Cohomology of $GL(2)$

A Thesis

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by

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This is to certify that this dissertation entitled Cohomology of $GL(2)$ towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents the research carried out by Shiva Chidambaram at Indian Institute of Science Education and Research under the supervision of Prof. A. Raghuram, Professor, Department of Mathematics , during the academic year 2014-2015.

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Dedicated to Amma, Appa and my loving friends.

Declaration

I hereby declare that the matter embodied in the report entitled Cohomology of $GL(2)$ are the results of the investigations carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Prof. A. Raghuram and the same has not been submitted elsewhere for any other degree.

Shiva Chidambaram

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Abstract

Let F be a number field. Let $G = GL(2)$ over F. Let $\mathbb{A}$ and $\mathbb{A}_f$ denote the ring of adeles and finite adeles of F respectively. Let K_∞ denote the maximal compact subgroup of $G_\infty = \prod_v G(F_v)$ thickened by the center, where the product runs over all archimedean places v of F and F_v denotes the completion of F at v. For a fixed open compact subgroup $K_f \subset G(\mathbb{A}_f)$, let $S^G_{K_f} = G(F)\backslash G(\mathbb{A})/K_f K_\infty$ be a locally symmetric space attached to G. Let $r[d]$ be an irreducible representation of G of highest weight d, and let $\mathscr{F}_d$ denote the corresponding sheaf on $S^G_{K_f}$. The goal of this project is to understand the cohomology $H^\bullet(S^G_{K_f}, \mathscr{F}_d)$ via its relation to the theory of automorphic forms on G. This relation arises due to the isomorphism $H^\bullet(S^G_{K_f}, \mathscr{F}_d) \simeq H^\bullet(\mathfrak{g}_\infty, K_\infty; C^\infty(G_F\backslash G_{\mathbb{A}}/K_f) \otimes R[d]^\vee)$. A specific problem is to understand the inner cohomology denoted $H^\bullet_!(S^G_{K_f}, \mathscr{F}_d)$, which by definition is the image of compactly supported cohomology in full cohomology:

$$H^\bullet_!(S^G_{K_f}, \mathscr{F}_d) := \mathrm{Im}\left(H^\bullet_c(S^G_{K_f}, \mathscr{F}_d) \longrightarrow H^\bullet(S^G_{K_f}, \mathscr{F}_d)\right).$$

It is known that inner cohomology contains cuspidal cohomology, which is genered by cusp forms on G. The problem is to classify inner cohomology classes which are not cuspidal. In this thesis, we deal with $F = \mathbb{Q}$ and $F = \mathbb{Q}(\sqrt{-n})$ where n is a square free positive integer. We give a description of the inner cohomology mentioned above in these two cases.

Contents

Chapter 1

Introduction

We start by mentioning a theorem in Eberhard Freitag's Hilbert Modular Forms ([8], Chapter III, Theorem 6.3). It gives a description of the Betti numbers b^q for an arbitrary congruence subgroup Γ of $SL(2,\mathbb{R})^n$. These are the numbers that give the dimensions of the sheaf cohomology groups $H^q(\Gamma) = H^q(\mathbb{H}^n/\Gamma, \mathbb{C})$ corresponding to the constant sheaf given by $\mathbb{C}$, considered as vector spaces over $\mathbb{C}$.

$$b^q = \dim_{\mathbb{C}} H^q(\Gamma). \tag{1.0.1}$$

If we denote the set of archimedean places of F by S, the completion of F at a place v by F_v, the adele ring of F by $\mathbb{A}$, the finite adele ring by $\mathbb{A}_f$, the direct product $\prod_{v\in S} GL(2, F_v)$ by $GL(2)_\infty$, and its maximal compact subgroup thickened by the center of $GL(2)_\infty$ by K_∞, the objects $H^q(\Gamma)$ are exactly isomorphic to certain sheaf cohomology groups $H^q(S^G_{K_f}, \mathscr{F}_d)$ for certain open compact subgroups K_f of $GL(2,\mathbb{A}_f)$ depending on Γ. To be precise, for an open compact subgroup $K_f \subseteq GL(2,\mathbb{A}_f)$, the space $S^G_{K_f}$ mentioned above, is actually the locally symmetric space $GL(2,F)\backslash GL(2,\mathbb{A})/K_f K_\infty$ attached to $G = GL(2)$, and $\mathscr{F}_d$ denotes the sheaf on $S^G_{K_f}$ derived from a highest weight irreducible representation of $GL(2)_\infty$, denoted $(r[d], R[d])$, with highest weight $d = (d_v)_{v\in S}$.

The theorem also gives the dimensions of various different subspaces of $H^q(\Gamma)$, namely the Eisenstien cohomology, square integrable cohomology, inner cohomology, cuspidal cohomology and universal cohomology. We know from Chapter III, Theorem 6.3 of [8], that for all degrees other than zero, the full cohomology breaks up as a direct sum of Eisenstein cohomology and square integrable cohomology; and the square integrable cohomology is the same as inner cohomology, which by definition is the image of compactly supported cohomology in full cohomology, for all these degrees. Moreover, inner cohomology at any degree q is the direct sum of universal and cuspidal cohomologies. We are interested particularly in the inner cohomology classes that are not cuspidal, i.e., the universal cohomology. In this thesis, we have tried to understand this universal part of the cohomology for some special cases of number fields F, namely, $F = \mathbb{Q}$ and $F = \mathbb{Q}(\sqrt{-n})$, $n \geq 0$. We take the approach of Waldspurger [12] using automorphic forms on $GL(2)$, making use of the following isomorphism of the sheaf cohomology group to a certain relative lie algebra cohomology, as explained in [5].

$$H^\bullet(S^G_{K_f}, \mathscr{F}_d) \simeq H^\bullet(\mathfrak{g}_\infty, K_\infty; C^\infty(G_F\backslash G_\mathbb{A})^{K_f} \otimes R[d]^\vee). \tag{1.0.2}$$

Let (π, E) be an irreducible automorphic representation of $GL(2)$ over F. We first find out when E has non-trivial Lie algebra cohomology. Propositon I.4 of [12] gives us a classification of irreducible cohomological representations for $GL(2,\mathbb{R})$ and $GL(2,\mathbb{C})$ and describes the cohomology groups as well. We also use Theorem I.5.1 of [12] that captures all the possible automorphic representations of $GL(2)$ over F, that could contribute to inner cohomology. This set includes only cuspidal and one-dimensional representations with infinity type isomorphic to one of a finite set of representations depending on the weight d. Further, cuspidals contribute to cuspidal cohomology, and hence we try to find the one-dimensional representations that contribute to inner cohomology. These exactly correspond to universal cohomology classes.

We start by stating the classification of irreducible admissible representations of $GL(2,\mathbb{R})$ and $GL(2,\mathbb{C})$, and then define Lie Algebra cohomology, also called $(\mathfrak{g}, K)$ cohomology, and talk about irreducible admissible cohomological representations and their cohomology groups in Chapter 2. This is basically a summary of certain results following [7], [11], [4]. In Chapter 3, we set up the basic notations to talk about the sheaf cohomology groups $H^\bullet(S^G_{K_f}, \mathscr{F}_d)$ and state their relation to the relative lie algebra cohomology groups given in 1.0.2. We also introduce the inner cohomology groups, define the spectrum and describe the set of irreducible automorphic representations that are candidates to contributing to inner cohomology, following [5], [10] and [12]. We discuss Theorem I.5.1 of [12] and also give an outline of the proof of it given there. In Chapter 4, we give a summary of results given in [8] that we mentioned in the beginning. Then, we go on to calculate the inner cohomology using our approach involving the language of automorphic forms for the case of $F = \mathbb{Q}$. In Chapter 5, we do the same for the case of an imaginary quadratic field.

Chapter 2

Cohomological Representations of Archimedean $GL(2)$

2.1 Classification of irreducible admissible $(\mathfrak{g}, K)$ modules

We follow [7] and [11] in this section for classifying irreducible admissible representations of $GL(2,\mathbb{R})$ and $GL(2,\mathbb{C})$ respectively. The classification is up to infinitesimal equivalence as $\left(gl(2,\mathbb{R}), O(2)\right)$ and respectively $\left(gl(2,\mathbb{C}), U(2)\right)$ modules.

Let $G = GL(2,\mathbb{R})$. Let K denote the maximal compact subgroup $O(2)$ of G, $K^0 = SO(2)$ and $\mathfrak{g}$ denote the Lie algebra $gl(2,\mathbb{R})$ of G. Let

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad R = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}. \tag{2.1.1}$$

be elements of the complexification $\mathfrak{g}_{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$. These form a basis of $\mathfrak{g}_{\mathbb{C}}$ as a complex vector space. Let $U(\mathfrak{g}_{\mathbb{C}})$ denote the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. The elements $\Delta = \frac{-1}{4}\left(H^2 + 2RL + 2LR\right)$ and Z lie in the center of $U(\mathfrak{g}_{\mathbb{C}})$. Since the irreducible unitary representations of K^0 are one dimensional and are parametrized by the integers, with $k \in \mathbb{Z}$ denoting the character $\sigma_k(k_\theta) = e^{ik\theta}$, where $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, we may denote the σ_k-isotypic component $V(\sigma_k)$ of a representation (π, V) simply by $V(k)$. The set of all integers k such that $V(k) \neq 0$ is called the set of K^0-types of V. With notations as in [12], the irreducible admissible $(\mathfrak{g}, K)$ modules for G are then classified based on the K^0-types and the action of Z and Δ.

1. The finite dimensional representations π, obtained by twisting the symmetric powers of the standard representation of $GL(2,\mathbb{R})$ on $\mathbb{C}^2$, by the characters of the form $\chi\circ\det$, where χ is a character of $\mathbb{R}^\times$.
   $\pi = \mathrm{Sym}^{h-2}\left(\mathbb{C}^2\right)\otimes\left(\det^{1+\frac{a-h}{2}}\mathrm{sgn}(\det)^\epsilon\right)$, where $a,h\in\mathbb{Z}$, $h\geq 2$, $a\equiv h \pmod 2$ and $\epsilon\in\{0,1\}$. A representation of this type will be denoted $(r[d],R[d])$, with $d=(h,a,\epsilon)$ as above. For d as above, we let $d'=(h,a,\epsilon')$ where $\epsilon'=\epsilon-1$ and $\check{d}=(h,-a,\epsilon)$. The representation $r[\check{d}]$ is in fact isomorphic to the dual $r[d]^\vee$ of the representation $r[d]$. The representation $r[d]$ is of K^0-type $\Sigma^0(h)=\{l\in\mathbb{Z}\mid l\equiv h \pmod 2, -h<l<h\}$. Z acts by the scalar a, while the Casimir element Δ acts by the scalar $\frac{h}{2}\left(1-\frac{h}{2}\right)$.

2. Let χ_1,χ_2 be characters of $\mathbb{R}^\times$ defined as $\chi_i(y)=\mathrm{sgn}(y)^{\epsilon_i}|y|^{s_i}$ for $i=1,2$. Let χ be the character of the Borel subgroup $B(\mathbb{R})$ defined as

$$\chi\begin{pmatrix} y_1 & x\\ 0 & y_2\end{pmatrix}=\chi_1(y_1)\chi_2(y_2). \tag{2.1.2}$$

   If $\chi_1\chi_2^{-1}\neq\mathrm{sgn}^\epsilon|\cdot|^{k-1}$ where $\epsilon\in\{0,1\}$ and k is an integer of the same parity as ϵ, then the K-finite vectors of the induced representation $\mathrm{Ind}_{B(\mathbb{R})}^G(\chi_1,\chi_2)$ of G form an irreducible admissible $(\mathfrak{g},K)$-module denoted $\pi(\chi_1,\chi_2)$. If $\epsilon=\epsilon_1+\epsilon_2 \pmod 2$, $\mu=s_1+s_2$ and $\lambda=\frac{1}{4}(s_1-s_2+1)(s_2-s_1+1)$, then the set of K^0-types of $\pi(\chi_1,\chi_2)$ is $\{k\in\mathbb{Z}\mid k\equiv\epsilon \pmod 2\}$, Z acts by μ and Δ by λ. These are called principal series representations and are denoted $P_\mu(\lambda,\epsilon)$. If $\mu=0$, they are denoted simply as $P(\lambda,\epsilon)$.

3. Let χ_1,χ_2 be characters of $\mathbb{R}^\times$ defined as

$$\begin{aligned}\chi_1(y)&=y^{\frac{a+h}{2}-1}|y|^{\frac{1}{2}}\mathrm{sgn}(y)^\epsilon,\\ \chi_2(y)&=y^{\frac{a-h}{2}+1}|y|^{\frac{-1}{2}}\mathrm{sgn}(y)^\epsilon.\end{aligned} \tag{2.1.3}$$

   for $h\geq 1, a\equiv h \pmod 2$ and $\epsilon\in\{0,1\}$. Then, $\chi_1\chi_2^{-1}=\mathrm{sgn}^{\epsilon'}|\cdot|^{h-1}$ where $\epsilon'\equiv h \pmod 2$. Let χ be defined as in Equation 2.1.2. Let $\mathfrak{H}$ denote the $(\mathfrak{g},K)$-module of K-finite vectors in $\mathrm{Ind}_{B(\mathbb{R})}^G(\chi_1,\chi_2)$, then $\mathfrak{H}$ has an irreducible invariant subspace $\mathfrak{H}_0$ with set of K^0-types equal to $\Sigma^\pm(h)=\{l\in\mathbb{Z}\mid l\equiv h \pmod 2, l\geq h \text{ or } l\leq -h\}$. We see that Z acts on this space by a and Δ by $\lambda=\frac{h}{2}\left(1-\frac{h}{2}\right)$. These are called discrete series representations if $h\geq 2$ and limits of discrete series if $h=1$ and are denoted

$\pi[d]$ where $d=(h,a,\epsilon)$. It is to be noted that $\pi[d]\simeq\pi[d']$. Further, the quotient $\mathfrak{H}/\mathfrak{H}_0$ is an irreducible admissible $(\mathfrak{g},K)$-module isomorphic to $r[d]$, i.e., we have an exact sequence of $(\mathfrak{g},K)$ modules:

$$0 \longrightarrow \mathfrak{H}_0 \longrightarrow \mathfrak{H} \longrightarrow r[d] \longrightarrow 0. \tag{2.1.4}$$

Now, let $G=GL(2,\mathbb{C})$. Then, G is the product of its center denoted $Z(\mathbb{C})\simeq\mathbb{C}^\times$ and $SL_2(\mathbb{C})$. This product is not direct and the two subgroups intersect in $\{\pm I\}$, which is the center of $SL_2(\mathbb{C})$.

