Asymptotic conservation laws for loop level soft photon theorems

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by

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Dedicated to Kaka (Shri Ram Deshpande)

Declaration

I hereby declare that this written submission entitled "Asymptotic conservation laws for loop level soft photon theorems" represents my idea in my own words and where others' ideas have been included; I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed. The work reported in this thesis is the original work done by me under the guidance of Dr. Nabamita Banerjee.

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Certificate

I/ We certify that the thesis entitled **"Asymptotic conservation laws for loop level soft photon theorems"** presented by Ms Sayali Atul Bhatkar represents her original work which was carried out by her at IISER, Pune under my/ our guidance and supervision during the period from Aug, 2016 to Feb, 2021. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other University or institutions. I/ We further certify that the above statements made by her in regard to this thesis are correct to the best of my/ our knowledge.

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Dr. Nabamita Banerjee,

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Co-Supervisor.

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Abstract

In the thesis, we study new asymptotic conservation laws in electromagnetism that could reproduce the loop effects in the soft expansion of QED amplitudes. We also investigate certain unexplored properties of asymptotic expansions of the classical as well as quantum gauge field.

Incorporating the effect of long range electromagnetic force (present in four spacetime dimensions) acting on the scattered particles, we analyse the new modes that arise in the asymptotic radiative field emitted in a generic classical scattering process. We show that there exist new asymptotic conservation laws ($Q_m; m = 1, 2$) that are obeyed by the classical radiative field. Building on the $m = 1, 2$ cases, we propose that there exists a conservation law for every m. The corresponding charges are made of modes of the asymptotic electromagnetic field that appear at $\mathcal{O}(e^{2m+1})$ and are expected to be conserved exactly.

The asymptotic behaviour of the gauge field is modified in the quantum theory due to use of Feynman boundary condition. We derive the analogue of the first of above asymptotic conservation laws upon imposing Feynman boundary condition on the radiative field. We also discuss new modes in the Feynman solution which are absent in the classical radiative solution. These modes lead to quantum corrections to the asymptotic charges.

We anticipate that the Q_m charges imply existence of m-loop soft theorems for every m. In particular we show that the Ward identity for the Q_1 charge is equivalent to the 1-loop exact subleading soft photon theorem for loop level QED amplitudes. We demonstrate this equivalence in the context of massless scalar QED in presence of dynamical gravity. This asymptotic charge is directly related to the dressing of fields due to long range forces. In presence of gravity, the new feature is that the soft photon also acquires a dressing due to long range gravitational force and contributes to the asymptotic charge.

We expect this story to hold beyond 1-loop as well. The Q_2 conservation law that we derived is expected to be related to the 2-loop exact soft photon theorem that has been obtained recently in the literature. It would be very interesting to explore this equivalence for $m > 1$.

Keywords : Asymptotic conservation laws, soft theorems.

Contents

Chapter 1

Introduction and Goal

Recently many interesting features have been explored in the infrared regime of gauge theories and gravitational theories. Soft theorems form an important part of infrared physics. A particle is said to be soft if its energy is much smaller than other energy scales of the process. Thus if we consider a quantum scattering process, a soft particle is such that its energy is much smaller than the energies/ masses of other (hard) particles. Scattering amplitudes involving soft particles display remarkable properties which are encoded in soft theorems. Let us focus on QED amplitudes. The leading order term in the soft limit of an $(n+1)$ point amplitude goes as inverse of the soft energy. At this order, the amplitude factorises into a soft factor times the n point amplitude without the soft photon. The information of the soft photon is contained entirely in the soft factor and the form of the soft factor is same for all processess. Hence it is called a soft theorem. The soft factor depends on the electric charges and momenta of the hard particles and the direction of the soft momentum. It is insensitive to the other properties of the scattered particles and also to the details of the short range forces that are responsible for scattering. Consequently it is independent of intermediate resonances that might be produced in the process. The form of the leading soft factor is true to all loop orders and is same in all spacetime dimensions ! It is uncorrected by non-minimal couplings or higher derivative corrections. This kind of universality makes soft theorems an interesting topic to study. This topic has been explored since 1950's by Bloch, Gell mann, Goldberger, Low, Weinberg and many others [1–24].

In the recent years, soft theorems have been studied for multiple soft particles as well [25–34] . This line of study has an interesting application. Laddha and Sen used multiple soft photon theorems to derive the low frequency spectrum of electromagnetic radiation emitted in a generic classical scattering process [35–39]. In the low frequency limit, the classical radiative field is universally given in terms of the charges and asymptotic momenta of the scattering particles and is independent of other details of the scattering. The universal terms in the 'soft' expansion of the classical field define classical soft theorems. The leading order term in the classical radiation in the low frequency limit goes like inverse of the frequency such that the coefficient of this term is the leading soft factor that appears in the quantum soft theorem. This is called the leading classical soft theorem and it has been amply discussed in the literature under the guise of the electromagnetic memory effect [40–46]. This setup involves a test charge placed in the asymptotic regime of the spacetime. This charge can be used to detect the late time radiation emitted in a process. The leading term in the late time radiation is related to the leading classical soft theorem and gives rise to a shift in the velocity (components tranverse to the radial direction) of the test charge. Thus the magnitude of this shift is controlled by the leading soft factor. It is fascinating that the soft factors are directly observable in an experiment!

The universality in the soft expansion of amplitudes (or the classical field) is manifestation of underlying symmetries of the theory. The leading soft photon theorem is related to asymptotic symmetry group called large gauge transformations. Large gauge transformations refer to the U(1) gauge transformations that do not die off at infinity. The usual 'small' gauge transformations vanish at infinity and the corresponding charges vanish identically. Thus these charges annihilate the physical states of the theory. On the other hand the large gauge transformations lead to non-zero charges. These charges have a non-trivial action on the physical states. It is well known that the vacuum state also transforms under this symmetry i.e. this symmetry is spontaneously broken. The leading soft theorem is exactly equivalent to the Ward identity of the large gauge transformations. The relation between the leading soft theorem and spontaneously broken large gauge transformations was studied by Ferrarri, Picasso and others in 1970's [47–52].

There has been a renewed interest in this equivalence since 2013 due to new insights provided by Strominger [53, 54]. This work was followed by many interesting papers that shed light on the relation between these two corners of the IR sector of gauge theories and gravity [55–72]. Strominger and his collaborators pioneered the study of equivalence between soft theorems and asymptotic conservation laws. [1] The asymptotic conservation law corresponding to the leading soft photon theorem was discussed in [56, 69, 73]. The subleading

[1]See eq (2.1) for the precise statement of an asymptotic conservation law.

term in the soft expansion of tree level amplitudes goes like ω^0 and was first studied in [2–4]. This coefficient consists of a universal piece plus a term that depends on the details of the non-minimal terms present in the theory [74]. Ward identity corresponding to the subleading photon theorem has been discussed in [62–64, 76]. The symmetry underlying this Ward identity or its relation to U(1) gauge group is not well understood yet. In [75], Campiglia and Laddha showed that the subleading soft photon theorem (for tree level amplitudes) is equivalent to an asymptotic conservation law. In fact, in the same paper Campiglia and Laddha showed that the classical radiative field at $\mathcal{O}(e)$ admits an infinite number of conservation laws. They also provided evidence that suggests that these conservation laws are equivalent to the infinite number of tree level soft theorems derived in [77, 78]. Thus, tree level soft theorems in QED can be related to asymptotic conservation laws.

It is well known that the loop level S-matrix becomes ill defined in four spacetime dimensions due to infrared divergences. Infrared divergences are a result of long range effects that arise due to presence of massless particles. To discuss effect of loop corrections on soft theorems, it is necessary to have a prescription to deal with these divergences. One approach is to study soft behaviour of regulated amplitudes and it has been shown that the leading soft factor does not receive any loop corrections. Beyond the leading order, the soft factors admit loop corrections in four spacetime dimensions [79–81]. Another rigorous approach to deal with these divergences is to use either dressed states [82–84] or a finite dressed S-matrix [85, 86]. Soft theorems have been studied using dressed states in [87–89]. There are a few unresolved questions yet. Physically we expect that the massless particles that make up the dressing should have a cutoff [90] but introduction of a cutoff scale breaks Lorentz invariance. Soft theorems need to be studied in presence of this cutoff. These questions should be addressed in the future.

In 2018, Sahoo and Sen studied the subleading term in soft expansion of loop amplitudes systematically using the regulating technique introduced in [92]. This subleading term in soft expansion is logarithmic in soft energy and it is intimately tied to the long range forces present in four spacetime dimensions [91]. They showed that this term is universal and also argued that it is 1-loop exact. This is a significant result. A natural question is to probe if this new soft theorem can be related to a new asymptotic conservation law. The first step in this direction was taken in [93]. Campiglia and Laddha constructed asymptotic charges corresponding to the Sahoo-Sen soft theorem for massive scalar QED. This is quite a remarkable result given the fact that the loop level soft factor has a very complicated structure [91].

The goal of this thesis is to study new conservation laws in electromagnetism that could reproduce the loop effects in the soft expansion of QED amplitudes. As a subgoal we will also investigate certain unexplored aspects of asymptotic expansions of the classical as well as quantum gauge field in presence of long range electromagnetic force. This thesis is based on the work carried out in [99–101]. The organisation of this thesis is as follows.

- In Chapter 2 we lay out the background for our main calculations. We discuss classical scattering of charged particles ignoring the effect of long range electromagnetic forces. Rederiving the Q_0 conservation law, we discuss its relation to the leading soft theorem.

- In Chapter 3, we incorporate the effect of long range electromagnetic force acting on the scattered particles and analyse the asymptotic radiative field emitted in a generic classical scattering process. We show that there exist new asymptotic conservation laws ($Q_m; m = 1, 2$) that are obeyed by the classical radiative field. We propose that such conservation laws should exist for every m.

- The asymptotic behaviour of the gauge field is expected to change in the quantum theory due to use of Feynman boundary condition. This modification is the focus of Chapter 4. We study the new modes that arise in the Feynman solution and show that the Q_1 conservation law is in fact violated by the Feynman solution. We show that the Feynman solution obeys a modified conservation law denoted by $\tilde{Q}_1$. We also discuss certain unfavourable features of the $\tilde{Q}_1$-charge.

- In Chapter 5, we revisit the classical Q_1 conservation law and quantise these charges. Upon quantisation these charges get extra 'quantum' contribution and we show that the corresponding Ward identity is equivalent to the 1-loop exact Sahoo-Sen soft photon theorem for massless scalar QED in presence of dynamical gravity.

We expect this story to hold beyond 1-loop as well. The Q_2 conservation law that we derived is expected to be related to a new 2-loop exact soft photon theorem. And indeed such a soft theorem was recently derived in [95]. The relation between Q_m charges and subsubleading universal modes in the soft expansion needs to investigated at higher loop orders ($m > 1$). It is clear that there is a rich structure in the IR physics at loop level that needs to explored further.

Chapter 2

Preliminaries

In this chapter we will setup our notations and review some important concepts that we need to use later on. The asymptotic conservation laws proposed by Strominger and his collaborators take following form in the classical theory :

$$Q^+[\lambda^+] \mid_{\mathscr{I}^+_-} \quad = \quad Q^-[\lambda^-] \mid_{\mathscr{I}^-_+} . \tag{2.1}$$

The future charge Q^+ is defined at $\mathscr{I}^+_-$ i.e. the $u \to -\infty$ sphere of the future null infinity. Similarly, the past charge Q^- is defined at $\mathscr{I}^-_+$ which is the $v \to \infty$ sphere of the past null infinity. λ^+ is an arbitrary function on 2-sphere and parametrises the charge. The parameter at the past is related to it via antipodal map : $\lambda^+(\hat{x}) = \lambda^-(-\hat{x})$.

Asymptotic conservation laws are related to soft theorems. In [56,59,73], the authors discussed the symmetry underlying the leading soft photon theorem and also showed that the corresponding charges Q_0 obey an asymptotic conservation law. In section 2.1, we will calculate the radial component of the asymptotic electric field generated in a classical scattering process and derive the Q_0 conservation law. Then we will turn to the quantum theory. In 2.2 we discuss the asympotic phase space of massless scalar field and photons. In 2.3, we will demonstrate the equivalence between the Q_0 asymptotic conservation law and the leading soft photon theorem for massless scalar QED. In section 2.4, we will discuss the new subleading soft photon theorem for loop amplitudes derived by Sahoo and Sen in 2018. One of the goals of this thesis is to understand if there exists a new conservation law related to this soft theorem.

2.1 The Q_0 conservation law

In this subsection we will calculate the asymptotic electromagnetic field generated by a general classical scattering process and show that certain modes are conserved.

Hence we need to find the asymptotic expansion of the radiative field near future null infinity denoted by $\mathscr{I}^+$. It is useful to work in retarded co-ordinate system. The flat metric takes following form in these co-ordinates ($u = t - r$)

$$ds^2 = -du^2 - 2dudr + r^2\ 2\gamma_{z\bar{z}}\ dzd\bar{z}; \quad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}.$$

$\mathscr{I}^+$ corresponds to the limit $r \to \infty$ with u finite. We use $\hat{x}$ or $(z, \bar{z})$ interchangeably to describe points on S^2. We will often use following parametrisation of a 4 dimensional spacetime point (Greek indices will be used to denote 4d cartesian components) :

$$x^\mu = rq^\mu + ut^\mu, \quad q^\mu = (1, \hat{x}), \quad t^\mu = (1, \vec{0}). \tag{2.2}$$

It is useful to note that q^μ is null. In terms of the stereographic co-ordinates, we have

$$q = \frac{1}{1+z\bar{z}}\{1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}\}.$$

Let us describe a general scattering setup. We have some n' number of charged particles coming in to interact. We denote the respective velocities by V_i^μ, charges by e_i and masses by m_i (for $i = 1 \cdots n'$). $(n - n')$ number of final charged particles are produced as a result of the interaction that eventually move away from each other. For outgoing particles we denote the velocities by V_i^μ, charges by e_i and masses by m_i (for $i = n' + 1 \cdots (n - n')$) respectively. We can divide the entire spacetime into two parts : a bulk region which is a sphere of radius R around the origin such that the non trivial interaction between the particles takes place within this sphere. In this region, the particles in general move on complicated trajectories depending on short range forces present between them. This short range interaction could be of any sort. The second region is the asymptotic region $r > L$ in which we can completely ignore the short range forces. In the asymptotic region, we need to include the effect of the long range electromagnetic interaction that starts at $\mathcal{O}(\frac{1}{r^2})$. We will carry out the calculations perturbatively in coupling e as well as in asymptotic parameters $1/r$ (or $1/t$).

In Lorenz gauge, the radiation can be obtained from the equation $\Box A_\mu = -j_\mu$. Using the retarded propagator, we get :

$$A_\sigma(x) = \frac{1}{2\pi}\int d^4x'\ \delta([x-x']^2)\ j_\sigma(x')\ \Theta(t-t'). \tag{2.3}$$

We have chosen the retarded root of δ-function i.e. $t > t'$. The retarded root is given by

$$t'_0 = t - |\vec{x} - \vec{x}'|. \tag{2.4}$$

The form of t'_0 at large r is $t'_0 = u + \mathcal{O}(\frac{1}{r})$. Thus the field $A_\sigma(r, u, \hat{x})$ at large r gets contribution from $t' \sim u$. The bulk sources correspond to the region $|r'| < R$ or $|t'| < R$ (as $c = 1$) and contribute to A_σ at $|u| < R$. It is a characteristic of the retarded propagator that the asymptotic field at large u does not get contribution from the bulk region $|t'| < R$. Thus we can focus only on the asymptotic ($t' > R$) trajectories.

Let us write down the form of the source current that describes our scattering event. First we restrict to the leading order in coupling e, hence we can ignore the effect of long range electromagetic interactions on the asymptotic trajectories. Therefore the particles are free in the asymptotic region. Thus an incoming particle has the trajectory:

$$x_i^\mu = [V_i^\mu \tau + d_i]\Theta(-T - \tau). \tag{2.5}$$

τ is an affine parameter, here T denotes the value of τ such that $r(-T) = R$ hence the short range forces can be ignored for $\tau < -T$. Similarly, an outgoing particle has the trajectory :

$$x_j^\mu = [V_j^\mu \tau + d_j]\Theta(\tau - T).$$

For ougoing particles, $r(T) = L$ hence the short range forces can be ignored for $\tau > T$. The form of the trajectories for $|\tau| < T$ is not known. The current is given by summing over all particles that participate in the scattering. The asymptotic part of this current can be written down as :

$$j_\sigma^{\text{asym}}(x') = \int d\tau \Big[\sum_{i=n'+1}^{n} e_i V_{i\sigma}\ \delta^4(x' - x_i)\ \Theta(\tau - T) + \sum_{i=1}^{n'} e_i V_{i\sigma}\ \delta^4(x' - x_i)\ \Theta(-T - \tau) \Big] \tag{2.6}$$

Here, we have labelled the incoming particles by i running from 1 to n' and outgoing particles by i running from $n' + 1$ to n.

The asymptotic radiative field generated by the scattering process can be obtained as follows

$$A_\sigma(x) = \frac{1}{2\pi}\int d^4x'\ \delta([x-x']^2)\ j_\sigma(x')\ \Theta(t-t')\ ,$$
$$= \frac{1}{2\pi}\int d\tau\ \Big[\sum_{i=n'+1}^{n} \frac{\delta(\tau-\tau_0)\ e_i V_{i\sigma}}{|2\tau+2V_i.(x-d_i)|}\ \Theta(\tau-T) + \sum_{i=1}^{n'} \frac{\delta(\tau-\tau_0)\ e_i V_{i\sigma}}{|2\tau+2V_i.(x-d_i)|}\ \Theta(-\tau-T)\Big]. \tag{2.7}$$

The retarded root of the delta function $\delta(|x-x'|^2)$ is given by :

$$\tau_0 = -V_i.(x-d_i) - \big[\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ \big]^{1/2}. \tag{2.8}$$

Hence, the total asymptotic field generated by the scattering process is given by

$$A_\sigma(x) = \sum_{i=n'+1}^{n} \frac{1}{4\pi} \frac{e_i V_{i\sigma}\ \Theta(\tau_0 - T)}{\sqrt{(V_i.x - V_i.d_i)^2 + (x-d_i)^2}} + \sum_{i=1}^{n'} \frac{1}{4\pi} \frac{e_i V_{i\sigma}\ \Theta(-T-\tau_0)}{\sqrt{(V_i.x - V_i.d_i)^2 + (x-d_i)^2}}\ . \tag{2.9}$$

Next let us use the limit $r \to \infty$ with $u < r$ in eq.(2.9). We have $\tau_0 = \frac{u}{|q.V_i|} + \mathcal{O}(1)$, hence (2.9) gives

$$A_\sigma(x)|_{\mathcal{J}^+} = -\frac{1}{4\pi r}\Big[\sum_{i=n'+1}^{n} \frac{e_i V_{i\sigma}}{V_i.q}\Theta(u-T) + \sum_{i=1}^{n'} \frac{1}{4\pi}\frac{e_i V_{i\sigma}}{V_i.q}\Theta(-T-u) + \ldots\Big] + \mathcal{O}(\frac{1}{r^2})\ . \tag{2.10}$$

At large values of u, we see that the $\frac{1}{r}$-term goes like u^0. '...' denote u-fall offs that are faster than any (negative) power law behaviour. The u^0-mode is related to the so called memory effect [43–45]. The memory effect refers to the observed shift in velocity of a test particle (placed at large r and large u) due to passage of electromagnetic radiation. As seen from above expression this shift is universal and is controlled by the leading soft factor. We will see in the forthcoming sections that the $\frac{u^0}{r}$-mode is uncorrected even when we go to higher orders in e. The subleading behaviour changes substantially as we go to next order in e.

Let us study the frequency space radiative field. Given (2.10) for A_μ, the the Fourier transformed function $\tilde{A}_\mu$ at small ω turns out to be :

$$\tilde{A}_\mu(\omega, r, \hat{x}) = \frac{e^{i\omega r}}{4\pi i r}\Big[\frac{1}{\omega}\sum_{i=1}^{n}\frac{\eta_i e_i V_{i\mu}}{V_i.q} + ...\Big] \quad \text{as } \omega \to 0. \tag{2.11}$$

We see that the radiative field behaves as $\frac{1}{\omega}$ as the frequency ω of the radiation is taken to 0. The coefficient of this term is proportional to the leading soft factor given in (2.38). This is the statement of the classical leading soft theorem. This shows the direct relation between the classical soft theorem and the memory effect.

Extending to higher orders in $\frac{1}{r}$, we find that the asymptotic expansion of the radiative field in (2.9) around $\mathcal{I}^+$ takes following form

$$A_\mu(x)|_{\mathcal{I}^+} = \frac{1}{4\pi}\sum_{\substack{m=0,n=1\\ m<n}}^{\infty}[A_\mu^{[n,-m]}(\hat{x})]\,\frac{u^m}{r^n} + ... \,, \quad |u| \to \infty. \tag{2.12}$$

Here '...' denote the terms that fall off faster than any power law. It should be noted that above expression is valid at $\mathcal{O}(e)$ as it was obtained ignoring the asymptotic electromagnetic force present between the scattering particles.

Next we will study asymptotic conservation law satisfied by a particular mode of the radial component of the electric field. As seen in (2.1) the future charge Q^+ is defined at $\mathcal{I}^+_-$. Let us calculate the asymptotic field strength produced by the scattering event using (2.10) around $u \to -\infty$:

$$F_{\mu\nu}(x)|_{u\to-\infty} = \frac{1}{4\pi}\sum_{i=1}^{n'}\frac{e_i\big[V_{i\mu}(x_\nu - d_{i\nu}) - (x_\mu - d_{i\mu})V_{i\nu}\big]}{[(V_i.x - V_i.d_i)^2 + (x-d_i)^2]^{3/2}} + ... \,. \tag{2.13}$$

We note that there are corrections to above expression when we go to higher orders in e and will be discussed in the next chapter. Using (2.2), the coefficient of the $\frac{1}{r^2}$-term can be written down.

$$F_{\mu\nu}(x)|_{u\to-\infty} = -\frac{1}{4\pi r^2}\sum_{i=1}^{n'}\frac{e_i}{(V_i.q)^3}\big[V_{i\mu}q_\nu - q_\mu V_{i\nu} + ...\big] + \mathcal{O}(\frac{1}{r^3}). \tag{2.14}$$

Radial component of the electric field is given by $E_r = F_{ur}$. Performing a co-ordinate transformation of above expression we get

$$F_{ur}(x)|_{u\to-\infty} = -\frac{1}{4\pi r^2}\ [\ \sum_{i=1}^{n'}\frac{e_i}{(V_i.q)^2} + ...\ ]\ + \mathcal{O}(\frac{1}{r^3})\ . \tag{2.15}$$

Next we need to derive the radial component of the electric field at past null infinity and then compare it with the expression at the future null infinity. Past null infinity is defined by the limit $r\to\infty$ with $v = t + r$ finite. In this co-ordinate system, 4 dimensional spacetime point can be parametrised as :

$$x^\mu = r\bar{q}^\mu + vt^\mu, \quad \bar{q}^\mu = (-1,\hat{x}), \quad t^\mu = (1,\vec{0}). \tag{2.16}$$

$\bar{q}^\mu$ is a null vector. We need to expand all the quantities around $\mathcal{I}^-$. Around $\mathcal{I}^-$, we have from (2.8) : $\tau_0 = -2r\ V_i.\bar{q} + \mathcal{O}(1)$. Using (2.9), we see that the contribution of the outgoing particles to the field at $\mathcal{I}^-$ comes with a factor of $\Theta(\tau_0 - T)$, hence it goes to 0. This is a consequence of retarded boundary conditions. From (2.9), we get :

$$A_\sigma(x)|_{\mathcal{I}^-} = \frac{1}{4\pi r}\Big[\sum_{i=1}^{n'}\frac{e_iV_{i\sigma}}{V_i.\bar{q}} + ...\Big] + \mathcal{O}(\frac{1}{r^2})\ . \tag{2.17}$$

Calculating the field strength, we get ($E_r = F_{vr}$)

$$F_{vr}(x)|_{v\to\infty} = -\frac{1}{4\pi r^2}\ [\ \sum_{i=1}^{n'}\frac{e_i}{(V_i.\bar{q})^2} + ...\ ]\ + \mathcal{O}(\frac{1}{r^3})\ . \tag{2.18}$$

Next we can write down the Q_0 conservation law [56,73]. Using (2.15) along with above equation, we get :

$$F_{ur}^{[u^0/r^2]}(\hat{x})|_{\mathcal{I}^+_-} = F_{vr}^{[v^0/r^2]}(-\hat{x})|_{\mathcal{I}^-_+}. \tag{2.19}$$

Here, $F_{ur}^{[u^0/r^2]}$ denotes the coefficient of $\frac{u^0}{r^2}$-term of F_{ur}. Hence it a function of the sphere co-ordinates $\hat{x}$. Similarly $F_{ur}^{[v^0/r^2]}$ denotes the coefficient of $\frac{v^0}{r^2}$-term of F_{vr}. The future charge is defined as $Q_0^+[\lambda^+] = \int d^2z\ \lambda^+(\hat{x})\ F_{ur}^{[u^0/r^2]}(\hat{x})$. Q_0^- is defined analogously. We have :

$$Q_0^+[\lambda^+]\ |\ _{\mathcal{I}^+_-} \quad = \quad Q_0^-[\lambda^-]\ |\ _{\mathcal{I}^-_+}. \tag{2.20}$$

λ^+ is an arbitrary function on 2-sphere and $\lambda^+(\hat{x}) = \lambda^-(-\hat{x})$. Thus this law tells us that the leading order mode in the radial component of the electric field is conserved in above asymptotic sense. This law is related to the leading soft theorem. The leading soft theorem is equivalent to the Ward identity for S-matrix : $Q_0^+ S - S Q_0^- = 0$ [56, 73] with Q_0's defined as given above. We will discuss this equivalence for massless scalar QED in Section 2.3.

2.2 Asymptotic dynamics of massless scalars and photons

We next turn to theory of a massless scalar ϕ minimally coupled to U(1) gauge field A_μ. So, our system is described by following action :

$$S = -\int d^4x \sqrt{-g}\, \Big[\frac{1}{4} F^2 + \eta^{\mu\nu} \left(D_\mu \phi\right)^* \left(D_\nu \phi\right)\Big], \tag{2.21}$$

where $D_\mu \phi = \partial_\mu \phi \;-\; ie A_\mu \phi$. Dynamics of massless scalar field is given by

$$\eta^{\mu\nu} D_\mu D_\nu \; \phi(x) \; = 0. \tag{2.22}$$

The free scalar field satisfies the box equation : $\Box\phi = 0$ where $\Box = -\partial_t^2 + \vec{\partial}^2$. Solution to this equation can be written as

$$\phi(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\omega} \; [b_p \; e^{ip.x} + d_p^\dagger \; e^{-ip.x} \;]. \tag{2.23}$$

Here p^μ is a massless momentum, hence $\omega = |\vec{p}|$. And $p.x = p_\mu x^\mu$ is the Lorentz 4-product. b_p, $d_p^\dagger$ are arbitrary complex functions that parametrise the free data of the massless scalar field.

We are interested in studying the asymptotic dynamics of massless fields. The future null infinity denoted by $\mathscr{I}^+$ corresponds to $r \to \infty$ keeping $u = t - r$ finite. $\mathscr{I}^+$ provides a natural home to define the asymptotic phase space of massless fields. Solution in (2.23) can be expanded around the future null infinity. Let us parametrise the massless momentum by $p = \omega(1, \hat{x}')$. In the retarded co-ordinates we get $p.x = -\omega(u + r - r\hat{x}.\hat{x}')$. In the large r limit we can use stationary phase approximation that sets $\hat{x}' = \hat{x}$. The leading order coefficient in asymptotic expansion for massless scalars turns out to be [56]

$$\phi(u, r, \hat{x}) = \frac{1}{r} \; \phi^1(u, \hat{x}) + \mathcal{O}(\frac{1}{r^2}) \; , \tag{2.24}$$

where

$$\phi^1(u,\hat{x}) = \frac{-i}{8\pi^2}\int_0^\infty d\omega\ [b(\omega,\hat{x})\ e^{-i\omega u} - d^\dagger(\omega,\hat{x})\ e^{i\omega u}\ ]. \tag{2.25}$$

Thus the leading order term in the asymptotic expansion of the massless scalar field is $\mathcal{O}(\frac{1}{r})$. The $\mathcal{O}(\frac{1}{r^2})$ terms get fixed in terms of ϕ^1 by the equation of motion. The asymptotic phase space for massless scalar fields is made up of complex functions $\{\phi^1(u,\hat{x}), \partial_u\phi^1(u,\hat{x})\}$ such that $\phi^1(u,\hat{x})|_{|u|\to\infty} = \mathcal{O}(\frac{1}{u^\epsilon})$. This condition ensures that the symplectic form and other charges (like Hamiltonian) acting on this phase space are well defined [94].

In the quantum theory, b is identified as the annihilation operator for particles while d is the annihilation operator for antiparticles. Thus there is a direct correspondence between the asymptotic fields and the Fock states in the quantum theory. The classical asymptotic field at $\mathcal{I}^+$ written in frequency space is a function of ω and angle $\hat{x}$ and gets mapped to a Fock state with momentum $p = \omega(1,\hat{x})$.

Next we turn to the gauge field. Let us first consider the homogenous part of the gauge field i.e. the free part. Choosing the $\partial_\mu A^\mu = 0$, the equations of motion reduce to $\Box A_\mu = 0$. Hence most of the analysis for the free part of the gauge field is similar to the massless scalar field. And indeed we get [56]

$$A_\mu(u,r,\hat{x}) = \frac{1}{r}\ A^1_\mu(u,\hat{x}) + \dots\ , \tag{2.26}$$

where using stationary phase approximation the leading order term is :

$$A^1_\mu(u,\hat{x}) = \frac{-i}{8\pi^2}\int_0^\infty d\omega\ [a_\mu(\omega,\hat{x})\ e^{-i\omega u} - a^\dagger_\mu(\omega,\hat{x})\ e^{i\omega u}\ ]. \tag{2.27}$$

Here the free data is parametrised by $a_\mu = \sum_{r=+,-}\epsilon^{*r}_\mu(\hat{x})a_r(\omega,\hat{x})$, where the sum runs over the two physical helicities. ϵ^r_μ is the polarisation vector for electromagnetic field. Let us transform to retarded co-ordinates and use following choice for polarisation vectors :

$$\epsilon^\mu_-(\hat{x}) = \frac{1}{\sqrt{2}}\frac{\partial}{\partial\bar{z}}[(1+z\bar{z})q^\mu], \quad \epsilon^\mu_+(\hat{x}) = \frac{1}{\sqrt{2}}\frac{\partial}{\partial z}[(1+z\bar{z})q^\mu]. \tag{2.28}$$

q^μ has been defined in (2.2). It can be checked that $q.\epsilon_\pm = 0$ and that $\epsilon^\mu_\pm$ are null. We have :

$$A^0_z(u,\hat{x}) = \frac{-i}{8\pi^2}\sqrt{\gamma_{z\bar{z}}}\int_0^\infty d\omega\ [a_+(\omega,\hat{x})\ e^{-i\omega u} - a^\dagger_-(\omega,\hat{x})\ e^{i\omega u}\ ] \tag{2.29}$$

and

$$A^0_{\bar{z}}(u,\hat{x}) = \frac{-i}{8\pi^2}\sqrt{\gamma_{z\bar{z}}}\int_0^\infty d\omega\ [a_-(\omega,\hat{x})\ e^{-i\omega u} - a^\dagger_+(\omega,\hat{x})\ e^{i\omega u}\ ]. \tag{2.30}$$

Thus $\{A^0_z(u,\hat{x}), A^0_{\bar{z}}(u,\hat{x})\}$ encode the free data of the U(1) gauge field. The asymptotic phase space for U(1) gauge field is made up of functions $\{A^0_z(u,\hat{x}), A^0_{\bar{z}}(u,\hat{x}),$ $\partial_u A^0_z(u,\hat{x}), \partial_u A^0_{\bar{z}}(u,\hat{x})\}$ such that [1]

$$A^0_B(u,\hat{x})|_{|u|\to\infty} = u^0 + \mathcal{O}(\frac{1}{u^\epsilon}). \tag{2.31}$$

This condition is similar to the one required for massless scalar field except for the u^0-mode. This mode is absent in the massless scalar field. It is required for photons since they have non-trivial zero mode (the $\frac{1}{\omega}$-mode). Similar to the case of massless scalar field, the free data of the U(1) gauge field gets identified with the creation/ annihilation operators of positive/ negative helicity photons in the quantum theory.

Let us discuss the asymptotic behaviour of inhomogenous solution to the equation of motion of the gauge field. We will see that the presence of massless sources leads to new modes in the gauge field. In presence of sources Maxwell's equations take following form

$$\Box A_\mu = -j_\mu, \quad j_\mu = ie\left(\phi D_\mu \phi^* - \phi^* D_\mu \phi\right). \tag{2.32}$$

Using the fall offs given in (2.24) for massless scalars, we get following asymptotic behaviour for the current components :

$$j_u = \frac{j^2_u(u,\hat{x})}{r^2} + ..., \qquad j_A = \frac{j^2_A(u,\hat{x})}{r^2} + ..., \qquad j_r = \frac{j^4_r(u,\hat{x})}{r^4} + ...\ . \tag{2.33}$$

[1] We denote the vector components on S^2 by capital latin alphabets i.e. $B = \{z,\bar{z}\}$.

The asymptotic expansion of the resultant gauge field components is given by :

$$\begin{aligned} A_r &= \frac{A_r^1(\hat{x})}{r} + A_r^{\log}(u,\hat{x})\frac{\log r}{r^2} + \dots , \\ A_u &= A_u^{\log}(u,\hat{x})\frac{\log r}{r} + \frac{A_u^1(u,\hat{x})}{r} + \dots , \\ A_A &= A_A^0(u,\hat{x}) + A_A^{\log}(u,\hat{x})\frac{\log r}{r} + \dots . \end{aligned} \tag{2.34}$$

The asymptotic expansion given in (2.34) leads to following fall offs for the field strength :

$$\begin{aligned} F_{ru} &= \frac{F_{ru}^2(u,\hat{x})}{r^2} + \dots , & F_{uA} &= F_{uA}^0(u,\hat{x}) + \dots , \\ F_{AB} &= F_{AB}^0(u,\hat{x}) + \dots , & F_{rA} &= \frac{F_{rA}^2(u,\hat{x})}{r^2} + \dots . \end{aligned} \tag{2.35}$$

It is important to note that F_{rA} actually starts at $F_{rA} = F_{rA}^{\log}(u,\hat{x})\frac{\log r}{r^2} + \dots$ due to presence of massless fields. Maxwell's equations imply $\partial_u F_{rA}^{\log} = 0$, so we can set this mode to 0 consistently.

