the $\mathcal{S}$-matrix was computed using monodromy properties of the hypergeometric function [16, 30, 15]. The $p = 3$ case was considered in [15] where the contour integral representation was used to compute the $\mathcal{S}$-matrix, taking the results from [34]. The integrals are the same as the ones appearing in [34] (section 2.3), except with the cross ratio z replaced by λ, a function of τ. This work is based on the proposal of [37] that all Wronskian index $\ell = 0$ characters can be represented by these Feigin-Fuchs type integrals. In [37], the conjecture is proved to be true for number of characters $p \leq 5$ theories. Here we provide some evidence for the conjecture for arbitrary p, by providing an algorithm to compute the $\mathcal{S}$-matrix for an arbitrary number of characters.

3.1 The Contour Integral Representation of l = 0 Characters

Here we express characters in terms of the modular λ function, which maps six copies of the fundamental domain to the complex plane. It is defined using Jacobi theta functions in terms of τ:

$$\lambda(\tau) = \left(\frac{\theta_2(\tau)}{\theta_3(\tau)}\right)^4 = 16\, q^{1/2}(1 - 8q^{1/2} + 44q + \dots) \tag{3.1}$$

The modular transformations in terms of λ are

$$\begin{aligned} \mathcal{T} &: \lambda \to \frac{\lambda}{\lambda - 1} \\ \mathcal{S} &: \lambda \to 1 - \lambda \end{aligned} \tag{3.2}$$

Let us introduce the contours for the integrals. For every pair of integers A, A' where $0 \le A \le n_1,\ 0 \le A' \le n_2$, we define the multi-variable integration contours $\mathcal{C}_A$ and $\mathcal{C}_{A'}$ as follows:

$$\begin{aligned} \int_{\mathcal{C}_A} \prod_{i=1}^{n_1} dt_i &= \int_1^\infty dt_{n_1} \cdots \int_1^\infty dt_{A+2} \int_1^\infty dt_{A+1} \int_0^\lambda dt_A \cdots \int_0^\lambda dt_2 \int_0^\lambda dt_1 \\ \int_{\mathcal{C}_{A'}} \prod_{j=1}^{n_2} d\tau_j &= \int_1^\infty d\tau_{n_2} \cdots \int_1^\infty d\tau_{A'+2} \int_1^\infty d\tau_{A'+1} \int_0^\lambda d\tau_{A'} \cdots \int_0^\lambda d\tau_2 \int_0^\lambda d\tau_1 \end{aligned} \tag{3.3}$$

and we order the contours as

$$\mathrm{Im}(t_i) > \mathrm{Im}(t_k) \text{ for } i > k, \qquad \mathrm{Im}(\tau_j) < \mathrm{Im}(\tau_l) \text{ for } j > l \tag{3.4}$$

to avoid contours from overlapping.

The integral proposed by [37] is thus:

$$\begin{aligned} J_{AA'}(a,b,\lambda) \equiv N_{AA'}\, (\lambda(1-\lambda))^\alpha \int_{\mathcal{C}_A} \prod_{i=1}^{n_1} dt_i \int_{\mathcal{C}_{A'}} \prod_{j=1}^{n_2} d\tau_j \\ \times \prod_{i=1}^{A} \Big[t_i(1-t_i)(\lambda - t_i)\Big]^a \prod_{i=A+1}^{n_1} \Big[t_i(t_i-1)(t_i-\lambda)\Big]^a \\ \times \prod_{j=1}^{A'} \Big[\tau_j(1-\tau_j)(\lambda - \tau_j)\Big]^b \prod_{j=A'+1}^{n_2} \Big[\tau_j(\tau_j-1)(\tau_j-\lambda)\Big]^b \\ \times \prod_{1 \le k < i \le n_1} (t_i - t_k)^{-2a/b} \prod_{1 \le l < j \le n_2} (\tau_j - \tau_l)^{-2b/a} \prod_{i,j} (t_i - \tau_j)^{-2} \end{aligned} \tag{3.5}$$

where α is a function of the parameters of the theory: a, b, n_1, n_2, as we shall see shortly.

The index AA' on J is a "composite" index, where $0 \le A \le n_1$, $0 \le A' \le n_2$. The number of

characters of the theory is given by $p = (n_1 + 1)(n_2 + 1)$. The index AA' each labels a different character, for example for 4 characters, a theory with $(n_1, n_2) = (1, 1)$, the different characters will be labelled as J_{00}, J_{01}, and J_{10}. Importantly, we have ordered the terms in the integrand such that they remain positive in their respective region of integration, except for the cross terms $(t_i - t_j)$ and $(\tau_i - \tau_j)$, where the contour ordering orders them such that the imaginary part is positive and negative for t_i and τ_j respectively.

Let us also define the unnormalized integrals $\hat{J}_{AA'}$:

$$J_{AA'}(a, b, \lambda) = N_{AA'}(\lambda(1-\lambda))^{\alpha}\ \hat{J}_{AA'}(a, b, \lambda) \tag{3.6}$$

Both the normalized and unnormalized integrals transform under the modular-$\mathcal{S}$ transformation,

$$\begin{aligned} \hat{J}_{AA'}(\lambda) &= \sum_{BB'} \hat{\mathcal{S}}_{AA'\ BB'} \hat{J}_{BB'}(1-\lambda) \\ J_{AA'}(\lambda) &= \sum_{BB'} \mathcal{S}_{AA'\ BB'} J_{BB'}(1-\lambda) \end{aligned} \tag{3.7}$$

since the factor $(\lambda(1-\lambda))^{\alpha}$ is invariant under the $\mathcal{S}$ transformation. Notice that the two matrices $\mathcal{S}$ and $\hat{\mathcal{S}}$ are related by

$$\begin{aligned} \mathcal{S}_{AA'BB'} &= \frac{N_{AA'}}{N_{BB'}} \hat{\mathcal{S}}_{AA'\ BB'} \\ \implies \mathcal{S} &= N \hat{\mathcal{S}} N^{-1}, \text{ where } N = \text{diag}(N_{AA'}) \end{aligned} \tag{3.8}$$

Note that under the $\hat{\mathcal{S}}$ transformation twice, λ returns to λ, hence $\hat{\mathcal{S}}^2 = \mathbb{1}$, and hence $\mathcal{S}^2 = \mathbb{1}$.

Characters are eigenfunctions of the $\mathcal{T}$ transformation, which is clear from the q expansion definition of the character:

$$\mathcal{T}\chi_i(\tau) = \chi_i(\tau + 1) = e^{2\pi i(-c/24 + h_i)} \chi_i(\tau) \tag{3.9}$$

and we see that these integrals also are eigenfunctions of the $\mathcal{T}$ transformation, with the phases giving the desired critical exponents. An easier way to see this is to look at the leading λ behaviour, which is twice the power of the leading q behaviour, as done in [37].

$$J_{AA'} = q^{\frac{1}{2}(\alpha + \Delta_{AA'})} \tag{3.10}$$

We get the identifications:

$$\begin{aligned}
\alpha &= \frac{1}{3}\left(-n_1(1+3a) - n_2(1+3b) + \frac{a}{b}n_1(n_1-1) + \frac{b}{a}n_2(n_2-1) + 2n_1n_2\right) \\
\Delta_{AA'} &= A(1+2a) + A'(1+2b) - \frac{a}{b}A(A-1) - \frac{b}{a}A'(A'-1) - 2AA' \\
\frac{1}{2}(\alpha + \Delta_{AA'}) &= -\frac{c}{24} + h_i
\end{aligned} \tag{3.11}$$

For the identity character, which we identify with the index $(A, A') = (0, 0)$ and thus $\Delta_{00} = 0$, which allows us to realize

$$c = -12\alpha, \quad h_i = \frac{1}{2}\Delta_{AA'} \tag{3.12}$$

Using the Riemann-Roch relation (1.152), we see that these allowed values of c and h_i are only valid for $\ell = 0$.

For the rest of this chapter, we will specialize to $n_2 = 0$, for which we are able to obtain results. Thus, we will drop the second index from the integrals and parameter $\Delta_{AA'}$, and denote them as J_A and Δ_A respectively. For brevity of notation, we shall also use the notation: $\mathfrak{s}(x) = \sin(\pi x)$ and $\mathfrak{c}(x) = \cos(\pi x)$.

3.2 Computation of the modular S-matrix

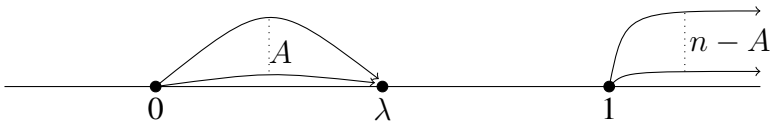
3.2.1 Monodromy of the integrals

In this section we shall compute the modular $\mathcal{S}$-matrix using the monodromy properties of the integrals, employing contour deformations. Here we systematise this computation and arrive at a description as a sum over paths, which renders the computation very explicit and allows us to go beyond the results of [15, 37] which were taken in turn from [38, 34]. Since we will be looking at the $n_2 = 0$ case, we rename n_1 as n, and hence corresponding integrals have the form:

$$\begin{aligned}
\hat{J}_A(\lambda) = &\int_1^\infty dt_n \int_1^\infty dt_{n-1} \cdots \int_0^\lambda dt_A \cdots \int_0^\lambda dt_1 \\
&\times \prod_{i=1}^{A} \left[t_i(1-t_i)(\lambda - t_i)\right]^a \prod_{i=A+1}^{n} \left[t_i(t_i-1)(t_i-\lambda)\right]^a \prod_{0\le k<i\le n} (t_i - t_k)^{2\rho}
\end{aligned}$$

Due to the absence of the τ_i integration variables, the parameter $-a/b$ is replaced by ρ. Notice that $\hat{J}_A$ has A contours running from 0 to λ and the remaining $n - A$ from 1 to ∞. Thus there are altogether $p = n + 1$ contour integrals, which are going to be candidates to describe a p-character

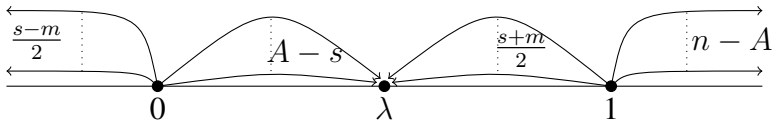
CFT. The integral in this set can be represented by the contour diagram:

$\hat{J}_A(\lambda) =$ 

(3.13)

To deform a contour, consider the semicircular contour which encloses the upper or lower half plane, and does not cross any other contours or branch cuts. Since we consider the values of the parameters such that the integrand falls of at infinity (and analytically continue our results for other values of the parameters), the contribution will come from the contour running along the real line. The integral along the entire real line is equal to zero, so we break up the integration regions into $(-\infty, 0)$, $(0, \lambda)$, $(\lambda, 1)$, and $(1, \infty)$. The contour integral in one region is equal to the negative of the sum in the rest of the regions. However, since we keep the integrand real in each region, this pulls out a phase from the integrand, which differs in the upper and lower half planes. We then take a linear combination of the upper and lower half plane deformed contours (both of which are equal to the original integral) such that the $(0, \lambda)$ region is cancelled out.