Suppose π is an irreducible admissible representation of $SL_2(\mathbb{C})$, and let ε be the character through which $\{\pm I\}$ acts. Let χ be an extension of ϵ to a quasi-character of $Z(\mathbb{C})$. Then there is a unique extension of π to a representation of $GL_2(\mathbb{C})$, on which $Z(\mathbb{C})$ acts via the character χ. Conversely, given an irreducible admissible representation of $GL(2,\mathbb{C})$ with central character χ, we can consider the restriction to $SL(2,\mathbb{C})$, which is also irreducible. Further, the original representation of $GL(2,\mathbb{C})$ is got back from this representation of $SL(2,\mathbb{C})$, by such a construction using the character χ. Hence, it is enough to classify irreducible admissible representations of $SL(2,\mathbb{C})$.

Let K denote the maximal compact subgroup $U(2,\mathbb{C})$ of unitary matrices in G and $\mathfrak{g}$ denote the real Lie algebra $gl(2,\mathbb{C})$ of G. Let us denote the complexification of $\mathfrak{g}$ by $\mathfrak{g}_{\mathbb{C}}$. We may identify $\mathfrak{g}_{\mathbb{C}}$ with $\mathfrak{g}\bigoplus\mathfrak{g}$ by the mapping

$$X\otimes 1+Y\otimes i \quad\longmapsto\quad (X+iY)\oplus(\bar{X}+i\bar{Y}).$$

This also identifies the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ with $U(\mathfrak{g})\bigotimes U(\mathfrak{g})$. Let Δ and Z be elements in the center of $U(gl(2,\mathbb{C}))$ defined above. Let $\Delta_1=\Delta\otimes 1,\Delta_2=1\otimes\Delta, Z_1=Z\otimes 1$ and $Z_2=1\otimes Z$ be elements of $U(\mathfrak{g})\bigotimes U(\mathfrak{g})\simeq U(\mathfrak{g}_{\mathbb{C}})$. Then, they are in fact elements in the center of $U(\mathfrak{g}_{\mathbb{C}})$. A representation of G is said to be admissible if its restriction to the group $SU(2,\mathbb{C})$ of unitary matrices of determinant 1, breaks up into a direct sum of irreducible representations of $SU(2,\mathbb{C})$ each occuring with finite multiplicity.

Let $n\geq 0$. Let V_n be the vector space of complex homogeneous polynomials in two variables of degree n. Consider $SU(2,\mathbb{C})$ acting on this space as follows.

$$\rho_n\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)P\left(\begin{pmatrix} z_1\\ z_2\end{pmatrix}\right)=P\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}^{-1}\begin{pmatrix} z_1\\ z_2\end{pmatrix}\right). \tag{2.1.5}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2,\mathbb{C})$. Then, (ρ_n, V_n) is an irreducible representation of $SU(2,\mathbb{C})$ of dimension $n+1$. In fact, any irreducible representation of $SU(2,\mathbb{C})$ is equivalent to ρ_n for some $n \geq 0$.

We classify all irreducible admissible representations of $GL(2,\mathbb{C})$ using the above classification of irreducible representations of $SU(2,\mathbb{C})$. Let

$$\mu_i(z) = z^{s_i + \frac{1}{2}(a_i - b_i)} \bar{z}^{s_i - \frac{1}{2}(a_i - b_i)}. \tag{2.1.6}$$

be quasi-characters of $\mathbb{C}^\times$, where $s_i \in \mathbb{C}$, a_i and b_i are non-negative integers with $a_1 \geq a_2$, $b_1 \geq b_2$ and $a_i b_i = 0$ for $i = 1,2$. Let $\mu(z) = \mu_1\mu_2^{-1}(z) = z^{s+\frac{1}{2}(a-b)}\bar{z}^{s-\frac{1}{2}(a-b)}$ with $s = s_1 - s_2, a = a_1 - a_2, b = b_1 - b_2$ such that $ab = 0$. This implies that either $a_1 = a_2 = 0$ or $b_1 = b_2 = 0$. The quasi-characters μ_1 and μ_2 together define a quasi-character of the Borel subgroup $B(\mathbb{C})$ of upper triangular matrices.

$$\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \longmapsto \mu_1(\alpha)\mu_2(\gamma).$$

Let us denote the induced representation of G obtained from the above character by $\mathscr{B}(\mu_1,\mu_2)$. The action of elements Δ_1, Δ_2, Z_1 and Z_2 of $Z(U(\mathfrak{g}_\mathbb{C}))$ on $\mathscr{B}(\mu_1,\mu_2)$ is given by scalars as described below.

$$\begin{aligned} \rho(\Delta_1) &\equiv \frac{1}{2}\left(\left(s + \frac{a-b}{2}\right)^2 - 1\right); \quad \rho(Z_1) \equiv s_1 + s_2 + \frac{1}{2}(a_1 - b_1 + a_2 - b_2). \\ \rho(\Delta_2) &\equiv \frac{1}{2}\left(\left(s + \frac{b-a}{2}\right)^2 - 1\right); \quad \rho(Z_2) \equiv s_1 + s_2 + \frac{1}{2}(b_1 - a_1 + b_2 - a_2). \end{aligned} \tag{2.1.7}$$

Further, $\mathscr{B}(\mu_1,\mu_2)$ is admissible and contains the representations ρ_n of $SU(2,\mathbb{C})$ if and only if $n \geq a+b$ and $n \equiv a+b \pmod 2$. Moreover, ρ_n occurs with multiplicity exactly 1 in $\mathscr{B}(\mu_1,\mu_2)$ in this case. We will denote the unique subspace of $\mathscr{B}(\mu_1,\mu_2)$ that is isomorphic to the representation ρ_n of $SU(2,\mathbb{C})$ by $\mathscr{B}(\mu_1,\mu_2,\rho_n)$.

1. If $\mu(z)$ is not of the form $z^p\bar{z}^q$ with $p,q \in \mathbb{Z}$ and $p,q \geq 1$ or $p,q \leq 1$, then $\mathscr{B}(\mu_1,\mu_2)$ is irreducible as a representation of G.

2. If $\mu(z) = z^p \bar{z}^q$ with $p, q \in \mathbb{Z}$, $p, q \geq 1$, there is a unique proper invariant subspace

$$V = \sum_{\substack{n \geq p+q \\ n \equiv p+q \pmod 2}} \mathscr{B}(\mu_1, \mu_2, \rho_n).$$

   We will denote the representation of G on V by $\sigma(\mu_1, \mu_2)$ and the representation of G on the quotient space $\mathscr{B}(\mu_1, \mu_2)/V$ by $\pi(\mu_1, \mu_2)$. $\sigma(\mu_1, \mu_2)$ is in fact equivalent to the representation $\mathscr{B}(\nu_1, \nu_2)$ of 1, for characters ν_1, ν_2 such that $\nu_1\nu_2 = \mu_1\mu_2$, $\nu_1\nu_2^{-1}(z) = z^p \bar{z}^{-q}$. On the other hand, $\pi(\mu_1, \mu_2)$ is an irreducible admissible finite dimensional representation of G.

3. If $\mu(z) = z^p \bar{z}^q$ with $p, q \in \mathbb{Z}$, $p, q \leq -1$, there is a unique proper invariant subspace

$$V = \sum_{\substack{|p-q| \leq n < p+q \\ n \equiv p+q \pmod 2}} \mathscr{B}(\mu_1, \mu_2, \rho_n).$$

   We will denote the representation of G on V by $\pi(\mu_1, \mu_2)$ and the representation of G on the quotient space $\mathscr{B}(\mu_1, \mu_2)/V$ by $\sigma(\mu_1, \mu_2)$. The representations $\pi(\mu_1, \mu_2)$ and $\sigma(\mu_1, \mu_2)$ are in fact isomorphic to the representations $\pi(\mu_2, \mu_1)$ and $\sigma(\mu_2, \mu_1)$ of 2.

Any irreducible admissible representation of $GL(2, \mathbb{C})$ is isomorphic to a representation $\pi(\mu_1, \mu_2)$ for some characters μ_1, μ_2 of $\mathbb{C}^\times$ as in Equation 2.1.6. Further, the finite dimensional irreducible admissible representations given above may also be realised as

$$\mathrm{Sym}^{h_1-1}(\mathbb{C}^2)\overline{\mathrm{Sym}}^{h_2-1}(\mathbb{C}^2) \otimes (\det^{1+\frac{a_1-h_1}{2}} \overline{\det}^{1+\frac{a_2-h_2}{2}}) \tag{2.1.8}$$

for integers h_i, a_i such that $h_i \geq 2, a_i \equiv h_i \pmod 2$. We will denote this representation by $r[d]$ and its space by $R[d]$, where $d = (h_1, h_2, a_1, a_2)$.

2.2 Relative Lie algebra cohomology or $(\mathfrak{g}, K)$-cohomology

Let us consider the reductive Lie group $G = GL(2, \mathbb{R})$ or $GL(2, \mathbb{C})$. Let $\mathfrak{g} = gl(2, \mathbb{R})$ or $gl(2, \mathbb{C})$ denote its corresponding Lie algebra of all $2x2$ matrices with real and respectively complex entries. Let $K^0 = O(2, \mathbb{R})$ or $U(2, \mathbb{C})$ be a maximal compact subgroup of G, depending on the two cases. Let Z denote the center of G and $K = K^0 Z$. Let $\mathfrak{k}$ denote the Lie algebra of K. As a real vector space, $\mathfrak{k}$ is two dimensional if $G = GL(2,\mathbb{R})$ and five dimensional if $G = GL(2,\mathbb{C})$.

We know that the adjoint action of K on $\mathfrak{g}$ given by

$$\mathrm{Ad}(k)X = kXk^{-1} \qquad \forall\, k \in K, X \in \mathfrak{g} \tag{2.2.1}$$

leaves the sub-algebra $\mathfrak{k}$ invariant. Thus, it induces an action on the quotient $\mathfrak{g}/\mathfrak{k}$. We complexify this space to get a representation of K on $\left(\mathfrak{g}/\mathfrak{k}\right)_{\mathbb{C}} = \mathfrak{g}/\mathfrak{k} \otimes \mathbb{C}$. For convenience, we will omit the $\mathbb{C}$ altogether and just denote the space as $\mathfrak{g}/\mathfrak{k}$ in the future. Note that, $\mathfrak{g}/\mathfrak{k}$ is now a two or three dimensional complex Lie algebra, according to whether we are working with $GL(2,\mathbb{R})$ or $GL(2,\mathbb{C})$. We will work with the following basis of the complex vector space $\mathfrak{g}/\mathfrak{k}$ for $GL(2,\mathbb{R})$ and $GL(2,\mathbb{C})$.

For $G = GL(2,\mathbb{R})$, we will work with the basis

$$z_1 = i \otimes X - 1 \otimes Y, \qquad z_2 = i \otimes X + 1 \otimes Y. \tag{2.2.2}$$

where $X = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and $Y = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. The action of $SO(2,\mathbb{R})$ on $\mathfrak{g}/\mathfrak{k}$ decomposes into a direct sum of characters $\Theta_2 \oplus \Theta_{-2}$ with respect to this basis, where Θ_n denotes the character $k_\theta \mapsto e^{-in\theta}$. In other words, $k_\theta \cdot z_1 = \mathrm{Ad}(k_\theta)z_1 = e^{-2i\theta}z_1 = \Theta_2(k_\theta)z_1$ and $k_\theta \cdot z_2 = \mathrm{Ad}(k_\theta)z_2 = e^{2i\theta}z_2 = \Theta_{-2}(k_\theta)z_2$. Further, if we let $w = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \in O(2,\mathbb{R})$, then $w \cdot z_1 = \mathrm{Ad}(w)z_1 = z_2$ and $w \cdot z_2 = \mathrm{Ad}(w)z_2 = z_1$. Moreover, it is obvious that the adjoint action by elements of $Z(\mathbb{R})$ is trivial. Thus, we know explicitly the action of $K = O(2,\mathbb{R})Z(\mathbb{R})$ on $\mathfrak{g}/\mathfrak{k}$.

For $G = GL(2,\mathbb{C})$, we will work with the following basis of $\mathfrak{g}/\mathfrak{k}$:

$$w_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \qquad w_3 = \begin{pmatrix} & i \\ -i & \end{pmatrix}. \tag{2.2.3}$$

In both cases, the action of K on $\mathfrak{g}/\mathfrak{k}$ induces an action of K on $\bigwedge^q \mathfrak{g}/\mathfrak{k}$. Let (π, V) be a $(\mathfrak{g}, K)$ module. Let $C^q(\mathfrak{g}, K; V) = \mathrm{Hom}_K(\bigwedge^q \mathfrak{g}/\mathfrak{k}, V)$, where K acts on $\bigwedge^q \mathfrak{g}/\mathfrak{k}$ as described above. Let us define cochain maps $d : C^q(\mathfrak{g},K;V) \longrightarrow C^{q+1}(\mathfrak{g},K;V)$ as

$$\begin{aligned} d\eta\left(X_0 \wedge X_1 \wedge \cdots \wedge X_q\right) &= \sum_{i=0}^{q} (-1)^i X_i \cdot \eta\left(X_0 \wedge \cdots \hat{X}_i \cdots \wedge X_q\right) \\ &\quad + \sum_{0\le i<j\le q} (-1)^{i+j} \eta\left([X_i, X_j] \wedge X_0 \wedge \cdots \hat{X}_i \cdots \hat{X}_j \cdots \wedge X_q\right). \end{aligned} \tag{2.2.4}$$

for $\eta \in C^q(\mathfrak{g},K;V)$, where $\hat{}$ over an argument means it should be excluded. It can be verified that the maps d are well defined and further that $d^2 = 0$, i.e., the composite map $C^q(\mathfrak{g},K;V) \longrightarrow C^{q+1}(\mathfrak{g},K;V) \longrightarrow C^{q+2}(\mathfrak{g},K;V)$ is zero for every q. So, this indeed defines a cochain complex which we will denote by $C^\bullet(\mathfrak{g},K;V)$.