The Maxwell's equations are given by $\nabla^\nu F_{\sigma\nu} = j_\sigma$ and imply following equations for the coefficients in (2.35) :

$$\begin{aligned} \partial_u F_{ru}^2 \ + \ \partial_u D^B A_B^0 \ &= \ j_u^2, \\ \partial_u F_{rA}^2 \ - \ \frac{1}{2}\partial_A F_{ru}^2 \ + \frac{1}{2} D^B F_{AB}^0 \ &= \ \frac{1}{2} j_A^2. \end{aligned} \tag{2.36}$$

We will need these equations in our subsequent calculations.

2.3 Leading soft theorem and the Q_0 conservation law

In this subsection, we will discuss the Ward identity associated with the Q_0 charge defined in (2.20) in the context of massless scalar QED and show that it is equivalent to the leading soft theorem. Soft theorems are universal statements about quantum amplitudes in the low energy limit. In the limit when energy of one of the scattering photons is taken to be small, the amplitude factorises into the lower point amplitude without soft photon times a universal soft factor :

$$\mathcal{A}_{n+1}(p_i,k) = \frac{S_0}{\omega}\ \mathcal{A}_n(p_i) \ + \dots . \tag{2.37}$$

$$S_0 = \sum_{i=1}^{n} \eta_i e_i \frac{\epsilon . p_i}{p_i . q}. \tag{2.38}$$

S_0 is the leading soft factor. Here e_i, p_i are respectively the charges and momenta of the hard particles and we have used the standard convention of writing soft factors with η_i such that $\eta_i = 1(-1)$ for outgoing (incoming) particles. ϵ is the polarisation vector of the soft photon and $k^\mu = \omega q^\mu$ is the soft momentum.

Let us rewrite the leading soft theorem using special variables that are well suited for asymptotic calculations. We parametrise massless momenta by

$$p_j = \omega_j q_j, \quad q_j = \frac{1}{1+z_j\bar{z}_j}\{1+z_j\bar{z}_j, z_j+\bar{z}_j, -i(z_j-\bar{z}_j), 1-z_j\bar{z}_j\}.$$

Similarly the soft momentum is parametrised by $k_\mu = \omega q_\mu$. The polarisation vectors are given by [56]

$$\epsilon_-^\mu = \frac{1}{\sqrt{2}}\frac{\partial}{\partial \bar{z}}[(1+z\bar{z})q^\mu], \quad \epsilon_+^\mu = \frac{1}{\sqrt{2}}\frac{\partial}{\partial z}[(1+z\bar{z})q^\mu]. \tag{2.39}$$

Using the expression for q_μ in (2.39) we get :

$$\epsilon_-^\mu = \frac{1}{\sqrt{2}}\{z, 1, i, z\}, \quad \epsilon_+^\mu = \frac{1}{\sqrt{2}}\{\bar{z}, 1, -i, \bar{z}\}. \tag{2.40}$$

Using above expressions we can rewrite the leading soft theorem for negative helicity photon as

$$\lim_{\omega\to 0}[\omega\ \mathcal{A}_{n+1}(\omega_i, z_i^A; \omega, z^A, -)] = \sqrt{\gamma^{z\bar{z}}}\sum_{i=1}^{n}\frac{\eta_i e_i}{\bar{z}-\bar{z}_i}\ \mathcal{A}_n(\omega_i, z_i^A)\ . \tag{2.41}$$

Here we have used the projector $\lim_{\omega\to 0}[\omega\ ...]$ to isolate the leading order term. Hence above expression is an exact statement.

Next we will show that the leading soft theorem i.e. (2.41) is equivalent to Ward identity for S-matrix : $Q_0^+ S - S Q_0^- = 0$ where Q_0's are defined in (2.20). Let us turn to the expression for the asymptotic charge Q_0. The future charge is given by (2.20)

$$\begin{aligned} Q_0^+[\lambda^+] &= \int d^2z\ \lambda^+(\hat{x})\ F_{ru}^{2,0}(\hat{x})|_{\mathcal{I}_-^+} \\ &= -\int du\ d^2z\ \lambda^+(\hat{x})\ \partial_u F_{ru}^{2,0}(\hat{x}) + \int d^2z\ \lambda^+(\hat{x})\ F_{ru}^{2,0}(\hat{x})|_{\mathcal{I}_+^+}. \end{aligned} \tag{2.42}$$

The second term vanishes for our case. It should be noted that this term is non-zero in presence of massive particles. Next we use (2.36) in the expression for the future charge to get

$$Q_0^+[\lambda^+] = \int du\ d^2z\ \lambda^+(\hat{x})\ [\partial_u D^B A_B^0\ -\ j_u^2]$$
$$:= Q_+^{\text{soft}}[\lambda] + Q_+^{\text{hard}}[\lambda]. \tag{2.43}$$

This defines the soft and hard parts of the asymptotic charge.

The soft charge is linear in the gauge field and leads to insertion of photon. Let us analyse the expression for soft charge.

$$Q_+^{\text{soft}}[\lambda] = \int du\ d^2z\ \lambda^+(\hat{x})\ \gamma^{z\bar{z}}\partial_u[\partial_z A_{\bar{z}}^0 + \partial_{\bar{z}} A_z^0]. \tag{2.44}$$

Using (2.14), it is seen that the retarded solution satisfies $F_{z\bar{z}}|_{\mathcal{I}^+_\pm} = 0$. Generically the field is required to satisfy the condition $\int du\ \partial_u F_{z\bar{z}} = 0$ [59]. This condition implies $\int du\ \partial_u \partial_z A_{\bar{z}}\ =\ \int du\ \partial_u \partial_{\bar{z}} A_z$. Substituting in the expression for soft charge, we get

$$Q_+^{\text{soft}}[\lambda] =\ 2\int du\ d^2z\ \lambda^+(\hat{x})\ \gamma^{z\bar{z}}\partial_u\partial_z A_{\bar{z}}^0. \tag{2.45}$$

It is useful to transform to the Fourier space basis $\tilde{A}_{\bar{z}}$.

$$Q_+^{\text{soft}}[\lambda] = -\ 2i\int d\omega\ d^2z\ \lambda^+(\hat{x})\ \delta(\omega)\ \gamma^{z\bar{z}}\partial_z(\omega\tilde{A}_{\bar{z}}^0). \tag{2.46}$$

It should be noted that the integral over ω ranges from $-\infty$ to ∞. It is clear that only zero energy modes contribute to above expression. Hence the name 'soft' charge. Also it is apparent from above expression that it is non zero only if $\tilde{A}_{\bar{z}}^0$ has a pole at $\omega = 0$.

To simplify our calculations we parametrise λ^+ in following way, $\lambda^+(\hat{x}) = \frac{1}{\bar{z}-\bar{z}_a}$. After an integration by parts we get

$$Q_+^{\text{soft}}[z_a] =\ 4\pi i\int d\omega\ \delta(\omega)\ \omega\tilde{A}_{\bar{z}}^0(\omega, \hat{x}_a). \tag{2.47}$$

Here we have used the identity : $\partial_z \frac{1}{\bar{z}-\bar{z}_a} = 2\pi\delta^2(z-z_a)$. Next we quantise above expression of the charge. The gauge field $A_{\bar{z}}^0$ can be expressed in terms of the creation and annihilation operators of photon. Using (2.30), we get

$$\tilde{A}^0_{\bar{z}}(\omega,\hat{x}) = -i\sqrt{2}\frac{a_-(\omega,\hat{x})}{4\pi(1+z\bar{z})} \quad ... \quad \omega > 0, \quad \tilde{A}^0_{\bar{z}}(\omega,\hat{x}) = i\sqrt{2}\frac{a^\dagger_+(-\omega,\hat{x})}{4\pi(1+z\bar{z})} \quad ... \quad \omega < 0. \tag{2.48}$$

Above expression hints that there is a discontinuity at $\omega = 0$ in the quantum theory. Later on we will see that it plays an important role in the analysis of the $\log\omega$ soft theorem. We use (2.48) in the expression for the soft charge. Since $\tilde{A}^0_{\bar{z}}(\omega)$ is discontinuous at $\omega = 0$, we need a prescription to define the limit $\omega \to 0$. We define the soft limit from positive side to get

$$Q^{\text{soft}}_+[z] = \sqrt{\gamma_{z\bar{z}}}\lim_{\omega\to 0^+}\omega\; a_-(\omega,\hat{x}). \tag{2.49}$$

Next we turn to the expression of the hard charge given in (5.20). Using $\lambda^+ = \frac{1}{\bar{z}-\bar{z}_a}$, we get

$$Q^{\text{hard}}_+[z_a] = -\int du\; d^2z\; \frac{j^2_u}{\bar{z}-\bar{z}_a}. \tag{2.50}$$

Since we have the complete expression for the asymptotic charge, we can write down the Ward identity for the charge. The Ward identity for S matrix is

$$\begin{aligned}\Big[\, Q_0\;,\; S\,\Big] &= 0,\\ \Rightarrow \Big(Q^{\text{soft}}_+\; S \;-\; S\; Q^{\text{soft}}_-\Big) &= -\Big(Q^{\text{hard}}_+\; S \;-\; S\; Q^{\text{hard}}_-\Big).\end{aligned}$$

Using (2.49) and (2.50), we get

$$\begin{aligned}&< \text{out}|\lim_{\omega\to 0^+}\sqrt{\gamma_{z\bar{z}}}\;\omega\; a_-(\omega,\hat{x})\; S\;|\text{in} >\\ &= \int du'\; d^2z'\;\frac{1}{\bar{z}'-\bar{z}}\; < \text{out}|\;\big[j^2_u\; S \;-\; S\; j^2_v\;\big]\;|\text{in} > .\end{aligned} \tag{2.51}$$

The action of above operators on the Fock states can be evaluated in a straightforward way :

$$< \text{out}|\lim_{\omega\to 0^+}\omega\; a_-(\omega,\hat{x})\; S\;|\text{in} > \;=\; \sqrt{\gamma^{z\bar{z}}}\sum_i\frac{\eta_i e_i}{\bar{z}-\bar{z}_i}\; < \text{out}|\; S\;|\text{in} > . \tag{2.52}$$

Comparing with (2.41) we see that this is the statement of leading soft theorem for outgoing negative helcity theorem. Thus we have proved that the leading soft theorem can be derived from the conservation law (2.20). These steps can be retraced to derive the conservation law from the soft theorem. This establishes the equivalence between the classical asymptotic conservation law for Q_0 and the leading soft theorem.

2.4 Subleading soft photon theorem

Let us first discuss the subleading term in the soft expansion of tree level amplitudes [2–4].

$$\mathcal{A}_{n+1}(p_i,k) = \left[\frac{S_0}{\omega} \; + S_1 \;\right] \mathcal{A}_n(p_i) \; + \mathcal{O}(\omega) \; . \tag{2.53}$$

The subleading term is $\mathcal{O}(\omega^0)$ and the coefficient is given by

$$S_1 = \sum_{i=1}^{n} e_i \frac{\epsilon_\mu k_\nu \; J_i^{\mu\nu}}{q.p_i} + \ldots \; .$$

We recall that e_i, p_i are respectively the charges and momenta of the hard particles and $\eta_i = 1(-1)$ for outgoing (incoming) particles. ϵ^μ is the polarisation vector of the soft photon and $k^\mu = \omega q^\mu$ is the soft momentum. $J_i^{\mu\nu} = p_i^\mu \partial_i^\nu - p_i^\nu \partial_i^\mu$ denotes the angular momentum of the i^{th} hard particle. '...' are the corrections due to non-minimal couplings that might be present in the theory. These corrections have been studied in [74]. This should be contrasted with the leading soft theorem that is uncorrected by non-minimal couplings. Ward identity corresponding to this soft photon theorem has been studied in [62–64, 76]. But the symmetry underlying these charges is not clear yet. The asymptotic conservation law underlying the tree level subleading soft photon theorem was discussed in [75].

The leading soft photon theorem and the corresponding Ward identity are true to all loop orders and hence are exact quantum statements. Beyond the leading order, there are non-trivial loop corrections in four spacetime dimensions [79–81]. The subleading term in the soft expansion of loop amplitudes was systematically studied in [91]. Using the fact that the infrared divergences for $(n+1)$ amplitude and n amplitude in QED cancel, they unambiguously identified the soft factor corresponding to the subleading $\log\omega$ mode and showed that it is universal. In this thesis, we will derive the asymptotic conservation law underlying this loop level soft theorem.

Let us state the soft expansion of loop amplitudes in QED. We have [91]

$$\mathcal{A}_{n+1}(p_i,k) = \frac{S_0}{\omega}\,\mathcal{A}_n(p_i)\ +\ S_{\log}\log\omega\ \mathcal{A}_n(p_i)\ + \dots\ , \tag{2.54}$$

here, $S_0 = \sum_i \eta_i e_i \frac{\epsilon.p_i}{p_i.q}$ is the leading soft factor and

$$\begin{aligned} S_{\log} &= \frac{i}{4\pi}\sum_{\substack{i,j\ i\neq j\\ \eta_i\eta_j=1}} e_i^2 e_j \frac{\epsilon^\mu q^\nu}{q.p_i}\ m_i^2 m_j^2\ \frac{[p_{i\mu}p_{j\nu}-p_{i\nu}p_{j\mu}]}{[(p_i.p_j)^2-m_i^2m_j^2]^{\frac{3}{2}}} \\ &\quad -\frac{1}{8\pi^2}\sum_{\substack{i,j\\ i\neq j}}\eta_i\eta_j e_i^2 e_j\frac{\epsilon^\mu q^\nu}{q.p_i}\ [p_{i\mu}\partial_{i\nu}-p_{i\nu}\partial_{i\mu}]\ \Big[\frac{p_i.p_j}{[(p_i.p_j)^2-m_i^2m_j^2]^{\frac{1}{2}}}\log\frac{p_i.p_j+\sqrt{(p_i.p_j)^2-m_i^2m_j^2}}{p_i.p_j-\sqrt{(p_i.p_j)^2-m_i^2m_j^2}}\Big]. \end{aligned} \tag{2.55}$$

Thus the form of the subleading soft factor is much more complicated than the leading soft factor. Hence it is quite amazing that the authors of [93] could reproduce this complicated factor from an asymptotic charge. This soft factor is related to the dressing of the hard particles under long range electromagnetic force. The first line of $S_{\log}$ constitutes the classical soft factor. This is the term that controls the low energy expansion of classical radiative field. It is a direct effect of late time acceleration of the hard particles under long range electromagnetic forces. The term in the second line is purely quantum. This mode is absent in the classical theory. This can be checked by explicitly calculating low energy radiative field emitted in a classical process [36, 91].

The $\log\omega$ soft theorem receives corrections in presence of dynamical gravity. The correction is as follows

$$\begin{aligned} S^{\rm grav}_{\log} =& \frac{-i}{8\pi}\sum_{\substack{i,j\ i\neq j\\ \eta_i\eta_j=1}} e_i\frac{\epsilon^\mu q^\nu}{q.p_i}[p_{i\mu}p_{j\nu}-p_{i\nu}p_{j\mu}]\ p_i.p_j\ \frac{[2(p_i.p_j)^2-3m_j^2m_i^2]}{[(p_i.p_j)^2-m_i^2m_j^2]^{\frac{3}{2}}} - \frac{i}{4\pi}\sum_{i,\ \eta_i=1} q.p_i\ \sum_j \eta_j e_j\frac{\epsilon.p_j}{q.p_j} \\ &-\frac{1}{16\pi^2}\sum_{\substack{i,j\\ i\neq j}}\eta_i\eta_j e_i\frac{\epsilon^\mu q^\nu}{q.p_i}[p_{i\mu}\partial_{i\nu}-p_{i\nu}\partial_{i\mu}]\Big[\frac{2(p_i.p_j)^2-m_i^2m_j^2}{[(p_i.p_j)^2-m_i^2m_j^2]^{\frac{1}{2}}}\log\frac{p_i.p_j+\sqrt{(p_i.p_j)^2-m_i^2m_j^2}}{p_i.p_j-\sqrt{(p_i.p_j)^2-m_i^2m_j^2}}\Big] \\ &+\frac{1}{8\pi^2}\sum_i \eta_i q.p_i\ \log\frac{(p_i.q)^2}{m_i^2}\ \sum_j\eta_j e_j\frac{\epsilon.p_j}{q.p_j}\ . \end{aligned} \tag{2.56}$$

Analogous to the electromagnetic soft factor, the first line of $S^{\rm grav}_{\log}$ constitutes the classical soft factor. It is a direct effect of late time acceleration of the particles under long range gravitational forces. The term in the second line is purely quantum and is absent in the classical theory. It is expected that the gravitational correction to the soft factor can be derived by including the effect of gravity in the analysis of [93].

The important takeaway point from above discussion is that the $\log \omega$ mode is intimately tied to the long range forces present in 4 spacetime dimensions. In the next section we will start by analysing the effect of long range electromagnetic force on classical dynamics.

Chapter 3

Asymptotic conservation laws in Classical electromagnetism

In Chapter 2 we obtained the asymptotic field generated by a scattering event ignoring the long range forces acting on the scattering particles. In this chapter we will include the effect of long range forces present between the scattering particles and study the new modes that arise in the asymptotic radiative field. Further we will also derive the asymptotic conservation laws obeyed by these new modes. We carry out the calculations perturbatively in coupling e and asymptotic parameter $\frac{1}{r}$. This chapter is based on our results derived in [99].

3.1 The Q_1 conservation law

In this section we will obtain the asymptotic radiative field keeping the first order correction in e. Therefore we need to take into account the leading order effect of electromagnetic long range force acting on scattering particles. Due to the presence of long range electromagnetic force, a particle continues to accelerate at late times and this gives rise to new modes in the asymptotic field starting at $\mathcal{O}(e^3)$. In particular F_{rA} gets a $\frac{\log u}{r^2}$ term because of the long range interaction between particles :

$$F_{rA}|_{u\to-\infty} = \frac{1}{r^2}\ [\ u\ F_{rA}^{[u/r^2]}(\hat{x})\ +\ \log u\ F_{rA}^{[\log u/r^2]}(\hat{x})\ +...] + \mathcal{O}(\frac{1}{r^3})\ . \tag{3.1}$$

Similarly around the past null infinity we have : [1]

$$F_{rA}|_{v\to\infty} = \frac{\log r}{r^2}\ [v^0\ F_{rA}^{[\log r/r^2]}(\hat{x})\ +...] + \mathcal{O}(\frac{1}{r^2})\ . \tag{3.2}$$

[1]The $\frac{\log r}{r^2}$-mode was missed in [93].

In this section we will derive the conservation law obeyed by these modes. [2]

$$F_{rA}^{[\log u/r^2]}(\hat{x})|_{\mathcal{I}^+_-} = F_{rA}^{[\log r/r^2]}(-\hat{x})|_{\mathcal{I}^-_+}. \tag{3.3}$$

This is the first of the conservation laws that we discuss. It is important to note that above modes are $\mathcal{O}(e^3)$ and we expect that above modes will not get corrected by higher order corrections in e.

Our goal is to derive above conservation law. In this process we will also rederive the leading tail to the memory term in the radiative field. Let us find the first order correction to equation of trajectory of particles in asymptotic regions i.e. to eqn (2.5). This calculation has been done in [91], we reproduce it here. The equation of trajectory of j^{th} outgoing particle is given by :

$$m_j \frac{\partial^2 x_j^\mu}{\partial \tau^2} = e_j \; F^{\mu\nu}(x_j(\tau)) \; V_{j\nu}. \tag{3.4}$$

We need to find the field experienced by j which is generated by other particles that interact with j. The field strength has been calculated in (2.13). The field strength given in (2.13) needs to be evaluated at the position of the particle i.e. $x = x_j(\tau)$. Using $x^\mu = x_j^\mu(\tau) = V_j^\mu \tau + d_j^\mu$ in (2.13) we see that the electromagnetic force F felt by the scattered particle takes following form :

$$F(\tau) = \sum_{m=2}^{\infty} \frac{c_m}{\tau^m}.$$

The $\mathcal{O}(\frac{1}{\tau^2})$-term depends only on the charges and asymptotic velocities of the interacting particles. Dipole interactions and higher order moments contribute to the terms for $m > 2$ and hence the $m > 2$ modes are sensitive to the charge distribution of the scattering objects. The leading order term in the field strength is given by

$$F_{\mu\nu}(x_j(\tau))|_{\tau\to\infty} = \frac{1}{4\pi\tau^2} \sum_{\substack{i=m+1,\\ i\neq j}}^{n} e_i \frac{(V_{i\mu}V_{j\nu} - V_{j\mu}V_{i\nu})}{[(V_i.V_j)^2 - 1]^{3/2}} + \mathcal{O}(\frac{1}{\tau^3}). \tag{3.5}$$

Here, we have not included any incoming particles as they do not contribute to the field at $\tau \to \infty$. So, the asymptotic trajectory of the j^{th} particle is given by

[2] We thank the authors of [93] for suggesting this new conservation law.

following equation :

$$m_j \frac{\partial^2 x_j^\mu}{\partial \tau^2} = -\frac{e_j}{4\pi\tau^2} \sum_{\substack{i=m+1,\\ i\neq j}}^{n} e_i \frac{(V_{i\mu} + V_{j\mu} V_i.V_j)}{[(V_i.V_j)^2 - 1]^{3/2}} + \mathcal{O}(\frac{1}{\tau^3}). \tag{3.6}$$

Solving above differential equation we see that the asymptotic trajectories of the particles are corrected with a logarithmic term.

$$x_i^\mu = V_i^\mu \ \tau + c_i^\mu \log\tau + d_i + \mathcal{O}(\frac{1}{\tau}). \tag{3.7}$$

where for outgoing particles we have :

$$c_i^\mu = \frac{1}{4\pi} \sum_{\substack{j=m+1,\\ j\neq i}}^{n} e_i e_j \frac{(V_i.V_j \ V_i^\mu + V_j^\mu)}{[(V_i.V_j)^2 - 1]^{3/2}}. \tag{3.8}$$

Above expression carries an extra minus sign compared to [91] because of difference in convention. For i^{th} incoming particle, j runs over the incoming particles :

$$c_i^\mu = \frac{1}{4\pi} \sum_{\substack{j=1,\\ j\neq i}}^{m} e_i e_j \frac{(V_i.V_j \ V_i^\mu + V_j^\mu)}{[(V_i.V_j)^2 - 1]^{3/2}}. \tag{3.9}$$

Next we find the asymptotic field produced by an outgoing particle i with the corrected trajectory given in (3.7). As a result of long range interactions, the current corresponding to an i^{th} particle is modified to $j_\sigma^{(i)}(x') = \int d\tau \ e_i \big[V_{i\sigma} + \frac{c_{i\sigma}}{\tau}\big] \ \delta^4(x' - x_i) \ \Theta(\tau - T)$. Using the retarded propagator we can write down the expression for classical radiation sourced by i^{th} asymptotically accelerating particle.

$$A_\sigma^{(i)}(x) = \frac{1}{2\pi} \int d\tau \ \delta([x - x_i(\tau)]^2) \ e_i \big[V_{i\sigma} + \frac{c_{i\sigma}}{\tau}\big] \ \Theta(t - t_i) \ \Theta(\tau - T). \tag{3.10}$$

This equation includes $\mathcal{O}(e^3)$ corrections to eqn (2.9). Solving the δ-function condition is highly difficult because of the logarithmic correction. We solve it perturbatively. The details of the calculation are relegated to Appendix A. We quote the solution to the delta function constraint from (A.5) :

$$\tau_1 = -V_i.(x - d_i) - \Big[\ (V_i.x - V_i.d_i)^2 + (x - d_i)^2 \ - 2(x - d_i).c_i \log\tau_0 \ \Big]^{1/2}. \tag{3.11}$$

Here, τ_0 is the zeroth order solution given in (2.8). We also have $\delta([x-x_i(\tau)]^2) = \frac{\delta(\tau-\tau_1)}{|2\tau+2V_i.(x-d_i)+\frac{2(x-d_i).c_i}{\tau}|}$. The result of the integral is :

$$A^{(i)}_\sigma(x) = \frac{1}{4\pi}\frac{\Theta(\tau_1 - T)\ e_i\big[V_{i\sigma} + \frac{c_{i\sigma}}{\tau_1}\big]}{\big[\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ \ - 2(x-d_i).c_i\log\tau_0\big]^{1/2} - \frac{(x-d_i).c_i}{\tau_1}}. \tag{3.12}$$

Since above expression is vaild only to $\mathcal{O}(e^3)$, we can expand the denominator to $\mathcal{O}(e^3)$ as well. Summing over all the incoming and outgoing particles, we get

$$\begin{aligned} A_\sigma(x) = & \frac{1}{4\pi}\sum_{i=n'+1}^{n}\frac{e_i}{X}\Theta(\tau_1 - T)\ \Big[\ V_{i\sigma}\big[1 + \frac{1}{X^2}(x-d_i).c_i\log\tau_0 + \frac{(x-d_i).c_i}{X\tau_1}\big] + \frac{c_{i\sigma}}{\tau_1}\Big] \\ & + \frac{1}{4\pi}\sum_{i=1}^{n'}\frac{e_i}{X}\Theta(-\tau_1 - T)\ \Big[\ V_{i\sigma}\big[1 + \frac{1}{X^2}(x-d_i).c_i\log\tau_0 + \frac{(x-d_i).c_i}{X\tau_1}\big] + \frac{c_{i\sigma}}{\tau_1}\Big]\ , \end{aligned} \tag{3.13}$$

$$\text{where } X = [\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ ]^{1/2}. \tag{3.14}$$

This is the asymptotic radiative field generated by the scattering process including the leading effect of long range electromagnetic force on scattering particles.

Next we study the asymptotic expansion of above expression. Focussing on the $\frac{1}{r}$-term of A_σ, we get :

$$\begin{aligned} A_\sigma(x)|_{\mathcal{I}^+} = & -\frac{1}{4\pi r}\sum_{i=n'+1}^{n} e_i\Theta(u-T)\Big[\ \frac{V_{i\sigma}}{q.V_i} - \frac{1}{u}\big[c_{i\sigma} - V_{i\sigma}\frac{q.c_i}{q.V_i}\big]\Big] \\ & - \frac{1}{4\pi r}\sum_{i=1}^{n'} e_i\Theta(-u-T)\Big[\ \frac{V_{i\sigma}}{q.V_i} - \frac{1}{u}\ \big[c_{i\sigma} - V_{i\sigma}\frac{q.c_i}{q.V_i}\big]\ \Big] + \dots\ . \end{aligned} \tag{3.15}$$

We can compare above fall offs to the leading order radiative fall offs in (2.10). It is interesting to note that including even the first order correction in e has altered the late time profile appreciably. The presence of the $\frac{1}{u}$-term leads to the so called tail memory effect [37, 39]. Like the leading u^0-mode the $\frac{1}{u}$-mode is also universal and insensitive to details of the bulk trajectories.

It is interesting to study the frequency space radiative field. Given (3.15), we can study its fourier transform. The Fourier transformed function has following behaviour at small ω [39] :

$$\tilde{A}_\mu(\omega, r, \hat{x}) = \frac{e^{i\omega r}}{4\pi i r}\Big[\ \frac{S^0_\mu}{\omega}\ +\ S^1_\mu\ \log\omega + \dots\Big]\ \text{ as } \omega \to 0. \tag{3.16}$$

This gives us the classical subleading soft theorem. Like the quantum subleading soft term discussed in (2.54) the classical subleading term is also logarithmic. The classical soft factor is

$$S^1_\mu = i\sum_{j=1}^{n}\eta_j e_j\ \Big[\frac{V_{j\mu}}{V_j.q}\ q.c_j - c_{j\mu}\ \Big]\ . \tag{3.17}$$

We note that the coefficient of $\log\omega$ in the classical field i.e. S_1 is only a part of $S_{\log}$ that appears in quantum soft theorem given in (2.55). As discussed earlier, a part of $S_{\log}$ vanishes in the classical theory.

Let us conclude this discussion after studying the form of subleading corrections to (3.15). We recall that the subleading terms in (3.7) were ignored. If we include the effect of these subleading terms, then (3.15) takes following form :

$$A_\mu(x) \sim \frac{1}{4\pi r}\Big[e\ u^0\ +\ e^3\sum_{n=1}^{\infty}\frac{1}{u^n}\ \Big] + \mathcal{O}(\frac{1}{r^2}),\quad u\to\pm\infty. \tag{3.18}$$

Thus at $\mathcal{O}(e^3)$, the radiative field is expected to have power law tails such that the leading tail is universal. In the next section we will discuss the $\mathcal{O}(e^5)$ corrections to above expression.

Next we turn to the $\frac{1}{r^2}$-term of A_σ and derive the conservation law that we briefly discussed in (3.3). In the previous section, we rederived the Q_0 conservation law by comparing the respective asymptotic expansions of the radiative field around future and past. We will follow similar strategy here. We need to expand all the terms in (3.13) around $\mathcal{I}^+$. Using (2.8), we have $\log\tau_0|_{\mathcal{I}^+}\sim\log u+\mathcal{O}(1)$. Then using (2.2) we find the leading order term in (3.11) :

$$\tau_1|_{\mathcal{I}^+} = -\frac{u}{q.V_i} - \frac{q.c_i}{q.V_i}\ \log u + \mathcal{O}(1). \tag{3.19}$$

Using (2.2) in (3.14), we get $X = -rq.V_i + \mathcal{O}(u)$. Substituting the limiting value of X in (3.13), it is seen that the leading term in $\frac{1}{r^2}$-term of A_σ is $\mathcal{O}(u)$ while the subleading term is $\mathcal{O}(\log u)$ as noted in (3.1). We can read off the coefficient of the $\mathcal{O}(\frac{\log u}{r^2})$ term in A_σ :

$$A_\sigma^{[\log u/r^2]}(x)|_{\mathcal{I}^+} = -\frac{1}{4\pi}\sum_{i=n'+1}^{n}\Theta(u-T)\ e_i\ V_{i\sigma}\ \frac{q.c_i}{(q.V_i)^3} - \frac{1}{4\pi}\sum_{i=1}^{n'}\Theta(-u-T)e_i\ V_{i\sigma}\ \frac{q.c_i}{(q.V_i)^3}. \tag{3.20}$$

From here on, we just need to transform co-ordinates to get to F_{rA}. We have : $A_r = q^\mu A_\mu$ and $A_A = r(\partial_A q^\mu)A_\mu$. Using it in $F_{rA} = \partial_r A_A - \partial_A A_r$, we get :

$$F^{[\log u/r^2]}_{rA}(x)|_{\mathcal{I}^+_-} = \frac{1}{4\pi}\sum_{i=1}^{n'} \frac{e_i\ q^\mu(\partial_A q^\nu)}{(q.V_i)^3}\ [V_{i\mu}c_{i\nu} - V_{i\nu}c_{i\mu}]. \tag{3.21}$$

Let us derive the field configuration at past null infinity and compare above expression with the coefficient of $\mathcal{O}(\frac{\log r}{r^2})$ term at the past. So, we expand A_σ in (3.13) around $\mathcal{I}^-$. Using (2.16) the leading order term in (3.11) is :

$$\tau_1|_{\mathcal{I}^-} = -2r\ V_i.\bar{q} + \frac{\bar{q}.c_i}{V_i.\bar{q}}\ \log r + \mathcal{O}(1). \tag{3.22}$$

Using (2.8), we get $\log \tau_0|_{\mathcal{I}^-} \sim \log r + \mathcal{O}(1)$. Substituting in (3.13), we write down the coefficient of the $\mathcal{O}(\frac{\log r}{r^2})$ term in A_σ :

$$A^{[\log r/r^2]}_{\sigma}(x)|_{\mathcal{I}^-} = \frac{1}{4\pi}\sum_{i=1}^{n'} e_i\ V_{i\sigma}\ \frac{\bar{q}.c_i}{(\bar{q}.V_i)^3}. \tag{3.23}$$

Performing co-ordinate transformation :

$$F^{[\log r/r^2]}_{rA}(x)|_{\mathcal{I}^-_+} = -\frac{1}{4\pi}\sum_{i=1}^{n'} \frac{e_i\ \bar{q}^\mu(\partial_A \bar{q}^\nu)}{(\bar{q}.V_i)^3}\ [V_{i\mu}c_{i\nu} - V_{i\nu}c_{i\mu}]. \tag{3.24}$$

Thus, from (3.21) and (3.24) we can indeed check that the modes are equal under antipodal idenfication. The apparently extra minus sign in (3.24) is compensated by the factors of q^μ. Finally we have shown that a generic scattering process obeys following conservation law :

$$F^{[\log u/r^2]}_{rA}(\hat{x})|_{\mathcal{I}^+_-} = F^{[\log r/r^2]}_{rA}(-\hat{x})|_{\mathcal{I}^-_+}. \tag{3.25}$$

The corresponding charges are defined as $Q^+_1 = \int d^2z\ F^{[\log u/r^2]}_{rA}(\hat{x})\ W^A(\hat{x})|_{\mathcal{I}^+_-}$ and $Q^-_1 = \int d^2z\ F^{[\log r/r^2]}_{rA}(-\hat{x})W^A(-\hat{x})|_{\mathcal{I}^-_+}$. The charges are parametrised by an S^2 vector field W^A and are expected to be conserved exactly.

Let us emphasise important points about this conservation law. The charges are related to the logarithmic modes in the radiative field which arise as a consequence of the long range interactions between the scattering particles. These modes appear at $\mathcal{O}(e^3)$ and are expected to be uncorrected at higher orders in e. It is natural to expect that this conservation law would also be related to a soft theorem. Using Maxwell's equations it can be shown that the

Q_1 charge is related to the $\log\omega$ soft mode. In Chapter 5, we will quantise the Q_1 charge and show that it reproduces the full $\log\omega$ soft photon theorem [91].

3.2 The Q_2 conservation law

In this section we will obtain the asymptotic radiative field keeping the subleading corrections in e. Including these corrections we show that the mode expansion of the $\frac{1}{r^3}$ term of F_{rA} around future null infinity is :

$$F_{rA}^{[1/r^3]}|_{u\to-\infty} = u^2\ F_{rA}^{[u^2/r^3]}(\hat{x})\ + u\log u\ F_{rA}^{[u\log u/r^3]}(\hat{x}) + (\log u)^2\ F_{rA}^{[(\log u)^2/r^3]}(\hat{x})\ + ...\ . \tag{3.26}$$

Expansion of F_{rA} around the past null infinity is given by :

$$F_{rA}|_{v\to\infty} = \frac{\log r}{r^2}\ [v^0\ F_{rA}^{\log r/r^2]}(\hat{x})\ + ...] + \frac{(\log r)^2}{r^3}\ [\ v^0\ F_{rA}^{[(\log r)^2/r^3]}(\hat{x})\ + ...] + \mathcal{O}(\frac{1}{r^2})\ . \tag{3.27}$$

In this section, we show that a generic classical scattering process obeys following conservation law :

$$F_{rA}^{[(\log u)^2/r^3]}(\hat{x})|_{\mathcal{I}^+_-} = -F_{rA}^{[(\log r)^2/r^3]}(-\hat{x})|_{\mathcal{I}^-_+}. \tag{3.28}$$

This is the second of the conservation laws that we derive.