The result of the above procedure is that the original contour has disappeared and is replaced by two contours, one in each of the segments $(1, \lambda)$ and $(0, -\infty)$. Now if we carry out the procedure s times, sequentially on the A contours in the $(0, \lambda)$ region, we will be left with only $A - s$ of these contours. The remaining s will be distributed in the regions $(0, -\infty)$ and $(1, \lambda)$. We introduce an integer m such that $\frac{s-m}{2}$ contours are in $(0, -\infty)$ and $\frac{s+m}{2}$ are in $(1, \lambda)$. Clearly m takes the values $-s, -s+2, \cdots, s-2, s$. The contour configuration for a fixed (s, m) will be labelled $\mathscr{V}_{s,m}$ and, as long as $s < A$, it looks like the following:

$\mathscr{V}_{s,m} =$ 

(3.14)

Once a particular $\mathscr{V}_{s,m}$ has been reached (with $s < A$), a further unfolding of a single contour in the $(\lambda, 0)$ region leads to either $\mathscr{V}_{s+1,m-1}$ or $\mathscr{V}_{s+1,m+1}$ depending on whether the unfolded contour is replaced by another contour in the $(0, -\infty)$ region or the $(\lambda, 1)$ region respectively. However, after A steps there will be no $0 \to \lambda$ contours left. Thereafter we must unfold the remaining $n - A$ contours in the $(1, \infty)$ region by deforming each contour in two ways and removing the contributions from the region $(\lambda, 0)$. We now seek a recursive relation between $V_{s,m}$ and its successor diagrams $\mathscr{V}_{s+1,m-1}$ and $\mathscr{V}_{s+1,m+1}$.

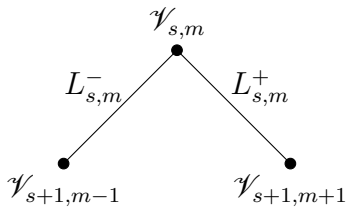
Thus we are led to define the following coefficients:

$$
\begin{aligned}
s < A: \quad L^{-}_{s,m} &= \frac{e^{i\pi\left(A-\frac{3s-m}{2}-1\right)\rho}\ \mathfrak{s}(3a+(n+A-\left(\frac{s-m}{2}\right)-2)\rho)}{\mathfrak{s}(2a+(n+m-1)\rho)} \\
L^{+}_{s,m} &= \frac{e^{i\pi\left(A-\frac{3s+m}{2}-1\right)\rho}\ \mathfrak{s}(a+(n-A+\left(\frac{s+m}{2}\right))\rho)}{\mathfrak{s}(2a+(n+m-1)\rho)} \\
s \geq A: \quad L^{-}_{s,m} &= -\frac{e^{i\pi\left(n-\frac{3s-m}{2}-1\right)\rho}\ \mathfrak{s}(a+\left(\frac{s-m}{2}\right)\rho)}{\mathfrak{s}(2a+(n+m-1)\rho)} \\
L^{+}_{s,m} &= \frac{e^{i\pi\left(n-\frac{3s+m}{2}-1\right)\rho}\ \mathfrak{s}(a+\left(\frac{s+m}{2}\right)\rho)}{\mathfrak{s}(2a+(n+m-1)\rho)}
\end{aligned}
\tag{3.15}
$$

In terms of these, we have the recursion relation:

$$
\mathscr{V}_{s,m} = L^{-}_{s,m}\mathscr{V}_{s+1,m-1} + L^{+}_{s,m}\mathscr{V}_{s+1,m+1} \tag{3.16}
$$

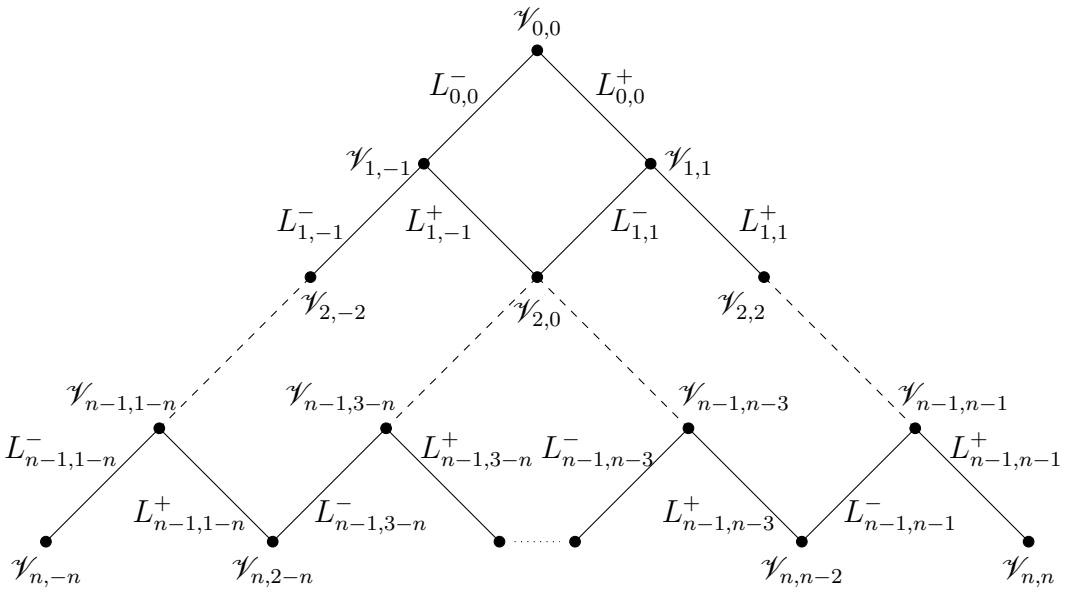
Therefore one unfolding can be represented as the following diagram



(3.17)

where both the links are directed downwards. Starting from $\hat{J}_A(\lambda)$ which can be identified as $\mathscr{V}_{0,0}$,

n-unfoldings can be represented as the following graph:



(3.18)

Notice that the vertices in the final row are the integrals $\hat{J}_A(1-\lambda)$ such that

$$\mathscr{Y}_{n,2A-n} = (-1)^n \hat{J}_A(1-\lambda) \tag{3.19}$$

where the sign on the RHS comes from the change of variables $t_i \to 1 - t_i$. The algorithm to compute the $\hat{\mathcal{S}}$ matrix is as follows:

1. For the row in the $\hat{\mathcal{S}}$-matrix corresponding to $\hat{J}_A$, identify $\mathscr{Y}_{0,0}$ with $\hat{J}_A(\lambda)$.
2. To compute the element $\hat{\mathcal{S}}_{A\,B}$, trace a path in the graph (3.18) starting from $\mathscr{Y}_{0,0}$ to $\mathscr{Y}_{n,2B-n}$ while multiplying the contribution from each link on the path.
3. Sum over the contributions from all such paths to obtain $\hat{\mathcal{S}}_{A\,B}$.

Now that we have shown how to compute $\hat{\mathcal{S}}$ in a simple algorithmic fashion, we go on to calculate the normalisations of the integrals $\hat{J}_A$.

3.2.2 Normalisation of the integrals

The normalisations N_A were originally introduced in (3.5). These are important to calculate as they contain important information about the degeneracy of the ground state of the characters, as we shall see, and are determined in terms of the parameters a and ρ. Recall that the second index of $N_{AA'}$ has been dropped because we are working with the class of integrals having $n_2 = 0$. Thus, the normalization of the $\mathcal{S}$-matrix from the from the unnormalized one calculated in the previous section is

$$\mathcal{S}_{AB} = \frac{N_A}{N_B}\,\hat{\mathcal{S}}_{AB} \tag{3.20}$$

Thus, once we compute the normalisations we will have finally determined the modular $\mathcal{S}$-matrix.

The normalization of the integral N_A is defined by demanding that the leading coefficient of q of the integral is the ground-state degeneracy D_A of the corresponding character, which is a non-negative integer, such that:

$$\chi_A(q) = D_A\, q^{\frac{1}{2}(\alpha+\Delta_A)}(1+\mathcal{O}(q)) \tag{3.21}$$

Note that for a genuine RCFT, we should have $D_0 = 1$ because the identity character has a non-degenerate ground state (this can be relaxed for theories of IVOA type [39] and for quasi-characters [30]).

To calculate the N_A, we only need the leading behaviour in q, corresponding to taking the $\lambda \to 0$ limit. To compute this, it will be convenient to first define the ordered integral:

$$\begin{aligned} I_A(\lambda) = N_A\,(\lambda(1-\lambda))^\alpha \int_1^\infty dt_n \int_1^{t_n} dt_{n-1} \cdots \int_1^{t_{A+2}} dt_{A+1} \int_0^\lambda dt_A \int_0^{t_A} dt_{A-1} \cdots \int_0^{t_2} dt_1 \\ \times \prod_{i=1}^{A} \big[t_i(1-t_i)(\lambda-t_i)\big]^a \prod_{i=A+1}^{n} \big[t_i(t_i-1)(t_i-\lambda)\big]^a \prod_{0\le k<i\le n} (t_i-t_k)^{2\rho} \end{aligned} \tag{3.22}$$

The integrand and pre-factors are identical to those in J_A, and the only difference is that the integration range for I_A has $t_1 < t_2 < \cdots < t_n$. The relation between $J_A(\lambda)$ to $I_A(\lambda)$ can be shown to be [34, 35]:

$$\begin{aligned} J_A(\lambda) &= \left(\prod_{k=1}^{A-1} e^{i\pi k\rho}\, \frac{\mathfrak{s}((k+1)\rho)}{\mathfrak{s}(\rho)} \prod_{l=1}^{n-A-1} e^{i\pi l\rho}\, \frac{\mathfrak{s}((l+1)\rho)}{\mathfrak{s}(\rho)} \right) I_A(\lambda) \\ &\equiv \Theta_A(\rho)\, I_A(\lambda) \end{aligned} \tag{3.23}$$

For $A = 0, 1$ the product over k is absent, and similarly for $A = n-1, n$ the product over l is absent.

In view of the ordered integrals, the factor $(t_i - t_k)$ is always non-negative. Thus we can place a modulus sign around it if we like. Thereafter we are free to extend the integral to be un-ordered. This will simply count the integration region $A!$ and $(n-A)!$ times. Hence we have:

$$\begin{aligned} I_A(\lambda) =& \frac{N_A}{A!(n-A)!}\,(\lambda(1-\lambda))^\alpha \int_1^\infty dt_n \cdots \int_1^\infty dt_{A+1} \int_0^\lambda dt_A \cdots \int_0^\lambda dt_1 \\ &\times \prod_{i=1}^{A} \big[t_i(1-t_i)(\lambda-t_i)\big]^a \prod_{i=A+1}^{n} \big[t_i(t_i-1)(t_i-\lambda)\big]^a \prod_{i<k} \big|t_i-t_k\big|^{2\rho} \end{aligned} \tag{3.24}$$

Next, we make the substitutions $t_i = \lambda u_i$ for $i = 1, 2, \cdots A$, and $t_i = \frac{1}{u_i}$ for $i = A+1, A+2, \cdots, n$.