Now, the q^{th} relative Lie algebra cohomology or $(\mathfrak{g},K)$ cohomology group $H^q(\mathfrak{g},K;V)$ is defined to be the q^{th} cohomology group of this cochain complex,

$$H^q(\mathfrak{g},K;V) = H^q\left(C^\bullet(\mathfrak{g},K;V)\right).$$

Let us try to compute $H^0(\mathfrak{g},K;V)$. By definition, it is the kernel of the map $d : C^0(\mathfrak{g},K;V) \longrightarrow C^1(\mathfrak{g},K;V)$. So, let $\eta \in C^0(\mathfrak{g},K;V)$ be such that $d\eta = 0$. So, for all $X \in \mathfrak{g}/\mathfrak{k}$, $d\eta(X) = 0$ i.e., $X \cdot \eta(1) = 0$. So, η is in fact a $(\mathfrak{g},K)$ module homomorphism. Thus, $H^0(\mathfrak{g},K;V) = \mathrm{Hom}_{(\mathfrak{g},K)}(\mathbb{C},V)$. For $q \ge 0$, we know that $H^q(\mathfrak{g},K;V) = \mathrm{Ext}^q_{(\mathfrak{g},K)}(\mathbb{C},V)$, where $\mathbb{C}$ denotes the trivial $(\mathfrak{g},K)$ module with $\mathfrak{g}$ acting as zero and K acting as identity. This is shown in Chapter I, Section 2 of [4] by taking a special projective resolution of $\mathbb{C}$, which we sketch here.

Let V be a K module. Then we get an action of $\mathfrak{k}$ on V by differentiating. Let $L = \mathfrak{g} \otimes_{\mathfrak{k}} V$ for $q \ge 0$ thinking of $\mathfrak{g}$ as a $\mathfrak{k}$ module by right translation. Then L affords a representation of $\mathfrak{g}$ by left translation on the first component $\mathfrak{g}$ of L. The action of $\mathfrak{k}$ on L obtained by restriction gives the usual tensor product representation $\mathfrak{g} \otimes_{\mathfrak{k}} V$ with $\mathfrak{k}$ acting on $\mathfrak{g}$ by the adjoint action. This is true because

$$Y \cdot (X \otimes v) = [Y,X] \otimes v + X \otimes Y \cdot v = Y.X \otimes v - X.Y \otimes v + X \otimes Y \cdot v = Y.X \otimes v$$

for $Y \in \mathfrak{k}, X \in \mathfrak{g}$ and $v \in V$.

Let $L_q = \mathfrak{g} \otimes_{\mathfrak{k}} \bigwedge^q(\mathfrak{g}/\mathfrak{k})$. Then, by the above construction, we get that L_q is a representation of $\mathfrak{g}$, and hence a $(\mathfrak{g},K)$ module. It is actually known ([4], Chapter I, Section 2.4, Lemma) that L_q's are projective objects in the category of $(\mathfrak{g},K)$ modules. Define for $q > 0$,

$\partial_q : L_q \longrightarrow L_{q-1}$ by

$$\begin{aligned}\partial_q(X \otimes X_1 \wedge \cdots \wedge X_q) = &\sum_{i=1}^{q} (-1)^{i-1} X_i \cdot X \otimes X_1 \wedge \cdots \hat{X}_i \cdots \wedge X_q \\ &+ \sum_{1 \le i < j \le q} (-1)^{i+j} X \otimes [X_i, X_j] \wedge X_1 \wedge \cdots \hat{X}_i \cdots \hat{X}_j \cdots \wedge X_q.\end{aligned} \tag{2.2.5}$$

Moreover, let $\epsilon : L_0 = \mathfrak{g} \longrightarrow \mathbb{C}$ be the augmentation map. Then the chain complex

$$\cdots \longrightarrow L_q \xrightarrow{\partial_q} L_{q-1} \xrightarrow{\partial_{q-1}} \cdots \xrightarrow{\partial_1} L_0 \xrightarrow{\epsilon} \mathbb{C} \longrightarrow 0$$

is exact and hence is the desired projective resolution of $\mathbb{C}$. Now, applying the functor $\mathrm{Hom}_{\mathfrak{g}}(\_, V)$, we get the complex

$$\mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, V) \xrightarrow{\epsilon} \mathrm{Hom}_{\mathfrak{g}}(L_0, V) \xrightarrow{\partial_1} \cdots \mathrm{Hom}_{\mathfrak{g}}(L_{q-1}, V) \xrightarrow{\partial_q} \mathrm{Hom}_{\mathfrak{g}}(L_q, V) \longrightarrow \cdots$$

Now, observing that $\mathrm{Hom}_{\mathfrak{g}}(L_q, V) \simeq \mathrm{Hom}_{\mathfrak{k}}(\bigwedge^q(\mathfrak{g}/\mathfrak{k}), V) \simeq \mathrm{Hom}_K(\bigwedge^q(\mathfrak{g}/\mathfrak{k}), V)$ tells us that the last complex is exactly the complex $C^\bullet(\mathfrak{g}, K; V)$ together with the maps d, that was used in defining the relative lie algebra cohomology. Thus, we get that

$$H^q(\mathfrak{g}, K; V) = \mathrm{Ext}^q_{(\mathfrak{g},K)}(\mathbb{C}, V). \tag{2.2.6}$$

2.3 Cohomological Representations of $GL(2,\mathbb{R})$ and $GL(2,\mathbb{C})$

An irreducible admissible $(\mathfrak{g}, K)$ module V of $GL(2,\mathbb{R})$ or $GL(2,\mathbb{C})$ is said to be of cohomological type or a cohomological representation if $H^\bullet(\mathfrak{g}, K; V \otimes F) \neq 0$ for some finite dimensional irreducible representation F of $GL(2,\mathbb{R})$ or respectively $GL(2,\mathbb{C})$.

We would like to have an explicit classification of all irreducible cohomological representations of $GL(2,\mathbb{R})$ and $GL(2,\mathbb{C})$. We will need a Lemma to help prove this classification result, which we state below. This is due to D. Wigner and is popularly known as Wigner's Lemma.

Lemma 1.
Let U, V be two $(\mathfrak{g}, K)$ modules. Let the center $Z\big(U(\mathfrak{g})\big)$ of the universal enveloping algebra $U(\mathfrak{g})$ act on U and V by the infinitesimal characters χ_U and χ_V. If $\chi_U \neq \chi_V$, then $\mathrm{Ext}^q_{(\mathfrak{g},K)}(U, V) = 0$ *for all $q \geq 0$.*

Proof. The proof we give here is from Chapter 1, Section 4 of [4]. It is trivial to see that $\mathrm{Ext}^0_{(\mathfrak{g},K)}(U,V) = \mathrm{Hom}_{(\mathfrak{g},K)}(U,V) = 0$ if the infinitesimal characters χ_U and χ_V are different. We know that for $q \geq 1$, the group $\mathrm{Ext}^q_{(\mathfrak{g},K)}(U,V)$ is canonically isomorphic to the group of all q length exact sequences with first term V and last term U upto an equivalence under the Baer sum. Since the infinitesimal characters are not equal, we may find an element $z \in Z\big(U(\mathfrak{g})\big)$ such that $\chi_U(z) = 0$ and $\chi_V(z) = 1$. Now, for any $S \in \mathrm{Ext}^q_{(\mathfrak{g},K)}(U,V)$ given by an exact sequence:

$$S \equiv \quad 0 \longrightarrow V \longrightarrow X_q \longrightarrow X_{q-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow U \longrightarrow 0,$$

the element z defines an endomorphism of S given by the action of z on each of the X_i's, U and V. So, the exact sequence $\chi_V(z)\cdot S$, which is the pushout of S by the map $V \xrightarrow{z} V$ is equivalent to the pullback $S\cdot\chi_U(z)$ of S by the map $U \xrightarrow{z} U$. Since z was chosen such that $\chi_U(z) = 0, \chi_V(z) = 1$, we get that $1.S \simeq S.0$, which implies that $S \simeq 0$. Thus, we have shown that $\mathrm{Ext}^q_{(\mathfrak{g},K)}(U,V) = 0$. $\square$

Let $d = (h,a,\epsilon)$ or $d = (h_1,a_1,h_2,a_2)$ and let $(r[d],R[d])$ denote the finite dimensional irreducible representation of $GL(2,\mathbb{R})$ or $GL(2,\mathbb{C})$ as defined in Section 2.1. The following theorem determines all the irreducible admissible representations of $GL(2,\mathbb{R})$ and respectively $GL(2,\mathbb{C})$ that are of cohomological type.

Theorem 1.
Let (π,V) be an irreducible $(\mathfrak{g},K)$ module for $GL(2,\mathbb{R})$ or respectively $GL(2,\mathbb{C})$ such that $H^\bullet(\mathfrak{g},K;V\otimes R[d]^\vee) \neq 0$. Then, $\pi \simeq r[d], r[d']$ or $\pi[d]$ for $GL(2,\mathbb{R})$ and $\pi \simeq r[d]$ or $\pi[d]$ for $GL(2,\mathbb{C})$.

Proof. We already know from Section 2.2 that $H^q(\mathfrak{g},K;V\otimes R[d]^\vee) = \mathrm{Ext}^q_{(\mathfrak{g},K)}(\mathbb{C}, V\otimes R[d]^\vee)$. Since $R[d]$ is finite dimensional, $V\otimes R[d]^\vee \simeq \mathrm{Hom}\big(R[d],V\big)$. Further, since $\mathrm{Hom}(X,\_)$ is right adjoint to $\_\otimes X$, we get that

$$\begin{aligned} H^q(\mathfrak{g},K;V\otimes R[d]^\vee) \simeq \mathrm{Ext}^q_{(\mathfrak{g},K)}\big(\mathbb{C},\mathrm{Hom}(R[d],V)\big) &\simeq \mathrm{Ext}^q_{(\mathfrak{g},K)}\big(\mathbb{C}\otimes R[d],V\big) \\ &\simeq \mathrm{Ext}^q_{(\mathfrak{g},K)}\big(R[d],V\big). \end{aligned} \tag{2.3.1}$$

Thus, by Wigner's Lemma, we get that for this cohomology to be non-zero, we want the infinitesimal characters $\chi_{R[d]}$ and χ_V to be equal, which tells us that Z and the Casimir element Δ act by the scalars a and $\frac{h}{2}\left(1-\frac{h}{2}\right)$ respectively. This proves the theorem since $r[d], r[d']$ and $\pi[d]$, and respectively $r[d]$ and $\pi[d]$ are the only irreducible admissible

$(\mathfrak{g}, K)$ modules for $GL(2,\mathbb{R})$ and respectively $GL(2,\mathbb{C})$ with the desired action of $Z\big(U(\mathfrak{g})\big)$. $\square$

It may be noted that the only infinite dimensional cohomological representations for $GL(2,\mathbb{R})$ are the discrete series representations.

Next, we state a well known proposition (Proposition I.4, [12]) that describes the cohomology groups $H^q(\mathfrak{g}, K; V \otimes R[d]^\vee)$ for different $(\mathfrak{g}, K)$ modules (π, V). We note by Theorem 1 that $H^q(\mathfrak{g}, K; V \otimes R[d]^\vee) = 0$ for $\pi \not\simeq r[d], r[d']$ or $\pi[d]$ for $GL(2,\mathbb{R})$ and $\pi \not\simeq r[d]$ or $\pi[d]$ for $GL(2,\mathbb{C})$.

Let (π, V) be a $(\mathfrak{g}, K)$ module for $GL(2,\mathbb{R})$ or $GL(2,\mathbb{C})$. In the following proposition, we will denote $C^n(\mathfrak{g}, K; V \otimes R[d]^\vee)$ simply by C^n and $H^n(\mathfrak{g}, K; V \otimes R[d]^\vee)$ by H^n for $n \geq 0$, exactly as in [12].

Proposition 1.

1. *Let $G = GL(2,\mathbb{R})$.*
   - (a) *If π is not isomorphic to $r[d]$, $r[d']$ or $\pi[d]$, then $H^n = \{0\}$ for all n.*
   - (b) *If $\pi \simeq r[d]$, $H^0 \simeq \mathbb{C}$ and $H^n = \{0\}$ for all $n \neq 0$.*
   - (c) *If $\pi \simeq r[d']$, $H^2 \simeq \mathbb{C}$ and $H^n = \{0\}$ for all $n \neq 2$.*
   - (d) *If $\pi \simeq \pi[d]$, $C^1 = H^1 \simeq \mathbb{C}$ and $C^n = H^n = \{0\}$ for all $n \neq 1$.*
2. *Let $G = GL(2,\mathbb{C})$.*
   - (a) *If π is not isomorphic to $r[d]$ or $\pi[d]$, then $H^n = \{0\}$ for all n.*
   - (b) *If $\pi \simeq r[d]$, $H^0 \simeq H^3 \simeq \mathbb{C}$ and $H^n = \{0\}$ for all $n \neq 0, 3$.*
   - (c) *If $\pi \simeq \pi[d]$, $C^1 = H^1 \simeq \mathbb{C}$, $C^2 = H^2 \simeq \mathbb{C}$ and $C^n = H^n = \{0\}$ for all $n \neq 1, 2$.*

Chapter 3

Cohomology of $GL(2)$

3.1 Basics

Let F be a number field. Let $\mathbb{A}$ denote its ring of adeles, $\mathbb{A}_f$ the ring of finite adeles, $\mathbb{A}^\times$ the group of ideles. Further, for all places v of F, let F_v denote the completion of F at v. Let S_1, S_2 denote the set of all real and respectively complex places of F. Let $S = S_1 \cup S_2$ denote the set of all archimedean places of F and $\bar{S}$ denote the set of all embeddings of F in $\mathbb{C}$. Let us fix a section from S to $\bar{S}$ so that v represents an embedding corresponding to the place v. Let $\bar{v}$ represent the conjugate of the embedding v for $v \in S_2$.