Next we will calculate the asymptotic field configuration to prove above conservation law. Let us first outline our steps. In the last section, we obtained the asymptotic field including $\mathcal{O}(e^3)$ corrections. The $\mathcal{O}(e^3)$ modes in the radiation arise due to acceleration of the charged particles under the long range electromagnetic force. This radiation backreacts on the particles. When we go to higher orders in e we need to include the effect of this backreaction. The backreaction leads to deviation in the asymptotic trajectories of the particles. Let us estimate the subleading correction to the equation of trajectory (3.6). The field strength can be calculated using (3.13) and then is evaluated at the position of j^{th} particle. Using (3.7) in (3.13), we get following modes in the field strength

$$F_{\mu\nu}(x_j(\tau)) \sim e\ \sum_{m=2}^{\infty}\frac{1}{\tau^m} + e^3\ \sum_{m=3}^{\infty}\frac{\log\tau}{\tau^m}. \tag{3.29}$$

We need to substitute above expansion in the equation of trajectory.

$$m_j \frac{\partial^2 x_j^\mu}{\partial \tau^2} = e_j \; F^{\mu\nu}(x_j(\tau)) \; V_{j\nu}. \tag{3.30}$$

Hence, we get

$$m_j \frac{\partial^2 x_j^\mu}{\partial \tau^2} \sim \frac{e^2}{\tau^2} + e^4 \frac{\log \tau}{\tau^3} + \frac{e^2}{\tau^3} + \cdots . \tag{3.31}$$

This gives us the leading $\mathcal{O}(e^4)$ correction to (3.6). The $\frac{\log \tau}{\tau^3}$ term gives rises to the subleading correction to the asymptotic trajectory. It takes following form

$$x_i^\mu = V_i^\mu \tau + c_i^\mu \log \tau + d_i + f_{i\sigma} \frac{\log \tau}{\tau}. \tag{3.32}$$

The exact form of $f_{i\mu}$ is not relevant for the conservation law and we will not discuss its derivation here. Nonetheless it is interesting to note that this term is universal and fixed in terms of charges and asymptotic velocities of the scattering particles. The explicit form of $f_{i\mu}$ is given in [95, 99].

Next we find the resultant correction to radiative field including the subsubleading correction to the trajectory. The field generated by i^{th} outgoing particle is given using the retarded Green function :

$$A_\sigma^{(i)}(x) = \frac{1}{2\pi} \int d^4x' \; \delta(\; (x - x')^2) \; j_\sigma^{(i)}(x') \; \Theta(t - t'). \tag{3.33}$$

Here we need to use the modified current calculated using the corrected trajectory given in (3.32). We will ignore the $\mathcal{O}(\frac{1}{\tau^2})$ terms in the current.

$$j_\sigma^{(i)}(x') = \int d\tau \; e_i \big[V_{i\sigma} + \frac{c_{i\sigma}}{\tau} - f_{i\sigma} \frac{\log \tau}{\tau^2} \big] \; \delta^4(x' - x_i) \; \Theta(\tau - T).$$

We have to solve $(x - x')^2 = 0$ to second order in coupling e. The solution is given in (A.12) in Appendix A. Next we will use :

$$\delta(\; (x - x')^2) = \frac{\delta(\tau - \tau_2)}{|2\tau + 2V_i.(x - d_i) + \frac{2(x - d_i).c_i}{\tau} - 2 f_i.(x - d_i) \frac{\log \tau}{\tau^2} + 2 \frac{f_i.(x - d_i)}{\tau^2} - 2c_i^2 \frac{\log \tau}{\tau}|}. \tag{3.34}$$

Let us first discuss the qualitative behaviour of the leading $\frac{1}{r}$ term in A_σ. The solution to the asymptotic field at this order turns out to be :

$$A_\sigma^{(i)}(x) = \frac{1}{4\pi} \frac{\Theta(\tau_2 - T)\ e_i \big[V_{i\sigma} + \frac{c_{i\sigma}}{\tau_2} - f_{i\sigma} \frac{\log \tau_2}{\tau_2^2}\big]}{|V_i.x + \frac{x.c_i}{\tau_2} - f_i.x \frac{\log \tau_2}{\tau_2^2} + \frac{f_i.x}{\tau_2^2} + \mathcal{O}(r^0)|}. \tag{3.35}$$

Here we have ignored the terms in the solution that contribute at $\mathcal{O}(\frac{1}{r^2})$ or higher. Next it remains to subtitute the value of τ_2. We get it from (A.13) :

$$\tau_2|_{\mathcal{I}^+} = -\frac{u + q.d_i}{q.V_i} - \frac{q.c_i}{q.V_i} \log \frac{u}{(-q.V_i)} + q.f_i \frac{\log u}{u} - \frac{(q.c_i)^2}{q.V_i} \frac{\log u}{u} + \mathcal{O}(\frac{1}{u}). \tag{3.36}$$

We will substitute above solution of τ_2 in (3.35) and find the $\frac{1}{r}$ term in A_σ.

$$\begin{aligned} A_\sigma^{(i)}(x)|_{\mathcal{I}^+} = &- \frac{e_i \Theta(\tau_2 - T)}{4\pi r} \Big[\frac{V_{i\sigma}}{q.V_i} \big[1 + \frac{q.c_i}{u} - (q.c_i)^2 \frac{\log u}{u^2} + q.f_i\ q.V_i \frac{\log u}{u^2}\big] \\ &- \frac{c_{i\sigma}}{u} \big[1 - q.c_i \frac{\log u}{u}\big] - f_{i\sigma}\ q.V_i \frac{\log u}{u^2} + \mathcal{O}(\frac{1}{u^2})\Big]. \end{aligned} \tag{3.37}$$

Summing over contributions from all particles, we get

$$\begin{aligned} A_\sigma(x)|_{\mathcal{I}^+} = &-\frac{1}{4\pi r}\Big[\sum_{i=n'+1}^{n} \Theta(u-T)\ \frac{e_i V_{i\sigma}}{q.V_i} - \sum_{i=1}^{n'} \Theta(-u-T)\ \frac{e_i V_{i\sigma}}{q.V_i} \Big] \\ &+ \frac{1}{4\pi r u} \Big[\sum_{i=n'+1}^{n} e_i\ \Theta(u-T)\ \big[c_{i\sigma} - V_{i\sigma} \frac{q.c_i}{q.V_i}\big] - \sum_{i=1}^{n'} \Theta(-u-T)\ e_i \big[c_{i\sigma} - V_{i\sigma} \frac{q.c_i^-}{q.V_i}\big] \Big] \\ &- \frac{1}{4\pi r} \frac{\log u}{u^2} \Big[\sum_{i=n'+1}^{n} e_i\ \Theta(u-T)\ \big[q.c_i\ c_{i\sigma} - V_{i\sigma} \frac{(q.c_i)^2}{q.V_i} + V_{i\sigma}\ f_i.q - q.V_i\ f_{i\sigma}\big] \\ &- \sum_{i=1}^{n'} e_i \Theta(-u-T)\ \big[q.c_i\ c_{i\sigma} - V_{i\sigma} \frac{(q.c_i)^2}{q.V_i} + V_{i\sigma}\ f_i.q - q.V_i\ f_{i\sigma}\big] \Big] + \ldots . \end{aligned} \tag{3.38}$$

Above expression gives the lower order terms in the late time radiation. As discussed earlier, the leading term is $\mathcal{O}(u^0)$ and gives rise to the so called memory effect. The $\mathcal{O}(\frac{1}{u})$ term is the leading order tail to the memory term [37, 39] discussed in (3.15). From above expression, we see that the subleading tail to the memory term is $\mathcal{O}(\frac{\log u}{u^2})$. If we go back to (3.18), we see that such a mode is absent if we restrict ourselves to $\mathcal{O}(e^3)$. Though we have not derived the form of $f_{i\sigma}$, it is important to note that it is universal [95,99]. Similar to the first two terms, the subleading tail is fixed in terms of the charges and the asymptotic velocities of the scattering particles. Interested readers can refer to [95, 99] for detailed structure of this term.

Let us go back to (3.35) to obtain the Q_2 conservation law. This charge is defined in terms of the coefficient of the $\mathcal{O}(\frac{\log u}{r^3})$ mode. Hence now we need to retain the subleading corrections in $\frac{1}{r}$.

$$A_\sigma^{(i)}(x) = \frac{1}{4\pi}\frac{e_i\big[V_{i\sigma} + \frac{c_{i\sigma}}{\tau_2} - f_{i\sigma}\frac{\log\tau_2}{\tau_2^2}\big]\ \Theta(\tau_2 - T)}{|\tau_2 + V_i.(x-d_i) + \frac{(x-d_i).c_i}{\tau_2} - f_i.x\frac{\log\tau_2}{\tau_2^2} + \frac{f_i.x}{\tau_2^2} - c_i^2\frac{\log\tau_2}{\tau_2}|}. \tag{3.39}$$

We recall that τ_2 is given in (3.36). At $\mathcal{I}^+$, the charge is expected to be defined in terms of $\frac{(\log u)^2}{r^3}$-mode of A_σ. Using (3.36), we have :

$$\frac{1}{\tau_2} = -\frac{q.V_i}{u}\Big[1 + \mathcal{O}(\frac{\log u}{u})\Big].$$

Let us dicuss the second term in the numerator i.e. $\frac{e_i}{|\tau_2+V_i.x+...|}\frac{c_{i\sigma}}{\tau_2}$. The expansion of this term is of following form :

$$\frac{1}{ur}\Big[1 + \frac{\log u}{u} + ... + \frac{1}{r}[u + \log u + u^0] + \frac{1}{r^2}[u^2 + u\log u + (\log u)^2 + ...] + ...\Big].$$

Thus the second term in the numerator does not contribute to $\frac{(\log u)^2}{r^3}$ mode. Next we turn to the third term in the numerator that can be expanded as follows

$$\frac{\log u}{u^2 r}\Big[1 + \frac{1}{r}[u + u^0] + \frac{1}{r^2}[u^2 + ...] + ...\Big].$$

Thus the last term in the numerator also does not contribute to $\frac{(\log u)^2}{r^3}$. Similarly, last four terms in the denominator do not contribute to $\frac{(\log u)^2}{r^3}$ and hence are irrelevant for subsequent analysis. So, we are left with following terms in (3.39) :

$$A_\sigma^{(i)}(x) \sim \frac{1}{4\pi}\frac{e_i V_{i\sigma}\Theta(\tau_2 - T)}{|\tau_2 + V_i.(x-d_i)|}. \tag{3.40}$$

We have used '$\sim$' instead of '$=$' as we are ignoring certain terms in A_σ that do not contribute to the charge given in (3.28). Next we need to substitute the value of τ_2 given in (3.36). Let us retain only the logarithmic terms in τ_2 that are of relevance to us. We get :

$$A_\sigma^{(i)}(x)|_{\mathcal{I}^+} \sim \frac{1}{4\pi}\frac{e_i\ V_{i\sigma}\ \Theta(u - T)}{X[1 - 2x.c_i\frac{\log u}{X^2} + c_i^2\frac{(\log u)^2}{X^2}]^{1/2}}. \tag{3.41}$$

We recall that $X = [\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ ]^{1/2}$. Hence, the limiting value of X is $X|_{\mathcal{I}^+} = -rq.V_i + \mathcal{O}(u)$. Substituting the value of X the coefficient of $\frac{(\log u)^2}{r^3}$ comes out to be :

$$A^{(i)}_\sigma(x)|_{\mathcal{I}^+} \sim -\frac{\Theta(u-T)}{8\pi}\frac{(\log u)^2}{r^3}\frac{V_{i\sigma}}{q.V_i}\Big[3\frac{(q.c_i)^2}{(q.V_i)^4} - \frac{c_i^2}{(q.V_i)^2}\Big]. \tag{3.42}$$

Next we need to sum over the contributions from all particles. Let us write down the coefficient at $\mathcal{I}^+_-$. The contribution around $\mathcal{I}^+_-$ is from the incoming particles, thus we get :

$$A^{[(\log u)^2/r^3]}_\sigma(\hat{x})|_{\mathcal{I}^+_-} = -\sum_{i=1}^{n'}\frac{e_i}{8\pi}\frac{V_{i\sigma}}{q.V_i}\Big[3\frac{(q.c_i)^2}{(q.V_i)^4} - \frac{c_i^2}{(q.V_i)^2}\Big]. \tag{3.43}$$

We just need to transform co-ordinates to go to F_{rA}. Thus :

$$F^{[(\log u)^2/r^3]}_{rA}(\hat{x})|_{u\to-\infty} = \sum_{i=1}^{n'}\frac{e_i}{4\pi}\frac{3(q.c_i)}{(q.V_i)^5}\, q^\mu(\partial_A q^\nu)\ [V_{i\mu}c_{i\nu} - V_{i\nu}c_{i\mu}]. \tag{3.44}$$

Let us carry out the corresponding calculation at past null infinity. At $\mathcal{I}^-$, the term of our interest is the $\frac{(\log r)^2}{r^3}$-mode of A_σ. Using (A.14), we follow earlier logic and analogous to (3.41) we get following expression at past null infinity we get :

$$A^{(i)}_\sigma(x)|_{\mathcal{I}^-} \sim \frac{e_i\ V_{i\sigma}}{X[1 - 2x.c_i\frac{\log r}{X^2} + c_i^2\frac{(\log r)^2}{X^2}]^{1/2}}.$$

Using (2.16) in $X = [\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ ]^{1/2}$, the limiting value at past null infinity turns out to be $X|_{\mathcal{I}^-} = r\bar{q}.V_i + \mathcal{O}(r^0)$. Expanding above expression, we obtain the $\frac{(\log r)^2}{r^3}$ term in A_σ :

$$A^{[(\log r)^2/r^3]}_\sigma(x)|_{\mathcal{I}^-} = \sum_{i=1}^{n'}\frac{e_i}{8\pi}\frac{V_{i\sigma}}{V_i.\bar{q}}\Big[\ 3\frac{(\bar{q}.c_i)^2}{(V_i.\bar{q})^4} - \frac{c_i^2}{(V_i.\bar{q})^2}\ \Big]. \tag{3.45}$$

Next we just need to use appropiate co-ordinate transformations to arrive at F_{rA}. We get :

$$F^{[(\log r)^2/r^3]}_{rA}(\hat{x})|_{v\to\infty} = -\sum_{i=1}^{n'}\frac{e_i}{4\pi}\frac{3(\bar{q}.c_i)}{(\bar{q}.V_i)^5}\,\bar{q}^\mu\ (\partial_A\bar{q}^\nu)\ [V_{i\mu}c_{i\nu} - V_{i\nu}c_{i\mu}]. \tag{3.46}$$

Using (3.44) and (3.46), we can write down the conservation law for these modes :

$$F_{rA}^{[(\log u)^2/r^3]}(\hat{x})|_{\mathcal{J}^+_-} = -F_{rA}^{[(\log r)^2/r^3}(-\hat{x})|_{\mathcal{J}^-_+}. \tag{3.47}$$

It is important to note the minus sign. The charges can be defined as $Q_2^+ = \int d^2z\ Y^A\ F_{rA}^{[(\log u)^2/r^3]}(\hat{x})|_{\mathcal{J}^+_-}$ and $Q_2^- = \int d^2z\ Y^A\ F_{rA}^{[(\log r)^2/r^3]}(-\hat{x})|_{\mathcal{J}^-_+}$. It is expected that the charges conserved exactly. Y^A is an S^2 vector field which is a free parameter.

Thus we have derived the second classical conservation law. Like the Q_1-charge, these charges are also related to long range interactions between the scattering particles. These modes appear at $\mathcal{O}(e^5)$ and are expected to be uncorrected at higher orders in e. We anticipate that this conservation law would also be related to a soft theorem. Since these charges are made of $\mathcal{O}(e^5)$ modes the corresponding soft mode should appear at 2-loop order. This conservation law is expected to be related to the 2-loop level soft theorem derived in [95].

3.3 Proposal for Q_m conservation laws

In Section 3.1, we included the leading effect of long range electromagnetic force on the scattering particles and showed that the asymptotic trajectories get modified as follows (to $\mathcal{O}(e^2)$)

$$x_{i\sigma} = V_{i\sigma}\tau + c_{i\sigma}\log\tau + d_{i\sigma} + \sum_n^\infty c_{i\sigma}^{(0,n)}\frac{1}{\tau^n}. \tag{3.48}$$

The radiative field produced by the scattering has new logarithmic modes that obey the Q_1 conservation law. This law given in (3.25) relates the coefficient of the $\frac{\log r}{r^2}$ mode at the past to the coefficient of the $\frac{\log u}{r^2}$ mode at the future.

In Section 3.2, we carried out similar calculations at subsubleading order in e. Including the subleading effect of long range electromagnetic force on the scattering particles, we showed that the asymptotic trajectories get modified as follows (to $\mathcal{O}(e^4)$)

$$x_{i\sigma} = V_{i\sigma}\tau + c_{i\sigma}\log\tau + d_{i\sigma} + \sum_n^\infty c_{i\sigma}^{(0,n)}\frac{1}{\tau^n} + \sum_n^\infty c_{i\sigma}^{(1,n)}\frac{\log\tau}{\tau^n}. \tag{3.49}$$

The radiative field has new modes that fall off as square of logarithm and obey the Q_2 conservation law. This law given in (3.47) relates the coefficient of the $\frac{(\log r)^2}{r^3}$ mode at the past to the coefficient of the $\frac{(\log u)^2}{r^3}$ mode at the future.

Thus we see that these laws have a nice structure which hints that such laws could exist for $m > 2$ as well. When we include the effect of long range forces on the trajectory, the full correction to the trajectory is of the form :

$$x_{i\sigma} = V_{i\sigma}\tau + c_{i\sigma}\log\tau + d_{i\sigma} + \sum_{\substack{m\leq n\\ m=0,n=1}}^{\infty} c_{i\sigma}^{(m,n)}\frac{(\log\tau)^m}{\tau^n}, \tag{3.50}$$

where $c_{i\sigma}^{(m,n)}$'s typically admit a series expansion in the coupling e. The leading logarithmic terms are produced only from the asymptotic trajectories and are not sensitive to the specifics of bulk dynamics. Other terms may depend on details of scattering. So, not all the $c_{i\sigma}^{(m,n)}$'s are universal.

The resultant radiative field includes various powers of logarithmic modes. Based on the $m = 1, 2$ cases, we propose that there exists an asymptotic conservation law for every m given by :

$$F_{rA}^{[(\log u)^m/r^{m+1}]}(\hat{x})|_{\mathcal{I}^+_-} = (-1)^{m+1}\ F_{rA}^{[(\log r)^m/r^{m+1}]}(-\hat{x})|_{\mathcal{I}^-_+}. \tag{3.51}$$

Here $F_{rA}^{[(\log u)^m/r^{m+1}]}$ denotes the coefficient of the $\frac{(\log u)^m}{r^{m+1}}$-mode in F_{rA}. Similarly $F_{rA}^{[(\log r)^m/r^{m+1}]}$ denotes the coefficient of the $\frac{(\log r)^m}{r^{m+1}}$-mode in F_{rA}. These modes are expected to appear at $\mathcal{O}(e^{2m+1})$ and should be conserved exactly. The m^{th} level future charge is defined as $Q_m^+ = \int d^2z\ Y_m^A(\hat{x})\ F_{rA}^{[(\log u)^m/r^{m+1}]}(\hat{x})|_{\mathcal{I}^+_-}$ and the past charge is defined as $Q_m^- = \int d^2z\ Y_m^A(-\hat{x})\ F_{rA}^{[(\log r)^m/r^{m+1}]}(-\hat{x})|_{\mathcal{I}^-_+}$.

Thus, we expect that classical electromagnetism admits a hierarchy of infinite number of asymptotic conservation laws. We have derived these laws for $m = 1, 2$. The corresponding charges Q_m are closely related to the long range electromagnetic force present between the scattering particles. A natural question to investigate further is the implication of these charges for the quantum theory. We anticipate that the Q_m charges imply existence of m-loop soft theorems for every m. In particular it is expected that the Q_1 charge would be related to the 1-loop exact $\log\omega$ soft photon theorem derived by Sahoo and Sen [91]. In chapter 5, we will establish this equivalence in the context of massless scalar QED coupled to gravity. Before going to this equivalence let us note that the conservation laws were obtained for retarded solutions of classical equation of the electromagnetic field. In the quantum theory we need to use Feynman boundary condition. It is not clear if the conservation laws (that we discussed in this chapter) continue to hold in the quantum theory. We will address this question in the next chapter.

In this chapter we also discussed the form of the tails in the late time radiative field that arise as a result of long range interaction between scattering particles in (3.38).

$$A_\mu \sim \frac{1}{4\pi r}\left[\ u^0 \ + \ \frac{1}{u} \ + \ \frac{\log u}{u^2}\ \right] + \mathcal{O}(\frac{1}{ru^2}), \quad u \to \pm\infty. \tag{3.52}$$

In the next chapter we will also discuss the analogue of (3.52) for the Feynman solution.

Chapter 4

Asymptotic conservation law with Feynman boundary condition

The Q_0 conservation law corresponding to the leading soft theorem has been discussed in (2.20). In the last chapter it was shown that there exist new asymptotic conservation laws for classical electromagnetism. Unlike the Q_0 conservation law these new laws are asymmetric. The first of these laws given in (3.25) relates the coefficient of the $\frac{\log r}{r^2}$ mode at the past to the coefficient of the $\frac{\log u}{r^2}$ mode at the future. This asymmetry is expected to change in the quantum theory due to the use of Feynman propagator. In this chapter we will derive the analogue of this asymptotic conservation law upon imposing Feynman boundary condition on the radiative field. We will also hightlight some important differences between asymptotic expansion of the classical field and that of the quantum gauge field. This chapter is based on our results derived in [100].

Let us describe our setup. We consider a scattering process in which some n number of charged particles come in to interact and eventually move away. The details of this process have been discussed in Section 2.1. In this chapter we will obtain the radiative field produced by scattering of n charged particles upon imposing Feynman boundary condition. The solution so obtained is complex in general. It is not an obervable quantity but serves as a prototype for the behaviour of the quantum gauge field. Also it must noted that the concept of point particle does not make sense quantum mechanically but it is a good enough approximation for our purposes.[1] Thus inspite of it being an unphysical

[1] A general wave packet can be approximated as a point particle to the zeroth order. The subleading corrections involve derivatives of delta function and do not affect the modes of our interest.

problem, this simple setup allows us to explore certain modes of the Feynman solution that have physical significance. We will also argue that presence of such modes is universal feature of the quantum gauge field. Including the effect of long range electromagnetic force, we will derive the analogue of the Q_1 law for Feynman solution. As earlier the calculations are carried out perturbatively in e and asymptotic parameter $\frac{1}{\tau}$.

4.1 Radiative field at $\mathcal{O}(e)$ with Feynman propagator

Let us calculate the asymptotic radiative field generated in scattering of charged particles to leading order in e. Hence we can ignore the long range electromagnetic force acting on the scattering particles. As discussed in (2.6), the scattering event is described by following current

$$j^{\text{asym}}_\sigma(x') = \int d\tau \Big[\sum_{i=n+1}^{2n} e_i V_{i\sigma}\ \delta^4(x' - x_i)\ \Theta(\tau - T) + \sum_{i=1}^{n} e_i V_{i\sigma}\ \delta^4(x' - x_i)\ \Theta(-\tau - T) \Big] \tag{4.1}$$

Here, we have labelled the incoming particles by i running from 1 to n and outgoing particles by i running from $n+1$ to $2n$. We do not have an explicit form of the bulk trajectories $x_i(\tau)$ for $|\tau| < T$. The radiative field is given by

$$A_\sigma(x) = \int d^4x'\ G(x, x')\ j_\sigma(x')\ , \tag{4.2}$$

using the usual momentum representation of the Feynman propagator we have,

$$G(x, x') \quad = \quad \int \frac{d^4p}{(2\pi)^4}\ \frac{e^{ip.(x-x')}}{p^2 - i\epsilon}.$$

We can perform the momentum integral according to the given ϵ-prescription and obtain the form of the propagator in the position space.

$$G(x; x') \quad = \quad \frac{1}{4\pi^2} \Big[\pi\delta_+(\ (x - x')^2) + \pi\delta_-(\ (x - x')^2) + \frac{i}{(x - x')^2} \Big]. \tag{4.3}$$

The subscript '+' denotes the retarded root of the δ-function constraint i.e. $t > t'$, while the subscript '-' denotes the advanced root of the δ-function constraint i.e. $t' > t$.

Since the first term in above expression is proportional to the retarded propagator, the electromagnetic field generated by this term is similar to the one obtained in the previous section. We denote the field generated by the first term in (4.3) by superscript '+'. We have discussed in the previous section using (2.4) that the asymptotic field gets contribution only from the asymptotic part of the current. So using (4.1), we get

$$A_\sigma^+(x) = \frac{1}{8\pi}\sum_{i=n+1}^{2n} \frac{e_i V_{i\sigma}\ \Theta(\tau_0^+ - T)}{\sqrt{(V_i.x - V_i.d_i)^2 + (x-d_i)^2}} + \frac{1}{8\pi}\sum_{i=1}^{n} \frac{e_i V_{i\sigma}\ \Theta(-\tau_0^+ - T)}{\sqrt{(V_i.x - V_i.d_i)^2 + (x-d_i)^2}} \ . \tag{4.4}$$

Here the retarded root is given by $\tau_0^+ = -(V_i.x - V_i.d_i) - \sqrt{(V_i.x - V_i.d_i)^2 + (x-d_i)^2}$. The asymptotic expansion of above expression around $\mathcal{I}^+$ is similar to (2.12).

$$A_\mu^+(x)|_{\mathcal{I}^+} = \frac{1}{8\pi}\sum_{\substack{m=0,n=1\\ m<n}}^{\infty} [A_\mu^{[n,-m]}(\hat{x})]^+\ \frac{u^m}{r^n} + \ldots\ , \tag{4.5}$$

where '...' denote the terms that fall off faster than any power law.

The second term in (4.3) is proportional to the advanced propagator, we will denote the field generated by this term by the superscript '$-$'. Due to the reasons similar to the retarded case, this term also does not get any contribution from the bulk current and it suffices to use (4.1). We have

$$A_\sigma^-(x) = \frac{1}{8\pi}\sum_{i=n+1}^{2n} \frac{e_i V_{i\sigma}\ \Theta(\tau_0^- - T)}{\sqrt{(V_i.x - V_i.d_i)^2 + (x-d_i)^2}} + \frac{1}{8\pi}\sum_{i=1}^{n} \frac{e_i V_{i\sigma}\ \Theta(-\tau_0^- - T)}{\sqrt{(V_i.x - V_i.d_i)^2 + (x-d_i)^2}} \ . \tag{4.6}$$

Here the advanced root is given by $\tau_0^- = -(V_i.x - V_i.d_i) + \sqrt{(V_i.x - V_i.d_i)^2 + (x-d_i)^2}$. We can expand above expression around $\mathcal{I}^+$, an important point to notice here is that $\tau_0^-|_{\mathcal{I}^+} = 2r|q.V_i| + \mathcal{O}(r^0)$. Substituting this value in above expression, the step function with the incoming particles goes like $\Theta(-r)$ hence the contribution of the incoming particles in above expression goes to 0. The asymptotic expansion takes following form,

$$A_\mu^-(x)|_{\mathcal{I}^+} = \frac{1}{8\pi}\sum_{\substack{m=0,n=1\\ m<n}}^{\infty} [A_\mu^{[n,-m]}(\hat{x})]^-\ \frac{u^m}{r^n} + \ldots\ . \tag{4.7}$$

The coefficients $[A]^-$ should be contrasted with $[A]^+$ in (4.5). $[A]^-$ are same throughout $\mathcal{I}^+$ from $u \to -\infty$ to $u \to \infty$ while the coefficients $[A]^+$ in the retarded solution take differents values at $u \to \pm\infty$ respectively.

Finally we turn to the contribution from the third term in (4.3) i.e. from

$\frac{i}{4\pi^2(x-x')^2}$, we denote it by superscript '*'. This term gets contribution from all of spacetime including the bulk.

$$A^*_\mu(x) = \frac{i}{4\pi^2} \int d^4x' \frac{j_\mu(x')}{(x-x')^2}.$$

We will use (4.1) for the asymptotic part of the current. The explicit form of the bulk current is not available. Using (4.1), we get

$$A^*_\sigma(x) = \frac{i}{4\pi^2}\Big[\int_T^\infty d\tau \sum_{i=n}^{2n} \frac{e_i V_{i\sigma}}{(x-V_i\tau-d_i)^2} + \int_{-\infty}^{-T} d\tau \sum_{i=1}^{n} \frac{e_i V_{i\sigma}}{(x-V_i\tau-d_i)^2} + \int_{r'<R} d^4x' \frac{j_\sigma(x')}{(x-x')^2}\Big]. \tag{4.8}$$

First we focus on the asymptotic contribution.

$$A^{*\text{asym}}_\sigma(x) = -\frac{i}{4\pi^2}\left[\int_T^\infty d\tau \sum_{j=n}^{2n} \frac{e_j V_{j\sigma}}{(\tau-\tau_0^+)(\tau-\tau_0^-)} + \int_{-\infty}^{-T} d\tau \sum_{j=1}^{n} \frac{e_j V_{j\sigma}}{(\tau-\tau_0^+)(\tau-\tau_0^-)}\right].$$

$\tau_0^\pm$ are the solutions to the equation $(x-V_i\tau-d_i)^2=0$ and the expressions are given in (B.1). The integral involving the outgoing particles has a divergence at the upper limit. Let us regulate it with an IR cutoff 'L'. Similarly we regulate the second integral with a cutoff '$-L$' to get

$$\begin{aligned} A^{*\text{asym}}_\sigma(x) = &\frac{i}{4\pi^2}\sum_{j=n+1}^{2n} \frac{e_j V_{j\sigma}}{\tau_0^- - \tau_0^+}\left[\log\frac{L-\tau_0^+}{T-\tau_0^+} - \log\frac{L-\tau_0^-}{T-\tau_0^-}\right] \\ &- \frac{i}{4\pi^2}\sum_{j=1}^{n} \frac{e_j V_{j\sigma}}{\tau_0^- - \tau_0^+}\left[\log\frac{L+\tau_0^+}{T+\tau_0^+} - \log\frac{L+\tau_0^-}{T+\tau_0^-}\right]. \end{aligned}$$

All the quantities appearing in the argument of the log function come with a modulus sign which we do not write down explicitly. We expand the square brackets in the limit $L\to\infty$ and see that the divergent pieces cancel. Hence the final expression is finite, we get

$$A^{*\text{asym}}_\sigma(x) = \frac{i}{4\pi^2}\sum_{j=n+1}^{2n} \frac{e_j V_{j\sigma}}{\tau_0^- - \tau_0^+} \log\frac{\tau_0^- - T}{\tau_0^+ - T} - \frac{i}{4\pi^2}\sum_{j=1}^{n} \frac{e_j V_{j\sigma}}{\tau_0^- - \tau_0^+} \log\frac{\tau_0^- + T}{\tau_0^+ + T}.$$

We will use (B.1) to substitute for $\tau_0^- - \tau_0^+$ and also rewrite above expression in a succinit form

$$A_\sigma^{*\text{asym}}(x) = \frac{i}{8\pi^2} \sum_{j=1}^{2n} \frac{\eta_j e_j V_{j\sigma}}{\sqrt{(V_j.x - V_j.d_j)^2 + (x-d_j)^2}} \log \frac{\tau_0^- - \eta_j T}{\tau_0^+ - \eta_j T}. \tag{4.9}$$

Here $\eta_j = 1(-1)$ for outgoing (incoming) particles. Next we will find the asymptotic expansion of above expression. Using (B.2) we have

$$\tau_0^+|_{\mathcal{I}^+} = \frac{u + q.d_i}{|q.V_i|} + \mathcal{O}(\frac{1}{r}), \quad \tau_0^-|_{\mathcal{I}^+} = 2r|q.V_i| + \mathcal{O}(r^0).$$

Thus we get

$$\Big[\log \frac{\tau_0^- - T}{\tau_0^+ - T}\Big]_{\mathcal{I}^+} = \log \frac{r}{u} + \mathcal{O}(1).$$

We find that there are logarithmic modes in the radiative field. This is an interesting result as such kind of modes are absent in the retarded solution at $\mathcal{O}(e)$ that was derived in (2.12). This tells us that the Feynman solution has certain features very different from the retarded solution. Let us write down the full asymptotic expansion of $A_\sigma^{*\text{asym}}$. Using (B.1), it is seen that

$$\log \tau_0^-|_{\mathcal{I}^+} \sim \log r + \sum_{\substack{m,n=0,\\ m\leq n.}}^{\infty} \frac{u^m}{r^n}.$$

Similarly

$$\log \tau_0^+|_{\mathcal{I}^+} \sim \log u + \sum_{\substack{n=0,\\ m=-\infty,\\ m\leq n.}}^{\infty} \frac{u^m}{r^n}.$$

We will write down the expansion for $A_\sigma^{*\text{asym}}$ by substituting above expressions in (4.9).

$$A_\sigma^{*\text{asym}}(x) \sim \log \frac{u}{r} \sum_{\substack{m=0,n=1\\ m<n}}^{\infty} \frac{u^m}{r^n} + \sum_{\substack{n=1,\\ m=-\infty,\\ m<n.}}^{\infty} \frac{u^m}{r^n}. \tag{4.10}$$

Next we turn to the bulk contribution i.e. the $r' < R$ term in (4.8). We do not have the explicit expression of the bulk current.

$$A_\sigma^{*\text{bulk}}(x) = \frac{i}{4\pi^2} \int_{r'<R} d^4x' \; j_\sigma(x') \, \frac{1}{(x-x')^2} \, . \tag{4.11}$$

Let us estimate the contribution of this integral around $\mathscr{I}^+$. $\frac{1}{(x-x')^2}|_{\mathscr{I}^+} \sim \frac{1}{r}\sum_{n=1}^{\infty}\frac{1}{u^n} + \mathcal{O}(\frac{1}{r^2})$. Hence the asymptotic expansion of $A_\sigma^{*\text{bulk}}(x)$ takes following form

$$A_\sigma^{*\text{bulk}}(x)|_{\mathscr{I}^+} \sim \sum_{\substack{n=1,\\ m=-\infty,\\ m<n-1.}}^{\infty} \frac{u^m}{r^n} \ . \tag{4.12}$$

Finally we write down the Feynman solution using (4.4),(4.6) and (4.9) :

$$\begin{aligned} A_\sigma(x) &= A_\sigma^+(x) + A_\sigma^-(x) + A_\sigma^*(x) \\ &= \frac{1}{8\pi}\sum_{i=1}^{2n}\frac{e_i V_{i\sigma}\ \Theta(\eta_i\tau_0^+ - T)}{\sqrt{(V_i.x - V_i.d_i)^2 + (x-d_i)^2}} + \frac{1}{8\pi}\sum_{i=n+1}^{2n}\frac{e_i V_{i\sigma}\ \Theta(\tau_0^- - T)}{\sqrt{(V_i.x - V_i.d_i)^2 + (x-d_i)^2}} \\ &+ \frac{i}{8\pi^2}\sum_{j=1}^{2n}\frac{\eta_j e_j V_{j\sigma}}{\sqrt{(V_j.x - V_j.d_j)^2 + (x-d_j)^2}}\ \log\frac{\tau_0^- - \eta_j T}{\tau_0^+ - \eta_j T} + A_\sigma^{*\text{bulk}}(x). \end{aligned} \tag{4.13}$$

As before, $\eta_j = 1(-1)$ for outgoing (incoming) particles. The first line and the first term in the second line are sensitive only to charges and asymptotic velocities of the scattering particles. We do not have an explicit form for $A_\sigma^{*\text{bulk}}$. In general this term will depend on the details of the scattering process and short ranges forces present between the particles. We are not interested in such non-universal terms. Let us write down the asymptotic expansion of the full solution in (4.13) (including $A_\sigma^{*\text{bulk}}$). It is given by

$$\begin{aligned} A_\sigma(x)|_{\mathscr{I}^+} = &\sum_{\substack{m=0,n=1\\ m<n}}^{\infty}\frac{u^m}{r^n}\ A_\sigma^{[n,-m]}(\hat{x}) + \log u \sum_{\substack{m=0,n=1\\ m<n}}^{\infty}\frac{u^m}{r^n}\ L_{1\sigma}^{[n,-m]}(\hat{x}) \\ &+ \log r\sum_{\substack{m=0,n=1\\ m<n}}^{\infty}\frac{u^m}{r^n}\ L_{2\sigma}^{[n,-m]}(\hat{x})\ + \sum_{m,n=1}^{\infty}\frac{A_\sigma^{[n,m]}(\hat{x})}{u^m r^n}. \end{aligned} \tag{4.14}$$

Let us compare above solution with the retarded solution we have in (2.12). The retarded solution has only $A_\sigma^{[n,-m]}$ kind of modes. The other kind of modes present in the Feynman solution that have log behaviour or fall off as negative powers of u are absent in the retarded solution (at $\mathcal{O}(e)$). This tells us that the asymptotic expansion of the Feynman solution has modes that are absent in the classical solution. We will refer to such modes as purely quantum modes.