The integral then becomes:

$$\begin{aligned} I_A(\lambda) =& \frac{N_A}{A!(n-A)!}\,\lambda^{\alpha+A\left(1+2a+\rho(A-1)\right)}(1-\lambda)^{\alpha}\int_0^1 du_n\cdots\int_0^1 du_1 \\ &\times\prod_{i=1}^{A}\Big[u_i(1-u_i)(1-\lambda u_i)\Big]^a\prod_{i=A+1}^{n}u_i^{-2-3a-2\rho(n-1)}\Big[(1-u_i)(1-\lambda u_i)\Big]^a \\ &\times\prod_{1\le i<k\le A}\big|u_i-u_k\big|^{2\rho}\prod_{A+1\le i<k\le n}\big|u_i-u_k\big|^{2\rho}\prod_{\substack{1\le i\le A,\\ A+1\le k\le n}}\big|1-\lambda u_i u_k\big|^{2\rho} \end{aligned} \tag{3.25}$$

Notice that the leading power of λ which has been factored out of the integral is Δ_A.

In the above form it is easy to compute the leading behaviour as $\lambda\to 0$, and one finds:

$$\begin{aligned} I_A(\lambda\to 0) &= \frac{N_A}{A!(n-A)!}\,\lambda^{\alpha+\Delta_A}\int_0^1 du_A\cdots\int_0^1 du_1\prod_{i=1}^{A}\Big[u_i(1-u_i)\Big]^a\prod_{1\le i<k\le A}\big|u_i-u_k\big|^{2\rho} \\ &\quad\times\int_0^1 du_n\cdots\int_0^1 du_{A+1}\prod_{i=A+1}^{n}u_i^{-2-3a-2\rho(n-1)}(1-u_i)^a\prod_{A+1\le i<k\le n}\big|u_i-u_k\big|^{2\rho} \\ &= \frac{N_A}{A!(n-A)!}\,(16\sqrt{q}\,)^{\alpha+\Delta_A}\,S_A\big(a+1,a+1,\rho\big)\,S_{n-A}\big(-1-3a-2\rho(n-1),a+1,\rho\big) \end{aligned} \tag{3.26}$$

In this limit the integral has factorised into two parts, and we have identified each of these as a Selberg integral. This integral is defined as:

$$\begin{aligned} S_m(\alpha,\beta,\gamma) &= \int_0^1 du_m\cdots\int_0^1 du_1\prod_{i=1}^{m}u_i^{\alpha-1}(1-u_i)^{\beta-1}\prod_{1\le i<j\le m}\big|u_i-u_j\big|^{2\gamma} \\ &= m!\prod_{k=1}^{m}\frac{\Gamma(k\gamma)}{\Gamma(\gamma)}\prod_{k=0}^{m-1}\frac{\Gamma(\alpha+k\gamma)\,\Gamma(\beta+k\gamma)}{\Gamma(\alpha+\beta+(m+k-1)\gamma)} \end{aligned} \tag{3.27}$$

Demanding this to be equal to the ground-state degeneracy, D_A of the corresponding character we fix the normalisation to be

$$N_A = \frac{16^{-(\alpha+\Delta_A)}\,A!(n-A)!\,D_A}{\Theta_A(\rho)\,S_A\big(a+1,a+1,\rho\big)\;S_{n-A}\big(-1-3a-2\rho(n-1),a+1,\rho\big)} \tag{3.28}$$

3.2.3 Final result

One can now insert the results from the previous two sections, namely (3.18) and (3.28), into (3.20) to obtain the modular $\mathcal{S}$ matrix for any desired case. Note that it will contain the undetermined factor D_A. This can be found in multiple ways: (i) if one knows the chiral algebra of the theory then the D_A would be known (our method can still provide a useful way to compute $\mathcal{S}$), (ii) in the spirit of [14, 16, 15] one can guess the D_A by requiring an integral q-series, (iii) following [24, 25] one can determine the D_A for coset theories via the bilinear coset relation.

3.3 Applications of the modular S matrix

Here we see what uses we can make of the $\mathcal{S}$ matrix. From Verlinde's formula we can obtain the fusion rules of CFTs which have unit multiplicity. We can also modify the $\mathcal{S}$-matrix suitably for larger than 1 multiplicity theories.

Using the following property of the $\mathcal{S}$ matrix:

$$S^\dagger M S = M \tag{3.29}$$

where $M = \text{diag}(M_i)$ is the matrix of multiplicities, we can provide constraints on the multiplicities, and the ground state degeneracy of characters in terms of the parameters a, ρ and n. This could be an interesting way to approach classification of RCFT. Indeed, we have made some progress in the area in the paper [35]. The computation of the $\mathcal{S}$ matrix also gives evidence for the proposal of the contour integral representation of characters, even for larger number of characters where the proposal is still a conjecture. Explicit computation of the $\mathcal{S}$ matrix for the $\text{SU}(2)_7$ WZW Model, an 8 character theory matches with the known result (1.144).

We have provided an algorithm which computes the $\mathcal{S}$ matrix term by term. However, for the first and last rows of the $\mathcal{S}$ matrix, we can give a "closed" form solution, since from our algorithm it follows that there is a unique path from the initial vertex to either the left-most or right-most vertex at the bottom. The corresponding solution for the two rows are thus:

$$\begin{aligned}
\mathcal{S}_{A0} &= (-1)^n \frac{N_A}{N_0} \prod_{s=0}^{A-1} \frac{e^{i\pi(A-2s-1)\rho}\,\mathfrak{s}(3a+(n+A-s-2)\rho)}{\mathfrak{s}(2a+(n-s-1)\rho)} \prod_{s=A}^{n} \frac{-e^{i\pi(n-2s-1)\rho}\,\mathfrak{s}(a+s\rho)}{\mathfrak{s}(2a+(n-s-1)\rho)} \\
\mathcal{S}_{An} &= (-1)^n \frac{N_A}{N_n} \prod_{s=0}^{A-1} \frac{e^{i\pi(A-2s-1)\rho}\,\mathfrak{s}(a+(n-A+s)\rho)}{\mathfrak{s}(2a+(n+s-1)\rho)} \prod_{s=A}^{n} \frac{e^{i\pi(n-2s-1)\rho}\,\mathfrak{s}(a+s\rho)}{\mathfrak{s}(2a+(n+s-1)\rho)}
\end{aligned} \tag{3.30}$$

which extends the results from [34], who were only able to provide the last row.

Chapter 4

Quasi-Characters for Three Character CFTs

This chapter will briefly, for reasons of space, deal with solutions of order 3 MLDE, in order to understand some steps required for a classification of all three character CFTs, the subject of our paper [32]. This is extensively discussed in my collaborator Palash Singh's thesis.

The MLDE method of classifying CFT does not specify any sort of chiral algebra the CFT possesses, and thus can be used to find theories which are not the standard minimal models or WZW models. Examples of such CFTs are coset models, the Monster and Baby Monster CFTs, etc. The problem of classifying 1 character (meromorphic) CFT is a solved problem using the techniques of MLDE [15], and so is the order 2 case [30].

Classifying using the MLDE involves solving the MLDE by fixing the free parameters in the MLDE, and studying whether the solution has a positive integral q-expansion. Candidate or admissible characters must have the following three properties:

1. The set of characters transform under modular transformations as a vector valued modular form to keep the partition function of the theory modular invariant.

2. The ground state of the identity character must be non-degenerate, since any legitimate QFT has a unique vacuum.

3. After suitably normalizing the identity character, the rest of the characters must have positive integer coefficients in their q-expansion.

Quasi-characters are an auxiliary construction required for creating more characters which solve MLDE with larger values of ℓ. It turns out that for $\ell > 6$, the number of parameters required to solve the differential equation become too large to uniquely determine by parameters of the theory like central charge and conformal dimensions. However, we can construct large ℓ characters by taking linear combination of solutions of lower ℓ. The solutions which are useful for this purpose are called quasi-characters. They also have integral coefficients in their q-expansion, but are not necessarily positive, since they do not correspond to any physical CFT. The constraint that the "identity" character must have a non-degenerate ground state is also lifted for these solutions.

4.1 Fusion Classes of Three Character CFTs

The first point to note is that there can in general be more primaries than characters. This happens for example whenever a primary is complex – in this case its complex conjugate has the same character. Thus we must be careful to account for multiplicities. For the case of interest to us, namely order 3, it turns out that there are cases with a total of three, four, five or nine primary fields. With three or four primaries there are nine possible fusion classes, classified in [8]. One of our cases does not fall within the above classification – while it has three characters, two of them correspond to complex primaries and hence the total number of primaries (counting each one and its complex conjugate as distinct) is five. We have labelled the fusion rules for this case as $\mathcal{B}_4^{(1)}$. Another one of our cases is a tensor product of a two character theory. The two character theory has a complex primary, so when taking the product with itself, one obtains 9 primaries.

By construction, characters are eigenvectors of the modular $\mathcal{T}$-transformation $\tau \to \tau + 1$ and transform under $\tau \to -\frac{1}{\tau}$ by a non-trivial modular $\mathcal{S}$-matrix. Their fusion rules can be found using the Verlinde formula [36]:

$$\mathcal{N}_{ijk} = \sum_m \frac{\mathcal{S}_{im}\mathcal{S}_{jm}\mathcal{S}^*_{km}}{\mathcal{S}_{0m}} \tag{4.1}$$

A general procedure to compute the modular $\mathcal{S}$-matrix has been provided in our previous paper [35] for a large class of characters having $\ell = 0$. This class includes, in particular, all $\ell = 0$ CFT's of order 3. However there is a subtlety in using $\mathcal{S}$ to compute the fusion rules. This is straightforward for theories without multiplicity, but we cannot directly use the Verlinde formula for cases where the multiplicity is greater than one. For these, we must use the technique in [15] to "expand" the $\mathcal{S}$-matrix to a larger matrix whose dimension is equal to the total number of primaries. This expanded matrix diagonalises the fusion rules of the primaries and can be used in the Verlinde formula.

4.2 The Quasi-Character Families in order 3 MLDE

The key ingredient for constructing these families is the bilinear relation and the novel coset construction. The bilinear relation of two theories is as follows. It is well known that the partition function of a $c = 24$ meromorphic CFT is given by the Klein j function plus a constant. By taking the "inner product" of characters from two different CFTs, sometimes one obtains the j function. Such theories are called coset pairs.

$$\sum_i \chi_i(\tau)\chi'_i(\tau) = j(\tau) + \mathcal{N} \tag{4.2}$$

In the order 2 case [30], the bilinear relation of [24] was able to construct an $\ell = 2$ coset pair of an $\ell = 0$ CFT. For order 3, this is not true. The coset of a $c = 24$ meromorphic CFT with an $\ell = 0$ three character CFT also has $\ell = 0$. This suggests that they could lie in the same quasi-character "family," since in the order 2 case, the quasi-character family all share the same ℓ. Using this as an inspiration, we conjectured using known $\ell = 0$ three character CFTs, a full quasi-character family

for each. In some cases we were able to combine some families, and were able to make infinitely many, shown in table 4.1.