For any algebraic group H defined over F, let $H_F = H(F)$, $H_\mathbb{A} = H(\mathbb{A})$, $H_f = H(\mathbb{A}_f)$, $H_v = H(F_v)$ for all places v of F and $H_\infty = \prod_{v\in S} H_v$. Let $G = GL(2)/F$, Z its center and B the subgroup of upper triangular matrices. Let $\mathfrak{g}_v$ denote the lie algebra of G_v for every $v \in S$ and $\mathfrak{g}_\infty = \bigoplus_{v\in S} \mathfrak{g}_v$. Let $K_v = O(2,\mathbb{R})Z(\mathbb{R})$ for a real place v and $K_v = U(2,\mathbb{C})Z(\mathbb{C})$ for a complex place v. Let $K_\infty = \prod_{v\in S} K_v$. Let $\mathfrak{k}_v$ denote the Lie algebra of K_v and $\mathfrak{k}_\infty = \bigoplus_{v\in S} \mathfrak{k}_v$. Let us denote the Hecke algebra of G by $\mathscr{H}$, and let $\mathscr{H}_F, \mathscr{H}_\mathbb{A}, \mathscr{H}_f, \mathscr{H}_v$ and $\mathscr{H}_\infty$ denote the respective Hecke algebras of $G_F, G_\mathbb{A}, G_f, G_v$ and G_∞. Now, given any open compact subgroup $K_f \subseteq G_f$, we attach a locally symmetric space

$$S^G_{K_f} = G(F)\backslash G(\mathbb{A})/K_\infty K_f. \tag{3.1.1}$$

Let $d = \left((h_v)_{v\in\bar{S}}, (a_v)_{v\in\bar{S}}, (\epsilon_v)_{v\in S_1}\right)$ be such that $h_v, a_v \in \mathbb{Z}$, $h_v \geq 2$, $a_v \equiv h_v \pmod 2$ for all v and $\epsilon_v \in \{0,1\}$ for all $v \in S_1$. Let $d_v = (h_v, a_v, \epsilon_v)$ if $v \in S_1$ and $(h_v, a_v, h_{\bar{v}}, a_{\bar{v}})$ if $v \in S_2$. Then for each $v \in S$, $r[d_v]$ denotes an irreducible finite dimensional representation as in

Section 2.1. Consider the following representation of G_∞, denoted by $r[d]$.

$$r[d] = \bigotimes_{v \in S} r[d_v]. \tag{3.1.2}$$

It is an irreducible finite dimensional representation of G_∞ since each of the $r[d_v]$ are irreducible and finite dimensional. We will denote the space of this representation by $R[d]$. The quantity d is said to be a dominant integral weight corresponding to the representation $(r[d], R[d])$. The contragredient $(r[d]^\vee, R[d]^\vee)$ is isomorphic to $\bigotimes_{v\in S} r[\check{d}_v]$. We will work with the contragredient representation for convenience.

The representation $r[d]^\vee$ of G_∞ defines a sheaf $\mathscr{F}_d$ of vector spaces on the space $S^G_{K_f}$ by

$$\mathscr{F}_d(U) = \left\{ s : p^{-1}(U) \longrightarrow R[d]^\vee \quad \Big| \quad s(gx) = r[d]^\vee(g)s(x) \quad \forall g \in G_F, x \in G_{\mathbb{A}} \right\}, \tag{3.1.3}$$

where $p : G_{\mathbb{A}}/K_\infty K_f \longrightarrow S^G_{K_f}$ is the projection map. The objects of interest to us are the sheaf cohomology groups $H^\bullet(S^G_{K_f}, \mathscr{F}_d)$. The sheaf cohomology groups $H^\bullet(S^G_{K_f}, \_)$ are by definition, the right derived functors of the left exact functor $H^0(S^G_{K_f}, \_)$. This is called the global sections functor and is defined as

$$H^0(X, \_)(\mathscr{G}) = H^0(X, \mathscr{G}) = \mathscr{G}(X)$$

for any topological space X and a sheaf $\mathscr{G}$ on X. The cohomology of $S^G_{K_f}$ with coefficients in the sheaf $\mathscr{F}_d$, denoted $H^\bullet(S^G_{K_f}, \mathscr{F}_d)$ are calculated by taking any injective resolution, say $0 \longrightarrow \mathscr{F}_d \xrightarrow{\epsilon} \mathscr{I}^0 \xrightarrow{d_0} \mathscr{I}^1 \xrightarrow{d_1} \cdots$ of the sheaf $\mathscr{F}_d$, applying the global sections functor, dropping the first term of the resulting cochain complex and computing cohomology. We get the following cochain complex after applying the global sections functor to the injective resolution $0 \longrightarrow \mathscr{F}_d \xrightarrow{\epsilon} \mathscr{I}^\bullet$ mentioned above.

$$0 \longrightarrow H^0(S^G_{K_f}, \mathscr{F}_d) \xrightarrow{\epsilon} H^0(S^G_{K_f}, \mathscr{I}^0) \xrightarrow{d_0} \cdots \xrightarrow{d_{q-1}} H^0(S^G_{K_f}, \mathscr{I}^q) \xrightarrow{d_q} H^0(S^G_{K_f}, \mathscr{I}^{q+1}) \cdots$$

After dropping the first term, we get

$$H^0(S^G_{K_f}, \mathscr{I}^0) \xrightarrow{d_0} H^0(S^G_{K_f}, \mathscr{I}^1) \xrightarrow{d_1} \cdots \xrightarrow{d_{q-1}} H^0(S^G_{K_f}, \mathscr{I}^q) \xrightarrow{d_q} H^0(S^G_{K_f}, \mathscr{I}^{q+1}) \cdots$$

So, the q^{th} cohomology group of $S^G_{K_f}$ with coefficients in $\mathscr{F}_d$ is given by

$$H^q(S^G_{K_f}, \mathscr{F}_d) = H^q(H^0(S^G_{K_f}, \mathscr{I}^\bullet)) = \ker d_q / \operatorname{Im} d_{q-1}.$$

Though the definition a priori involves a choice of an injective resolution of $\mathscr{F}_d$, it can be shown $H^q(S^G_{K_f}, \mathscr{F}_d)$ is independent of this choice ([13], Chapter III, Theorem 1.1A).

3.2 Relation to Lie algebra cohomology

It is a basic proposition in the theory of Sheaf Cohomology that we may use any acyclic resolution of $\mathscr{F}_d$ to compute the cohomology groups, instead of the injective resolution $0 \longrightarrow \mathscr{F}_d \xrightarrow{\epsilon} \mathscr{I}^\bullet$ used in the definition in previous section. In particular, we may use the de-Rham complex. It is the following resolution [9] of the constant sheaf $\mathbb{R}_M$ on a manifold M, using the sheaves Ω^k_M of smooth k-forms, that assign to any open set U of M, the set $\Omega^k_M(U)$ of smooth k-forms on U.

$$0 \longrightarrow \mathbb{R}_M \xrightarrow{i} \Omega^0_M \xrightarrow{d_0} \Omega^1_M \xrightarrow{d_1} \cdots \tag{3.2.1}$$

Tensoring the above resolution by a sheaf $\mathscr{G}$ gives an acyclic resolution of $\mathscr{G}$. If we use such a resolution of the sheaf $\mathscr{F}_d$ defined in Equation 3.1.3, the resulting cochain complex, obtained after applying the global sections functor and discarding the first term, can be reinterpreted in terms of the complex computing relative Lie algebra cohomologies. This yields us the following isomorphism

$$H^\bullet(S^G_{K_f}, \mathscr{F}_d) \simeq H^\bullet(\mathfrak{g}_\infty, K_\infty; C^\infty(G_F\backslash G_\mathbb{A})^{K_f} \otimes R[d]^\vee), \tag{3.2.2}$$

where $C^\infty(G_F\backslash G_\mathbb{A})$ represents the space of all smooth functions on $G_\mathbb{A}$ that are left invariant under G_F and $C^\infty(G_F\backslash G_\mathbb{A})^{K_f}$ represents the subspace of functions that are right invariant under K_f. The space $C^\infty(G_F\backslash G_\mathbb{A})$ contains the space $C^\infty_{\text{cusp}}(G_F\backslash G_\mathbb{A})$ of all cusp forms on $G_\mathbb{A}$. The inclusion $C^\infty_{\text{cusp}}(G_F\backslash G_\mathbb{A}) \hookrightarrow C^\infty(G_F\backslash G_\mathbb{A})$ induces an injection in cohomology [3]. This defines cuspidal cohomology $H^\bullet_{\text{cusp}}(S^G_{K_f}, \mathscr{F}_d)$. We have the following commutative diagram

$$\begin{tikzcd} H^\bullet(S^G_{K_f},\mathscr{F}_d) \arrow[r, "\simeq"] & H^\bullet(\mathfrak{g}_\infty,K_\infty;C^\infty(G_F\backslash G_{\mathbb{A}})^{K_f}\otimes R[d]^\vee) \\ H^\bullet_{\mathrm{cusp}}(S^G_{K_f},\mathscr{F}_d) \arrow[r, "\simeq"] \arrow[u, hook] & H^\bullet(\mathfrak{g}_\infty,K_\infty;C^\infty_{\mathrm{cusp}}(G_F\backslash G_{\mathbb{A}})^{K_f}\otimes R[d]^\vee) \arrow[u, hook] \end{tikzcd}$$

We know by multiplicity one theorem ([7], Chapter III, Theorem 3.3.6) for the action of $G_{\mathbb{A}}$ on the space of cusp forms $C^\infty_{\mathrm{cusp}}(G_F\backslash G_{\mathbb{A}})$, that any cuspidal automorphic representation of $G_{\mathbb{A}}$ appears in the direct sum decomposition of $C^\infty_{\mathrm{cusp}}(G_F\backslash G_{\mathbb{A}})$ with multiplicity one. Using this direct sum decomposition, we get that

$$H^\bullet_{\mathrm{cusp}}(S^G_{K_f},\mathscr{F}_d) \quad \simeq \bigoplus_{\pi=\bigotimes'_v \pi_v} H^\bullet(\mathfrak{g}_\infty,K_\infty;\pi_\infty\otimes R[d]^\vee)\otimes\pi_f, \tag{3.2.3}$$

where the direct sum is over all irreducible cuspidal automorphic representations π, with π_∞ and π_f denoting the respective spaces of the representations $\pi_\infty = \bigotimes_{v\in S}\pi_v$ of G_∞ and $\pi_f = \bigotimes'_{v\notin S}\pi_v$ of G_f. We say that an irreducible cuspidal automorphic representation π contributes to the cuspidal cohomology $H^\bullet(S^G_{K_f},\mathscr{F}_d)$ if $H^\bullet(\mathfrak{g}_\infty,K_\infty;\pi_\infty\otimes R[d]^\vee)\neq 0$.

3.3 Inner Cohomology

Let X be a locally compact topological space and $\mathscr{F}$ be a sheaf of abelian groups on X. Let U be an open subset of X and s be a section of $\mathscr{F}$ over U i.e., $s\in\mathscr{F}(U)$. We denote the set of all points $x\in U$ such that $s|_x\neq 0$ in the stalk $\mathscr{F}_x$ by $\mathrm{Supp}(s)=|s|$. It is clear that $|s|$ is a closed subset of U. We may work with the functor $H^0_c(X,\_)$ of taking global sections with compact support, which is again left exact. Now, cohomology with compact supports is defined to be the right derived functors of $H^0_c(X,\_)$. Clearly $H^0_c(X,\mathscr{G})$ is a subgroup of $H^0(X,\mathscr{G})$ for every sheaf $\mathscr{G}$ on X. Thus, we get a map from $H^q_c(X,\mathscr{F})\longrightarrow H^q(X,\mathscr{F})$ for every $q\geq 0$. This map is in general not injective and the image of this map is called the inner cohomology and denoted $H^q_!(X,\mathscr{F})$.

$$H^\bullet_!(X,\mathscr{F}) \quad = \quad \mathrm{Im}\left(H^\bullet_c(X,\mathscr{F})\longrightarrow H^\bullet(X,\mathscr{F})\right). \tag{3.3.1}$$

We are particularly interested in the inner cohomology groups $H^\bullet_!(S^G_{K_f},\mathscr{F}_d)$ which by definition are $\operatorname{Im}\left(H_c^\bullet(S^G_{K_f},\mathscr{F}_d) \longrightarrow H^\bullet(S^G_{K_f},\mathscr{F}_d)\right)$. It is known ([12], Section I.5, Theorem I.5.1 (2)) that inner cohomology contains cuspidal cohomology.

$$H^q_{\text{cusp}}(S^G_{K_f},\mathscr{F}_d) \subseteq H^q_!(S^G_{K_f},\mathscr{F}_d) \qquad \forall q \geq 0. \tag{3.3.2}$$

Futher we have the following commutative diagram which tells us that the inner cohomology $H^\bullet_!(S^G_{K_f},\mathscr{F}_d)$ is isomorphic to the image $\tilde{H}^\bullet$ of the map $H^\bullet(\mathfrak{g}_\infty, K_\infty; C_c^\infty(G_F\backslash G_{\mathbb{A}})^{K_f} \otimes R[d]^\vee) \longrightarrow H^\bullet(\mathfrak{g}_\infty, K_\infty; C^\infty(G_F\backslash G_{\mathbb{A}})^{K_f} \otimes R[d]^\vee)$ induced by the inclusion of the space $C_c^\infty(G_F\backslash G_{\mathbb{A}})$ of smooth functions on $G_F\backslash G_{\mathbb{A}}$ that are compactly supported in $Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}$, inside $C^\infty(G_F\backslash G_{\mathbb{A}})$.

$$\begin{tikzcd}
H^\bullet(S^G_{K_f},\mathscr{F}_d) \arrow[r, "\simeq"] & H^\bullet(\mathfrak{g}_\infty, K_\infty; C^\infty(G_F\backslash G_{\mathbb{A}})^{K_f} \otimes R[d]^\vee) \\
H_c^\bullet(S^G_{K_f},\mathscr{F}_d) \arrow[u] \arrow[r, "\simeq"] & H^\bullet(\mathfrak{g}_\infty, K_\infty; C_c^\infty(G_F\backslash G_{\mathbb{A}})^{K_f} \otimes R[d]^\vee) \arrow[u]
\end{tikzcd}$$

In the above diagram, the vertical arrows are in general not injective. We have a decomposition of $H^\bullet_!(S^G_{K_f},\mathscr{F}_d)$ similar to 3.2.3.

$$H^\bullet_!(S^G_{K_f},\mathscr{F}_d) \quad \simeq \bigoplus_{\pi\in \mathrm{Coh}(G,K_f,d)} H^\bullet_!(S^G_{K_f},\mathscr{F}_d)(\pi). \tag{3.3.3}$$

The set of isomorphism classes of representations π which occur in the above decomposition is called the spectrum and denoted $\mathrm{Coh}(G,K_f,d)$ [10]. All cuspidal automorphic representations that contribute to cuspidal cohomology belong to the spectrum $\mathrm{Coh}(G,K_f,d)$ because of 3.3.2. Hence we will be concerned with those $\pi \in \mathrm{Coh}(G,K_f,d)$ that are not cuspidal. A theorem of Waldspurger [12] gives us a class of representations $A_1(\omega,d)$ that contains the spectrum $\mathrm{Coh}(G,K_f,d)$. We will shortly define this class $A_1(\omega,d)$ of representations. So, our task of finding the non-cuspidal part of the spectrum is reduced to finding out which of the representations in the class $A_1(\omega,d)$ are non-cuspidal and belong to $\mathrm{Coh}(G,K_f,d)$, in other words, contribute to the inner cohomology $H^\bullet_!(S^G_{K_f},\mathscr{F}_d)$.