Let us comment on some important differences between the Feynman solution and the retarded solution. The leading order term of (4.13) is $\mathcal{O}(\frac{\log r}{r})$. If we study (2.12), we see that such kind of modes are completely absent in the retarded solution! The retarded solution discussed in (2.12) starts at $\mathcal{O}(\frac{1}{r})$. The $\frac{1}{r}$-component of the Feynman solution given in (4.13) takes following form

$$A_\sigma(x)|_{\mathcal{J}^+} \sim \frac{1}{4\pi r}\ [\ \log u + u^0 + \sum_{n=1}^{\infty} \frac{1}{u^n} + ...\ ]\ . \tag{4.15}$$

Above expression should be contrasted with (2.10). The $\frac{\log u}{r}$ mode is absent in the classical field. It is very important to note that this mode violates the Ashtekar-Struebel fall offs for the radiative field [94]. The consequences of this fact need to be studied in the future. This mode will play an important role in the quantum part of the $\log\omega$ soft factor in Chapter 5. Here we have derived the $\frac{\log u}{r}$ mode in a toy process of scattering of point charged particles. Next we will argue that the presence of such a mode is a general feature of the quantum gauge field.

The $\frac{\log u}{r}$ mode in A_σ

Let us write down the coefficient of the $\frac{\log u}{r}$ mode. Using the radiative field calculated in (4.13), we see that this mode arises from the second line of (4.13). The contribution from the first term in the second line of (4.13) is given by

$$A^{[1,\log]}_\sigma(x) = \frac{i}{8\pi^2}\ \sum_{j=1}^{2n} \frac{\eta_j e_j V_{j\sigma}}{V_j . q}\ . \tag{4.16}$$

Here $A^{[1,\log]}_\sigma$ has been used to denote the coefficient of the $\frac{\log u}{r}$ mode. We see that this mode is proportional to the leading soft factor. This hints that this mode is tied to the $\frac{1}{\omega}$-mode. We will next derive this $\frac{\log u}{r}$ mode in an alternative way that brings out its relation to the $\frac{1}{\omega}$-mode [93].

Let us study the quantum gauge field $\hat{A}_\mu$. In momentum space the expansion of $\hat{A}_\mu$ is given by

$$\hat{A}_\sigma(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2|\vec{p}|}\ [a_\sigma(p)\ e^{ip.x} + a^\dagger_\sigma(p)\ e^{-ip.x}\ ]. \tag{4.17}$$

Here, $a_\sigma(p) = \sum_{r=+,-} \epsilon^{*r}_\sigma\ a^r(p)$ such that $a^r(p)$ is identified as the annihilation operator for the respective helicity photons and $\epsilon^r_\sigma(\hat{x})$ is the polarisation vector. The leading order term around $\mathcal{J}^+$ is $\mathcal{O}(\frac{1}{r})$ and its coefficient $\hat{A}^{[1]}_\mu(u, \hat{x})$ is given by (2.27)

$$\hat{A}^{[1]}_{\sigma}(u,\hat{x}) = -\frac{i}{8\pi^2}\int_0^{\infty} d\omega\ [a_\sigma(\omega,\hat{x})\ e^{-i\omega u} - a^\dagger_\sigma(\omega,\hat{x})\ e^{i\omega u}\ ]. \tag{4.18}$$

Above expression can be rewritten as

$$\begin{aligned} \hat{A}^{[1]}_{\sigma}(u,\hat{x}) &= -\frac{i}{8\pi^2}\int_{-\infty}^{\infty} d\omega\ \tilde{A}_\sigma\ e^{-i\omega u}\ , \\ \text{here}\ \ \tilde{A}_\sigma &= [\ a_\sigma(\omega,\hat{x})\ \Theta(\omega) - a^\dagger_\sigma(-\omega,\hat{x})\Theta(-\omega)\ ]. \end{aligned} \tag{4.19}$$

We define a function $\hat{A}^{[1]+}_{\mu}(u,\hat{x})$ that has contribution from only positive frequencies i.e.

$$\hat{A}^{[1]+}_{\mu}(u,\hat{x}) = -\frac{i}{8\pi^2}\int_0^{\infty} d\omega\ \tilde{A}_\mu(\omega,\hat{x})\ e^{-i\omega u}.$$

We know that around $\omega \sim 0$, the behaviour of the radiative data is given by $\tilde{A}_\mu(\omega,\hat{x}) = \frac{1}{\omega}\tilde{A}^{+0}_{\mu}(\hat{x}) + \dots$. This low energy behaviour dictates the large u behaviour. The interval $\omega \in [0,u^{-1}]$ gives rise to a log mode.

$$\begin{aligned} \hat{A}^{[1]+}_{\mu}(u,\hat{x}) &= -\frac{i}{8\pi^2}\int_0^{\infty} d\omega\ [\frac{1}{\omega}\tilde{A}^{+0}_{\mu}(\hat{x}) + \dots]\ e^{-i\omega u}, \\ &= -\frac{i}{8\pi^2}\int_0^{u^{-1}} d\omega\ [\frac{1}{\omega}\tilde{A}^{+0}_{\mu}(\hat{x}) + \dots], \\ &= -\frac{i}{8\pi^2}\ \log(u^{-1})\ \tilde{A}^{+0}_{\mu}(\hat{x}) + \dots\ . \end{aligned} \tag{4.20}$$

'...' denote terms that are subleading at large u. Simlarly for negative frequencies, we have :

$$\hat{A}^{[1]-}_{\mu}(u,\hat{x}) = \frac{i}{8\pi^2}\ \log(u^{-1})\ \tilde{A}^{-0}_{\mu}(\hat{x}) + \dots\ . \tag{4.21}$$

Collecting the positve and negative frequency terms we get :

$$\begin{aligned} \hat{A}^{[1,\log]}_{\mu}(\hat{x}) &= \frac{i}{8\pi^2}\big[\ \tilde{A}^{+0}_{\mu}(\hat{x})\ -\ \tilde{A}^{-0}_{\mu}(\hat{x})\big]\ . \\ &= \frac{i}{8\pi^2}\big[\lim_{\omega\to 0^+}\ \omega\tilde{A}_\mu(\omega,\hat{x}) - \lim_{\omega\to 0^-}\ \omega\tilde{A}_\mu(\omega,\hat{x})\big]\ . \end{aligned}$$

Above expression tells us that the $\log u$ term is governed by the discontinuity in $\omega\tilde{A}_\mu$ as $\omega \to 0$. This term is absent in the classical radiative field wherein we have to use retarded propagator. For such solutions, $\omega\tilde{A}_z$ is continuous at

$\omega = 0$ [35] and the coefficient of $\log|u|$ term vanishes. We will see that it is non-trivial quantum mechanically. This is because of the fact in the quantum field, the positive frequencies involve annihilation operator while negative frequencies involve creation operator as seen in (4.19). We get

$$\hat{A}_\mu^{[1,\log]}(\hat{x}) = \frac{i}{8\pi^2}\big[\lim_{\omega\to 0^+}\omega a_\mu(\omega,\hat{x}) - \lim_{\omega\to 0^+}\omega a_\mu^\dagger(\omega,\hat{x})\big]. \tag{4.22}$$

Thus the $\log u$ mode is tied to the leading soft mode[2]. We can evaluate the insertion of above operator using leading soft theorem.

$$\begin{aligned} < \mathrm{out}|\hat{A}_\mu^{[1,\log]}(\hat{x})S|\mathrm{in}> &= \frac{i}{8\pi^2}\big[\epsilon_\mu^+\epsilon_\nu^- + \epsilon_\mu^-\epsilon_\nu^+\big]\sum_{j=1}^{2n}\frac{\eta_j e_j V_j^\nu}{V_j.q} < \mathrm{out}|S|\mathrm{in}>, \\ &= \frac{i}{8\pi^2}\sum_{j=1}^{2n}\frac{\eta_j e_j V_{j\mu}}{V_j.q} < \mathrm{out}|S|\mathrm{in}> . \end{aligned} \tag{4.23}$$

Here, $|\mathrm{in}> = |1,2,...,n'>$ and $<\mathrm{out}| = <n'+1,...,2n|$. We see that when the quantum operator in inserted between generic states, the coefficient of the log u mode matches with our expression obtained from Feynman radiative solution in (4.16). It is clear from above derivation that the existence of the $\frac{\log u}{r}$ mode is tied to the $\frac{1}{\omega}$-mode. Since the $\frac{1}{\omega}$-mode is universal we expect that the $\frac{\log u}{r}$ mode is also universal.

In [100], we also established that the $\frac{1}{ur}$-mode in (4.15) is related to the tree level subleading soft mode (ω^0). This hints that all the $\frac{1}{u^n r}$-modes in (4.15) should be controlled by the ω^{n-1} soft modes. This also tells us that though we obtained (4.15) for a toy example the presence of $\log u$ and $\frac{1}{u^n r}$ modes is a general feature of the quantum gauge field. Hence, we expect that the quantum gauge field should in general contain modes as given in (4.14).

4.2 Effect of long range forces on asymptotic trajectories

At large distances, the electromagnetic force present between the scattering particles falls off as $\mathcal{O}(\frac{1}{r^2})$ and gives rise to logarithmic correction to the straight line trajectory at late times as discussed in (3.7). In this section we will obtain

[2] It is interesting to note that this $\log u$ mode has appeared in equation (A.2) of [97].

the the explicit form of this logarithmic correction which arises due to the Feynman solution.

We need to find the leading order term in the asymptotic electromagnetic field strength using (4.13). This leading $\mathcal{O}(\frac{1}{r^2})$-mode has an imaginary piece. Thus the corrected equation of trajectory will also have an imaginary piece! This should be compared with the Faddeev-Kulish dressing of scalar fields under electromagnetic force. The logarithmic correction to the trajectory of a particle is in one to one correspondence with the logarithmic dressing of the scalar field [84] as we will discuss below.

The equation of trajectory of j^{th} outgoing particle is given by :

$$m_j \frac{\partial^2 x_j^\mu}{\partial \tau^2} = e_j \; F^{\mu\nu}(x_j(\tau)) \; V_{j\nu}. \tag{4.24}$$

Here, we need to find the field strength generated at the position of the outgoing particle i.e. at $x = x_j(\tau)$ using (4.13). It will get contribution from other particles that interact with the j^{th} particle. Since the outgoing j^{th} particle approaches $\mathcal{I}^+_+$ asymptotically, we need to evaluate (4.13) around $u \to \infty$. It is important to note some subtle points. As seen from (4.13), the contribution from the first line to the field around $u \to \infty$ contains contribution only from outgoing particles. While the second line of (4.13) contains contribution from both incoming and outgoing particles. The leading order field at large τ is given by

$$\begin{aligned} F_{\mu\nu}(x_j(\tau))|_{\mathcal{I}^+_+} &= \frac{1}{4\pi\tau^2} \sum_{\substack{i=n+1,\\ i\neq j}}^{2n} e_i \frac{(V_{i\mu}V_{j\nu} - V_{j\mu}V_{i\nu})}{[(V_i.V_j)^2-1]^{3/2}} \\ &+ \sum_{\substack{i=1,\\ i\neq j}}^{2n} \frac{i\eta_i e_i}{8\pi^2\tau^2} \frac{(V_{i\mu}V_{j\nu} - V_{j\mu}V_{i\nu})}{[(V_i.V_j)^2-1]^{3/2}} \Big[\log \frac{-V_i.V_j + \sqrt{(V_i.V_j)^2-1}}{-V_i.V_j - \sqrt{(V_i.V_j)^2-1}} + 2V_i.V_j \; \sqrt{(V_i.V_j)^2-1} \Big] + \mathcal{O}(\frac{1}{\tau^3}). \end{aligned} \tag{4.25}$$

Substituting (4.25) in (4.24), the leading order correction to the asymptotic trajectories of the particles is

$$x_j^\mu = V_j^\mu \; \tau + (c_j^\mu + i \boldsymbol{c}_j^\mu) \log\tau + d_j + \mathcal{O}(\frac{1}{\tau}),$$

where we get (for outgoing particles)

$$
\begin{aligned}
c_j^\mu &= \frac{1}{4\pi} \sum_{\substack{i=n+1,\\ i\neq j}}^{2n} e_i e_j \frac{(V_{i\mu} + V_{j\mu} V_i.V_j)}{[(V_i.V_j)^2 - 1]^{3/2}}, \\
\boldsymbol{c}_j^\mu &= \frac{1}{8\pi^2} \sum_{\substack{i=1,\\ i\neq j}}^{2n} \eta_i e_i e_j \frac{(V_{i\mu} + V_{j\mu} V_i.V_j)}{[(V_i.V_j)^2 - 1]^{3/2}} \Big[\log \frac{-V_i.V_j + \sqrt{(V_i.V_j)^2 - 1}}{-V_i.V_j - \sqrt{(V_i.V_j)^2 - 1}} + 2V_i.V_j \ \sqrt{(V_i.V_j)^2 - 1} \Big].
\end{aligned}
\tag{4.26}
$$

Thus we see that the logarithmic correction to the trajectory has two parts. The real part matches with eqn (3.8) that arose due to the retarded solution. The imaginary correction is an artefact of the Feynman solution. Let us compare the logarithmic correction in the trajectory with the logarithmic dressing of the scalar field [84]. c_j^μ is related to the Φ term in the dressing of scalar field as given in eqn (11) of [84] while $\boldsymbol{c}_j^\mu$ is related to the $\mathbf{R}$ term in the dressing of scalar field as given in eqn (10) of [84]. For j^{th} incoming particle, the corresponding terms are given by

$$
\begin{aligned}
c_j^\mu &= -\frac{1}{4\pi} \sum_{\substack{i=1,\\ i\neq j}}^{n} e_i e_j \frac{(V_{i\mu} + V_{j\mu} V_i.V_j)}{[(V_i.V_j)^2 - 1]^{3/2}} \\
\boldsymbol{c}_j^\mu &= \frac{1}{8\pi^2} \sum_{\substack{i=1,\\ i\neq j}}^{2n} \eta_i e_i e_j \frac{(V_{i\mu} + V_{j\mu} V_i.V_j)}{[(V_i.V_j)^2 - 1]^{3/2}} \Big[\log \frac{-V_i.V_j + \sqrt{(V_i.V_j)^2 - 1}}{-V_i.V_j - \sqrt{(V_i.V_j)^2 - 1}} + 2V_i.V_j \ \sqrt{(V_i.V_j)^2 - 1} \Big].
\end{aligned}
\tag{4.27}
$$

It should be noted that in the first term, the sum over i includes only the incoming particles.

For conciseness, let us define $C_j^\mu = c_j^\mu + i\boldsymbol{c}_j^\mu$ so that

$$
x_j^\mu = V_j^\mu \ \tau + C_j^\mu \log \tau + d_j + \mathcal{O}(\frac{1}{\tau}). \tag{4.28}
$$

Because of this $\mathcal{O}(e^2)$ correction to the asymptotic trajectories, the current given in (4.1) also gets corrected.

$$
\begin{aligned}
j_\sigma^{\text{asym}}(x') \ &= \ \int d\tau \ \Big[\sum_{j=n+1}^{2n} e_j \big[V_{j\sigma} + \frac{C_{j\mu}}{\tau} \big] \ \delta^4(x' - x_j) \ \Theta(\tau - T) \\
&\quad + \sum_{j=1}^{n} e_j \big[V_{j\sigma} + \frac{C_{j\mu}}{\tau} \big] \ \delta^4(x' - x_j) \ \Theta(-\tau - T) \Big].
\end{aligned}
\tag{4.29}
$$

Next we will find the electromagnetic field generated by above current.

4.3 Radiative field at $\mathcal{O}(e^3)$ with Feynman propagator

In this section we will obtain the asymptotic radiative field keeping the leading order effect of long range electromagnetic force acting on the particles. Using (4.29) we get

$$A_\sigma(x) = \int_T^\infty d\tau \sum_{j=n+1}^{2n} G(x,x_j)\ e_j\big[V_{j\sigma} + \frac{C_{j\sigma}}{\tau}\big] + \int_{-\infty}^{-T} d\tau \sum_{j=1}^{n} G(x,x_j)\ e_j\big[V_{j\sigma} + \frac{C_{j\sigma}}{\tau}\big]$$
$$+ \int_{r'<R} d^4x'\ G(x,x')\ j_\sigma^{\rm bulk}(x')\ .$$

We recall the expression of the Feynman propagator

$$G(x;x') \;=\; \frac{1}{4\pi^2}\Big[\pi\delta_+(\ (x-x')^2) + \pi\delta_-(\ (x-x')^2) + \frac{i}{(x-x')^2}\Big]. \quad (4.30)$$

Let us first write down the contribution from the first two terms of (4.30). As disscussed in the beginning of Section 4.2, these terms get contribution only from asymptotic sources and it suffices to use (4.29). We cannot solve the δ-function condition exactly because of the logarithmic correction. We solve it perturbatively in Appendix 4 and quote the solution to the delta function constraint from (B.5)

$$\tau_1^\pm = -V_i.(x-d_i) \mp \big[\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ \ - 2(x-d_i).C_i \log\tau_0^\pm\ \big]^{1/2}. \quad (4.31)$$

Here, $\tau_0^\pm$ is the zeroth order solution given in (B.1). Hence we get

$$A_\sigma^+(x) + A_\sigma^-(x)$$
$$= \frac{1}{4\pi}\int d\tau \sum_{i=n+1}^{2n} [\delta_+(\ (x-x')^2) + \delta_+(\ (x-x')^2)\ ]\ e_i\big[V_{i\sigma} + \frac{C_{j\sigma}}{\tau}\big]\ \Theta(\tau - T)\ \ +\ \ \text{in.}$$
$$= \frac{1}{4\pi}\int d\tau \sum_{i=n+1}^{2n} \frac{\delta(\tau-\tau_1^+) + \delta(\tau-\tau_1^-)}{|2\tau + 2V_i.(x-d_i) + \frac{2}{\tau}C_i.(x-d_i)|}\ e_i\big[V_{i\sigma} + \frac{C_{j\sigma}}{\tau}\big]\ \Theta(\tau-T)\ \ +\ \ \text{in.}$$

We have not written the contribution if the incoming particles explicitly. Above expression is vaild only to $\mathcal{O}(e^3)$. Expanding the roots in (4.31) to $\mathcal{O}(e^3)$, we have :

$$\tau_1^{\pm} = -V_i.(x-d_i) \mp \left[\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ \right]^{1/2} \pm (x-d_i).C_i \frac{\log \tau_0^{\pm}}{X}.$$

Hence we get

$$\begin{aligned}
A^{+}_{\sigma}(x) + A^{-}_{\sigma}(x) = & \frac{1}{8\pi} \sum_{i=n+1}^{2n} \frac{\Theta(u-T)\ e_i}{X} \Big[\ V_{i\sigma}\big[1 + \frac{(x-d_i).C_i}{X^2} \log \tau_0^{+} + \frac{(x-d_i).C_i}{X\tau_0^{+}}\big] + \frac{C_{i\sigma}}{\tau_0^{+}}\Big] \\
& + \frac{1}{8\pi} \sum_{i=1}^{n} \frac{\Theta(-u-T)\ e_i}{X} \Big[\ V_{i\sigma}\big[1 + \frac{(x-d_i).C_i}{X^2} \log \tau_0^{+} + \frac{(x-d_i).C_i}{X\tau_0^{+}}\big] + \frac{C_{i\sigma}}{\tau_0^{+}}\Big] \\
& + \frac{1}{8\pi} \sum_{i=n+1}^{2n} \frac{e_i}{X} \Big[\ V_{i\sigma}\big[1 + \frac{(x-d_i).C_i}{X^2} \log \tau_0^{-} - \frac{(x-d_i).C_i}{X\tau_0^{-}}\big] + \frac{C_{i\sigma}}{\tau_0^{-}}\Big]\ ,
\end{aligned} \tag{4.32}$$

where $X = [\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ ]^{1/2}$.

We can study the expansion of above expression around $\mathcal{I}^+$. Using (B.1) and (B.14) we have

$$\begin{aligned}
& \big[A^{+}_{\sigma}(x) + A^{-}_{\sigma}(x)\big]|_{\mathcal{I}^+} \\
& = \sum_{\substack{n=1,\\ m=-\infty,\\ m<n.}}^{\infty} A^{[n,-m]}_{\mu}(\hat{x})\ \frac{u^m}{r^n}\ +\ \log u \sum_{\substack{n=2,m=0,\\ m<n-1}}^{\infty} A^{[n,-m]}_{1\mu}(\hat{x})\ \frac{u^m}{r^n}\ +\ \log r \sum_{\substack{n=2,m=0,\\ m<n-1}}^{\infty} A^{[n,-m]}_{2\mu}(\hat{x})\ \frac{u^m}{r^n} + \ldots\ .
\end{aligned} \tag{4.33}$$

Above expression should be compared with (4.5) and (4.7). The logarithmic modes present in above expression appear only at $\mathcal{O}(e^3)$ and are a direct consequence of the long range electromagnetic forces present between the scattering particles. These modes are absent in (4.5) and (4.7). '...' denote terms that fall off faster than any power law.

Let us turn to the contribution from the third term of (4.30).

$$A^{*}_{\sigma}(x) = \frac{i}{4\pi^2} \int d^4x'\ \frac{j_\sigma(x')}{(x-x')^2}\ . \tag{4.34}$$

Using (4.29), we write down the asymptotic contribution to above expression. The integral needs to be regulated with an IR cutoff 'R'.

$$\begin{aligned}A_\sigma^{*\text{asym}}(x) &= \frac{i}{4\pi^2}\Bigg[\int_T^R \sum_{j=n+1}^{2n} \frac{d\tau\ e_j[V_{j\sigma}+\frac{C_{j\sigma}}{\tau}]}{(x-V_j\tau-d_j-C_j\log\tau)^2} + \int_{-R}^{-T}\sum_{j=1}^{n}\frac{d\tau\ e_j[V_{j\sigma}+\frac{C_{j\sigma}}{\tau}]}{(x-V_j\tau-d_j-C_j\log\tau)^2}\Bigg],\\ &= \frac{i}{4\pi^2}\int_T^R d\tau \sum_{j=n+1}^{2n}\Big[\frac{e_j[V_{j\sigma}+\frac{C_{j\sigma}}{\tau}]}{(x-V_j\tau-d_j)^2} + \frac{2(x-d_j).C_j}{(x-V_j\tau-d_j)^4}\log\tau\Big] \quad + \quad \text{in.}\end{aligned}$$

In above expression we have not written the contribution of the incoming particles explicitly to avoid clutter. Let us rewrite the expression using $\tau_0^\pm$.

$$\begin{aligned}&A_\sigma^{*\text{asym}}(x)\\ &= \frac{i}{4\pi^2}\int_T^R d\tau \sum_{j=n+1}^{2n}\Big[e_j\frac{[V_{j\sigma}+\frac{C_{j\sigma}}{\tau}]}{\tau_0^- - \tau_0^+}\Big[\frac{1}{\tau-\tau_0^+} - \frac{1}{\tau-\tau_0^-}\Big] \ + \ e_jV_{j\sigma}\frac{2(x-d_j).C_j\ \log\tau}{(\tau-\tau_0^-)^2(\tau-\tau_0^+)^2}\Big] \quad + \quad \text{in.}\end{aligned} \tag{4.35}$$

$\tau_0^\pm$ given in (B.1) are the solutions to the equation $(x-V_i\tau-d_i)^2 = 0$. Above integrals have been discussed in Appendix B. The final expression of (4.35) is given in (B.13). To this expression, we add the contribution of (4.32) to get

$$
\begin{aligned}
A_\sigma(x) &= \frac{i}{4\pi^2}\sum_{j=1}^{2n}\frac{\eta_j e_j V_{j\sigma}}{\tau_0^- - \tau_0^+}\Big[\log\frac{1}{\tau_0^+ - \eta_j T} - \log\frac{1}{\tau_0^- - \eta_j T}\Big] \\
&+ \frac{i}{4\pi^2}\sum_{j=1}^{2n}\frac{\eta_j e_j C_{j\sigma}}{(\tau_0^- - \tau_0^+)}\ \Big[\ \frac{1}{\tau_0^+}\log\frac{T}{\tau_0^+ - \eta_j T}\ -\ \frac{1}{\tau_0^-}\log\frac{T}{\tau_0^- - \eta_j T}\ \Big] \\
&+ \frac{i}{4\pi^2}\sum_{j=1}^{2n}\frac{2\eta_j e_j V_{j\sigma}(x-d_j).C_j}{(\tau_0^- - \tau_0^+)^2}\log(T)\Big[\frac{1}{T-\tau_0^+} + \frac{1}{T-\tau_0^-}\Big] \\
&+ \frac{i}{4\pi^2}\sum_{j=1}^{2n}\frac{2\eta_j e_j V_{j\sigma}(x-d_j).C_j}{(\tau_0^- - \tau_0^+)^2}\Big[\ \frac{1}{\tau_0^-}\log\frac{T}{\tau_0^- - \eta_j T} + \frac{1}{\tau_0^+}\log\frac{T}{\tau_0^+ - \eta_j T}\ \Big] \\
&+ \frac{i}{4\pi^2}\sum_{j=1}^{2n}\frac{4\eta_j e_j V_{j\sigma}(x-d_j).C_j}{(\tau_0^- - \tau_0^+)^3}\Big[\ln\tau_0^-\ln(\tau_0^- - \eta_j T) - \ln\tau_0^+\ln(\tau_0^+ - \eta_j T) + \frac{[\ln^2\tau_0^+ - \ln^2\tau_0^-]}{2} \\
&- \frac{i}{4\pi^2}\sum_{j=1}^{2n}\frac{4\eta_j e_j V_{j\sigma}(x-d_j).C_j}{(\tau_0^- - \tau_0^+)^3}\Big[\mathrm{Li}_2(1-\frac{\eta_j T}{\tau_0^-}) - \mathrm{Li}_2(1-\frac{\eta_j T}{\tau_0^+})\Big] \\
&+ \frac{1}{8\pi}\sum_{i=n+1}^{2n}\frac{\Theta(u-T)\ e_i}{X}\Big[\ V_{i\sigma}\big[1+\frac{(x-d_i).C_i}{X^2}\log\tau_0^+ + \frac{(x-d_i).C_i}{X\tau_0^+}\big] + \frac{C_{i\sigma}}{\tau_0^+}\Big] \\
&+ \frac{1}{8\pi}\sum_{i=1}^{n}\frac{\Theta(-u-T)\ e_i}{X}\Big[\ V_{i\sigma}\big[1+\frac{(x-d_i).C_i}{X^2}\log\tau_0^+ + \frac{(x-d_i).C_i}{X\tau_0^+}\big] + \frac{C_{i\sigma}}{\tau_0^+}\Big] \\
&+ \frac{1}{8\pi}\sum_{i=n+1}^{2n}\frac{e_i}{X}\Big[\ V_{i\sigma}\big[1+\frac{(x-d_i).C_i}{X^2}\log\tau_0^- - \frac{(x-d_i).C_i}{X\tau_0^-}\big] + \frac{C_{i\sigma}}{\tau_0^-}\Big]\ + A_\sigma^{*\mathrm{bulk}}(x).
\end{aligned}
\tag{4.36}
$$

$X = [\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ ]^{1/2}$. This is the result of this section. $A_\sigma^{*\mathrm{bulk}}(x)$ denotes the contribution of the bulk sources to (4.34). As this term depends on the the detailed form of bulk trajectories, the exact form of this term cannot be obtained. This term has been estimated in (4.12).

We can write down the asymptotic expansion of above solution around the future null infinity. Using (4.33), (B.15) and (4.12), we get

$$
\begin{aligned}
A_\sigma(x)|_{\mathcal{I}^+} = &\ (\log r)^2\sum_{\substack{n=2,\\ m=0,\\ m<n-1.}}^{\infty}[A^{[n,-m]}_{\ell_1\mu}(\hat{x})]\,\frac{u^m}{r^n} + \log r\sum_{\substack{n=1,\\ m=0,\\ m<n.}}^{\infty}[A^{[n,-m]}_{\ell_2\mu}(\hat{x})]\,\frac{u^m}{r^n} + (\log u)^2\sum_{\substack{n=2,\\ m=0,\\ m<n-1.}}^{\infty}[A^{[n,-m]}_{\ell_3\mu}(\hat{x})] \\
&+ \log u\sum_{\substack{n=1,\\ m=-\infty,\\ m<n.}}^{\infty}[A^{[n,-m]}_{\ell_4\mu}(\hat{x})]\,\frac{u^m}{r^n} + \sum_{\substack{n=1,\\ m=-\infty,\\ m<n.}}^{\infty}[A^{[n,-m]}_{\mu}(\hat{x})]\,\frac{u^m}{r^n}\ .
\end{aligned}
\tag{4.37}
$$

It is interesting to study the $\mathcal{O}(e^3)$ corrections to $\frac{1}{r}$ term of A_σ :

$$A_\sigma(x)|_{\mathcal{J}^+} \sim \frac{1}{r} \left[\log u + u^0 + \sum_{m=1}^{\infty} \frac{\log u}{u^m} + \sum_{n=1}^{\infty} \frac{1}{u^n} + \dots \right] . \tag{4.38}$$

This expression should be compared with its retarded analogue given in (3.18). The $\frac{\log u}{u^m}$-modes are absent in (3.18). These are new 'quantum' modes that appear at $\mathcal{O}(e^3)$ as a result of long range electromagnetic interactions between the scattering particles. In [100], we have demonstrated that the $\frac{\log u}{ur}$- mode is controlled by the universal soft $\log\omega$-mode. This tells us that the such modes are universally present in the quantum gauge field. Using (4.36), it can be shown that the $\log u$ and the u^0 modes are not modified at $\mathcal{O}(e^3)$. It should be noted that the coefficients of the $\frac{1}{u^n}$-modes are modified at $\mathcal{O}(e^3)$.

4.4 The $\tilde{Q}_1$ conservation law

In (4.36) we obtained the asymptotic field including $\mathcal{O}(e^3)$ corrections. In this section we will obtain an asymptotic conservation law obeyed by certain modes in the asymptotic field.

F_{rA} calculated using (4.36) takes following form

$$F_{rA}|_{\mathcal{J}^+_-} = \frac{\log r}{r^2}\ F_{rA}^{[\log r]}(\hat{x}) + \frac{1}{r^2} \left[\ u \log u\ F_{rA}^{[u\log u]}(\hat{x}) + (\log u)^2\ F_{rA}^{[(\log u)^2]}(\hat{x}) \right.$$
$$\left. + u\ F_{rA}^{[u]}(\hat{x})\ +\ \log u\ F_{rA}^{[\log u]}(\hat{x}) + \dots \right] . \tag{4.39}$$

Let us compare above expression with (3.1). (3.1) is the corresponding expansion for the retarded solution.

$$F_{rA}^{\text{ret}}|_{\mathcal{J}^+_-} = \frac{1}{r^2} \left[\ u\ F_{rA}^{[u/r^2]}(\hat{x})\ +\ \log u\ F_{rA}^{[\log u/r^2]}(\hat{x})\ + \dots\right] + \mathcal{O}(\frac{1}{r^3}) .$$

Thus the Feynman solution contains modes like $\frac{\log r}{r^2}$, $\frac{(\log u)^2}{r^2}$ at future that are absent in the retarded solution. The Feynman solution around the past null infinity gives

$$F_{rA}|_{\mathcal{J}^-_+} = \frac{\log r}{r^2}\ F_{rA}^{[\log r]}(\hat{x}) + \frac{1}{r^2} \left[\ v \log v\ F_{rA}^{[v\log v]}(\hat{x}) + (\log v)^2\ F_{rA}^{[(\log v)^2]}(\hat{x}) + \right.$$
$$\left. v\ F_{rA}^{[v]}(\hat{x})\ +\ \log v\ F_{rA}^{[\log v]}(\hat{x}) + \dots \right] . \tag{4.40}$$

From (4.39) and (4.40), we see that the expansion around future and past is symmetric. This is expected for Feynman propagator. In contrast, the retarded solution is not symmetric as seen in (3.1) and (3.2).