Each family is arranged as follows. The theory listed in the second column is the $k = 0$ theory, with its corresponding c and h_i values. The $k = 1$ theory is its coset dual, constructed from a meromorphic $c = 24$ theory, as described above. We were able to rule out certain congruences of the label k in most of the families, by checking for integrality of the q-series and analyticity of the modular $\mathcal{S}$-matrix computed using the algorithm in the previous chapter. Each quasi-character family is classified by its fusion class and the formula for the critical exponents. An interesting thing to note is that we find quasi-characters even for $r < 0$ in the cases of the $B_{r,1}$ and $D_{r,1}$ families.

Another interesting property is that we find that the bilinear relation is satisfied by pairs of quasi-characters as well. Particularly, the k and $1 - k$ members of the family are coset pairs, where the constant $\mathcal{N}$ is not necessarily an integer. There is however, a small caveat. Since the solutions of the MLDE are up to a sign, we a priori do not know the sign of the ground state degeneracy D_i for quasi-characters. Even modular invariance of the "partition function" created by the quasi-characters does not help since the degeneracy will be squared. However, the bilinear relation will only hold true only for a particular sign of the product $D_i D_i'$, and thus puts a nontrivial relation on the signs of the ground state degeneracies for the two quasi-character solutions.

One more property we see is that, on computing the modular $\mathcal{S}$ matrix using our algorithm in the previous chapter, the entire family shares the same, or a few $\mathcal{S}$ matrices, up to a sign of D_i. So we have two relations, the bilinear relation and the common modular $\mathcal{S}$ matrix to help us fix the sign of the quasi-character.

4.3 Constructing Admissible Characters from Quasi-Characters in order 3

4.3.1 Adding quasi-characters

To construct solutions of $p = 3$ MLDE with larger values of ell, we can take linear combinations of quasi-character to construct them. Let's see how this can be done. Consider two sets of (quasi-) characters with central charges c, c' and conformal dimensions h_i, h_i'. They have the general form:

$$\begin{aligned} \chi_i &= q^{\alpha_i}\left(a_0^{(i)} + a_1^{(i)} q + a_2^{(i)} q^2 + \cdots\right) \\ \chi_i' &= q^{\alpha_i'}\left(a_0'^{(i)} + a_1'^{(i)} q + a_2'^{(i)} q^2 + \cdots\right) \end{aligned} \tag{4.3}$$

where $\alpha_i = -\frac{c}{24} + h_i$. The coefficients $a_n^{(\prime)}$ are integers, but need not be positive. For adding quasi-characters to be well defined, they must satisfy these two conditions:

- the sum must have a well defined $\mathcal{S}$ matrix
- the q-expansion must be in integer powers of q

Fusion Class	Theories	c	h_1	h_2	Remark
$\mathcal{A}_2^{(3)}$	$\mathcal{M}_{2,7}$	$\frac{304}{7}k - \frac{68}{7}$	$\frac{20}{7}k - \frac{3}{7}$	$\frac{18}{7}k - \frac{2}{7}$	$k \neq 4 \bmod 7$
	$\mathcal{M}_{2,5} \otimes \mathcal{M}_{2,5}$	$\frac{208}{5}k - \frac{44}{5}$	$\frac{14}{5}k - \frac{2}{5}$	$\frac{12}{5}k - \frac{1}{5}$	$k \neq 3 \bmod 5$
$\mathcal{A}_3^{(4)}$	$G_{2,1} \otimes G_{2,1}$	$\frac{64}{5}k + \frac{28}{5}$	$\frac{6}{5}k + \frac{2}{5}$	$\frac{2}{5}k + \frac{4}{5}$	$k \neq 3 \bmod 5$
	$F_{4,1} \otimes F_{4,1}$	$\frac{16}{5}k + \frac{52}{5}$	$\frac{4}{5}k + \frac{3}{5}$	$-\frac{2}{5}k + \frac{6}{5}$	$k \neq 3 \bmod 5$
$\mathcal{B}_4^{(1)}$	$A_{4,1}$	$16k + 4$	$\frac{6}{5}k + \frac{2}{5}$	$\frac{4}{5}k + \frac{3}{5}$	$k \in \frac{\mathbb{Z}}{2}, 2k \neq 1 \bmod 5$
$\mathcal{A}_2^{(1)}$	$B_{r,1}$	$(23 - 2r)k + \frac{2r+1}{2}$	$\frac{(15-2r)}{8}k + \frac{2r+1}{16}$	$k + \frac{1}{2}$	$r = 0, 1$ incl.
$\mathcal{A}_3^{(1)}$	$E_{7,1} \otimes E_{7,1}$	$-4k + 14$	$\frac{1}{2}k + \frac{3}{4}$	$-k + \frac{3}{2}$	
	$D_{r,1}$	$2(12 - r)k + r$	$\frac{(8-r)}{4}k + \frac{r}{8}$	$k + \frac{1}{2}$	$r = 2 \bmod 4$
$\mathcal{B}_3^{(4)}$	$D_{r,1}$	$2(12 - r)k + r$	$\frac{(8-r)}{4}k + \frac{r}{8}$	$k + \frac{1}{2}$	$r \neq 2 \bmod 4$ $r \neq 0 \bmod 8$
$\mathcal{B}_8^{(1)}$	$A_{2,1} \otimes A_{2,1}$	$16k + 4$	$\frac{4}{3}k + \frac{1}{3}$	$\frac{2}{3}k + \frac{2}{3}$	$k \in \frac{\mathbb{Z}}{2}, 2k \neq 1 \bmod 3$

Table 4.1: Infinitely many sets of quasi-character families for each fusion class. Here $k, r \in \mathbb{Z}$ subject to the restrictions above, unless stated otherwise.

The sum, therefore looks like

$$\widetilde{\chi}_i = \chi'_i + N\chi_i = q^{\widetilde{\alpha}_i}\left(\widetilde{a}_0^{(i)} + \widetilde{a}_1^{(i)}q + \widetilde{a}_2^{(i)}q^2 + \cdots\right) \tag{4.4}$$

For the first requirement to hold, the two quasi-characters must have the same $\mathcal{S}$ matrix, and hence must be in the same quasi-character family. For the second requirement to hold, their central charges must differ by a multiple of 24:

$$c' - c = 24m \tag{4.5}$$

Examples are worked out in [32].

4.3.2 The Wronskian index of a sum of quasi-characters

To calculate the Wronskian index ℓ of a sum of quasi-characters, let us assume without loss of generality that $c' > c$, and hence $m > 0$. The exponents $\widetilde{\alpha}_i$ are the more singular of the two two

exponents α_i and α'_i. Since $\alpha_0^{(\prime)} = -\frac{c^{(\prime)}}{24}$, $\alpha_0 > \alpha'_0$, and hence

$$\begin{aligned} \widetilde{\alpha}_0 &= \alpha'_0 \\ \widetilde{\alpha}_{1,2} &= \widetilde{\alpha}_0 + \min(h'_{1,2}, h_{1,2} + m) \end{aligned} \tag{4.6}$$

One can show if $h'_1 < h_1 + m$, then $h_2 > h_2 + m$ and visa versa. Thus, using the Riemann-Roch relation (1.152), we get the results for the Wronskian index of $\widetilde{\chi}_i$ listed in table 4.2.

$\min(h'_1, h_1 + m)$	$\min(h'_2, h_2 + m)$	ℓ
h'_1	$h_2 + m$	$6(h'_2 - h_2 - m)$
$h_1 + m$	h'_2	$6(h'_1 - h_1 - m)$
$h_1 + m$	$h_2 + m$	$6m$

Table 4.2: Possible values of the Wronskian index ℓ

4.3.3 Admissible characters with large Wronskian Index

Now that we have seen that we can obtain quasi-characters of $\ell > 0$, by adding quasi-characters of $\ell = 0$. We can obtain admissible characters by tuning the parameter N which we used in the linear combination, to "cure" the negative signs of the quasi-character. However, this requires studying the asymptotic sign of the coefficients of the quasi-character, which leads us to classifying them in terms of their behaviour, as given below:

- Class A: The q-expansion has only non-negative coefficients.
- Class B: After a finite power of the q expansion, there are only positive coefficients (assuming the ground state degeneracy is chosen positive).
- Class C: After a finite power of the q expansion there are only negative coefficients (again assuming the ground state degeneracy is chosen positive).
- Class D: The asymptotic sign of the coefficients does not stabilize but oscillates between positive and negative values.

Note that each of the A,B,C behaviours is defined in a convention where the first term is chosen positive, corresponding to a positive degeneracy D_i. If instead one chooses $D_i < 0$, which is sometimes required, then the behaviour flips in an obvious way.

Once again, for examples refer to our paper [32]. Such admissible characters have been shown to be characters real CFTs through the coset construction with $c = 32$ meromorphic CFTs, whose coset pairs with three characters are $\ell = 0$ and $\ell = 6$.

Chapter 5

Non-Abelian Anyons and the Quantum Hall Effect

Here we shall talk about systems in 2+1 dimensions, for example a 2 dimensional electron gas confined to a plane. Such systems are studied by experimentalists in the lab, so it is worthwhile understanding the physics behind these systems, and interesting phenomena which occur here. We shall also understand how RCFT can by used to probe these systems. [40, 41, 42]

Let us first define an anyon. Consider the wave function of two identical particles, and exchange the particles in a counter clockwise direction

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = e^{i\theta}\psi(\mathbf{r}_2, \mathbf{r}_1) \tag{5.1}$$

On exchanging the particles, the wave function picks up a phase. Here θ is called the statistical angle. Clearly when $\theta = 0$, the wave function describes two bosons, and when $\theta = \pi$, the wave function describes two fermions. Any other phase describes particles known as abelian anyons.

When one has N particles in the system, then a more rich structure arises. These particles will be representations of the discrete group known as the Braid group $\mathcal{B}_N$.

5.1 The Braid Group

The Braid group $\mathcal{B}_N$ is the discrete group whose elements are different braidings of N strings. Braiding two strings is the act of taking one string and exchanging it with another, either clockwise or counterclockwise. Clockwise and counterclockwise braidings are not the same, and are in fact, the inverse operation of each other. Clearly, the operation of braiding is the generator of the Braid group. The generators are σ_i, which is the act of braiding the i^{th} string with the $(i+1)^{\text{th}}$ string counterclockwise, and σ_i^{-1} is the clockwise exchange. Hence, the index i takes values $1, \cdots, N-1$

for N strings. The generators obey the relations:

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i \qquad \forall |i-j| \geq 2 \\ \sigma_i\sigma_{i+1} &\neq \sigma_{i+1}\sigma_i \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i^2 &\neq 1 \end{aligned} \tag{5.2}$$

The last relation makes the Braid group $\mathcal{B}_N$ have infinite elements, and differentiates it from the permutation group $\mathfrak{S}_N$. An interesting subgroup of the Braid group is called the Pure Braid group, and it is the subgroup where the strings start and end in the same order.