We will introduce the class $A_1(\omega,d)$ and state the relevant theorem of Waldspurger in the next few pages. We follow [12] and give a sketch of the proof of Theorem 3. The full proof can be found in [12].

Let ω be a continuous quasi-character of $F^\times\backslash\mathbb{A}^\times$ such that ω_v is equal to the central quasi-character of $r[d_v]$, where d is as in Section 3.1 and ω_v denotes the local character of ω at the place v, i.e.,

$$\begin{aligned} \omega_v(z) &= z^{a_v}, \qquad \text{if } v \in S_1, \\ \omega_v(z) &= z^{a_v}\overline{z}^{a_{\overline{v}}}, \qquad \text{if } v \in S_2. \end{aligned}$$

for $z \in F_v^{\times}$. There exists such a ω if and only if the following conditions hold ([12], Section I.5).

- If $S_1 \neq \emptyset$, $a_v = a$ for all $v \in S$.
- If $S_1 = \emptyset$, $a_{\sigma\circ v} + a_{\sigma\circ\overline{v}}$ is constant for all $\sigma \in \mathrm{Aut}(\mathbb{C}), v \in \mathrm{S}$.

We assume that the above conditions are satisfied. Let $\mathscr{A}(\omega)$ be the space of automorphic forms on $G_F\backslash G_{\mathbb{A}}$, with central character ω, and let $\mathscr{A}_0(\omega)$ be the subspace of cuspidal automorphic forms. The Hecke algebra $\mathscr{H}_{\mathbb{A}}$ acts on the two spaces by right translation. Let $\mathscr{A}_1(\omega)$ be the direct sum of $\mathscr{A}_0(\omega)$ and the one dimensional invariant subspaces of $\mathscr{A}(\omega)$. Let $A_0(\omega)$ and $A_1(\omega)$ be respectively the sets of irreducible subspaces of $\mathscr{A}_0(\omega)$ and $\mathscr{A}_1(\omega)$. We can think of $A_0(\omega)$ and $A_1(\omega)$ as sets of irreducible non-isomorphic automorphic representations, as a result of multiplicity one theorem.

Let $(\pi, V) \in A_1(\omega)$. Then it is a simple admissible module or alternatively called as an irreducible admissible representation of $\mathscr{H}_{\mathbb{A}}$. We have the following well known theorem ([7], Chapter III, Theorem 3.3.3) that gives us an easier way to look at such representations.

Theorem 2.
Tensor Product Theorem: Let (π, V) be an irreducible admissible representation of $\mathscr{H}_{\mathbb{A}}$. Then, for each place v of F, we have an irreducible admissible representation (π_v, V_v) of $\mathscr{H}_v$, and V_v contains a non-zero $GL(2,\mathfrak{o}_v)$ fixed vector v_v^0 for almost all non-Archimedean places v, such that π is isomorphic to the restricted tensor product of the representations π_v with respect to the v_v^0. It is denoted $\pi \simeq \bigotimes' \pi_v$.

Thus $(\pi, V) \in A_1(\omega)$ can be written as a restricted tensor product of representations π_v. For d as in Section 3.1, let $A_0(\omega, d)$ and $A_1(\omega, d)$ be the set of all representations (π, V) in $A_0(\omega)$ and respectively in $A_1(\omega)$ such that $\pi_v \simeq r[d_v], r[d_v']$ or $\pi[d_v]$ for all $v \in S_1$ and $\pi_v \simeq r[d_v]$ or $\pi[d_v]$ for all $v \in S_2$. It should be noted that $A_0(\omega, d) = A_1(\omega, d)$ unless $h_v = 2$ for all $v \in \bar{S}$.

Let us denote by $\mathscr{C}(\omega, K_f)$ the space of smooth functions ϕ on $G_F\backslash G_{\mathbb{A}}/K_f$ such that $\phi(zg) = \omega(z)\phi(g)$ for all $z \in Z_{\mathbb{A}}, g \in G_{\mathbb{A}}$. Let $\mathscr{C}_B(\omega, K_f)$ be the space of smooth functions on $B_F\backslash G_{\mathbb{A}}/K_f$ satisfying the same property, where B denotes the group of upper triangular matrices. Clearly $\mathscr{C}(\omega,K_f)\subseteq\mathscr{C}_B(\omega,K_f)$. Moreover, the sub-algebra $\mathscr{H}_{\mathbb{A}}^{K_f}$ of K_f bi-invariant elements in $\mathscr{H}_{\mathbb{A}}$ acts on the spaces $\mathscr{C}(\omega,K_f)$ and $\mathscr{C}_B(\omega,K_f)$.

Let $\mathscr{C}'_B(\omega,K_f)$ be the quotient of $\mathscr{C}_B(\omega,K_f)$ by the subspace of elements that vanish on some Siegel domain for $G_F\backslash G_{\mathbb{A}}/K_\infty K_f$. We will denote $H^q(\mathfrak{g}_\infty,K_\infty;\mathscr{C}_c(\omega,K_f)\otimes R[d]^\vee)$ by H_c^q, $H^q(\mathfrak{g}_\infty,K_\infty;\mathscr{C}(\omega,K_f)\otimes R[d]^\vee)$ by H^q, $H^q(\mathfrak{g}_\infty,K_\infty;\mathscr{C}_B(\omega,K_f)\otimes R[d]^\vee)$ by H_B^q and $H^q(\mathfrak{g}_\infty,K_\infty;\mathscr{C}'_B(\omega,K_f)\otimes R[d]^\vee)$ by $H_{B'}^q$. Then, it is known ([14], Lemma 2.3) that the following sequence

$$0\longrightarrow\mathscr{C}_c(\omega,K_f)\longrightarrow\mathscr{C}(\omega,K_f)\longrightarrow\mathscr{C}'_B(\omega,K_f)\longrightarrow 0 \tag{3.3.4}$$

is exact and gives the following long exact sequence

$$H_c^0\longrightarrow H^0\longrightarrow H_{B'}^0\longrightarrow\cdots\longrightarrow H_c^q\longrightarrow H^q\longrightarrow H_{B'}^q\longrightarrow H_c^{q+1}\longrightarrow\cdots \tag{3.3.5}$$

in cohomology. The maps $H^q\longrightarrow H_{B'}^q$ are called restriction maps. Thus, the inner cohomology $\tilde{H}^q$ can also be defined as kernel of this restriction map. It is also known that this is the same as $\ker(H^q\longrightarrow H_B^q)$.

Now, let $(\pi,E)\in A_1(\omega)$. Let $\pi=\bigotimes'\pi_v$ and let E_v represent the space of π_v for every place v. Let $E_f=\bigotimes'_{v\notin S}E_v$. Let E^{K_f} and $E_f^{K_f}$ denote respectively the subspaces of elements that are fixed by K_f. The inclusion $E^{K_f}\hookrightarrow\mathscr{C}(\omega,K_f)$ defines by functoriality a map from $H^q(\mathfrak{g}_\infty,K_\infty;E^{K_f}\otimes R[d]^\vee)\longrightarrow H^q$. We let $H^q(E)$ denote the image of this map. We are now ready to state the theorem of Waldspurger [12].

Theorem 3.

1. *There exists a set* $\Xi\subseteq A_1(\omega,d)$ *such that*
   - *if* $E\in\Xi$, *then* $E_f^{K_f}\neq 0$ *and the* $\mathscr{H}_f^{K_f}$ *module* $H^q(E)$ *is isomorphic to* $m\cdot E_f^{K_f}=E_f^{K_f}\oplus E_f^{K_f}\oplus\cdots\oplus E_f^{K_f}$ *for some integer* $m\geq 1$.
   - *there is a direct sum decomposition*

$$\tilde{H}=\bigoplus_{E\in\Xi}H^q(E). \tag{3.3.6}$$

2. *If* $E\in A_0(\omega,d)$ *and* $E_f^{K_f}\neq 0$, *then* $E\in\Xi$ *and* $H^q(E)\simeq E_f^{K_f}$.

Proof. We give only a sketch of the proof. It involves looking at a similar direct sum decomposition of the square integrable cohomology $H^q_{(2)} = H^q(\mathfrak{g}_\infty, K_\infty; \mathscr{C}_2(\omega, K_f) \otimes R[d]^\vee)$, where $\mathscr{C}_2(\omega, K_f)$ denotes the subspace consisting of those $\phi \in \mathscr{C}(\omega, K_f)$ such that for any finite sequence $X_1, X_2, \cdots X_i$ in $\mathfrak{g}_\infty$, $X_1 \cdot X_2 \cdot \ldots \cdot X_i \cdot \phi$ is square integrable on $G_F Z_{\mathbb{A}} \backslash G_{\mathbb{A}}$. Since $\mathscr{C}_c(\omega, K_f) \subseteq \mathscr{C}_2(\omega, K_f)$, this decomposition directly provides a direct sum decomposition similar to 3.3.6. Further, writing $E = \bigotimes' E_v$, $E_\infty = \bigotimes_{v\in S} E_v$ and $E_f = \bigotimes'_{v\in S} E_v$, we get that $E^{K_f} = E_\infty \otimes E_f^{K_f}$ and

$$H^q(\mathfrak{g}_\infty, K_\infty; E^{K_f} \otimes R[d]^\vee) = E_f^{K_f} \otimes H^q(\mathfrak{g}_\infty, K_\infty; E_\infty \otimes R[d]^\vee).$$

Using Künneth's theorem, we get that

$$H^q(\mathfrak{g}_\infty, K_\infty; E^{K_f} \otimes R[d]^\vee) = E_f^{K_f} \otimes \left(\bigoplus_{\sum_{v\in S} q_v = q} \otimes_v H^{q_v}(\mathfrak{g}_v, K_v; E_v \otimes R[d]^\vee) \right). \tag{3.3.7}$$

Now, Proposition 1 tells us that if $\pi \notin A_1(\omega, d)$, the right side of the above equation vanishes and hence $H^q(\mathfrak{g}_\infty, K_\infty; E^{K_f} \otimes R[d]^\vee) = 0$, which implies $H^q(E) = 0$. This justifies $\Xi \subseteq A_1(\omega, d)$ and the isomorphism $H^q(E) \simeq m \cdot E_f^{K_f}$.

If (π, E) is a cuspidal automorphic representation, the map $E^{K_f} \longrightarrow \mathscr{C}(\omega, K_f) \longrightarrow \mathscr{C}_B(\omega, K_f)$ actually factors through $\mathscr{C}^0_B(\omega, K_f)$ where $\mathscr{C}^0_B(\omega, K_f)$ denotes the subspace consisting of $\phi \in \mathscr{C}_B(\omega, K_f)$ such that

$$\int_{F\backslash A} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx = 0, \qquad \text{for all } g \in G_{\mathbb{A}}.$$

Hence the map $H^q(E) \longrightarrow H^q \longrightarrow H^q_B$ factors through $H^q(\mathfrak{g}_\infty, K_\infty; \mathscr{C}^0_B(\omega, K_f) \otimes R[d]^\vee)$, which is zero for all $q \geq 0$ by [14]. □

We deduce from the above theorem that the only representations π appearing in the spectrum $\mathrm{Coh}(G, K_f, d)$ are either cuspidal or one-dimensional. Furthermore, all such representations belong to the set $A_1(\omega, d)$. Thus, we can deduce that unless $h_v = 2$ for all $v \in \bar{S}$, where h_v's are as in Section 3.1, the inner cohomology is exactly equal to the cuspidal cohomology. This is because $A_1(\omega, d) = A_0(\omega, d)$ unless $h_v = 2$ for all $v \in \bar{S}$. Since we are only interested in the non-cuspidal part of the spectrum, we will assume from here onward that $h_v = 2$ for all $v \in \bar{S}$, i.e., that $r[d]$ is a one-dimensional representation.

Chapter 4

Inner cohomology of $GL(2)$ over a totally real field

4.1 A summary of known results

This section is a summary of results in [8], concerning the inner cohomology $\tilde{H}^\bullet = \mathrm{Im}(H_c^\bullet \longrightarrow H^\bullet) = \ker(H^\bullet \longrightarrow H_B^\bullet)$ of Section 3.3 for a totally real number field F with degree of the extension $[F:\mathbb{Q}] = n$. The dimension of the underlying manifold $S_{K_f}^G$ is $2n$. We state the theorem of Poincaré duality for the manifold $S_{K_f}^G$.

Theorem 4.
Poincaré Duality: Let $\mathscr{F}$ be a sheaf defined on $S_{K_f}^G$ and let $\mathscr{F}^\vee$ denote its dual sheaf. Then, there exists a non-degenerate bilinear pairing

$$H^q(S_{K_f}^G, \mathscr{F}) \times H_c^{2n-q}(S_{K_f}^G, \mathscr{F}^\vee) \quad \longrightarrow \quad \mathbb{C} \tag{4.1.1}$$

for any integer q, which tells us that $H_c^{2n-q}(S_{K_f}^G, \mathscr{F}^\vee) \simeq (H^q(S_{K_f}^G, \mathscr{F}))^*$. In particular if they are finite dimensional, $\dim H^q(S_{K_f}^G, \mathscr{F}) = \dim H_c^{2n-q}(S_{K_f}^G, \mathscr{F}^\vee)$.