In this section we will show that above modes obey an conservation equation given by

$$[F_{rA}^{[\log u]}(\hat{x}) - F_{rA}^{[\log r]}(\hat{x})]\ |_{\mathcal{I}^+_-} = [-F_{rA}^{[\log v]}(-\hat{x}) + F_{rA}^{[\log r]}(-\hat{x})]\ |_{\mathcal{I}^-_+}. \tag{4.41}$$

Above equation should be compared with (3.25) which is obeyed by the retarded solution. It is important to note that (3.25) is violated by the Feynman solution.

Modes at Future null infinity

Let us find the full $\frac{\log u}{r^2}$-mode of A_σ at the future null infinity. We need to expand all the terms in the Feynman solution given in (4.36) around $\mathcal{I}^+$. There are many terms that contribute to the $\frac{\log u}{r^2}$-mode. We will list them. From the first line of (4.36), using (B.19) we get

$$\frac{i}{8\pi^2} \sum_{j=1}^{2n} \eta_j \frac{e_j V_{j\sigma}}{(V_j.q)^2}\ [\ \frac{q.d_j}{(V_j.q)} + V_j.d_j\ ]. \tag{4.42}$$

From the second line of (4.36), we get using (B.17) and (B.19)

$$-\frac{i}{16\pi^2} \sum_{j=1}^{2n} \eta_j e_j C_{j\sigma}\ [\ \frac{1}{(V_j.q)^2} - 1\ ]. \tag{4.43}$$

The third line of (4.36) does not have a log u term. From the fourth line of (4.36), we get using (B.17) and (B.19)

$$-\frac{i}{8\pi^2} \sum_{j=1}^{2n} \eta_j \frac{e_j V_{j\sigma}}{V_j.q} \Big[\ q.C_j[-\frac{3}{2(V_j.q)^2} - \frac{V_j^0}{(V_j.q)} + \frac{1}{2}\ ] + C_j^0\ \Big]. \tag{4.44}$$

Using (B.19), the fifth line of (4.36) gives

$$-\frac{i}{8\pi^2} \sum_{j=1}^{2n} \eta_j e_j V_{j\sigma} \frac{q.C_j}{(q.V_j)^3}\ \ln|q.V_j|. \tag{4.45}$$

As shown in Appendix 4, the sixth line of (4.36) and $A_\sigma^{*\text{bulk}}$ do not have any logarithmic modes. We turn to the seventh and eighth lines of (4.36). Using (B.7), we have $\log \tau_0|_{\mathcal{I}^+} \sim \log u + \mathcal{O}(u^0)$. Using (2.2), we get $X = -rq.V_i + \mathcal{O}(r^0)$.

Substituting the limiting value of X, we can read off the coefficient of the $\mathcal{O}(\frac{\log u}{r^2})$ term in the seventh and eighth lines of (4.36) :

$$-\frac{1}{8\pi}\sum_{i=n+1}^{2n}\Theta(u-T)\ e_i\ V_{i\sigma}\ \frac{q.C_i}{(q.V_i)^3}-\frac{1}{8\pi}\sum_{i=1}^{n}\Theta(-u-T)e_i\ V_{i\sigma}\ \frac{q.C_i}{(q.V_i)^3} \quad (4.46)$$

We have the full coefficient of the $\frac{\log u}{r^2}$ term.

$$\begin{aligned} A_\sigma^{[\log u/r^2]}|_{\mathcal{I}^+_-} &= -\frac{1}{8\pi}\sum_{i=1}^{n} e_i\ V_{i\sigma}\ \frac{q.C_i}{(q.V_i)^3}+\frac{i}{8\pi^2}\ \sum_{j=1}^{2n}\eta_j\frac{e_jV_{j\sigma}}{(V_j.q)^2}\ [\ \frac{q.d_j}{(V_j.q)}+V_j.d_j\ ] \\ &-\frac{i}{16\pi^2}\sum_{j=1}^{2n}\eta_je_jC_{j\sigma}\ [\ \frac{1}{(V_j.q)^2}-1\ ]-\frac{i}{8\pi^2}\sum_{j=1}^{2n}\eta_je_jV_{j\sigma}\frac{q.C_j}{(q.V_j)^3}\ \ln|q.V_j| \\ &-\frac{i}{8\pi^2}\sum_{j=1}^{2n}\eta_j\frac{e_jV_{j\sigma}}{V_j.q}\ \Big[\ q.C_j[-\frac{3}{2(V_j.q)^2}-\frac{V_j^0}{(V_j.q)}+\frac{1}{2}\ ]+C_j^0\ \Big]. \end{aligned} \quad (4.47)$$

Next we need to write down the coefficient of the $\frac{\log r}{r^2}$-term in A_σ. From the first line of (4.36), using (B.19) we get

$$-\frac{i}{8\pi^2}\ \sum_{j=1}^{2n}\eta_j\frac{e_jV_{j\sigma}}{(V_j.q)^2}\ [\ \frac{q.d_j}{(V_j.q)}+V_j.d_j\ ]. \quad (4.48)$$

From the second line of (4.36), using (B.18) and (B.19) we get

$$\frac{i}{16\pi^2}\sum_{j=1}^{2n}\eta_j\frac{e_jC_{j\sigma}}{(q.V_j)^2}. \quad (4.49)$$

The third line of (4.36) does not have a log r term. From the fourth line of (4.36), using (B.18) and (B.19) we get

$$\frac{i}{16\pi^2}\sum_{j=1}^{2n}\eta_j\frac{e_jV_{j\sigma}}{(V_j.q)^3}\ q.C_j. \quad (4.50)$$

The fifth line of (4.36) contributes as follows

$$-\frac{i}{8\pi^2}\sum_{j=1}^{2n}\eta_je_jV_{j\sigma}\frac{q.C_j}{(q.V_j)^3}\ \ln(2|q.V_j|). \quad (4.51)$$

Substituting $X = -rq.V_i + \mathcal{O}(r^0)$ in the eighth line of (4.36), we get

$$-\frac{1}{8\pi}\sum_{i=n+1}^{2n} e_i\ V_{i\sigma}\ \frac{q.C_i}{(q.V_i)^3}. \tag{4.52}$$

We have the full coefficient of the $\frac{\log r}{r^2}$ term.

$$A_\sigma^{[\log r/r^2]}(x)|_{\mathcal{J}^+} = -\frac{1}{8\pi}\sum_{i=n+1}^{2n} e_i\ V_{i\sigma}\ \frac{q.C_i}{(q.V_i)^3} - \frac{i}{8\pi^2}\ \sum_{j=1}^{2n}\eta_j\frac{e_jV_{j\sigma}}{(V_j.q)^2}\ [\ \frac{q.d_i}{(V_i.q)} + V_j.d_j\ ]$$
$$+\frac{i}{16\pi^2}\sum_{j=1}^{2n}\eta_j\frac{e_jC_{j\sigma}}{(q.V_j)^2} + \frac{i}{16\pi^2}\sum_{j=1}^{2n}\eta_j\frac{e_jV_{j\sigma}}{(V_j.q)^3}\ q.C_j - \frac{i}{8\pi^2}\sum_{j=1}^{2n}\eta_je_jV_{j\sigma}\frac{q.C_j}{(q.V_j)^3}\ \ln(2|q.V_j|). \tag{4.53}$$

Modes at Past null infinity

Next we need to derive the field configuration at past null infinity and then compare the two expressions. Analogous to (4.36), around $\mathscr{I}^-$ we have

$$
\begin{aligned}
A_\sigma(x) &= \frac{i}{4\pi^2}\sum_{j=1}^{2n}\frac{\eta_j e_j V_{j\sigma}}{\tau_0^- - \tau_0^+}\Big[\log\frac{1}{\tau_0^+ - \eta_j T} - \log\frac{1}{\tau_0^- - \eta_j T}\Big] \\
&+ \frac{i}{4\pi^2}\sum_{j=1}^{2n}\frac{\eta_j e_j C_{j\sigma}}{(\tau_0^- - \tau_0^+)}\ \Big[\ \frac{1}{\tau_0^+}\log\frac{T}{\tau_0^+ - \eta_j T}\ -\ \frac{1}{\tau_0^-}\log\frac{T}{\tau_0^- - \eta_j T}\ \Big] \\
&+ \frac{i}{4\pi^2}\sum_{j=1}^{2n}\frac{2\eta_j e_j V_{j\sigma}(x-d_j).C_j}{(\tau_0^- - \tau_0^+)^2}\log T\Big[\frac{1}{\tau_0^+ - \eta_j T} + \frac{1}{\tau_0^- - \eta_j T}\Big] \\
&+ \frac{i}{4\pi^2}\sum_{j=1}^{2n}\frac{2\eta_j e_j V_{j\sigma}(x-d_j).C_j}{(\tau_0^- - \tau_0^+)^2}\Big[\ \frac{1}{\tau_0^-}\log\frac{T}{\tau_0^- - \eta_j T} + \frac{1}{\tau_0^+}\log\frac{T}{\tau_0^+ - \eta_j T}\ \Big] \\
&+ \frac{i}{4\pi^2}\sum_{j=1}^{2n}\frac{4\eta_j e_j V_{j\sigma}(x-d_j).C_j}{(\tau_0^- - \tau_0^+)^3}\Big[-\ln\tau_0^+\ln(\tau_0^+ - \eta_j T) + \ln\tau_0^-\ln(\tau_0^- - \eta_j T) + \frac{1}{2}[\ln^2\tau_0^+ - \ln^2\tau_0^-]\Big] \\
&- \frac{i}{4\pi^2}\sum_{j=1}^{2n}\frac{4\eta_j e_j V_{j\sigma}(x-d_j).C_j}{(\tau_0^- - \tau_0^+)^3}\Big[\mathrm{Li}_2\Big(-\frac{\eta_j T - \tau_0^-}{\tau_0^-}\Big) - \mathrm{Li}_2\Big(-\frac{\eta_j T - \tau_0^+}{\tau_0^+}\Big)\Big] \\
&+ \frac{1}{8\pi}\sum_{i=n+1}^{2n}\frac{\Theta(v-T)\ e_i}{X}\Big[\ V_{i\sigma}\big[1 + \frac{(x-d_i).C_i}{X^2}\log\tau_0^- - \frac{(x-d_i).C_i}{X\tau_0^-}\big] + \frac{C_{i\sigma}}{\tau_0^-}\Big] \\
&+ \frac{1}{8\pi}\sum_{i=1}^{n}\frac{\Theta(-v-T)\ e_i}{X}\Big[\ V_{i\sigma}\big[1 + \frac{(x-d_i).C_i}{X^2}\log\tau_0^- - \frac{(x-d_i).C_i}{X\tau_0^-}\big] + \frac{C_{i\sigma}}{\tau_0^-}\Big] \\
&+ \frac{1}{8\pi}\sum_{i=1}^{n}\frac{e_i}{X}\Big[\ V_{i\sigma}\big[1 + \frac{(x-d_i).C_i}{X^2}\log\tau_0^+ + \frac{(x-d_i).C_i}{X\tau_0^+}\big] + \frac{C_{i\sigma}}{\tau_0^+}\Big]\ .
\end{aligned}
\tag{4.54}
$$

Here, $X = [\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ ]^{1/2}$.

Let us write down the full coefficient of the $\frac{\log v}{r^2}$-mode. We need to take the limit $r \to \infty$ with $v = t + r$ finite. In this co-ordinate system, 4 dimensional spacetime point can be parametrised as given in (2.16). From the first line of (4.54), we get using (B.23)

$$\frac{i}{8\pi^2}\ \sum_{j=1}^{2n}\eta_j\frac{e_j V_{j\sigma}}{(V_j.\bar{q})^2}\ [\ \frac{\bar{q}.d_j}{(V_j.\bar{q})} + V_j.d_j\ ]. \tag{4.55}$$

From the second line of (4.54), using (B.22) and (B.23) we get

$$-\frac{i}{16\pi^2}\sum_{j=1}^{2n}\eta_j e_j C_{j\sigma}\ [\ \frac{1}{(V_j.\bar{q})^2} - 1\ ]. \tag{4.56}$$

The third line of (4.54) does not have a log v term. From the fourth line of (4.54), using (B.22) and (B.23) we get

$$-\frac{i}{8\pi^2}\sum_{j=1}^{2n}\eta_j\frac{e_jV_{j\sigma}}{V_j.\bar{q}}\Big[\,\bar{q}.C_j[\,\frac{V_j^0}{(V_j.\bar{q})}-\frac{3}{2(\bar{q}.V_i)^2}+\frac{1}{2}\,]-C_j^0\,\Big]. \tag{4.57}$$

The fifth line of (4.54) gives

$$-\quad\frac{i}{8\pi^2}\sum_{j=1}^{2n}\eta_je_jV_{j\sigma}\frac{\bar{q}.C_j}{(\bar{q}.V_j)^3}\;\ln|\bar{q}.V_j|\;. \tag{4.58}$$

From seventh line of (4.54) we get

$$\frac{1}{8\pi}\sum_{i=n+1}^{2n}\Theta(v-T)\;e_i\;V_{i\sigma}\;\frac{\bar{q}.C_i}{(\bar{q}.V_i)^3}+\frac{1}{8\pi}\sum_{i=1}^{n}\Theta(-v-T)e_i\;V_{i\sigma}\;\frac{\bar{q}.C_i}{(\bar{q}.V_i)^3}. \tag{4.59}$$

Hence we have

$$\begin{aligned}A_\sigma^{[\log v/r^2]}(x)=&+\frac{1}{8\pi}\sum_{i=n+1}^{2n}e_i\;V_{i\sigma}\;\frac{\bar{q}.C_i}{(\bar{q}.V_i)^3}+\frac{i}{8\pi^2}\;\sum_{j=1}^{2n}\eta_j\frac{e_jV_{j\sigma}}{(V_j.\bar{q})^2}\;[\;\frac{\bar{q}.d_j}{(V_j.\bar{q})}+V_j.d_j\;]\\&-\frac{i}{16\pi^2}\sum_{j=1}^{2n}\eta_je_jC_{j\sigma}\;[\;\frac{1}{(V_j.\bar{q})^2}-1\;]-\frac{i}{8\pi^2}\sum_{j=1}^{2n}\eta_je_jV_{j\sigma}\frac{\bar{q}.C_j}{(\bar{q}.V_j)^3}\;\ln|\bar{q}.V_j|\\&-\frac{i}{8\pi^2}\sum_{j=1}^{2n}\eta_j\frac{e_jV_{j\sigma}}{V_j.\bar{q}}\;\Big[\;\bar{q}.C_j[-\frac{3}{2(V_j.\bar{q})^2}+\frac{V_j^0}{(V_j.\bar{q})}+\frac{1}{2}\;]-C_j^0\;\Big].\end{aligned} \tag{4.60}$$

Next we turn to the $\frac{\log r}{r^2}$-mode. From the first line of (4.54), we get

$$-\frac{i}{8\pi^2}\;\sum_{j=1}^{2n}\eta_j\frac{e_jV_{j\sigma}}{(V_j.\bar{q})^2}\;[\;\frac{\bar{q}.d_j}{(V_j.\bar{q})}+V_j.d_j\;]. \tag{4.61}$$

From the second line of (4.54), we get

$$\frac{i}{16\pi^2}\sum_{j=1}^{2n}\eta_je_jC_{j\sigma}\;\frac{1}{(V_j.\bar{q})^2}. \tag{4.62}$$

The third line of (4.54) does not have a log r term. From the fourth line of (4.54), we get

$$\frac{i}{16\pi^2}\sum_{j=1}^{2n}\eta_j\frac{e_jV_{j\sigma}}{(V_j.\bar{q})^3}\;\bar{q}.C_j. \tag{4.63}$$

The fifth line of (4.54) gives

$$-\frac{i}{8\pi^2}\sum_{j=1}^{2n}\eta_j e_j V_{j\sigma}\frac{\bar{q}.C_j}{(\bar{q}.V_j)^3}\ \ln|2\bar{q}.V_j|\ . \tag{4.64}$$

We get following contribution from the last line of (4.54)

$$\frac{1}{8\pi}\sum_{i=1}^{n} e_i\ V_{i\sigma}\ \frac{\bar{q}.C_i}{(\bar{q}.V_i)^3}. \tag{4.65}$$

The total coefficient is

$$\begin{aligned}A_\sigma^{[\log r/r^2]}(x)|_{\mathcal{I}^-} &= \frac{1}{8\pi}\sum_{i=1}^{n} e_i\ V_{i\sigma}\ \frac{\bar{q}.C_i}{(\bar{q}.V_i)^3} - \frac{i}{8\pi^2}\ \sum_{j=1}^{2n}\eta_j\frac{e_jV_{j\sigma}}{(V_j.\bar{q})^2}\ [\ \frac{\bar{q}.d_j}{(V_j.\bar{q})} + V_j.d_j\ ]\\ &+\frac{i}{16\pi^2}\sum_{j=1}^{2n}\eta_j e_j C_{j\sigma}\ \frac{1}{(V_j.\bar{q})^2} + \frac{i}{16\pi^2}\sum_{j=1}^{2n}\eta_j\frac{e_jV_{j\sigma}}{(V_j.\bar{q})^3}\ \bar{q}.C_j - \frac{i}{8\pi^2}\sum_{j=1}^{2n}\eta_j e_j V_{j\sigma}\frac{\bar{q}.C_j}{(\bar{q}.V_j)^3}\ \ln|2\bar{q}.V_j|\ .\end{aligned} \tag{4.66}$$

From (4.47), (4.53), (4.60) and (4.66) we can indeed check that following modes are equal under antipodal idenfication.

$$[A_\sigma^{[\log u/r^2]}(\hat{x}) - A_\sigma^{[\log r/r^2]}(\hat{x})]\ |_{\mathcal{I}^+_-} = [A_\sigma^{[\log v/r^2]}(-\hat{x}) - A_\sigma^{[\log r/r^2]}(-\hat{x})]\ |_{\mathcal{I}^-_+}. \tag{4.67}$$

Using co-ordinate transformation, it can be shown that the quantum gauge field is expected to obey following conservation equation :

$$[F_{rA}^{[\log u/r^2]}(\hat{x}) - F_{rA}^{[\log r/r^2]}(\hat{x})]\ |_{\mathcal{I}^+_-} = [-F_{rA}^{[\log v/r^2]}(-\hat{x}) + F_{rA}^{[\log r/r^2]}(-\hat{x})]\ |_{\mathcal{I}^-_+}. \tag{4.68}$$

Compared to (4.67), the RHS of above expression has extra minus sign as it has an extra factor of $\partial_A\bar{q}^\mu$ due to co-ordinate transformation. Finally we have derived the $\tilde{Q}_1$-conservation equation such that the future charge is defined by $\tilde{Q}_1^+ = \int d^2z\ Y^A(\hat{x})\ [F_{rA}^{[\log u/r^2]}(\hat{x}) - F_{rA}^{[\log r/r^2]}(\hat{x})]|_{\mathcal{I}^+_-}$ and the past charge by $\tilde{Q}_1^- = \int d^2z\ Y^A(-\hat{x})\ [-F_{rA}^{[\log v/r^2]}(-\hat{x}) + F_{rA}^{[\log r/r^2]}(-\hat{x})]|_{\mathcal{I}^-_+}$. It should be possible to prove (4.68) in general by following analysis of [78] albeit with Feynman boundary condition.

In this chapter we have obtained the radiative field produced by scattering of n charged point particles using Feynman propagator. This problem is unphysical but the Feynman radiative solution so derived is useful to illustrate interesting features of the quantum gauge field. This chapter shows that the asymptotic expansion of the quantum gauge field contains new modes that are absent in the retarded classical field. Let us summarise two key differences that are relevant for our analysis. As discussed in (3.18), the $\frac{1}{r}$ term in retarded A_μ at $\mathcal{O}(e^3)$ takes following form :

$$A_\mu^{\text{ret}}(x)|_{\mathcal{J}^+} = \frac{1}{4\pi r}[\ u^0\ +\ \sum_{n=1}^{\infty}\frac{1}{u^n}\ ]\ + \mathcal{O}(\frac{1}{r^2})\ . \tag{4.69}$$

In (4.38) we showed that the $\frac{1}{r}$-term in the Feynman solution at $\mathcal{O}(e^3)$ has following behaviour

$$A_\sigma^{\text{feyn}}(x)|_{\mathcal{J}^+} \sim\ \frac{1}{r}\ [\ \log u + u^0 + \sum_{m=1}^{\infty}\frac{\log u}{u^m} + \sum_{n=1}^{\infty}\frac{1}{u^n} + ...\ ]\ + \mathcal{O}(\frac{1}{r^2})\ .$$

Here, '...' denote terms that fall off faster than any power law in u. Thus the $\frac{\log u}{r}$-mode is a purely quantum mode and it will play an important role in the analysis of Chapter 5.

Incorporating the effect of long range electromagnetic force on the scattering particles, we obtained the Feynman solution including the $\mathcal{O}(e^3)$ corrections in (4.36). In Chapter 3, we had shown that the analogous modes in the retarded solution obey the conservation law in (3.25) such that the coefficient of the $\frac{\log r}{r^2}$ mode at the past is related to the coefficient of the $\frac{\log u}{r^2}$ mode at the future. This law is violated by the Feynman solution. We discussed the modified asymptotic conservation equation obeyed by the $\mathcal{O}(e^3)$ logarithmic modes in the Feynman solution. This equation has been derived in (4.68) and relates the difference in the coefficients of the $\frac{\log u}{r^2}$ and $\frac{\log r}{r^2}$ modes in F_{rA} at $\mathcal{J}^+_-$ to the difference in the coefficients of the $\frac{\log r}{r^2}$ and $\frac{\log v}{r^2}$ in F_{rA} at $\mathcal{J}^-_+$. $\tilde{Q}_1$ charges are defined accordingly.

$\tilde{Q}_1$ charges are expected to be related to the soft $\log\omega$-mode. Let us discuss some features of $\tilde{Q}_1$ that make $\tilde{Q}_1$ an unfavourable candidate to be the asymptotic charge. The second term in (4.47), (4.53), (4.60) and (4.66) respectively are $\mathcal{O}(e)$ and not related to long range interaction between scattering particles, hence these terms in $\tilde{Q}_1$ are certainly not related to the $\log\omega$-mode. Some of the contributions to $\tilde{Q}_1$ are not Lorentz invariant. A close inspection tells us that many $\mathcal{O}(e^3)$ terms in $\tilde{Q}_1$ are not related to the $\log\omega$-mode. All these 'irrelevant' terms that contribute to $\tilde{Q}_1$ are expected to drop out of the final Ward identity such that the $\log\omega$ soft theorem is reproduced. This has not been checked explicitly yet.

In the next chapter we will show that there is a better prescription to define the asymptotic charge corresponding to the $\log \omega$ soft theorem. The prescription is to use the Q_1 charges we studied in (3.25) in Chapter 3 and then quantise the expressions. Though the classical Q_1-law in (3.25) is violated in the quantum theory, we will show that this law can be used to reproduce the full $\log \omega$ soft theorem in (2.54) including the purely quantum terms. This will be demonstrated in Chapter 5.

Chapter 5

The $\log \omega$ soft theorem and the Q_1 conservation law

In this chapter, we will start with the Q_1-conservation law derived in (3.25). This conservation law relates the coefficient of the $\frac{\log r}{r^2}$-mode at the past to the coefficient of the $\frac{\log u}{r^2}$-mode at the future :

$$F_{rA}^{2,\log}(\hat{x}) \mid_{\mathcal{I}^+_-} \quad = \quad F_{rA}^{\log,0}(-\hat{x}) \mid_{\mathcal{I}^-_+}. \tag{5.1}$$

We will construct the associated charges for massless scalar QED and show that the quantised charges reproduce the full $\log \omega$ soft theorem. This chapter is based on our calculations that appeared in [101].

Let us state the subleading soft theorem for loop amplitudes in presence of massless scattering particles [91]

$$\mathcal{A}_{n+1}(p_i, k) = \frac{S_0}{\omega}\ \mathcal{A}_n(p_i) \ + \ (S_{\log} + S^{\text{grav}}_{\log})\ \log \omega\ \mathcal{A}_n(p_i) \ + \dots\ , \tag{5.2}$$

$S^{\text{grav}}_{\log}$ is the correction to the soft factor in presence of dynamical gravity. Let us first focus on the purely electromagnetic term.

$$S_{\log} = -\frac{1}{4\pi^2} \sum_{i,j;i\neq j} \eta_i \eta_j e_i \frac{\epsilon_\mu q_\rho}{p_i.q}\ \frac{e_i e_j}{p_i.p_j} (p_j^\rho p_i^\mu - p_i^\rho p_j^\mu) \tag{5.3}$$

Above expression is obtained by taking massless limit of (2.55). We recall that e_i, p_i are respectively the charges and momenta of the hard particles and $\eta_i = 1(-1)$ for outgoing (incoming) particles. ϵ^μ is the polarisation vector of the soft photon and $k^\mu = \omega q^\mu$ is the soft momentum. An interesting observation is that the classical part of the soft factor which is non zero for the massive case goes to 0 in the massless limit. We will see that this result comes out naturally from the asymptotic charge as well.

As we discussed earlier in (2.56) the $\log\omega$ soft factor gets corrected by gravitational couplings. For massless scattering particles this correction is given by

$$\begin{aligned} S_{\log}^{\text{grav}} &= \frac{i}{4\pi}\sum_{\substack{i,j;i\neq j\\ \eta_i\eta_j=1}} e_i\frac{\epsilon_\mu q_\rho}{p_i.q}(p_j^\rho p_i^\mu - p_i^\rho p_j^\mu) - \frac{i}{4\pi}\sum_i \eta_i e_i\frac{\epsilon.p_i}{p_i.q}\sum_{j,\eta_j=1} q.p_j \\ &- \frac{1}{4\pi^2}\sum_{i,j;i\neq j}\eta_i\eta_j e_i\frac{\epsilon_\mu q_\rho}{p_i.q}\ (p_j^\rho p_i^\mu - p_i^\rho p_j^\mu)\ \log[p_i.p_j]\ ^{1} + \frac{1}{4\pi^2}\sum_i \eta_i e_i\frac{\epsilon.p_i}{p_i.q}\sum_j \eta_j q.p_j\ \log\ p_j.q. \end{aligned} \tag{5.4}$$

The first line in above expression also appears in the soft expansion of the classical radiative field [91]. The second line is absent in the classical theory and represents purely quantum contribution. It was noted in [91] that if we assume the momenta of the hard particles is $\mathcal{O}(\hbar^0)$, neither the classical nor the quantum terms in (5.4) have any power of $\hbar$. Thus, an intriguing aspect of the 'quantum' terms is that these terms are independent of $\hbar$ and do not trivially vanish in the limit $\hbar \to 0$.

The presence of such purely quantum terms is a significant feature of this soft theorem which is absent in the case of leading soft theorem. So we wish to highlight how the asymptotic charge reproduces the quantum contribution to the soft theorem without going into the details of the calculation. We start by constructing the classical expression for the Q_1 charge in massless scalar QED. This charge is very closely related to the long range forces present in four spacetime dimensions and gets contribution from dressing of free fields due to long range forces. The leading order dressing of massless scalar field is given by (5.8):

$$\phi(x) = -\frac{i e^{ieA_r^1(\hat{x})\log r}}{8\pi^2 r}\int d\omega\ [b(\omega,\hat{x})\ e^{-i\omega u} - d^\dagger(\omega,\hat{x})\ e^{i\omega u}].$$

A_r^1 defined in (2.34) is the electromagnetic dressing. This dressing contributes to the charge via (5.31) and the contribution can be schematically written as $Q_1 \sim \hat{S}_1\ A_r^1$, where $\hat{S}_1$ closely resembles the tree level subleading soft operator.

[1] Similar to the purely electromagnetic term, there could be a potential gravitational term :

$$-\frac{1}{4\pi^2}\sum_{i,j;i\neq j}\eta_i\eta_j e_i\frac{\epsilon_\mu q_\rho}{p_i.q}\ (p_j^\rho p_i^\mu - p_i^\rho p_j^\mu) \tag{5.5}$$

that vanishes because of momentum conservation.

Then we quantise the charge. In the quantum theory the A_r^1 mode gets additional contribution i.e. $A_r^1 = \overset{\text{class}}{A_r^1} + \overset{\text{quan}}{A_r^1}$. The classical mode is obtained by evolving the sources with retarded propagator (and turns out to be trivial). $\overset{\text{quan}}{A_r^1}$ is absent in the classical theory and is related to the quantum $\frac{\log u}{r}$-mode that is present in the quantum photon field and has been discussed in (4.22). Classical radiative fields are continous in $\omega \to 0$ hence the $\frac{\log u}{r}$-mode is absent in the classical theory. $\overset{\text{quan}}{A_r^1}$ reproduces the quantum corrections to the soft theorem.

Another important point to be noted is that there are certain divergences that appear due to the presence of massless particles. The quantum contributions to the dressing have divergent pieces arising from collinear configurations. The divergent part of electromagnetic dressing is a constant and does not contribute to the charge. Thus, the charge is rendered finite.

Gravitational corrections to the asymptotic charge have been discussed in Section 5.4.

5.1 Dressing of the massless scalar field

In this section we will study the effect of long range forces on massless scalar field and find the corresponding contribution to the asymptotic charge given in (5.1). In absence of long range forces, asymptotic fields satisfy free equations of motion. Including the correction to the asymptotic dynamics due to long range interactions leads to dressing of the free fields. We will show that the long range electromagnetic force results in a new mode in the asymptotic radiative field that falls off as $1/u$.

For massive fields the effect of long range forces is obtained perturbatively by studying asymptotic potential order by order around $t \to \infty$. This leads to the well known Faddeev-Kulish dressing of massive scalars [84]. For massless scalars, the asymptotic states live at null infinity. So, we will study the corrections to the free equation of motion at null infinity. Massless scalars satisfy following equation :

$$\eta^{\mu\nu} D_\mu D_\nu \; \phi(x) \; = 0. \tag{5.6}$$

Let us expand above equation around future null infinity. Using the fall offs given in (2.34), we find that the leading order equation is (at $\mathcal{O}(\frac{1}{r^2})$) :

$$-2\partial_u\partial_r\phi - \frac{2}{r}\partial_u\phi = -2ie\frac{A^1_r(\hat{x})}{r}\partial_u\phi. \tag{5.7}$$

Thus, the leading order effect of long range forces on the massless field is given by A^1_r. The solution of above equation is given by :

$$\phi(x) = -\frac{ie^{ieA^1_r(\hat{x})\log\frac{r}{r_0}}}{8\pi^2 r}\int d\omega\ [b(\omega,\hat{x})\ e^{-i\omega u} - d^\dagger(\omega,\hat{x})\ e^{i\omega u}], \tag{5.8}$$

where, b and $d^\dagger$ are the free data for massless scalar. First we will restrict ourselves to classical dynamics. Upon quantisation, b can be interpreted as the annihilation operator for free particles while d would become the annihilation operator for free antiparticles (see (2.25)). r_0 depends on scales of short range interactions, hence $r_0 << r$. For our analysis we can set $r_0 = 1$ to avoid clutter. From (5.8), we see that the effect of long range electromagnetic force is to associate a cloud of photons to a free massless scalar particle. The dressing factor A^1_r is analogous to the Fadeev-Kulish dressing of a free massive scalar particle. Next we find the resultant correction to the U(1) current. Dressing of scalar field leads to a new logarithmic fall off in the current (2.33) :

$$j_A \;=\; j^{\log}_A\,\frac{\log r}{r^2} + \frac{j^2_A}{r^2} + \ldots\,, \tag{5.9}$$

where

$$j^{\log}_A = 2e^2\ \partial_A A^1_r\ |\phi^1|^2. \tag{5.10}$$

We also have :

$$j_r = j^{\log}_r\frac{\log r}{r^4} + \frac{j^4_r}{r^4} + \ldots\quad, j_u \;=\; \frac{j^2_u}{r^2} \;+\; j^{\log}_u\,\frac{\log r}{r^3} + \ldots\,.$$

We can change to Cartesian co-ordinates and get following fall offs for the U(1) current

$$j_\mu \;=\; \frac{j^2_\mu}{r^2} \;+\; j^{\log}_\mu\,\frac{\log r}{r^3} + \ldots\,. \tag{5.11}$$

Let us find the the gauge field generated by the new logarithmic fall off in the current. In Lorenz gauge, we have $\Box A_\mu = -j_\mu$. Using the retarded propagator, the solution to the gauge field is given by :

$$A_\sigma(x) = \frac{1}{2\pi}\int d^4x'\ \delta(\ (x-x')^2)\ \Theta(t-t')\ j_\sigma(x'). \tag{5.12}$$

We will substitute the new logarithmic modes of the current in above expression and find the resultant contribution to the field. The details of the calculation have been relegated to Appendix C. We show that the log modes give rise to a $\frac{1}{u}$ term in A^0_A such that the coefficient is given by (C.11) :

$$A^{0,1}_{\bar{z}}(\hat{x}) = \frac{1}{4\pi}\frac{\sqrt{2}}{1+z\bar{z}}\int_{-\infty}^{\infty} du' \int_{S^2} d^2z'\ \frac{\epsilon^\mu_- q^\sigma}{q.q'}\ q'_{[\mu}D'^A q'_{\sigma]}\ j^{\log}_A. \tag{5.13}$$

This $\frac{1}{u}$-term has been discussed in the context of scattering of point particles in Chapter 3. In above expression we have used the following basis for polarisation vectors [56]:

$$\epsilon^\mu_- = \frac{1}{\sqrt{2}}\frac{\partial}{\partial\bar{z}}[(1+z\bar{z})q^\mu],\quad \epsilon^\mu_+ = \frac{1}{\sqrt{2}}\frac{\partial}{\partial z}[(1+z\bar{z})q^\mu]. \tag{5.14}$$

The expression for A_z can be obtained from the expression for $A_{\bar{z}}$ by replacing ϵ_- by ϵ_+. Next we need to show how above $\frac{1}{u}$-mode contributes to the asymptotic charge defined using (5.1). From (2.36), we have,

$$\partial_u^2 F^2_{rA}\ +\ \frac{1}{2}\partial_u\partial_A D^B A^0_B\ -\frac{1}{2}\partial_u^2 D^B F^0_{AB}\ =\ \frac{1}{2}\partial_u j^2_A. \tag{5.15}$$

So, for $A = z$ component, we get :

$$\partial_u^2 F^2_{rz} = -D_z D^{\bar{z}}\partial_u A^0_{\bar{z}} + \frac{1}{2}\partial_u j^2_z \tag{5.16}$$

Above equation relates $\frac{1}{u}$ term in $A_{\bar{z}}$ to $\frac{\log u}{r^2}$ term in F^2_{rz}. From above equation we get the precise relations :

$$F^{2,\log}_{rz} = -\gamma^{z\bar{z}}\ D^2_z A^{0,1}_{\bar{z}} \text{ and } F^{2,\log}_{r\bar{z}} = -\gamma^{z\bar{z}}\ D^2_{\bar{z}} A^{0,1}_z. \tag{5.17}$$

These relations will be needed when we write down the expression for the asymptotic charge.