The relation of the Braid group to anyons is that the Braid group will act on the world line of the anyons. Therefore, the anyons must be *representations* of the Braid group. The Braid group is a non-abelian group, as seen by the generator algebra, and so in general the representations of the group will also be non-abelian. However, the 1 dimensional irreducible representation of the group will be abelian, since numbers commute. The abelian representation of the group is given by exponentials $e^{im\theta}$, where θ is the statistical angle of the anyon. When there are more than one type of abelian anyon species in the system, then the statistical angle will be given by a symmetric matrix θ_{ab}, where a and b label the anyon species, and θ_{ab} gives us the statistical angle of exchanging anyons a and b counterclockwise.

Particles which have to be represented by higher dimensional irreps of the Braid group will have non-abelian braiding statistics, and will be called non-abelian anyons. This can occur when an eigenstate of the system has some degeneracy. The degenerate subspace of the Hilbert space will have dimension equal to the degeneracy. We will see that braiding particles will cause nontrivial unitary rotations in this degenerate subspace, and the dimension of the representation will be equal to the degeneracy of the eigenstate.

5.2 Abelian Anyons

Before understanding non-abelian anyons, let us quickly understand some physics of the abelian case. We have already seen that wavefunctions that describe anyons will pick up a phase under counterclockwise exchange, with statistical angle θ. However, one can construct other "species" of anyons by considering bound states of anyons. Here we mean bound states in the loose sense, as long as the particles are much closer to each other than other particles in the system, then the collection of particles is considered a bound state. Once in the bound state, we will ignore any internal dynamics of the bound state giving rise to any phases internally.

A bound state of m particles being exchanged counterclockwise with another bound state of n particles will pick up a phase of

$$(e^{im\theta})^n = e^{imn\theta} \tag{5.3}$$

since each particle in the n-particle bound state will contribute a phase of $e^{im\theta}$, but will happen n times. Clearly, if there is anyon of statistics $\theta = \frac{\pi}{m}$, then by creating bound states we can obtain

anyons of the following statistics:

$$\theta = \frac{4\pi}{m}, \frac{9\pi}{m}, \cdots \frac{(m-1)^2\pi}{m} \tag{5.4}$$

Fusing particles is just the process of bringing the particles together, to create new particles. For abelian anyons it's very simple: fuse an m and n particle bound state to get an $m+n$ particle bound state. The statistical angle of the new particle will be $(m+n)^2\theta$. Thus, we write the fusion rules of particles, labelled by their statistical angle as:

$$m^2\theta \times n^2\theta = (m+k)^2\theta \tag{5.5}$$

The story for non-abelian anyons is quite different, which we shall see now.

5.3 Non-Abelian Anyons

In the abelian case, two anyons fused to form just a single other anyon. However, in the non-abelian case, we can allow for two particles to fuse into multiple different particles. We can neatly encapsulate this in the formula

$$\phi_a \times \phi_b = \sum_c \mathcal{N}_{ab}^c \phi_c \tag{5.6}$$

where ϕ_a's are the different types of quasi-particle in the system. For abelian anyons, $\mathcal{N}_{k\,k'}^{k''} = \delta_{k+k',k''}$. A simple set of fusion rules is the Ising fusion rules (whose fields are σ and ψ):

$$\sigma \times \sigma = \mathbb{1} \quad \sigma \times \psi = \sigma \quad \psi \times \psi = \mathbb{1} \tag{5.7}$$

Notice that the σ particle can fuse to give different final particles, either the vacuum or the particle ψ. This can lead to interesting results. Consider taking the vacuum expectation value of four σ particles $\langle\sigma\sigma\sigma\sigma\rangle$. Their allowed fusion channels are by the fields $\mathbb{1}$ and ψ, so when two σ's fuse to give $\mathbb{1}$, so do the other two. Then the two $\mathbb{1}$ fields can fuse to give the final state which is the vacuum. Or both the pairs could have fused to give ψ fields and both of which fuse to give the vacuum. In CFT, we identified these as the two conformal blocks which contribute to the four point function. Here we say that the system has topological charge $= \mathbb{1}$. However, if we consider the correlation function $\langle\psi|\sigma\sigma\sigma\sigma\rangle$, then the four σ particles will combine to form ψ in their respective fusion channels, and it is said that this configuration has topological charge $= \psi$.

While computing these correlation functions, we saw that we are required to consider the contributions from the different fusion channels. For $\langle\sigma\sigma\sigma\sigma\rangle$, the two fusion channels are via $\mathbb{1}$ and ψ, which we can call $\Psi_{\mathbb{1}}$ and Ψ_ψ respectively. In general, four fields can fuse through multiple fusion channels, Ψ_a. However, we considered the fusion of the first two and last two pairs to determine the fusion channels. We could have chosen to first fuse the 1st and 3rd σ's, and 2nd and 4th σ's, which would have given us the fusion channels $\widetilde{\Psi}_a$. Clearly, these are two different bases

for fusion, and thus are related by the change of basis matrix, which we call the F matrix:

$$\widetilde{\Psi}_a = F_{ab}\Psi_b \tag{5.8}$$

where F_{ab} is short for $[F_l^{ijk}]_{ab}$ which encodes the following information:

$$\begin{aligned} i, j &\xrightarrow{\text{Fuse}} a, \text{ where all allowed } a \text{ makes the first basis} \\ a, k &\xrightarrow{\text{Fuse}} l \end{aligned} \tag{5.9}$$

$$\begin{aligned} j, k &\xrightarrow{\text{Fuse}} b, \text{ where all allowed } b \text{ makes the second basis} \\ b, i &\xrightarrow{\text{Fuse}} l \end{aligned} \tag{5.10}$$

On fusing, the particles may have braided around each other, and picked up a phase, but would not have changed the fusion channel. This information is contained in the R-matrix, R_c^{ab}, which is the phase picked up when a and b are exchanged counterclockwise, and fuse to give c.

Thus, to fully specify the braiding statistics of a system of anyons, we need to specify :

- Particle species
- Fusion rules
- F-matrices
- R-matrices

5.4 The Quantum Hall Effect

A physical system in which non-abelian anyons can be realized is a fractional quantum Hall system. To understand the quantum Hall effect, let's start with the classical system, using the Drude model.

5.4.1 The Classical Hall Effect

The system is a 2+1d electron gas, confined to the x, y plane. There is a strong transverse magnetic field $\vec{B} = B\hat{z}$, and one can turn on a small electric field $\vec{E} = E\hat{x}$. When the electric field is not present, the electrons will satisfy the equation $m\vec{a} = -e\,\vec{v} \times \vec{B}$, whose solutions are circular trajectories given by

$$x(t) = X - R\sin(\omega_B t + \phi) \quad y(t) = Y - R\cos(\omega_B t + \phi) \tag{5.11}$$

where $\omega_B = \frac{eB}{m}$ is the cyclotron frequency, (X, Y) is the center of the circle of the trajectory with radius R. In the Drude model, one turns on the electric field, adds a friction term, and studies the

system at equilibrium ($\vec{a} = 0$):

$$m\vec{a} = -e(\vec{E} + \vec{v} \times \vec{B}) - \frac{m\vec{v}}{\tau} \tag{5.12}$$

where τ is the scattering time. On applying the equilibrium condition, substituting $-ne\vec{v}$ (n is the current density) for $\vec{J}$, the current density we get :

$$\begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{pmatrix} \vec{J} = \frac{e^2 n\tau}{m}\vec{E} \tag{5.13}$$

Comparing with Ohm's law $J = \sigma E$, we have the conductivity tensor, and its inverse the resistivity tensor ρ:

$$\sigma = \frac{e^2 n\tau}{m}\frac{1}{1+\omega_B^2\tau}\begin{pmatrix} 1 & -\omega_B\tau \\ \omega_B\tau & 1 \end{pmatrix} \quad \rho = \frac{m}{e^2 n\tau}\begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{pmatrix} = \frac{1}{ne}\begin{pmatrix} \frac{m}{e\tau} & B \\ -B & \frac{m}{e\tau} \end{pmatrix} \tag{5.14}$$

The off-diagonal entries in the conductivity tensor: $\sigma_{xy} = \frac{-e^2\omega_B n\tau^2}{m(1+\omega_B^2\tau^2)}$ is responsible for the Hall effect, electric current in the y direction on application of electric and magnetic fields in the x and z directions respectively.

What is seen in the quantum Hall effect is extremely curious. The transverse resistivity ρ_{xy} has experimentally found to take the values $\frac{h}{\nu e^2}$, where $\nu \in \mathbb{Z}$, and the longitudinal resistivity $\rho_{xx} = 0$. The magnetic field takes the value $B = \frac{hn}{\nu e}$ at the center of the plateau of the ρ_{xy} vs B. The parameter ν can take particular fractional values if the disorder in the system is increased, which we shall see soon.

5.4.2 Landau Levels

We shall now quantize the system. The lagrangian of this system is

$$L = \frac{1}{2}m\,\dot{x}^i\dot{x}_i - e\dot{x}^i A_i \tag{5.15}$$

which is gauge invariant. The canonical momentum $p_i = \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}_i - eA_i$ and the Hamiltonian is

$$H = \frac{1}{2m}(p_i + eA_i)^2 \tag{5.16}$$

Notice that the canonical momentum p_i is not gauge invariant, but the quantity $m\dot{x}_i$ is. We shall call this mechanical momentum. However, mechanical momentum does not have the canonical Poisson bracket structure, since its Poisson bracket with itself is nonvanishing.

$$\{m\dot{x}_i, m\dot{x}_j\} = -e\,\epsilon_{ijk}B_k \tag{5.17}$$

Quantizing this system amounts to exchanging the poisson brackets for commutators, and making the phase space variables x, p operators. Calling the mechanical momentum operator as $\Pi_i = p_i + eA_i$, the commutator reads

$$[\Pi_x, \Pi_y] = -ie\hbar B \tag{5.18}$$

We can see we can construct raising and lowering operators from the mechanical momentum:

$$a^\dagger = \frac{\Pi_x - i\Pi_y}{\sqrt{2e\hbar B}} \quad a = \frac{\Pi_x + i\Pi_y}{\sqrt{2e\hbar B}} \tag{5.19}$$

which satisfies the standard commutation relation $[a, a^\dagger] = i\hbar$, and we can rewrite the hamiltonian in terms of the raising and lowering operators:

$$H = \frac{1}{2m}\Pi \cdot \Pi = \hbar\omega_B \left(a^\dagger a + \frac{1}{2}\right) \tag{5.20}$$

All this shows that the 2d electron gas system is behaving like a 1D harmonic oscillator. We shall see that the extra degree of freedom goes into a macroscopic degeneracy at each energy level of the "harmonic oscillator." To see this, we have to specify a gauge.