Since the cohomology groups for negative degrees are defined to be zero, this in particular tells us that the cohomology groups $H^q(S_{K_f}^G, \mathscr{F})$ are zero for $q \geq 2n$.

There is a subspace called the Eisenstein cohomology $H_{\mathrm{Eis}}^q(S_{K_f}^G, \mathscr{F}) \subseteq H^q(S_{K_f}^G, \mathscr{F})$ for every $q \geq 0$ such that after the identification 3.2.2, it maps isomorphically onto the image of the map $H^q(\mathfrak{g}_\infty, K_\infty; \mathscr{C}(\omega, K_f) \otimes R[d]^\vee) \longrightarrow H^q(\mathfrak{g}_\infty, K_\infty; \mathscr{C}_B(\omega, K_f) \otimes R[d]^\vee)$. Further for $q > 0$, there is another subspace called the square integrable cohomology

$H^q_{\rm squ}(S^G_{K_f},\mathscr{F}) \subseteq H^q(S^G_{K_f},\mathscr{F})$ such that

$$H^q(S^G_{K_f},\mathscr{F}) \quad = \quad H^q_{\rm Eis}(S^G_{K_f},\mathscr{F}) \quad \oplus \quad H^q_{\rm squ}(S^G_{K_f},\mathscr{F}). \tag{4.1.2}$$

It is known that $H^0(S^G_{K_f},\mathscr{F}) = H^0_{\rm Eis}(S^G_{K_f},\mathscr{F}) = H^0_{\rm squ}(S^G_{K_f},\mathscr{F}) \simeq \mathbb{C}$. Further, using the short exact sequence

$$0 \longrightarrow \mathscr{C}_c(\omega,K_f) \longrightarrow \mathscr{C}(\omega,K_f) \longrightarrow \mathscr{C}'_B(\omega,K_f) \longrightarrow 0$$

of Langlands [14], we get the long exact sequence 3.3.5. Together with 4.1.2, this tells us that for $q>0$, $H^q_{\rm squ}(S^G_{K_f},\mathscr{F})$ is indeed the image of the map $H^q_c \longrightarrow H^q$. That is, $H^q_{\rm squ}(S^G_{K_f},\mathscr{F}) = H^q_!(S^G_{K_f},\mathscr{F})$ for all $q>0$. Further, using the method of Matsushima and Shimura ([8], Chapter III, Section 1), we get a decomposition

$$H^q_{\rm squ}(S^G_{K_f},\mathscr{F}) \quad = \quad H^q_{\rm univ}(S^G_{K_f},\mathscr{F}) \quad \oplus \quad H^q_{\rm cusp}(S^G_{K_f},\mathscr{F}) \tag{4.1.3}$$

of the square integrable cohomology into a direct sum of its universal and cuspidal parts.

Using Poincaré duality and the surjectivity of the restriction map for degrees q from n to $2n-2$, the following result ([8], Chapter III, Section 6, Theorem 6.3) for the dimensions of the different cohomologies is obtained. The dimensions of the cohomologies $H^q(S^G_{K_f},\mathscr{F})$, $H^q_{\rm Eis}(S^G_{K_f},\mathscr{F})$, $H^q_{\rm squ}(S^G_{K_f},\mathscr{F})$, $H^q_{\rm inn}(S^G_{K_f},\mathscr{F})$, $H^q_{\rm univ}(S^G_{K_f},\mathscr{F})$ and $H^q_{\rm cusp}(S^G_{K_f},\mathscr{F})$ are denoted by $b^q, b^q_{\rm Eis}, b^q_{\rm squ}, b^q_{\rm inn}, b^q_{\rm univ}$ and $b^q_{\rm cusp}$ respectively.

Theorem 5.

1. $b^0 = b^0_{Eis} = b^0_{squ} = 1; \quad b^0_{inn} = b^0_{univ} = b^0_{cusp} = 0.$

2. $b^{2n} = b^{2n}_{Eis} = b^{2n}_{squ} = b^{2n}_{inn} = b^{2n}_{univ} = b^{2n}_{cusp} = 0.$

3. *For* $0<q<2n, \quad b^q = b^q_{Eis} + b^q_{squ}, \quad b^q_{squ} = b^q_{inn} = b^q_{univ} + b^q_{cusp}$ *where*

   (a) $b^q_{Eis} = \begin{cases} 0, & \text{if } 0<q<n \\ h\cdot\binom{n-1}{q-n}, & \text{if } n\le q<2n-1 \\ h-1, & \text{if } q=2n-1 \end{cases}$

   (b) $b^q_{univ} = \begin{cases} \binom{n}{q/2} & \text{if } q \text{ is even} \\ 0, & \text{if } q \text{ is odd} \end{cases}$

(c) $b^q_{cusp} = \begin{cases} 0, & \text{if } q \neq n \\ \sum\limits_{k+l=q} h^{k,l}_{cusp}, & \text{if } q = n \end{cases}$

where h denotes the number of cusp classes of Γ *and*

$$h^{k,l}_{cusp} = \sum_{\substack{b \subseteq \{1,2,\ldots,n\} \\ |b| \neq k}} \dim \left[\Gamma^b, (2,2,\ldots,2)\right]_0$$

A cusp class is an orbit of a point in $\mathbb{P}^n(\mathbb{R})$, *whose stabilizer in* Γ *contains a non-trivial unipotent matrix.*

4.2 The case $F = \mathbb{Q}$

The results stated in the previous section are obtained by geometric means. In this section, we wish to reproduce them in the specific case $F = \mathbb{Q}$ by a direct approach using only algebraic and analytical considerations. We make use of Theorem 3. This tells us that the only non-cuspidal representations π in the spectrum $\mathrm{Coh}(G, K_f, d)$ are one dimensional automorphic representations and belong to the set $A_1(\omega, d)$. So, we check which of these representations contribute to inner cohomology.

Let (π, E) be an one dimensional automorphic representation of $G(F)$ with central character ω. So, $E = \mathbb{C}\xi$ where ξ is a continuous quasi-character of $G(F)\backslash G(\mathbb{A})$ which agrees with ω on $Z(\mathbb{A})$. Any such quasi-character factors through the determinant map. So, we may let $\xi = \chi \circ \det$ where $\chi : F^\times \backslash \mathbb{A}^\times \longrightarrow \mathbb{C}^\times$ is a continuous quasi-character of the group of ideles. It is easily checked that the action π of $G(\mathbb{A})$ on the space E by right translation, gives rise to the same character ξ. Let π be the restricted tensor product over all places v of F, of the representations π_v. Then, $\pi_v \simeq \xi_v$ for all v, where ξ_v denotes the restriction of ξ to G_v. Let us denote the only archimedean place of F by ∞. Then, since we are concerned with only those (π, E) in $A_1(\omega, d)$, we get that $\pi_\infty \simeq r[d]$ or $r[d']$, i.e., $\xi_\infty \simeq r[d]$ or $r[d']$. Moreover, for an open compact subgroup $K_f \subseteq G(\mathbb{A}_f)$, $E^{K_f} \neq 0$ if and only if ξ is right invariant under K_f, in which case, $E^{K_f} = E$.

Using Equation 3.3.7 and Proposition 1, we obtain that the cohomology groups $H^q(\mathfrak{g}_\infty, K_\infty; E^{K_f} \otimes R[d]^\vee)$ of all degrees other than $q = 0$ and $q = 2$ are trivial. Further, if $\pi_\infty \simeq r[d]$, then $H^2(\mathfrak{g}_\infty, K_\infty; E^{K_f} \otimes R[d]^\vee) = 0$ and $H^0(\mathfrak{g}_\infty, K_\infty; E^{K_f} \otimes R[d]^\vee) \simeq E_f^{K_f}$ as $\mathfrak{H}_f^{K_f}$ modules and if $\pi_\infty \simeq r[d']$, then $H^0(\mathfrak{g}_\infty, K_\infty; E^{K_f} \otimes R[d]^\vee) = 0$ and $H^2(\mathfrak{g}_\infty, K_\infty; E^{K_f} \otimes R[d]^\vee) \simeq E_f^{K_f}$ as $\mathfrak{H}_f^{K_f}$ modules.

Thus, we need to check if $H^q(E)$ maps to 0 in $H^q(\mathfrak{g}_\infty, K_\infty; \mathscr{C}_B(\omega, K_f)\otimes R[d]^\vee)$, only for degrees $q = 0, 2$. By definition,

$$\begin{aligned} H^0(\mathfrak{g}_\infty, K_\infty; V) &= \ker\Big(C^0\left(\mathfrak{g}_\infty, K_\infty; V\right) \longrightarrow C^1\left(\mathfrak{g}_\infty, K_\infty; V\right)\Big) \\ &\subseteq C^0\left(\mathfrak{g}_\infty, K_\infty; V\right) \end{aligned} \tag{4.2.1}$$

Hence the induced map

$$\begin{aligned} H^0(\mathfrak{g}_\infty, K_\infty; E^{K_f}\otimes R[d]^\vee) &\longrightarrow H^0(\mathfrak{g}_\infty, K_\infty; \mathscr{C}(\omega, K_f)\otimes R[d]^\vee) \\ &\longrightarrow H^0(\mathfrak{g}_\infty, K_\infty; \mathscr{C}_B(\omega, K_f)\otimes R[d]^\vee) \end{aligned} \tag{4.2.2}$$

obtained from the inclusions $E^{K_f} \hookrightarrow \mathscr{C}(\omega, K_f) \hookrightarrow \mathscr{C}_B(\omega, K_f)$ is injective. Thus, if $\pi_\infty \simeq r[d]$, $H^0(E) \neq 0$ does not map to zero in $H^0(\mathfrak{g}_\infty, K_\infty; \mathscr{C}_B(\omega, K_f)\otimes R[d]^\vee)$, i.e, π doesn't contribute to inner cohomology at degree 0. So, none of the one dimensional irreducible automorphic representations of $G(\mathbb{A})$ contribute to inner cohomology at degree 0.

Now, suppose $\pi_v \simeq r[d']$. Since $\mathfrak{g}_\infty/\mathfrak{k}_\infty$ is a two dimensional complex Lie algebra, $\bigwedge^2(\mathfrak{g}_\infty/\mathfrak{k}_\infty)$ is one dimensional and $z_1 \wedge z_2$ is a basis. Therefore, $C^2(\mathfrak{g}_\infty, K_\infty; E^{K_f}\otimes R[d]^\vee)$ is also a one dimensional space. In fact, it can be easily seen that $C^2(\mathfrak{g}_\infty, K_\infty; E^{K_f}\otimes R[d]^\vee) = \mathbb{C}\Psi$ where $\Psi(\alpha z_1 \wedge z_2) = \alpha\xi\otimes 1$ is a K_∞ module homomorphism. Further, since we know $H^2(E) \neq 0$, we get that $H^2(\mathfrak{g}_\infty, K_\infty; E^{K_f}\otimes R[d]^\vee) = C^2(\mathfrak{g}_\infty, K_\infty; E^{K_f}\otimes R[d]^\vee)$. Now, the question is when does $H^2(E)$ map to 0 in $H^2(\mathfrak{g}_\infty, K_\infty; \mathscr{C}_B(\omega, K_f)\otimes R[d]^\vee)$. This translates to the question of existence of a $\phi \in C^1(\mathfrak{g}_\infty, K_\infty; \mathscr{C}_B(\omega, K_f)\otimes R[d]^\vee)$ such that $d\phi = \Psi$.

Since ϕ is a K_∞ module homomorphism, it is completely determined by its value on either z_1 or z_2. Let $\phi(z_1) = \eta\otimes 1$ for some $\eta \in \mathscr{C}_B(\omega, K_f)$. Let $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2) \subseteq K_\infty$, $w = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in K_\infty$ and $c_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in K_\infty$ for $x \in \mathbb{R}^\times$. Then, $\phi(k_\theta\cdot z_1) = k_\theta\cdot\phi(z_1)$ if and only if $\rho(k_\theta)\eta = e^{-2i\theta}\eta$, and $\phi(c_x\cdot z_1) = c_x\cdot\phi(z_1)$ if and only if $\rho(c_x)\eta = x^a\eta$. Further, $\phi(z_2) = \phi(w\cdot z_1) = w\cdot\phi(z_1) = \rho(w)\eta\otimes(-1)^{\epsilon-\frac{a}{2}}$. Then,

$$\phi\big(k_\theta\cdot(pz_1 + qz_2)\big) = \phi(e^{-2i\theta}pz_1 + e^{2i\theta}qz_2) = e^{-2i\theta}p\eta\otimes 1 + e^{2i\theta}q(-1)^{\epsilon-\frac{a}{2}}\rho(w)\eta\otimes 1$$

$$\begin{aligned} k_\theta \cdot \phi(pz_1 + qz_2) = k_\theta \cdot \left(p\eta \otimes 1 + q\rho(w)\eta \otimes (-1)^{\epsilon - \frac{a}{2}}\right) &= p\rho(k_\theta)\eta \otimes 1 + q(-1)^{\epsilon-\frac{a}{2}}\rho(k_\theta w)\eta \otimes 1 \\ &= pe^{-2i\theta}\eta \otimes 1 + q(-1)^{\epsilon-\frac{a}{2}}\rho(wk_{-\theta})\eta \otimes 1 \\ &= pe^{-2i\theta}\eta \otimes 1 + q(-1)^{\epsilon-\frac{a}{2}}e^{2i\theta}\rho(w)\eta \otimes 1 \end{aligned}$$

$$\begin{aligned} c_x \cdot \phi(pz_1 + qz_2) &= p\rho(c_x)\eta \otimes x^{-a} + q\rho(c_x w)\eta \otimes (-1)^{\epsilon-\frac{a}{2}}x^{-a} \\ &= px^a\eta \otimes x^{-a} + q\rho(w)\rho(c_x)\eta \otimes (-1)^{\epsilon-\frac{a}{2}}x^{-a} \\ &= \phi(pz_1 + qz_2) = \phi\left(c_x \cdot (pz_1 + qz_2)\right) \end{aligned}$$

which shows that ϕ as defined above, does indeed belong to $C^1(\mathfrak{g}_\infty, K_\infty; \mathscr{C}_B(\omega, K_f) \otimes R[d]^\vee)$ when η satisfies

$$\begin{aligned} \rho(k_\theta)\eta &= e^{-2i\theta}\eta \quad \text{for all } k_\theta \in SO(2) \\ \rho(c_x)\eta &= x^a\eta \quad \text{for all } x \in \mathbb{R}^\times \end{aligned} \tag{4.2.3}$$