5.2 The classical asymptotic charge for massless scalar QED

Let us start with the asymptotic conservation law given in (5.1)

$$F_{rA}^{2,\log}(\hat{x}) \quad = \quad F_{rA}^{\log,0}(-\hat{x}). \tag{5.18}$$

We recall that the LHS is the coefficient of the $\frac{\log u}{r^2}$-mode present at the future. Similarly the RHS is the coeffficient of the $\frac{\log r}{r^2}$ mode living at the past. We multiply above equation with an arbitrary parameter V^A and integrate over the sphere to get

$$\int_{\mathcal{I}^+_-} d^2z\ V^A(\hat{x})F_{rA}^{2,\log}(\hat{x}) \quad = \quad \int_{\mathcal{I}^-_+} d^2z\ V^A(-\hat{x})F_{rA}^{\log,0}(-\hat{x}). \tag{5.19}$$

The charge at the future is defined by $Q_1^+[V_+^A] = -\int d^2z\ V_+^A F_{rA}^{\log,0} \mid_{\mathcal{I}^+_-}$. The past charge is defined similarly. Our claim is that upon quantisation, these charges reproduce the outgoing soft photon theorem given in (5.2). There exists a $\log v$ mode at past if there is incoming soft photon. Similar conservation law (that involves advanced propagator) relating $\frac{\log v}{r^2}$ mode at $\mathcal{I}^-_+$ to $\frac{\log r}{r^2}$ mode at $\mathcal{I}^+_-$ can be used to reproduce the incoming soft theorem. In this section we will restrict to the classical theory.

Let us study the future charge :

$$\begin{aligned} Q_1^+[V] &= -\int d^2z\ V^A F_{rA}^{2,\log} \mid_{\mathcal{I}^+_-}, \\ &= u^2\partial_u^2 \int d^2z\ V^A F_{rA}^{2} \mid_{u\to-\infty}. \end{aligned}$$

The u-operator isolates the coefficient of the $\log u$ term of F_{rA}^2. We can rewrite the future charge as an integral over entire future null infinity minus the term at $\mathcal{I}^+_+$.

$$\begin{aligned} Q_1^+[V] = &\ -\int_{-\infty}^{\infty} du' \int d^2z\ V^A\partial_u\,[u^2\partial_u^2 F_{rA}^2] - \int d^2z\ V^A F_{rA}^{2,\log} \mid_{\mathcal{I}^+_+}, \\ := &\ Q_+^{\text{soft}}[V] + Q_+^{\text{hard}}[V]. \end{aligned} \tag{5.20}$$

This defines the soft and hard parts of asymptotic charge. We can simplify the soft charge expression further. Using Maxwell's equations given in (2.36) for

$\partial_u F^2_{rA}$, we get :

$$Q^{\text{soft}}_+ = -\frac{1}{2}\int_{-\infty}^{\infty} du' \int d^2z' \ V^A \partial_u \ \left[u^2 \partial_u [\partial_A F^2_{ru} - D^B F^0_{AB} + j^2_A] \ \right]. \tag{5.21}$$

j^2_A does not have a $\frac{1}{u}$-term, so j^2_A also drops out of above expression and we get

$$\begin{aligned} Q^{\text{soft}}_+ &= \int_{-\infty}^{\infty} du' \int d^2z' \ \Big[V^z(\hat{x}')\partial_u \ [u^2 \partial_u D_z D^{\bar{z}} A^0_{\bar{z}}(u,\hat{x}')] + z' \leftrightarrow \bar{z}'\Big], \\ &= \int_{-\infty}^{\infty} du' \int d^2z' \ \Big[D'^2_z V^z \ \gamma^{z\bar{z}} \ \partial_u \ [u^2 \partial_u A^0_{\bar{z}}(u,\hat{x}')] + z' \leftrightarrow \bar{z}'\Big]. \end{aligned} \tag{5.22}$$

The last line was derived using integration by parts. Next it is instructive to go to the frequency space :

$$Q^{\text{soft}}_+ = \int d^2z' \ \Big[D'^2_z V^z \ \gamma^{z\bar{z}} \ \lim_{\omega\to 0} \omega \ \partial^2_\omega \ \omega \ \tilde{A}^0_{\bar{z}}(\omega,\hat{x}') + z' \leftrightarrow \bar{z}'\Big].$$

It should be recalled that as yet above expression is classical and the limit is well defined. The ω-derivatives isolate the coefficient of soft $\log\omega$ mode of $\tilde{A}^0_{\bar{z}}$. This shows that the Q_1-charge is indeed related to the soft $\log\omega$ mode.

Next let us turn to the expression of future hard charge :

$$Q^{\text{hard}}_+ = -\int d^2z' \ V^A \ F^{2,\log}_{rA}(\hat{x}').$$

Using (5.17) in the expression for the hard charge, it can be written in terms of coefficient of the $\frac{1}{u}$ mode.

$$Q^{\text{hard}}_+ = \int d^2z' \ V^z \ \gamma^{z\bar{z}} \ D^2_z A^{0,1}_{\bar{z}}(\hat{x}') + \int d^2z' \ V^{\bar{z}} \gamma^{z\bar{z}} \ D^2_{\bar{z}} A^{0,1}_z(\hat{x}') \tag{5.23}$$

To avoid unnecessary cluttering of equations we will work with $V^{\bar{z}} = 0$. Then we can integrate by parts to get following equation :

$$Q^{\text{hard}}_+ = \int d^2z' \ D'^2_z V^z \ \gamma^{z\bar{z}} \ A^{0,1}_{\bar{z}}(\hat{x}'), \tag{5.24}$$

Using (5.13) and (5.10) we get :

$$A^{0,1}_{\bar{z}}(\hat{x}) = \frac{\sqrt{\gamma_{z\bar{z}}}}{4\pi}\int_{-\infty}^{\infty} du' \int d^2z' \ \frac{q^\mu \ \epsilon^\sigma_-}{q.q'} \ q'_{[\sigma}\partial'_{q^\mu]} \ [\ 2e^2 A^1_r(\hat{x}')|\phi^1(\hat{x}')^2 \ ]. \tag{5.25}$$

Equations (5.24) and (5.25) provide us the expression of the future hard charge. Thus the hard part of the Q_1-charge is related to the dressing of the massless scalar field under long range electromagnetic force.

Let us turn to the the expression of the past charge. We have :

$$Q_1^-[V] = -\int d^2z\ V^A F_{rA}^{\log,0} \mid_{\mathcal{I}_+^-}.$$

We know from (C.14) that $F_{rA}^{\log,0}$ depends only on particle currents i.e. it has no contribution from radiation. Thus, at past the charge is entirely made of hard modes.

$$Q_1^-[V] = -\int d^2z\ V^A F_{rA}^{\log,0} \mid_{\mathcal{I}_+^-} \quad := \ Q_-^{\text{hard}}[V].$$

The conservation law that we have started with in (5.19) involves only outgoing soft radiation. Using (C.14), the charge at past can be recast as :

$$Q_1^- = -\int d^2z'\ D_z'^2 V^z\ \gamma^{z\bar{z}}\ B^{\log}(\hat{x}'), \tag{5.26}$$

where,

$$B^{\log}(\hat{x}) = \frac{1}{4\pi}\frac{\sqrt{2}}{1+z\bar{z}}\int_{-\infty}^{\infty} dv' \int_{S^2} d^2z'\ \frac{q^\sigma \epsilon_-^\mu}{q.\bar{q}'}\ \bar{q}'_{[\mu}\partial'_{\bar{q}^\sigma]}\ [\ +2e^2 A_r^1(-\hat{x}')|\phi^1(-\hat{x}')|^2\ ]. \tag{5.27}$$

To summarise we have the obtained the classical expression of the Q_1-charge. It consists of contribution from soft as well as hard modes. The expression of soft charge is given in (5.29). This operator isolates soft $\log\omega$ mode. The expression of hard charge is given in (5.24) and (5.25). The hard charge is given in terms of dressing A_r^1.

5.3 The Ward identity for massless scalar QED

In this section we will turn to the quantum theory. We will use the classical expression for asymptotic charge Q_1 from the previous section and quantise it. The Ward identity for S matrix for the asymptotic charge is

$$\begin{aligned} &\Big[\ Q_1\ ,\ S\ \Big] = 0, \\ \Rightarrow &\Big(Q_+^{\text{soft}}\ S\ -\ S\ Q_-^{\text{soft}}\ \Big) = -\Big(Q_+^{\text{hard}}\ S\ -\ S\ Q_-^{\text{hard}}\ \Big). \end{aligned}$$

Quantising the soft charge we express the gauge field in terms of Fock operators (2.30) :

$$\tilde{A}^0_{\bar{z}}(\omega,\hat{x}) = -i\sqrt{2}\frac{a_-(\omega,\hat{x})}{4\pi(1+z\bar{z})} \;\;...\;\; \omega > 0, \quad \tilde{A}^0_{\bar{z}}(\omega,\hat{x}) = i\sqrt{2}\frac{a^\dagger_+(-\omega,\hat{x})}{4\pi(1+z\bar{z})} \;\;...\;\; \omega < 0. \tag{5.28}$$

So we get :

$$Q^{\text{soft}}_+ = -\frac{i}{4\pi}\int d^2z' \left[D'^2_z V^z \sqrt{\gamma'^{z\bar{z}}} \lim_{\omega\to 0^+} \omega\, \partial^2_\omega\, \omega \;\; a_-(\omega,\hat{x}') + z' \leftrightarrow \bar{z}'\right]. \tag{5.29}$$

Thus, the action of Q^{soft}_+ involves insertion of zero energy photon modes. We have defined the soft limit from positive side which is consistent with the fact that we have only outgoing radiative modes in (5.19). Hence $Q^{\text{soft}}_- = 0$ as we discussed in the last section.

Using (5.24) and (5.26), we get

$$-\Big(Q^{\text{hard}}_+\; S \;-\; S\; Q^{\text{hard}}_-\Big) \;=\; -\int d^2z'\; D'^2_z V^z\; \gamma^{z\bar{z}}\; \Big(A^{0,1}_{\bar{z}}(\hat{x}')\; S \;-\; S\; B^{\log}(\hat{x}')\Big). \tag{5.30}$$

Next we need to evaluate the action of above operators on a Fock state. From (5.25), we have following expression for $A^{0,1}_{\bar{z}}$.

$$A^{0,1}_{\bar{z}}(\hat{x}) = \frac{\sqrt{\gamma_{z\bar{z}}}}{2\pi}\int_{-\infty}^{\infty} du' \int d^2z'\; \frac{q^\mu\;\epsilon^\sigma_-}{q.q'}\; q'_{[\sigma}\partial'_{q^\mu]}\; e^2 A^1_r(\hat{x}')|\phi^1(\hat{x}')^2|\;. \tag{5.31}$$

It is interesting to note that the hard charge resembles tree level subleading soft operator acting on A^1_r. The action of (5.31) on an outgoing Fock state can be easily evaluated.

$$<\text{out}|\; Q^{\text{hard}}_+ = <\text{out}|\; 4\pi \sum_{i\;\epsilon\;out} U^{\sigma\mu}(q_i)\; e_i^2 q_{i[\sigma}\partial_{q_i^\mu]}\frac{A^1_r(z_i^A)}{\omega_i}\;]. \tag{5.32}$$

where we have defined

$$U^{\sigma\mu}(q_i) = \int d^2z'\; D'^2_z V^z(z,z')\; \frac{\sqrt{\gamma'^{z\bar{z}}}}{16\pi^2}\; \frac{\epsilon^\sigma_- q'^\mu}{q'.q_i}, \tag{5.33}$$

to make the expressions compact. Similarly we can use (5.27) to get the action of the past hard charge on an incoming state. Then we need to substitute for A^1_r.

This mode has a classical and a quantum part : $A_r^1 = \overset{\text{class}}{A_r^1} + \overset{\text{quan}}{A_r^1}$. The classical mode is obtained by evolving the sources with retarded propagator. We obtain the quantum contribution in a slightly roundabout way. The quantum gauge field contributes to $\overset{\text{quan}}{A_r^1}$ via (5.42). We will find these modes explicitly and write down the Ward identity.

Classical part

Let us find the classical contribution to A_r^1 using retarded propagator. We know that the solution for gauge field in Lorenz gauge is given by

$$A_\mu(x^\mu)|_{class} = \frac{1}{2\pi}\int d^4x' \ \delta\big(\ (x-x')^2 \ \big)\Theta(t-t') \ j_\mu(x'), \tag{5.34}$$

where we have used the retarded propagator. We need the $\frac{1}{r}$ component of above expression to find classical part of A_r^1. Taking large r limit we get :

$$A_\mu(u,r,\hat{x})|_{class} = -\frac{1}{4\pi r}\int_{-\infty}^{\infty} du' \int d^2z' \ \frac{j_\mu^2(\hat{x}',u')}{q.q'}.$$

It can be checked that above expression is consistent with the fall offs mentioned in (2.34). In particular we have :

$$\overset{\text{class}}{A_r^1}(\hat{x}) = \frac{1}{4\pi r}\int_{-\infty}^{\infty} du' \int d^2z' \ j_u^2(\hat{x}',u'). \tag{5.35}$$

This part of $A_r^1(x)$ is just a constant (i.e. independent of $u, \hat{x}$). Thus, the classical electromagnetic dressing is trivial.

Let us go back to the action of the hard charge (5.32)

$$< \text{out}| \ \Big[Q^{\text{hard}} \ , \ S \ \Big]_{class} |\text{in}> = -\sum_i e_i \ U^{\sigma\mu}(q_i) \quad < \text{out}| \ \Big[q_{i[\sigma}\partial_{q_i^{\mu}]} \ \overset{\text{class}}{A_r^1}(\hat{x}_i) \ , \ S \ \Big] \ |\text{in}> . \tag{5.36}$$

Using (5.35), we see that above term vanishes. This reflects the absence of classical $\log\omega$ term in soft electromagnetic radiation (in absence of gravitational coupling) [91].

Quantum part

Next we want to check if there is a part of A^1_r that has not been captured by the retarded propagator. Let us work in $r, u \to \infty$ limit $(u < r)$ where the sources have died down and we can use the homogenous solution. It is useful to work with Herdegen's representation [96] of homogenous A_μ. This is a way to write a generic homogenous solution for gauge field in Lorenz gauge in terms of free data A^0_A. ($A^0_A = \lim_{r\to\infty} A_A$).

$$\overset{hom}{A_\mu}(x) = -\frac{1}{(2\pi)}\int d^2z' \, \frac{1+z'\bar{z}'}{\sqrt{2}} \, \Big[\epsilon^-_\mu \ \dot{A}^0_z(u=-x\cdot q', \hat{q}') + \epsilon^+_\mu \ \dot{A}^0_{\bar{z}}(u=-x\cdot q', \hat{q}') \ \Big]. \tag{5.37}$$

q'^μ is defined according to (2.2). From above expression it can be seen that $A^0_B \sim \overset{\log}{A_B} \log u$ gives rise to a $\frac{1}{r}$ term in A_μ. We had discussed the presence of such a $\log u$ mode in (4.15). Let us write down the A^1_r term by co-ordinate transformation. We will denote it with a 'quan' overtext.

$$\overset{quan}{A^1_r}(x) \ = \frac{1}{2\pi} \ \int d^2z' \ \frac{1+z'\bar{z}'}{\sqrt{2}}\frac{1}{q'.q} \ [\epsilon^-.q \ \overset{\log}{A_z}(\hat{x}') + \epsilon^+.q \ \overset{\log}{A_{\bar{z}}}(\hat{x}')]. \tag{5.38}$$

In (4.15), we had discussed that the quantum gauge field admits following asymptotic expansion

$$A_A \ = \ \overset{\log}{A_A} \ \log|u| + \ \mathcal{O}(u^0) \ + \ldots \ , \quad u \to \pm\infty, \tag{5.39}$$

here, $\overset{\log}{A_A}$ can be obtained using a co-ordinate transformation in (4.22). We get

$$\overset{\log}{A_z}(u, \hat{x}) = \frac{i}{8\pi^2}\frac{\sqrt{2}}{1+z\bar{z}} \ \lim_{\omega\to 0^+} \ \omega[a_+(\omega,\hat{x}) - a^\dagger_-(\omega,\hat{x})] \ . \tag{5.40}$$

And

$$\overset{\log}{A_{\bar{z}}}(\hat{x}) = \frac{i}{8\pi^2}\frac{\sqrt{2}}{1+z\bar{z}} \ \lim_{\omega\to 0^+} \ \omega[a_-(\omega,\hat{x}) + a^\dagger_+(\omega,\hat{x})]. \tag{5.41}$$

The $\log u$ mode is tied to the discontinuity of the quantum gauge field. The classical field is continuous as $\omega \to 0$ hence this mode vanishes classically. An important point to note is that we are not introducing a new independent mode in the quantum system, the free data for classical system is sufficient to describe the quantized system as well. For scalars, $\omega\tilde{\phi}$ is trivial as $\omega \to 0$. Hence there is no $\log|u|$ term for scalars.

The $\overset{quan}{A_r^1}$ operator has a non-trivial action when inserted in the expression for charge. We will use the leading soft theorem to evaluate action of (5.40) and (5.41) and substitute for $\overset{\log}{A}_B$ in the expression of $\overset{quan}{A_r^1}$ given in (5.38). Action of A_r^1 on a generic put state is given by :

$$
\begin{aligned}
&< \text{out}|\ \overset{quan}{A_r^1}(x)\ S\ |\text{in}> \\
&= i < \text{out}| \int \frac{d^2z'}{16\pi^3}\ \Big[\frac{\epsilon^-.q}{q'.q}\ \sum_j \eta_j e_j \frac{\epsilon^+.p_j}{q'.p_j}\ +\ \frac{\epsilon^+.q}{q'.q}\ \sum_j \eta_j e_j \frac{\epsilon^-.p_j}{q'.p_j}\Big]\ S\ |\text{in}> .
\end{aligned}
\tag{5.42}
$$

Next we need to do the sphere integral. We have relegated this calculation to Appendix C and we will quote the results here. The finite part of the integral is (C.21) :

$$
< \text{out}|\ \overset{quan}{A_r^1}(\hat{x})\ S\ |\text{in}> = -\frac{i}{4\pi^2}\ \sum_j \eta_j e_j\ \log(q.p_j). \tag{5.43}
$$

Let use this expression to evaluate the action of hard charge and obtain the quantum contribution. Using (5.43) for A_r^1 in (5.32), we get

$$
\begin{aligned}
&< \text{out}|\ \Big[Q^{\text{hard}}\ ,\ S\ \Big]_{quan}\ |\text{in}> \\
&= -\frac{i}{\pi}\ \sum_{i,j;i\neq j} \eta_i\eta_j \frac{e_i^2 e_j}{\omega_i}\ U^{\sigma\mu}(q_i)\ \ q_{i[\sigma}\partial_{q_i^\mu]}\ \log[\frac{2(p_j.q_i)}{m_j^2}]\ ]\ \ \mathcal{A}_n.
\end{aligned}
\tag{5.44}
$$

Here, we have $\mathcal{A}_n = <\text{out}|\ S\ |\text{in}>$. $U^{\sigma\mu}$ has been defined in (5.33). The term that depends on m_j is the divergent piece in A_r^1. It is killed by the derivative operator $q_{i[\sigma}\partial_{q_i^\mu]}$ and the final expression is finite.

The Ward identity

Collecting together (5.36) and (5.44) we get the complete action of the hard charge and we can write down the Ward identity. Thus, the S-matrix needs to satisfy following Ward identity for a generic V^z that lives on S^2 :

$$
\Big[Q^{\text{soft}}(V^z)\ ,\ S\ \Big] = -\frac{i}{\pi}\ \sum_{i,j;i\neq j} \eta_i\eta_j e_i\ U^{\sigma\mu}(q_i)\ \frac{e_i e_j}{q_j\cdot q_i}(q_{j\mu}q_{i\sigma}\ -\ q_{j\sigma}q_{i\mu}\ )\ . \tag{5.45}
$$

$Q^{\text{soft}}(V)$ defined in (5.29) inserts soft modes of photon. Dependence on V^z is via the U defined in (5.33). Ward identity involving $V^{\bar{z}}$ can be written down similarly.

The Sahoo-Sen soft theorem

Let us derive the Sahoo-Sen soft theorem from above Ward identity. To derive negative helicity soft theorem we choose :

$$V^z(z,z') = \sqrt{2}(1+z'\bar{z}')\frac{z-z'}{\bar{z}-\bar{z}'}, \quad V^{\bar{z}} = 0. \tag{5.46}$$

Performing the sphere $(z', \bar{z}')$ integral in (5.29), we get :

$$Q_+^{\text{soft}} = -i \lim_{\omega \to 0} \omega \ \partial_\omega^2 \ \omega \ \ a_-(\omega, \hat{x}). \tag{5.47}$$

Next we will use (5.46) in the expression for hard charge (5.45). The sphere integral in the expression for U in (5.33) can be done easily. So, the Ward identity can be recast as :

$$\lim_{\omega \to 0} \omega \ \partial_\omega^2 \ \omega \ \mathcal{A}_{n+1} \quad = -\frac{1}{4\pi^2} \sum_{i,j;i\neq j} \eta_i \eta_j e_i \frac{\epsilon_\mu k_\rho}{p_i.k} \ \frac{e_i e_j}{p_i.p_j} (p_j^\rho p_i^\mu - p_i^\rho p_j^\mu) \ \mathcal{A}_n. \tag{5.48}$$

This is exactly the statement of the $\log \omega$ soft theorem (5.2) given by Sahoo and Sen for massless scalar QED (without dynamical gravity). In this analysis we have derived the soft theorem from the Ward identity. The Ward identity (with $V^{\bar{z}} = 0$) can be derived from the soft theorem by multiplying both sides of the statement of soft theorem with $\int d^2z \ D_{\bar{z}}^2 V^z(z) \ \frac{\sqrt{\gamma_{z\bar{z}}}}{16\pi^2}$. Thus, we can conclude that the Ward identity (5.68) is exactly equivalent to the Sahoo-Sen soft photon theorem for massless scalar QED.

5.4 Corrections in presence of dynamical gravity

In this section we will briefly discuss the gravitational corrections to the Ward identity. We will not give the entire derivation but only highlight the important points. Interested readers can refer to [101] for details.

Let us write down the asymptotic behaviour of the gravitational field. We will work in the perturbative linear gravity regime where gravitational dynamics is confined to perturbations around flat space time : $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. In the de Donder gauge $\partial_\mu \bar{h}^{\mu\nu} = 0$, where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\sigma{}_\sigma$. In this gauge, the metric field satisfies $\Box\bar{h}_{\mu\nu} = -2T_{\mu\nu}$ and admits following expansion : [2]

$$h_{rr} = \frac{h^1_{rr}(\hat{x})}{r} + h^{\log}_{rr}(u,\hat{x})\frac{\log r}{r^2} + \ldots\,, \qquad h_{ur} = \frac{h^1_{ur}(u,\hat{x})}{r} + h^{\log}_{ur}(u,\hat{x})\frac{\log r}{r^2} + \ldots\,,$$
$$h_{uu} = h^{\log}_{uu}(u,\hat{x})\frac{\log r}{r} + \frac{h^1_{uu}(u,\hat{x})}{r} + \ldots\,, \qquad h_{rA} = h^0_{rA}(\hat{x}) + h^{\log}_{rA}(u,\hat{x})\frac{\log r}{r^2} + \ldots\,,$$
$$h_{uA} = h^0_{uA}(u,\hat{x}) + h^{\log}_{uA}(u,\hat{x})\frac{\log r}{r} + \ldots\,, \qquad h_{AB} = r\; h^{-1}_{AB}(u,\hat{x}) + \log r\; h^{\log}_{AB}(u,\hat{x}) + \ldots\,. \tag{5.49}$$

Gravitational dressing of massless scalar field

First we will discuss the dressing of massless scalar field in presence of dynamical gravity. The equation of motion of the scalar field is given by $g^{\mu\nu}D_\mu D_\nu\phi = 0$. We will use (5.49) in above equation for the scalar field and obtain the solution. The leading order gravitational correction to (5.8) is given by :

$$\phi(x) = -\frac{i e^{ieA^1_r(\hat{x})\log r}}{8\pi^2 r}\int d\omega\; [b(\omega,\hat{x})\; e^{-i\omega u}e^{i\omega\log r\frac{h^1_{rr}(\hat{x})}{2}} \;-d^\dagger(\omega,\hat{x})\; e^{i\omega u}e^{-i\omega\log r\frac{h^1_{rr}(\hat{x})}{2}}\;], \tag{5.50}$$

The dressing of the massless scalar field contributes to the charge via (5.13). The gravitational correction to (5.13) is given by

$$A^{0,1}_{\bar{z}}(\hat{x})|_{\text{scal}} = \frac{\sqrt{\gamma_{z\bar{z}}}}{4\pi}\int_{-\infty}^{\infty} du'\int d^2z'\;\frac{q^\mu\;\epsilon^\sigma_-}{q.q'}\;q'_{[\sigma}\partial'_{q^\mu]}\;[-\;\frac{1}{2}h^1_{rr}(\hat{x}')j^2_u(\hat{x}')\;\;+2e^2A^1_r(\hat{x}')|\phi^1(\hat{x}')^2\;]. \tag{5.51}$$

We have added a subscript 'scal' to highlight the fact that this contribution arises from scalar field dressing. There is an additional contribution to above mode from the dressing of the gauge field.

Gravitational dressing of U(1) gauge field

Next we discuss the effect of long range gravitational force on gauge fields. We start with the homogenous equation $\Box A^{hom}_\mu = 0$. Asymptotically such a

[2] Some of the coefficients are independent of u, this follows from the de Donder gauge condition itself.

solution exhibits following form (2.27)

$$A_\sigma^{hom}(u,r,\hat{x}) = -\frac{i}{8\pi^2 r}\int d\omega\ [a_\sigma(\omega,\hat{x})\ e^{-i\omega u} - a_\sigma^\dagger(\omega,\hat{x})\ e^{i\omega u}\ ], \tag{5.52}$$

where $a_\sigma = \sum_{r=+,-}\epsilon_\sigma^{*r}\ a_r$. Let us turn on the sources. Choosing the generalised Lorenz gauge $\nabla_\mu A^\mu = 0$, Maxwell's equations reduce to :

$$\nabla^2 A_\mu = -j_\mu + R_\mu{}^\nu A_\nu. \tag{5.53}$$

$R_{\mu\nu}$ is the Ricci tensor. j_μ is the U(1) current. The gravitational corrections can be expanded in perturbative gravity regime. Ignoring the U(1) current, we have

$$\Box A_\sigma = j_\sigma^{\text{grav}}, \tag{5.54}$$

where we have defined :

$$j_\sigma^{\text{grav}} = h^{\mu\nu}\partial_\mu\partial_\nu A_\sigma + \eta^{\mu\nu}\Gamma^\rho_{\mu\nu}\partial_\rho A_\sigma + 2\eta^{\mu\nu}\Gamma^\rho_{\mu\sigma}\partial_\nu A_\rho + \eta^{\mu\nu}\ A_\lambda\ \partial_\mu\Gamma^\lambda_{\nu\sigma} + [\partial_\mu\Gamma^\mu_{\nu\sigma} - \partial_\nu\Gamma^\mu_{\mu\sigma}]A^\nu + \mathcal{O}(G^2).$$

Next j_σ^{grav} can be evaluated on the zeroth order solution. Using (5.49) and (5.52), we see that the source has following behaviour around future null infinity

$$j_\sigma^{\text{grav}}(x) = \frac{1}{r^2}h^1_{rr}\partial_u^2 A^1_\sigma\ +\ \mathcal{O}(\frac{1}{r^3}). \tag{5.55}$$

The $\mathcal{O}(\frac{1}{r^3})$ terms in $j_\sigma^{\text{grav}}(x)$ produce subleading corrections, hence are not relevant for our analysis. The leading order gravitational current gives rise to following dressing of the gauge field

$$A_\sigma(u,r,\hat{x}) = -\frac{i}{8\pi^2 r}\int d\omega\ [a_\sigma(\omega,\hat{x})\ e^{-i\omega u}e^{i\omega\log(r\omega)\frac{h^1_{rr}(\hat{x})}{2}}\ - a_\sigma^\dagger(\omega,\hat{x})\ e^{i\omega u}e^{-i\omega\log(r\omega)\frac{h^1_{rr}(\hat{x})}{2}} \tag{5.56}$$

This expression represents the effect of gravitational field on outgoing photons. Thus, we see that the $\log r$ dressing of photons is exactly similar to the $\log r$ dressing of massless scalars. There is an additional $\log\omega$ dressing of the soft photons. This mode contributes to the charge. By co-ordinate transformation of (5.56) we get analogous to (5.51)

$$A_{\bar{z}}^{0,1}(\hat{x})|_{\text{grav dress}} = -\frac{1}{8\pi}\frac{\sqrt{2}}{1+z\bar{z}}\ h^1_{rr}(\hat{x})\ \lim_{\omega\to 0}\omega a_-(\omega,\hat{x}). \tag{5.57}$$

The gravitational contribution to the charge depends on h^1_{rr}. For $A^{0,1}_z$, we have to replace negative helicity operators with positive helicity operators. For incoming soft photon, above analysis needs to be repeated at past null infinity.

The Ward identity

We know that the Ward identity for S matrix for the Q_1 asymptotic charge can be written down as :

$$\Big[\, Q_1 \;,\; S \,\Big] = 0,$$
$$\Rightarrow \Big(Q^{\rm soft}_{+} \; S \;-\; S \; Q^{\rm soft}_{-} \Big) = -\Big(Q^{\rm hard}_{+} \; S \;-\; S \; Q^{\rm hard}_{-} \Big).$$

Analogous to (5.30), we get

$$Q^{\rm soft}_{+} \; S \; = - \int d^2z' \; D'^2_z V^z \; \gamma^{z\bar z} \; \Big(A^{0,1}_{\bar z}(\hat x') \; S \;-\; S \; B^{\log}(\hat x') \Big). \tag{5.58}$$

We discuss the gravitational corrections to soft charge in Appendix C in (C.29) and show that these corrections vanish.

Next we need to evaluate the action of above operators on a Fock state. From (5.51) and (5.57), we get the expression of $A^{0,1}_{\bar z}$.

$$\begin{aligned} A^{0,1}_{\bar z}(\hat x) = \; & \frac{\sqrt{\gamma_{z\bar z}}}{4\pi} \int_{-\infty}^{\infty} du' \int d^2z' \; \frac{q^\mu \; \epsilon^\sigma_-}{q.q'} \; q'_{[\sigma}\partial'_{q^\mu]} \; [- \; \frac{1}{2} h^1_{rr}(\hat x') j^2_u(\hat x') \; + 2e^2 A^1_r(\hat x') |\phi^1(\hat x')^2 \;] \\ & - \frac{\sqrt{\gamma_{z\bar z}}}{8\pi} \; h^1_{rr}(\hat x) \; \lim_{\omega\to 0} \omega a_-(\omega, \hat x). \end{aligned} \tag{5.59}$$

It is interesting to note that the first line resembles tree level subleading soft operator acting on $h^1_{rr} + A^1_r$. Similarly the second line is h^1_{rr} times the leading soft operator. The action of (5.59) on an outgoing Fock state gives

$$\begin{aligned} <\text{out}| \; Q^{\rm hard}_{+} = \; & <\text{out}| \; 4\pi \sum_{i \;\epsilon\; out} U^{\sigma\mu}(q_i) \; q_{i[\sigma}\partial_{q^\mu_i]} \; [\; \frac{e_i}{2} h^1_{rr}(z_i) \; + \; e^2_i \frac{A^1_r(z_i)}{\omega_i} \;] \\ & - <\text{out}| \int d^2z' \; D'^2_{\bar z} V^z(z') \; \frac{\sqrt{\gamma'_{z\bar z}}}{8\pi} \; \sum_i \frac{e_i \; \epsilon^- . \; p_i}{q'.p_i} \; h^1_{rr}(z'). \end{aligned} \tag{5.60}$$

where as before we have defined

$$U^{\sigma\mu}(q_i) = \int d^2z' \; D'^2_z V^z(z,z') \; \frac{\sqrt{\gamma'^{z\bar z}}}{16\pi^2} \; \frac{\epsilon^\sigma_- q'^\mu}{q'.q_i}, \tag{5.61}$$

to make the expressions compact. The gravitational correction to the the past hard charge is given by

$$B^{\log}(\hat{x}) =$$
$$\frac{1}{4\pi}\frac{\sqrt{2}}{1+z\bar{z}}\int_{-\infty}^{\infty} dv' \int_{S^2} d^2z' \ \frac{q^\sigma \epsilon_-^\mu}{q.\bar{q}'} \ \bar{q}'_{[\mu}\partial'_{\bar{q}^\sigma]} \ [- \ \frac{1}{2}h^1_{rr}(-\hat{x}')j^2_u(-\hat{x}') \ + 2e^2 A^1_r(-\hat{x}')|\phi^1(-\hat{x}')^2 \ ]. \tag{5.62}$$

Classical part

Let us write down the gravitational term. The classical part of h^1_{rr} calculated using the retarded propagator is

$$\overset{\text{class}}{h^1_{rr}}(\hat{x}) = -\frac{1}{2\pi r}\int_{-\infty}^{\infty} du' \int d^2z' \ q.q' \ T^2_{uu}(\hat{x}', u'). \tag{5.63}$$

Hence we get

$$< \text{out}| \Big[Q^{\text{hard}} \ , \ S \ \Big]_{class} |\text{in} >$$
$$= - \sum_{i,j;\eta_i\eta_j=1} e_i \ U^{\sigma\mu}(q_i) \ \ q_{i[\sigma}\partial_{q_i^\mu]} \ (p_j.q_i) \ \ \mathscr{A}_n + \sum_i \eta_i e_i \ q_{i\sigma} \ U^{\sigma\mu} \sum_{j;\eta_j=1} p_{j\mu} \ \ \mathscr{A}_n \tag{5.64}$$

Here, we have defined $\mathscr{A}_n =< \text{out}| \ S \ |\text{in} >$.