Landau Gauge

Landau gauge is the gauge choice $\vec{A} = xB\hat{y}$. The hamiltonian in this gauge reads

$$H = \frac{1}{2m}\left(p_x^2 + (p_y + eBx)^2\right) \tag{5.21}$$

which is translationally invariant in the y direction. To find energy eigenstates, we can plug in the ansatz for the wave function $\psi_k(x, y) = e^{iky} f_k(x)$. The energy of this wavefunction is

$$H\psi_k(x, y) = \frac{1}{2m}(p_x^2 + (\hbar k + eBx)^2) f_k(x) = \left(\frac{1}{2m}p_x^2 + \frac{m\omega_B^2}{2}(x + l_B^2 k)^2\right) f_k(x) \tag{5.22}$$

As expected, the x dependent wavefunction is an eigenstate of the harmonic oscillator hamiltonian, centered at $x = -kl_B^2 = \frac{\hbar k}{eB}$. Here $l_B = \sqrt{\frac{\hbar}{eB}}$ is the "magnetic length". The energy levels however do not depend on the momentum k, and is

$$E_n = \hbar\omega_B \left(n + \frac{1}{2}\right) \tag{5.23}$$

These energy levels are called Landau levels. Thus each Landau level has infinite degeneracy, for an infinite plane, and no momentum cutoff, since all k are allowed at each n. The eigenstates of the

hamiltonian in this gauge are thus

$$\psi_{n,k} \sim e^{iky} H_n(x + kl_B^2)\, e^{\frac{-x+kl_B^2}{2l_B^2}} \tag{5.24}$$

where H_n are the hermite polynomials.

If we consider a finite plane, as one would expect in a physical setting, we can estimate the degeneracy at each energy level. Consider the plane to be a rectangle of sides L_x, L_y. Allowed momentum are now discrete, quantized in units of $\frac{2\pi}{L_y}$. In the x direction, the wave function will be peaked at $-kl_B^2$, where $x \in [0, L_x]$. Therefore, k can take values $[-\frac{L_x}{l_B^2}, 0]$. Thus the degeneracy will be

$$\mathcal{N}_{\text{deg}} = \frac{L_y}{2\pi} \int_{-\frac{L_x}{l_B^2}}^{0} dk = \frac{eBA}{h} \equiv \frac{BA}{\Phi_0} \tag{5.25}$$

where $\Phi_0 = \frac{h}{e}$ is called the flux quantum. Notice that the degeneracy is proportional to the area of the sample, and the applied magnetic field.

Landau gauge with external Electric Field

On adding an external electric field, to drive current through the sample, the Hamiltonian changes by a linear term

$$H = \frac{1}{2m}(p_x^2 + (p_y^2 + eBx)^2) - eEx \tag{5.26}$$

This will not change the form of the solutions of the wave function, but will give an E dependent shift:

$$\psi(x, y) = \psi_{n,k}\left(x - \frac{mE}{eB^2}, y\right) \tag{5.27}$$

The energy of these states also change, and the degeneracy due to k is lifted:

$$E_{n,k} = \hbar\omega_B\left(n + \frac{1}{2}\right) + eEkl_B^2 - \frac{mE^2}{2B^2} \tag{5.28}$$

We can see that we get the classical value of drift velocity of electrons immediately from this, by calculating the group velocity $|v| = \frac{\partial\omega}{\partial k} = \frac{E}{B}$.

Symmetric Gauge

This gauge choice is

$$\vec{A} = -\frac{1}{2}\vec{r} \times \vec{B} = -\frac{yB}{2}\hat{x} + \frac{xB}{2}\hat{y} \tag{5.29}$$

which is rotationally invariant (around the z axis), which is a symmetry of the system.

This gauge is useful for defining an "alternate" momentum operator:

$$\widetilde{\Pi}_i = p_i - eA_i \tag{5.30}$$

as opposed to $\Pi_i = p_i + eA_i$. Note that $\widetilde{\Pi}$ is not gauge invariant. The commutators of the momentum and alternate momentum operator are:

$$\begin{aligned} [\Pi_x, \Pi_y] = -ie\hbar B \quad [\widetilde{\Pi}_x, \widetilde{\Pi}_y] = ie\hbar B \\ [\Pi_i, \widetilde{\Pi}_i] = 2ie\hbar\partial_i A_i \\ [\Pi_i, \widetilde{\Pi}_j] = ie\hbar(\partial_i A_j - \partial_j A_i) \end{aligned} \tag{5.31}$$

In the symmetric gauge, these commutators simplify to be $[\Pi_i, \widetilde{\Pi}_j] = 0$. Since these operators are now decoupled, we can define a new set of ladder operators:

$$b = \frac{\widetilde{\Pi}_x + i\widetilde{\Pi}_y}{\sqrt{2e\hbar B}} \quad b^\dagger = \frac{\widetilde{\Pi}_x - i\widetilde{\Pi}_y}{\sqrt{2e\hbar B}} \tag{5.32}$$

This shows that we can use the ladder operators a and b to define our states, with respect to a ground state $|0,0\rangle$:

$$|n,m\rangle = \frac{(a^\dagger)^n\,(b^\dagger)^m}{\sqrt{n!\,m!}}\,|0,0\rangle \tag{5.33}$$

where the quantum number n still labels the Landau level, and thus the energy $E_n = \hbar\omega_B(n + \frac{1}{2})$.

5.4.3 The Lowest Landau Level

The ground states in the symmetric gauge are known as the Lowest Landau Level states:

$$a\,|0,m\rangle = 0 \tag{5.34}$$

To find the wave function of these states, we can write the lowering operator as a differential operator, in the coordinates $z = x + iy$.

$$a = -i\sqrt{2}\left(l_B\bar{\partial} + \frac{z}{4l_B}\right); \quad a^\dagger = -i\sqrt{2}\left(l_B\partial - \frac{\bar{z}}{4l_B}\right) \tag{5.35}$$

We see that the zero eigenfunction of the a operator is

$$\psi_{LLL,m} \sim \left(\frac{z}{2l_B}\right)^m e^{-\frac{|z|^2}{4l_B^2}} \tag{5.36}$$

The quantum number m is the quantized angular momentum of the solution. To see this, we can define the angular momentum operator $J = \hbar(z\partial - \bar{z}\bar{\partial})$.

$$J\psi_{LLL,m} = \hbar m\Psi_{LLL,m} \tag{5.37}$$

Thus, we see that the lowest Landau level describes wavefunctions of angular momentum m, and is peaked at radius $r = \sqrt{2m}l_B$, which we can calculate by finding the maxima of the wavefunction,

by setting $z = re^{i\theta}$ and differentiating with respect to r.

5.5 The Integer Quantum Hall Effect

The integer QHE is the phenomenon in which the transverse Hall conductivity is quantized in units of $\frac{e^2}{h} \equiv \frac{e}{\Phi_0}$, and longitudinal conductivity vanishes. We can explain this through understanding the current in terms of the quantum operators we understood in the previous section.

First, let's consider the effect of putting a chemical potential at low temperature in the system. The system is essentially a free Fermi gas in an external magnetic field, whose effect is to discretize the energy levels to the Landau levels E_i. Each Landau level has a degeneracy $g = eBA/h$. Let the total number of electrons be N, and each Landau level houses n_i electrons. Thus, we can write the partition function as

$$\begin{aligned} Z = \sum_{\text{configs}} e^{-\beta(E-\mu N)} = \prod_i \sum_{n_i=1}^{g} e^{\beta(\mu-E_i)n_i} = \prod_i \frac{e^{\beta(\mu-E_i)(1+g)}-1}{e^{\beta(\mu-E_i)}-1} \\ \implies \ln Z = \sum_i \ln \frac{e^{\beta(\mu-E_i)(1+g)}-1}{e^{\beta(\mu-E_i)}-1} \end{aligned} \tag{5.38}$$

The occupation number of an energy level $\langle n_i \rangle$ is given by

$$\begin{aligned} \langle n \rangle = \frac{1}{\beta}\frac{\partial \ln Z}{\partial \mu} \\ \implies \langle n_i \rangle = \frac{(1+g)e^{\beta(\mu-E_k)(1+g)}}{e^{\beta(\mu-E_k)(1+g)}-1} - \frac{e^{\beta(\mu-E_k)}}{e^{\beta(\mu-E_k)}-1} \\ \implies \lim_{\beta\to\infty} \langle n_i \rangle = \begin{cases} 0 & E_k > \mu \\ g & E_k < \mu \end{cases} \end{aligned} \tag{5.39}$$

which recovers the fact that at low temperatures, free fermions will completely fill up all energy levels up to the chemical potential. Therefore, if $E_\nu < \mu < E_{\nu+1}$, all ν Landau levels will be completely filled.

The current vector $I_i = -e\dot{x}_i$ can be written in terms of the mechanical momentum operator: $I_i = -\frac{e}{m}\Pi_i = -\frac{e}{m}(p_i + eA_i)$. The measured current will thus be

$$\langle I_i \rangle = -\frac{e}{m} \sum_{\text{filled}} \langle \psi | p_i + eA_i | \psi \rangle \tag{5.40}$$

where $|\psi\rangle$ are the occupied states in the system. Working in the Landau gauge, $\vec{A} = xB\hat{y}$, these states have the wave function (5.24).

Studying the components of the current we see

$$I_x = -\frac{e}{m}\sum_{n=1}^{\nu}\int dk\,\langle\psi_{n,k}|-i\hbar\partial_x|\psi_{n,k}\rangle = 0 \tag{5.41}$$

$$\begin{aligned} I_y &= -\frac{e}{m}\sum_{n=1}^{\nu}\int dk\,\langle\psi_{n,k}|\hbar k + exB|\psi_{n,k}\rangle \\ &= -\frac{e}{m}\sum_{n=1}^{\nu}\int dk\left(\hbar k + eB\left(\frac{-\hbar k}{eB}+\frac{mE}{eB^2}\right)\right) \\ &= -e\nu\frac{E}{B}\int dk = -\frac{e^2\nu EA}{h} = -\frac{e\nu EA}{\Phi_0} \end{aligned} \tag{5.42}$$

where we have used the fact that the states in this gauge are harmonic oscillators with potential centered at $\frac{-\hbar k}{eB}+\frac{mE}{eB^2}$, and the degeneracy due to finite sample size is $\frac{eBA}{\Phi_0}$. From this, we see that $\vec{E}=E\hat{x} \implies \vec{J}=-\frac{e\nu E}{\Phi_0}\hat{y}$ and thus $\sigma_{xx}=0, \sigma_{xy}=\frac{e\nu}{\Phi_0}$, which is exactly the values measured in experiment.

5.6 Effects of Disorder in the Quantum Hall system

We have derived the observations seen in the Quantum Hall experiment in the previous section, but we did not take into the fact that impurities and temperature exist, and might completely destroy any quantization of resistivity we calculate. To check that the plateaus survive even if the sample is not perfectly flat or clean, we can use perturbation theory! Let the disorder in the system be modelled by some potential V. The hierarchy of energy scales in the system is $\beta^{-1} << V << \hbar\omega_B$. Thus, we are still at very low temperatures, but the sample is still quite clean. The perturbations won't disturb the energy gap between Landau levels.