Now, the question of when there exists a ϕ such that $d\phi = \Psi$ can be formulated in terms of η.

$$\begin{aligned} d\phi(z_1 \wedge z_2) = \Psi(z_1 \wedge z_2) = \xi \otimes 1 \quad &\Longleftrightarrow \quad z_1 \cdot \phi(z_2) - z_2 \cdot \phi(z_1) = \xi \otimes 1 \quad \left(\because [z_1, z_2] = 0\right) \\ &\Longleftrightarrow \quad z_1 \cdot \left(\rho(w)\eta \otimes (-1)^{\epsilon-\frac{a}{2}}\right) - z_2 \cdot \eta \otimes 1 = \xi \otimes 1 \\ &\Longleftrightarrow \quad (-1)^{\epsilon-\frac{a}{2}} z_1 \cdot \rho(w)\eta - z_2 \cdot \eta = \xi \end{aligned}$$

We may take η to be a function of the form $\sum_{i=1}^{n} \eta_i$, where for each i, η_i is a function on $GL(2, \mathbb{A})$ defined as

$$\eta_i\left((g_v)_v\right) = \eta_{i,\infty}(g_\infty)\prod_p \eta_{i,p}(g_p)$$

where the product runs over primes p and $\eta_{i,\infty}$ is a function on $GL(2, \mathbb{R})$ and $\eta_{i,p}$ is a function on $GL(2, \mathbb{Q}_p)$ for every prime p. Now, in the equation

$$(-1)^{\epsilon-\frac{a}{2}} z_1 \cdot \rho(w)\eta - z_2 \cdot \eta = \xi \tag{4.2.4}$$

the action on η only affects the infinite components. Further, ξ being a quasi-character can be written as $\xi_\infty \prod_p \xi_p$, where ξ_∞ is a quasi-character of $GL(2, \mathbb{R})$ and ξ_p is a quasi-character of $GL(2,\mathbb{Q}_p)$ for every prime p. Hence, without loss of generality we may take

$$\eta\big((g_v)_v\big) = \eta_\infty(g_\infty)\prod_p \xi_p(g_p) \tag{4.2.5}$$

Let $b = \begin{pmatrix}\alpha & \beta\\ 0 & \gamma\end{pmatrix} \in B(\mathbb{Q})$. Since we want $\eta \in \mathscr{C}_B(\omega, K_f)$, $\eta(bg) = \eta(g)$ for all $g \in G_{\mathbb{A}}$, which implies

$$\eta_\infty(b_\infty g_\infty)\cdot\prod_p \xi_p(b_p)\xi_p(g_p) = \eta(bg) \quad\implies\quad \prod_p \xi_p(b_p)\eta_\infty(b_\infty g_\infty) = \eta_\infty(g_\infty)$$

Since ξ is a function on $GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A})$, ξ is invariant under left action by elements of $GL(2,\mathbb{Q})$ and hence $B(\mathbb{Q})$ in particular. This gives $\xi_\infty(b_\infty)\prod_p \xi_p(b_p) = 1$. Hence, we get

$$\eta_\infty(b_\infty g_\infty) = \xi_\infty(b_\infty)\eta_\infty(g_\infty) \quad \forall\, g_\infty \in G_{\mathbb{R}}$$

We know moreover that $\xi_\infty = \det^{1+\frac{a-h}{2}}\operatorname{sgn}(\det)^{\epsilon-1}$ since we have assumed that $\pi_v \simeq r[d']$ at the archimedean place v. Thus, we get

$$\eta_\infty(b_\infty g_\infty) = (\alpha\gamma)^{1+\frac{a-h}{2}}\operatorname{sgn}(\alpha\gamma)^{\epsilon-1}\eta_\infty(g_\infty) = (\alpha\gamma)^{\frac{a}{2}}\operatorname{sgn}(\alpha\gamma)^{\epsilon-1}\eta_\infty(g_\infty) \tag{4.2.6}$$

Further, since we require η_∞ to be a smooth function, the above equation holds even for $b_\infty = \begin{pmatrix}\alpha & \beta\\ 0 & \gamma\end{pmatrix} \in B(\mathbb{R})$. We know that every element of G has a representation in one of the following forms depending on the sign of its determinant.

$$g = \begin{pmatrix}u & \\ & u\end{pmatrix}\begin{pmatrix}y^{1/2} & xy^{-1/2}\\ & y^{-1/2}\end{pmatrix}k_\theta \tag{4.2.7}$$

$$g = \begin{pmatrix}u & \\ & u\end{pmatrix}\begin{pmatrix}y^{1/2} & xy^{-1/2}\\ & y^{-1/2}\end{pmatrix}k_\theta\begin{pmatrix}1 & \\ & -1\end{pmatrix}, \qquad k_\theta = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} \tag{4.2.8}$$

where $x, y, u, \theta \in \mathbb{R}$ and $u, y > 0$. Thus, using Equations 4.2.3 and 4.2.5, we get

$$\eta_\infty(g) = \begin{cases} u^a e^{-2i\theta}\eta_\infty(I), & \text{if } g = \begin{pmatrix} u & \\ & u \end{pmatrix}\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} k_\theta \\ (-1)^{\epsilon-1-\frac{a}{2}} u^a e^{2i\theta}\eta_\infty(I), & \text{if } g = \begin{pmatrix} u & \\ & u \end{pmatrix}\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} k_\theta \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \end{cases} \tag{4.2.9}$$

Thus, we also have

$$\rho(w)\eta_\infty(g) = \begin{cases} (-1)^{\epsilon-1-\frac{a}{2}} u^a e^{2i\theta}\eta_\infty(I), & \text{if } g = \begin{pmatrix} u & \\ & u \end{pmatrix}\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} k_\theta \\ u^a e^{-2i\theta}\eta_\infty(I), & \text{if } g = \begin{pmatrix} u & \\ & u \end{pmatrix}\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} k_\theta \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \end{cases} \tag{4.2.10}$$

Let $X = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and $Y = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ be elements of $\mathfrak{g}$. Let us note the following identities.

$$\exp(tX) = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}, \quad \exp(tY) = \begin{pmatrix} p(t) & q(t) \\ q(t) & p(t) \end{pmatrix}, \quad \text{where } p(t) = \frac{e^t + e^{-t}}{2}, q(t) = \frac{e^t - e^{-t}}{2}$$

$$k_\theta \exp(tX) = \begin{pmatrix} y_t^{1/2} & x_t y_t^{-1/2} \\ & y_t^{-1/2} \end{pmatrix} k_{\theta_t} \tag{4.2.11}$$

where

$$y_t = \left(\sin^2\theta e^{2t} + \cos^2\theta e^{-2t}\right)^{-1}, x_t = \frac{y_t - e^{2t}}{\tan\theta e^{2t}}, \theta_t = \tan^{-1}(\tan\theta e^{2t})$$

$$k_\theta \exp(tY) = \begin{pmatrix} u_t^{1/2} & v_t u_t^{-1/2} \\ & u_t^{-1/2} \end{pmatrix} k_{\phi_t} \tag{4.2.12}$$

where

$$u_t = \left(p(2t) - \sin 2\theta q(2t)\right)^{-1}, v_t = \frac{\cos 2\theta\left(\cos\theta(p(3t)-p(t)) - \sin\theta(q(3t)-q(t))\right)}{2(-\sin\theta p(t) + \cos\theta q(t))}$$

$$\phi_t = \tan^{-1}\left(\frac{\sin\theta p(t) - \cos\theta q(t)}{\cos\theta p(t) - \sin\theta q(t)}\right).$$

Then, if we use the above identities 4.2.11 and 4.2.12, we have for g as in 4.2.7,

$$\begin{aligned}(X\cdot\eta_\infty)(g) &= \frac{d}{dt}\bigg|_{t=0}\rho\left(\exp(tX)\right)\eta_\infty(g) = \frac{d}{dt}\bigg|_{t=0}\eta_\infty\left(g\exp(tX)\right)\\ &= \frac{d}{dt}\bigg|_{t=0}u^a e^{-2i\theta_t}\eta_\infty(I)\\ &= -2i\sin 2\theta e^{-2i\theta}u^a\eta_\infty(I)\end{aligned} \tag{4.2.13}$$

$$\begin{aligned}(Y\cdot\eta_\infty)(g) &= \frac{d}{dt}\bigg|_{t=0}\rho\left(\exp(tY)\right)\eta_\infty(g) = \frac{d}{dt}\bigg|_{t=0}\eta_\infty(g\exp(tY))\\ &= \frac{d}{dt}\bigg|_{t=0}u^a e^{-2i\phi_t}\eta_\infty(I)\\ &= 2i\cos 2\theta e^{-2i\theta}u^a\eta_\infty(I)\end{aligned} \tag{4.2.14}$$

Similarly, for g as in 4.2.8, we get

$$(X\cdot\eta_\infty)(g) = (-1)^{\epsilon-\frac{a}{2}-1}2i\sin 2\theta e^{2i\theta}u^a\eta_\infty(I) \tag{4.2.15}$$

$$(Y\cdot\eta_\infty)(g) = (-1)^{\epsilon-\frac{a}{2}-1}2i\cos 2\theta e^{2i\theta}u^a\eta_\infty(I) \tag{4.2.16}$$

Since $z_1 = i\otimes X - 1\otimes Y$ and $z_2 = i\otimes X + 1\otimes Y$, we get that

$$z_1\cdot\eta_\infty(g) = \begin{cases}-2iu^a\eta_\infty(I), & \text{if } g \text{ is as in 4.2.7}\\ (-1)^{\epsilon-\frac{a}{2}}2iu^a\eta_\infty(I), & \text{if } g \text{ is as in 4.2.8}\end{cases} \tag{4.2.17}$$

$$z_2 \cdot \eta_\infty(g) = \begin{cases} 2ie^{-4i\theta} u^a \eta_\infty(I), & \text{if } g \text{ is as in 4.2.7} \\ (-1)^{\epsilon - \frac{a}{2} - 1} 2ie^{4i\theta} u^a \eta_\infty(I), & \text{if } g \text{ is as in 4.2.8} \end{cases} \tag{4.2.18}$$

Let us now call the left side of the differential equation 4.2.4 as ζ. By the form of η assumed in 4.2.5, we get that $\zeta\left((g_v)_v\right) = \zeta_\infty(g_\infty) \prod_p \xi_p(g_p)$, where

$$\begin{aligned} \zeta_\infty(g_\infty) &= \left((-1)^{\epsilon - \frac{a}{2}} z_1 \cdot \rho(w)\eta_\infty - z_2 \cdot \eta_\infty\right)(g_\infty) \\ &= \left((-1)^{\epsilon - \frac{a}{2}} \rho(w)\left(z_2 \cdot \eta_\infty\right) - z_2 \cdot \eta_\infty\right)(g_\infty) \\ &= (-1)^{\epsilon - \frac{a}{2}} \left(z_2 \cdot \eta_\infty\right)(gw) - \left(z_2 \cdot \eta_\infty\right)(g_\infty) \end{aligned}$$

which means that

$$\zeta_\infty(g_\infty) = \begin{cases} -2i\left(e^{4i\theta} + e^{-4i\theta}\right) u^a \eta_\infty(I), & \text{if } g_\infty \text{ is as in 4.2.7} \\ (-1)^{\epsilon - \frac{a}{2}} 2i\left(e^{4i\theta} + e^{-4i\theta}\right) u^a \eta_\infty(I), & \text{if } g_\infty \text{ is as in 4.2.8} \end{cases} \tag{4.2.19}$$

On the other hand, ξ_∞ being a quasi-character of $GL(2,\mathbb{R})$ factors through the determinant map. In fact, $\xi_\infty \simeq r[d']$, i.e., $\xi_\infty = \det^{\frac{a}{2}} \mathrm{sgn}(\det)^{\epsilon - 1}$. Since ζ_∞ is clearly not invariant under elements of $SL(2,\mathbb{R})$, for example, $\zeta_\infty(k_\theta) = -2i\left(e^{4i\theta} + e^{-4i\theta}\right)\eta_\infty(I) \neq -4i\eta_\infty(I) = \zeta_\infty(I)$ for $k_\theta \in SO(2,\mathbb{R}) \subset SL(2,\mathbb{R})$, we can conclude that $\zeta_\infty \neq \xi_\infty$ and hence $\zeta \neq \xi$. This means that Equation 4.2.4 is not satisfied for any suitable η, which shows that there does not exist any $\phi \in C^1(\mathfrak{g}_\infty, K_\infty; \mathscr{C}_B(\omega, K_f) \otimes R[d]^\vee)$ such that $d\phi = \Psi$, which means that Ψ does not map to zero in $H^2(\mathfrak{g}_\infty, K_\infty; \mathscr{C}_B(\omega, K_f) \otimes R[d]^\vee)$. This shows that $H^2(E)$ does not map to zero in $H^2(\mathfrak{g}_\infty, K_\infty; \mathscr{C}_B(\omega, K_f) \otimes R[d]^\vee)$.

So, (π, E) does not contribute to inner cohomology at degree 2 as well. So, none of the one dimensional irreducible automorphic representations of $G(\mathbb{A})$ contribute to inner cohomology at degree 2.

Further, we have already mentioned that $H^q(\mathfrak{g}_\infty, K_\infty; E^{K_f} \otimes R[d]^\vee) = 0$ for a one-dimensional automorphic representations (π, E) of $G(\mathbb{A})$, for all degrees $q \neq 0, 2$. Putting all these facts together, we get that for the case $F = \mathbb{Q}$, inner cohomology is the same as cuspidal cohomology in all degrees.