Quantum part

Next we turn to the quantum contribution to h^1_{rr} from (C.22) and (C.23). Substituting in the expression for hard charge we get :

$$< \text{out}| \Big[Q^{\text{hard}} \ , \ S \ \Big]_{quan} |\text{in} >$$
$$= -\frac{i}{\pi}\sum_{i,j;i\neq j} \eta_i\eta_j e_i \ U^{\sigma\mu}(q_i) \ \ q_{i[\sigma}\partial_{q_i^\mu]} \ \ p_j.q_i \ \ \log[\frac{2(p_j.q_i)}{m_j^2}] \ \mathscr{A}_n$$
$$+ \, i\int d^2z' \ D'^2_{\bar{z}}V^z(z') \ \frac{\sqrt{\gamma'_{z\bar{z}}}}{16\pi^3} \ \sum_i \eta_i \frac{e_i \ \epsilon^- . \ p_i}{q'.p_i} \sum_j \eta_j q'.p_j \log[\frac{2(p_j.q')}{m_j^2}] \ \ \mathscr{A}_n. \tag{5.65}$$

The terms that depend on m_j are divergent. The two divergent terms in above expression cancel each other. Thus, we get a finite action of the charge. The full quantum term is given by :

$$
\begin{aligned}
&< \text{out}| \Big[Q^{\text{hard}}\ ,\ S\ \Big]_{quan}\ |\text{in} > \\
&= -\frac{i}{\pi}\sum_{i,j;i\neq j}\eta_i\eta_j e_i\ U^{\sigma\mu}(q_i)\ \ q_{i[\sigma}\partial_{q_i^{\mu}]}\ \Big[p_j.q_i\ \ \log(p_j.q_i) + \frac{e_ie_j}{\omega_i}\log(p_j.q_i)\ \Big]\ \mathscr{A}_n \\
&+\frac{i}{\pi}\sum_i \eta_i e_i\ q_{i\sigma}\ \sum_j \eta_j\tilde{U}_j^{\sigma\mu}\ p_{j\mu}\ \mathscr{A}_n.
\end{aligned}
\tag{5.66}
$$

here we have defined

$$
\tilde{U}_j^{\sigma\mu}(q_i) = \int d^2z'\ D_z'^2V^z(z,z')\ \frac{\sqrt{\gamma'^{z\bar{z}}}}{16\pi^2}\ \frac{\epsilon_-^{\sigma}q'^{\mu}}{q'.q_i}\log(q'.p_j). \tag{5.67}
$$

Collecting together (5.64) and (5.66) we get the complete action of the hard charge and we can write down the full Ward identity including the purely electromagnetic term. Thus, the S-matrix needs to satisfy following Ward identity for a generic V^z that lives on S^2 :

$$
\Big[Q^{\text{soft}}(V^z)\ ,\ S\ \Big] = -\ \mathscr{C}_{\text{hard}}(V^z)\ \ S. \tag{5.68}
$$

$Q^{\text{soft}}(V)$ defined in (5.29) inserts soft modes of photon. We have :

$$
\begin{aligned}
\mathscr{C}_{\text{hard}}(V^z) = &\sum_i \eta_i e_i\ q_{i\sigma}\ U^{\sigma\mu}\sum_{j;\eta_j=1}p_{j\mu}\ -\sum_{i,j;\eta_i\eta_j=1}e_i\ U^{\sigma\mu}(q_i)\ (p_{j\mu}q_{i\sigma}\ -\ p_{j\sigma}q_{i\mu}\ ) \\
&-\frac{i}{\pi}\sum_{i,j;i\neq j}\eta_i\eta_j e_i\ U^{\sigma\mu}(q_i)\ \Big[\frac{e_ie_j}{q_j.q_i}(q_{j\mu}q_{i\sigma}\ -\ q_{j\sigma}q_{i\mu}\ ) + (p_{j\mu}q_{i\sigma}\ -\ p_{j\sigma}q_{i\mu}\ )\ \log(-p_j.p_i)\ \Big]^3 \\
&+\ \frac{i}{\pi}\sum_i \eta_i e_i\ q_{i\sigma}\ \sum_j \eta_j\tilde{U}_j^{\sigma\mu}\ p_{j\mu}\ .
\end{aligned}
\tag{5.69}
$$

Dependence on V^z is via the U's defined in (5.61) and (5.67). Ward identity involving $V^{\bar{z}}$ can be written down similarly.

The Sahoo-Sen soft theorem

Let us derive the Sahoo-Sen soft theorem from above Ward identity. To derive negative helicity soft theorem we choose as before

$$
V^z(z,z') = \sqrt{2}(1+z'\bar{z}')\frac{z-z'}{\bar{z}-\bar{z}'},\quad V^{\bar{z}} = 0. \tag{5.70}
$$

[3] The first term in (5.64) produces a term that vanishes due to conservation of momenta. This is the term discussed in (5.5).

Performing the sphere $(z', \bar{z}')$ integral in (5.29), we get :

$$Q_+^{\text{soft}} = -i \lim_{\omega \to 0} \omega\ \partial_\omega^2\ \omega\ \ a_-(\omega, \hat{x}). \tag{5.71}$$

Next we will use (5.70) in the expression for hard charge given in (5.69). The sphere integral in the expression for U (5.61) and for $\tilde{U}$ in (5.67) can be done as earlier. We get :

$$\begin{aligned}
\mathscr{C}_{\text{hard}} = & \frac{1}{4\pi} \sum_i \eta_i e_i \frac{\epsilon.p_i}{p_i.k} \sum_{j;\eta_j=1} k.p_j \ \ - \frac{1}{4\pi} \sum_{\substack{i,j;i\neq j \\ \eta_i\eta_j=1}} e_i \frac{\epsilon_\mu k_\rho}{p_i.k} (p_j^\rho p_i^\mu - p_i^\rho p_j^\mu) \\
& - \frac{i}{4\pi^2} \sum_{i,j;i\neq j} \eta_i\eta_j e_i \frac{\epsilon_\mu k_\rho}{p_i.k} \left[\frac{e_i e_j}{p_i.p_j} (p_j^\rho p_i^\mu - p_i^\rho p_j^\mu) + (p_j^\rho p_i^\mu - p_i^\rho p_j^\mu) \log[p_i.p_j] \right] \\
& + \frac{i}{4\pi^2} \sum_i \eta_i e_i \frac{\epsilon.p_i}{p_i.k} \sum_j \eta_j k.p_j \ \log\ p_j.q\ .
\end{aligned} \tag{5.72}$$

So, the Ward identity can be recast as :

$$\begin{aligned}
& \lim_{\omega\to 0} \omega\ \partial_\omega^2\ \omega\ \mathscr{A}_{n+1} \\
& = \Bigg[- \frac{i}{4\pi} \sum_i \eta_i e_i \frac{\epsilon.p_i}{p_i.k} \sum_{j;\eta_j=1} k.p_j \ \ + \frac{i}{4\pi} \sum_{\substack{i,j;i\neq j \\ \eta_i\eta_j=1}} e_i \frac{\epsilon_\mu k_\rho}{p_i.k} (p_j^\rho p_i^\mu - p_i^\rho p_j^\mu) \\
& - \frac{1}{4\pi^2} \sum_{i,j;i\neq j} \eta_i\eta_j e_i \frac{\epsilon_\mu k_\rho}{p_i.k} \left[\frac{e_i e_j}{p_i.p_j} (p_j^\rho p_i^\mu - p_i^\rho p_j^\mu) + (p_j^\rho p_i^\mu - p_i^\rho p_j^\mu) \log[p_i.p_j] \right] \\
& + \frac{1}{4\pi^2} \sum_i \eta_i e_i \frac{\epsilon.p_i}{p_i.k} \sum_j \eta_j k.p_j \ \log\ p_j.q \Bigg] \mathscr{A}_n.
\end{aligned} \tag{5.73}$$

This is exactly the Sahoo-Sen soft theorem (5.2) for massless scalar QED including the gravitational correction given in (5.4). The Ward identity (with $V^{\bar{z}} = 0$) can be derived from the soft theorem by multiplying both sides of the statement of soft theorem with $\int d^2z\ D_{\bar{z}}^2 V^z(z)\ \frac{\sqrt{\gamma_{z\bar{z}}}}{16\pi^2}$. Hence we can conclude that the Ward identity (5.68) is exactly equivalent to the Sahoo-Sen soft photon theorem for massless scalar QED coupled to dynamical gravity.

Thus in this chapter, we have demonstrated that the 1-loop exact $\log\omega$ soft photon theorem [91] can be recast as Ward identity of an asymptotic charge for massless scalar QED in presence of dynamical gravity. We used the Q_1 classical law given in (5.1) to contruct the asymptotic charge. This asymptotic charge is directly related to the dressing of fields due to long range forces. In presence of gravity, the new feature is that the soft photon also acquires a dressing due to long range gravitational force and contributes to the asymptotic charge.

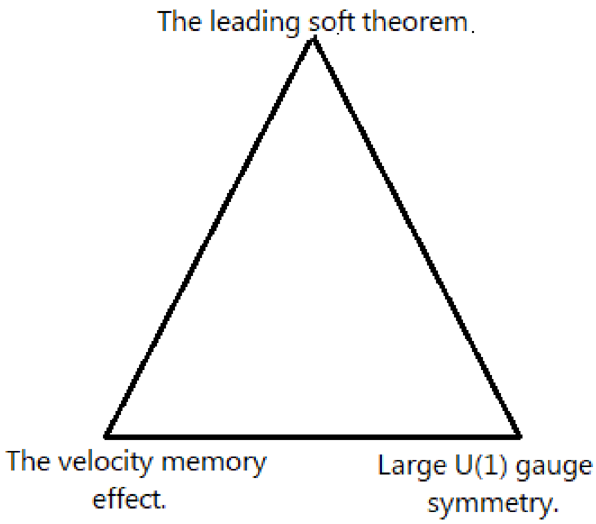
Upon quantisation the Q_1 charge reproduces the full $\log \omega$ soft theorem including the quantum corrections. This is inspite of the fact that the law (5.1) is itself violated in the quantum theory. We expect that this prescription can be used to define the asymptotic charges for higher loop order soft theorems as well. Starting from the classical charges for $m \geq 2$ given in (3.51), it is expected that the quantised Q_m charges should reproduce (a part of)[4] the $\omega^{m-1}(\log \omega)^m$ soft modes including purely quantum terms.

[4]Beyond subleading order there are remainder terms that are not controlled by these Q_m-charges

Chapter 6

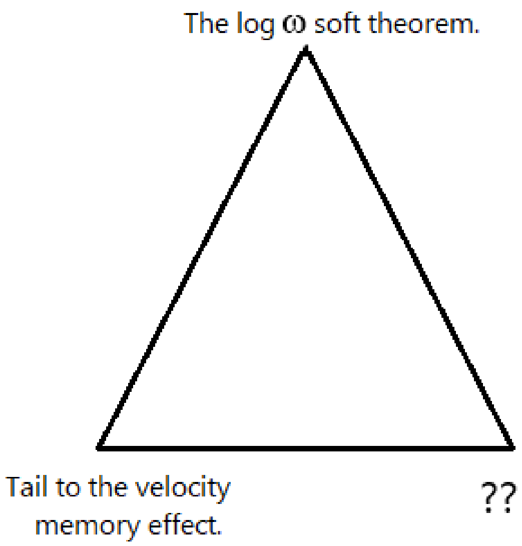
Summary and Outlook

Recent investigations have shed light on rich structure of IR physics of gauge theories and gravity. These results have been very elegantly cast as 'IR triangles' by Strominger and his collaborators. Let us consider the first of these triangles for electromagnetism.



In the soft limit, the leading $\frac{1}{\omega}$-mode in QED amplitudes is universal as seen in (2.53). This is the statement of the leading soft photon theorem and forms the first corner of this triangle. The second corner is the velocity memory effect. This is a classically observable effect in which the passage of electromagnetic radiation waves produces a permanent shift in the velocity of a test charge. The third corner is that of asymptotic symmetries. In this case we have large gauge transformations that form a subgroup of the U(1) group that acts non trivially on the physical states. These three corners bring out the universal features of IR physics and are interrelated.

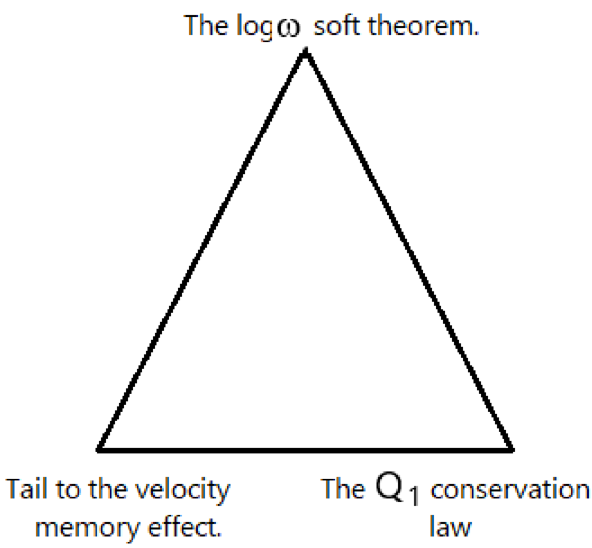
In 2018 Sahoo and Sen derived the subleading soft photon theorem for loop amplitudes [91] given in (2.54). The memory effect corresponding to this soft theorem was discussed in [37]. So it seems likely that there exists a Strominger's triangle for this case as well.



In this thesis we have explored the third corner of above triangle.

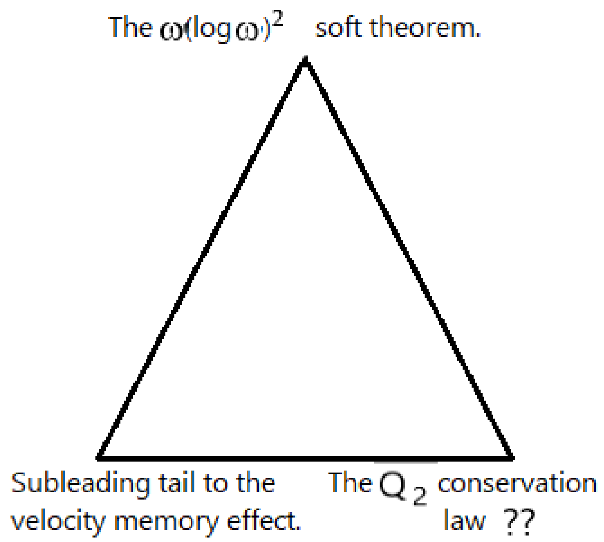
Incorporating the effect of long range electromagnetic force on scattering particles we showed that the classical radiative field obeys the Q_1 conservation law given in (3.25). While it seems natural to use above charges to reproduce the $\log \omega$ soft theorem there are certain things that needed to be clarified. The asymptotic behaviour of the gauge field is modified in the quantum theory due to use of Feynman boundary condition. We discussed the purely quantum modes in the Feynman solution which are absent in the classical radiative solution in (4.38). These modes lead to quantum corrections to the asymptotic charges. Incorporating the effect of long range electromagnetic force in (4.36), it was shown that the Q_1 conservation law is violated by the Feynman solution. The Feynman solution satisfies a modified version of this law given in (4.68).

Nonetheless we showed that the classical Q_1 conservation law given in (5.19) can be used to reproduce the full $\log \omega$ soft theorem. Upon quantisation the Q_1 charge gets contribution from an additional mode (absent in the classical theory) which gives rise to the quantum corrections to the $\log \omega$ soft theorem. We proved that the $\log \omega$ soft photon theorem for massless scalar QED coupled to gravity is equivalent to the Ward identity corresponding to the Q_1 charge. This leads us to following IR triangle



It must however be noted that the nature or existence of a well defined symmetry associated to this conservation law is not clear at this point. The action of the soft part of Q_1 charge is trivial on the asymptotic phase space that is normally constructed (as described in Section 2.2). This phase space needs to be extended so that the Q_1 charge has a well defined action on the extended phase space. This question is under investigation.

We expect this story to hold for $m > 1$. In this thesis we have also shown that the classical radiative field obeys the Q_2 conservation law given in (3.47). These charges are made up of $\mathcal{O}(e^5)$ modes and hence are expected to be related to a 2-loop soft photon theorem. And indeed such a theorem has been derived in [95]. We expect an IR triangle for the $m = 2$ case with following corners.



The equivalence between the Q_2-law and the $\omega(\log\omega)^2$ soft theorem is also under investigation.

We have proposed that the classical radiative field should satisfy an infinite number of conservation laws as given in (3.51). We expect that upon quantisation these Q_m charges should get related to the universal $\omega^{m-1}(\log\omega)^m$ modes in soft expansion of loop amplitudes giving rise to an IR triangle at every m. This is quite intriguing. We believe that this proposal for Q_m-conservation law can be proved for generic m by incorporating the effects of long range force in the analysis of [75]. Several questions are in order about the conserved charges Q_m. Most importantly, it needs to be checked if indeed all the $\{Q_m, m \geq 1\}$ charges are independent or if they are related. What is the underlying symmetry? Do these charges correspond to new kind of large gauge transformations? These questions promise to bring out interesting aspects of low energy physics of QED.

It is clear that infrared regime of loop level QED has a rich structure. This structure needs to explored further. Similar questions should be probed in the context of gravity and QCD to uncover the structure underlying infrared physics of these theories at loop level.

Appendix A

Appendix for Chapter 3

Perturbative solution

The retarded Green function for d-Alembertian operator has $\delta([x-x_i(\tau)]^2)$. We will find the solution of this delta function perturbatively in coupling e. Here, $x_i^\mu(\tau)$ is the equation of trajectory that gets corrected as we go to higher orders in e.

At zeroth order, we have free particles :

$$x_i^\mu = V_i^\mu \tau + d_i.$$

Hence, the root of delta function $\delta([x-x_i]^2)$ is given by :

$$\tau_0 = -V_i.(x-d_i) - \left[\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ \right]^{1/2}. \tag{A.1}$$

The sign of the square root has been chosen to ensure retarded boundary condition i.e. $t > t_i$. Now, let us study above expression in the limit $r \to \infty$ with u finite. Around $\mathcal{I}^+$, using (2.2) we get

$$\tau_0|_{\mathcal{I}^+} = -\frac{u}{q.V_i} + \mathcal{O}(1). \tag{A.2}$$

Let us take the limit $r \to \infty$ limit in (A.1) keeping v finite. Using (2.16), we get :

$$\tau_0|_{\mathcal{I}^-} = -2r\ V_i.\bar{q} + \mathcal{O}(1).$$

Next we include the leading order effect of long range electromagnetic force. We know that the first order correction to the trajectory is given by (3.7) :

$$x_i^\mu = V_i^\mu\ \tau + c_i^\mu \log\tau + d_i.$$

Using the corrected trajectory, the solution of delta function $\delta([x-x_i(\tau)]^2)$ is given by :

$$\tau^2 + 2\tau V_i.(x-d_i) - (x-d_i)^2 = -2(x-d_i).c_i \log\tau + c_i^2(\log\tau)^2. \qquad \text{(A.3)}$$

We have used $V_i.c_i = 0$. Noting that c^μ is $\mathcal{O}(e^2)$, the RHS of above equation can be treated as a perturbation. Hence we substitute the zeroth order solution (A.1) in RHS of (A.3) that leads to following equation for τ :

$$\tau^2 + 2\tau V_i.(x-d_i) - (x-d_i)^2 = -2(x-d_i).c_i \log\tau_0. \qquad \text{(A.4)}$$

We ignored the c_i^2 term as it is $\mathcal{O}(e^4)$. Now, above equation is just a quadratic equation in τ and the solution is given by :

$$\tau_1 = -V_i.(x-d_i) - \left[\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ - 2(x-d_i).c_i\log\tau_0\ \right]^{1/2}. \qquad \text{(A.5)}$$

We have used a subscript 1 to denote that it includes the first order perturbative effects. We can expand the squareroot to $\mathcal{O}(e^2)$:

$$\tau_1 = -V_i.(x-d_i) - \left[\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\right]^{1/2}\ + \frac{(x-d_i).c_i}{X}\log\tau_0\ . \qquad \text{(A.6)}$$

Here, we have defined $X = [\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ ]^{1/2}$. Thus, the first order solution is the zeroth order solution plus a perturbation :

$$\tau_1 = \tau_0 + \frac{(x-d_i).c_i}{X}\log\tau_0\ . \qquad \text{(A.7)}$$

Expanding around $\mathcal{I}^+$, we get :

$$\tau_1|_{\mathcal{I}^+} = -\frac{u}{q.V_i} - \frac{q.c_i}{q.V_i}\ \log u + \mathcal{O}(1). \qquad \text{(A.8)}$$

Thus, in $u \to \pm\infty$ limit, the correction to τ_0 is suppressed by $\frac{\log u}{u}$ in addition to the suppression due to e^2 factor. Expanding (A.6) around $\mathcal{I}^-$, we get :

$$\tau_1|_{\mathcal{I}^-} = -2r\ V_i.\bar{q} + \frac{\bar{q}.c_i}{V_i.\bar{q}}\ \log r + \mathcal{O}(r^0). \qquad \text{(A.9)}$$

Second order in perturbation

Let us repeat above steps after including second order effects of long range forces. The subsubleading correction to the trajectory is (3.32)

$$x_{i\sigma} = V_{i\sigma}\tau + c_{i\sigma}\log\tau + d_{i\sigma} + f_{i\sigma}\frac{\log\tau}{\tau},$$

here $f_i \sim \mathcal{O}(e^4)$. Hence at $\mathcal{O}(e^4)$, $\delta([x - x_i(\tau)]^2)$ implies following equation for τ :

$$\tau^2 + 2\tau V_i.(x-d_i) - (x-d_i)^2 = -2(x-d_i).c_i\log\tau_1 - 2(x-d_i).f_i\frac{\log\tau_1}{\tau_1} + c_i^2(\log\tau_1)^2. \tag{A.10}$$

We have used the fact that $V_i.c_i = V_i.f_i = 0$. Here, we have substituted the first order solution for the terms in the RHS. τ_1 is given in (B.5). The second order solution is

$$\begin{aligned}\tau_2 = &- V_i.(x-d_i) - \Big[\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ \ - 2(x-d_i).c_i\log\tau_1 \\ &- 2(x-d_i).f_i\frac{\log\tau_1}{\tau_1} + c_i^2(\log\tau_1)^2\Big]^{1/2}.\end{aligned} \tag{A.11}$$

We can expand the squareroot :

$$\begin{aligned}\tau_2 = \tau_0 + (x-d_i).c_i\frac{\log\tau_0}{X}\ -\ c_i^2\frac{(\log\tau_0)^2}{2X} + (x-d_i).f_i\ \frac{\log\tau_0}{X\tau^0} + (x.c_i - d_i.c_i)^2\frac{(\log\tau_0)^2}{2X^3} \\ + (x.c_i - d_i.c_i)^2\frac{\log\tau_0}{\tau_0\ X^2}.\end{aligned} \tag{A.12}$$

We have used (B.6) for τ_1 to derive above expression. And as before $X = [\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ ]^{1/2}$. Now, let us study above expression in the limit $r \to \infty$ with u finite. We have :

$$\tau_2|_{\mathcal{J}^+} = -\frac{u + q.d_i}{q.V_i} - \ \frac{q.c_i}{q.V_i}\ \log\frac{u}{(-q.V_i)} + q.f_i\ \frac{\log u}{u} - \frac{(q.c_i)^2}{q.V_i}\ \frac{\log u}{u} + \mathcal{O}(\frac{1}{u}). \tag{A.13}$$

The $\mathcal{O}(\frac{1}{r})$ term in τ_2 starts at $\mathcal{O}(u^2)$. This produces $\mathcal{O}(\frac{u^2}{r^3})$-term in A_μ (see (3.39)). We see from (A.12) that there is a $\mathcal{O}(\frac{(\log u)^2}{r})$ term, this contributes to the $\mathcal{O}(\frac{(\log u)^2}{r^3})$-term in A_μ. We can expand (A.12) in large r limit keeping v finite to get :

$$\tau_2|_{\mathcal{J}^-} = -2r\ V_i.\bar{q} + \ \frac{\bar{q}.c_i}{V_i.\bar{q}}\ \log r + \frac{(\bar{q}.c_i)^2}{(V_i.\bar{q})^3}\frac{(\log r)^2}{2r} - \frac{c_i^2}{\bar{q}.V_i}\frac{(\log r)^2}{2r} + \mathcal{O}(v). \tag{A.14}$$

Appendix B

Appendix for Chapter 4

Perturbative solution

Let us study both the roots of the $\delta([x-x_i]^2)$ perturbatively in coupling e. Here, $x_i^\mu(\tau)$ is the equation of trajectory that gets corrected as we go to higher orders in e. At zeroth order, we have free particles :

$$x_i^\mu = V_i^\mu \tau + d_i.$$

The roots of delta function $\delta([x-x_i]^2)$ are given by

$$\tau_0^\pm = -V_i.(x-d_i) \mp \left[\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ \right]^{1/2}. \tag{B.1}$$

τ_0^+ satisifies retarded boundary condition while τ_0^- satisifies advanced boundary condition. Let us study above expression in the limit $r \to \infty$ with u finite. Thus, around $\mathcal{I}^+$, using (2.2) we get :

$$\tau_0^+|_{\mathcal{I}^+} = \frac{u + q.d_i}{|q.V_i|} + \mathcal{O}(\frac{1}{r}), \quad \tau_0^-|_{\mathcal{I}^+} = 2r|q.V_i| + \mathcal{O}(r^0). \tag{B.2}$$

Now we take $r \to \infty$ limit of (B.1) keeping v finite, using (2.16), we get :

$$\tau_0^+|_{\mathcal{I}^-} = -2r\ V_i.\bar{q} + \mathcal{O}(r^0), \quad \tau_0^-|_{\mathcal{I}^-} = \frac{v + \bar{q}.d_i}{\bar{q}.V_i} + \mathcal{O}(\frac{1}{r}).$$

Next we include the leading order effect of long range electromagnetic force. We know that the first order correction to the trajectory is given by (4.28) :

$$x_i^\mu = V_i^\mu\ \tau + C_i^\mu \log\tau + d_i.$$

Using the corrected trajectory, the solution of delta function $\delta([x-x_i]^2)$ is given by :

$$\tau^2 + 2\tau V_i.(x-d_i) - (x-d_i)^2 = -2(x-d_i).C_i \log\tau + C_i^2(\log\tau)^2. \qquad \text{(B.3)}$$

Here we have used the fact that $V_i.C_i = 0$. Noting that C_i^μ is $\mathcal{O}(e^2)$, the RHS of above equation can be treated as a perturbation. Hence we substitute the zeroth order solution (B.1) in RHS of (C.4) that leads to following equation for τ :

$$\tau^2 + 2\tau V_i.(x-d_i) - (x-d_i)^2 = -2(x-d_i).C_i \log\tau_0^\pm. \qquad \text{(B.4)}$$

We ignored the C_i^2 term as it is $\mathcal{O}(e^4)$. Now, above equation is just a quadratic equation in τ and the solution is given by :

$$\tau_1^\pm = -V_i.(x-d_i) \mp \left[\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ \ -2(x-d_i).C_i\log\tau_0^\pm\ \right]^{1/2}. \qquad \text{(B.5)}$$

We have used a subscript 1 to denote that it includes the first order perturbative effects. We can expand the squareroot to $\mathcal{O}(e^2)$:

$$\tau_1^\pm = -V_i.(x-d_i) \mp \left[\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\right]^{1/2}\ \pm \frac{(x-d_i).C_i}{X}\log\tau_0^\pm\ . \qquad \text{(B.6)}$$

Here, we have defined $X = [\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ ]^{1/2}$ and $\tau_0^\pm$ are given in (B.1). Expanding around $\mathcal{I}^+$, we get :

$$\begin{aligned}
\tau_1^+|_{\mathcal{I}^+} &= -\frac{u+q.d_i}{q.V_i} - \frac{q.C_i}{q.V_i}\ \log u + \mathcal{O}(1), \\
\tau_1^-|_{\mathcal{I}^+} &= 2rq.V_i + \frac{q.C_i}{q.V_i}\ \log r + \mathcal{O}(r^0).
\end{aligned} \qquad \text{(B.7)}$$

Expanding (B.6) around $\mathcal{I}^-$, we get :

$$\begin{aligned}
\tau_1^+|_{\mathcal{I}^-} &= -2r\ V_i.\bar{q} + \frac{\bar{q}.C_i}{V_i.\bar{q}}\ \log r + \mathcal{O}(r^0), \\
\tau_1^-|_{\mathcal{I}^-} &= \frac{v+\bar{q}.d_i}{\bar{q}.V_i} - \frac{\bar{q}.C_i}{\bar{q}.V_i}\ \log v + \mathcal{O}(1).
\end{aligned} \qquad \text{(B.8)}$$

Integrals in section 4.3

The indefinite integrals in the first term of (4.35) are simple.

$$\begin{aligned}&\int dx\ [1+\frac{1}{x}\ ]\ [\ \frac{1}{x-\tau_0^+}-\frac{1}{x-\tau_0^-}\ ]\\ =\ &\ln\frac{x-\tau_0^+}{x-\tau_0^-}\ +\ \frac{1}{\tau_0^+}\ \log\frac{x-\tau_0^+}{x}-\ \frac{1}{\tau_0^-}\ \log\frac{x-\tau_0^-}{x}.\end{aligned} \tag{B.9}$$

The indefinite integral in the second term of (4.35) is given by

$$\begin{aligned}&\int\frac{dx\ \ \log x}{(x-\tau_0^-)^2(x-\tau_0^+)^2}\\ =\ &\frac{2}{(\tau_0^- -\tau_0^+)^3}\left[\ln\tau_0^+\ln(x-\tau_0^+)-\ln\tau_0^-\ln(x-\tau_0^-)\right]\\ &+\frac{2}{(\tau_0^- -\tau_0^+)^3}\left[\mathrm{Li}_2\left(-\frac{x-\tau_0^-}{\tau_0^-}\right)-\mathrm{Li}_2\left(-\frac{x-\tau_0^+}{\tau_0^+}\right)\right]-\frac{\ln x}{(\tau_0^+-\tau_0^-)^2}\left[\frac{1}{(x-\tau_0^+)}+\frac{1}{(x-\tau_0^-)}\right]\\ &-\frac{1}{(\tau_0^+-\tau_0^-)^2}\left[\ \frac{1}{\tau_0^-}\log\frac{x}{(x-\tau_0^-)}+\frac{1}{\tau_0^+}\log\frac{x}{(x-\tau_0^+)}\ \right]\end{aligned} \tag{B.10}$$

Above integral is to be integrated from T to R for outgoing particles. Let us consider the upper limit and show that the divergent terms (in the $R\to\infty$ limit) indeed cancel and also find if there is any finite contribution.

$$\frac{2}{(\tau_0^- -\tau_0^+)^3}\left[\ln\tau_0^+\ln R-\ln\tau_0^-\ln R+\mathrm{Li}_2\left(-\frac{R-\tau_0^-}{\tau_0^-}\right)-\mathrm{Li}_2\left(-\frac{R-\tau_0^+}{\tau_0^+}\right)\right]+\mathcal{O}(\frac{\ln R}{R}). \tag{B.11}$$

Let us use following property of the dilogarithm function (for $x<-1$) [98].

$$\mathrm{Li}_2(x)=-\frac{\pi^2}{6}-\frac{1}{2}\log(1-x)\ [2\log(-x)-\log(1-x)]\ +\ \mathrm{Li}_2(\frac{1}{1-x}).$$

Thus we have

$$\begin{aligned}\mathrm{Li}_2\left(-\frac{R-\tau_0^+}{\tau_0^+}\right)\ &=\ -\frac{\pi^2}{6}-\frac{1}{2}\log(\frac{R}{\tau_0^+})\ [2\log(\frac{R}{\tau_0^+}-1)-\log(\frac{R}{\tau_0^+})]\ +\ \mathrm{Li}_2(\frac{\tau_0^+}{R})\\ &=-\frac{\pi^2}{6}-\frac{1}{2}\log^2(\frac{R}{\tau_0^+})\ +\ \mathcal{O}(\frac{1}{R}).\end{aligned}$$

Hence (B.11) is equal to

$$\frac{1}{(\tau_0^- -\tau_0^+)^3}[\ln^2\tau_0^+-\ln^2\tau_0^-]+\mathcal{O}(\frac{\ln R}{R}).$$

Now we can write down the result of the definite integral.

$$\int_T^R \frac{dx\ \log x}{(x-\tau_0^-)^2(x-\tau_0^+)^2} =$$
$$\frac{2}{(\tau_0^- - \tau_0^+)^3}\Big[\ln\tau_0^+ \ln\frac{1}{\tau_0^+ - T} - \ln\tau_0^- \ln\frac{1}{\tau_0^- - T} + \frac{1}{2}[\ln^2\tau_0^+ - \ln^2\tau_0^-]\Big]$$
$$-\frac{2}{(\tau_0^- - \tau_0^+)^3}\Big[\mathrm{Li}_2\Big(-\frac{T-\tau_0^-}{\tau_0^-}\Big) - \mathrm{Li}_2\Big(-\frac{T-\tau_0^+}{\tau_0^+}\Big)\Big] + \frac{\ln T}{(\tau_0^+ - \tau_0^-)^2}\Big[\frac{1}{(T-\tau_0^+)} + \frac{1}{(T-\tau_0^-)}\Big]$$
$$+\frac{1}{(\tau_0^+ - \tau_0^-)^2}\Big[\frac{1}{\tau_0^-}\log\frac{T}{\tau_0^- - T} + \frac{1}{\tau_0^+}\log\frac{T}{\tau_0^+ - T}\Big] \tag{B.12}$$

Hence we can write down the result of the both integrals in (4.35).

$$A_\sigma^{*\mathrm{asym}}(x)$$
$$= \frac{i}{4\pi^2}\sum_{j=n+1}^{2n}\frac{e_j V_{j\sigma}}{\tau_0^- - \tau_0^+}\Big[\log\frac{1}{\tau_0^+ - T} - \log\frac{1}{\tau_0^- - T}\Big]$$
$$+ \frac{i}{4\pi^2}\sum_{j=n+1}^{2n}\frac{e_j C_{j\sigma}}{(\tau_0^- - \tau_0^+)}\Big[\frac{1}{\tau_0^+}\log\frac{T}{\tau_0^+ - T} - \frac{1}{\tau_0^-}\log\frac{T}{\tau_0^- - T}\Big]$$
$$+ \frac{i}{4\pi^2}\sum_{j=n+1}^{2n}\frac{2e_j V_{j\sigma}(x-d_j).C_j}{(\tau_0^- - \tau_0^+)^2}\log T\Big[\frac{1}{T-\tau_0^+} + \frac{1}{T-\tau_0^-}\Big]$$
$$+ \frac{i}{4\pi^2}\sum_{j=n+1}^{2n}\frac{2e_j V_{j\sigma}(x-d_j).C_j}{(\tau_0^- - \tau_0^+)^2}\Big[\frac{1}{\tau_0^-}\log\frac{T}{\tau_0^- - T} + \frac{1}{\tau_0^+}\log\frac{T}{\tau_0^+ - T}\Big]$$
$$+ \frac{i}{4\pi^2}\sum_{j=n+1}^{2n}\frac{4e_j V_{j\sigma}(x-d_j).C_j}{(\tau_0^- - \tau_0^+)^3}\Big[-\ln\tau_0^+\ln(\tau_0^+ - T) + \ln\tau_0^-\ln(\tau_0^- - T) + \frac{1}{2}[\ln^2\tau_0^+ - \ln^2\tau_0^-]\Big]$$
$$- \frac{i}{4\pi^2}\sum_{j=n+1}^{2n}\frac{4e_j V_{j\sigma}(x-d_j).C_j}{(\tau_0^- - \tau_0^+)^3}\Big[\mathrm{Li}_2(-\frac{T-\tau_0^-}{\tau_0^-}) - \mathrm{Li}_2(-\frac{T-\tau_0^+}{\tau_0^+})\Big] + \text{ in.} \tag{B.13}$$