First, let's recall the classical equations of motion of an electron in a transverse magnetic field:

$$x(t) = X - R\sin(\omega_B t+\phi) \quad y(t) = Y - R\cos(\omega_B t+\phi) \tag{5.43}$$

Notice that X, Y are constants of motion, so if we use these as quantum operators, they will commute with the unperturbed Hamiltonian:

$$[X,H] = i\hbar\dot{X} = 0 \quad [Y,H] = i\hbar\dot{Y} = 0 \tag{5.44}$$

These operators give the alternate momentum operators $\widetilde{\Pi}$ some physical meaning:

$$X = -\frac{\widetilde{\Pi}_y}{eB} \quad Y = \frac{\widetilde{\Pi}_x}{eB} \implies [X,Y] = il_B^2 \tag{5.45}$$

Now, adding the disorder potential to the Hamiltonian, the center of mass operators X, Y will

now change with time:

$$\begin{aligned}
[X, H+V] = [X, V] = [X, Y]\frac{\partial V}{\partial Y} = il_B^2 \frac{\partial V}{\partial Y} \\
[Y, H+V] = [Y, V] = [Y, X]\frac{\partial V}{\partial X} = -il_B^2 \frac{\partial V}{\partial X} \\
\implies (\dot{X}, \dot{Y}) \cdot \vec{\nabla} V = 0
\end{aligned} \tag{5.46}$$

This means that the center of mass drifts along equipotential surfaces of the disorder potential! Extended equipotential surfaces in the bulk are much rarer than localized ones, thus the wave functions are localized rather than extending from one end to the other. Extended equipotential surfaces are found more towards the edge, and all the current is carried through these surfaces.

5.7 The Fractional Quantum Hall Effect

In the previous section, we argued that by quantizing the 2d electron gas in an external electric and magnetic field, one can accurately explain the observation that the transverse resistivity must be quantized in integers. However, experiments have seen resistivities in quantum Hall systems where the value of ν is not integer, but is fractional! This seems to be in conflict with our success in explaining the integer quantum Hall effect in the previous section, however we should take a closer look.

5.7.1 The Laughlin Wavefunction

Laughlin proposed the following wave function to explain the FQHE, after also considering inter-electron interactions [43].

$$\psi(z_i) = \prod_{i<j} (z_i - z_j)^m e^{-\frac{\sum_i |z_i|^2}{4l_B^2}} \tag{5.47}$$

where the prefactor encodes repulsion between electrons. Note that the wave function vanishes when two electrons are at the same position, with a zero of order m. Since this wave function describes a collection of fermions, it must be anti-symmetric, hence m must be odd.

Let us compute the filling fraction of this wavefunction. Focus on the particle at z_1. From the prefactor, it will get $m(N-1)$ powers, where N is the total number of electrons. Note that the maximum angular momentum of the electron will also be given by the power of z_i, which for z_1 will also be $m(N-1)$. We saw that the peak of the wave function of the lowest Landau level in symmetric gauge was $\sqrt{2m}l_B$. Thus, the radius of the "droplet" will be $R \sim \sqrt{2mN}l_B$, and the corresponding area will be $A \sim 2\pi m N l_B^2$. The total number of particles the system can hold is

$$N = \nu \frac{AB}{\Phi_0} = \nu \frac{2\pi m N \hbar B}{eB\Phi_0} \implies \nu = \frac{1}{m} \tag{5.48}$$

Thus, this wave function does describe a system where one can get fractional filling of Landau

levels.

More evidence that this wavefunction is indeed physical, we can see the $m = 1$ wavefunction. To build a wavefunction describing N fermions, we can use the Slater determinant $\psi(z_i) = \det[\psi_i(z_j)]$. Let us fill the lowest Landau level with electrons of momenta $0, 1, \cdots, N-1$. If we take ψ_k to be the LLL wavefunction with angular momentum $k-1$: $\psi_k(z) = z^{k-1} e^{-\frac{|z|^2}{4l_B^2}}$, the total wavefunction obtained is

$$\psi(z_i) = \begin{vmatrix} z_1^0 & \cdots & z_N^0 \\ \vdots & & \vdots \\ z_1^{N-1} & \cdots & z_N^{N-1} \end{vmatrix} e^{-\frac{\sum_i |z_i|^2}{4l_B^2}} = \prod_{i<j} (z_i - z_j)\, e^{-\frac{\sum_i |z_i|^2}{4l_B^2}} \tag{5.49}$$

which is indeed the $m = 1$ Laughlin wavefunction.

5.7.2 Quasiholes in the Laughlin Wavefunction

Excitations of the Laughlin wavefunction are the quasi-holes and quasi-particles of the system. An obvious way to introduce holes is by introducing zeroes into the wavefunction at the position of the holes. So, for M holes:

$$\psi(z_i; \eta_j) = \prod_{j=1}^{M} \prod_{i=1}^{N} (z_i - \eta_j) \prod_{k<l} (z_k - z_l)^m\, e^{-\sum_i \frac{|z_i|^2}{4l_B^2}} \tag{5.50}$$

Heuristically, one can see that a hole at position η_j carries charge $+e/m$ since if m of them came together at one point, it would mimic the factor of an electron $(z_i - z_j)^m$, but would physically show the deficit of an electron, since η_j are not dynamical variables of real particles in the system, but are just parameters of the system.

We shall compute the statistics and fractional charge exactly in the next chapter using the corresponding Chern-Simons theory.

5.7.3 The Moore-Read Wavefunction

The Moore-Read wave function is also a wave function to describe filling fraction $\nu = \frac{1}{m}$, but for even m [44]. The wavefunction is very similar to Laughlin's, but with an extra factor:

$$\psi(z_i) = \text{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^m\, e^{-\frac{\sum_i |z_i|^2}{4l_B^2}} \tag{5.51}$$

where Pf is the Pfaffian, and is defined for antisymmetric even dimensional matrices:

$$\text{Pf}(M) = \frac{1}{2^{d/2}(d/2)!} \epsilon^{i_1 i_2 \cdots i_d} M_{i_1 i_2} \cdots M_{i_{d-1} i_{d-2}} \tag{5.52}$$

Since the pfaffian is an antisymmetric function of z_i and z_j, for the wavefunction to be antisymmetric, m must be even.

Introducing holes in the wavefunction is similar to the Laughlin case. Consider the 1 hole case, exactly like the Laughlin case,

$$\psi(z_i;\eta) = \prod_{i=1}^{N}(z_i-\eta)\,\mathrm{Pf}\left(\frac{1}{z_i-z_j}\right)\prod_{k<l}(z_k-z_l)^m\, e^{-\frac{\sum_i |z_i|^2}{4l_B^2}} \tag{5.53}$$

To add multiple holes, it's a little more complicated, we have to replace the pfaffian in the ground state with the following function:

$$\mathrm{Pf}\left(\frac{\prod_{l=1,n=1}^{M}(z_i-\eta_{k_l})(z_j-\eta_{m_n})+(k\leftrightarrow m)}{z_i-z_j}\right) \tag{5.54}$$

Note that there are even number of holes $2M$. Also note that if there are two holes at the same place, then the wave function is equal to 1 hole. Thus the holes have half the charge of the Laughlin holes, i.e. $e/2m$.

The number of ways of inserting holes in the Pfaffian like this is $\frac{1}{2}(2M)!/(M!)^2$, but the dimension of the Hilbert space with with $2M$ holes is 2^{M-1}. For the proof, refer to [45].

The quasi-holes in the Moore-Read wavefunction are the Ising anyons described previously in this chapter. The link is through the TQFT description of the Quantum Hall effect, Chern-Simons theory. In particular, the wavefunction of both the ground state and the holes of the Moore-Read wavefunction can be derived from the edge states of the $SU(2)_2$ Chern-Simons theory, which is the $SU(2)_2$ WZW model! Similarly the Laughlin wavefunction and holes are derived from the abelian U(1) Chern-Simons theory, which will be shown in the following chapter.

Chapter 6

Chern-Simons Theories

Chern-Simons theories are 2+1 d topological quantum field theories, which means that they do not have any dynamics, but the non-triviality comes from certain topological aspects of the theory. They are gauge theories, but the kinetic term is not given by the square of the field strength. In fact, the action is only order 1 in derivatives! Chern-Simons theories turn out to be the effective field theory descriptions of the Quantum Hall effect. On coupling to matter and sources, we shall see that it is a very illuminating description of the Quantum Hall system. [46, 40]

6.1 Abelian Chern Simons

6.1.1 The Integer Quantum Hall effect

First we shall attempt to explain the Integer QHE using abelian Chern-Simons theory, coupled to a charge carrier current. The Abelian Chern-Simons action is

$$S_{CS} = \frac{k}{4\pi} \int d^3x \, \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \tag{6.1}$$

where A_μ is a U(1) gauge field. This action is gauge invariant, under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \omega$, the action changes only by a total derivative. One can couple it to a current with the term

$$S_{\text{coupling}} = \int d^3x \, A_\mu J^\mu \tag{6.2}$$

The equations of motion with the coupling are:

$$\frac{k}{4\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho} = J^\mu \tag{6.3}$$

where $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$ is the field strength of the gauge field. In 2+1 d, we identify the electric field $E_i = F_{i0}$ and the magnetic field $B = \epsilon^{ij} \partial_i A_j = \frac{1}{2} \epsilon^{ij} F_{ij}$. Note that magnetic field in 2+1 dimensions becomes a scalar.

If we identify J^0 with charge density ρ, and J^i is current density, then from the equations of motion we have the relations

$$\begin{aligned} J^0 &= \rho = \frac{k}{2\pi} B \\ J^i &= \frac{k}{2\pi} \epsilon^{ij} E_j \end{aligned} \tag{6.4}$$

The first relates magnetic flux to electric charge density, called "flux attachment". The second looks likes Ohm's law: $J^i = \sigma^{ij} E_j$. Thus, we get the relation

$$\sigma^{ij} = \frac{k}{2\pi} \epsilon^{ij} \tag{6.5}$$

From our study of the Integer QHE through the Landau levels treatment, we obtain $\sigma_{xx} = 0$, and $\sigma_{xy} = \frac{\nu e^2}{h}$ where $\nu \in \mathbb{Z}$. Thus we identify

$$k = \frac{\nu e^2}{\hbar} \tag{6.6}$$

Since for the Quantum Chern-Simons theory, the level k is quantized in integers (for the exact same reasons as the quantization of level of the WZW model, under a large gauge transformation, it happens to pick up the same "winding number" term which is the Wess-Zumino term, thus quantizing the level k for a well defined path integral), this identification makes sense. Further more, since the theory is a U(1) gauge theory, it is abelian and the generators need not be normalized to satisfy the algebra since there isn't any. Hence we can scale the units of quantization, in this case we must scale it in units of $\frac{e^2}{\hbar}$. To remove these units, we can scale the gauge field by a factor of e, and the level will once again be purely integral (upon $\hbar$), independent of the coupling constant e.