Chapter 5

The case of Imaginary Quadratic Fields

Let $F = \mathbb{Q}[\sqrt{-n}]$ for a positive square free integer n. Let (π, E) be an one dimensional automorphic representation of $G(F)$ with central character ω. So, $E = \mathbb{C}\xi$ where ξ is a continuous quasi-character of $G(F)\backslash G(\mathbb{A})$ which agrees with ω on $Z(\mathbb{A})$. Any such quasi-character factors through the determinant map. So, we may let $\xi = \chi \circ \det$ where $\chi : F^\times\backslash\mathbb{A}^\times \longrightarrow \mathbb{C}^\times$ is a continuous quasi-character of the group of ideles. It is easily checked that the action π of $G(\mathbb{A})$ on the space E by right translation, gives rise to the same character ξ.

Let ∞ denote the only archimedean place of F, corresponding to the two conjugate embeddings $\mu, \bar{\mu} : F \hookrightarrow \mathbb{C}, \mu(a+b\sqrt{-n}) = a+b\sqrt{n}i, \bar{\mu}(a+b\sqrt{-n}) = a-b\sqrt{n}i$. Let π be the restricted tensor product over all places v of F, of the representations π_v. Then, $\pi_v \simeq \xi_v$ for all v, where ξ_v denotes the restriction of ξ to G_v. Then, since we are concerned with only those (π, E) in $A_1(\omega, d)$, we get that $\pi_\infty \simeq r[d]$, i.e., $\xi_\infty \simeq r[d]$. Moreover, for an open compact subgroup $K_f \subseteq G(\mathbb{A}_f)$, $E^{K_f} \neq 0$ if and only if ξ is right invariant under K_f, in which case, $E^{K_f} = E$.

By Equation 3.3.7 and Proposition 1, we get that if $\pi_\infty \not\simeq r[d]$ or $\pi[d]$, all the cohomology groups $H^q(\mathfrak{g}_\infty, K_\infty; E^{K_f} \otimes R[d]^\vee)$ are trivial. Further, since (π, E) is a finite dimensional representation, π_∞ has to be finite dimensional and hence $\pi_\infty \not\simeq \pi[d]$. Thus for nontrivial cohomology we require $\pi_\infty \simeq r[d]$ and in this case, the two cohomology groups $H^0(\mathfrak{g}_\infty, K_\infty; E^{K_f} \otimes R[d]^\vee)$ and $H^3(\mathfrak{g}_\infty, K_\infty; E^{K_f} \otimes R[d]^\vee)$ are equivalent to $E_f^{K_f}$ as $\mathfrak{H}_f^{K_f}$ modules, while the cohomology groups at other degrees are zero.

So, we need to check if $H^q(E)$ maps to 0 in $H^q(\mathfrak{g}_\infty, K_\infty; \mathscr{C}_B(\omega, K_f) \otimes R[d]^\vee)$ only for degrees $q = 0,3$. Since $H^0(\mathfrak{g}_\infty, K_\infty; V) \subseteq C^0\left(\mathfrak{g}_\infty, K_\infty; V\right)$, we still have that the maps

$$\begin{aligned} H^0(\mathfrak{g}_\infty, K_\infty; E^{K_f} \otimes R[d]^\vee) \longrightarrow H^0(\mathfrak{g}_\infty, K_\infty; \mathscr{C}(\omega, K_f) \otimes R[d]^\vee) \\ \longrightarrow H^0(\mathfrak{g}_\infty, K_\infty; \mathscr{C}_B(\omega, K_f) \otimes R[d]^\vee) \end{aligned} \quad (5.0.1)$$

obtained from the inclusions $E^{K_f} \hookrightarrow \mathscr{C}(\omega, K_f) \hookrightarrow \mathscr{C}_B(\omega, K_f)$ are all injective. Thus, $H^0(E) \neq 0$ does not map to zero in $H^0(\mathfrak{g}_\infty, K_\infty; \mathscr{C}_B(\omega, K_f) \otimes R[d]^\vee)$, i.e, π doesn't contribute to inner cohomology at degree 0. Thus, none of the one dimensional irreducible automorphic representations of $G(\mathbb{A})$ contribute to inner cohomology at degree 0. Furthermore, it is known that cuspidal cohomology does not occur in degree 0. Hence, we get

$$H^0_!(S^G_{K_f}, \mathscr{F}_d) = 0 \quad (5.0.2)$$

In order to consider the case $q = 3$, we will make use of Poincaré duality and a long exact sequence similar to 3.3.5. The manifold $S^G_{K_f}$ can be compactified using Borel Serre compactification [10]. Let us denote the resulting smooth manifold by $\bar{S}^G_{K_f}$. It is of the same dimension as $S^G_{K_f}$. Further it has a boundary $\partial \bar{S}^G_{K_f}$. Then, $\bar{S}^G_{K_f} = S^G_{K_f} \cup \partial \bar{S}^G_{K_f}$. The boundary has a differentiable structure of corners and is a manifold of one less dimension. It is stratified into boundary components, that are indexed by each proper rational parabolic subgroup $P \subset G$. The sheaf $\mathscr{F}_d$ defined in 3.1.3 naturally extends to a sheaf of vector spaces on the compact manifold $\bar{S}^G_{K_f}$. We also have that $H^\bullet(\bar{S}^G_{K_f}, \mathscr{F}_d) \simeq H^\bullet(S^G_{K_f}, \mathscr{F}_d)$, the isomorphism being obtained by the natural restriction $\bar{S}^G_{K_f} \longrightarrow S^G_{K_f}$. We also have the following long exact sequence of sheaf cohomology groups

$$\cdots H^i_c(S^G_{K_f}, \mathscr{F}_d) \xrightarrow{i^*} H^i(\bar{S}^G_{K_f}, \mathscr{F}_d) \xrightarrow{r^*} H^i(\partial \bar{S}^G_{K_f}, \mathscr{F}_d) \xrightarrow{\delta *} H^{i+1}_c(S^G_{K_f}, \mathscr{F}_d) \longrightarrow \cdots \quad (5.0.3)$$

We will denote the cohomology groups $H^i_c(S^G_{K_f}, \mathscr{F}_d), H^i(\bar{S}^G_{K_f}, \mathscr{F}_d), H^i(\partial \bar{S}^G_{K_f}, \mathscr{F}_d)$ from now onward by H^i_c, H^i and $H^i(\partial)$ respectively.

In our case of $G = GL(2)$ over an imaginary quadratic field F, the manifold $S^G_{K_f}$ is of dimension 3 and hence its boundary $\partial \bar{S}^G_{K_f}$ is of dimension 2. Hence, we know that $H^i(S^G_{K_f}, \mathscr{F}_d) = H^i(\bar{S}^G_{K_f}, \mathscr{F}_d) = 0$ for all $i > 3$, and $H^i(\partial \bar{S}^G_{K_f}, \mathscr{F}_d) = 0$ for all $i > 2$. Consequently the long exact sequence 5.0.3 becomes

$$H_c^0 \xrightarrow{i^*} H^0 \xrightarrow{r^*} H^0(\partial) \xrightarrow{\delta^*} H_c^1 \xrightarrow{i^*} H^1 \xrightarrow{r^*} H^1(\partial) \xrightarrow{\delta^*} H_c^2$$
$$\xrightarrow{i^*} H^2 \xrightarrow{r^*} H^2(\partial) \xrightarrow{\delta^*} H_c^3 \xrightarrow{i^*} H^3 \xrightarrow{r^*} 0 \tag{5.0.4}$$

The exactness of the above sequence tells us that the inner cohomology $H_!^3(S_{K_f}^G, \mathscr{F}_d) = \mathrm{Im}(H_c^3 \xrightarrow{i*} H^3)$ is equal to all of H^3.

Further, the map $H_c^0 \xrightarrow{i^*} H^0$ is also known to be injective. This is because the sheaves Ω_c^k of smooth compactly supported k-forms on $S_{K_f}^G$ inject into the sheaves Ω^k of smooth k-forms on $S_{K_f}^G$. We have already seen that the cohomology groups may be computed by using a resolution of the sheaf $\mathscr{F}_d$ as in 3.2.1 and computing cohomology of the cochain complex obtained after applying the global sections functor and discarding the first term. Since H^0 of a cochain complex $C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots$ is simply the kernel of the map d^0, we know that H^0 is a subspace of C^0. Using this together with the injectivity of $\Omega_c^0 \hookrightarrow \Omega^0$, we get the desired result that the map $i^* : H_c^0(S_{K_f}^G, \mathscr{F}_d) \longrightarrow H^0(S_{K_f}^G, \mathscr{F}_d)$ is injective.

Since we have already shown that $H_!^0(S_{K_f}^G, \mathscr{F}_d) = 0$ (Equation 5.0.2), we conclude that $H_c^0(S_{K_f}^G, \mathscr{F}_d) = 0$.

Now, we know by Poincaré duality that there exists a non-degenerate bilinear pairing

$$H_c^0(S_{K_f}^G, \mathscr{F}_d) \times H^3(S_{K_f}^G, \mathscr{F}_d) \longrightarrow \mathbb{C}$$

Since we know that $H_c^0(S_{K_f}^G, \mathscr{F}_d) = 0$, we get that $H^3(S_{K_f}^G, \mathscr{F}_d) = 0$.

We had already noted that $H_!^3(S_{K_f}^G, \mathscr{F}_d) = H^3(S_{K_f}^G, \mathscr{F}_d)$ using the long exact sequence 5.0.4. Thus, we get that $H_!^3(S_{K_f}^G, \mathscr{F}_d) = 0$.

We dilineate below the different steps of the proof.

1. $i^* : H_c^0(S_{K_f}^G, \mathscr{F}_d) \longrightarrow H^0(S_{K_f}^G, \mathscr{F}_d)$ is injective.

2. $H_!^0(S_{K_f}^G, \mathscr{F}_d) = \mathrm{Im}\, i^* = 0$.

3. From 1 and 2, we get $H_c^0(S_{K_f}^G, \mathscr{F}_d) = 0$.

4. From 3, using Poincaré duality, we get $H^3(S_{K_f}^G, \mathscr{F}_d) = 0$.

5. From 4, we get that $H_!^3(S_{K_f}^G, \mathscr{F}_d) = 0$.

We would like to point out that though $H^3_!(S^G_{K_f}, \mathscr{F}_d) = H^3(S^G_{K_f}, \mathscr{F}_d) = 0$, the cohomology with compact supports $H^3_c(S^G_{K_f}, \mathscr{F}_d) \neq 0$ and one dimensional automorphic representations contribute to this cohomology. This can be seen as follows. We know that $H^3_c(S^G_{K_f}, \mathscr{F}_d) \simeq H^0(S^G_{K_f}, \mathscr{F}_d)$ by Poincaré duality. Hence, it is sufficient to show the existence of one dimensional automorphic representations, i.e., quasi-characters ξ of $G_F \backslash G_{\mathbb{A}}$, that contribute to $H^0(S^G_{K_f}, \mathscr{F}_d) \simeq H^0(\mathfrak{g}_\infty, K_\infty; \mathscr{C}(\omega, K_f) \otimes R[d]^\vee)$. $C^0(\mathfrak{g}_\infty, K_\infty; \mathscr{C}(\omega, K_f) \otimes R[d]^\vee)$ consists of all K_∞ module homomorphisms from $\mathbb{C} \longrightarrow \mathscr{C}(\omega, K_f) \otimes R[d]^\vee$. For any quasi-character ξ such that $\xi_\infty \simeq r[d]$, we get such a K_∞ module homomorphism that takes 1 to $\xi \otimes 1$. Further, since ξ factors through the determinant, and $\exp(w_1), \exp(w_2)$ and $\exp(w_3)$ are all in $SL(2)$, we get that $w_1 \cdot \xi = w_2 \cdot \xi = w_3 \cdot \xi = 0$, which imply that any such K_∞ module homomorphism is mapped to the zero homomorphism in $C^1(\mathfrak{g}_\infty, K_\infty; \mathscr{C}(\omega, K_f) \otimes R[d]^\vee)$. This tells us that all one dimensional automorphic representations π of G with $\pi_\infty \simeq r[d]$ contribute to $H^0(S^G_{K_f}, \mathscr{F}_d)$ and hence to $H^3_c(S^G_{K_f}, \mathscr{F}_d)$.

We have already mentioned that $H^q(\mathfrak{g}_\infty, K_\infty; E^{K_f} \otimes R[d]^\vee) = 0$ for degree $q \neq 0, 3$. Thus, we get that in degrees 1 and 2, inner cohomology is the same as cuspidal cohomology.

Chapter 6

Results

A summary of results concerning the inner cohomology groups for $G = GL(2)$ over $\mathbb{Q}$ is given below.

- $H^0_!(S^G_{K_f}, \mathscr{F}_d) = 0.$
- $H^1_!(S^G_{K_f}, \mathscr{F}_d) = H^1_{\text{cusp}}(S^G_{K_f}, \mathscr{F}_d).$
- $H^2_!(S^G_{K_f}, \mathscr{F}_d) = 0.$
- $H^q(S^G_{K_f}, \mathscr{F}_d) = 0 \quad$ and hence, $H^q_!(S^G_{K_f}, \mathscr{F}_d) = 0 \quad$ for $q > 2$.

A summary of results concerning the inner cohomology groups for $G = GL(2)$ over an imaginary quadratic field is given below.

- $H^0_!(S^G_{K_f}, \mathscr{F}_d) = 0.$
- $H^q_!(S^G_{K_f}, \mathscr{F}_d) = H^q_{\text{cusp}}(S^G_{K_f}, \mathscr{F}_d) \quad$ for $q = 1, 2$.
- $H^3_!(S^G_{K_f}, \mathscr{F}_d) = 0.$
- $H^q(S^G_{K_f}, \mathscr{F}_d) = 0 \quad$ and hence, $H^q_!(S^G_{K_f}, \mathscr{F}_d) = 0 \quad$ for $q > 3$.

In both cases that we considered, viz., the field of rational numbers and imaginary quadratic fields, we have obtained that there is no universal cohomology, and hence inner cohomology is equal to cuspidal cohomology. This is not true for a general number field. For example, if we consider a real quadratic field, i.e., $F = \mathbb{Q}(\sqrt{n})$ for a square-free integer $n \geq 1$, statement 3b of Theorem 5 tells us that inner cohomology at degree 2 is two dimensional. In particular, it is non-trivial. I hope to study the inner cohomology for any totally imaginary field F using similar methods, and compute expressions similar to those in Theorem 5 for the dimensions of the universal cohomology for different degrees.

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