Let us study the expansion of various terms in above expression. Using (B.1), it is seen that

$$\tau_0^+|_{\mathcal{I}^+} \sim u\,[\,1 + \frac{1}{u} + \sum_{\substack{0\le m\le n,\\ n=1}}^{\infty}\frac{u^m}{r^n}\,].$$
$$\tau_0^-|_{\mathcal{I}^+} \sim r + u + r^0u^0 + \sum_{\substack{0\le m\le n+1,\\ n=1}}^{\infty}\frac{u^m}{r^n}\,.$$

$$\log\tau_0^-|_{\mathcal{I}^+} \sim \log r + \sum_{\substack{m,n=0,\\ m\leq n.}}^{\infty}\frac{u^m}{r^n}.$$

$$\log\tau_0^+|_{\mathcal{I}^+} \sim \log u + \sum_{\substack{n=0,\\ m=-\infty,\\ m\leq n.}}^{\infty}\frac{u^m}{r^n}. \tag{B.14}$$

$$\begin{aligned}
\frac{1}{(\tau_0^- - \tau_0^+)}\frac{1}{\tau_0^+}\log\frac{T}{(T-\tau_0^+)} &\sim [\log u + \mathcal{O}(1)]\sum_{\substack{m=-\infty,\\ m\leq n,\\ n=1}}^{\infty}\frac{u^m}{r^n}.\\
\frac{1}{(\tau_0^- - \tau_0^+)}\frac{1}{\tau_0^-}\log\frac{T}{(T-\tau_0^-)}\,] &\sim [\mathcal{O}(1) + \log r]\sum_{\substack{0<m<n-1,\\ n=2}}^{\infty}\frac{u^m}{r^n}.\\
\frac{(x-d_j).C_j}{(\tau_0^+ - \tau_0^-)^2}\frac{1}{(T-\tau_0^+)} &\sim \sum_{\substack{m=-\infty,\\ m<n,\\ n=1}}^{\infty}\frac{u^m}{r^n}.\\
\frac{(x-d_j).C_j}{(\tau_0^+ - \tau_0^-)^2}\frac{1}{\tau_0^+}\log\frac{T}{(T-\tau_0^+)} &\sim [\log u + \mathcal{O}(1)]\sum_{\substack{m=-\infty,\\ m\leq n,\\ n=1}}^{\infty}\frac{u^m}{r^n}.\\
\frac{(x-d_j).C_j}{(\tau_0^+ - \tau_0^-)^2}\frac{1}{\tau_0^-}\log\frac{T}{(T-\tau_0^-)} &\sim [\mathcal{O}(1) + \log r]\sum_{\substack{0<m<n-1,\\ n=2}}^{\infty}\frac{u^m}{r^n}.\\
\frac{(x-d_j).C_j}{(\tau_0^+ - \tau_0^-)^3}\ln\tau_0^+\ln(T-\tau_0^+) &\sim \frac{1}{r^2}\Big[(\log u)^2\,[1+\sum_{\substack{0\leq m\leq n,\\ n=1}}^{\infty}\frac{u^m}{r^n}] + [\mathcal{O}(1)+\log u]\,[1+\sum_{\substack{m=-\infty,\\ m\leq n,\\ n=0}}^{\infty}\frac{u^m}{r^n}]\Big].\\
\frac{(x-d_j).C_j}{(\tau_0^+ - \tau_0^-)^3}\ln\tau_0^-\ln(T-\tau_0^-) &\sim \frac{1}{r^2}\,[\mathcal{O}(1) + (\log r)^2 + \log r]\,[1+\sum_{\substack{0\leq m\leq n,\\ n=1}}^{\infty}\frac{u^m}{r^n}].\\
\frac{(x-d_j).C_j}{(\tau_0^+ - \tau_0^-)^3}\,\mathrm{Li}_2\Big(-\frac{T-\tau_0^+}{\tau_0^+}\Big) &\sim \frac{1}{r^2}\sum_{\substack{n=0,\\ m=-\infty,\\ m\leq n.}}\frac{u^m}{r^n}\;.\\
\frac{(x-d_j).C_j}{(\tau_0^+ - \tau_0^-)^3}\,\mathrm{Li}_2\Big(-\frac{T-\tau_0^-}{\tau_0^-}\Big) &\sim \frac{1}{r^2}\sum_{\substack{m,n=0,\\ m\leq n.}}\frac{u^m}{r^n}\;.
\end{aligned} \tag{B.15}$$

Terms relevant for section 4.4

To find the coefficients of $\frac{\log u}{r^2}$ and $\frac{\log r}{r^2}$ modes in A_σ, we need to calculate some lower order terms in the asymptotic expansion of (4.36) explicitly. Here we list the asymptotic expansions of various quantities that appear in (4.36).

Around $\mathscr{I}^+$

Let us start with the retarded root

$$\tau_0^+ = -V_i.(x-d_i) - \left[\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ \right]^{1/2}.$$

Around future null infinity, we get using (2.2)

$$\begin{aligned}
\tau_0^+|_{\mathscr{I}^+} &= -V_i.x + V_i.d_i + rV_i.q\ \big[1 - 2\frac{uV_i^0 + V_i.d_i}{rV_i.q} + \frac{(uV_i^0 + V_i.d_i)^2}{r^2(V_i.q)^2} + \frac{(x-d_i)^2}{r^2(V_i.q)^2}\big]^{\frac{1}{2}} \\
&= -\frac{u + q.d_i}{(V_i.q)} - \frac{u^2}{2r(V_i.q)} - \frac{u^2V_i^0}{r(V_i.q)^2} - \frac{u^2}{2r(V_i.q)^3} + \mathcal{O}(\frac{u}{r}). \qquad \text{(B.16)}
\end{aligned}$$

Hence we have

$$\frac{1}{\tau_0^+} = -\frac{(V_i.q)}{u}\left[1 - \frac{q.d_i}{u} + \mathcal{O}(\frac{1}{u^2}) - \frac{u}{r}[\frac{1}{2} + \frac{V_i^0}{(V_i.q)} + \frac{1}{2(V_i.q)^2}] + \mathcal{O}(\frac{u^0}{r})\right]. \qquad \text{(B.17)}$$

Next we turn to the advanced root.

$$\tau_0^- = -V_i.(x-d_i) + \left[\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ \right]^{1/2}.$$

Around future null infinity, we get using (2.2)

$$\begin{aligned}
\tau_0^- &= -V_i.x + V_i.d_i - rV_i.q\ \big[1 - 2\frac{uV_i^0 + V_i.d_i}{rV_i.q} + \frac{(uV_i^0 + V_i.d_i)^2}{r^2(V_i.q)^2} + \frac{(x-d_i)^2}{r^2(V_i.q)^2}\big]^{\frac{1}{2}} \\
&= -2rV_i.q + 2uV_i^0 + 2V_i.d_i + \frac{u + q.d_i}{(V_i.q)} + \mathcal{O}(\frac{1}{r}). \\
\frac{1}{\tau_0^-} &= -\frac{1}{2(V_i.q)r}\left[1 + \frac{uV_i^0 + V_i.d_i}{r(V_i.q)} + \frac{u + q.d_i}{2r(V_i.q)^2} + \mathcal{O}(\frac{1}{r^2})\right]. \qquad \text{(B.18)}
\end{aligned}$$

Also we can write down the asymptotic expansion of following term.

$$\begin{aligned}
\frac{2}{\tau_0^- - \tau_0^+} &= \frac{1}{r|V_i.q|}\ \big[1 - 2\frac{uV_i^0 + V_i.d_i}{rV_i.q} + \frac{(uV_i^0 + V_i.d_i)^2}{r^2(V_i.q)^2} + \frac{(x-d_i)^2}{r^2(V_i.q)^2}\big]^{-\frac{1}{2}}, \\
&= -\frac{1}{r}\frac{1}{V_i.q}\ \big[1 + \frac{1}{r}\frac{uV_i^0}{V_i.q} + \frac{1}{r}\frac{V_i.d_i}{V_i.q} + \frac{1}{r}\frac{u + d_i.q}{(V_i.q)^2} + \mathcal{O}(\frac{1}{r^2})\ \big]. \qquad \text{(B.19)}
\end{aligned}$$

.

Around $\mathscr{I}^-$

We start with the advanced root .

$$\tau_0^- = -V_i.(x-d_i) + \left[\ (V_i.x - V_i.d_i)^2 + (x-d_i)^2\ \right]^{1/2}.$$

Around past null infinity, we get using (2.16)

$$\begin{aligned}
\tau_0^- &= -V_i.x + V_i.d_i + rV_i.\bar{q}\ \Big[1 - 2\frac{vV_i^0 + V_i.d_i}{rV_i.\bar{q}} + \frac{(vV_i^0 + V_i.d_i)^2}{r^2(V_i.\bar{q})^2} + \frac{(x-d_i)^2}{r^2(V_i.\bar{q})^2}\Big]^{\frac{1}{2}} \\
&= \frac{v - \bar{q}.d_i}{(V_i.\bar{q})} - \frac{v^2}{2r(V_i.\bar{q})} + \frac{v^2V_i^0}{r(V_i.\bar{q})^2} - \frac{v^2}{2r(V_i.\bar{q})^3} + \mathcal{O}(\frac{v}{r}). \qquad \text{(B.20)}
\end{aligned}$$

Hence we have

$$\frac{1}{\tau_0^-} = \frac{(V_i.\bar{q})}{v}\Big[1 + \frac{\bar{q}.d_i}{v} + \mathcal{O}(\frac{1}{v^2}) + \frac{v}{r}\Big[\ \frac{1}{2} - \frac{V_i^0}{(V_i.\bar{q})} + \frac{1}{2(V_i.\bar{q})^2}\ \Big] + \mathcal{O}(\frac{v^0}{r})\Big] \qquad \text{(B.21)}$$

Similarly for the retarded root we get

$$\begin{aligned}
\tau_0^+ &= -V_i.x + V_i.d_i - rV_i.\bar{q}\ \Big[1 - 2\frac{vV_i^0 + V_i.d_i}{rV_i.\bar{q}} + \frac{(vV_i^0 + V_i.d_i)^2}{r^2(V_i.\bar{q})^2} + \frac{(x-d_i)^2}{r^2(V_i.\bar{q})^2}\Big]^{\frac{1}{2}} \\
&= -2rV_i.\bar{q} + 2vV_i^0 + 2V_i.d_i - \frac{v - \bar{q}.d_i}{(V_i.\bar{q})} + \mathcal{O}(\frac{1}{r}) \\
\frac{1}{\tau_0^+} &= -\frac{1}{2(V_i.\bar{q})r}\Big[1 + \frac{vV_i^0 + V_i.d_i}{r(V_i.\bar{q})} - \frac{v - \bar{q}.d_i}{2r(V_i.\bar{q})^2} + \mathcal{O}(\frac{1}{r^2})\Big] \qquad \text{(B.22)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{2}{\tau_0^- - \tau_0^+} &= \frac{1}{rV_i.\bar{q}}\ \Big[1 - 2\frac{vV_i^0 + V_i.d_i}{rV_i.\bar{q}} + \frac{(vV_i^0 + V_i.d_i)^2}{r^2(V_i.\bar{q})^2} + \frac{(x-d_i)^2}{r^2(V_i.\bar{q})^2}\Big]^{-\frac{1}{2}}, \\
&= \frac{1}{r}\frac{1}{V_i.\bar{q}}\ \Big[1 + \frac{1}{r}\frac{vV_i^0}{V_i.\bar{q}} + \frac{1}{r}\frac{V_i.d_i}{V_i.\bar{q}} - \frac{1}{r}\frac{v - d_i.\bar{q}}{(V_i.\bar{q})^2} + \mathcal{O}(\frac{1}{r^2})\ \Big]. \qquad \text{(B.23)}
\end{aligned}$$

Appendix C

Appendix for Chapter 5

Calculating the $\frac{1}{u}$-mode in A^0_A

Dressing of scalars under long range forces lead to logarithmic modes in the current :

$$j_A \ = j_A^{\log} \ \frac{\log r}{r^2} + \frac{j_A^2}{r^2} \ + ... \ .$$

We also have :

$$j_r = j_r^{\log} \frac{\log r}{r^4} + \frac{j_r^4}{r^4} + ... \ \ , j_u \ = \ \frac{j_u^2}{r^2} \ + \ j_u^{\log} \ \frac{\log r}{r^3} + ... \ .$$

For the Cartesian components of the U(1) current we have :

$$j_\mu \ = \ \frac{j_\mu^2}{r^2} \ + \ j_\mu^{\log} \ \frac{\log r}{r^3} + ... \ . \tag{C.1}$$

We will substitute above current source in :

$$A_\sigma(x) = \frac{1}{2\pi} \int d^4x' \ \delta(\ (x-x')^2) \ \Theta(t-t') \ j_\sigma(x'). \tag{C.2}$$

Let us take the limit $r \to \infty$ keeping u finite :

$$\begin{aligned}
&A_\sigma(u, r, \hat{x}) \\
&= \frac{1}{4\pi r} \int_{-\infty}^{\infty} du' \int_0^{\infty} dr' \int_{S^2} \frac{d^2z'}{-q.q'} \ \delta(r' + \frac{u-u'}{q.q'}) \ \Big[j_\sigma^2(u', z') \ + j_\sigma^{\log}(u', z') \ \frac{\log r'}{r'} + j_\sigma^3(u', z') \ \frac{1}{r'} + ... \Big], \\
&= \frac{1}{4\pi r} \int_{-\infty}^{\infty} du' \int_{S^2} d^2z' \left[\frac{j_\sigma^2(u', z')}{-q.q'} \ + j_\sigma^{\log}(u', z') \ \frac{\log(u-u')}{u-u'} + \frac{[j_\sigma^3(u', z') - j_\sigma^{\log}(u', z') \ \log(-q.q')]}{u-u'} \right].
\end{aligned} \tag{C.3}$$

We are interested in studying the u-behaviour in $u \to \infty$ limit. In (C.3), the j_σ^2 term contributes to u^0 term as $u \to \infty$. The next dominant fall off in $u \to \infty$

limit is $\frac{\log u}{u}$. It comes from the region $u' << u$. Thus, we have :

$$A_\sigma(u,r,\hat{x}) = \frac{1}{4\pi r}\int_{-\infty}^{\infty} du' \int_{S^2} d^2z' \left[\frac{j^2_\sigma(u',z')}{-q.q'} + j^{\log}_\sigma(u',z')\,\frac{\log u}{u} + ...\right]. \quad \text{(C.4)}$$

First we rewrite the coefficient in retarded co-ordinates (Recalling that $q^\mu = (1,\hat{x})$.) :

$$j^{\log}_\sigma = -q_\sigma j^{\log}_u + \gamma^{AB}\partial_B q_\sigma\, j^{\log}_A. \quad \text{(C.5)}$$

$j^{\log}_u$ can be eliminated using the conservation equation of current :

$$j^{\log}_u = \partial_u j^{\log}_r - D^A j^{\log}_A. \quad \text{(C.6)}$$

Substituting in the expression for $j^{\log}_\sigma$:

$$j^{\log}_\sigma = -q_\sigma \partial_u j^{\log}_r + D^A[q_\sigma\, j^{\log}_A]. \quad \text{(C.7)}$$

Thus, $j^{\log}_\sigma$ is a total derivative. When (C.7) is substituted in (C.4), the $D^A[q_\sigma\, j^{\log}_A]$ term vanishes trivially due to sphere integral. Using the logarithmic fall off of the gauge field : $A_r = \frac{A^1_r}{r} + A^{\log}_r \frac{\log r}{r^2} + ...$ in the expression of U(1) current we get

$$j^{\log}_r = -2e^2 A^{\log}_r |\phi^1|^2. \quad \text{(C.8)}$$

Using (C.8) let us study the behaviour of $j^{\log}_r$ as $|u| \to \infty$. Following the logic of [63], we use the fall off $\phi \sim \frac{1}{u^{1+\epsilon}}$ as $|u| \to \infty$[1]. Now, let us find the u-fall off of $A^{\log}_r$. Using the gauge condition we have : $\partial_u A^{\log}_r = -A^{\log}_u$. Then $A^{\log}_u$ can be related to the current by Maxwell's equation : $2\partial_u A^{\log}_u = j^2_u$. Hence, $A^{\log}_r$ can have a $\mathcal{O}(u)$ term as $|u| \to \infty$. Using these u-fall offs in the expression (C.8) we get $j^{\log}_r \to 0$ as $|u| \to \infty$. Thus, the first term in (C.7) also gives a vanishing contribution. Hence the coefficient of $\frac{\log u}{u}$ vanishes.

The next term falls off as $1/u$ and this is the term that is relevant for loop level charge. Let us rewrite the $1/u$-term in a nice form. To start with, we have :

$$A_\sigma(u,r,\hat{x}) = \frac{1}{4\pi r u}\int_{-\infty}^{\infty} du' \int_{S^2} d^2z' \left[j^3_\sigma(z') - j^{\log}_\sigma(z')\ \log(-q.q')\right].$$

[1]This is more restrictive than the fall offs discussed in Section 2.2 as given in [94]

We manipulate j^3_σ in similar fashion :

$$j^3_\sigma = -q_\sigma j^3_u + \gamma^{AB}\partial_B q_\sigma\ j^2_A \ = \ -q_\sigma \partial_u j^4_r - q_\sigma j^{\log}_u + D^A[\ q_\sigma\ j^2_A], \tag{C.9}$$

and we get :

$$A_\sigma(u,r,\hat{x}) = \frac{1}{4\pi r u}\int_{-\infty}^{\infty} du' \int_{S^2} d^2z' \left[- q'_\sigma[j^{\log}_u + \partial'_u j^4_r] + \left[q'_\sigma \partial_u j^{\log}_r - D'^A[q'_\sigma\ j^{\log}_A]\right]\ \log(-q.q')\right].$$

We again substitute for $j^{\log}_u$ using (C.6). Upto total sphere derivative terms, above expression can be rewritten as :

$$A_\sigma(u,r,\hat{x})$$
$$= \frac{1}{4\pi r u}\int_{-\infty}^{\infty} du' \int_{S^2} d^2z' \left[\ q'_\sigma[D'^A j^{\log}_A + j^{\log}_A\ D'^A \log(-q.q')] - q'_\sigma\ \partial'_u[j^{\log}_r + j^4_r - j^{\log}_r \log(-q.q')]\ \right].$$

We have already checked that $j^{\log}_r \to 0$ as $|u| \to \infty$. Using similar logic it can be shown that $j^4_r \to 0$ as $|u| \to \infty$. We can rewrite the first term as

$$A_\sigma(u,r,\hat{x}) = \frac{1}{4\pi r u}\int_{-\infty}^{\infty} du' \int_{S^2} d^2z'\ q^\mu\ \frac{q'_{[\sigma} D'^A q'_{\mu]}}{q.q'}\ j^{\log}_A, \tag{C.10}$$

where, $q_{[\mu}\ D^A q_{\nu]} = q_\mu\ D^A q_\nu - q_\nu\ D^A q_\mu$. Finally we perform a co-ordinate transformation (using (2.2)) to get :

$$A^{0,1}_{\bar{z}}(u,\hat{x}) = \frac{1}{4\pi}\frac{\sqrt{2}}{1+z\bar{z}}\int_{-\infty}^{\infty} du' \int_{S^2} d^2z'\ \frac{\epsilon^\sigma_- q^\mu}{q.q'}\ q'_{[\sigma} D'^A q'_{\mu]}\ j^{\log}_A. \tag{C.11}$$

Thus we have showed that the log dressing give rise to a $1/u$ term in A^0_A such that the coefficient is given by (C.11). The expression for A_z can be obtained from the expression for $A_{\bar{z}}$ by replacing ϵ_- by ϵ_+. The polarisation vectors are given in (5.14).

Hard charge at past null infinity :

We recall that the charge at past is defined in terms of following mode $\frac{\log r}{r^2} F_{rA}(\hat{x})|_{\mathcal{I}^-_+}$ (5.1). To study this mode first we will expand Maxwell's equations in large r at finite v and take $v \to \infty$ limit in the solution. Around $\mathcal{I}^-$, the gauge field equation $\Box A_\mu = -j_\mu$ is :

$$[2\partial_v\partial_r + \frac{2}{r}\partial_v + \frac{2}{r}\partial_r + \partial_r^2 + \frac{D^2}{r^2}]\ A_\sigma = -j_\sigma.$$

Using the asymptotic expansion for current :

$$j_\mu = j^2_\mu \frac{1}{r^2} + j^{\log}_\mu \frac{\log r}{r^3} + \dots\ ,$$

we get :

$$A_\mu = A^{\log 1}_\mu \frac{\log r}{r} + A^1_\mu \frac{1}{r} + A^{\log 2}_\mu \frac{\log r}{r^2} + \dots\ .$$

The coefficients satisfy :

$$\begin{aligned} 2\partial_v A^{\log 1}_\sigma &= -j^2_\sigma, \\ -2\partial_v A^{\log 2}_\sigma + D^2 A^{\log 1}_\sigma &= -j^{\log}_\sigma. \end{aligned}$$

The $\frac{\log r}{r^2}$ term in F_{rA} comes from $A^{\log 2}_\sigma$. $A^{\log 1}_\sigma$ is $\mathcal{O}(e)$ term and does not contribute to F_{rA}. So, we ignore it henceforth :

$$A^{\log 2}_\sigma(x) = \frac{1}{2}\int_{-\infty}^{v} dv' \overset{\log}{j_\sigma}(v', \hat{x}). \tag{C.12}$$

In above solution, we have chosen the integration constant such that $A^{\log 2}_\sigma \to 0$ as $v \to -\infty$. With a co-ordinate transformation, we get :

$$F^{\log 0}_{rz}|_{\mathcal{I}^-_+} = -\frac{1}{2}\partial_z \bar{q}^\mu \int_{-\infty}^{\infty} dv' \overset{\log}{j_\mu}(v', \hat{x}). \tag{C.13}$$

This can be rewritten as :

$$\begin{aligned} F^{\log 0}_{rz}|_{\mathcal{I}^-_+} &= -\frac{1}{2}\int_{-\infty}^{\infty} dv' \overset{\log}{j_z}(v', \hat{x}), \\ &= -\frac{1}{2}\int_{-\infty}^{\infty} dv' d^2z'\ \delta^2(\hat{x}+\hat{x}') j_z(v', -\hat{x}'), \\ &= \frac{1}{4\pi}\ \int_{-\infty}^{\infty} dv' \int d^2z'\ D_z^2\ \Big[\frac{q^\nu\partial_{\bar{z}}q^\mu}{q.\bar{q}'}\ \bar{q}'_{[\mu}D'^A\bar{q}'_{\nu]}\ \overset{\log}{j_A}(v', -\hat{x}')\Big]. \end{aligned}$$

In the last line we have used an identity. Above expression can be rewritten as

$$F^{\log 0}_{rz}|_{\mathcal{I}^-_+} = \frac{1}{4\pi}\frac{\sqrt{2}}{1+z\bar{z}}\int_{-\infty}^{\infty} dv' \int_{S^2} d^2z'\ D_z^2\ \Big[\frac{\epsilon^\mu_- q^\nu}{q.\bar{q}'}\ \bar{q}'_{[\mu}D^A\bar{q}'_{\nu]}\ j^{\log}_A(v', -\hat{x}')\Big]. \tag{C.14}$$

The polarisation vectors are given in (5.14). $\bar{q}$ has been defined in (2.16).

Quantum modes in h^1_{rr} and A^1_r

Let us start with the expression for $\overset{quan}{A^1_r}$ given in (5.42) :

$$\begin{aligned} &< \text{out}|\overset{quan}{A^1_r}(x)\ S\ |\text{in}> \\ &= i < \text{out}|\int \frac{d^2 z'}{16\pi^3}\Big[\frac{\epsilon^-.q}{q'.q}\ \sum_j \eta_j e_j \frac{\epsilon^+.p_j}{q'.p_j}\ +\ \frac{\epsilon^+.q}{q'.q}\ \sum_j \eta_j e_j \frac{\epsilon^-.p_j}{q'.p_j}\Big]\ S\ |\text{in}> . \end{aligned} \tag{C.15}$$

Using completeness relations for polarisation tensors :

$$\frac{\epsilon^-.q}{q'.q}\ \sum_j \eta_j e_j \frac{\epsilon^+.p_j}{q'.p_j}\ +\ \frac{\epsilon^+.q}{q'.q}\ \sum_j \eta_j e_j \frac{\epsilon^-.p_j}{q'.p_j} = \sum_j \eta_j e_j \frac{q.p_j}{q.q'\ \ q'.p_j}. \tag{C.16}$$

Thus, in (C.15), we need to do following integral :

$$\begin{aligned} I = \int d^2 z'\ \frac{1}{q.q'\ \ q'.p_j} &= \int d^2 z'\ \int_0^1 dx\ \frac{1}{[\hat{q}'.(x\hat{q}+(1-x)\omega_j\hat{q}_j)-x-(1-x)\omega_j]^2}, \\ &= 2\pi\int_0^1 dx\ \frac{1}{[x(1-x)\omega_j(1-\hat{q}_j.\hat{q})]}. \end{aligned} \tag{C.17}$$

But I is divergent. These are collinear divergences that appear as we are dealing with massless particles. We will see that the diverging terms cancel and the charge is finite. Let us regulate the integral by introducing a regulator m_j by making p_j massive. Repeating previous steps for a massive p_j, we get :

$$I = \frac{4\pi}{q.p_j}\int_0^1 dx\ \frac{1}{[2x(1-x)+\frac{m_j^2}{q.p_j}(1-x)^2]}.$$

Thus, I still has divergence coming from $x = 1$. But we will see that $x = 1$ term vanishes due to conservation of momentum.

$$I = \frac{4\pi}{q.p_j}\ \frac{m_j^2-2q.p_j}{(-2q.p_j)}\ \Big[\lim_{x\to 1}\log(1-x)+\log[\frac{m_j^2}{m_j^2-2q.p_j}]\ \Big] \tag{C.18}$$

Let us study above expression in the limit when the regulator is taken to 0 :

$$\lim_{m_j\to 0} I = \frac{4\pi}{q.p_j}\ \Big[\lim_{x\to 1}\log(1-x)+\log[m_j^2]\ -\log[-2q.p_j]\ +...\Big]. \tag{C.19}$$

Here, '...' denote terms that vanish when regulator is set to 0. The infinite piece is as follows :

$$< \text{out}|\overset{quan}{A_r^1}(x)\ S\ |\text{in} > |_{inf} = \frac{i}{4\pi^2}\sum_j \eta_j e_j \Big[\lim_{x\to 1}\log(1-x) + \log[-\frac{m_j^2}{2}]\ \Big]$$
$$= \frac{i}{4\pi^2}\sum_j \eta_j e_j \log m_j^2. \tag{C.20}$$

Here, the first piece vanishes due to conservation of charge. We could have regulated the $x = 1$ divergence right from the beginning by introducing a mass for the null vector q^μ and gotten the same result for I. Thus, the infinite piece is a constant. We have for the finite part :

$$< \text{out}|\overset{quan}{A_r^1}(\hat{x})\ S\ |\text{in} > = -\frac{i}{4\pi^2}\ \sum_j \eta_j e_j\ \ \log(q.p_j). \tag{C.21}$$

Next we will repeat the calculation for the metric field. The infinite piece is as follows [101]

$$< \text{out}|\overset{quan}{h_{rr}^1}(x)\ S\ |\text{in} > |_{inf} = \frac{i}{2\pi^2}\sum_j \eta_j (q.p_j)\log[m_j^2]. \tag{C.22}$$

The finite piece is :

$$< \text{out}|\overset{quan}{h_{rr}^1}(x)\ S\ |\text{in} > = -\frac{i}{2\pi^2}\ \sum_j \eta_j (q.p_j)\ \ \log(q.p_j). \tag{C.23}$$

The soft charge in presence of gravity

Here we write down Maxwell's equations in presence of gravitational fluctuations given by (5.49).

Let us study the $\nabla^\mu F_{u\mu} = j_u$ equation. Expanding the equation around $r \to \infty$, at $\mathcal{O}(\frac{1}{r^2})$ we get :

$$\partial_u \overset{2}{F}_{ru} + \partial_u D^B A_B^0 - j_u^2 = -\gamma^{CB}\ h_{Cr}^0\ \partial_u F_{uB}^0. \tag{C.24}$$

In the equation $\nabla^\mu F_{A\mu} = 0$, there appears a gravity correction even at $\mathcal{O}(\frac{1}{r})$:

$$\partial_u \overset{1}{F}_{rA} - h^1_{rr}\ \partial_u \overset{0}{F}_{Au} = 0. \tag{C.25}$$

This implies $\log r$ dressing of A_A that has also been discussed in (5.56). $\nabla^\mu F_{A\mu} = 0$ at $\mathcal{O}(\frac{1}{r^2})$ gives :

$$\begin{aligned}
&2\partial_u \overset{2}{F}_{rA} - \partial_A \overset{2}{F}_{ru} + D^B F^0_{AB} - j^2_A \\
&= h^1_{rr}\ \partial_u \overset{1}{F}_{Au} + h^2_{rr}\ \partial_u \overset{0}{F}_{Au} - \gamma^{CB}\ h^0_{Cr}\ \partial_u F^0_{AB} - \gamma^{CB}\ h^0_{Cr}\ D_B F^0_{Au} - \gamma^{BC} h^{-1}_{AB} F^0_{uC} \\
&+ \partial_u h^2_{rr}\ \overset{0}{F}_{Au} + \frac{1}{2} h^1_{rr}\ \overset{0}{F}_{Au} - \frac{1}{2}\gamma^{BC} h^{-1}_{BC} F^0_{Au} - \gamma^{BC} D_B h^0_{Ar}\ F^0_{uC} - D^B h^0_{Br}\ F^0_{Au} - 2h^{1\ ur}\ F^0_{Au}.
\end{aligned} \tag{C.26}$$

We use above equation to substitute for $\partial_u \overset{2}{F}_{rA}$ in (5.20) i.e. in

$$Q^{\text{soft}}_+ = -\int du\ d^2z'\ V^A \partial_u\ \left[u^2 \partial_u^2 \overset{2}{F}_{rA}\right], \tag{C.27}$$

and we get (5.21) i.e.

$$Q^{\text{soft}}_+ = -\frac{1}{2}\int du\ d^2z'\ V^A \partial_u\ \left[u^2 \partial_u [\partial_A \overset{2}{F}_{ru} - D^B F^0_{AB} + j^2_A]\ \right] + ..., \tag{C.28}$$

where "..." refers to the gravity corrections that come from RHS of (C.26) and (C.24). We will analyse them one by one. Out of the metric components appearing in Maxwell's equations only h^2_{rr} and h^{-1}_{AB} depend on u, rest of them are u-independent. This simplifies the analysis for most of the terms.

Term $h^1_{rr}\ \partial_u F^1_{Au}$

$$Q^{\text{cor}}_1 = -\frac{1}{2}\int du\ d^2z\ V^z \partial_u\ \left[u^2 \partial_u [h^1_{rr}\ \partial_u \overset{1}{F}_{uz}]\right]\ +\ z \leftrightarrow \bar{z}. \tag{C.29}$$

Using Bianchi identities we can simplify above expression to :

$$Q^{\text{cor}}_1 = -\frac{1}{2}\int d^2z\ V^z \int du\ h^1_{rr}\ \partial_u\ \left[u^2 \partial_u^2 D_z^2 A^0_{\bar{z}}\right] +\ z \leftrightarrow \bar{z}. \tag{C.30}$$

The operator picks out difference between boundary values of $\log u$ piece of A_A which is 0.

Term $h^2_{rr}\ \partial_u F^0_{uA}$

$$Q_2^{\rm cor} = -\frac{1}{2}\int du\ d^2z\ V^A \partial_u\ \Big[u^2\partial_u[h^2_{rr}\ \partial_u \overset{0}{F}_{uA}]\Big]. \tag{C.31}$$

h^2_{rr} has atmost a $\mathcal{O}(u)$ term. Using the u-behaviour of $\overset{0}{F}_{uA}$ we see that this term is also 0.

Term $\gamma^{BC}h^{-1}_{BA}\ F^0_{uC}$

$$Q_3^{\rm cor} = \frac{1}{2}\int du\ d^2z\ V^A \partial_u\ \Big[u^2\partial_u[\gamma^{BC}h^{-1}_{BA}\ F^0_{uC}]\Big].$$

This term vanishes trivially for classical fall offs of F^0_{uC}. For the quantum $\log u$ fall offs we get 2 terms for A=z (the analysis is similar for $A=\bar{z}$) :

$$Q_3^{\rm cor} = -\frac{1}{2}\int du\ d^2z\ V^z\gamma^{\bar{z}z}\ \partial_u h^{-1}_{zz}\ A^{0,\log}_{\bar{z}} + \frac{1}{2}\int du\ d^2z\ V^z\gamma^{\bar{z}z}\ \partial_u A^0_{\bar{z}}\ h^{-1,\log}_{zz}\ . \tag{C.32}$$

Upto unimportant overall factors that are common to both terms, the first integrand is : $\lim_{\omega\to 0}\omega[c_+(\omega)+c^\dagger_-(\omega)]\ \lim_{\omega\to 0}\omega[a_-(\omega)-a^\dagger_+(\omega)]$. Similarly the second integrand is : $\lim_{\omega\to 0}\omega[c_+(\omega)-c^\dagger_-(\omega)]\ \lim_{\omega\to 0}\omega[a_-(\omega)+a^\dagger_+(\omega)]$. Thus, $Q_3^{\rm cor}=0$.

Term $h^0_{rC}\ \partial_u F^0_{AB}$

$$\begin{aligned} Q_4^{\rm cor} &= \frac{1}{2}\int du\ d^2z\ V^A \partial_u\ \Big[u^2\partial_u[\gamma^{CB}\ h^0_{Cr}\ \partial_u \overset{0}{F}_{AB}]\Big], \\ &= \frac{1}{2}\int du\ d^2z\ \gamma^{CB}\ h^0_{Cr}\ V^A\partial_u\ [u^2\partial_u^2\partial_{(B}A^0_{A)}]. \end{aligned} \tag{C.33}$$

This is similar to (C.30) and vanishes by same logic. The analysis for rest of the terms is exactly similar.

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