6.1.2 The Fractional Quantum Hall effect

To describe the Fractional QHE, we introduce a new "emergent" gauge field a_μ. Perhaps this introduces the "electron interaction" which is required to modify the Landau level system to obtain the Laughlin states in the previous discussion. Let us consider the current J^μ we coupled to the U(1) gauge field A_μ. If we describe this current in terms of the emergent gauge field a_μ, we can write

$$J^\mu = \frac{e^2}{4\pi\hbar} \epsilon^{\mu\nu\rho} f_{\nu\rho} \tag{6.7}$$

where $f_{\mu\nu}$ is the field strength of the gauge field a_μ. Let us set the (integer quantized) level of this gauge field to be m, and since we are in the lowest Landau level, we can set $k = 0$. Thus the effective action we have is

$$S = \frac{e^2}{4\pi\hbar} \int d^3x \epsilon^{\mu\nu\rho} (A_\mu f_{\nu\rho} + m\, a_\mu \partial_\nu a_\rho) = \frac{e^2}{4\pi\hbar} \int d^3x \epsilon^{\mu\nu\rho} (2A_\mu \partial_\nu a_\rho + m\, a_\mu \partial_\nu a_\rho) \tag{6.8}$$

Integrating out a_μ by replacing it with its equation of motion : $a_\mu = \frac{1}{m}A_\mu$, we get the following action:

$$S = \frac{e^2}{4\pi\hbar m}\int d^3x\, \epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda = \frac{me^2}{4\pi\hbar}\int d^3x\, \epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \tag{6.9}$$

From here, we see that the analogous computation as the previous section gives us the Hall conductivity:

$$\sigma_{xy} = \frac{1}{m}\frac{e^2}{h} \implies \nu = \frac{1}{m} \tag{6.10}$$

which is the desired filling fraction.

6.2 Edge Modes and obtaining the Quantum Hall Wave functions

From the action derived previously for the fractional filling, let us see how we can recover the Quantum Hall wavefunctions from these. For this, we must introduce a boundary to the Quantum Hall system. Consider the boundary to be the line $y = 0$, with $y < 0$ being the bulk of the Quantum Hall state. The boundary term of the Chern-Simons action is $\epsilon^{ij}a_i\delta a_j$, where i, j run over x, t. For the equation of motion $f_{\mu\nu} = 0$, we have to set a linear combination of a_x and a_t to 0. We choose the gauge:

$$a_t - va_x = 0 \tag{6.11}$$

such that the boundary term vanishes. Under the change of variables: $t' = t, x' = x + vt, y' = y$, the gauge choice becomes $a'_{t'} = 0$ which implies that the new equations of motion $f'_{x'y'} = 0 \implies a'_i = \partial_i\phi$. The boundary action then looks like, in the original coordinates

$$S = \frac{m}{4\pi}\int d^2x \left(\partial_t\phi\partial_x\phi - v(\partial_x\phi)^2\right) \tag{6.12}$$

whose equations of motion are

$$\partial_t\rho(x,t) - v\partial_t\rho(x,t) = 0 \tag{6.13}$$

where $2\pi\rho = \partial_x\phi$. These are chiral waves, since they propagate as $x + vt$, waves which propagate as $x - vt$ do not satisfy the equation of motion. This means that the U(1) Chern Simons theory with a boundary has a chiral boson on the boundary.

On compactifying the boundary into a circle of radius $R = \sqrt{2mN}l_B$, and coordinate σ, we have the action

$$S = \frac{m}{4\pi}\int dtd\sigma \left(\partial_t\phi\,\partial_\sigma\phi - v(\partial_\sigma\phi)^2\right) \tag{6.14}$$

Decomposing the field ϕ into Fourier modes:

$$\phi(\sigma,t) = \frac{1}{\sqrt{L}}\sum_{n=-\infty}^{\infty}\phi_n(t)e^{\frac{2\pi i\sigma}{L}n} \tag{6.15}$$

and $\rho_n = \frac{ik_n}{2\pi}\phi_n$, where $k_n = \frac{2\pi n}{L}$ is the momentum of the mode. The action in terms of the Fourier modes, after integrating out delta functions in σ and removing boundary terms in t, we get:

$$S = -\frac{m}{2\pi}\int dt \sum_{n=1}^{\infty}\left(ik_n\dot{\phi}_n\phi_{-n} + vk_n^2\phi_n\phi_{-n}\right) \tag{6.16}$$

Thus the conjugate momentum of ϕ_n is $-\frac{imk_n}{2\pi}\phi_{-n}$, and one can write down the commutation relation:

$$[\phi_n, \phi_k] = \frac{2\pi}{m\,k_n}\delta_{n,-k} \tag{6.17}$$

which is the U(1) Kac-Moody algebra. In terms of fields, the equal time commutator is

$$[\phi(\sigma), \phi(\sigma')] = \frac{i\pi}{m}\text{sgn}(\sigma - \sigma') \tag{6.18}$$

The operator we will associate with the electron is the vertex operator

$$\Psi(\sigma, t) = \ :e^{im\phi(\sigma,t)}: \tag{6.19}$$

This has the correct charge since on calculating the commutator of Ψ and $\Psi^\dagger$ with ρ, it inserts and annihilates an object of unit charge respectively. Similarly, we also obtain the following relations:

$$\begin{aligned}[\Psi(\sigma), \Psi(\sigma')] = 0 \implies m \text{ is even} \\ \{\Psi(\sigma), \Psi(\sigma')\} = 0 \implies m \text{ is odd}\end{aligned} \tag{6.20}$$

which implies that m must be odd for Ψ to describe a fermion. Similarly, the vertex operator $\Psi_{\text{qh}} = :e^{i\phi}:$ corresponds to the quasi-hole and quasi-particle with charge $\pm\frac{e}{m}$. We see that on commuting this operator, we get a statistical phase of $e^{i\pi/m}$, hence describing anyonic excitations.

From now, we shall pass to complex coordinates: $z = e^{-i(\frac{2\pi}{L}\sigma + it)}$. Recall this is the map from the cylinder to the plane. Let us take the correlator of N electron operators, and insert a neutralizing background charge operator, to respect the shift symmetry ($\phi \to \phi + \alpha$) of the chiral boson action.

$$\left\langle \Psi(z_1)\cdots\Psi(z_N)e^{-\rho_0\int_\gamma d^2z'\phi(z')}\right\rangle = \prod_{i<j}(z_i - z_j)^m e^{-\rho_0\sum_{i=1}^N\int_\gamma d^2z'\log(z_i - z')} \tag{6.21}$$

which uses the fact that the free boson propagator goes as $\langle\phi(z)\phi(w)\rangle = -\frac{1}{m}\log(z-w)$. After a gauge transformation for the integral in the exponential, taking the leading order terms, and identifying $\rho_0 = \frac{1}{2\pi l_B^2}$ we obtain the Laughlin wave function:

$$\left\langle \Psi(z_1)\cdots\Psi(z_N)e^{-\rho_0\int_\gamma d^2z'\phi(z')}\right\rangle = \prod_{i<j}(z_i - z_j)^m e^{\sum_{i=1}^N\frac{|z_i|^2}{4l_B^2}}$$

On inserting the quasi-particle and quasi-hole operators, we can easily obtain the quasi-hole

wavefunctions as well.

We have seen in this chapter that Chern-Simons theory can describe the dynamics and wave functions of the Quantum Hall effect. We were able to recover the filling fractions obtained through the Landau level treatment as well. This link to Chern-Simons theory facilitates a link to RCFT as well. We were able to recover U(1) Kac-Moody modes as edge-modes on the boundary. In fact this is a common feature and extends to the non-abelian case, where the theory on the boundary is a WZW model of the same group and level as the Chern-Simons theory. In the abelian Chern-Simons case, the theory that lives on the boundary is the chiral boson, an RCFT with $c = 1$. To describe Non-Abelian anyons, i.e. to recover the Moore-Read wavefunction, we use the vertex operators of a chiral boson theory and a free fermion theory, which is the $SU(2)_2$ WZW model.

The upshot of this discussion is that the Chern-Simons theory is the effective low energy theory which is able to describe the interesting dynamics of the system by integrating out the complicated interactions of the strongly coupled electron gas in the system. The pure Chern-Simons theory is topological, but we showed by coupling to a quasi-particle current, we were able to construct all the observables of the quantum Hall system, and the topological properties of anyons, through the wave functions. Since Chern-Simons theories are dual to boundary WZW models, we can use properties of RCFT to help us understand aspects of the Quantum Hall system further.

Conclusion

This thesis covers many topics, including a brief review of CFTs and WZW models, computation of correlation functions, modular invariance and modular bootstrap techniques for classification of RCFT, and a brief review of the Quantum Hall effect. The underlying physics governing all these topics is RCFT. The goal was to understand some established RCFT like minimal models and WZW models, and then explore the classification of RCFT, extending the work of [15, 24, 25, 30], etc, and then explore its applications in physical systems like the Quantum Hall Effect.

We started by reviewing important basics of CFTs in two dimensions, the central theme of the thesis. We covered both the basic construction of the Virasoro algebra and the Kac-Moody algebra in the context of WZW models. Modular invariance of CFTs defined on a torus was introduced as well, which allowed us to define characters and the partition function. We then looked at the characters of the SU(2) WZW model, and derived the modular $\mathcal{S}$-matrix of the same. Then the Modular Linear Differential Equation method of classifying RCFTs was discussed.

The following chapter detailed some computations of correlation functions of WZW Models, and we introduced the Dotsenko-Fateev method for calculating correlators of minimal models. The Dotsenko-Fateev construction shows that the correlators can be expressed in terms of a contour integral due to the introduction of "screening operators" [34]. In [37] it was proposed that these integrals are the characters for RCFT with $\ell = 0$. This proposal was explored by the computation of the modular $\mathcal{S}$ matrix using the monodromy properties of the integral, as in our paper [35].

Following this we briefly discussed the classification problem for three characters and introduced quasi-characters for the order 3 MLDE, and how they can be used to construct new admissible characters. This work is described in the paper [32].

From here, we explore the physical system of the quantum Hall effect. First we understood some physics of anyons and the braid group. Then we studied the 2d electron gas with a transverse magnetic field, and constructed the Landau levels and the Landau wave function. A couple of quantum Hall states were discussed, including the Laughlin and Moore-Read states, and their respective quasi-hole excitations. We showed that one could also describe the system using a topological QFT, the Chern-Simons theory. We explored the connection of the U(1) Chern-Simons theory and explained both the integer and (abelian) fractional quantum Hall effect. We were also able to construct the Laughlin state and its quasi-hole excitations using the Edge modes of the Chern-Simons theory. Here is the connection to RCFT: The edge theory of a Chern Simons theory is a WZW model. Using free field techniques similar to the Dotsenko-Fateev construction of correlators of minimal models, we were able to construct the Quantum Hall wavefunctions using the RCFT dual to the Chern-Simons theory used to describe the quantum Hall state.

There are many directions this work can go in. For the contour integral representation of characters, one can try to generalize the integrals to include the chemical potential z, once the chiral algebra is known. To classify four character CFTs and further, one can look for the appropriate coset theories and perform a similar construction for possible quasi-character families. It is not yet known that quasi-character solutions exist for the order 4 and higher MLDEs. A new direction in MLDE would be to use cusp forms to create indicial equations for MLDE of any ℓ, and see what this implies for the classification problem. An interesting observation was made in the quasi-character families of the order 3 MLDE, is that there exist quasi-characters for SO($-r$), where r is positive. It will be interesting to see what properties these negative dimension Kac-Moody algebras have.

The most immediate direction is to understand if other quantum Hall states, perhaps with different filling fractions, can be constructed by our classification of RCFT. So far, only a few special CFTs which have free fields have been used to construct these wavefunctions. It will be interesting to understand the universality of these wave-functions and what other quantum Hall states lie in the same universality classes, which are tractable to take to the thermodynamic limit for numerical simulations.

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